Can we make Impermanent Loss Impermanent Again? (DRAFT)

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1 Introduction

We suggest a method whereby liquidity providers (LPs) on a constant function market maker might avoid losing money due to arbitrageurs. The model adapts the classic setting from Blackwell [1956], treating the profitability question as equivalent to asking whether arbitrageurs can *exclude* LPs from certain payoff regions. We find cases where the LP could be excluded from profitability if they use any standard strategy, but if they instead adopt a strategy called "veiling" they can avoid being excluded.

2 LPs and Traders

Assume that LPing is a game where you (the "LP") put resources into a pool and then continuously accept fee-paying trades. You want to assure yourself some minimal payoff (inclusive of the fees you charge). However, as in Milionis et al. [2022], a trader may be able to *exclude* you from achieving that payoff under certain market conditions.

Below is a simple model in which a trader can normally prevent the LP from obtaining positive payoffs. We then show how an LP can avoid this by using veiling, e.g. to set their fees.

Setup. Consider the following two-player game:

- The LP (row player) has two pure strategies in setting fees, denoted $i = High, Low$.
- The trader (column player) has two pure strategies in investment, denoted $j = Balanced, Uninformed$.
- The one-stage payoffs to the players are given in the bimatrix

	$Balanced$	$Uninformed$
$High$	(3, 1)	(−2, 2)
Low	(−3, 3)	(2, 0)

If the trader is more "balanced" (i.e. between informed and uninformed trades) and LP sets High fees, they extract value while limiting their exploitative trading. Whereas if the trader is more "uninformed" and LP set High fees, uninformed traders are priced out and the mix is more extractive.

Low fees against a "balanced" trader, means more exploitation from arbitrageurs with less revenue from uninformed flow to cover the loss. If you set Low fees against an "uninformed" trader, you keep them happy and get flow that is profitable overall.

2.1 Profitability and Exclusion

We treat the impermanent loss question here as being about whether the LP can achieve a target payoff region $T > 0$ in this game. If the LP can't reach T, they are unprofitable in expectation.

Formally, following Blackwell [1956], we say that a set $T \subset \mathbb{R}$ (of possible payoffs to the LP) is *excludable* if there is a strategy $y^{(0)}$ for the arbitrageur such that, no matter what strategy x the LP chooses, the resulting expected payoff

$$c\big(x, y^{(0)}\big) \ \notin \ T.$$

In our game we find that the LP is excluded from profitability region T under normal play. But if they can implement a "Veiled Strategy" — i.e., a commitment strategy that conditions on a non-convergent random sequence — they are not excluded.

