Adaptative balancing techniques applied to parallel mechanisms

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SUMMARY

KEYWORDS:

1 Introduction and literature review

Balancing is an important issue related to the design of mechanical systems in general, and also parallel mechanisms, in particular. In fact, the performance of parallel mechanisms associated to specific applications depends on the choice of the balancing method, namely, either static or dynamic, either passive or active, whether it is valid for a given trajectory or even for any motion.

In a statically balanced mechanism, the potential energy is invariant. Hence, the actuator torques/forces are null at any configuration [6]. On the other hand, in a dynamically balanced mechanism, the shaking forces and shaking moments at its frame are reduced or even eliminated. As a consequence, the mechanism components are less susceptible to vibration, wear and fatigue [4], improving its life.

Passive balancing means that the original mechanism architecture is modified by using some techniques. The most common techniques are the addition of counterweights [**? ? ? ? ? ? ? ? ? ?**], the use of counter-rotating inertias [**? ? ? ?**] and the redistribution of masses [**? ?**]. Alternatively, other works propose the addition of extra links [**? ? ?**]. However, for high speed manipulators, these techniques might cause the increase of the actuator torques and the size of links and joints. Hence, in order to avoid such undesirable effects, some authors [**? ? ? ? ? ?**] recommend the use of elastic springs attached to the mechanism.

For the active balancing [**? ? ? ? ? ?**], more complex modifications are needed to implement it. For instance, the counterweights position in the moving links might be altered according to the end-effector load or the given trajectory. Hence, additional actuators/sensors and control are usually employed to reach satisfactory performance levels. Arakelian and Smith [**?**] employ a computer control of the motion of a inertia flywheel connected to the mechanism.

Moreover, Coelho et al. [**?**] and Moradi et al. [**?**] use balancing to achieve the decoupling of dynamic equations. Consequently, this action simplifies the control of manipulators due to the fact that the actuators can be controlled independently.

The main contribution of this work is to highlight the importance of the dynamic modelling process in order to achieve the compensation conditions associated to the chosen balancing technique. Due to the fact that parallel mechanisms have highly complex structures, the use of dynamic formalisms that employ redundant generalized coordinates, in association with the successive coupling of additional balancing elements to the original system model, can bring remarkable benefits. Additionally, this book chapter also discusses the impact of the dynamic model, developed in accordance with the methodology shown here, for the control strategy of parallel mechanisms.

This chapter is organized as follows. Section 2 treats of the proposed methodology, the dynamic modelling is described in section 3, section 4 presents the control, section 5 shows a numerical example and simulations. Finally, the conclusions are presented in section 6.

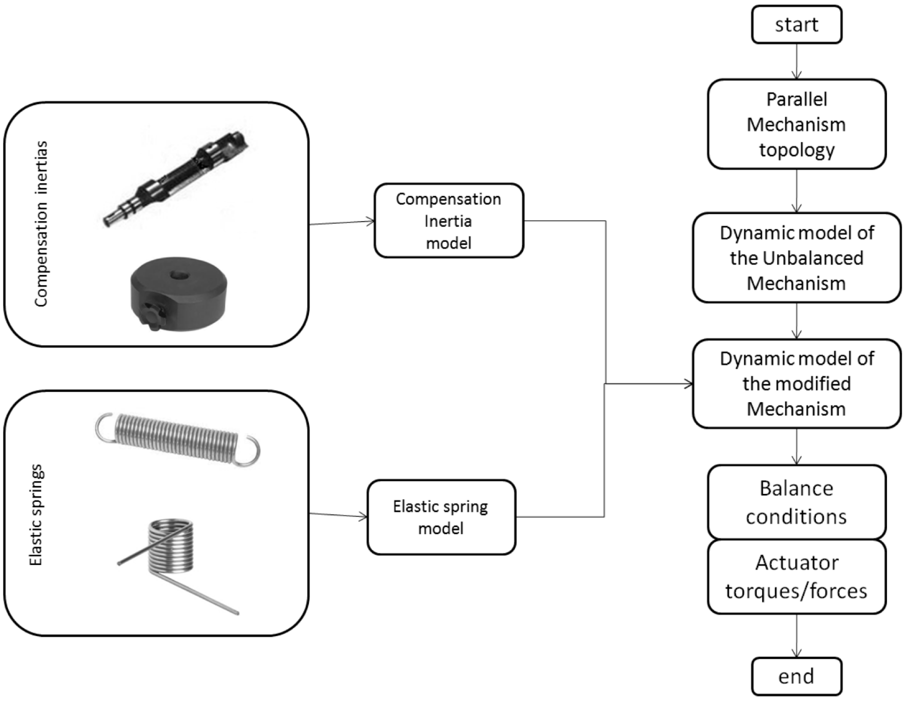

Figure 1: Dynamic modelling process

2 Theoretical background

2.1 Basic structure of a dynamic model

Let $\mathscr{S}_n$ be a multibody mechanical system with a finite number $\nu^{\#}(\mathscr{S}_n)$ of degrees of freedom. Consider the definition of a set of $\nu_q(\mathscr{S}_n)$ generalized coordinates $q_{n,i}$, $i \in \mathscr{I}_q(\mathscr{S}_n)$, for the description of every possible configuration of such system. A set of coordinates is useful for such description if and only if, an arbitrary assignment of values for these coordinates, correspond either to a finite number of non-neighboring configurations or to no configuration. In the first case, the values assigned to the coordinates are called *compatible*. It is important to notice that descriptions in which compatible values of coordinates correspond to more than one configuration can be avoided by adding new coordinates to the original set, so that compatible values in this latter set of variables uniquely describe any possible configuration of the system.

Consider a particular compatible set of values for the generalized coordinates and suppose that some of these values are varied infinitesimally. The maximum number of independent variations still leading to compatible values corresponds to the local number of degrees of freedom associated to the respective configurations. The maximum local number of degrees of freedom among all the possible configurations of the system is the number of degrees of freedom of $\mathscr{S}_n$ (which, by hypothesis, is finite). A particular set of generalized coordinates is called *redundant* if the number of coordinates is greater than the number of degrees of freedom of $\mathscr{S}_n$, i.e., if $\nu_q(\mathscr{S}_n) > \nu^{\#}(\mathscr{S}_n)$. In such cases there must be as much independent invariants as the excess of generalized coordinates over the number of degrees of freedom of the system, i.e., $\nu_q(\mathscr{S}_n) - \nu^{\#}(\mathscr{S}_n)$ independent invariants, imposing conditions that limit the maximum number of independent variations (and, consequently, time derivatives) of the generalized coordinates that still lead to compatible values for these variables. Denoting $\mathbb{q}_n = [q_{n,i}]$, these invariants can be expressed by the so called *constraint equations*, once they represent the kinematic constraints of system $\mathscr{S}_n$, which can be presented in the following form:

$$\psi_{n,r}(t, \mathbb{q}_n, \dot{\mathbb{q}}_n) = 0 \qquad \text{for} \qquad r \in \{1, 2, \dots, \nu_q(\mathscr{S}_n) - \nu^{\#}(\mathscr{S}_n)\} \tag{1}$$

If $\psi_{n,r}(t, \mathbb{q}_n, \mathsf{d}\mathbb{q}_n)$ is an exact differential, the corresponding constraint is classified as *holonomic*. In these cases, there is a function $h_{n,r}(t, \mathbb{q}_n)$ such that $\mathsf{d}h_{n,r} = \psi_{n,r}(t, \mathbb{q}_n, \mathsf{d}\mathbb{q}_n) = 0$, i.e., $h_{n,r}(t, \mathbb{q}_n)$ is an invariant representing a constraint which depends exclusively on the instantaneous configurations of the system, but not on its instantaneous motion. When $\psi_{n,r}(t, \mathbb{q}_n, \mathsf{d}\mathbb{q}_n)$ is an inexact differential, the corresponding constraint is called *nonholonomic*. Actually, the class of nonholonomic constraints include constraints that cannot even be expressed by equations. These latter cases, however, are out of the scope of this work.

Now, consider that, in order to replace time derivatives of generalized coordinates in the description of instantaneous motions of the system, a set of $\nu_p(\mathscr{S}_n)$ quasi-velocities $p_{n,j},\ j \in \mathscr{I}_p(\mathscr{S}_n)$, is defined. The most common example of the use of quasi-velocities occurs in Newton-Euler equations: angular velocity components are tipically used to describe instantaneous rotations of a rigid body instead of time derivatives of the generalized coordinates associated to the instantaneous orientations of this body (which can be, Euler angles, quaternion components, etc.). However, time integrals of the components of angular velocity, in general, cannot be used to describe instantaneous configurations of a rigid body. That is, although it may be convenient to use a particular quasi-velocity in the description of instantaneous motions of a mechanical system, it is not required that its time integral have any physical sense.

Once quasi-velocities replace time derivatives of generalized coordinates, the following set of independent equations defining such transformation of variables must be provided:

$$\phi_{n,s}(t, \mathbb{q}_n, \dot{\mathbb{q}}_n, \mathbb{p}_n) = 0 \qquad \text{for} \qquad s \in \{1, 2, \ldots, \max(\nu_q(\mathscr{S}_n),\ \nu_p(\mathscr{S}_n))\} \tag{2}$$

Suppose that all the $\phi_{n,s}(t, \mathbb{q}_n, \dot{\mathbb{q}}_n, \mathbb{p}_n)$ are class $\mathscr{C}^1$ functions and that for all states $(t, \mathbb{q}_n, \dot{\mathbb{q}}_n, \mathbb{p}_n)$ satisfying both systems of equations (1, 2) the conditions of the Theorem of Implicit Functions are satisfied, so that it is possible to use (some of the) equations (2) to obtain the following solutions (valid in some neighborhood of such states):

$$\dot{q}_{n,i} = \dot{q}^*_{n,i}(t, \mathbb{q}_n, \mathbb{p}_n) \qquad \forall\, i \in \mathscr{I}_q(\mathscr{S}_n) \tag{3}$$

$$p_{n,j} = p^*_{n,j}(t, \mathbb{q}_n, \dot{\mathbb{q}}_n) \qquad \forall\, j \in \mathscr{I}_p(\mathscr{S}_n) \tag{4}$$

Now, consider the following system of equations:

$$\begin{cases} \psi_{n,r}(t, \mathbb{q}_n, \dot{\mathbb{q}}^*_n(t, \mathbb{q}_n, \mathbb{p}_n)) = 0 & \text{for} \quad r \in \{1, 2, \ldots, \nu_q(\mathscr{S}_n) - \nu^{\#}(\mathscr{S}_n)\} \\ \phi_{n,s}(t, \mathbb{q}_n, \dot{\mathbb{q}}^*_n(t, \mathbb{q}_n, \mathbb{p}_n), \mathbb{p}_n) = 0 & \text{for} \quad s \in \{1, 2, \ldots, \max(\nu_q(\mathscr{S}_n),\ \nu_p(\mathscr{S}_n))\} \end{cases} \tag{5}$$

Once the expressions of $\dot{q}^*_{n,i}(t, \mathbb{q}_n, \mathbb{p}_n)$ are obtained from local solutions of (some of the) equations (2), at least $\nu_q(\mathscr{S}_n)$ equations must be eliminated from the system (5) in order to obtain a system of independent equations. Indeed, for any possible state, system (5) will not have more than $\nu_c(\mathscr{S}_n) = \nu_p(\mathscr{S}_n) - \nu^{\#}(\mathscr{S}_n)$, and when it has less than $\nu_c(\mathscr{S}_n)$ independent equations, the correspoding state is called *singular*. It is clear that the time derivative of the independent equations in system (5) can be expressed in the following form (replacing $\dot{q}_{n,i} = \dot{q}^*_{n,i}(t, \mathbb{q}_n, \mathbb{p}_n)$, when necessary):

$$c_{n,r}(t, \mathbb{q}_n, \mathbb{p}_n, \dot{\mathbb{p}}_n) = \sum_j A_{n,rj}(t, \mathbb{q}_n, \mathbb{p}_n)\, \dot{p}_{n,j} + b_{n,r}(t, \mathbb{q}_n, \mathbb{p}_n) = 0 \qquad \text{for} \qquad r \in \{1, 2, \ldots, \nu_c(\mathscr{S}_n)\} \tag{6}$$

It is also possible to express these equations in the following matrix form:

$$\mathbb{c}_n(t, \mathbb{q}_n, \mathbb{p}_n, \dot{\mathbb{p}}_n) = \mathbb{A}_n(t, \mathbb{q}_n, \mathbb{p}_n)\, \dot{\mathbb{p}}_n + \mathbb{b}_n(t, \mathbb{q}_n, \mathbb{p}_n) = \mathbb{0} \tag{7}$$

In order to determine completely all the quasi-accelerations $\dot{p}_{n,j}$ associated to a given state of the mechanical system, it is necessary to form a system of equations in which equations (6) are taken along with the dynamic equations of the model.

