# Adaptative balancing techniques applied to parallel mechanisms

Tarcisio Antonio Hess Coelho [a], Renato Maia Matarazzo Orsino [b], André Garnier Coutinho [a]

[a] *Department of Mechatronics and Mechanical Systems Engineering, Escola Politecnica, University of Sao Paulo, Brazil. E-mail: tarchess@usp.br*
[b] *Department of Mechanical Engineering, Escola Politecnica, University of Sao Paulo, Brazil.*

**SUMMARY**

KEYWORDS:

### *0.1 Controle por modos deslizantes extendido*

As seen in subsections 2.1 and 2.2, it's very convenient to use redundant coordinates to perform parallel mechanism dynamic modeling. Thus, we propose in this subsection a control law for systems described by redundant coordinates.

Let $\mathscr{M}$ be a multibody mechanical system whose mathematical model is given by equations (**??**, **??**). Suppose that each $\mathbb{f}_n$ is an affine function of the control inputs $u_{n,k}$ in which the coefficients of the $u_{n,k}$ may depend on the instantaneous configuration of the system. Suppose additionally that that all the $\mathbb{A}_n$ are independent of the quasi-velocities $p_{n,j}$ and all the $\mathbb{M}_n$, $\mathbb{g}_n$, $\mathbb{A}_n$ and $\mathbb{b}_n$ are independent of the $u_{n,k}$. Under these conditions, matrices $\mathbb{C}_n$ will not depend on any quasi-velocity, $\mathbb{d}$ can be expressed as an affine function of the control inputs and $\mathbb{c}$ is independent of them. Considering that the number of control inputs in a mechanical system is exactly equal to the number of degrees of freedom of $\mathscr{M}$, it may be possible to solve equations (**??**) in order to express each $u_{n,k}$ as an explicit function of state variables and physical parameters of $\mathscr{M}$. For the sake of brevity, omitting the indexes $n$ of the subsystems of $\mathscr{M}$, it can be stated that, in such cases, for $k \in \{1, \ldots, \nu^{\#}(\mathscr{M})\}$:

$$u_k = \sum_r C_{kr}(t, \mathbb{q}) \Big( \sum_j M_{rj}(t, \mathbb{q})\, \dot{p}_j + w_r(t, \mathbb{q}, \mathbb{p}) + z_r(t, \mathbb{q}) \Big) \tag{1}$$

That is equivalent to:

$$u_k = \sum_r M'_{kr}(t, \mathbb{q})\, \dot{p}_r + w'_k(t, \mathbb{q}, \mathbb{p}) + z'_k(t, \mathbb{q}) \tag{2}$$

It can be shown that each $w'_k(t, \mathbb{q}, \mathbb{p})$ can be expressed as a sum of a bilinear and a linear function on the quasi-velocities, i.e., there are functions $D'_{krs}(t, \mathbb{q})$ and $B'_{kr}(t, \mathbb{q})$ such that:

$$w'_k(t, \mathbb{q}, \mathbb{p}) = \sum_r \sum_s D'_{krs}(t, \mathbb{q})\, p_r\, p_s + \sum_r B'_{kr}(t, \mathbb{q})\, p_r \tag{3}$$

Adaptive balancing types will be defined in the next section based on equation (2).

For keeping the sake of brevity, the indexes $n$ of the subsystems of $\mathscr{M}$ will be omitted from this point in this subsection.

From the control perspective, it's convenient to use $\dot{\mathbb{q}}$ instead of $\mathbb{p}$, because the quasi-velocities $\mathbb{p}$ usually are not integrable, what prevents the position feedback in the direction of these coordinates, then we will do so.

Based on equations (1) and (**??**), consider the dynamic model of a multi-body mechanical system described by the following equations:

$$\begin{cases} \mathbb{C}^{\mathsf{T}}(t,\mathbb{q})\Big(\mathbb{M}(t,\mathbb{q})\ddot{\mathbb{q}} + \mathbb{w}(t,\mathbb{q},\dot{\mathbb{q}}) + \mathbb{z}(t,\mathbb{q})\Big) = \mathbb{u} \\ \mathbb{A}(t,\mathbb{q})\ddot{\mathbb{q}} + \mathbb{b}(t,\mathbb{q},\dot{\mathbb{q}}) = \mathbb{0} \end{cases} \tag{4}$$

Rewriting in a compact matrix way:

$$\begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \ddot{\mathbb{q}} = \begin{bmatrix} \mathbb{u} - \mathbb{C}^{\mathsf{T}}(\mathbb{w}+\mathbb{z}) \\ -\mathbb{b} \end{bmatrix} \tag{5}$$

We want to find a control law such that, in closed loop, $\ddot{\mathbb{q}} = \mathbb{v}$, being $\mathbb{v}$ a control input matrix column. For this to happen, the following control law is used:

$$\mathbb{u} = \mathbb{C}^{\mathsf{T}}(\mathbb{M}\mathbb{v} + \mathbb{w} + \mathbb{z}) \tag{6}$$

