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Gauss–Legendre algorithm

The **Gauss–Legendre algorithm** is an algorithm to compute the digits of π. It is notable for being rapidly convergent, with only 25 iterations producing 45 million correct digits of π. However, it has some drawbacks (for example, it is computer memory-intensive) and therefore all record-breaking calculations for many years have used other methods, almost always the Chudnovsky algorithm. For details, see Chronology of computation of π.

The method is based on the individual work of Carl Friedrich Gauss (1777–1855) and Adrien-Marie Legendre (1752–1833) combined with modern algorithms for multiplication and square roots. It repeatedly replaces two numbers by their arithmetic and geometric mean, in order to approximate their arithmetic-geometric mean.

The version presented below is also known as the **Gauss–Euler**, **Brent–Salamin** (or **Salamin–Brent**) **algorithm**;[1] it was independently discovered in 1975 by Richard Brent and Eugene Salamin. It was used to compute the first 206,158,430,000 decimal digits of π on September 18 to 20, 1999, and the results were checked with Borwein's algorithm.

Algorithm

1. Initial value setting:

$$a_0 = 1 \qquad b_0 = \frac{1}{\sqrt{2}} \qquad t_0 = \frac{1}{4} \qquad p_0 = 1.$$

2. Repeat the following instructions until the difference of a_n and b_n is within the desired accuracy:

$$a_{n+1} = \frac{a_n + b_n}{2},$$

$$b_{n+1} = \sqrt{a_n b_n},$$

$$t_{n+1} = t_n - p_n(a_n - a_{n+1})^2,$$

$$p_{n+1} = 2p_n.$$

3. π is then approximated as:

$$\pi \approx \frac{(a_{n+1} + b_{n+1})^2}{4t_{n+1}}.$$

The first three iterations give (approximations given up to and including the first incorrect digit):

$$3.140\ldots$$
$$3.14159264\ldots$$
$$3.14159265358979323\,82\ldots$$

The algorithm has quadratic convergence, which essentially means that the number of correct digits doubles with each iteration of the algorithm.

Mathematical background

Limits of the arithmetic–geometric mean

The arithmetic–geometric mean of two numbers, a_0 and b_0, is found by calculating the limit of the sequences

$$a_{n+1} = \frac{a_n + b_n}{2},$$
$$b_{n+1} = \sqrt{a_n b_n},$$

which both converge to the same limit.
If $a_0 = 1$ and $b_0 = \cos\varphi$ then the limit is $\frac{\pi}{2K(\sin\varphi)}$ where $K(k)$ is the complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

If $c_0 = \sin\varphi$, $c_{i+1} = a_i - a_{i+1}$, then

$$\sum_{i=0}^{\infty} 2^{i-1} c_i^2 = 1 - \frac{E(\sin\varphi)}{K(\sin\varphi)}$$

where $E(k)$ is the complete elliptic integral of the second kind:

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta \text{ and } K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

Gauss knew of both of these results.[2] [3] [4]

Legendre’s identity

Legendre proved the following identity:

$$K(\cos\theta)E(\sin\theta) + K(\sin\theta)E(\cos\theta) - K(\cos\theta)K(\sin\theta) = \frac{\pi}{2}, \text{ for all } \theta.$$ [2]

Elementary proof with integral calculus

The Gauss-Legendre algorithm can be proven to give results converging to π using only integral calculus. This is done here[5] and here.[6]

See also

- Numerical approximations of π

References

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3. Salamin, Eugene, *Computation of pi*, Charles Stark Draper Laboratory ISS memo 74–19, 30 January 1974, Cambridge, Massachusetts
4. Salamin, Eugene (1976), "Computation of pi Using Arithmetic–Geometric Mean", *Mathematics of Computation*, vol. 30, no. 135, pp. 565–570, doi:10.2307/2005327 (https://doi.org/10.2307%2F2005327), ISSN 0025-5718 (https://www.worldcat.org/issn/0025-5718), JSTOR 2005327 (https://www.jstor.org/stable/2005327)
5. Lord, Nick (1992), "Recent Calculations of π: The Gauss-Salamin Algorithm", *The Mathematical Gazette*, **76** (476): 231–242, doi:10.2307/3619132 (https://doi.org/10.2307%2F3619132), JSTOR 3619132 (https://www.jstor.org/stable/3619132), S2CID 125865215 (https://api.semanticscholar.org/CorpusID:125865215)
6. Milla, Lorenz (2019), *Easy Proof of Three Recursive π-Algorithms*, arXiv:1907.04110 (https://arxiv.org/abs/1907.04110)