AMM LP Excluded if Predictable

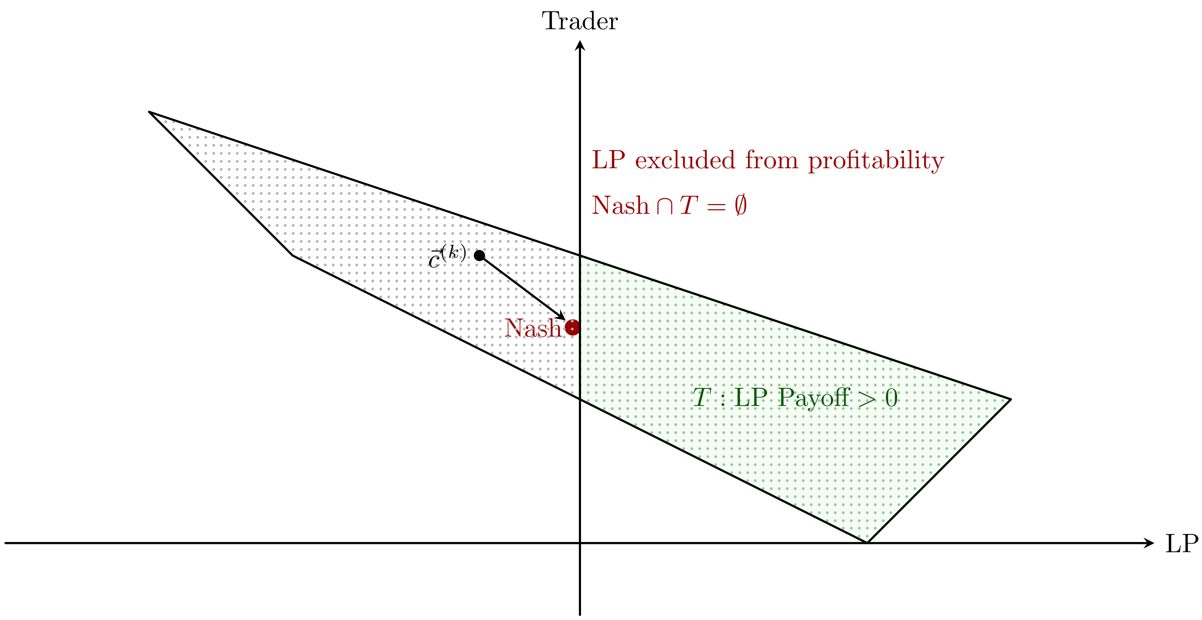

Figure 1: As in Blackwell, the LP starts from some point $\vec{c}^{(k)}$ and seeks a strategy that reaches the "profitability" region T. This figure shows they have no (mixed or pure) strategy which crosses the hyperplane tangent to the shortest distance between $\vec{c}^{(k)}$ and T then region T, and thus T is "excludable" by the adversary/arbitrageur. The above depicts exclusion according to a rational response from the arbitrageur as in the game 2.

2.2 Case 1: A Standard Mixed Strategy

We first illustrate that if the LP uses a *standard* mixed strategy converging to a single probability p over $\{1, 2\}$, then the arbitrageur can find a counter-strategy that excludes T.

$$\text{LP's payoff} \ = \ 3\,p\,q \ + \ (-2)\,p\,(1-q) \ + \ (-3)\,(1-p)\,q \ + \ 2\,(1-p)\,(1-q),$$

$$\text{Arbitrageur's payoff} \ = \ 1\,p\,q \ + \ 2\,p\,(1-q) \ + \ 3\,(1-p)\,q \ + \ 0\,(1-p)\,(1-q).$$

Thus, the unique mixed Nash equilibrium in the one-shot game occurs at

$$p^* = 0.75, \quad q^* = 0.40,$$

with the LP's equilibrium payoff equal to 0 and the arbitrageur's payoff equal to 1.5. Thus the LP's payoff is not in T.

