A new approach for designing dynamic balanced serial mechanisms

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Abstract

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1 Introduction and literature review

2 Mothodology

2.1 Dynamic Model

O modelo dinâmico de um mecanismo serial pode ser escrito da seguinte maneira:

$$\mathbb{M}^{\#}(\mathbb{q}^{\#})\ddot{\mathbb{q}}^{\#} + \mathbb{v}^{\#}(\mathbb{q}^{\#},\dot{\mathbb{q}}^{\#}) + \mathbb{g}^{\#}(\mathbb{q}^{\#}) = \mathbb{u} \tag{1}$$

Sendo $\mathbb{q}^{\#}$ um conjunto de coordenadas generalizadas independentes, cujos elementos são os deslocamentos relativos das juntas e $\mathbb{u}$ os esforços generalizados nas direções das quasi-velocidades independentes $\mathbb{p}^{\#} = \dot{\mathbb{q}}^{\#}$.

Para realizar o balanceamento dinâmico de um mecanismo serial, utilizando abordagem proposta, é necessário primeiro obter o modelo do mecanismo desbalanceado. Como em um mecanismo serial é possível de expressar todas as velocidades lineares absolutas dos centros de massa das barras e todas as velocidades angulares absolutas das barras em função de $\mathbb{q}^{\#}$ e $\dot{\mathbb{q}}^{\#}$, o modelo dinâmico pode ser obtido sem grandes dificuldades utilizando métodos de mecánica analítica, como Lagrange, Kane e Orsino, aliados a programas ou bibliotecas de linguagens de programação que são capazes de utilizar manipulação simbólica, como o Mathematica e o SymPy.

2.2 Static Balancing

Depois de obtido o modelo dinâmico, realiza-se o balanceamento estático encontrando as posições dos centros de massa das barras que fazem com que $\mathbb{g}^{\#} = \mathbb{0}$. Isso é possível para mecanismos com juntas apenas rotativas e mecanismos com juntas prismáticas cuja direções são ortogonais à gravidade. O posicionamento dos centros de massa é realizado mecanicamente com o prologamento das barras do mecanismo e adição de contra-pesos.

2.3 Dynamic Balancing

O balanceamento dinâmico é obtido acoplando discos girantes ao modelo do mecanismo estáticamente balanceado. Isso é feito utilizando a técnica de acoplamento de subsistemas do Método Orsino.

Seja $\mathscr{M}_0$ um subsistema mecânico constituido por um mecanismo serial estaticamente balanceado, cuja equação de movimento é dada pela equação (1), com $\mathbb{g}^{\#} = \mathbb{0}$. Seja $\mathscr{M}_i$ um subsistema mecânico constituido de um disco girante que será acoplado ao mecanismo, cuja equação de movimento é dada por:

$$\mathbb{M}_i^{\#}\dot{\mathbb{p}}^{\#} + \mathbb{v}_i^{\#} + \mathbb{g}_i^{\#} = \mathbb{u}_i \tag{2}$$

Sendo $\mathbb{p}^{\#}$ um conjunto de quasi-velocidades independentes, cujos elementos são as componentes não nulas do vetor velocidade angular absoluta do disco, escrito em uma base solidária ao disco, e $\mathbb{v}_i^{\#} = \mathbb{g}_i^{\#} = \mathbb{u}_i = \mathbb{0}$. Nesse modelo, são considerados apenas os efeitos das inércias rotativas, sendo os efeitos da massa do disco considerados na massa e nos momentos de inércia da barra em que o disco for acoplado, efetando também no cálculo do posicionamento dos contra-pesos, de modo que o mecanismo continue estáticamente balanceado.

