

A new approach for designing dynamic balanced serial mechanisms

André Garnier Coutinho^a, Tarcisio Antonio Hess Coelho^b

^a *Department of Mechatronics and Mechanical Systems Engineering, Escola Politecnica, University of Sao Paulo, Brazil. E-mail: andre.garnier.coutinho@usp.br*

^b *Department of Mechatronics and Mechanical Systems Engineering, Escola Politecnica, University of Sao Paulo, Brazil. E-mail: tarchess@usp.br*

Abstract

Balancing is an important issue related to the design of mechanical systems in general, and also parallel mechanisms, in particular. In fact, the performance of parallel mechanisms associated to specific applications depends on the choice of the balancing method, namely, either static or dynamic, either passive or active, whether it is valid for a given trajectory or even for any motion. The main contribution of this work is to highlight the importance of the dynamic modelling process in order to achieve the compensation conditions associated to the chosen balancing technique. Due to the fact that parallel mechanisms have highly complex structures, the use of dynamic formalisms that employ redundant generalized coordinates, in association with the successive coupling of additional balancing elements to the original system model, can bring remarkable benefits. Additionally, this book chapter also discusses the impact of the dynamic model, developed in accordance with the methodology shown here, for the control strategy of parallel mechanisms. Finally, the simulation results demonstrates how effective is the presented methodology for the planar 5-bar with revolute joints (5R).

KEYWORDS: Dynamic balancing, serial mechanisms

1 Introduction and literature review

2 Methodology

2.1 Dynamic Model

O modelo dinâmico de um mecanismo serial pode ser escrito da seguinte maneira:

$$\mathbb{M}^{\#}(\mathbf{q}^{\#})\ddot{\mathbf{q}}^{\#} + \mathbf{v}^{\#}(\mathbf{q}^{\#}, \dot{\mathbf{q}}^{\#}) + \mathbf{g}^{\#}(\mathbf{q}^{\#}) = \mathbf{u} \quad (1)$$

Sendo $\mathbf{q}^{\#}$ um conjunto de coordenadas generalizadas independentes, cujos elementos são os deslocamentos relativos das juntas e \mathbf{u} os esforços generalizados nas direções das quasi-velocidades independentes $\mathbf{p}^{\#} = \dot{\mathbf{q}}^{\#}$.

Para realizar o balanceamento dinâmico de um mecanismo serial, utilizando abordagem proposta, é necessário primeiro obter o modelo do mecanismo desbalanceado. Como em um mecanismo serial é possível de expressar todas as velocidades lineares absolutas dos centros de massa das barras e todas as velocidades angulares absolutas das barras em função de $\mathbf{q}^{\#}$ e $\dot{\mathbf{q}}^{\#}$, o modelo dinâmico pode ser obtido sem grandes dificuldades utilizando métodos de mecânica analítica, como Lagrange, Kane e Orsino, aliados a programas ou bibliotecas de linguagens de programação que são capazes de utilizar manipulação simbólica, como o Mathematica e o SymPy.

2.2 Static Balancing

Depois de obtido o modelo dinâmico, realiza-se o balanceamento estático encontrando as posições dos centros de massa das barras que fazem com que $\mathbf{g}^{\#} = \mathbf{0}$. Isso é possível para mecanismos com juntas apenas rotativas

e mecanismos com juntas prismáticas cuja direções são ortogonais à gravidade. O posicionamento dos centros de massa é realizado mecanicamente com o prologamento das barras do mecanismo e adição de contra-pesos.

2.3 Dynamic Balancing

O balanceamento dinâmico é obtido acoplando discos girantes ao modelo do mecanismo estáticamente balanceado. Isso é feito utilizando a técnica de acoplamento de subsistemas do Método Orsino.

