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Introduction to Cryptography *CPSC 418* *Fall 2016*
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HOME WORK #2

Problem	Marks
1	
2	
3	
4	
5	
6	
7	
Total	

Problem 1. Conditional entropy

1.a)
$\boldsymbol{H(M|C) = \sum_{c \in C} p(C) \sum_{m \in M} p(M|C) log_2(\frac{1}{p(M|C)})}$
$\boldsymbol{H(M|C) = \sum p(C) \sum p(M|C) log_2(\frac{1}{p(M|C)})}$
$\boldsymbol{= 4 * \frac{1}{4}(\frac{1}{2} log_2(2) + \frac{1}{2} log_2(2))}$
$\boldsymbol{= 1}$

1.b)
$\boldsymbol{H(M|C) = \sum p(C) \sum p(M|C) log_2(\frac{1}{p(M|C)})}$

Since the cryptosystem provides perfect secrecy, $\boldsymbol{p(M|C) = p(M)}$.

$\boldsymbol{= \sum p(C) \sum p(M) log_2(\frac{1}{p(M)})}$

We know $\boldsymbol{\sum p(M) log_2(\frac{1}{p(M)}) = log_2(\frac{1}{p(M)})}$, when a cryptosystem provides perfect secrecy.

$= \sum \boldsymbol{p(C) log_2(\frac{1}{p(M)})}$

With perfect secrecy, every M is equiprobable, so every C is equiprobable. Since $|\boldsymbol{C}| = |\boldsymbol{M}|$ (as every unique C comes from encrypting some unique M), so we have p(C) = p(M).

Thus,

$= \sum \boldsymbol{p(M) log_2(\frac{1}{p(M)})}$
$= \boldsymbol{H(M)}$

1.c)
No, since $\boldsymbol{p(M|C) = \frac{1}{2} \neq \frac{1}{4} = p(M)}$.

$\longrightarrow$ $\mathcal{A}$nswer

Problem 2. Binary polynomial arithmetic

2.a.i)
$\boldsymbol{x^3}$
$\boldsymbol{x^3 + 1}$
$\boldsymbol{x^3 + x}$
$\boldsymbol{x^3 + x + 1}$
$\boldsymbol{x^3 + x^2}$
$\boldsymbol{x^3 + x^2 + 1}$
$\boldsymbol{x^3 + x^2 + x}$
$\boldsymbol{x^3 + x^2 + x + 1}$

2.a.ii)
$\boldsymbol{x^3 = x * x * x}$
$\boldsymbol{x^3 + 1 = (x + 1)(x^2 - x + 1)}$
$\boldsymbol{x^3 + x = x(x^2 + x)}$
$\boldsymbol{x^3 + x + 1 =}$ irreducible
$\boldsymbol{x^3 + x^2 = x^2(x + 1)}$
$\boldsymbol{x^3 + x^2 + 1 =}$ irreducible
$\boldsymbol{x^3 + x^2 + x = x(x^2 + x + 1)}$
$\boldsymbol{x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1)}$

2.a.iii)
Let $\boldsymbol{A(x)}$ be a degree 3 polynomial. If $\boldsymbol{A(x)}$ is reducible, then it must the product of a degree 1 polynomial and some other polynomial(s) (of degree 2 or 1). In either case, there is a polynomial of degree 1 as a factor. The two possible polynomials of degree 1 are $\boldsymbol{P_1 = x + 1}$ and $\boldsymbol{P_2 = x}$. If A is reducible, then either $\boldsymbol{P_1}$ or $\boldsymbol{P_2}$ is a factor of A. Notice $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ are respectively equal to zero when $\boldsymbol{x = -1}$ or $\boldsymbol{x = 0}$. If $\boldsymbol{A(x)}$ is reducible with $\boldsymbol{P_1}$ as a factor, then $\boldsymbol{A(-1) = 0}$. If $\boldsymbol{A(x)}$ is reducible with $\boldsymbol{P_2}$ as a factor, then $\boldsymbol{A(0) = 0}$. Otherwise $\boldsymbol{A(x)}$ is irreducible.

Let $A_1(x) = x^3 + x + 1$, then
$A_1(0) = 0 + 0 + 1 = 1$ and $A_1(-1) = -1 - 1 + 1 = -1$

Let $A_2(x) = x^3 + x^2 + 1$, then
$A_2(0) = 0 + 0 + 1 = 1$ and $A_2(-1) = -1 + 1 + 1 = 1$

Neither $A_1(x)$ or $A_2(x)$ have P_1 or P_2 as factors, and are therefore irreducible.