Supondo que irão ser acoplados n discos ao mecanismo, são feitas as seguintes definições:

$$\mathbb{M}' = \begin{bmatrix} \mathbb{M}^{\#} & \mathbb{0} & \dots & \mathbb{0} \\ \mathbb{0} & \mathbb{M}_1^{\#} & \dots & \mathbb{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \dots & \mathbb{M}_n^{\#} \end{bmatrix} \tag{3}$$

$$\mathbb{v}' = \begin{bmatrix} \mathbb{v}^{\#\mathsf{T}} & \mathbb{v}_1^{\#\mathsf{T}} & \dots & \mathbb{v}_n^{\#\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \tag{4}$$

$$\mathbb{g}' = \begin{bmatrix} \mathbb{g}^{\#\mathsf{T}} & \mathbb{g}_1^{\#\mathsf{T}} & \dots & \mathbb{g}_n^{\#\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \tag{5}$$

$$\mathbb{p}^{\circ} = \begin{bmatrix} \mathbb{p}_1^{\#\mathsf{T}} & \dots & \mathbb{p}_n^{\#\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \tag{6}$$

$$\mathbb{p} = \begin{bmatrix} \mathbb{p}^{\#\mathsf{T}} & \mathbb{p}^{\circ\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \tag{7}$$

Seja $\underline{\mathbb{p}}^{\circ}$ o vetor $\mathbb{p}^{\circ}$ escrito em função de $\mathbb{q}^{\#}$ e $\mathbb{p}^{\#}$, ou seja:

$$\mathbb{p}^{\circ} = \underline{\mathbb{p}}^{\circ}(\mathbb{q}^{\#}, \mathbb{p}^{\#}) \tag{8}$$

Definimos a seguinte matriz das restrições cinemáticas:

$$\mathbb{C} = \begin{bmatrix} \mathbb{1} \\ \dfrac{\partial \underline{\mathbb{p}}^{\circ}}{\partial \mathbb{p}^{\#}} \end{bmatrix} \tag{9}$$

O modelo dinâmico do mecanismo serial com os discos acoplados, é dado por:

$$\mathbb{M}'^{\#}(\mathbb{q}^{\#})\ddot{\mathbb{q}}^{\#} + \mathbb{v}'^{\#}(\mathbb{q}^{\#}, \dot{\mathbb{q}}^{\#}) + \mathbb{g}'^{\#}(\mathbb{q}^{\#}) = \mathbb{u} \tag{10}$$

Sendo:

$$\mathbb{M}'^{\#} = \mathbb{C}^{\mathsf{T}}\mathbb{M}'\mathbb{C} \tag{11}$$

$$\mathbb{v}'^{\#} = \mathbb{C}^{\mathsf{T}}(\mathbb{M}'\dot{\mathbb{C}}\dot{\mathbb{q}}^{\#} + \mathbb{v}') \tag{12}$$

$$\mathbb{g}'^{\#} = \mathbb{C}^{\mathsf{T}}\mathbb{g}' \tag{13}$$

O balanceamento dinâmico é obtido encontrando as relações entre os parâmetros do sistema que tornam a matriz $\mathbb{M}'^{\#}$ diagonal e o vetor $\mathbb{v}'^{\#}$ nulo.

3 Applying the technique

For a 3-DOF serial mechanism:

$$\mathbb{M}^{\#} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \tag{14}$$

$$\mathbb{v}^{\#} = \begin{bmatrix} D_{111} & D_{122} & D_{133} \\ D_{211} & D_{222} & D_{233} \\ D_{311} & D_{322} & D_{333} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \dot{q}_3^2 \end{bmatrix} + 2 \begin{bmatrix} D_{112} & D_{113} & D_{123} \\ D_{212} & D_{213} & D_{223} \\ D_{312} & D_{313} & D_{323} \end{bmatrix} \begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \dot{q}_2\dot{q}_3 \end{bmatrix} \tag{15}$$

$$\mathbb{g}^{\#} = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}^{\top} \tag{16}$$

$$\mathbb{q}^{\#} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^{\top} \tag{17}$$

$$\mathbb{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{\top} \tag{18}$$

For a rotational joint we call $q_i = \theta_i$ and $u_i = \tau_i$, and for a prismatic joint we call $q_i = d_i$ and $u_i = f_i$.

