Università Commerciale "Luigi Bocconi"

MAFINRISK
Master of Quantitative Finance and Risk Management

XVIII Cycle

Group 5 Assignment

Numerical Methods

10052 Numerical Methods
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Outline of the work

In this introduction, we provide a brief outline of how our work has been structured.
The assignment aims to fairly evaluate an Executive Stock Option (ESO), which is a very customizable derivatives instrument granted to companies' top management as part of their compensation, under different assumptions.
Our baseline assumptions are the following:

- We consider a company X, whose stock price process is denoted as $\{S_t\}_{t>0}$ and is assumed to follow a Geometric Brownian Motion. The initial stock price is $S_0 = 100$.
- The risk-free rate, r, is equal to 1% for any maturity
- The volatility of S, σ, is 15%.
- The annual continuous dividend yield provided by the stock, q, is 5%.
- The ESO is issued at time $t = 0$ and has a maturity T of 5 years.
- The strike price of the ESO contract, K, is equal to $S_0 = 100$.

A brief section is devoted to all of the 8 points of the instructions; for each of them, we have provided:

- the new assumptions introduced or ruled out (if any);
- a concise explanation of the line of reasoning followed;
- the results obtained for the ESO price, according to the different hypothesis, considered;
- when appropriate, also some graphs are included.

To facilitate the reader, we report here the links to the different tasks: Task 1, Task 2, Task 3, Task 4, Task 5, Task 6, Task 7, Task 8. A Python script and an Excel file (the latter with only the first 4 Tasks) are also provided.[1]

1 Task 1

In Task 1, the goal is to evaluate the ESO assuming its payoff is the one of a plain Vanilla call option. Since all the needed values are available, we have simply applied the B&S closed-form solution, namely:

$$C(S,t) = S_t e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

where $d_1 = \frac{\ln \frac{S_t}{K} + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, and $d_2 = d_1 - \sigma\sqrt{T}$.

Employing these formulas we have obtained a final price of the ESO of 4.8620.
In addition, for the sake of completeness and to provide a logical "bridge" with the following tasks, we have decided to run the same problem employing Monte Carlo estimation. The number of simulations to be performed is such that it delivers an MC radius equal to approximately 10% of the point estimate. [2]

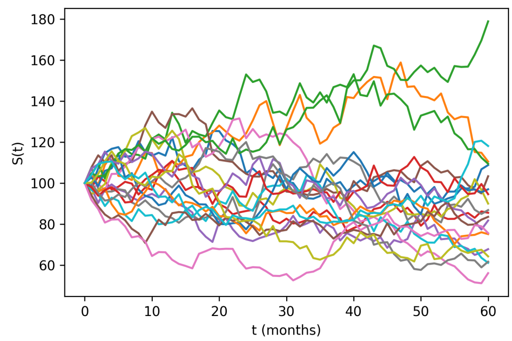

Figure 1: Task 1 simulated GBM stock paths

In Table 1 we have reported both the B&S closed-form result and the one arising from one MC simulation.

B&S	4.8620
Point estimate	4.7743
MC radius	0.4662
UB	5.2405
LB	4.3077
MC radius / point estimate	0.0976

Table 1: Task 1 B&S closed form and simulation results (N = 3000)

[1] The .py file is organized in multiple sections: the first contains the packages, the functions imported and the key variables used; moreover, one section is dedicated to each of the tasks of the assignment. Please, note that some variables are repeated at the beginning of each section (like `NSim`), to allow the reader to run each session independently and to avoid variables dimensions conflicts.
[2] This rule of thumb has been applied throughout the whole analysis we carried out.

2 Task 2

In this second Task, the following hypothesis is introduced: there is a $\gamma = 10\%$ annual probability that the holder leaves the company, whatever the stock price at that time. The holder of the ESO can leave only on a monthly basis starting from $t = \frac{1}{60}$ (monthly monitoring). [3] The monthly probability that the holder leaves or not the firm has been modelled by using an exponential distribution with $\lambda = 0.1054$. If the holder leaves the Company before T, the ESO becomes worthless and so the manager does not receive anything; if there is no early exit, then the ESO pays the standard payoff of a plain vanilla call option. Now, imagine a binomial tree, in which an "up" movement implies that the holder does not leave the company (with probability $1 - p = 1 - 0.8742\%$) while the "down" path represents the alternative that the holder leaves the company, without cashing in any payoff (this with probability $p = 0.8742\%$). Given this scheme across all the five years, the fair value of the ESO turns out to simply be the plain vanilla one multiplied by the total compounded monthly probability (over 5 years or 60 months) to remain in the company, namely, $(1 - 0.8742\%)^{60} = 59.0490\%$: considering the closed-form B&S price from the previous point, we obtained a fair value of the option equal to $4.8620 * 0.59049 = 2.8709$. Intuitively, the ESO is worth less since it guarantees a potential payoff in fewer scenarios than the plain vanilla one, therefore it can be deemed to be cheaper than the previous task.