Consider that $\mathscr{S}_n$ is an multiple rigid-body mechanical system, composed of a set of constrained rigid bodies $\mathcal{B}_{n,s},\ s \in \mathscr{I}_{\mathcal{B}}(\mathscr{S}_n)$ and a set of actuators providing control inputs $u_{n,k},\ k \in \mathscr{I}_u(\mathscr{S}_n)$. The following notation is adopted:

- $\mathsf{b}^\star_{n,s}$ represents the center of mass of $\mathcal{B}_{n,s}$;
- $\boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}$ and $\boldsymbol{a}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}$ are respectively the velocity and the acceleration of $\mathsf{b}^\star_{n,s}$ measured relatively to an inertial reference frame $\mathcal{N}$;

- $\boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}_{n,s}|\mathcal{N}}$ are respectively the angular velocity and the angular acceleration of $\mathcal{B}_{n,s}$ measured relatively to an inertial reference frame $\mathcal{N}$;
- $m_{\mathcal{B}_{n,s}}$ and $\boldsymbol{I}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}}$ are respectively the mass of $\mathcal{B}_{n,s}$ and its inertia tensor relative to point $\mathsf{b}^\star_{n,s}$;
- $\boldsymbol{f}_{\mathcal{B}_{n,s}}$ and $\boldsymbol{m}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}}$ are respectively the resultant force and resultant moment measured relatively to pole $\mathsf{b}^\star_{n,s}$, including control inputs effects and excluding constraint effects, acting on body $\mathcal{B}_{n,s}$.

Applying the Principle of Virtual Power to $\mathcal{S}_n$, it can be stated that:

$$\sum_s \Bigg(\delta \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}} \cdot \Big(\boldsymbol{f}_{\mathcal{B}_{n,s}} - m_{\mathcal{B}_{n,s}}\, \boldsymbol{a}_{\mathsf{b}^\star_{n,s}|\mathcal{N}} \Big) + \delta \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}} \cdot \Big(\boldsymbol{m}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}} - \boldsymbol{I}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}} \cdot \dot{\boldsymbol{\omega}}_{\mathcal{B}_{n,s}|\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}} \times \big(\boldsymbol{I}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}} \cdot \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}} \big) \Big) \Bigg) = 0 \tag{8}$$

Consider that at a given time t', all the values of the generalized coordinates $q_{n,i}(t')$ and of the quasi-velocities $p_{n,j}(t')$ are known, implying that $\delta \mathbb{q}_n(t') = \mathbb{0}$ and $\delta \mathbb{p}_n(t') = \mathbb{0}$. Thus, given the expressions of $\boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}$ and $\boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}$ in terms of $q_{n,i}$ and $p_{n,j}$, i.e., $\boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}} = \boldsymbol{v}^*_{\mathsf{b}^\star_{n,s}|\mathcal{N}}(t, \mathbb{q}_n, \mathbb{p}_n)$ and $\boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}} = \boldsymbol{\omega}^*_{\mathcal{B}_{n,s}|\mathcal{N}}(t, \mathbb{q}_n, \mathbb{p}_n)$, it can also be stated that $\delta \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}(t') = \mathbf{0}$ and $\delta \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}(t') = \mathbf{0}$. Consider the expansion in Taylor series of $\delta \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}(t'+\tau)$ and $\delta \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}(t'+\tau)$:

$$\delta \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}(t'+\tau) = \delta \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}(t') + \tau\, \delta \boldsymbol{a}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}(t') + \boldsymbol{O}(\tau^2) = \tau \sum_j \frac{\partial \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}}{\partial p_{n,j}}(t')\, \delta \dot{p}_{n,j}(t') + \boldsymbol{O}(\tau^2) \tag{9}$$

$$\delta \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}(t'+\tau) = \delta \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}(t') + \tau\, \delta \dot{\boldsymbol{\omega}}_{\mathcal{B}_{n,s}|\mathcal{N}}(t') + \boldsymbol{O}(\tau^2) = \tau \sum_j \frac{\partial \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}}{\partial p_{n,j}}(t')\, \delta \dot{p}_{n,j}(t') + \boldsymbol{O}(\tau^2) \tag{10}$$

At time instant $t'+\tau$, equation (8) can be rewritten in the following form, with all the first member evaluated at t':

$$\sum_j \delta \dot{p}_{n,j} \Big(f_{n,j}(t, \mathbb{q}_n, \mathbb{p}_n, \mathbb{u}_n) + g_{n,j}(t, \mathbb{q}_n, \mathbb{p}_n) - \sum_r M_{n,jr}(t, \mathbb{q}_n, \mathbb{p}_n)\, \dot{p}_{n,r} \Big) = \frac{O(\tau^2)}{\tau} \tag{11}$$

In equation (11), $f_{n,j}$ is the *j-th generalized force of system $\mathcal{S}_n$*, $g_{n,j}$ is the *j-th generalized gyroscopic inertia force of system $\mathcal{S}_n$* and $M_{n,jr}$ is the *generalized inertia associated to $\dot{p}_{n,j}$ and $\dot{p}_{n,r}$*. Their expressions satisfy the following identities:

$$f_{n,j}(t, \mathbb{q}_n, \mathbb{p}_n, \mathbb{u}_n) = \sum_s \left(\frac{\partial \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}}{\partial p_{n,j}} \cdot \boldsymbol{f}_{\mathcal{B}_{n,s}} + \frac{\partial \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}}{\partial p_{n,j}} \cdot \boldsymbol{m}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}} \right) \tag{12}$$

$$g_{n,j}(t, \mathbb{q}_n, \mathbb{p}_n) - \sum_r M_{n,jr}(t, \mathbb{q}_n, \mathbb{p}_n)\, \dot{p}_{n,r} = -\sum_s \Bigg(\frac{\partial \boldsymbol{v}_{\mathsf{b}^\star_{n,s}|\mathcal{N}}}{\partial p_{n,j}} \cdot \Big(m_{\mathcal{B}_{n,s}}\, \boldsymbol{a}_{\mathsf{b}^\star_{n,s}|\mathcal{N}} \Big) + \frac{\partial \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}}}{\partial p_{n,j}} \cdot \Big(\boldsymbol{I}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}} \cdot \dot{\boldsymbol{\omega}}_{\mathcal{B}_{n,s}|\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}} \times \big(\boldsymbol{I}_{\mathcal{B}_{n,s}|\mathsf{b}^\star_{n,s}} \cdot \boldsymbol{\omega}_{\mathcal{B}_{n,s}|\mathcal{N}} \big) \Big) \Bigg) \tag{13}$$

Taking the limit $\tau \to 0$ in equation (11), using matrix notation, it can be stated that, at time instant t':

$$\delta \dot{\mathbb{p}}_n^{\mathsf{T}} \left(\mathbb{f}_n + \mathbb{g}_n - \mathbb{M}_n\, \dot{\mathbb{p}}_n \right) = 0 \tag{14}$$

Also, at time instant t', equation (7) implies that:

$$\mathbb{A}_n\, \delta \dot{\mathbb{p}}_n = \mathbb{0} \tag{15}$$

Obviously, if it is possible to find a matrix $\mathbb{C}_n$ such that $\mathbb{A}_n\, \mathbb{C}_n = \mathbb{0}$, then $\delta \dot{\mathbb{p}}_n = \mathbb{C}_n\, \mathbb{\pi}_n$ will satisfy the condition imposed by equation (15) for any column-matrix $\mathbb{\pi}_n$ (with adequate dimensions). If, at time instant t', $\mathscr{S}_n$ is in a non-singular state, then equation (15) imposes $\nu_c(\mathscr{S}_n) = \nu_p(\mathscr{S}_n) - \nu^\#(\mathscr{S}_n)$ conditions for $\nu_p(\mathscr{S}_n)$ variations $\delta\dot{p}_{n,j}$. Thus, the most general solution for equation (15) is to express all the $\nu_p(\mathscr{S}_n)$ variations $\delta\dot{p}_{n,j}$ in terms of $\nu^\#(\mathscr{S}_n)$ arbitrary parameters. This can be achieved by finding a full rank, $\nu_p(\mathscr{S}_n)$ by $\nu^\#(\mathscr{S}_n)$ matrix $\mathbb{C}_n$ such that $\mathbb{A}_n\,\mathbb{C}_n = \mathbb{0}$, i.e., matrix $\mathbb{C}_n$ must be an *orthogonal complement* of matrix $\mathbb{A}_n$. Therefore, replacing $\delta\dot{\mathbb{p}}_n = \mathbb{C}_n\,\mathbb{\pi}_n$ in equation (14):

$$\mathbb{\pi}_n^\mathsf{T}\,\mathbb{C}_n^\mathsf{T}\,(\mathbb{f}_n + \mathbb{g}_n - \mathbb{M}_n\,\dot{\mathbb{p}}_n) = 0 \tag{16}$$

Considering that $\mathbb{C}_n$ is full rank and that $\mathbb{\pi}_n$ is column-matrix with $\nu^\#(\mathscr{S}_n)$ entries, the only possible way to ensure that this equation is satisfied is to state that:

$$\mathbb{d}_n = \mathbb{C}_n^\mathsf{T}\,(\mathbb{f}_n + \mathbb{g}_n - \mathbb{M}_n\,\dot{\mathbb{p}}_n) = \mathbb{0} \tag{17}$$

The system composed by equations (7, 17) completely determines, for a given time t' when all the state variables $q_{n,i}(t')$ and $p_{n,j}(t')$ are known, all the associated quasi-accelerations $\dot{p}_{n,j}(t')$, given the values of the control inputs $u_{n,k}(t')$. Thus, equations (7, 17) are a dynamic model of the multibody mechanical system $\mathscr{S}_n$. Finally, it is worth mentioning that, although the derivation of equation (17) was performed for a multiple rigid-body mechanical system, every multibody mechanical system with a finite number of degrees of freedom can be expressed by a system of equations in the form of (7, 17).