3.1 3-DOF RRR planar serial mechanism

$$\begin{cases} D_1 = g[(m_1 l_{g_1} + m_2 l_1 + m_3 l_1)\mathsf{c}(\theta_1) + (m_2 l_{g_2} + m_3 l_2)\mathsf{c}(\theta_1+\theta_2) + m_3 l_{g_3}\mathsf{c}(\theta_1+\theta_2+\theta_3)] \\ D_2 = g[(m_2 l_{g_2} + m_3 l_2)\mathsf{c}(\theta_1+\theta_2) + m_3 l_{g_3}\mathsf{c}(\theta_1+\theta_2+\theta_3)] \\ D_3 = g[m_3 l_{g_3}\mathsf{c}(\theta_1+\theta_2+\theta_3)] \end{cases} \tag{19}$$

Static balancing:

$$\begin{cases} D_1 = 0 \\ D_2 = 0 \\ D_3 = 0 \end{cases} \Rightarrow \begin{cases} l_{g_1} = -\frac{l_1(m_2+m_3)}{m_1} \\ l_{g_2} = -\frac{l_2 m_3}{m_2} \\ l_{g_3} = 0 \end{cases} \tag{20}$$

Static balanced mechanism:

$$\begin{cases} D_{11} = J_{x_1} + J_{x_2} + J_{x_3} + m_2 l_1^2 + m_3(l_1^2 + l_2^2) + \frac{l_1^2(m_2+m_3)^2}{m_1} + \frac{l_2^2 m_3^2}{m_2} \\ D_{22} = J_{x_2} + J_{x_3} + m_3 l_2^2 + \frac{l_2^2 m_3^2}{m_2} \\ D_{33} = J_{x_3} \\ D_{12} = D_{22} \\ D_{13} = D_{23} = D_{33} \\ \mathbb{v}^{\#} = \mathbb{0} \\ \mathbb{g}^{\#} = \mathbb{0} \end{cases} \tag{21}$$

Coupling 4 discs:

Planar disc models:

$$\mathbb{M}_i^{\#} = \begin{bmatrix} J_{x_{i+3}} \end{bmatrix}; \ \mathbb{p}_i^{\#} = \begin{bmatrix} \omega_{x_{i+3}} \end{bmatrix}, \ i = 1,2,3,4 \tag{22}$$

Quasi-velocities constraints:

$$\begin{cases} \omega_{x_4} = \omega_{x_1} + \dot{\theta}_2 \\ \omega_{x_5} = \omega_{x_1} + \beta\dot{\theta}_2 \\ \omega_{x_6} = \omega_{x_2} + \dot{\theta}_3 \\ \omega_{x_7} = \omega_{x_2} + \gamma\dot{\theta}_3 \end{cases} \Rightarrow \begin{cases} \omega_{x_4} = \dot{\theta}_1 + \dot{\theta}_2 \\ \omega_{x_5} = \dot{\theta}_1 + \beta\dot{\theta}_2 \\ \omega_{x_6} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \\ \omega_{x_7} = \dot{\theta}_1 + \dot{\theta}_2 + \gamma\dot{\theta}_3 \end{cases} \Rightarrow \underline{\mathbb{p}}^{\circ} = \begin{bmatrix} \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 + \beta\dot{\theta}_2 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \\ \dot{\theta}_1 + \dot{\theta}_2 + \gamma\dot{\theta}_3 \end{bmatrix} \tag{23}$$

$$\mathbb{C} = \begin{bmatrix} \mathbb{1} \\ \dfrac{\partial \underline{\mathbb{p}}^{\circ}}{\partial \mathbb{p}^{\#}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & \beta & 0 \\ 1 & 1 & 1 \\ 1 & 1 & \gamma \end{bmatrix} \tag{24}$$

$$\begin{cases} D'_{11} = D_{11} + J_{x_4} + J_{x_5} + J_{x_6} + J_{x_7} \\ D'_{22} = D_{22} + J_{x_4} + J_{x_5}\beta^2 + J_{x_6} + J_{x_7} \\ D'_{33} = D_{33} + J_{x_6} + J_{x_7}\gamma^2 \\ D'_{12} = D_{12} + J_{x_4} + J_{x_5}\beta + J_{x_6} + J_{x_7} \\ D'_{13} = D_{13} + J_{x_6} + J_{x_7}\gamma \\ D'_{23} = D'_{13} \\ \mathbb{v}'^{\#} = \mathbb{0} \end{cases} \tag{25}$$