Seja \mathcal{M}_0 um subsistema mecânico constituído por um mecanismo serial estaticamente balanceado, cuja equação de movimento é dada pela equação (1), com $\mathfrak{g}^\# = \mathbb{0}$. Seja \mathcal{M}_i um subsistema mecânico constituído de um disco girante que será acoplado ao mecanismo, cuja equação de movimento é dada por:

$$\mathbb{M}_i^\# \dot{\mathfrak{p}}^\# + \mathfrak{v}_i^\# + \mathfrak{g}_i^\# = \mathfrak{u}_i \quad (2)$$

Sendo $\mathfrak{p}^\#$ um conjunto de quasi-velocidades independentes, cujos elementos são as componentes não nulas do vetor velocidade angular absoluta do disco, escrito em uma base solidária ao disco, e $\mathfrak{v}_i^\# = \mathfrak{g}_i^\# = \mathfrak{u}_i = \mathbb{0}$. Nesse modelo, são considerados apenas os efeitos das inércias rotativas, sendo os efeitos da massa do disco considerados na massa e nos momentos de inércia da barra em que o disco for acoplado, afetando também no cálculo do posicionamento dos contra-pesos, de modo que o mecanismo continue estáticamente balanceado.

Supondo que irão ser acoplados n discos ao mecanismo, são feitas as seguintes definições:

$$\mathbb{M}' = \begin{bmatrix} \mathbb{M}^\# & \mathbb{0} & \dots & \mathbb{0} \\ \mathbb{0} & \mathbb{M}_1^\# & \dots & \mathbb{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \dots & \mathbb{M}_n^\# \end{bmatrix} \quad (3)$$

$$\mathfrak{v}' = \begin{bmatrix} \mathfrak{v}^\#^\top & \mathfrak{v}_1^\#^\top & \dots & \mathfrak{v}_n^\#^\top \end{bmatrix}^\top \quad (4)$$

$$\mathfrak{g}' = \begin{bmatrix} \mathfrak{g}^\#^\top & \mathfrak{g}_1^\#^\top & \dots & \mathfrak{g}_n^\#^\top \end{bmatrix}^\top \quad (5)$$

$$\mathfrak{p}^\circ = \begin{bmatrix} \mathfrak{p}_1^\#^\top & \dots & \mathfrak{p}_n^\#^\top \end{bmatrix}^\top \quad (6)$$

$$\mathfrak{p} = \begin{bmatrix} \mathfrak{p}^\#^\top & \mathfrak{p}^\circ^\top \end{bmatrix}^\top \quad (7)$$

Seja $\underline{\mathfrak{p}}^\circ$ o vetor \mathfrak{p}° escrito em função de $\mathfrak{q}^\#$ e $\mathfrak{p}^\#$, ou seja:

$$\mathfrak{p}^\circ = \underline{\mathfrak{p}}^\circ(\mathfrak{q}^\#, \mathfrak{p}^\#) \quad (8)$$

Definimos a seguinte matriz das restrições cinemáticas:

$$\mathbb{C} = \begin{bmatrix} \mathbb{1} \\ \frac{\partial \underline{\mathfrak{p}}^\circ}{\partial \mathfrak{p}^\#} \end{bmatrix} \quad (9)$$

O modelo dinâmico do mecanismo serial com os discos acoplados, é dado por:

$$\mathbb{M}'^\#(\mathfrak{q}^\#) \ddot{\mathfrak{q}}^\# + \mathfrak{v}'^\#(\mathfrak{q}^\#, \dot{\mathfrak{q}}^\#) + \mathfrak{g}'^\#(\mathfrak{q}^\#) = \mathfrak{u} \quad (10)$$

Sendo:

$$\mathbb{M}'^\# = \mathbb{C}^\top \mathbb{M}' \mathbb{C} \quad (11)$$

$$\mathbf{v}'^\# = \mathbb{C}^\top (\mathbb{M}' \dot{\mathbb{C}} \dot{\mathbf{q}}^\# + \mathbf{v}') \quad (12)$$

$$\mathbf{g}'^\# = \mathbb{C}^\top \mathbf{g}' \quad (13)$$

O balanceamento dinâmico é obtido encontrando as relações entre os parâmetros do sistema que tornam a matriz $\mathbb{M}'^\#$ diagonal e o vetor $\mathbf{v}'^\#$ nulo.

