Adaptative balancing techniques applied to parallel mechanisms

Tarcisio Antonio Hess Coelho [a], Renato Maia Matarazzo Orsino [b], André Garnier Coutinho [a]

[a] *Department of Mechatronics and Mechanical Systems Engineering, Escola Politecnica, University of Sao Paulo, Brazil. E-mail: tarchess@usp.br*

[b] *Department of Mechanical Engineering, Escola Politecnica, University of Sao Paulo, Brazil.*

SUMMARY

KEYWORDS:

0.1 Introduction to sliding modes control

In this subsection, a brief introduction to the sliding modes control will be done. The theme will be explored only to perform second order systems control, without parametric uncertainties, to not escape the scope of the chapter.

Consider a dynamical system given by the following differential equation:

$$\ddot{x} = u \tag{1}$$

A curve in the error phase plan, called sliding surface, can be defined:

$$s(e, \dot{e}) = -(\dot{e} + \lambda e) = 0,\ \lambda > 0 \tag{2}$$

Being $e = x^\diamond - x$ the error signal and $x^\diamond$ reference signal. Note that if the system is on the sliding surface, we have:

$$\dot{e} + \lambda e = 0 \Rightarrow e(t) = c\,\mathsf{e}^{-\lambda t} \tag{3}$$

Thus, the error drops exponentially to zero, with time constant $1/\lambda$.

To find a control law that brings the system to the sliding surface, we start from the definition of s:

$$s = -(\dot{e} + \lambda e)$$

Differentiating with respect to time:

$$\dot{s} = -(\ddot{e} + \lambda \dot{e}) = \ddot{x} - \ddot{x}^\diamond - \lambda \dot{e} \tag{4}$$

Substituting (1) into (4):

$$\dot{s} = u - \ddot{x}^\diamond - \lambda \dot{e} \tag{5}$$

Using the following control law:

$$u = \ddot{x}^\diamond + \lambda \dot{e} - k\,\mathsf{sign}(s),\ k > 0 \tag{6}$$

He have:

$$\dot{s} = -k\,\mathsf{sign}(s) \tag{7}$$

Suppose that the system starts at $s(0) = s_0 > 0$. Solving the ODE for $s > 0$:

$$\dot{s} = -k \Rightarrow s = -kt + c$$
$$s(0) = s_0 \Rightarrow c = s_0$$
$$\therefore s = s_0 - kt,\ s > 0$$

According to the solution found, when $t \to t_s = \frac{|s_0|}{k}$, $s \to 0$. Solving the ODE for $s(t_s) = 0$:

$$\dot{s} = 0 \Rightarrow s = c$$
$$s(t_s) = 0 \Rightarrow c = 0$$

Therefore, the solution of the ODE for $s(0) = s_0 > 0$ is given by:

$$s(t) = \begin{cases} s_0 - kt, & t < t_s \\ 0, & t \geq t_s \end{cases} \tag{8}$$

An analogous result is found solving the ODE for $s(0) = s_0 < 0$:

$$s(t) = \begin{cases} s_0 + kt, & t < t_s \\ 0, & t \geq t_s \end{cases} \tag{9}$$

Thus, it can be concluded that the ODE (7) converges to $s = 0$, regardless of the initial condition. Therefore, we have that the control law (6) makes the system represented by (1) follow the reference signal, because the error signal converges to zero.

0.2 Extended sliding modes control techniques

As seen in subsections 2.1 and 2.2, it's very convenient to use redundant coordinates to perform parallel mechanism dynamic modeling. Thus, we propose in this subsection a control law for systems described by redundant coordinates.

For the sake of brevity, the indexes n of the subsystems of $\mathscr{M}$ will be omitted in this subsection. Let $\mathscr{M}$ be a multibody mechanical system whose mathematical model is given by equations (**??**, **??**). Suppose that each $\mathbb{f}$ is an affine function of the control inputs u_k in which the coefficients of the u_k may depend on the instantaneous configuration of the system. Suppose additionally that that all the $\mathbb{A}$ are independent of the quasi-velocities p_j and all the $\mathbb{M}$, $\mathbb{g}$, $\mathbb{A}$ and $\mathbb{b}$ are independent of the u_k. Under these conditions, matrices $\mathbb{C}$ will not depend on any quasi-velocity, $\mathbb{d}$ can be expressed as an affine function of the control inputs and $\mathbb{c}$ is independent of them. Considering that the number of control inputs in a mechanical system is exactly equal to the number of degrees of freedom of $\mathscr{M}$:

$$\mathbb{u} = \mathbb{C}^{\mathsf{T}}(t, \mathbb{q})\Big(\mathbb{M}(t, \mathbb{q})\dot{\mathbb{p}} + \mathbb{w}(t, \mathbb{q}, \mathbb{p}) + \mathbb{z}(t, \mathbb{q})\Big) \tag{10}$$

From the control perspective, it is convenient to define $\mathbb{p} = \dot{\mathbb{q}}$, in order to have a position feedback control.