2.2 Dynamic coupling of subsystems

Let $\mathscr{M}$ be a multibody mechanical system which can be interpreted as an assemble of constrained subsystems $\mathscr{S}_n$, $n \in \mathscr{I}_\mathscr{S}(\mathscr{M})$. Consider that the dynamic model of each of the subsystems $\mathscr{S}_n$, when not constrained to the others, is given by the following pair of matrix equations:

$$\mathbb{c}_n = \mathbb{A}_n\,\dot{\mathbb{p}}_n + \mathbb{b}_n = \mathbb{0} \tag{18}$$

$$\mathbb{d}_n = \mathbb{C}_n^\mathsf{T}(\mathbb{f}_n + \mathbb{g}_n - \mathbb{M}_n\,\dot{\mathbb{p}}_n) = \mathbb{0} \tag{19}$$

The system of equations (18), $\mathbb{c}_n = \mathbb{0}$, represents the (*internal*) kinematic constraints of the subsystems $\mathscr{S}_n$, while the system of equations (19), $\mathbb{d}_n = \mathbb{0}$, describes its dynamics when no *external* kinematic constraints are present. Together, equations (18, 19) are called the *free subsystem $\mathscr{S}_n$ model*, which can be expressed in terms of the following variables and parameters:

- Generalized coordinates $q_{n,i}$ and quasivelocities $p_{n,j}$ describing every admissible state of $\mathscr{S}_n$ considering the absence of external kinematic constraints;
- Control inputs $u_{n,k}$ describing the effect of external actuations on $\mathscr{S}_n$;
- Fixed and adjustable physical parameters $a_{n,l}$ of $\mathscr{S}_n$.

When the subsystems are assembled to a single multibody mechanical system $\mathscr{M}$, additional kinematical constraints are imposed. Consider that these further constraints can be expressed by the following matrix equation:

$$\widehat{\mathbb{c}} = \sum_n \widehat{\mathbb{A}}_n\,\dot{\mathbb{p}}_n + \widehat{\mathbb{b}} = \mathbb{0} \tag{20}$$

The satisfaction of all the kinematic constraints expressed by (18, 20) in a particular (admissible) state of the system $\mathscr{M}$ implies that:

$$\mathbb{A}_n\,\delta\dot{\mathbb{p}}_n = \mathbb{0} \qquad\qquad \forall\, n \in \mathscr{I}_\mathscr{S}(\mathscr{M}) \tag{21}$$

$$\sum_n \widehat{\mathbb{A}}_n\,\delta\dot{\mathbb{p}}_n = \mathbb{0} \tag{22}$$

As shown in the previous section, a solution for each of the equations (21) is already known. Once $\mathbb{A}_n \mathbb{C}_n = \mathbb{0}$, $\forall\, n \in \mathscr{I}_{\mathcal{S}}(\mathscr{M})$, for every column-matrix $\mathbb{\pi}_n$ with as much elements as the number of degrees of freedom of the free subsystem $\mathcal{S}_n$ model, equations (21) are identically satisfied for:

$$\delta\dot{\mathbb{p}}_n = \mathbb{C}_n\, \mathbb{\pi}_n \qquad\qquad \forall\, n \in \mathscr{I}_{\mathcal{S}}(\mathscr{M}) \tag{23}$$

Replacing (23) in (22):

$$\sum_n \widehat{\mathbb{A}}_n\, \mathbb{C}_n\, \mathbb{\pi}_n = \mathbb{0} \tag{24}$$

Thus, suppose that it is possible to define matrices $\widehat{\mathbb{C}}_n$, $\forall\, n \in \mathscr{I}_{\mathcal{S}}(\mathscr{M})$, such that for every column-matrix $\mathbb{\pi}$ with as much elements as the number of degrees of freedom of the whole system $\mathscr{M}$, equation (24) is identically satisfied for:

$$\mathbb{\pi}_n = \widehat{\mathbb{C}}_n\, \mathbb{\pi} \qquad \text{or} \qquad \delta\dot{\mathbb{p}}_n = \mathbb{C}_n\, \widehat{\mathbb{C}}_n\, \mathbb{\pi} \qquad \forall\, n \in \mathscr{I}_{\mathcal{S}}(\mathscr{M}) \tag{25}$$

For the sake of simplicity in notation, suppose that $\mathscr{I}_{\mathcal{S}}(\mathscr{M}) = \{1, 2, \ldots, \nu_{\mathcal{S}}\}$. In this case, equations (21, 22) can be put in the following matrix form:

$$\begin{bmatrix} \mathbb{A}_1 & \mathbb{0} & \ldots & \mathbb{0} \\ \mathbb{0} & \mathbb{A}_2 & \ldots & \mathbb{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \ldots & \mathbb{A}_{\nu_{\mathcal{S}}} \\ \widehat{\mathbb{A}}_1 & \widehat{\mathbb{A}}_2 & \ldots & \widehat{\mathbb{A}}_{\nu_{\mathcal{S}}} \end{bmatrix} \begin{bmatrix} \delta\dot{\mathbb{p}}_1 \\ \delta\dot{\mathbb{p}}_2 \\ \vdots \\ \delta\dot{\mathbb{p}}_{\nu_{\mathcal{S}}} \end{bmatrix} = \mathbb{0} \tag{26}$$

Using equation (25) and considering that the choice of $\mathbb{\pi}$ is arbitrary, it can be stated that:

$$\begin{bmatrix} \mathbb{A}_1 & \mathbb{0} & \ldots & \mathbb{0} \\ \mathbb{0} & \mathbb{A}_2 & \ldots & \mathbb{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \ldots & \mathbb{A}_{\nu_{\mathcal{S}}} \\ \widehat{\mathbb{A}}_1 & \widehat{\mathbb{A}}_2 & \ldots & \widehat{\mathbb{A}}_{\nu_{\mathcal{S}}} \end{bmatrix} \begin{bmatrix} \mathbb{C}_1 & \mathbb{0} & \ldots & \mathbb{0} \\ \mathbb{0} & \mathbb{C}_2 & \ldots & \mathbb{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \ldots & \mathbb{C}_{\nu_{\mathcal{S}}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbb{C}}_1 \\ \widehat{\mathbb{C}}_2 \\ \vdots \\ \widehat{\mathbb{C}}_{\nu_{\mathcal{S}}} \end{bmatrix} = \mathbb{0} \tag{27}$$

Once $\mathbb{A}_n \mathbb{C}_n = \mathbb{0}$, $\forall\, n \in \mathscr{I}_{\mathcal{S}}(\mathscr{M})$, the satisfaction of equation (27) is ensured if and only if:

$$\begin{bmatrix} \widehat{\mathbb{A}}_1\mathbb{C}_1 & \widehat{\mathbb{A}}_2\mathbb{C}_2 & \ldots & \widehat{\mathbb{A}}_{\nu_{\mathcal{S}}}\mathbb{C}_{\nu_{\mathcal{S}}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbb{C}}_1 \\ \widehat{\mathbb{C}}_2 \\ \vdots \\ \widehat{\mathbb{C}}_{\nu_{\mathcal{S}}} \end{bmatrix} = \mathbb{0} \tag{28}$$

Adopting the notation:

$$\widehat{\mathbb{A}} = \begin{bmatrix} \widehat{\mathbb{A}}_1\mathbb{C}_1 & \widehat{\mathbb{A}}_2\mathbb{C}_2 & \ldots & \widehat{\mathbb{A}}_{\nu_{\mathcal{S}}}\mathbb{C}_{\nu_{\mathcal{S}}} \end{bmatrix} \qquad \text{and} \qquad \widehat{\mathbb{C}} = \begin{bmatrix} \widehat{\mathbb{C}}_1 \\ \widehat{\mathbb{C}}_2 \\ \vdots \\ \widehat{\mathbb{C}}_{\nu_{\mathcal{S}}} \end{bmatrix}$$

equation (28) can be rewritten in the following form:

$$\widehat{\mathbb{A}}\, \widehat{\mathbb{C}} = \mathbb{0} \tag{29}$$

As each matrix $\mathbb{C}_n$ is an othogonal complement of the respective $\mathbb{A}_n$, so is matrix $\widehat{\mathbb{C}}$ an orthogonal complement of $\widehat{\mathbb{A}}$. Thus, equation (29) proves that the same algorithm that is used to obtain matrices $\mathbb{C}_n$ can be used to obtain $\widehat{\mathbb{C}}$ and, consequently, $\widehat{\mathbb{C}}_1$, $\widehat{\mathbb{C}}_2$, …, $\widehat{\mathbb{C}}_{\nu_{\mathcal{S}}}$.

Applying the Principle of Virtual Power for the system $\mathscr{M}$, it can be stated that:

$$\sum_n \delta\dot{\mathbb{p}}_n^{\mathsf{T}}(\mathbb{f}_n + \mathbb{g}_n - \mathbb{M}_n\,\dot{\mathbb{p}}_n) = 0 \tag{30}$$

Using equation (25):

$$\mathbb{\pi}^{\mathsf{T}}\left(\sum_n \widehat{\mathbb{C}}_n^{\mathsf{T}}\mathbb{C}_n^{\mathsf{T}}(\mathbb{f}_n + \mathbb{g}_n - \mathbb{M}_n\,\dot{\mathbb{p}}_n)\right) = \mathbb{\pi}^{\mathsf{T}}\left(\sum_n \widehat{\mathbb{C}}_n^{\mathsf{T}}\,\mathbb{d}_n\right) = 0 \tag{31}$$

Again, once the choice of $\mathbb{\pi}$ is arbitrary, equation (31) implies that:

$$\mathbb{d} = \sum_n \widehat{\mathbb{C}}_n^{\mathsf{T}}\,\mathbb{d}_n = \mathbb{0} \tag{32}$$

Adopting the notation:

$$\mathbb{c} = \begin{bmatrix} \mathbb{c}_1 \\ \mathbb{c}_2 \\ \vdots \\ \mathbb{c}_{\nu_{\mathscr{S}}} \\ \widehat{\mathbb{c}} \end{bmatrix} \qquad \text{and} \qquad \widehat{\mathbb{d}} = \begin{bmatrix} \mathbb{d}_1 \\ \mathbb{d}_2 \\ \vdots \\ \mathbb{d}_{\nu_{\mathscr{S}}} \end{bmatrix}$$

the mathematical model of the system $\mathscr{M}$ is given by the following pair of matrix equations:

$$\mathbb{c} = \mathbb{0} \tag{33}$$
$$\mathbb{d} = \widehat{\mathbb{C}}^{\mathsf{T}}\,\widehat{\mathbb{d}} = \mathbb{0} \tag{34}$$

This approach for obtaining mathematical models of multibody mechanical systems based on already known free subsystem models is a basis for the development of a methodology for adaptive balancing of mechanical systems.

2.3 Introduction to sliding modes control

In this subsection, a brief introduction to the sliding modes control will be done. The theme will be explored only to perform second order systems control, without parametric uncertainties, to not escape the scope of the chapter.