LP Not Excluded if (Boundedly) Unpredictable

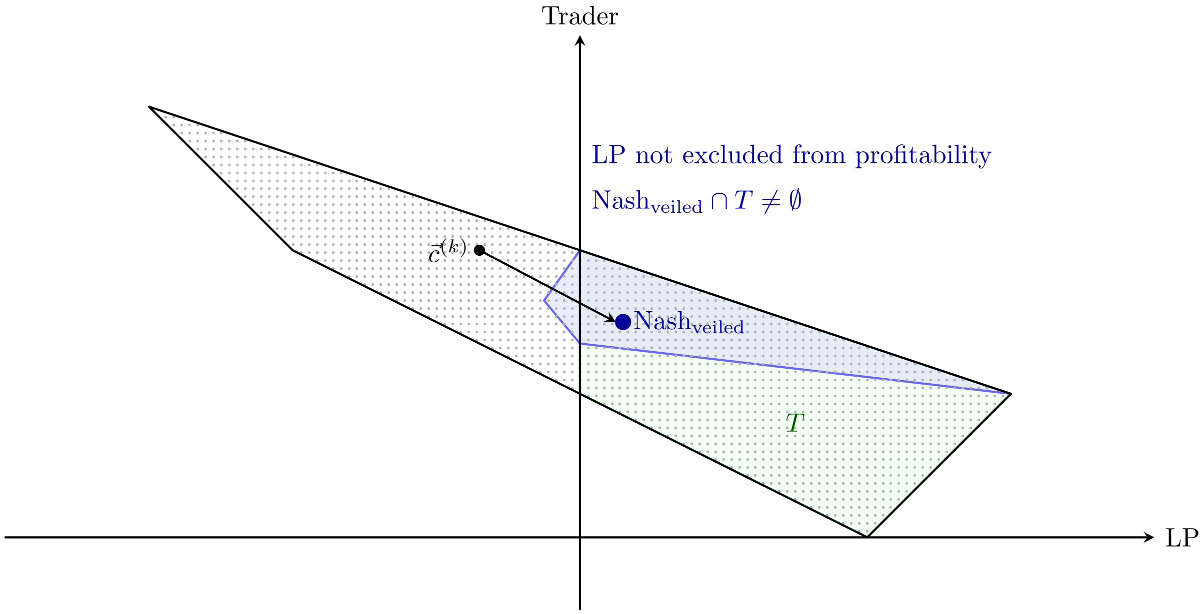

Figure 2: As was depicted in figure 1, the LP starts from $\vec{c}^{(k)}$ and attempts to reach "profitability" region T. By playing a non-convergent strategy, they are able to prevent exclusion in many possible states. As in the previous figure, the visualization corresponds to the game in 2.

2.3 Case 2: A Veiled (Non-convergent) Strategy

Now consider that the LP employs a strategy that does not converge to a single p. Such a strategy can employed by conditioning an action on a draw from an encrypted non-convergent sequence, e.g. Machine II (see A for more details.) A commitment to resolve based on such a draw is said to be "veiling" one's action (we here consider that the LP veils their fees.)

Figure 2 provides a visual intuition for payoffs under veiling, but to precisely characterize the efficient gains we require recourse to a multi-preference utility function. Keeney and Raiffa [1993] establish that under minimal conditions (weak monotonicity and independence in payoffs) each player will evaluate endpoints of these payoffs, and weight them additively or multiplicatively. For simplicity we assume an additive evaluation where both the LP and the trader consider the endpoints (as perfect substitutes.[1]

In a veiled equilibrium, the LP sets their lower probability of charging a high fee at $\underline{p}$ and their upper probability of a high fee at $\overline{p}$. It then suffices for the trader to play a simple mixed strategy, but for clarity we will have them best respond with a veiled strategy as well, with lower probability of "balanced" trading at $\underline{q}$ and upper probability $\overline{q}$.

A pareto optimal veiled equilibrium is observed with the following probabilities:

[1]Which is to say that they exhibit no special preference for or against uncertainty itself.

$$\underline{p} = 0.5, \quad \overline{p} = 1.0 \tag{1}$$
$$\underline{q} = 0.35, \quad \overline{q} = 1.0 \tag{2}$$

Some (very tedious) corner checks are relegated to appendix B which establish this as an equilibrium, resulting in payoffs:

$$\pi_{LP}^{veil} = 1.375 > \pi_{LP}^{Nash} = 0 \tag{3}$$
$$\pi_{Trader}^{veil} = 1.5 = \pi_{Trader}^{Nash} = 0 \tag{4}$$

The LP has raised their payoff from 0 to 1.375 by veiling, while the trader is no worse off (receiving 1.5 just as they had under the mixed strategy Nash.) See 2 for visual intuition about the set of outcomes created under this veiled strategy equilibrium.