3 Task 3

Here we followed the same scheme of the previous one with the following add-on: if the holder leaves, she/he immediately cashes in $(S_{t^*} - K)^+$, i.e. the payoff she/he would get in case of early exercising. One point to be stressed is the following: if the holder of the ESO leaves the company before the maturity of the contract and so cash in the payoff at that period t^*, then this payoff is discounted back to zero; the same applies at maturity T when the holder receives the standard payoff of the plain vanilla call option ($(S_T - K)^+$). It follows that the ESO fair price at $t = 0$ is given by the average of these discounted values.
Figures 2 and 3 depict, respectively, the behaviour of the simulated stock prices and the consequent *potential* "payoffs": note that, in the latter plot, the majority of the paths are identical to the corresponding simulated stock prices. This fact is related to the way in which we decided to model the problem: if at each time the holder does not leave the company, the *potential payoff* is simply set to be equal to the stock price at the same time; otherwise, when the holder decides to leave, the *potential payoff* plummets to zero (if $S_t < k$) or to a small positive value (if $S_t > k$), which represent its true payoff[4] (before decaying to zero).

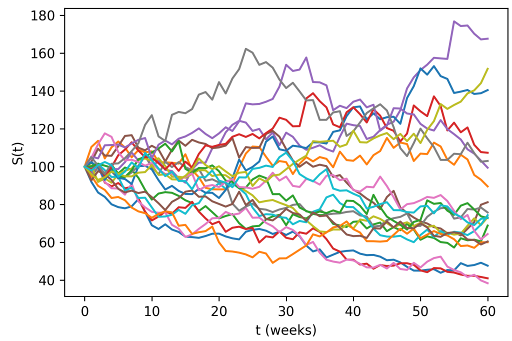

Figure 2: Task 3 simulated GBM stock paths

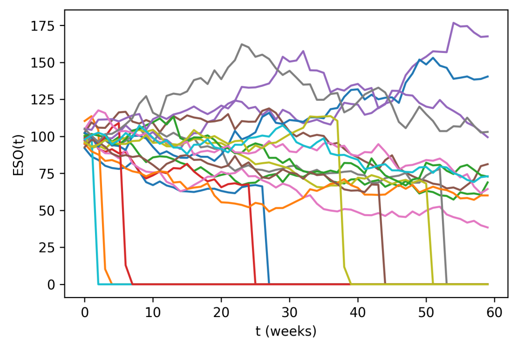

Figure 3: Task 3 ESO potential payoffs evolution

Given the specific framework we are considering, it is immediate to expect an ESO fair price higher than the one obtained in Task 2; this is simply motivated by the fact that the holder cash in a payoff even in the case she/he decides to leave the company.
In Table 2, we report one result of the simulation performed.

[3] Since $T = 5$, the contract lasts for 60 months.
[4] In this regard, please note the slanted behaviour of the orange and red payoff lines included between $t = 0$ and $t = 10$)

Point estimate	4.6022
MC radius	0.4380
UB	5.0446
LB	4.1597
MC radius / point estimate	0.0961

Table 2: Task 3 simulation results (N = 2500)

4 Task 4

In Task 4 a further distinction is introduced. The core problem is the same as the one in Task 3 but, this time, one has to take into account if the holder of the ESO does leave as a *Good Leaver* or as a *Bad Leaver* (of course, in the scenario that she/he decides to leave). This last distinction depends on the price of the stock at time t^*, i.e. the time of exit:

- if $S_{t^*} > 1.5 S_0$, then the leaver is considered as "Good" and the payoff of the ESO at that time is $2 \cdot (S_{t^*} - K)^+$;
- if $S_{t^*} < 1.5 S_0$, then the leaver is considered as "Bad" and the payoff of the ESO at that time is $0.7 \cdot (S_{t^*} - K)^+$.