Consider a dynamical system given by the following differential equation:

$$\ddot{x} = u \tag{35}$$

A curve in the error phase plan, called sliding surface, can be defined:

$$s(e, \dot{e}) = -(\dot{e} + \lambda e) = 0,\ \lambda > 0 \tag{36}$$

Being $e = x^{\diamond} - x$ the error signal and $x^{\diamond}$ reference signal. Note that if the system is on the sliding surface, we have:

$$\dot{e} + \lambda e = 0 \Rightarrow e(t) = c\,\mathrm{e}^{-\lambda t} \tag{37}$$

Thus, the error drops exponentially to zero, with time constant $1/\lambda$.

To find a control law that brings the system to the sliding surface, we start from the definition of s:

$$s = -(\dot{e} + \lambda e)$$

Differentiating with respect to time:

$$\dot{s} = -(\ddot{e} + \lambda\dot{e}) = \ddot{x} - \ddot{x}^{\diamond} - \lambda\dot{e} \tag{38}$$

Substituting (35) into (38):

$$\dot{s} = u - \ddot{x}^{\diamond} - \lambda\dot{e} \tag{39}$$

Using the following control law:

$$u = \ddot{x}^{\diamond} + \lambda\dot{e} - k\,\mathsf{sign}(s),\ k > 0 \tag{40}$$

He have:

$$\dot{s} = -k\,\mathsf{sign}(s) \tag{41}$$

Suppose that the system starts at $s(0) = s_0 > 0$. Solving the ODE for $s > 0$:

$$\dot{s} = -k \Rightarrow s = -kt + c$$
$$s(0) = s_0 \Rightarrow c = s_0$$
$$\therefore s = s_0 - kt,\ s > 0$$

According to the solution found, when $t \to t_s = \frac{|s_0|}{k}$, $s \to 0$. Solving the ODE for $s(t_s) = 0$:

$$\dot{s} = 0 \Rightarrow s = c$$
$$s(t_s) = 0 \Rightarrow c = 0$$

Therefore, the solution of the ODE for $s(0) = s_0 > 0$ is given by:

$$s(t) = \begin{cases} s_0 - kt, & t < t_s \\ 0, & t \geq t_s \end{cases} \tag{42}$$

An analogous result is found solving the ODE for $s(0) = s_0 < 0$:

$$s(t) = \begin{cases} s_0 + kt, & t < t_s \\ 0, & t \geq t_s \end{cases} \tag{43}$$

Thus, it can be concluded that the ODE (41) converges to $s = 0$, regardless of the initial condition. Therefore, we have that the control law (40) makes the system represented by (35) follow the reference signal, because the error signal converges to zero.

2.4 Extended sliding modes control techniques

As seen in subsections 2.1 and 2.2, it's very convenient to use redundant coordinates to perform parallel mechanism dynamic modeling. Thus, we propose in this subsection a control law for systems described by redundant coordinates.

For the sake of brevity, the indexes n of the subsystems of $\mathscr{M}$ will be omitted in this subsection. Let $\mathscr{M}$ be a multibody mechanical system whose mathematical model is given by equations (33, 34). Suppose that each $\mathbb{f}$ is an affine function of the control inputs u_k in which the coefficients of the u_k may depend on the instantaneous configuration of the system. Suppose additionally that that all the $\mathbb{A}$ are independent of the quasi-velocities p_j and all the $\mathbb{M}$, $\mathbb{g}$, $\mathbb{A}$ and $\mathbb{b}$ are independent of the u_k. Under these conditions, matrices $\mathbb{C}$ will not depend on any quasi-velocity, $\mathbb{d}$ can be expressed as an affine function of the control inputs and $\mathbb{c}$

is independent of them. Considering that the number of control inputs in a mechanical system is exactly equal to the number of degrees of freedom of $\mathscr{M}$:

$$\mathbb{u} = \mathbb{C}^\mathsf{T}(t,\mathbb{q})\Big(\mathbb{M}(t,\mathbb{q})\dot{\mathbb{p}} + \mathbb{w}(t,\mathbb{q},\mathbb{p}) + \mathbb{z}(t,\mathbb{q})\Big) \tag{44}$$

From the control perspective, it is convenient to define $\mathbb{p} = \dot{\mathbb{q}}$, in order to have a position feedback control.

Based on equations (44) and (18), consider the dynamic model of a multi-body mechanical system described by the following equations:

$$\begin{cases} \mathbb{C}^\mathsf{T}(t,\mathbb{q})\Big(\mathbb{M}(t,\mathbb{q})\ddot{\mathbb{q}} + \mathbb{w}(t,\mathbb{q},\dot{\mathbb{q}}) + \mathbb{z}(t,\mathbb{q})\Big) = \mathbb{u} \\ \mathbb{A}(t,\mathbb{q})\ddot{\mathbb{q}} + \mathbb{b}(t,\mathbb{q},\dot{\mathbb{q}}) = \mathbb{0} \end{cases} \tag{45}$$

Rewriting in a compact matrix way:

$$\begin{bmatrix} \mathbb{C}^\mathsf{T}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \ddot{\mathbb{q}} = \begin{bmatrix} \mathbb{u} - \mathbb{C}^\mathsf{T}(\mathbb{w}+\mathbb{z}) \\ -\mathbb{b} \end{bmatrix} \tag{46}$$

We want to find a control law such that, in closed loop, $\ddot{\mathbb{q}} = \mathbb{v}$, being $\mathbb{v}$ a control input matrix column. For this to happen, the following control law is used:

$$\mathbb{u} = \mathbb{C}^\mathsf{T}(\mathbb{M}\mathbb{v} + \mathbb{w} + \mathbb{z}) \tag{47}$$

As we want $\ddot{\mathbb{q}} = \mathbb{v}$ and $\ddot{\mathbb{q}}$ has restrictions, $\mathbb{v}$ must respect the same restrictions, i.e.:

$$\mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \tag{48}$$

Applying the control law (47) and the rectritions (48) in (46), we have:

$$\begin{bmatrix} \mathbb{C}^\mathsf{T}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \ddot{\mathbb{q}} = \begin{bmatrix} \mathbb{C}^\mathsf{T}(\mathbb{M}\mathbb{v}+\mathbb{w}+\mathbb{z}) - \mathbb{C}^\mathsf{T}(\mathbb{w}+\mathbb{z}) \\ \mathbb{A}\mathbb{v} \end{bmatrix} = \begin{bmatrix} \mathbb{C}^\mathsf{T}\mathbb{M}\mathbb{v} \\ \mathbb{A}\mathbb{v} \end{bmatrix} = \begin{bmatrix} \mathbb{C}^\mathsf{T}\mathbb{M} \\ \mathbb{A} \end{bmatrix} \mathbb{v}$$

Once the matrix $\begin{bmatrix} \mathbb{C}^\mathsf{T}\mathbb{M} \\ \mathbb{A} \end{bmatrix}$ is non-singular:

$$\ddot{\mathbb{q}} = \mathbb{v} \tag{49}$$

Let $\mathbb{v}'$ be given by the sliding modes control law:

$$\mathbb{v}' = \ddot{\mathbb{q}}_n^\diamond + \lambda\dot{\mathbb{e}} + k\,\mathsf{sign}(\dot{\mathbb{e}} + \lambda\mathbb{e}) \tag{50}$$

Being $\mathbb{e} = \mathbb{q}_n^\diamond - \mathbb{q}$ the error signal and $\mathbb{q}_n^\diamond$ the reference signal. If there wasn't any retrictions, we could do $\mathbb{v} = \mathbb{v}'$:

$$\ddot{\mathbb{q}} = \mathbb{v} \Rightarrow \ddot{\mathbb{e}} + \lambda\dot{\mathbb{e}} + k\,\mathsf{sign}(\dot{\mathbb{e}} + \lambda\mathbb{e}) = \mathbb{0} \Leftrightarrow \dot{\mathbb{s}} = -k\,\mathsf{sign}(\mathbb{s})$$

This would ensure that $\mathbb{e} \to \mathsf{0}$ when $t \to \infty$ for any initial condition, as seen in the last subsection.

As we have resctriction on $\mathbb{v}$, we look for the closest possible $\mathbb{v}$ of $\mathbb{v}'$ by solving the following optimization problem:

$$\begin{aligned} \min_{\mathbb{v}} \quad & (\mathbb{v}-\mathbb{v}')^\mathsf{T}\mathbb{M}(\mathbb{v}-\mathbb{v}') \\ \text{s.t.} \quad & \mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \end{aligned} \tag{51}$$

As $\mathbb{M}$ is positive-semidefinite, we have $(\mathbb{v}-\mathbb{v}')^\mathsf{T}\mathbb{M}(\mathbb{v}-\mathbb{v}') \geq \mathsf{0}$ for any $\mathbb{v}$.

Applying the method of Lagrange multipliers, it can be said that the following problem is equivalent to:

$$\min_{\mathbb{v},\mathbb{\lambda}} \quad L = (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + (\mathbb{A}\mathbb{v} + \mathbb{b})^{\mathsf{T}}\mathbb{\lambda} \tag{52}$$

To solve the problem, the Lagrangian function must be stationary:

$$\begin{aligned}\delta L = 0 &\Rightarrow \delta\mathbb{v}^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + (\mathbb{v} - \mathbb{v}')^{\mathsf{T}}\mathbb{M}\delta\mathbb{v} + (\mathbb{A}\delta\mathbb{v})^{\mathsf{T}}\mathbb{\lambda} + (\mathbb{A}\mathbb{v} + \mathbb{b})^{\mathsf{T}}\delta\mathbb{\lambda} = 0 \\ &\Rightarrow \delta\mathbb{v}^{\mathsf{T}}\Big((\mathbb{M} + \mathbb{M}^{\mathsf{T}})(\mathbb{v} - \mathbb{v}') + \mathbb{A}^{\mathsf{T}}\mathbb{\lambda}\Big) + \delta\mathbb{\lambda}^{\mathsf{T}}(\mathbb{A}\mathbb{v} + \mathbb{b}) = 0\end{aligned}$$

As $\mathbb{M}$ is symmetric and $\delta\mathbb{v}$ and $\delta\mathbb{\lambda}$ are arbitrary, we have:

$$\begin{cases} 2\mathbb{M}(\mathbb{v} - \mathbb{v}') + \mathbb{A}^{\mathsf{T}}\mathbb{\lambda} = \mathbb{0} \\ \mathbb{A}\mathbb{v} + \mathbb{b} = \mathbb{0} \end{cases} \tag{53}$$

As $\mathbb{C}$ is the orthogonal complement of $\mathbb{A}$, pre-multiplying the first equation of (53) by $\mathbb{C}^{\mathsf{T}}$, we have:

$$2\mathbb{C}^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') + \mathbb{C}^{\mathsf{T}}\mathbb{A}^{\mathsf{T}}\mathbb{\lambda} = \mathbb{0} \Rightarrow \mathbb{C}^{\mathsf{T}}\mathbb{M}(\mathbb{v} - \mathbb{v}') = \mathbb{0}$$

$$\therefore \mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v} = \mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{v}' \tag{54}$$

Thus, we have that the control law that makes the closed loop system as close as possible of $\ddot{\mathbb{q}} = \mathbb{v}'$, according to the optimization criterion adopted, is:

$$\mathbb{u} = \mathbb{C}^{\mathsf{T}}(\mathbb{M}\mathbb{v}' + \mathbb{w} + \mathbb{z}) \tag{55}$$