Appendix

A Non-convergent Sequences via Machine II

Definition 1 (Machine II Sequence). *A sequence of binary outputs*

$$(x_k)_{k=1}^{\infty} \quad \textit{where each } x_k \in \{0,1\},$$

is said to be of Machine II type *if, for each fraction p in the unit interval $[0,1]$ and for every positive integer N, there is a (contiguous) block of length* at least N *whose average value (empirical frequency of 1's) is arbitrarily close to p.*

In other words, one can always find a block of the form $(x_{m+1}, x_{m+2}, \ldots, x_{m+r})$, of length $r \geq N$, such that

$$\frac{1}{r}\sum_{\ell=m+1}^{m+r} x_\ell \ \approx \ p \quad \textit{(to within any desired tolerance).}$$

B Deriving the Pareto Optimal Veiled Equilibrium

Let us denote individual payoffs as follows:

$$a = 3 \qquad \text{(Row payoff: Up, Left)} \tag{5}$$
$$b = -2 \qquad \text{(Row payoff: Up, Right)} \tag{6}$$
$$c = -3 \qquad \text{(Row payoff: Down, Left)} \tag{7}$$
$$d = 2 \qquad \text{(Row payoff: Down, Right)} \tag{8}$$
$$w = 1 \qquad \text{(Column payoff: Up, Left)} \tag{9}$$
$$x = 2 \qquad \text{(Column payoff: Up, Right)} \tag{10}$$
$$y = 3 \qquad \text{(Column payoff: Down, Left)} \tag{11}$$
$$z = 0 \qquad \text{(Column payoff: Down, Right)} \tag{12}$$

The LP's payoff function can be factorized to this form:

$$\pi_1(p,q) = A - B(p-C)(q-D) \tag{13}$$

For this game, we compute the parameters:

$$\text{row\_determinant} = a - b - c + d = 3 - (-2) - (-3) + 2 = 10 \tag{14}$$
$$C = \frac{c-d}{\text{row\_determinant}} = \frac{-3-2}{10} = -\frac{1}{2} \tag{15}$$
$$D = \frac{b-d}{\text{row\_determinant}} = \frac{-2-2}{10} = -\frac{2}{5} \tag{16}$$
$$B = -\text{row\_determinant} = -10 \tag{17}$$
$$A = d + B \cdot C \cdot D = 2 + (-10) \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{2}{5}\right) = 2 + 2 = 4 \tag{18}$$

Thus, the Row player's payoff function is:

$$\pi_1(p,q) = 4 - 10\left(p - \left(-\frac{1}{2}\right)\right)\left(q - \left(-\frac{2}{5}\right)\right) \tag{19}$$
$$= 4 - 10\left(p + \frac{1}{2}\right)\left(q + \frac{2}{5}\right) \tag{20}$$

Similarly, the Column player's payoff function is $\pi_2(\underline{p},\underline{q}) = E + F(p-G)(q-H)$ with:

$$\text{col\_determinant} = w - x - y + z = 1 - 2 - 3 + 0 = -4 \tag{21}$$
$$G = \frac{y-z}{\text{col\_determinant}} = \frac{3-0}{-4} = -\frac{3}{4} \tag{22}$$
$$H = \frac{x-z}{\text{col\_determinant}} = \frac{2-0}{-4} = -\frac{1}{2} \tag{23}$$
$$F = \text{col\_determinant} = -4 \tag{24}$$
$$E = z - F \cdot G \cdot H = 0 - (-4) \cdot \left(-\frac{3}{4}\right) \cdot \left(-\frac{1}{2}\right) = 0 - 1.5 = -1.5 \tag{25}$$

The Column player's payoff function is:

$$\pi_2(\underline{p},\underline{q}) = -1.5 - 4\left(p - \left(-\frac{3}{4}\right)\right)\left(q - \left(-\frac{1}{2}\right)\right) \tag{26}$$
$$= -1.5 - 4\left(p + \frac{3}{4}\right)\left(q + \frac{1}{2}\right) \tag{27}$$

$$\pi_1(p,q) = A - B(p-C)(q-D) \tag{28}$$

B.1 Veiled Equilibrium Analysis

For the specific equilibrium of interest, we have:

$$\underline{p} = 0.5, \quad \overline{p} = 1.0 \tag{29}$$
$$\underline{q} = 0.35, \quad \overline{q} = 1.0 \tag{30}$$

