Financial Mathematics

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Contents

1. Basic notions: Financial transactions, future and present values, interest and discount rate, accumulation and discount factors;
2. Financial laws (exponential and linear);
3. Present and future values of annuities and some applications (loans, savings plans);
4. Net present value and Internal rate or return;

What does "financial" mean?

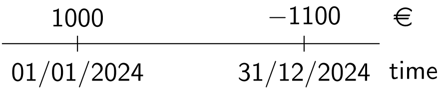
John Talbot offering a manuscript to Margaret of Anjou, on her betrothal to Henry VI in 1445, after being almost ruined by the ransom put on him. London, British Library

- Middle French: "end, ending; pardon, remission; payment, expense; settlement of a debt", from *finer* "to end, settle a dispute or debt".
- Gradually brought into English: "ransom" (mid-XV c.), "taxation" (late XV c.), in the sense of "management of money, science of monetary business" first recorded in English in 1770.

Financial transactions: example (1)

A simple financial transaction:

1. Time 01/01/2024 $\Longrightarrow$ An amount of 1000 € is received;
2. Time 31/12/2024 $\Longrightarrow$ An amount $1000 + 100$ € is given back;



What is the role of those 100 extra euros?

Financial transactions (1)
Definition

A *financial transaction* is an agreement between two counterparts to exchange cash flows.

- Time's role is fundamental. The deferral/anticipation of money availability is the reason why the transaction exists;
- Each amount of money is stated in a given account unit and is referred to a specific time;
- The amounts are fixed or contingent on a specified quantity (e. g. 100 euros multiplied a given market rate).

Remark:

The financial transaction has to be described by the cash-flow amounts with respective payment dates.

Financial transactions (2)
Time Value of Money

Consider the following statements:

1. Given a certain sum of money, an economic agent will prefer to have it available sooner rather than later.
2. Since an economic agent aims to maximize profit, and assuming that he can hold a certain sum of money at no cost, he will not make an investment that would lead to a loss.

From these considerations, we can **assume** that:

> *The cost of a financial transaction that postpones the maturity of a debt or brings forward the maturity of a receivable is positive, while the price for postponing a receivable or advancing a debt is negative.*

Financial transactions (3): Zero-coupon bond

Zero-coupon bonds (ZCBs) are financial contracts that guarantee the holder, on the part of the issuer, the payment of a specific sum F at a later date (*maturity*). To acquire this right, the investor must pay a certain price P at issue.

1. Time 01/01/2024 $\Longrightarrow$ An amount of 960 € is invested (price, outflow);
2. Time 31/12/2024 (maturity) $\Longrightarrow$ An amount 1000 € is received (face value or par value, inflow);

$-P = -960$	$F = 1000$	€
01/01	31/12	time

Remark:

The difference I between face value F and price P is called *interest*. So, face value can be thought as price (960 €) + interest (40 €).

Financial transactions (4): Interests in a two period operation

Interest represents a compensation to the lender for waiting (in the previous example, one year) for the reimbursement of the amount due.

The ratio

$$i = \frac{I}{P}$$

is called *interest rate of the operation* (on two dates).

It is common, in relation to deferral, to think of interest as the price of time — the reward received for having agreed to postpone the settlement of a debt. in this sense, it also represents the return on the investment.

Financial transactions (5): A four-period investment/borrowing operation

We borrow a certain amount of money S (*principal*) and we pay back a part R of it quarterly (*installments*).

1. Time 31/12/2024 $\Longrightarrow$ An amount of 1000 € is received;
2. Time 31/03/2025 $\Longrightarrow$ An amount 380 € is paid out;
3. Time 30/06/2025 $\Longrightarrow$ An amount 380 € is paid out;
4. Time 31/09/2025 $\Longrightarrow$ An amount 380 € is paid out;

1000	−380	−380	−380	€
12/24	03/25	06/25	09/25	time

Question:

In each installment, how much of the payment is interest?

Financial transactions (6): A four-period investment/borrowing operation

Conventionally, each installment is split between an *interest portion* I and a *principal portion* C.

1. Time 31/03/2025 $\implies R_{31/03} = 380 = C_{31/03} + I_{31/03}$;
2. Time 30/06/2025 $\implies R_{30/06} = 380 = C_{30/06} + I_{30/06}$;
3. Time 30/09/2025 $\implies R_{30/09} = 380 = C_{30/09} + I_{30/09}$.