Dynamic balancing:

$$\begin{cases} D'_{12} = 0 \\ D'_{13} = 0 \end{cases} \Rightarrow \begin{cases} \beta = -\frac{J_{x_2} + J_{x_3} + J_{x_4} + J_{x_6} + J_{x_7} + m_3 l_2^2 + \frac{m_3^2 l_2^2}{m_2}}{J_{x_5}} \\ \gamma = -\frac{J_{x_3} + J_{x_6}}{J_{x_7}} \end{cases} \tag{26}$$

Dynamic balanced mechanism:

$$\begin{cases} \tau_1 = k_1 \ddot{\theta}_1 \\ \tau_2 = k_2 \ddot{\theta}_2 \\ \tau_3 = k_3 \ddot{\theta}_3 \end{cases} \tag{27}$$

Being:

$$\begin{cases} k_1 = J_{x_1} + J_{x_2} + J_{x_3} + J_{x_4} + J_{x_5} + J_{x_6} + J_{x_7} + m_2 l_1^2 + m_3(l_1^2 + l_2^2) + \frac{l_1^2 (m_2 + m_3)^2}{m_1} + \frac{l_2^2 m_3^2}{m_2} \\ k_2 = J_{x_2} + J_{x_3} + J_{x_4} + J_{x_6} + J_{x_7} + m_3 l_2^2 + \frac{l_2^2 m_3^2}{m_2} + \frac{\left(J_{x_2} + J_{x_3} + J_{x_4} + J_{x_6} + J_{x_7} + m_3 l_2^2 + \frac{m_3^2 l_2^2}{m_2}\right)^2}{J_{x_5}} \\ k_3 = \frac{(J_{x_3} + J_{x_6})(J_{x_3} + J_{x_6} + J_{x_7})}{J_{x_7}} \end{cases} \tag{28}$$

3.2 3-DOF RRR spatial serial mechanism

$$\begin{cases} D_1 = 0 \\ D_2 = g[(m_2 l_{g_2} + m_3 l_2)\mathsf{c}(\theta_2) + m_3 l_{g_3}\mathsf{c}(\theta_2 + \theta_3)] \\ D_3 = g[m_3 l_{g_3}\mathsf{c}(\theta_2 + \theta_3)] \end{cases} \tag{29}$$

Static balancing:

$$\begin{cases} D_2 = 0 \\ D_3 = 0 \end{cases} \Rightarrow \begin{cases} l_{g_2} = -\frac{l_2 m_3}{m_2} \\ l_{g_3} = 0 \end{cases} \tag{30}$$

Static balanced mechanism:

$$\begin{cases} D_{11} = J_{y_2}\mathsf{s}^2(\theta_2) + J_{y_3}\mathsf{s}^2(\theta_2+\theta_3) + J_{z_1} + J_{z_2}\mathsf{c}^2(\theta_2) + J_{z_3}\mathsf{c}^2(\theta_2+\theta_3) + m_3(l_1 + l_2\mathsf{c}(\theta_2))^2 + \frac{(m_2 l_1 - m_3 l_2 \mathsf{c}(\theta_2))^2}{m_2} \\ D_{22} = J_{x_2} + J_{x_3} + m_2 l_2^2 + \frac{l_2^2 m_3^2}{m_2} \\ D_{33} = J_{x_3} \\ D_{12} = D_{13} = 0 \\ D_{23} = D_{33} \\ D_{211} = -\frac{1}{2}\Big(\big(J_{y_2} - J_{z_2}\big)\mathsf{s}(2\theta_2) + \big(J_{y_3} - J_{z_3} - m_3 l_2^2(1 + \frac{m_3}{m_2})\big)\mathsf{s}(2\theta_2 + 2\theta_3)\Big) \\ D_{311} = \frac{1}{2}\Big(\big(J_{z_3} - J_{y_3}\big)\mathsf{s}(2\theta_2 + 2\theta_3)\Big) \\ D_{111} = D_{122} = D_{133} = D_{222} = D_{233} = D_{322} = D_{333} = 0 \\ D_{112} = -D_{211} \\ D_{113} = -D_{311} \\ D_{123} = D_{212} = D_{213} = D_{223} = D_{312} = D_{313} = D_{323} = 0 \\ \mathbb{g} = \mathbb{0} \end{cases} \tag{31}$$