Based on equations (10) and (**??**), consider the dynamic model of a multi-body mechanical system described by the following equations:

$$\begin{cases} \mathbb{C}^{\mathsf{T}}(t, \mathbb{q})\Big(\mathbb{M}(t, \mathbb{q})\ddot{\mathbb{q}} + \mathbb{w}(t, \mathbb{q}, \dot{\mathbb{q}}) + \mathbb{z}(t, \mathbb{q})\Big) = \mathbb{u} \\ \mathbb{A}(t, \mathbb{q})\ddot{\mathbb{q}} + \mathbb{b}(t, \mathbb{q}, \dot{\mathbb{q}}) = \mathbb{0} \end{cases} \tag{11}$$

Rewriting in a compact matrix way:

$$\begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \ddot{\mathbb{q}} = \begin{bmatrix} \mathbb{u} - \mathbb{C}^{\mathsf{T}}(\mathbb{w} + \mathbb{z}) \\ -\mathbb{b} \end{bmatrix} \tag{12}$$

We want to find a control law such that, in closed loop, $\ddot{\mathbb{q}} = \mathbb{v}$, being $\mathbb{v}$ a control input matrix column. For this to happen, the following control law is used:

$$\mathbb{u} = \mathbb{C}^{\mathsf{T}}(\mathbb{M}\mathbb{v} + \mathbb{w} + \mathbb{z}) \tag{13}$$

As we want $\ddot{\mathbb{q}} = \mathbb{v}$ and $\ddot{\mathbb{q}}$ has restrictions, $\mathbb{v}$ must respect the same restrictions, i.e.:

$$\mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \tag{14}$$

Applying the control law (13) and the rectritions (14) in (12), we have:

$$\begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \ddot{\mathbb{q}} = \begin{bmatrix} \mathbb{C}^{\mathsf{T}}(\mathbb{M}\mathbb{v} + \mathbb{w} + \mathbb{z}) - \mathbb{C}^{\mathsf{T}}(\mathbb{w} + \mathbb{z}) \\ \mathbb{A}\mathbb{v} \end{bmatrix} = \begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v} \\ \mathbb{A}\mathbb{v} \end{bmatrix} = \begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \mathbb{v}$$

Once the matrix $\begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M} \\ \mathbb{A} \end{bmatrix}$ is non-singular:

$$\ddot{\mathbb{q}} = \mathbb{v} \tag{15}$$

Let $\mathbb{v}'$ be given by the sliding modes control law:

$$\mathbb{v}' = \ddot{\mathbb{q}}_n^{\diamond} + \lambda\dot{\mathbb{e}} + k\,\mathsf{sign}(\dot{\mathbb{e}} + \lambda\mathbb{e}) \tag{16}$$

Being $\mathbb{e} = \mathbb{q}_n^{\diamond} - \mathbb{q}$ the error signal and $\mathbb{q}_n^{\diamond}$ the reference signal. If there wasn't any retrictions, we could do $\mathbb{v} = \mathbb{v}'$:

$$\ddot{\mathbb{q}} = \mathbb{v} \Rightarrow \ddot{\mathbb{e}} + \lambda\dot{\mathbb{e}} + k\,\mathsf{sign}(\dot{\mathbb{e}} + \lambda\mathbb{e}) = \mathbb{0} \Leftrightarrow \dot{\mathbb{s}} = -k\,\mathsf{sign}(\mathbb{s})$$

This would ensure that $\mathbb{e} \to \mathsf{0}$ when $t \to \infty$ for any initial condition, as seen in the last subsection.

As we have resctriction on $\mathbb{v}$, we look for the closest possible $\mathbb{v}$ of $\mathbb{v}'$ by solving the following optimization problem:

$$\begin{aligned} \min_{\mathbb{v}} \quad & (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') \\ \text{s.t.} \quad & \mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \end{aligned} \tag{17}$$

As $\mathbb{M}$ is positive-semidefinite, we have $(\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') \geq \mathsf{0}$ for any $\mathbb{v}$.

Applying the method of Lagrange multipliers, it can be said that the following problem is equivalent to:

$$\min_{\mathbb{v},\mathbb{\lambda}} \quad L = (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + (\mathbb{A}\mathbb{v} + \mathbb{b})^{\mathsf{T}}\mathbb{\lambda} \tag{18}$$

To solve the problem, the Lagrangian function must be stationary:

$$\begin{aligned} \delta L = \mathsf{0} &\Rightarrow \delta\mathbb{v}^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}\delta\mathbb{v} + (\mathbb{A}\delta\mathbb{v})^{\mathsf{T}}\mathbb{\lambda} + (\mathbb{A}\mathbb{v} + \mathbb{b})^{\mathsf{T}}\delta\mathbb{\lambda} = \mathsf{0} \\ &\Rightarrow \delta\mathbb{v}^{\mathsf{T}}\Big((\mathbb{M} + \mathbb{M}^{\mathsf{T}})(\mathbb{v} - \mathbb{v}') + \mathbb{A}^{\mathsf{T}}\mathbb{\lambda}\Big) + \delta\mathbb{\lambda}^{\mathsf{T}}(\mathbb{A}\mathbb{v} + \mathbb{b}) = \mathsf{0} \end{aligned}$$

As $\mathbb{M}$ is symmetric and $\delta\mathbb{v}$ and $\delta\mathbb{\lambda}$ are arbitrary, we have:

$$\begin{cases} 2\mathbb{M}(\mathbb{v} - \mathbb{v}') + \mathbb{A}^{\mathsf{T}}\mathbb{\lambda} = \mathbb{0} \\ \mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \end{cases} \tag{19}$$

As $\mathbb{C}$ is the orthogonal complement of $\mathbb{A}$, pre-multiplying the first equation of (19) by $\mathbb{C}^{\mathsf{T}}$, we have:

$$2\mathbb{C}^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + \mathbb{C}^{\mathsf{T}}\mathbb{A}^{\mathsf{T}}\mathbb{\lambda} = \mathbb{0} \Rightarrow \mathbb{C}^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') = \mathbb{0}$$

$$\therefore \mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v} = \mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v}' \tag{20}$$

Thus, we have that the control law that makes the closed loop system as close as possible of $\ddot{\mathbb{q}} = \mathbb{v}'$, according to the optimization criterion adopted, is:

$$\mathbb{u} = \mathbb{C}^{\mathsf{T}}(\mathbb{M}\mathbb{v}' + \mathbb{w} + \mathbb{z}) \tag{21}$$