C is intended as a partial repayment of the debt, while I remunerates the time value of money.
It's natural to think about the *remaining balance* or *outstanding debt* D after an installment has been paid, s.t.:

$$D_{31/12} = S = 1,000 \text{€};$$
$$D_{31/03} = S - C_{31/03};$$
$$D_{30/06} = D_{31/03} - C_{30/06} = S - C_{31/03} - C_{30/06};$$
$$D_{30/09} = D_{30/06} - C_{30/09} = 0.$$

Financial transactions (7): A four-period investment/borrowing operation

Since we have not established a rule for decomposing the installment into principal and interest portions, both of the following possibilities are, in principle, equally valid.

Decomposition 1

	12/24	03/25	06/25	09/25	
		-46.67	-46.67	-46.66	I
		-333.33	-333.33	-333.34	C
	$1,000$	-380	-380	-380	€
time	12/24	03/25	06/25	09/25	

Decomposition 2

	12/24	03/25	06/25	09/25	
		-10	-10	-140	I
		-380	-380	-240	C
	$1,000$	-370	-370	-380	€
time	12/24	03/25	06/25	09/25	

Financial transactions (8): Interests in a four-period operation

⇒ We want to extend the notion of interest rate to the case of the multi-period operation we have just described.

⇒ Note that each interest portion refers to the three-month period immediately preceding it; thus, it can be considered as remuneration for the deferral of the debt corresponding to that period.

⇒ For this reason, it is customary to express the interest portion proportionally to the outstanding debt at the beginning of each relevant period.

$$i_k = \frac{I_k}{D_{k-1}}; \quad k = 31/03, 30/06, 30/09$$

The rates $i_{31/03}$, $i_{30/06}$, $i_{30/09}$ are called *period rates*.

Financial transactions (9): Interests in a four-period operation

⇒ We can now calculate the interest with its corresponding period interest rate for each of the decompositions.

Decomposition 1

Time	D_k	R_k	I_k	i_k
31/12	1,000	-	-	-
31/03	666.67	380	46.67	0.047
30/06	333.34	380	46.67	0.070
30/09	0	380	46.67	0.140

Decomposition 2

Time	D_k	R_k	I_k	i_k
31/12	1,000	-	-	-
31/03	630	370	10	0.010
30/06	260	370	10	0.016
30/09	0	380	120	0.462

Financial transactions: Notation (1)

$\Rightarrow$ We now want to develop a concise language to describe financial transactions.

Representation

A financial operation is represented by a sequence of amounts, each with its own time of payment.

Moreover, to fully describe a transaction, the contract must specify:

- the currency unit (we'll assume the same currency unit for all cash-flows);
- how to measure time;
- how to approximate calculations;
- ...

Financial transactions: Notation (2)

Notation

- With sign convention, amounts are represented by symbols such as:

$$a_0, a_1, \dots, a_m$$
$$x_0, x_1, \dots, x_m$$
$$S, -b_1, -b_2, \dots, -b_m$$

- Times (duration, from a given initial date): $t_0, t_1, \dots, t_m$ (with $t_0 \leq t_1 \leq \dots \leq t_m$)

Example: 1/1/2022, 1/4/2022, 1/7/2022, 1/1/2023, 1/1/2024.
$\Rightarrow$ How can we represent these dates in terms of duration from the starting date 1/1/2022?

- time in months: $t_0 = 0,\ t_1 = 3,\ t_2 = 6,\ t_3 = 12,\ t_4 = 24$
- time in quarters: $t_0 = 0,\ t_1 = 1,\ t_2 = 2,\ t_3 = 4,\ t_4 = 8$
- time in years: ?

Financial transactions: Conventions on time representation (1)

The way of measuring the fraction of a year between two dates must be specified on contract and is usually defined by certain conventions. We need to specify:

1. how to count the number of days between two dates. Usually, the first day is excluded and the last one is included.
2. how many days are there in a year.

The convention is usually summarized in the form $\frac{\text{num}}{\text{den}}$, where the "numerator" is the way of counting the number of days between two dates, while the "denominator" indicates the number of days in a year.