3 Adaptive balancing techniques

3.1 Definitions

Let $\mathscr{M}$ be a multibody mechanical system whose mathematical model is given by equations (33, 34). Suppose that each $\mathbb{f}_n$ is an affine function of the control inputs $u_{n,k}$ in which the coefficients of the $u_{n,k}$ may depend on the instantaneous configuration of the system. Suppose additionally that that all the $\mathbb{A}_n$ are independent of the quasi-velocities $p_{n,j}$ and all the $\mathbb{M}_n$, $\mathbb{g}_n$, $\mathbb{A}_n$ and $\mathbb{b}_n$ are independent of the $u_{n,k}$. Under these conditions, matrices $\mathbb{C}_n$ will not depend on any quasi-velocity, $\mathbb{d}$ can be expressed as an affine function of the control inputs and $\mathbb{c}$ is independent of them. Considering that the number of control inputs in a mechanical system is exactly equal to the number of degrees of freedom of $\mathscr{M}$, it may be possible to solve equations (34) in order to express each $u_{n,k}$ as an explicit function of state variables and physical parameters of $\mathscr{M}$. For the sake of brevity, omitting the indexes n of the subsystems of $\mathscr{M}$, it can be stated that, in such cases, for $k \in \{1, \ldots, \nu^{\#}(\mathscr{M})\}$:

$$u_k = \sum_r M'_{kr}(t,\mathbb{q})\,\dot{p}_r + w'_k(t,\mathbb{q},\mathbb{p}) + z'_k(t,\mathbb{q}) \tag{56}$$

It can be shown that each $w'_k(t,\mathbb{q},\mathbb{p})$ can be expressed as a sum of a bilinear and a linear function on the quasi-velocities, i.e., there are functions $D'_{krs}(t,\mathbb{q})$ and $B'_{kr}(t,\mathbb{q})$ such that:

$$w'_k(t,\mathbb{q},\mathbb{p}) = \sum_r\sum_s D'_{krs}(t,\mathbb{q})\,p_r\,p_s + \sum_r B'_{kr}(t,\mathbb{q})\,p_r \tag{57}$$

Adaptive balancing types are defined based on equation (56).

Definition 3.1. *Let $\mathscr{M}$ be a $\nu^{\#}(\mathscr{M})$ degrees of freedom multibody mechanical system with $\nu^{\#}(\mathscr{M})$ control inputs. Suppose that, except from singular states, it is possible to express the dynamic equations of the model of $\mathscr{M}$ in the form (56). The following types of adaptive balancing can be defined:*

- *Static adaptive balancing: when, by a particular choice of the values of the adjustable physical parameters a_l, denoted by a_l', it is possible to make all the $z_k'(t, \mathbb{q}) = 0$ for every configuration $(t, \mathbb{q})$ corresponding to a nonsingular state; in such cases, in every static equilibrium state of $\mathscr{M}$, all the control inputs will be null.*

- *Gyroscopic adaptive balancing: when, by a particular choice of the values of the adjustable physical parameters a_l, denoted by a_l', it is possible to make all the $z_k'(t, \mathbb{q}) = 0$, all the $D_{krs}'(t, \mathbb{q}) = 0$, and all the $B_{kr}'(t, \mathbb{q}) = 0$ for every configuration $(t, \mathbb{q})$ corresponding to a nonsingular state; in such cases the control inputs will be proportional to the quasi-accelerations of the system, independing of the values of the quasi-velocities.*

- *Decoupling adaptive balancing: when, by a particular choice of the values of the adjustable physical parameters a_l, denoted by a_l', and by some reordering of indices, it is possible to make all the $z_k'(t, \mathbb{q}) = 0$, all the $D_{krs}'(t, \mathbb{q}) = 0$, all the $B_{kr}'(t, \mathbb{q}) = 0$ and $M_{kr}'(t, \mathbb{q}) = 0$ whenever $k \neq r$, for every configuration $(t, \mathbb{q})$ corresponding to a nonsingular state; in such cases, each control input will be proportional to a particular quasi-acceleration.*

The adjective "fully" will be used whenever a particular choice of adjustable parameters leads to the following equations, for $k \in \{1, \ldots, \nu^{\#}(\mathscr{M})\}$:

$$u_k = \sum_r M_{kr}''\, \dot{p}_r + \sum_r \sum_s D_{krs}''\, p_r\, p_s + \sum_r \sum_s B_{kr}''\, p_r \tag{58}$$

with M_{kr}'', D_{krs}'' and B_{kr}'' independent of the configuration $(t, \mathbb{q})$ of the system. In such cases, when some $D_{krs}'' \neq 0$ or some $B_{kr}'' \neq 0$, it corresponds to a fully static adaptive balancing; when all the $D_{krs}'' = 0$ and all the $B_{kr}'' = 0$, it corresponds to a fully gyroscopic adaptive balancing and when all the $D_{krs}'' = 0$, all the $B_{kr}'' = 0$ and $M_{kr}'' = 0$ whenever $k \neq r$, it corresponds to a fully decoupling adaptive balancing.

3.2 Adaptive balancing of serial and parallel mechanisms

Let $\mathscr{U}$ be an "unbalanced" serial or parallel mechanism. A serial mechanism can be interpreted as an assemble of two subsystems: an open loop kinematic chain (representing the actual mechanism) and a payload. On the other hand, a parallel mechanism will be conceived as an assemble of the following subsystems: open loop kinematic chains (representing the active and passive chains of the acutal mechanism) and an end-effector rigidly attached to a payload.

Let $\mathscr{S}_n$ denote mechanical systems with adjustable parameters that are assembled to $\mathscr{U}$, in order to obtain a multibody mechanical system $\mathscr{M}$ that can be balanced according to some of the strategies shown in Definition 3.1.

Consider the results presented in the following propositions.

Proposition 3.2. *The dynamic coupling of any static adaptive balanced subsystems is a static adaptive balanced multibody mechanical system.*

Proof. Consider some static adaptive balanced subsystems $\mathscr{S}_n$. For each of these subsystems there is a particular matrix $\mathbb{C}_n$ such that:

$$\mathbb{d}_n = \mathbb{u}_n - \mathbb{M}_n'(t, \mathbb{q}_n)\, \dot{\mathbb{p}}_n - \mathbb{w}_n'(t, \mathbb{q}_n, \mathbb{p}_n) \tag{59}$$

In this equation, $\mathbb{w}_n'(t, \mathbb{q}_n, \mathbb{p}_n)$ denotes the column-matrix whose entries are $w_{n,k}'(t, \mathbb{q}_n, \mathbb{p}_n)$.

Consider that these subsystems are assembled. The dynamic equations of the multidoby mechanical system $\mathscr{M}$ thus obtained is given by

$$\mathbb{d} = \sum_n \widehat{\mathbb{C}}_n^{\mathsf{T}} \left(\mathbb{u}_n - \mathbb{M}_n'(t, \mathbb{q}_n)\, \dot{\mathbb{p}}_n - \mathbb{w}_n'(t, \mathbb{q}_n, \mathbb{p}_n)\right) = \mathbb{0} \tag{60}$$

Supposing that all but $\nu^{\#}(\mathscr{M})$ of the $u_{n,k}$ are identically zero in the system $\mathscr{M}$, equations (60) can be manipulated to obtain explicit equations for the remaining $\nu^{\#}(\mathscr{M})$. It is evident that such equations will have the same form as equations (56) with all the $z'_k(t, \mathbb{q}) = 0$. This concludes the proof. $\square$

Analogously, it can also be proved that:

Proposition 3.3. *The dynamic coupling of any gyroscopic adaptive balanced subsystems is a gyroscopic adaptive balanced multibody mechanical system.*

Considering the results presented on Propositions 3.2 and 3.3 the following strategy for balancing $\mathscr{U}$ can be stated:

- If $\mathscr{U}$ is a serial mechanism, constraint a subsystem $\mathscr{S}_n$ with adjustable parameters to each of its links (or to pairs of links) and verify which values of parameters have do be chosen in order to achieve a desired balancing. If there are not enough parameters, some of the subsystems $\mathscr{S}_n$ can be substituted or others can be added.

- If $\mathscr{U}$ is a parallel mechanism, consider it as an assemble of open loop kinematic chains and an end-effector rigidly attached to a payload. Consider each open loop kinematic chain as a serial mechanism and attach to its end-effector part of the inertia of the end-effector and the payload of the original mechanism. (this inertia redistribuition has to be able to reproduce adequately the inertial effects of the end-effector and payload on each chain of the mechanism). The next step is to balance each of these serial mechanisms according to the strategy previously presented, and obtain the model of the balanced parallel mechanism by recoupling the models of the balanced versions of these subsystems (according to the same constraints already existing in the original mechanism).

4 Case study: adaptive balancing of a 5-bar mechanism

4.1 Preliminaries

Consider a planar two degrees of freedom 5-bar mechanism with revolute joints, also called 5R, which can be conceived as two RR mechanisms constrained by a revolute joint. Thus, considering the strategy presented on Section 3.2, the first step to balance a 5R mechanism is to balance a RR mechanism.

Consider the RR mechanism presented in Figure 2. It is constitued by two articulated rigid arms $\mathcal{B}_{\mathrm{RR},1}$ and $\mathcal{B}_{\mathrm{RR},2}$ ($m_{\mathrm{RR},\mathcal{B}_s}$ denotes the mass of $\mathcal{B}_{\mathrm{RR},s}$, $\mathtt{b}^{\star}_{\mathrm{RR},s}$ denotes the centre of mass of $\mathcal{B}_{\mathrm{RR},s}$ and $I_{\mathrm{RR},\mathcal{B}_s}$ denotes moments of inertia of $\mathcal{B}_{\mathrm{RR},s}$ relative to an axis passing through $\mathtt{b}^{\star}_{\mathrm{RR},s}$ perpendicularly to its plane of motion, $s \in \{1, 2\}$). Points $\mathtt{p}_{\mathrm{RR},1}$ and $\mathtt{p}_{\mathrm{RR},2}$ are the geometric centres of the revolute joints and $\mathtt{p}_{\mathrm{RR},3}$ represents the geometric center of the end-effector of the mechanism (which is rigidly attached to its second arm). The main geometric parameters of this system are: the distance between points $\mathtt{p}_{\mathrm{RR},1}$ and $\mathtt{p}_{\mathrm{RR},2}$, denoted by $a_{\mathrm{RR},1}$ and the distance between points $\mathtt{p}_{\mathrm{RR},2}$ and $\mathtt{p}_{\mathrm{RR},3}$, denoted by $a_{\mathrm{RR},2}$. Two additional adimensional parameters $\gamma_{\mathrm{RR},1}$ and $\gamma_{\mathrm{RR},2}$ are defined to express the ratio of the distance between $\mathtt{p}_{\mathrm{RR},1}$ and $\mathtt{b}^{\star}_{\mathrm{RR},1}$ to $a_{\mathrm{RR},1}$ and the ratio of the distance between $\mathtt{p}_{\mathrm{RR},2}$ and $\mathtt{b}^{\star}_{\mathrm{RR},2}$ to $a_{\mathrm{RR},2}$, respectively.

