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Chudnovsky algorithm

The **Chudnovsky algorithm** is a fast method for calculating the digits of π, based on Ramanujan's π formulae. It was published by the Chudnovsky brothers in 1988.[1]

It was used in the world record calculations of 2.7 trillion digits of π in December 2009,[2] 10 trillion digits in October 2011,[3][4] 22.4 trillion digits in November 2016,[5] 31.4 trillion digits in September 2018–January 2019,[6] 50 trillion digits on January 29, 2020,[7] 62.8 trillion digits on August 14, 2021,[8] and 100 trillion digits on March 21, 2022.[9]

Algorithm

The algorithm is based on the negated Heegner number $d = -163$, the *j*-function $j\left(\frac{1+i\sqrt{163}}{2}\right) = -640320^3$, and on the following rapidly convergent generalized hypergeometric series:[2]

$$\frac{1}{\pi} = 12\sum_{q=0}^{\infty} \frac{(-1)^q (6q)!(545140134q + 13591409)}{(3q)!(q!)^3 (640320)^{3q+\frac{3}{2}}}$$

A detailed proof of this formula can be found here:[10]

For a high performance iterative implementation, this can be simplified to

$$\frac{(640320)^{\frac{3}{2}}}{12\pi} = \frac{426880\sqrt{10005}}{\pi} = \sum_{q=0}^{\infty} \frac{(6q)!(545140134q + 13591409)}{(3q)!(q!)^3 (-262537412640768000)^q}$$

There are 3 big integer terms (the multinomial term M_q, the linear term L_q, and the exponential term X_q) that make up the series and π equals the constant C divided by the sum of the series, as below:

$$\pi = C\left(\sum_{q=0}^{\infty} \frac{M_q \cdot L_q}{X_q}\right)^{-1}$$, where:

$$C = 426880\sqrt{10005},$$
$$M_q = \frac{(6q)!}{(3q)!(q!)^3},$$
$$L_q = 545140134q + 13591409,$$
$$X_q = (-262537412640768000)^q.$$

The terms M_q, L_q, and X_q satisfy the following recurrences and can be computed as such:

$$L_{q+1} = L_q + 545140134 \qquad \text{where } L_0 = 13591409$$
$$X_{q+1} = X_q \cdot (-262537412640768000) \qquad \text{where } X_0 = 1$$
$$M_{q+1} = M_q \cdot \left(\frac{(12q+2)(12q+6)(12q+10)}{(q+1)^3}\right) \text{ where } M_0 = 1$$

The computation of M_q can be further optimized by introducing an additional term K_q as follows:

$$K_{q+1} = K_q + 12 \qquad \text{where } K_0 = -6$$
$$M_{q+1} = M_q \cdot \left(\frac{K_{q+1}^3 - 16K_{q+1}}{(q+1)^3}\right) \text{ where } M_0 = 1$$

Note that

$$e^{\pi\sqrt{163}} \approx 640320^3 + 743.99999999999925\ldots \text{ and}$$
$$640320^3 = 262537412640768000$$
$$545140134 = 163 \cdot 127 \cdot 19 \cdot 11 \cdot 7 \cdot 3^2 \cdot 2$$
$$13591409 = 13 \cdot 1045493$$

This identity is similar to some of Ramanujan's formulas involving π,[2] and is an example of a Ramanujan–Sato series.

The time complexity of the algorithm is $O\left(n(\log n)^3\right)$.[11]

See also

- Borwein's algorithm
- Numerical approximations of π
- Ramanujan–Sato series

References

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