- $\Rightarrow$ $\frac{\text{ACT}}{\text{ACT}}$ uses the effective number of days between two dates divided by 365 or 366.
- $\Rightarrow$ $\frac{\text{ACT}}{360}$ the effective number of days is divided by 360.
- $\Rightarrow$ $\frac{30}{360}$ each month is considered composed of 30 days. This is equivalent to the formula to calculate the days between two dates with the formula:

$$360 \cdot (Y_2 - Y_1) + 30 \cdot (M_2 - M_1) - (D_2 - D_1)$$

Financial transactions: Conventions on time representation (2)

As an example, lets calculate the fraction of year passing between 1/1/2022 and 1/3/2022 with these three conventions.

- $\Rightarrow$ $\frac{ACT}{ACT} \implies \frac{31+28}{365} = 0.1616$. If the period spans a leap year, the calculation must be split, with the portion falling in the leap year computed using 366 days and the remainder using 365 days.
- $\Rightarrow$ $\frac{ACT}{360} \implies \frac{31+28}{360} = 0.1639$
- $\Rightarrow$ $\frac{30}{360} \implies \frac{30+30}{360} = \frac{360(2022-2022)+30(3-1)+(1-1)}{360} = \frac{60}{360} = 0.1666$

The results can differ across conventions. The choice of convention is often associated with the type of contract (for example $\frac{ACT}{ACT}$ is usually used for government bonds, while $\frac{30}{360}$ is often adopted for mortgages).

Financial transactions: General notation

We consider a financial transaction, with payments $a_0, a_1, \ldots, a_m$ due, respectively, at times $t_0, t_1, \ldots, t_m$, with $t_0 < t_1 < \cdots < t_m$. This transaction can be represented writing its *cash flow stream*:

$$\underbrace{\{a_0, a_1, \ldots, a_m\}}_{\mathbf{a}} / \underbrace{\{t_0, t_1, \ldots, t_m\}}_{\mathbf{t}} = \mathbf{a}/\mathbf{t}$$

We can use this notation to represent the previous financial transactions:

- Zero-coupon bond $\Rightarrow$ $\{-960, 1000\}/\{0, 1\}$ amount in euro, time in years.
- investment/borrowing $\Rightarrow$ $\{1000, -380, -380, -380\}/\{0, 1, 2, 3\}$ amount in euro, time in quarters;

Financial transactions: Graphical & Table representation

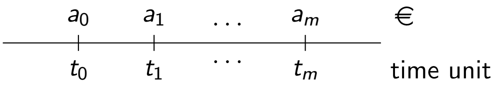
Table

Time	Cashflow
t_0	a_0
t_1	a_1
$\dots$	$\dots$
t_m	a_m

or

Time	t_0	t_1	$\dots$	t_m
Cashflow	a_0	a_1	$\dots$	a_m

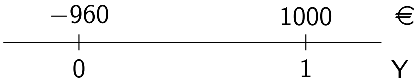
Time-axis (or time-diagram)



Financial transactions: Graphical & Table representation (examples)

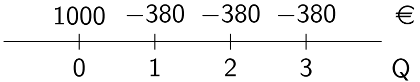
Zero Coupon Bond

Time	Cashflow
0	−960
1	1000



Investment/borrowing

Time	Cashflow
0	1000
1	−380
2	−380
3	−380



Financial transactions: Therminology (1)

(Pure) Investment transaction

Initial outflow (price of the investment), followed by incomes only

-1000	1100	€
0	3	years

(Pure) Financing transaction

Initial income (principal), followed by outflows (installments or payments) only

5000	-100	-100	$\ldots$	-100	€
0	1	2	$\cdots$	55	months

Financial transactions: Therminology (2)

Spot and forward transactions

The arrangement is defined at the current time t_0

Spot transaction: the first flow is due at time t_0

Time	t_0	t_1	$\dots$	t_m
Cashflow	a_0	a_1	$\dots$	a_m

Forward transaction: the first flow is due at time t_1

Time	t_0	t_1	$\dots$	t_m
Cashflow	0	a_1	$\dots$	a_m

Financial transactions: Terminology (3)

Example

$V(0) = \pm 1000 \qquad V(1) = \mp 1100$ €

0 — 1 — Y

One cash-flow incoming, one outgoing

For the investor:

- Transaction

$$\{-V(0), +V(1)\}/\{0, 1\}$$

- Long position, investment

For the debtor:

- Transaction

$$\{+V(0), -V(1)\}/\{0, 1\}$$

- Short position, financing

Remark

The amount of the cash-flows and their time are known in advance $\Longrightarrow$ The transaction is (assumed to be) certain, risk-free.