Converting to deviations from Nash $(p^*, q^*) = (0.75, 0.4)$:

$$x = \underline{p} - p^* = 0.5 - 0.75 = -0.25 \tag{31}$$
$$X = \overline{p} - p^* = 1.0 - 0.75 = 0.25 \tag{32}$$
$$y = \underline{q} - q^* = 0.35 - 0.4 = -0.05 \tag{33}$$
$$Y = \overline{q} - q^* = 1.0 - 0.4 = 0.6 \tag{34}$$

Notice that $x \leq 0 \leq X$ and $y \leq 0 \leq Y$, which means this strategy profile spans the Nash equilibrium.

B.1.1 Equilibrium Conditions for Veiled Strategies

First, calculate the row and column adjustments:

$$\text{row\_adjust} = C - p^* = -\frac{1}{2} - 0.75 = -1.25 \tag{35}$$
$$\text{col\_adjust} = H - q^* = -\frac{1}{2} - 0.4 = -0.9 \tag{36}$$

Row player's equilibrium condition:

$$\frac{x + \text{row\_adjust}}{X + \text{row\_adjust}} \geq \max\left(\frac{y}{Y}, \frac{Y}{y}\right) \quad \text{and} \quad \max\left(\frac{y}{Y}, \frac{Y}{y}\right) \geq -1 \tag{37}$$
$$\frac{-0.25 + (-1.25)}{0.25 + (-1.25)} \geq \max\left(\frac{-0.05}{0.6}, \frac{0.6}{-0.05}\right) \tag{38}$$
$$\frac{-1.5}{-1.0} \geq \max\left(\frac{-0.05}{0.6}, \frac{0.6}{-0.05}\right) \tag{39}$$
$$1.5 \geq \max(-0.083, -12) \tag{40}$$
$$1.5 \geq -0.083 \quad \checkmark \tag{41}$$

Column player's equilibrium condition:

$$\frac{Y - \text{col\_adjust}}{y - \text{col\_adjust}} \geq \max\left(\frac{x}{X}, \frac{X}{x}\right) \quad \text{and} \quad \max\left(\frac{x}{X}, \frac{X}{x}\right) \geq -1 \tag{42}$$
$$\frac{0.6 - (-0.9)}{-0.05 - (-0.9)} \geq \max\left(\frac{-0.25}{0.25}, \frac{0.25}{-0.25}\right) \tag{43}$$
$$\frac{1.5}{0.85} \geq \max(-1, -1) \tag{44}$$
$$1.765 \geq -1 \quad \checkmark \tag{45}$$

Both conditions are satisfied, confirming this is a valid veiled equilibrium.

B.2 Payoffs in the veiled Equilibrium

Using the Hurwicz criterion with $\alpha = 0.5$ (arithmetic mean between best and worst cases), we calculate the expected payoffs.

For the Row player, payoffs at the four extreme points are:

$$\pi_1(\underline{p},\underline{q}) = \pi_1(0.5, 0.35) = 0.5 \cdot 0.35 \cdot 3 + 0.5 \cdot 0.65 \cdot (-2) + 0.5 \cdot 0.35 \cdot (-3) + 0.5 \cdot 0.65 \cdot 2 \tag{46}$$
$$= 0.525 - 0.65 - 0.525 + 0.65 = 0 \tag{47}$$
$$\pi_1(\underline{p},\overline{q}) = \pi_1(0.5, 1.0) = 0.5 \cdot 1.0 \cdot 3 + 0.5 \cdot 0 \cdot (-2) + 0.5 \cdot 1.0 \cdot (-3) + 0.5 \cdot 0 \cdot 2 \tag{48}$$
$$= 1.5 + 0 - 1.5 + 0 = 0 \tag{49}$$
$$\pi_1(\overline{p},\underline{q}) = \pi_1(1.0, 0.35) = 1.0 \cdot 0.35 \cdot 3 + 1.0 \cdot 0.65 \cdot (-2) + 0 \cdot 0.35 \cdot (-3) + 0 \cdot 0.65 \cdot 2 \tag{50}$$
$$= 1.05 - 1.3 + 0 + 0 = -0.25 \tag{51}$$
$$\pi_1(\overline{p},\overline{q}) = \pi_1(1.0, 1.0) = 1.0 \cdot 1.0 \cdot 3 + 1.0 \cdot 0 \cdot (-2) + 0 \cdot 1.0 \cdot (-3) + 0 \cdot 0 \cdot 2 \tag{52}$$
$$= 3 + 0 + 0 + 0 = 3 \tag{53}$$