3 Applying the technique

For a 3-DOF serial mechanism:

$$\mathbb{M}^\# = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \quad (14)$$

$$\mathbf{v}^\# = \begin{bmatrix} D_{111} & D_{122} & D_{133} \\ D_{211} & D_{222} & D_{233} \\ D_{311} & D_{322} & D_{333} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \dot{q}_3^2 \end{bmatrix} + 2 \begin{bmatrix} D_{112} & D_{113} & D_{123} \\ D_{212} & D_{213} & D_{223} \\ D_{312} & D_{313} & D_{323} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \dot{q}_2 \\ \dot{q}_1 \dot{q}_3 \\ \dot{q}_2 \dot{q}_3 \end{bmatrix} \quad (15)$$

$$\mathbf{g}^\# = [D_1 \quad D_2 \quad D_3]^\top \quad (16)$$

$$\mathbf{q}^\# = [q_1 \quad q_2 \quad q_3]^\top \quad (17)$$

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3]^\top \quad (18)$$

For a rotational joint we call $q_i = \theta_i$ and $u_i = \tau_i$, and for a prismatic joint we call $q_i = d_i$ and $u_i = f_i$.

3.1 3-DOF RRR planar serial mechanism

$$\begin{cases} D_1 = g[(m_1 l_{g1} + m_2 l_1 + m_3 l_1) \mathbf{c}(\theta_1) + (m_2 l_{g2} + m_3 l_2) \mathbf{c}(\theta_1 + \theta_2) + m_3 l_{g3} \mathbf{c}(\theta_1 + \theta_2 + \theta_3)] \\ D_2 = g[(m_2 l_{g2} + m_3 l_2) \mathbf{c}(\theta_1 + \theta_2) + m_3 l_{g3} \mathbf{c}(\theta_1 + \theta_2 + \theta_3)] \\ D_3 = g[m_3 l_{g3} \mathbf{c}(\theta_1 + \theta_2 + \theta_3)] \end{cases} \quad (19)$$

Static balancing:

$$\begin{cases} D_1 = 0 \\ D_2 = 0 \\ D_3 = 0 \end{cases} \Rightarrow \begin{cases} l_{g1} = -\frac{l_1(m_2+m_3)}{m_1} \\ l_{g2} = -\frac{l_2 m_3}{m_2} \\ l_{g3} = 0 \end{cases} \quad (20)$$

Static balanced mechanism:

$$\begin{cases} D_{11} = J_{x_1} + J_{x_2} + J_{x_3} + m_2 l_1^2 + m_3(l_1^2 + l_2^2) + \frac{l_1^2(m_2+m_3)^2}{m_1} + \frac{l_2^2 m_3^2}{m_2} \\ D_{22} = J_{x_2} + J_{x_3} + m_3 l_2^2 + \frac{l_2^2 m_3^2}{m_2} \\ D_{33} = J_{x_3} \\ D_{12} = D_{22} \\ D_{13} = D_{23} = D_{33} \\ \mathbb{V}^\# = \mathbb{0} \\ \mathbb{G}^\# = \mathbb{0} \end{cases} \quad (21)$$

Coupling 4 discs:

Planar disc models:

$$\mathbb{M}_i^\# = [J_{x_{i+3}}] ; \mathbb{P}_i^\# = [\omega_{x_{i+3}}], \quad i = 1, 2, 3, 4 \quad (22)$$

Quasi-velocities constraints:

$$\begin{cases} \omega_{x_4} = \omega_{x_1} + \dot{\theta}_2 \\ \omega_{x_5} = \omega_{x_1} + \beta \dot{\theta}_2 \\ \omega_{x_6} = \omega_{x_2} + \dot{\theta}_3 \\ \omega_{x_7} = \omega_{x_2} + \gamma \dot{\theta}_3 \end{cases} \Rightarrow \begin{cases} \omega_{x_4} = \dot{\theta}_1 + \dot{\theta}_2 \\ \omega_{x_5} = \dot{\theta}_1 + \beta \dot{\theta}_2 \\ \omega_{x_6} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \\ \omega_{x_7} = \dot{\theta}_1 + \dot{\theta}_2 + \gamma \dot{\theta}_3 \end{cases} \Rightarrow \underline{\mathbb{P}}^\circ = \begin{bmatrix} \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 + \beta \dot{\theta}_2 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \\ \dot{\theta}_1 + \dot{\theta}_2 + \gamma \dot{\theta}_3 \end{bmatrix} \quad (23)$$