Financial transactions: Coupon bonds (1)

A (fixed) coupon bond is a contract that entitles the holder to receive a series of m periodic payments. The first $m-1$ payments are each equal to a fixed amount $I > 0$, while the final payment is $C + I$. The amount C is referred to as the bond's *face value* (or *nominal value*, or *par value*.

$-P$	I	I	$\dots$	$I + C$	€
t^*	$t^* + \tau$	$t^* + 2\tau$	$\dots$	$t^* + m\tau$	t

Assuming that the payment period is τ, a coupon bond issued at time t^* can be represented from the investor point of view as:

$$\{-P, I, I, \dots, I + C\}/\{t^*, t^* + \tau, t^* + 2\tau, \dots, t^* + m\tau\}$$

Financial transactions: Coupon bonds (2)

The coupon bond is said to be quoted:

- Above par: if $P > C$;
- At par: if $P = C$;
- Below par if $P < C$.

Often, the coupon I is expressed as a percentage i_C (*coupon rate*) of the face value: $I = i_C \cdot C$. Thus, the transaction can be represented as follows:

$$\{-P, i_C \cdot C, i_C \cdot C, \ldots, (1 + i_C) \cdot C\}/\{t^*, t^* + \tau, t^* + 2\tau, \ldots, t^* + m\tau\}$$

By summing the coupons I paid in a year and dividing by the face value C, one obtains the so-called *annual nominal rate of the bond*.

Financial transactions: Coupon bonds (3)

[illegible]	Lithuania Fx 3,5% Jul31 Eur		3,50	[illegible]	
XS2576364371	Latvia Fx 3,5% Jan28 Eur		3,50	17/01/2028	
XS1970549561	Romania Tf 3,5% Ap34 Eur		3,50	03/04/2034	
XS1241085353	Gs Intl Mc Nv27 Eur	99,80	3,472	26/11/2027	
XS2679922828	World Bank Fx 3,45% Sep38 Eur		3,45	13/09/2038	
XS1341083555	Imi Collezio Mc Ge26 Eur	100,342	3,423	26/01/2026	
XS3124345631	Bulgaria Fx 3,375% Jul35 Eur	98,24	3,375	18/07/2035	
EU000A2SCAK5	Efsf Tf 3,375% Ag38 Eur		3,375	30/08/2038	

Some bond prices form the website of Italian Stock Exchange

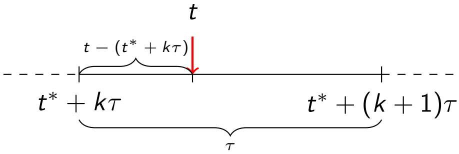
Suppose now that we want to buy a bond *after* the issue date. Since a certain amount towards the coupon payment has passed, the price should reflect that.

Financial transactions: Coupon bonds (4)

Suppose that, at time $t > t^*$, we want to buy a coupon bond. It must result:

$$t^* + k\tau \leq t \leq t^* + (k+1)\tau; \qquad k = 0, \ldots, m-1$$

We can graphically represent the situation as follows:



We then define the *accrued interest* as:

$$A = I \frac{t - (t^* + k\tau)}{(t^* + (k+1)\tau) - (t^* + k\tau)} = I \frac{t - (t^* + k\tau)}{\tau}$$

It represents the portion of the coupon that has already accrued, assuming that coupons are allocated proportionally over time.

Financial transactions: Coupon bonds (5)

⇒ In financial markets, the quoted price Q does not take into account the accrued interest. This is done to isolate the pure market component of the price, without it being affected by the passage of time since the last coupon. This "pure market" price is called *clean price* or *dry price*.

⇒ In order to purchase the bond, we must pay the holder the accrued interest: therefore, the effective price P will be the sum of clean price Q and accrued interest A. This is called *dirty price* or *tel quel price*.

⇒Therefore, we have:

$$P = Q + A$$

Financial transactions: Coupon bonds (6)

Consider the Italian treasury bond "Btp-1mg31 6%", with maturity 1/5/31 and issued above par at 102.40 €.
It pays semiannual coupons with a coupon rate $i_C = 3\%$, starting from 1/11/99. The contract states that the convention to be used to measure time is $\frac{\text{ACT}}{\text{ACT}}$.
On the 24th of September 2025, the bond is quoted at 116.36 € on the Milan Stock Exchange. How much should we pay this bond?