As we want $\ddot{\mathbb{q}} = \mathbb{v}$ and $\ddot{\mathbb{q}}$ has restrictions, $\mathbb{v}$ must respect the same restrictions, i.e.:

$$\mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \tag{7}$$

Applying the control law (6) and the rectritions (7) in (5), we have:

$$\begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \ddot{\mathbb{q}} = \begin{bmatrix} \mathbb{C}^{\mathsf{T}}(\mathbb{M}\mathbb{v} + \mathbb{w} + \mathbb{z}) - \mathbb{C}^{\mathsf{T}}(\mathbb{w}+\mathbb{z}) \\ \mathbb{A}\mathbb{v} \end{bmatrix} = \begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v} \\ \mathbb{A}\mathbb{v} \end{bmatrix} = \begin{bmatrix} \mathbb{C}^{\mathsf{T}}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \mathbb{v}$$

$$\therefore \ddot{\mathbb{q}} = \mathbb{v} \tag{8}$$

Let $\mathbb{v}'$ be given by the sliding modes control law:

$$\mathbb{v}' = \ddot{\mathbb{q}}_n^{\diamond} + \lambda\dot{\mathbb{e}} + k\,\mathsf{sign}(\dot{\mathbb{e}} + \lambda\mathbb{e}) \tag{9}$$

Being $\mathbb{e} = \mathbb{q}_n^{\diamond} - \mathbb{q}$ the error signal and $\mathbb{q}_n^{\diamond}$ the reference signal. If there wasn't any retrictions, we could do $\mathbb{v} = \mathbb{v}'$ :

$$\ddot{\mathbb{q}} = \mathbb{v} \Rightarrow \ddot{\mathbb{e}} + \lambda\dot{\mathbb{e}} + k\,\mathsf{sign}(\dot{\mathbb{e}} + \lambda\mathbb{e}) = \mathbb{0} \Leftrightarrow \dot{\mathbb{s}} = -k\,\mathsf{sign}(\mathbb{s})$$

This would ensure that $\mathbb{e} \to 0$ when $t \to \infty$ for any initial condition, as seen in the last subsection.

As we have resctriction on $\mathbb{v}$, we look for the closest possible $\mathbb{v}$ of $\mathbb{v}'$ by solving the following optimization problem:

$$\begin{aligned} \min_{\mathbb{v}} \quad & (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') \\ \text{s.t.} \quad & \mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \end{aligned} \tag{10}$$

As $\mathbb{M}$ is positive-semidefinite, we have $(\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') \geq 0$ for any $\mathbb{v}$.

Applying the method of Lagrange multipliers, it can be said that the following problem is equivalent to:

$$\min_{\mathbb{v},\mathbb{\lambda}} \quad L = (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + (\mathbb{A}\mathbb{v} + \mathbb{b})^{\mathsf{T}}\mathbb{\lambda} \tag{11}$$

To solve the problem, the Lagrangian function must be stationary:

$$\delta L = 0 \Rightarrow \delta\mathbb{v}^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}\delta\mathbb{v} + (\mathbb{A}\delta\mathbb{v})^{\mathsf{T}}\mathbb{\lambda} + (\mathbb{A}\mathbb{v} + \mathbb{b})^{\mathsf{T}}\delta\mathbb{\lambda} = 0$$

$$\Rightarrow \delta\mathbb{v}^{\mathsf{T}}\left((\mathbb{M}+\mathbb{M}^{\mathsf{T}})(\mathbb{v}-\mathbb{v}')+\mathbb{A}^{\mathsf{T}}\mathbb{\lambda}\right)+\delta\mathbb{\lambda}^{\mathsf{T}}(\mathbb{A}\mathbb{v}+\mathbb{b})=0$$

As $\mathbb{M}$ is symmetric and $\delta\mathbb{v}$ and $\delta\mathbb{\lambda}$ are arbitrary, we have:

$$\begin{cases} 2\mathbb{M}(\mathbb{v}-\mathbb{v}')+\mathbb{A}^{\mathsf{T}}\mathbb{\lambda}=\mathbb{0} \\ \mathbb{A}\mathbb{v}+\mathbb{b}=\mathbb{0} \end{cases} \tag{12}$$

As $\mathbb{C}$ is the orthogonal complement of $\mathbb{A}$, pre-multiplying the first equation of (12) by $\mathbb{C}^{\mathsf{T}}$, we have:

$$2\mathbb{C}^{\mathsf{T}}\mathbb{M}(\mathbb{v}-\mathbb{v}')+\mathbb{C}^{\mathsf{T}}\mathbb{A}^{\mathsf{T}}\mathbb{\lambda}=\mathbb{0} \Rightarrow \mathbb{C}^{\mathsf{T}}\mathbb{M}(\mathbb{v}-\mathbb{v}')=\mathbb{0}$$

$$\therefore \mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v}=\mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v}' \tag{13}$$

Thus, we have that the control law that makes the closed loop system as close as possible of $\ddot{\mathbb{q}}=\mathbb{v}'$, according to the optimization criterion adopted, is:

$$\mathbb{u}=\mathbb{C}^{\mathsf{T}}(\mathbb{M}\mathbb{v}'+\mathbb{w}+\mathbb{z}) \tag{14}$$

**Acknowledgments**