Table 3 reports one estimate (MC simulation with N = 5000) of the ESO price at time zero, keeping into account this additional feature. It seems reasonable to get a point estimate greater than the previous task, since in this framework the options guarantees, on average, higher payoffs.

Point estimate	4.8423
MC radius	0.5077
UB	4.3295
LB	5.3552
MC radius / point estimate	0.1058

Table 3: Task 4 simulation results (N = 5000)

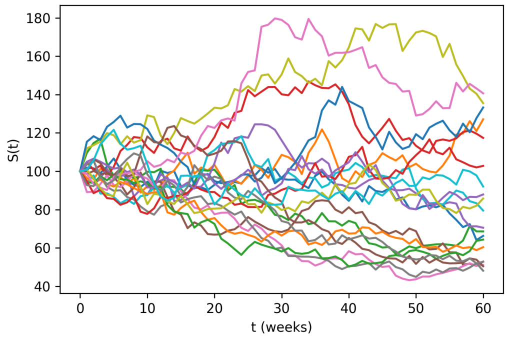

Figure 4: Task 4 simulated GBM stock paths

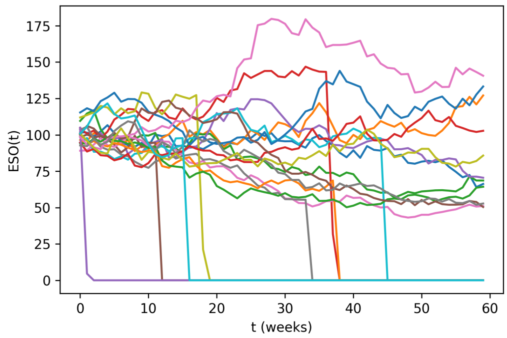

Figure 5: Task 4 ESO potential payoffs evolution

5 Task 5

In this section we cover Task 5; the main difference here is given by the fact that we disregard the assumption of constant volatility σ and interest rate r: in particular, the latter is explained by a deterministic constantly growing trend across the 5 years (starting from 1% to 2% at the end of the fifth year, with weekly monitoring), while stock and volatility are both stochastic and modelled by the Heston model. In order to jointly simulate these processes, we generated a bivariate random normal and then computed at each timestep the two variables, by taking into consideration the corresponding risk-free rate at the same time. Once the stock paths have been obtained, we simply computed the final payoff of the plain vanilla call option and discounted it at time zero.
The market model we used can be summarized by the following dynamics:

$$\begin{cases} \frac{dS_t}{S_t} = (r_t - q)dt + \sqrt{V_t}dW_s^{\mathbb{Q}}(t) \\ dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_V^{\mathbb{Q}}(t) \\ dr_t = \frac{r_T - r_0}{T}dt \end{cases}$$

where $d(W_s^{\mathbb{Q}}, W_V^{\mathbb{Q}}) = \rho dt$, $\kappa = 0.5$, $V_0 = 0.15^2$, $\theta = 0.2^2$, $\xi = 0.05$, $\rho = 0.8$, $r_0 = 0.01$ and $r_T = 0.02$.
Figures 6 and 7 show respectively the deterministic path followed by the interest rate and one simulated path of the weekly volatility (RHS) and the corresponding stock price process (LHS); it is immediate to see how highly correlation among the two processes ($\rho = 0.8$).

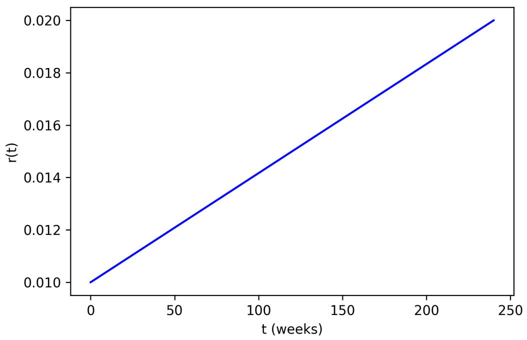

Figure 6: Risk-free rate

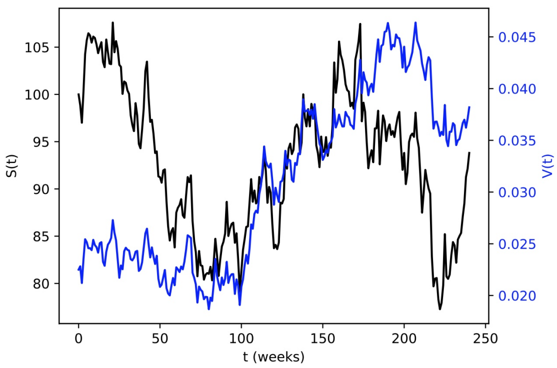

Figure 7: Heston model stock (LHS) and stochastic volatility (RHS)