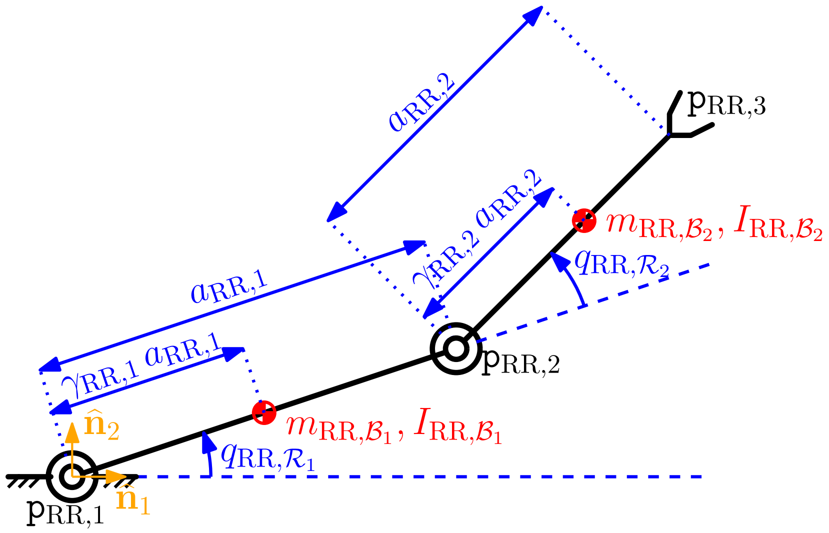

Figure 2: RR mechanism

In order to model this mechanism, consider a coordinate system $\mathtt{N}$ fixed to an inertial reference frame $\mathcal{N}$: the origin of $\mathtt{N}$ will be denoteb by $\mathtt{n}_0$ and its orthonormal vector basis is given by the unit vectors $\hat{\boldsymbol{n}}_1$ (horizontal direction in the plane of the motions of the arms), $\hat{\boldsymbol{n}}_2$ (vertical) and $\hat{\boldsymbol{n}}_3 = \hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2$. The following generalized coordinates are defined:

- $q_{\mathrm{RR},\mathcal{R}_1}$ and $q_{\mathrm{RR},\mathcal{R}_2}$ (see Figure 2) describing the relative motions associated to the revolute joints: $q_{\mathrm{RR},\mathcal{R}_1}$ is the angle between the horizontal and the line joining $\mathtt{p}_{\mathrm{RR},1}$, and $\mathtt{p}_{\mathrm{RR},2}$ and $q_{\mathrm{RR},\mathcal{R}_2}$ is the angle between this latter line and the line joining $\mathtt{p}_{\mathrm{RR},2}$ and $\mathtt{p}_{\mathrm{RR},3}$.
- $q_{\mathrm{RR},\mathtt{p}_r,1}$ and $q_{\mathrm{RR},\mathtt{p}_r,2}$, for $r \in \{1,2,3\}$, such that $\boldsymbol{r}_{\mathtt{p}_{\mathrm{RR},r}\,|\,\mathtt{n}_0} = q_{\mathrm{RR},\mathtt{p}_r,1}\,\hat{\boldsymbol{n}}_1 + q_{\mathrm{RR},\mathtt{p}_r,2}\,\hat{\boldsymbol{n}}_2$.

The following quasi-velocities are defined:

- $p_{\mathrm{RR},\mathcal{R}_1} = \dot{q}_{\mathrm{RR},\mathcal{R}_1}$ and $p_{\mathrm{RR},\mathcal{R}_2} = \dot{q}_{\mathrm{RR},\mathcal{R}_2}$.
- $p_{\mathrm{RR},\mathcal{B}_s,1}$, $p_{\mathrm{RR},\mathcal{B}_s,2}$ and $p_{\mathrm{RR},\mathcal{B}_s,3}$, for $s \in \{1,2\}$, such that $\boldsymbol{v}_{\mathtt{b}^\star_{\mathrm{RR},s}\,|\,\mathcal{N}} = p_{\mathrm{RR},\mathcal{B}_s,1}\,\hat{\boldsymbol{n}}_1 + p_{\mathrm{RR},\mathcal{B}_s,2}\,\hat{\boldsymbol{n}}_2$ and $\boldsymbol{\omega}_{\mathcal{B}_{\mathrm{RR},s}\,|\,\mathcal{N}} = p_{\mathrm{RR},\mathcal{B}_s,3}\,\hat{\boldsymbol{n}}_3$.

The following column-matrices can also be defined:

$$\mathbb{q}_{\mathrm{RR}} = \begin{bmatrix} q_{\mathrm{RR},\mathcal{R}_1} & q_{\mathrm{RR},\mathcal{R}_2} & q_{\mathrm{RR},\mathtt{p}_2,1} & q_{\mathrm{RR},\mathtt{p}_2,2} & q_{\mathrm{RR},\mathtt{p}_3,1} & q_{\mathrm{RR},\mathtt{p}_3,2} \end{bmatrix}^{\mathsf{T}} \tag{61}$$

$$\mathbb{p}_{\mathrm{RR}} = \begin{bmatrix} p_{\mathrm{RR},\mathcal{R}_1} & p_{\mathrm{RR},\mathcal{R}_2} & p_{\mathrm{RR},\mathcal{B}_1,1} & p_{\mathrm{RR},\mathcal{B}_1,2} & p_{\mathrm{RR},\mathcal{B}_1,3} & p_{\mathrm{RR},\mathcal{B}_2,1} & p_{\mathrm{RR},\mathcal{B}_2,2} & p_{\mathrm{RR},\mathcal{B}_2,3} \end{bmatrix}^{\mathsf{T}} \tag{62}$$

The transformation of variables relating the time derivatives of generalized coordinates to the quasi-velocities of the model can be given by the following identities:

$$\begin{cases} p_{\mathrm{RR},\mathcal{R}_1} = \dot{q}_{\mathrm{RR},\mathcal{R}_1} \\ p_{\mathrm{RR},\mathcal{R}_2} = \dot{q}_{\mathrm{RR},\mathcal{R}_2} \\ \boldsymbol{v}_{\mathtt{b}^\star_{\mathrm{RR},1}\,|\,\mathcal{N}} = (1-\gamma_{\mathrm{RR},1})\dot{\boldsymbol{r}}_{\mathtt{p}_{\mathrm{RR},1}\,|\,\mathtt{n}_0} + \gamma_{\mathrm{RR},1}\,\dot{\boldsymbol{r}}_{\mathtt{p}_{\mathrm{RR},2}\,|\,\mathtt{n}_0} \\ \boldsymbol{v}_{\mathtt{b}^\star_{\mathrm{RR},2}\,|\,\mathcal{N}} = (1-\gamma_{\mathrm{RR},2})\dot{\boldsymbol{r}}_{\mathtt{p}_{\mathrm{RR},2}\,|\,\mathtt{n}_0} + \gamma_{\mathrm{RR},2}\,\dot{\boldsymbol{r}}_{\mathtt{p}_{\mathrm{RR},3}\,|\,\mathtt{n}_0} \end{cases} \tag{63}$$

Solving these equations for the time derivatives of generalized coordinates, it can be stated that:

$$
\begin{cases}
\dot{q}^*_{\mathrm{RR},\mathcal{R}_1} = p_{\mathrm{RR},\mathcal{R}_1} \\
\dot{q}^*_{\mathrm{RR},\mathcal{R}_2} = p_{\mathrm{RR},\mathcal{R}_2} \\
\dot{q}^*_{\mathrm{RR},\mathsf{p}_1,1} = 0 \\
\dot{q}^*_{\mathrm{RR},\mathsf{p}_1,2} = 0 \\
\dot{q}^*_{\mathrm{RR},\mathsf{p}_2,1} = \dfrac{p_{\mathrm{RR},\mathcal{B}_1,1}}{\gamma_{\mathrm{RR},\mathcal{B}_1}} \\
\dot{q}^*_{\mathrm{RR},\mathsf{p}_2,2} = \dfrac{p_{\mathrm{RR},\mathcal{B}_1,2}}{\gamma_{\mathrm{RR},\mathcal{B}_1}} \\
\dot{q}^*_{\mathrm{RR},\mathsf{p}_3,1} = \dfrac{p_{\mathrm{RR},\mathcal{B}_2,1}}{\gamma_{\mathrm{RR},\mathcal{B}_2}} - \dfrac{\left(1-\gamma_{\mathrm{RR},\mathcal{B}_2}\right) p_{\mathrm{RR},\mathcal{B}_1,1}}{\gamma_{\mathrm{RR},\mathcal{B}_1}\gamma_{\mathrm{RR},\mathcal{B}_2}} \\
\dot{q}^*_{\mathrm{RR},\mathsf{p}_3,2} = \dfrac{p_{\mathrm{RR},\mathcal{B}_2,2}}{\gamma_{\mathrm{RR},\mathcal{B}_2}} - \dfrac{\left(1-\gamma_{\mathrm{RR},\mathcal{B}_2}\right) p_{\mathrm{RR},\mathcal{B}_1,2}}{\gamma_{\mathrm{RR},\mathcal{B}_1}\gamma_{\mathrm{RR},\mathcal{B}_2}}
\end{cases}
\tag{64}
$$

The holonomic constraint equations for this model can be given by the following identities:

$$
\begin{cases}
\dot{\boldsymbol{r}}_{\mathsf{p}_{\mathrm{RR},2}\,|\,\mathsf{p}_{\mathrm{RR},1}} = \boldsymbol{\omega}_{\mathcal{B}_{\mathrm{RR},1}\,|\,\mathcal{N}} \times \boldsymbol{r}_{\mathsf{p}_{\mathrm{RR},2}\,|\,\mathsf{p}_{\mathrm{RR},1}} \\
\dot{\boldsymbol{r}}_{\mathsf{p}_{\mathrm{RR},3}\,|\,\mathsf{p}_{\mathrm{RR},2}} = \boldsymbol{\omega}_{\mathcal{B}_{\mathrm{RR},2}\,|\,\mathcal{N}} \times \boldsymbol{r}_{\mathsf{p}_{\mathrm{RR},3}\,|\,\mathsf{p}_{\mathrm{RR},2}} \\
\boldsymbol{\omega}_{\mathcal{B}_{\mathrm{RR},1}\,|\,\mathcal{N}} = p_{\mathrm{RR},\mathcal{R}_1}\,\hat{\boldsymbol{n}}_3 \\
\boldsymbol{\omega}_{\mathcal{B}_{\mathrm{RR},2}\,|\,\mathcal{N}} = \left(p_{\mathrm{RR},\mathcal{R}_1} + p_{\mathrm{RR},\mathcal{R}_2}\right)\hat{\boldsymbol{n}}_3
\end{cases}
\tag{65}
$$