2.b.i)
Since $x^4 + x + 1 \equiv 0 \pmod{x^4 + x + 1}$
$x^4 \equiv x + 1 \pmod{x^4 + x + 1}$
$x^5 \equiv x^2 + x \pmod{x^4 + x + 1}$
$x^6 \equiv x^3 + x^2 \pmod{x^4 + x + 1}$

$f(x)g(x) = (x^2 + 1)(x^3 + x^2 + 1)$
$= x^5 + x^3 + x^4 + x^2 + x^2 + 1$
$= x^5 + x^3 + x^4 + 1$
$= (x^2 + x) + (x + 1) + x^3 + 1$
$= x^3 + x^2$

2.b.ii)
Since $x^4 + x + 1 \equiv 0 \pmod{x^4 + x + 1}$
$x^4 + x \equiv 1 \pmod{x^4 + x + 1}$
$x(x^3 + 1) \equiv 1 \pmod{x^4 + x + 1}$

So given $f(x) = x$, then $f^-1(x) = (x^3 + 1)$

d.i)
$y * [ay^3 + by^2 + cy + d)]$
$= ay^4 + by^3 + cy^2 + dy$
$= by^3 + cy^2 + dy + a$ (since $y^4 = 1$)

d.ii)
Prove that in this arithmetic, $y^i = y^j$ for any integer $i \geq 0$, where $j \equiv i \pmod 4$ with $0 \leq j \leq 3$

Base cases:
$i = 0$: $y^i = y^j$ since $0 \equiv 0 \pmod 4$ and $j == 0$
$i = 1$: $y^i = y^j$ since $1 \equiv 1 \pmod 4$ and $0 \leq j \leq 3$
$i = 2$: $y^i = y^j$ since $2 \equiv 2 \pmod 4$ and $0 \leq j \leq 3$
$i = 3$: $y^i = y^j$ since $3 \equiv 3 \pmod 4$ and $j == 3$

Induction Hypothesis:
Assume that $y^i = y^j$ for all $i \in \mathbb{Z}$, where $i \geq 0$ such that $j = i \pmod 4$ and $0 \leq j \leq 3$.

Suppose $4 \leq k$, where $k \in \mathbb{Z}$.
Since $(k + 1) \geq 0$ and $(k + 1) \in \mathbb{Z}$, by the induction hypothesis, we have $y^{k+1} = y^j$ where $j = (k + i) \pmod 4$ and $0 \leq j \leq 3$.

d.iii)
Let $ay^3 + by^2 + cy + d$ represent any 4-byte vector as polynomial.

Base case:
$i = 0$: $y^0(ay^3 + by^2 + cy + d) = ay^3 + by^2 + cy + d$
No bytes have been shifted, so this is a circular left shift of 0 bytes. Using the proof from *d.ii* we get that this is a shift of $j = 0$ bytes.
$i = 1$: $y^1(ay^3 + by^2 + cy + d)$ Using the proofs from d.i and d.ii, this is a left circular shift of $j = 1$ bytes.

Induction Hypothesis:
Assume for $i \geq 0$ that $y^i(ay^3 + by^2 + cy + d) = ay^{3+i} + by^{2+i} + cy^{1+i} + dy^i$ is a left circular shift of j bytes where $j = i \pmod 4$ and $j \geq 0$.

Suppose $k \in \mathbb{Z}$ and $k \geq 2$.

$y^{k+1}(ay^3 + by^2 + cy + d) = y^k(y^1(ay^3 + by^2 + cy + d))$
$= y^k(ay^{3+1} + by^{2+1} + cy^{1+1} + dy^1)$
$= ay^{3+(k+1)} + by^{2+(k+1)} + cy^{1+(k+1)} + dy^{k+1}$ (by IH)

Since $k + 1 \geq 0$ and $y^{k+1} = y^j$ where $j = k + 1 \pmod 4$ and $0 \leq j \leq 3$, the multiplication of any 4-byte vector with y^{k+1} is a left circular shift of j bytes.

$\longrightarrow$ $\mathcal{A}$nswer

Problem 3. Arithmetic with the constant polynomial of MixColumns in AES

3.a)
$c(1) = 1$
$c(2) = x$
$c(3) = x + 1$

b.i) $(01)(b) = 1(b_7x^7 ... + b_1x + b_0)$
$d_i = b_i$

b.ii)
$x^8 \equiv x^4 + x^3 + x + 1 \pmod{x^8 + x^4 + x^3 + x + 1}$
$(02)(b) = x(b_7x^7 + ... + b_1x + b_0)$
$= b_7x^8 + b_6x^7 ... + b_1x^2 + b_0x)$
$= b_7(x^4 + x^3 + x + 1) + b_6x^7 + ... + b_1x^2 + b_0x)$
$d = b_6x^7 + b_5x^6 + b_4x^5 + (b_7 + b_3)x^4 + (b_7 + b_2)x^3 + b_1x^2 + (b_7 + b_0)x + b_7$
$d_7 = b_6$
$d_6 = b_5$
$d_5 = b_4$
$d_4 = b_7 \oplus b_3$
$d_3 = b_7 \oplus b_2$