Coupling 2 discs:

Spatial disc models:

$$\mathbb{M}_i^{\#} = \begin{bmatrix} J_{x_{i+3}} & 0 & 0 \\ 0 & J_{y_{i+3}} & 0 \\ 0 & 0 & J_{z_{i+3}} \end{bmatrix} ; \ \mathbb{p}_i^{\#} = \begin{bmatrix} \omega_{x_{i+3}} \\ \omega_{y_{i+3}} \\ \omega_{z_{i+3}} \end{bmatrix}, \ i = 1, 2 \tag{32}$$

Quasi-velocities constraints:

$$\begin{cases} [\boldsymbol{\omega}_4]_{\mathsf{B}_4} = [\mathbb{1}]_{\mathsf{B}_4 \mid \mathsf{B}_2}[\boldsymbol{\omega}_2]_{\mathsf{B}_2} + \begin{bmatrix} \dot{\theta}_3 \\ 0 \\ 0 \end{bmatrix} \\ [\boldsymbol{\omega}_5]_{\mathsf{B}_5} = [\mathbb{1}]_{\mathsf{B}_5 \mid \mathsf{B}_2}[\boldsymbol{\omega}_2]_{\mathsf{B}_2} + \begin{bmatrix} \beta\dot{\theta}_3 \\ 0 \\ 0 \end{bmatrix} \end{cases} \Rightarrow \begin{cases} \omega_{x_4} = \dot{\theta}_2 + \dot{\theta}_3 \\ \omega_{y_4} = (\dot{\theta}_1\mathsf{s}(\theta_2))\mathsf{c}(\theta_3) + (\dot{\theta}_1\mathsf{c}(\theta_2))\mathsf{s}(\theta_3) \\ \omega_{z_4} = -(\dot{\theta}_1\mathsf{s}(\theta_2))\mathsf{s}(\theta_3) + (\dot{\theta}_1\mathsf{c}(\theta_2))\mathsf{c}(\theta_3) \\ \omega_{x_5} = \dot{\theta}_2 + \beta\dot{\theta}_3 \\ \omega_{y_5} = (\dot{\theta}_1\mathsf{s}(\theta_2))\mathsf{c}(\beta\theta_3) + (\dot{\theta}_1\mathsf{c}(\theta_2))\mathsf{s}(\beta\theta_3) \\ \omega_{z_5} = -(\dot{\theta}_1\mathsf{s}(\theta_2))\mathsf{s}(\beta\theta_3) + (\dot{\theta}_1\mathsf{c}(\theta_2))\mathsf{c}(\beta\theta_3) \end{cases} \tag{33}$$

$$\Rightarrow \underline{\mathbb{p}}^{\circ} = \begin{bmatrix} \dot{\theta}_2 + \dot{\theta}_3 \\ \dot{\theta}_1\mathsf{s}(\theta_2 + \theta_3) \\ \dot{\theta}_1\mathsf{c}(\theta_2 + \theta_3) \\ \dot{\theta}_2 + \beta\dot{\theta}_3 \\ \dot{\theta}_1\mathsf{s}(\theta_2 + \beta\theta_3) \\ \dot{\theta}_1\mathsf{c}(\theta_2 + \beta\theta_3) \end{bmatrix} \tag{34}$$