This leads to the following constraint equations:

$$
\begin{cases}
\psi_{\mathrm{RR},1} = \gamma_{\mathrm{RR},\mathcal{B}_1}\, p_{\mathrm{RR},\mathcal{B}_1,3}\left(q_{\mathrm{RR},\mathsf{p}_2,2} - q_{\mathrm{RR},\mathsf{p}_1,2}\right) + p_{\mathrm{RR},\mathcal{B}_1,1} = 0 \\
\psi_{\mathrm{RR},2} = \gamma_{\mathrm{RR},\mathcal{B}_1}\, p_{\mathrm{RR},\mathcal{B}_1,3}\left(q_{\mathrm{RR},\mathsf{p}_1,1} - q_{\mathrm{RR},\mathsf{p}_2,1}\right) + p_{\mathrm{RR},\mathcal{B}_1,2} = 0 \\
\psi_{\mathrm{RR},3} = \gamma_{\mathrm{RR},\mathcal{B}_1}\left(\gamma_{\mathrm{RR},\mathcal{B}_2}\, p_{\mathrm{RR},\mathcal{B}_2,3}\left(q_{\mathrm{RR},\mathsf{p}_3,2} - q_{\mathrm{RR},\mathsf{p}_2,2}\right) + p_{\mathrm{RR},\mathcal{B}_2,1}\right) - p_{\mathrm{RR},\mathcal{B}_1,1} = 0 \\
\psi_{\mathrm{RR},4} = \gamma_{\mathrm{RR},\mathcal{B}_1}\left(\gamma_{\mathrm{RR},\mathcal{B}_2}\, p_{\mathrm{RR},\mathcal{B}_2,3}\left(q_{\mathrm{RR},\mathsf{p}_2,1} - q_{\mathrm{RR},\mathsf{p}_3,1}\right) + p_{\mathrm{RR},\mathcal{B}_2,2}\right) - p_{\mathrm{RR},\mathcal{B}_1,2} = 0 \\
\psi_{\mathrm{RR},5} = p_{\mathrm{RR},\mathcal{R}_1} - p_{\mathrm{RR},\mathcal{B}_1,3} = 0 \\
\psi_{\mathrm{RR},6} = p_{\mathrm{RR},\mathcal{R}_1} + p_{\mathrm{RR},\mathcal{R}_2} - p_{\mathrm{RR},\mathcal{B}_2,3} = 0
\end{cases}
\tag{66}
$$

Calculating the time derivatives of the invariants $\psi_{\mathrm{RR},s}$, it can be stated that:

$$
\mathbb{A}_{\mathrm{RR}} = \begin{bmatrix}
0 & 0 & 1 & 0 & \gamma_{\mathrm{RR},\mathcal{B}_1}\left(q_{\mathrm{RR},\mathsf{p}_2,2} - q_{\mathrm{RR},\mathsf{p}_1,2}\right) & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \gamma_{\mathrm{RR},\mathcal{B}_1}\left(q_{\mathrm{RR},\mathsf{p}_1,1} - q_{\mathrm{RR},\mathsf{p}_2,1}\right) & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & \gamma_{\mathrm{RR},\mathcal{B}_1} & 0 & \gamma_{\mathrm{RR},\mathcal{B}_1}\gamma_{\mathrm{RR},\mathcal{B}_2}\left(q_{\mathrm{RR},\mathsf{p}_3,2} - q_{\mathrm{RR},\mathsf{p}_2,2}\right) \\
0 & 0 & 0 & -1 & 0 & 0 & \gamma_{\mathrm{RR},\mathcal{B}_1} & \gamma_{\mathrm{RR},\mathcal{B}_1}\gamma_{\mathrm{RR},\mathcal{B}_2}\left(q_{\mathrm{RR},\mathsf{p}_2,1} - q_{\mathrm{RR},\mathsf{p}_3,1}\right) \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\tag{67}
$$

A full rank matrix $\mathbb{C}_{\mathrm{RR}}$ satisfying the condition $\mathbb{A}_{\mathrm{RR}}\,\mathbb{C}_{\mathrm{RR}} = \mathbb{0}$ is the following:

$$
\mathbb{C}_{\mathrm{RR}} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\gamma_{\mathrm{RR},\mathcal{B}_1}\left(q_{\mathrm{RR},\mathsf{p}_1,2} - q_{\mathrm{RR},\mathsf{p}_2,2}\right) & 0 \\
\gamma_{\mathrm{RR},\mathcal{B}_1}\left(q_{\mathrm{RR},\mathsf{p}_2,1} - q_{\mathrm{RR},\mathsf{p}_1,1}\right) & 0 \\
1 & 0 \\
q_{\mathrm{RR},\mathsf{p}_1,2} + \left(\gamma_{\mathrm{RR},\mathcal{B}_2} - 1\right) q_{\mathrm{RR},\mathsf{p}_2,2} - \gamma_{\mathrm{RR},\mathcal{B}_2} q_{\mathrm{RR},\mathsf{p}_3,2} & \gamma_{\mathrm{RR},\mathcal{B}_2}\left(q_{\mathrm{RR},\mathsf{p}_2,2} - q_{\mathrm{RR},\mathsf{p}_3,2}\right) \\
q_{\mathrm{RR},\mathsf{p}_1,1} + \left(\gamma_{\mathrm{RR},\mathcal{B}_2} - 1\right) q_{\mathrm{RR},\mathsf{p}_2,1} - \gamma_{\mathrm{RR},\mathcal{B}_2} q_{\mathrm{RR},\mathsf{p}_3,1} & \gamma_{\mathrm{RR},\mathcal{B}_2}\left(q_{\mathrm{RR},\mathsf{p}_2,1} - q_{\mathrm{RR},\mathsf{p}_3,1}\right) \\
1 & 1
\end{bmatrix}
\tag{68}
$$

Suppose that there are actuators in both revolute joints, providing control torques $u_{\mathrm{RR},1}$ and $u_{\mathrm{RR},2}$ in the joints $\mathcal{R}_{\mathrm{RR},1}$ and $\mathcal{R}_{\mathrm{RR},2}$, respectively. Suppose also that the gravitational acceleration is given by $-g\hat{\boldsymbol{n}}_2$. Using equations (12, 13) it can also be stated that:

$$\mathbb{M}_{\mathrm{RR}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{\mathrm{RR},\mathcal{B}_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{\mathrm{RR},\mathcal{B}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\mathrm{RR},\mathcal{B}_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{\mathrm{RR},\mathcal{B}_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{\mathrm{RR},\mathcal{B}_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathrm{RR},\mathcal{B}_2} \end{bmatrix} \tag{69}$$

$$\mathbb{g}_{\mathrm{RR}} = \mathbb{0} \tag{70}$$

$$\mathbb{f}_{\mathrm{RR}} = \begin{bmatrix} u_{\mathrm{RR},1} \\ u_{\mathrm{RR},2} \\ 0 \\ -g\, m_{\mathrm{RR},\mathcal{B}_1} \\ 0 \\ 0 \\ -g\, m_{\mathrm{RR},\mathcal{B}_2} \\ 0 \end{bmatrix} \tag{71}$$

Now, consider the mathematical modelling of the balancing masses (BM). When unconstrained to the RR mechanism, they can be conceived as punctual masses that can freely execute a planar motion. Each balancing mass will be considered as a particle $\mathcal{B}_{\mathrm{BM},1}$ whose mass is equal to $m_{\mathrm{BM},\mathcal{B}_1}$. The point in the plane representing $\mathcal{B}_{\mathrm{BM},1}$ will be denoted by $\mathsf{p}_{\mathrm{BM},1}$. Suppose also that $\boldsymbol{r}_{\mathsf{p}_{\mathrm{BM},1}|\mathsf{n}_0} = q_{\mathrm{BM},\mathsf{p}_1,1}\, \hat{\boldsymbol{n}}_1 + q_{\mathrm{BM},\mathsf{p}_1,2}\, \hat{\boldsymbol{n}}_2$ and $\boldsymbol{v}_{\mathsf{p}_{\mathrm{BM},1}|\mathcal{N}} = \dot{\boldsymbol{r}}_{\mathsf{p}_{\mathrm{BM},1}|\mathsf{n}_0} = p_{\mathrm{BM},\mathsf{p}_1,1}\, \hat{\boldsymbol{n}}_1 + p_{\mathrm{BM},\mathsf{p}_1,2}\, \hat{\boldsymbol{n}}_2$. The dynamical model of system BM is defined by the following matrices:

$$\mathbb{C}_{\mathrm{BM}} = \mathbb{1} \tag{72}$$

$$\mathbb{M}_{\mathrm{BM}} = \begin{bmatrix} m_{\mathrm{BM},\mathcal{B}_1} & 0 \\ 0 & m_{\mathrm{BM},\mathcal{B}_1} \end{bmatrix} \tag{73}$$

$$\mathbb{g}_{\mathrm{BM}} = \mathbb{0} \tag{74}$$

$$\mathbb{f}_{\mathrm{BM}} = \begin{bmatrix} 0 \\ -g\, m_{\mathrm{BM},\mathcal{B}_1} \end{bmatrix} \tag{75}$$

In this description, the system BM has no (internal) kinematic constraints.

4.2 Adaptive balancing of a RR mechanism

Consider a multibody mechanical system denoted by RR_ that is constituted by an RR mechanism, two balancing masses (one attached to each of the arms of the RR mechanism) and a payload, as represented in Figure 4.

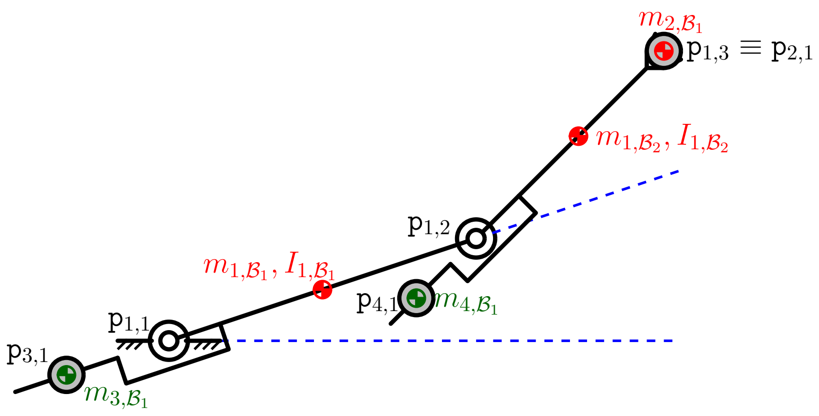

Figure 3: RR mechanism

The modelling process leading to the dynamical equations of system RR_ is synthesized in the diagram of Figure 4. Four subsystems will be considered: subsystem 1 will be a RR mechanism whose model is given by equations (68 – 71); subsystems 2 is a payload rigidly attached to the point $\mathrm{p}_{1,3}$ of subsystem 1 and will be modelled identically to a balancing mass (BM), whose model is given by equations (72 – 75); subsystems 3 and 4 are balancing masses (BM) rigidly attached to the arms $\mathcal{B}_{1,1}$ and $\mathcal{B}_{2,1}$ of subsystem 1, satisfying the relations $\boldsymbol{r}_{\mathrm{p}_{3,1}|\mathrm{p}_{1,1}} = -\gamma_{3,\mathcal{B}_1} \boldsymbol{r}_{\mathrm{p}_{1,2}|\mathrm{p}_{1,1}}$ and $\boldsymbol{r}_{\mathrm{p}_{4,1}|\mathrm{p}_{1,2}} = -\gamma_{4,\mathcal{B}_1} \boldsymbol{r}_{\mathrm{p}_{1,3}|\mathrm{p}_{1,2}}$.