In Table 4 one can appreciate how the ESO (in this case plain vanilla call) increased its value with respect to Section 1: this can be ascribed to two factors: a long call is both *rho* and *vega* positive, meaning that has positive sensitivity with respect to the risk-free rate and the volatility. In our case the risk-free rate is always increasing while the volatility is not constant and so much more volatile; this is for sure a plausible explanation for the higher option price we got.

Point estimate	7.3126
MC radius	0.7912
UB	8.1039
LB	6.5213
MC radius / point estimate	0.1082

Table 4: Task 5.1 simulation results (N = 2500)

We extended the new Heston framework to the Section 2 scheme; following the same procedure delineated there, we computed the *weekly* probability that the holder leaves the company by the same exponential distribution (equal to 0.2192%). As in the previous case, the ESO value at time zero is obtained by multiplying the plain vanilla simulated price with the total compounded probability of remaining in the firm, that is, $(1 - 0.2192\%)^{240}$, where 240 is the number of weeks in five years.

Point estimate	4.5612
MC radius	0.5091
UB	5.0704
LB	4.0521
MC radius / point estimate	0.0659

Table 5: Task 5.2 simulation results (N = 2500)

Lastly, we applied the same algorithm described in Section 3, with the only difference that, at every period, risk-free rate and volatility are not constant. As expected, the time-zero ESO price delivers on average higher values.
By looking at the 20 "payoffs" paths depicted in Figure 8 [5], there are some cases in which the holder leaves the company in advance. Those are the ones where the "payoffs" (both zeros payoffs when $S_t < K$ and positive payoffs when $S_t > K$; in the latter case the graph shows a slightly slanted behaviour, as it happens approximately at the 50^{th} week) decay suddenly toward zero.

[5] Note that those represented in the figure are *potential* and not *actual* payoffs; those are the results of the method according to which we decided to address and model the problem. If the holder does not leave at a given time t, then the "payoff" matrix in the python script assumes the simulated value of the stock at that time.

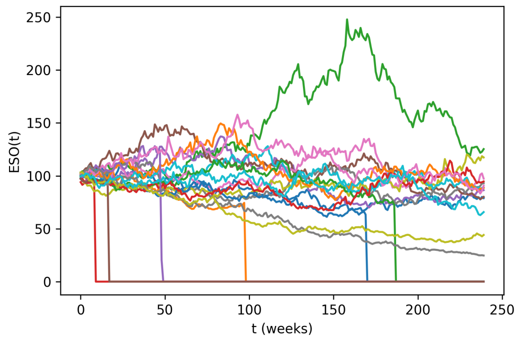

Figure 8: Task 5.3 ESO potential payoffs evolution

Point estimate	7.0029
MC radius	0.6749
UB	7.6778
LB	6.3279
MC radius / point estimate	0.0963

Table 6: Task 5.3 simulation results (N = 3000)

The results obtained in this Heston variant of previous tasks seem to be coherent with what we would have expected: the ESO prices we got are higher than the "standard" scenario (constant interest rate and volatility) ones, confirming that the option becomes more valuable as the volatility is allowed to change in a stochastic way.

6 Task 6

In Task 6 we returned to the same simple hypothesis of Task 1 (the stock follows a simple GBM, ignoring the further assumptions of the subsequent tasks), this time by considering the option to be an *American* one. We have addressed this problem employing the binomial (backward) approach; after having obtained u, d and p (respectively the up-movement factor, the down-movement factor and the probability of the up-movement) we computed the stock price over all the nodes over the 5 years time horizon and with monthly steps. In the end, by simply discounting back the payoffs at each node and, every time, picking the highest value between the discounted payoff and the alternative payoff coming from the early exercise opportunity, we got an American ESO fair price of 7.1290.