$$\mathbb{C} = \begin{bmatrix} \mathbb{1} \\ \frac{\partial \mathbb{p}^\circ}{\partial \mathbb{p}^\#} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ \mathsf{s}(\theta_2+\theta_3) & 0 & 0 \\ \mathsf{c}(\theta_2+\theta_3) & 0 & 0 \\ 0 & 1 & \beta \\ \mathsf{s}(\theta_2+\beta\theta_3) & 0 & 0 \\ \mathsf{c}(\theta_2+\beta\theta_3) & 0 & 0 \end{bmatrix} \tag{35}$$

$$\begin{cases} D'_{11} = D_{11} + J_{y_4}\mathsf{s}^2(\theta_2+\theta_2) + J_{y_5}\mathsf{s}^2(\beta\theta_2+\theta_2) + J_{z_4}\mathsf{c}^2(\theta_2+\theta_2) + J_{z_5}\mathsf{c}^2(\beta\theta_2+\theta_2) \\ D'_{22} = D_{22} + J_{x_4} + J_{x_5} \\ D'_{33} = D_{33} + J_{x_4} + J_{x_5}\beta^2 \\ D'_{12} = D'_{13} = 0 \\ D'_{23} = D_{23} + J_{x_4} + J_{x_5}\beta \\ D'_{211} = D_{211} \\ D'_{311} = D_{311} \\ D'_{111} = D'_{122} = D'_{133} = D'_{222} = D'_{233} = D'_{322} = D'_{333} = 0 \\ D'_{112} = D_{112} + \frac{1}{4}\Big((J_{y_4} - J_{z_4})\mathsf{s}(2\theta_2+2\theta_3) + (J_{y_5} - J_{z_5})\mathsf{s}(2\beta\theta_2+2\theta_3)\Big) \\ D'_{113} = D_{113} + \frac{1}{4}\Big((J_{y_4} - J_{z_4})\mathsf{s}(2\theta_2+2\theta_3) + (J_{y_5} - J_{z_5})\mathsf{s}(2\beta\theta_2+2\theta_3)\Big) \\ D'_{123} = D'_{212} = D'_{213} = D'_{223} = D'_{312} = D'_{313} = D'_{323} = 0 \end{cases} \tag{36}$$

Dynamic balancing:

$$\begin{cases} D'_{23} = 0 \\ D'_{211} = 0 \\ D'_{311} = 0 \\ D'_{112} = 0 \\ D'_{113} = 0 \end{cases} \Rightarrow \begin{cases} \beta = -\frac{J_{x_3}+J_{x_4}}{J_{x_5}} \\ J_{y_2} = J_{z_2} \\ J_{y_3} = J_{z_3} + m_3 l_2^2(1 + \frac{m_3}{m_2}) \\ J_{y_4} = J_{z_4} \\ J_{y_5} = J_{z_5} \end{cases} \tag{37}$$

Dynamic balanced mechanism:

$$\begin{cases} \tau_1 = k_1\ddot{\theta}_1 \\ \tau_2 = k_2\ddot{\theta}_2 \\ f_3 = k_3\ddot{d}_3 \end{cases} \tag{38}$$

Being:

$$\begin{cases} k_1 = J_{z_1} + J_{z_2} + J_{z_3} + J_{z_4} + J_{z_5} + m_2 l_1^2 + m_3(l_1^2 + l_2^2) + \frac{l_1^2 m_3^2}{m_2} \\ k_2 = J_{x_2} + J_{x_3} + J_{x_4} + J_{x_5} + m_3 l_2^2 + \frac{l_1^2 m_3^2}{m_2} \\ k_3 = \frac{(J_{x_3}+J_{x_4})(J_{x_3}+J_{x_4}+J_{x_5})}{J_{x_5}} \end{cases} \tag{39}$$