$$\mathbb{C} = \begin{bmatrix} \mathbb{1} \\ \frac{\partial \underline{\mathbb{P}}^\circ}{\partial \mathbb{P}^\#} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & \beta & 0 \\ 1 & 1 & 1 \\ 1 & 1 & \gamma \end{bmatrix} \quad (24)$$

$$\begin{cases} D'_{11} = D_{11} + J_{x_4} + J_{x_5} + J_{x_6} + J_{x_7} \\ D'_{22} = D_{22} + J_{x_4} + J_{x_5} \beta^2 + J_{x_6} + J_{x_7} \\ D'_{33} = D_{33} + J_{x_6} + J_{x_7} \gamma^2 \\ D'_{12} = D_{12} + J_{x_4} + J_{x_5} \beta + J_{x_6} + J_{x_7} \\ D'_{13} = D_{13} + J_{x_6} + J_{x_7} \gamma \\ D'_{23} = D'_{13} \\ \mathbb{V}'^\# = \mathbb{0} \end{cases} \quad (25)$$

Dynamic balancing:

$$\begin{cases} D'_{12} = 0 \\ D'_{13} = 0 \end{cases} \Rightarrow \begin{cases} \beta = -\frac{J_{x_2} + J_{x_3} + J_{x_4} + J_{x_6} + J_{x_7} + m_3 l_2^2 + \frac{m_3^2 l_2^2}{m_2}}{J_{x_5}} \\ \gamma = -\frac{J_{x_3} + J_{x_6}}{J_{x_7}} \end{cases} \quad (26)$$

Dynamic balanced mechanism:

$$\begin{cases} \tau_1 = k_1 \ddot{\theta}_1 \\ \tau_2 = k_2 \ddot{\theta}_2 \\ \tau_3 = k_3 \ddot{\theta}_3 \end{cases} \quad (27)$$

Being:

$$\begin{cases} k_1 = J_{x_1} + J_{x_2} + J_{x_3} + J_{x_4} + J_{x_5} + J_{x_6} + J_{x_7} + m_2 l_1^2 + m_3 (l_1^2 + l_2^2) + \frac{l_1^2 (m_2 + m_3)^2}{m_1} + \frac{l_2^2 m_3^2}{m_2} \\ k_2 = J_{x_2} + J_{x_3} + J_{x_4} + J_{x_6} + J_{x_7} + m_3 l_2^2 + \frac{l_2^2 m_3^2}{m_2} + \frac{(J_{x_2} + J_{x_3} + J_{x_4} + J_{x_6} + J_{x_7} + m_3 l_2^2 + \frac{m_3^2 l_2^2}{m_2})^2}{J_{x_5}} \\ k_3 = \frac{(J_{x_3} + J_{x_6})(J_{x_3} + J_{x_6} + J_{x_7})}{J_{x_7}} \end{cases} \quad (28)$$

3.2 3-DOF RRR spatial serial mechanism

$$\begin{cases} D_1 = 0 \\ D_2 = g[(m_2 l_{g_2} + m_3 l_2) \mathbf{c}(\theta_2) + m_3 l_{g_3} \mathbf{c}(\theta_2 + \theta_3)] \\ D_3 = g[m_3 l_{g_3} \mathbf{c}(\theta_2 + \theta_3)] \end{cases} \quad (29)$$

Static balancing:

$$\begin{cases} D_2 = 0 \\ D_3 = 0 \end{cases} \Rightarrow \begin{cases} l_{g_2} = -\frac{l_2 m_3}{m_2} \\ l_{g_3} = 0 \end{cases} \quad (30)$$