⇒ Since the last coupon was paid on 1/5/2025, we can calculate the accrued interest as:

$$A = 3 \cdot \frac{146}{365} = 3 \cdot 0.7935 = 2.38\,€$$

Therefore, the dirty price is:

$$P = Q + A = 116.36 + 2.38 = 118.74\,€$$

Sum of financial transactions (1)

Remark:

A financial transaction is completely described by its cash flow stream associated to its schedule. Therefore, two financial operations are *equivalent* if they are described by the same nonzero cash flows on the same schedule.

$\Rightarrow$ Suppose to have a financial transactions $\mathbf{x}/\mathbf{t}$, and a schedule $\mathbf{t}'$ s.t. $\mathbf{t} \subset \mathbf{t}'$. It's always possible to define a new financial transaction $\mathbf{x}'/\mathbf{t}'$ on the extended schedule $\mathbf{t}'$ imposing the cash flows to be the same on the elements of the smaller schedule and 0 otherwise. This new transaction is equivalent to the first one.

Example:

$$\mathbf{x}/\mathbf{t} = \{-10, 8, 8\}/\{0, 1, 2\}; \quad \mathbf{t}' = \{0, 0.5, 1, 2, 3\}$$

$$\Longrightarrow \mathbf{x}'/\mathbf{t}' = \{-10, 0, 8, 8, 0\}/\{0, 0.5, 1, 2, 3\}$$

Transactions $\mathbf{x}/\mathbf{t}$ and $\mathbf{x}'/\mathbf{t}'$ are equivalent.

Sum of financial transactions (2)

$\Rightarrow$ Given two financial transactions $\mathbf{x}/\mathbf{t}$ and $\mathbf{x}'/\mathbf{t}'$ we can define an extended schedule $\mathbf{t}'' = \mathbf{t} \cup \mathbf{t}'$ and extend both transactions to $\mathbf{t}''$.

We can the define the transaction $\mathbf{x}''/\mathbf{t}''$, *sum* of $\mathbf{x}/\mathbf{t}$ and $\mathbf{x}'/\mathbf{t}'$, as the transaction defined on the union schedule whose cash flows are the sum of the flows $\mathbf{x}$ and $\mathbf{x}'$ on each schedule point. We will write:

$$\mathbf{x}''/\mathbf{t}'' = \mathbf{x}/\mathbf{t} \oplus \mathbf{x}'/\mathbf{t}'$$

Example:

$$\mathbf{x}/\mathbf{t} = \{-10, 8, 8\}/\{0, 1, 2\}; \quad \mathbf{x}/\mathbf{t} = \{-3, 2, 2\}/\{0, 2, 4\};$$

$$\Longrightarrow \mathbf{x}''/\mathbf{t}'' = \{-10-3, 8, 8+2, 2\}/\{0, 1, 2, 4\} = \{-13, 8, 10, 2\}/\{0, 1, 2, 4\}$$

Sum of financial transactions (3)

It's possible to use the machinery we just introduced to decompose a financial transaction. We'll describe two main decomposition categories:

- *With respect to a date*: Given a financial transaction $\mathbf{x}''/\mathbf{t}''$ and a reference date t^*, we split the schedule vector into two, separating what happens before the reference date from what happens after it.
  Example:
  $$\mathbf{x}''/\mathbf{t}'' = \{10, -4, -4, -4\}/\{0, 1, 2, 3\}; \quad t^* = 1.5$$
  $$\Longrightarrow \mathbf{x}/\mathbf{t} = \{10, -4\}/\{0, 1\} \quad \mathbf{x}'/\mathbf{t}' = \{-4, -4\}/\{2, 3\}$$

- *With respect to the algebraic signs of the cash flows*: separating positive and negative cash-flows, we can obtain the cash inflow and the cash outflowflow.
  Example:
  $$\mathbf{x}''/\mathbf{t}'' = \{10, -4, -4, -4\}/\{0, 1, 2, 3\}$$
  $$\Longrightarrow \mathbf{a}/\mathbf{t} = \{10\}/\{0\} \quad \mathbf{l}/\mathbf{t}' = \{-4, -4, -4\}/\{1, 2, 3\}$$

The value function (1)

Suppose to have a financial transaction on two dates, represented as

$$\{-S, S+I\}/\{0,1\}$$

This means that, strictly speaking, the counterparts have agreed to consider the sum S at time 0 equivalent to the sum sum $S+I$ at time 1.