3.3 3-DOF RRP spatial serial mechanism (SCARA)

$$\begin{cases} D_1 = g[(m_1 l_{g_1} + m_2 l_1 + m_3 l_1)\mathsf{c}(\theta_1) + m_2 l_{g_2}\mathsf{c}(\theta_1+\theta_2)] \\ D_2 = g[m_2 l_{g_2}\mathsf{c}(\theta_1+\theta_2)] \\ D_3 = 0 \end{cases} \tag{40}$$

Static balancing:

$$\begin{cases} D_1 = 0 \\ D_2 = 0 \end{cases} \Rightarrow \begin{cases} l_{g_1} = -\frac{l_1(m_2+m_3)}{m_1} \\ l_{g_2} = 0 \end{cases} \tag{41}$$

Static balanced mechanism:

$$\begin{cases} D_{11} = J_{x_1} + J_{x_2} + J_{x_3} + m_2 l_1^2 + m_3 l_1^2 + \frac{l_1^2(m_2+m_3)^2}{m_1} \\ D_{22} = J_{x_2} + J_{x_3} \\ D_{33} = m_3 \\ D_{12} = D_{22} \\ D_{13} = D_{23} = 0 \\ \mathbb{v}^{\#} = \mathbb{0} \\ \mathbb{g}^{\#} = \mathbb{0} \end{cases} \tag{42}$$

Coupling 2 discs:

Planar disc models:

$$\mathbb{M}_i^{\#} = \left[J_{x_{i+3}}\right];\ \mathbb{p}_i^{\#} = \left[\omega_{x_{i+3}}\right],\ i = 1,2 \tag{43}$$

Quasi-velocities constraints:

$$\begin{cases} \omega_{x_4} = \omega_{x_1} + \dot{\theta}_2 \\ \omega_{x_5} = \omega_{x_1} + \beta\dot{\theta}_2 \end{cases} \Rightarrow \begin{cases} \omega_{x_4} = \dot{\theta}_1 + \dot{\theta}_2 \\ \omega_{x_5} = \dot{\theta}_1 + \beta\dot{\theta}_2 \end{cases} \Rightarrow \overline{\mathbb{p}} = \begin{bmatrix} \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 + \beta\dot{\theta}_2 \end{bmatrix} \tag{44}$$

$$\mathbb{C} = \begin{bmatrix} \mathbb{1} \\ \dfrac{\partial \mathbb{p}^{\circ}}{\partial \mathbb{p}^{\#}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & \beta & 0 \end{bmatrix} \tag{45}$$

$$\begin{cases} D'_{11} = D_{11} + J_{x_4} + J_{x_5} \\ D'_{22} = D_{22} + J_{x_4} + J_{x_5}\beta^2 \\ D'_{33} = D_{33} \\ D'_{12} = D_{12} + J_{x_4} + J_{x_5}\beta \\ D'_{13} = 0 \\ D'_{23} = 0 \\ \mathbb{v}'^{\#} = \mathbb{0} \end{cases} \tag{46}$$

Dynamic balancing:

$$D'_{12} = 0 \Rightarrow \beta = -\frac{J_{x_2} + J_{x_3} + J_{x_4}}{J_{x_5}} \tag{47}$$

Dynamic balanced mechanism:

$$\begin{cases} \tau_1 = k_1 \ddot{\theta}_1 \\ \tau_2 = k_2 \ddot{\theta}_2 \\ f_3 = k_3 \ddot{d}_3 \end{cases} \tag{48}$$

Being:

$$\begin{cases} k_1 = J_{x_1} + J_{x_2} + J_{x_3} + J_{x_4} + J_{x_5} + m_2 l_1^2 + m_3 l_1^2 + \frac{l_1^2 (m_2+m_3)^2}{m_1} \\ k_2 = \frac{(J_{x_2}+J_{x_3}+J_{x_4})(J_{x_2}+J_{x_3}+J_{x_4}+J_{x_5})}{J_{x_5}} \\ k_3 = m_3 \end{cases} \tag{49}$$

4 Conclusions

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