Static balanced mechanism:

$$\begin{cases} D_{11} = J_{y_2} s^2(\theta_2) + J_{y_3} s^2(\theta_2 + \theta_3) + J_{z_1} + J_{z_2} c^2(\theta_2) + J_{z_3} c^2(\theta_2 + \theta_3) + m_3 (l_1 + l_2 c(\theta_2))^2 + \frac{(m_2 l_1 - m_3 l_2 c(\theta_2))^2}{m_2} \\ D_{22} = J_{x_2} + J_{x_3} + m_2 l_2^2 + \frac{l_2^2 m_3^2}{m_2} \\ D_{33} = J_{x_3} \\ D_{12} = D_{13} = 0 \\ D_{23} = D_{33} \\ D_{211} = -\frac{1}{2} \left((J_{y_2} - J_{z_2}) s(2\theta_2) + (J_{y_3} - J_{z_3} - m_3 l_2^2 (1 + \frac{m_3}{m_2})) s(2\theta_2 + 2\theta_3) \right) \\ D_{311} = \frac{1}{2} \left((J_{z_3} - J_{y_3}) s(2\theta_2 + 2\theta_3) \right) \\ D_{111} = D_{122} = D_{133} = D_{222} = D_{233} = D_{322} = D_{333} = 0 \\ D_{112} = -D_{211} \\ D_{113} = -D_{311} \\ D_{123} = D_{212} = D_{213} = D_{223} = D_{312} = D_{313} = D_{323} = 0 \\ \mathfrak{g} = 0 \end{cases} \quad (31)$$

Coupling 2 discs:

Spatial disc models:

$$\mathbb{M}_i^\# = \begin{bmatrix} J_{x_{i+3}} & 0 & 0 \\ 0 & J_{y_{i+3}} & 0 \\ 0 & 0 & J_{z_{i+3}} \end{bmatrix}; \quad \mathbb{P}_i^\# = \begin{bmatrix} \omega_{x_{i+3}} \\ \omega_{y_{i+3}} \\ \omega_{z_{i+3}} \end{bmatrix}, \quad i = 1, 2 \quad (32)$$

Quasi-velocities constraints:

$$\begin{cases} [\boldsymbol{\omega}_4]_{B_4} = [\mathbb{1}]_{B_4 | B_2} [\boldsymbol{\omega}_2]_{B_2} + \begin{bmatrix} \dot{\theta}_3 \\ 0 \\ 0 \end{bmatrix} \\ [\boldsymbol{\omega}_5]_{B_5} = [\mathbb{1}]_{B_5 | B_2} [\boldsymbol{\omega}_2]_{B_2} + \begin{bmatrix} \beta \dot{\theta}_3 \\ 0 \\ 0 \end{bmatrix} \end{cases} \Rightarrow \begin{cases} \omega_{x_4} = \dot{\theta}_2 + \dot{\theta}_3 \\ \omega_{y_4} = (\dot{\theta}_1 s(\theta_2)) c(\theta_3) + (\dot{\theta}_1 c(\theta_2)) s(\theta_3) \\ \omega_{z_4} = -(\dot{\theta}_1 s(\theta_2)) s(\theta_3) + (\dot{\theta}_1 c(\theta_2)) c(\theta_3) \\ \omega_{x_5} = \dot{\theta}_2 + \beta \dot{\theta}_3 \\ \omega_{y_5} = (\dot{\theta}_1 s(\theta_2)) c(\beta \theta_3) + (\dot{\theta}_1 c(\theta_2)) s(\beta \theta_3) \\ \omega_{z_5} = -(\dot{\theta}_1 s(\theta_2)) s(\beta \theta_3) + (\dot{\theta}_1 c(\theta_2)) c(\beta \theta_3) \end{cases} \quad (33)$$

$$\Rightarrow \underline{\mathbb{p}}^\circ = \begin{bmatrix} \dot{\theta}_2 + \dot{\theta}_3 \\ \dot{\theta}_1 \mathbf{s}(\theta_2 + \theta_3) \\ \dot{\theta}_1 \mathbf{c}(\theta_2 + \theta_3) \\ \dot{\theta}_2 + \beta \dot{\theta}_3 \\ \dot{\theta}_1 \mathbf{s}(\theta_2 + \beta \theta_3) \\ \dot{\theta}_1 \mathbf{c}(\theta_2 + \beta \theta_3) \end{bmatrix} \quad (34)$$