$d_2 = b_1$
$d_1 = b_7 \oplus b_0$
$d_0 = b_7$

b.iii)
$(03)(b) = (x+1)(b_7x^7 + ... + b_1x + b_0)$
$= (b_7x^8 + b_6x^7 + ... + b_1x^2 + b_0x) + (b_7x^7 + ... + b_1x + b_0)$
$= b_7(x^4 + x^3 + x + 1) + (b_6 \oplus b_7)x^7 + ... + (b_0 \oplus b_1)x + b_0)$
$d_7 = b_6 \oplus b_7$
$d_6 = b_5 \oplus b_6$
$d_5 = b_4 \oplus b_5$
$d_4 = b_3 \oplus b_4 \oplus b_7$
$d_3 = b_2 \oplus b_3 \oplus b_7$
$d_2 = b_1 \oplus b_2$
$d_1 = b_0 \oplus b_1 \oplus b_7$
$d_0 = b_0 \oplus b_7$

c.i)
$y^4 \equiv 1 \pmod{y^4+1}$
$y^5 \equiv y \pmod{y^4+1}$
$y^6 \equiv y^2 \pmod{y^4+1}$

$t(y) = c(y)s(y) \pmod{y^4+1}$
$= [(03)y^3 + (01)y^2 + (01)y + (02)](s_3y^3 + s_2y^2 + s_1y + s_0) \pmod{y^4+1}$
$= (03)(s_3y^6 + s_2y^5 + s_1y^4 + s_0y^3)$
$+(01)(s_3y^5 + s_2y^4 + s_1y^3 + s_0y^2)$
$+(01)(s_3y^4 + s_2y^3 + s_1y^2 + s_0y)$
$+(02)(s_3y^3 + s_2y^2 + s_1y + s_0) \pmod{y^4+1}$

$= (03)s_3y^6$
$+((03)s_2 + (01)s_3)y^5$
$+((03)s_1 + (01)s_2 + (01)s_3)y^4$
$+((03)s_0 + (01)s_1 + (01)s_2 + (02)s_3)y^3$
$+((01)s_0 + (01)s_1 + (02)s_2)y^2$
$+((01)s_0 + (02)s_1)y$
$+(02)s_0 \pmod{y^4+1}$

$= (03)s_3y^2$
$+((03)s_2 + (01)s_3)y$
$+((03)s_1 + (01)s_2 + (01)s_3)$
$+((03)s_0 + (01)s_1 + (01)s_2 + (02)s_3)y^3$
$+((01)s_0 + (01)s_1 + (02)s_2)y^2$
$+((01)s_0 + (02)s_1)y$
$+(02)s_0 \pmod{y^4+1}$

$= ((03)s_0 + (01)s_1 + (01)s_2 + (02)s_3)y^3$
$+((01)s_0 + (01)s_1 + (02)s_2 + (03)s_3)y^2$
$+((01)s_0 + (02)s_1 + (03)s_2 + (01)s_3)y$
$+((02)s_0 + (03)s_1 + (01)s_2 + (01)s_3) \pmod{y^4+1}$

c.ii)

$$\boldsymbol{C} = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

$\longrightarrow$ $\mathcal{A}$nswer

Problem 4. Error propagation in block cipher modes

Let $\boldsymbol{C_i}$ denote the encryption of the i-th message block, $\boldsymbol{M_i}$.
Let $\boldsymbol{P_i}$ denote the plaintext obtained from the decryption of $\boldsymbol{C_i}$.

a.i)
ECB: Only $\boldsymbol{P_i}$ is affected since the decryption of each block is just a simple block substitution(independent of each other).
ii)
CBC: $\boldsymbol{P_i}$ and $\boldsymbol{P_{i+1}}$ are affected since $\boldsymbol{C_i}$ will be combined with $\boldsymbol{P_{i+1}}$ via an exclusive-OR.
iii)
OFB: Only $\boldsymbol{P_i}$ is affected since any given decryption state has been generated exclusively from the previous state (eventually originating from the IV).
iv)
CFB: When using only one register, $\boldsymbol{P_i}$ and $\boldsymbol{P_{i+1}}$ are affected since only the last $\boldsymbol{k}$ previous ciphertext blocks (in this case $\boldsymbol{k = 1}$) are kept in the register for decryption.
v)
CTR: Only $\boldsymbol{P_i}$ is affected since $\boldsymbol{CTR_i}$ (the source of the output block that gets XORed with $\boldsymbol{C_i}$) is simply an independent counter value.

b)
All message blocks following (and including) $\boldsymbol{M_i}$ are affected, since in CBC mode $\boldsymbol{C_i}$ is used in the encryption of $\boldsymbol{C_{i+1}}$ (and then $\boldsymbol{C_{i+1}}$ gets used when encrypting $\boldsymbol{C_{i+2}}$ and so on).

$\longrightarrow$ $\mathcal{A}$nswer

Submitted by Brian Yee - 00993104 on October 26, 2016.