$\Rightarrow$ Therefore, the financial contract implicitly states an intertemporal equivalence law between sums of money.

$\Rightarrow$ We can record this law the form of a function of time, defined in 0 and 1 (the contract execution is possible only in those two dates). This function of the schedule dates $V(t)$ is called value function. The tabular representation of the function is:

t	$V(t)$
0	S
1	$S+I$

The value function (2)

Let's now extend this notion to multi-period financial transactions. Consider the following composite transaction:

⇒ At time 0, an amount S is invested by a financial institution;

⇒ At time 1, the debtor can decide to:

1. Pay in the entire sum, with interest I. The resulting financial operation is:

$$\{-S, S + I\}/\{0, 1\}$$

2. Defer the principal reimbursement for one year, paying an additional interest I', so that results:

$$\{-S, S + I + I'\}/\{0, 2\}$$

⇒ We have implicitly defined two value functions, one for each possible choice of the debtor.

The value function (3)

The tabular representations of the two value functions are:

t	$V_1(t)$
0	S
1	$S+I$

t	$V_2(t)$
0	S
2	$S+I+I'$

However, we can see the second choice as a composition of two basic operations:

$$\{-S, S+I\}/\{0,1\} \oplus \{-(S+I), (S+I)+I'\}/\{1,2\}$$

$\Rightarrow$ Therefore, this contract involves three different time periods: this implicitly defines an intertemporal equivalence law $V(t)$ on three dates, whose representation is:

t	$V_2(t)$
0	S
1	$S+I$
2	$S+I+I'$

The value function (4)

Remark:

The composite transaction is not defined on a *fixed* time schedule. However, all potential outcomes have been examined and agreed upon at time 0. The function $V(t)$ summarizes all such alternative outcomes.

Basic assumptions about the value function:

1. For simplicity, and without loss of generality, we assume that the value function is positive-valued and defined on positive times:

$$V(t) > 0; \quad t \geq 0$$

2. To incorporate the time-value of money, we assume that the value function is strictly increasing:

$$t_1 < t_2 \Rightarrow V(t_1) < V(t_2)$$

Value function and cash flows (1)

Please note that the value function of a financial transaction and its cash flows are two different concepts. To see it with an example, let's go back to the investment/borrowing operation:

$$\{1000, -380, -380, -380\}/\{0, 1, 2, 3\}$$

Let's suppose that interest is described by Decomposition 1, such that:

$$\{1000, -333.33 - 46.67, -333.33 - 46.67, -333.34 - 46.66\}/\{0, 1, 2, 3\}$$

For the sake of brevity, let's measure time in quarters, starting from 31/12/2024. From this interest, we computed the following period interest rates:

$$i_1 = 0.047; \quad i_2 = 0.070; \quad i_3 = 0.140$$

We can now try to use these rates to build a composite operation of the same type we've used to introduce the value function.

Value function and cash flows (2)

The investment consists of $1,000$ € at time 0. Therefore:

$$V(0) = 1,000$$

At time 1, since the period interest rate is $i_1 = 0.047$, the period interest is 46.67 €. The borrower can decide to repay his entire debt, or to extend the payment term by another quarter, paying an additional interest. If he chooses the second option, the composite transaction becomes:

$$\{1,000, -1,046.67\}/\{1,046.67, -1,046.67 + I_2\}$$

Now, what is the interest pertaining to the second period? To preserve coherence with interest rate i_2, it must be $I_2 = 1,046.67 * i_2 = 73.27$. Therefore, the composite transaction becomes:

$$\{1,000, -1,046.67\}/\{0,1\} \oplus \{1,046.67, 1,119.43\}/\{0,1\}$$

Value function and cash flows (3)

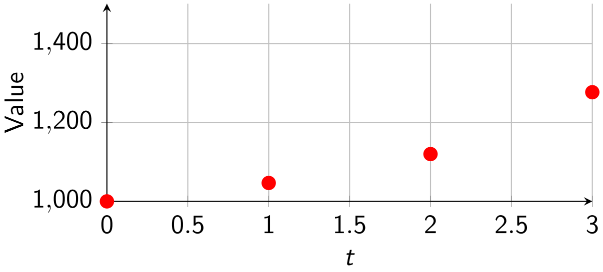
We now have the second point of the value function:

$$V(0) = 1,000; \quad V(1) = 1,046.67$$

We can repeat the same procedure: holding the period interest rates fixed and supposing to delay the principal payment until time 3, we find the remaining points of the value function:

$$V(2) = 1,119.93; \quad V(3) = 1,276.72$$

The graph of the function is:



Value function and cash flows (4)

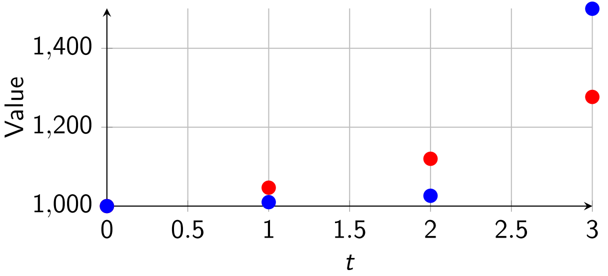
What happens if we repeat the same procedure using interest from Decomposition 2?

$$\{1000, -370 - 10, -370 - 10, -260 - 120\}/\{0, 1, 2, 3\}$$

$$i_1 = 0.010; \quad i_2 = 0.016; \quad i_3 = 0.462$$

$$V(0) = 1,000; \quad V(1) = 1,010; \quad V(2) = 1,026.16; \quad V(3) = 1,500.25$$

The graph of the function, superimposed on the previous one, is:



Value function and cash flows (5)

$\Rightarrow$ The same cash flows, with a different agreement on interest calculation, led to two different value functions.

$\Rightarrow$ To describe the value function associated to a transaction, the cash flow stream is not enough: the value function is linked to period interest rates.

$\Rightarrow$ Since the value function is expressed in monetary terms, its initial value is also required to fully describe it. In the previous example, starting from $V(0) = 500$, with the same period interest rate, would have produced two new value functions, whose value would have been exactly half of the starting ones.

Interest rate: definitions in terms of value function

We can now express some new quantities in a more formal way, in terms of the value function

Interest

$$I(t, t+\tau) = V(t+\tau) - V(t)$$

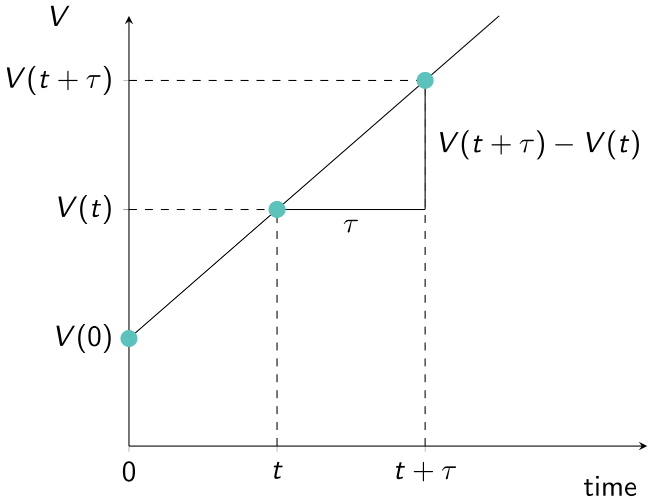
Interest rate

$$i(t, t+\tau) = \frac{V(t+\tau) - V(t)}{V(t)}$$

Accumulation factor

$$m(t, t+\tau) = \frac{V(t+\tau)}{V(t)}$$

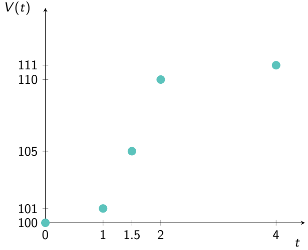
Interest rate: definitions in terms of value function



Interest rate: example

Suppose that a transaction is summarized by the value function in this table:

t	V(t)
0	100
1	101
1.5	105
2	110
4	111



We could calculate:

$$I(0,1) = V(1) - V(0) = 1 \quad I(2,4) = V(4) - V(2) = 1$$

$$i(0,1) = \frac{V(1) - V(0)}{V(0)} = 1\% \quad i(2,4) = \frac{V(4) - V(2)}{V(2)} = 0.909\%$$

Discount rate: definitions in terms of value function

Discount

$$D(t, t+\tau) = V(t+\tau) - V(t) = I(t, t+\tau)$$

Discount rate

$$d(t, t+\tau) = \frac{V(t+\tau) - V(t)}{V(t+\tau)}$$

Discount factor

$$v(t, t+\tau) = \frac{V(t)}{V(t+\tau)}$$