$$\mathbb{C} = \begin{bmatrix} \underline{\mathbb{1}} \\ \frac{\partial \underline{\mathbb{p}}^\circ}{\partial \underline{\mathbb{p}}^\#} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ \mathbf{s}(\theta_2 + \theta_3) & 0 & 0 \\ \mathbf{c}(\theta_2 + \theta_3) & 0 & 0 \\ 0 & 1 & \beta \\ \mathbf{s}(\theta_2 + \beta \theta_3) & 0 & 0 \\ \mathbf{c}(\theta_2 + \beta \theta_3) & 0 & 0 \end{bmatrix} \quad (35)$$

$$\begin{cases} D'_{11} = D_{11} + J_{y_4} \mathbf{s}^2(\theta_2 + \theta_2) + J_{y_5} \mathbf{s}^2(\beta \theta_2 + \theta_2) + J_{z_4} \mathbf{c}^2(\theta_2 + \theta_2) + J_{z_5} \mathbf{c}^2(\beta \theta_2 + \theta_2) \\ D'_{22} = D_{22} + J_{x_4} + J_{x_5} \\ D'_{33} = D_{33} + J_{x_4} + J_{x_5} \beta^2 \\ D'_{12} = D'_{13} = 0 \\ D'_{23} = D_{23} + J_{x_4} + J_{x_5} \beta \\ D'_{211} = D_{211} \\ D'_{311} = D_{311} \\ D'_{111} = D'_{122} = D'_{133} = D'_{222} = D'_{233} = D'_{322} = D'_{333} = 0 \\ D'_{112} = D_{112} + \frac{1}{4} \left((J_{y_4} - J_{z_4}) \mathbf{s}(2\theta_2 + 2\theta_3) + (J_{y_5} - J_{z_5}) \mathbf{s}(2\beta \theta_2 + 2\theta_3) \right) \\ D'_{113} = D_{113} + \frac{1}{4} \left((J_{y_4} - J_{z_4}) \mathbf{s}(2\theta_2 + 2\theta_3) + (J_{y_5} - J_{z_5}) \mathbf{s}(2\beta \theta_2 + 2\theta_3) \right) \\ D'_{123} = D'_{212} = D'_{213} = D'_{223} = D'_{312} = D'_{313} = D'_{323} = 0 \end{cases} \quad (36)$$

Dynamic balancing:

$$\begin{cases} D'_{23} = 0 \\ D'_{211} = 0 \\ D'_{311} = 0 \\ D'_{112} = 0 \\ D'_{113} = 0 \end{cases} \Rightarrow \begin{cases} \beta = -\frac{J_{x_3} + J_{x_4}}{J_{x_5}} \\ J_{y_2} = J_{z_2} \\ J_{y_3} = J_{z_3} + m_3 l_2^2 \left(1 + \frac{m_3}{m_2}\right) \\ J_{y_4} = J_{z_4} \\ J_{y_5} = J_{z_5} \end{cases} \quad (37)$$

Dynamic balanced mechanism:

$$\begin{cases} \tau_1 = k_1 \ddot{\theta}_1 \\ \tau_2 = k_2 \ddot{\theta}_2 \\ f_3 = k_3 \ddot{d}_3 \end{cases} \quad (38)$$

Being:

$$\begin{cases} k_1 = J_{z_1} + J_{z_2} + J_{z_3} + J_{z_4} + J_{z_5} + m_2 l_1^2 + m_3 (l_1^2 + l_2^2) + \frac{l_1^2 m_3^2}{m_2} \\ k_2 = J_{x_2} + J_{x_3} + J_{x_4} + J_{x_5} + m_3 l_2^2 + \frac{l_1^2 m_3^2}{m_2} \\ k_3 = \frac{(J_{x_3} + J_{x_4})(J_{x_3} + J_{x_4} + J_{x_5})}{J_{x_5}} \end{cases} \quad (39)$$