7 Task 7

In the present Task, we add a partial backdating feature to the framework delineated in the previous Section. More specifically, we assume that the holder of the ESO can look back at the values of the stock price taken in the first year, namely from time $t = \frac{1}{12}$ to $t = \frac{12}{12} = 1$, and choose one of these prices to be the strike of its ESO contract. Note that the ESO contract life is now starting at t=1 and ending at T=5, namely it lasts for 4 years (or equivalently for 48 months).
We addressed this problem by assuming that the ESO holder rationally (in order to maximize her/his profit) chooses the strike to be the *lowest* stock value touched in the first year.
Furthermore, we devised as many *4-years binomial trees* (same length) as the number of simulations carried out: particular, we created a 3D structure (which in python script is called `S_bin_m`) where the horizontal layers are binomial trees and the vertical dimension is the number of simulations. The initial value of each of them is the simulated S at $t = 1$, namely, the final value of the stock path obtained by a GBM simulation. Afterwards, an identical 3D matrix (`call_7_m`) has been created to store the binomial tree payoffs for each American option. Note that, as a result of the partial backdating feature, every simulation yields a different strike, therefore every binomial tree is calculated with reference to its specific strike. For computing the ESO fair price at t=0, we discounted back each of the binomial tree payoffs as made in task 6, i.e. by considering at each node the maximum value between the discounted payoff and the payoff from early exercising.
Finally, all the initial ESO values at $t = 1$ (so at the first year, when each option starts) are discounted at $t = 0$ to obtain the fair value of the contract at inception.
In Table 7 we reported one result for Task 7; note how we achieved a ratio between the MC radius and the point estimate of approximately 10% by means of only 125 simulations. This is due to the fact that the result is based on 125 different binomial trees starting from t=1, this allowing to reach a fast convergence to the true ESO price.

Point estimate	10.5549
MC radius	1.0896
UB	11.6446
LB	9.4653
MC radius / point estimate	0.1032

Table 7: Task 7 simulation results (N = 125)

As we can see from the above table, the fair price of the ESO is quite larger than the price obtained in Task 6. This fact can be motivated by the fact that leaving the possibility of choosing the strike (the lowest of the values taken by the stock at each simulation until $t = 1$) to the holder, gives her/him a noteworthy advantage in terms of potential profits.

8 Task 8

This scenario is a further evolution of the previous one: the holder can decide also *when* to issue the option within the first year; moreover, he can always decide which strike price to consider[6]. As a result of this additional feature, assuming a monthly monitoring, the option can starts from $t = 0$ until $t = \frac{12}{12} = 1$.
More specifically, the holder first chooses when to backdate the option: she/he has the convenience to issue it when the stock price is at its highest level (i.e. at the maximum of each simulated 1-year path, at time t_{Max}) to maximize its profit. Then, she/he observes the minimum price of the stock from time zero to t_{Max}. The randomness of the GBM may create a strongly increasing stock path: in this case, the strike K would be set equal always to the minimum value, which is equal to the initial value $S_0 = 100$.
Given this framework, it follows that all the conceivable options would be issued at least *at-the-money* (as in Task 7), since, even in the worst scenario of strongly decreasing stock process, the ESO would start at time $t = 0$, where both the stock price and the strike would be set equal to 100, namely S_0.
Extremely important in Task 8 is the temporal component. We addressed it by further enhancing the logic used in Task 7: now the 3D matrix representing the different binomial trees can have different lengths, depending on when the ESO is set to be initiated. After having correctly populated the operational structures (the matrices in the Python structures), the procedure reduces to be the same as in the latter Task. The only novelty is represented by the different discounting period to be used when moving the option payoffs back to time 0, which also depend on when the ESO is initiated. In Table 8 we report one result for the ESO price computed as mentioned above.

Point estimate	12.8711
MC radius	1.2419
UB	14.1130
LB	11.6292
MC radius / point estimate	0.0964

Table 8: Task 8 simulation results (N = 90)

As expected, now the ESO is worth even more than the ESO in Task 7. This is due mainly to the "double arbitrage possibility" granted to the holder (she/he can choose both when to issue the ESO and which strike to set). On average, the simulated trajectories required to obtain an MC radius/point estimate ratio of approximately 10% are less than those in Task 7: a good degree of convergence is reached with just 90 simulations.

[6]Note that (obviously) the strike price must be a price preceding the inception of the ESO