Thus:

$$\text{row\_min} = \min(0, 0, -0.25, 3) = -0.25 \tag{54}$$
$$\text{row\_max} = \max(0, 0, -0.25, 3) = 3 \tag{55}$$

With $\alpha = 0.5$, the Row player's expected Hurwicz payoff is:

$$\pi_1^H = \alpha \cdot \text{row\_max} + (1-\alpha) \cdot \text{row\_min} \tag{56}$$
$$= 0.5 \cdot 3 + 0.5 \cdot (-0.25) \tag{57}$$
$$= 1.5 - 0.125 = 1.375 \tag{58}$$

Similarly for the Column player:

$$\pi_2(\underline{p},\underline{q}) = \pi_2(0.5, 0.35) = 0.5 \cdot 0.35 \cdot 1 + 0.5 \cdot 0.65 \cdot 2 + 0.5 \cdot 0.35 \cdot 3 + 0.5 \cdot 0.65 \cdot 0 \tag{59}$$
$$= 0.175 + 0.65 + 0.525 + 0 = 1.35 \tag{60}$$
$$\pi_2(\underline{p},\overline{q}) = \pi_2(0.5, 1.0) = 0.5 \cdot 1.0 \cdot 1 + 0.5 \cdot 0 \cdot 2 + 0.5 \cdot 1.0 \cdot 3 + 0.5 \cdot 0 \cdot 0 \tag{61}$$
$$= 0.5 + 0 + 1.5 + 0 = 2 \tag{62}$$
$$\pi_2(\overline{p},\underline{q}) = \pi_2(1.0, 0.35) = 1.0 \cdot 0.35 \cdot 1 + 1.0 \cdot 0.65 \cdot 2 + 0 \cdot 0.35 \cdot 3 + 0 \cdot 0.65 \cdot 0 \tag{63}$$
$$= 0.35 + 1.3 + 0 + 0 = 1.65 \tag{64}$$
$$\pi_2(\overline{p},\overline{q}) = \pi_2(1.0, 1.0) = 1.0 \cdot 1.0 \cdot 1 + 1.0 \cdot 0 \cdot 2 + 0 \cdot 1.0 \cdot 3 + 0 \cdot 0 \cdot 0 \tag{65}$$
$$= 1 + 0 + 0 + 0 = 1 \tag{66}$$

Thus:

$$\text{col\_min} = \min(1.35, 2, 1.65, 1) = 1 \tag{67}$$
$$\text{col\_max} = \max(1.35, 2, 1.65, 1) = 2 \tag{68}$$

With $\alpha = 0.5$, the Column player's expected Hurwicz payoff is:

$$\pi_2^H = \alpha \cdot \text{col\_max} + (1-\alpha) \cdot \text{col\_min} \tag{69}$$
$$= 0.5 \cdot 2 + 0.5 \cdot 1 \tag{70}$$
$$= 1 + 0.5 = 1.5 \tag{71}$$

References

David Blackwell. An analog of the minimax theorem for vector payoffs. 1956.

Ralph L Keeney and Howard Raiffa. *Decisions with multiple objectives: preferences and value trade-offs*. Cambridge university press, 1993.

Jason Milionis, Ciamac C Moallemi, Tim Roughgarden, and Anthony Lee Zhang. Automated market making and loss-versus-rebalancing. *arXiv preprint arXiv:2208.06046*, 2022.