3.3 3-DOF RRP spatial serial mechanism (SCARA)

$$\begin{cases} D_1 = g[(m_1 l_{g_1} + m_2 l_1 + m_3 l_1) \mathbf{c}(\theta_1) + m_2 l_{g_2} \mathbf{c}(\theta_1 + \theta_2)] \\ D_2 = g[m_2 l_{g_2} \mathbf{c}(\theta_1 + \theta_2)] \\ D_3 = 0 \end{cases} \quad (40)$$

Static balancing:

$$\begin{cases} D_1 = 0 \\ D_2 = 0 \end{cases} \Rightarrow \begin{cases} l_{g_1} = -\frac{l_1(m_2+m_3)}{m_1} \\ l_{g_2} = 0 \end{cases} \quad (41)$$

Static balanced mechanism:

$$\begin{cases} D_{11} = J_{x_1} + J_{x_2} + J_{x_3} + m_2 l_1^2 + m_3 l_1^2 + \frac{l_1^2(m_2+m_3)^2}{m_1} \\ D_{22} = J_{x_2} + J_{x_3} \\ D_{33} = m_3 \\ D_{12} = D_{22} \\ D_{13} = D_{23} = 0 \\ \mathbf{v}^\# = \mathbf{0} \\ \mathbf{g}^\# = \mathbf{0} \end{cases} \quad (42)$$

Coupling 2 discs:

Planar disc models:

$$\mathbb{M}_i^\# = [J_{x_{i+3}}]; \quad \mathbb{P}_i^\# = [\omega_{x_{i+3}}], \quad i = 1, 2 \quad (43)$$

Quasi-velocities constraints:

$$\begin{cases} \omega_{x_4} = \omega_{x_1} + \dot{\theta}_2 \\ \omega_{x_5} = \omega_{x_1} + \beta \dot{\theta}_2 \end{cases} \Rightarrow \begin{cases} \omega_{x_4} = \dot{\theta}_1 + \dot{\theta}_2 \\ \omega_{x_5} = \dot{\theta}_1 + \beta \dot{\theta}_2 \end{cases} \Rightarrow \bar{\mathbb{P}} = \begin{bmatrix} \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 + \beta \dot{\theta}_2 \end{bmatrix} \quad (44)$$

$$\mathbb{C} = \begin{bmatrix} \mathbb{1} \\ \frac{\partial \bar{\mathbb{P}}^\circ}{\partial \bar{\mathbb{P}}^\#} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & \beta & 0 \end{bmatrix} \quad (45)$$

$$\begin{cases} D'_{11} = D_{11} + J_{x_4} + J_{x_5} \\ D'_{22} = D_{22} + J_{x_4} + J_{x_5} \beta^2 \\ D'_{33} = D_{33} \\ D'_{12} = D_{12} + J_{x_4} + J_{x_5} \beta \\ D'_{13} = 0 \\ D'_{23} = 0 \\ \mathbf{v}'^\# = \mathbf{0} \end{cases} \quad (46)$$

Dynamic balancing:

$$D'_{12} = 0 \Rightarrow \beta = -\frac{J_{x_2} + J_{x_3} + J_{x_4}}{J_{x_5}} \quad (47)$$

Dynamic balanced mechanism:

$$\begin{cases} \tau_1 = k_1 \ddot{\theta}_1 \\ \tau_2 = k_2 \ddot{\theta}_2 \\ f_3 = k_3 \ddot{d}_3 \end{cases} \quad (48)$$

Being:

$$\begin{cases} k_1 = J_{x_1} + J_{x_2} + J_{x_3} + J_{x_4} + J_{x_5} + m_2 l_1^2 + m_3 l_1^2 + \frac{l_1^2 (m_2 + m_3)^2}{m_1} \\ k_2 = \frac{(J_{x_2} + J_{x_3} + J_{x_4})(J_{x_2} + J_{x_3} + J_{x_4} + J_{x_5})}{J_{x_5}} \\ k_3 = m_3 \end{cases} \quad (49)$$

4 Conclusions

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