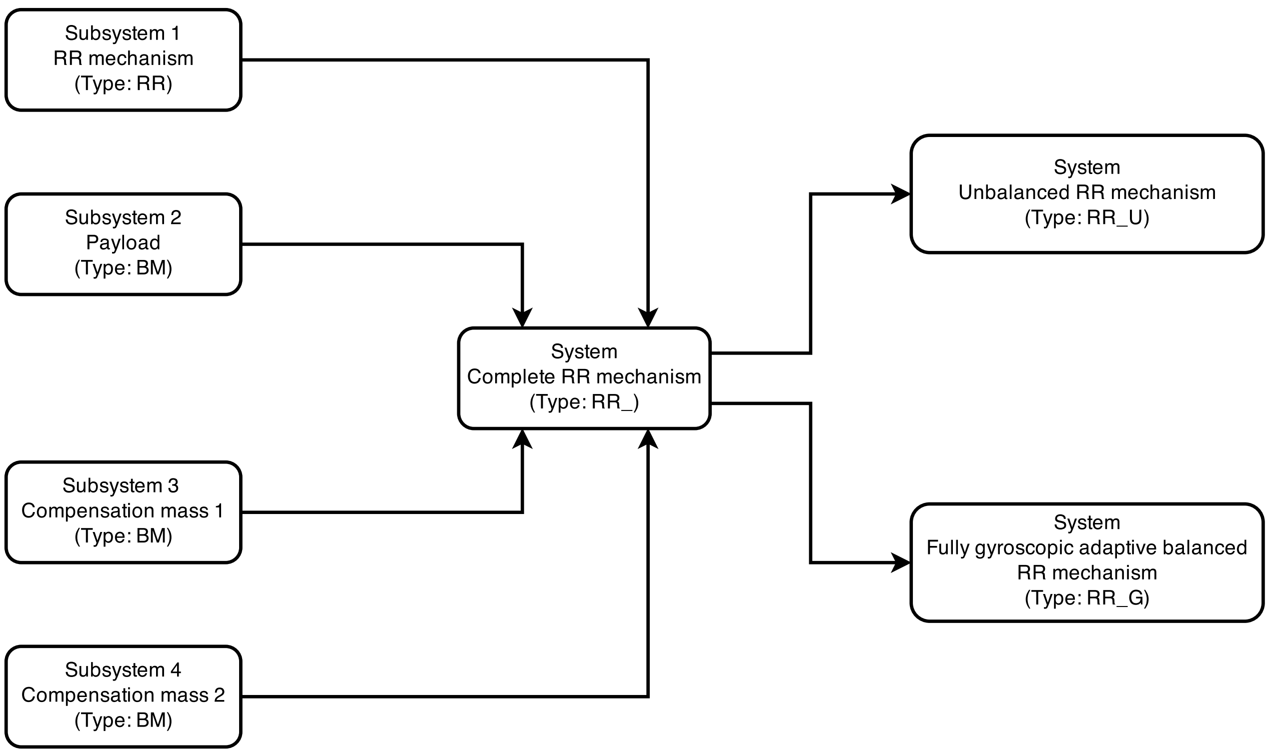

Figure 4: Dynamic modelling of the complete RR mechanism

The generalized coordinates and quasi-velocities of the model of system RR_ are the following:

$$\begin{cases} \mathbb{q}_{\mathrm{RR\_}} = \begin{bmatrix} \mathbb{q}_1^\mathsf{T} & \mathbb{q}_2^\mathsf{T} & \mathbb{q}_3^\mathsf{T} & \mathbb{q}_4^\mathsf{T} \end{bmatrix}^\mathsf{T} \\ \mathbb{q}_1 = \begin{bmatrix} q_{1,\mathcal{R}_1} & q_{1,\mathcal{R}_2} & q_{1,\mathrm{p}_1,1} & q_{1,\mathrm{p}_1,2} & q_{1,\mathrm{p}_2,1} & q_{1,\mathrm{p}_2,2} & q_{1,\mathrm{p}_3,1} & q_{1,\mathrm{p}_3,2} \end{bmatrix}^\mathsf{T} \\ \mathbb{q}_2 = \begin{bmatrix} q_{2,\mathrm{p}_1,1} & q_{2,\mathrm{p}_1,2} \end{bmatrix}^\mathsf{T} \\ \mathbb{q}_3 = \begin{bmatrix} q_{3,\mathrm{p}_1,1} & q_{3,\mathrm{p}_1,2} \end{bmatrix}^\mathsf{T} \\ \mathbb{q}_4 = \begin{bmatrix} q_{4,\mathrm{p}_1,1} & q_{4,\mathrm{p}_1,2} \end{bmatrix}^\mathsf{T} \end{cases} \tag{76}$$

$$\begin{cases} \mathbb{p}_{\mathrm{RR\_}} = \begin{bmatrix} \mathbb{p}_1^\mathsf{T} & \mathbb{p}_2^\mathsf{T} & \mathbb{p}_3^\mathsf{T} & \mathbb{p}_4^\mathsf{T} \end{bmatrix}^\mathsf{T} \\ \mathbb{p}_1 = \begin{bmatrix} p_{1,\mathcal{R}_1} & p_{1,\mathcal{R}_2} & p_{1,\mathcal{B}_1,1} & p_{1,\mathcal{B}_1,2} & p_{1,\mathcal{B}_1,3} & p_{1,\mathcal{B}_2,1} & p_{1,\mathcal{B}_2,2} & p_{1,\mathcal{B}_2,3} \end{bmatrix}^\mathsf{T} \\ \mathbb{p}_2 = \begin{bmatrix} p_{2,\mathcal{B}_1,1} & p_{2,\mathcal{B}_1,2} \end{bmatrix}^\mathsf{T} \\ \mathbb{p}_3 = \begin{bmatrix} p_{3,\mathcal{B}_1,1} & p_{3,\mathcal{B}_1,2} \end{bmatrix}^\mathsf{T} \\ \mathbb{p}_4 = \begin{bmatrix} p_{4,\mathcal{B}_1,1} & p_{4,\mathcal{B}_1,2} \end{bmatrix}^\mathsf{T} \end{cases} \tag{77}$$

Thus, the holonomic constraint equations describing how the payload and balancing masses are constrained to the RR mechanism can be expressed in the following form:

$$\begin{cases} \widehat{\psi}_{\mathrm{RR\_},1} = \left(\gamma_{1,\mathcal{B}_2} - 1\right) p_{1,\mathcal{B}_1,1} + \gamma_{1,\mathcal{B}_1} \left(p_{1,\mathcal{B}_2,1} - \gamma_{1,\mathcal{B}_2}\, p_{2,\mathcal{B}_1,1}\right) = 0 \\ \widehat{\psi}_{\mathrm{RR\_},2} = \left(\gamma_{1,\mathcal{B}_2} - 1\right) p_{1,\mathcal{B}_1,2} + \gamma_{1,\mathcal{B}_1} \left(p_{1,\mathcal{B}_2,2} - \gamma_{1,\mathcal{B}_2}\, p_{2,\mathcal{B}_1,2}\right) = 0 \\ \widehat{\psi}_{\mathrm{RR\_},3} = \gamma_{3,\mathcal{B}_1}\, p_{1,\mathcal{B}_1,1} + \gamma_{1,\mathcal{B}_1}\, p_{3,\mathcal{B}_1,1} = 0 \\ \widehat{\psi}_{\mathrm{RR\_},4} = \gamma_{3,\mathcal{B}_1}\, p_{1,\mathcal{B}_1,2} + \gamma_{1,\mathcal{B}_1}\, p_{3,\mathcal{B}_1,2} = 0 \\ \widehat{\psi}_{\mathrm{RR\_},5} = \gamma_{4,\mathcal{B}_1} \left(\gamma_{1,\mathcal{B}_1}\, p_{1,\mathcal{B}_2,1} - p_{1,\mathcal{B}_1,1}\right) + \gamma_{1,\mathcal{B}_2} \left(\gamma_{1,\mathcal{B}_1}\, p_{4,\mathcal{B}_1,1} - p_{1,\mathcal{B}_1,1}\right) = 0 \\ \widehat{\psi}_{\mathrm{RR\_},6} = \gamma_{4,\mathcal{B}_1} \left(\gamma_{1,\mathcal{B}_1}\, p_{1,\mathcal{B}_2,2} - p_{1,\mathcal{B}_1,2}\right) + \gamma_{1,\mathcal{B}_2} \left(\gamma_{1,\mathcal{B}_1}\, p_{4,\mathcal{B}_1,2} - p_{1,\mathcal{B}_1,2}\right) = 0 \end{cases} \tag{78}$$

The first time derivative of these equations corresponds to the external kinematic constraint equations of system RR_ , $\widehat{\mathbb{c}}_{\mathrm{RR\_}} = \mathbb{0}$.

Following the procedure described in Section 2.2, a matrix $\widehat{\mathbb{C}}_{\mathrm{RR\_}}$ satisfying the condition $\widehat{\mathbb{A}}_{\mathrm{RR\_}}\, \widehat{\mathbb{C}}_{\mathrm{RR\_}} = \mathbb{0}$ is given by:

$$\widehat{\mathbb{C}}_{\mathrm{RR\_}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ q_{1,\mathrm{p}_1,2} - q_{1,\mathrm{p}_3,2} & q_{1,\mathrm{p}_2,2} - q_{1,\mathrm{p}_3,2} \\ q_{1,\mathrm{p}_3,1} - q_{1,\mathrm{p}_1,1} & q_{1,\mathrm{p}_3,1} - q_{1,\mathrm{p}_2,1} \\ 1 & 1 \\ \gamma_{3,\mathcal{B}_1} \left(q_{1,\mathrm{p}_2,2} - q_{1,\mathrm{p}_1,2}\right) & 0 \\ \gamma_{3,\mathcal{B}_1} \left(q_{1,\mathrm{p}_1,1} - q_{1,\mathrm{p}_2,1}\right) & 0 \\ q_{1,\mathrm{p}_1,2} - \left(\gamma_{4,\mathcal{B}_1} + 1\right) q_{1,\mathrm{p}_2,2} + \gamma_{4,\mathcal{B}_1}\, q_{1,\mathrm{p}_3,2} & \gamma_{4,\mathcal{B}_1} \left(q_{1,\mathrm{p}_3,2} - q_{1,\mathrm{p}_2,2}\right) \\ q_{1,\mathrm{p}_1,1} - \left(\gamma_{4,\mathcal{B}_1} + 1\right) q_{1,\mathrm{p}_2,1} + \gamma_{4,\mathcal{B}_1}\, q_{1,\mathrm{p}_3,1} & \gamma_{4,\mathcal{B}_1} \left(q_{1,\mathrm{p}_3,1} - q_{1,\mathrm{p}_2,1}\right) \end{bmatrix} \tag{79}$$

Therefore, the dynamical model of system RR_ corresponds to the equation $\mathbb{d}_{\mathrm{RR\_}} = \widehat{\mathbb{C}}_{\mathrm{RR\_}}^\mathsf{T}\, \widehat{\mathbb{d}}_{\mathrm{RR\_}} = \mathbb{0}$.

It is remarkable, however, that such a mathematical model has a high number of variables among generalized coordinates and quasi-velocities. Many of them are useful only to simplify the modelling process, being unnecessary to keep then in the model once the dynamical equations are obtained. In order to eliminate some of the variables of the model, consider the following identities obtained from mathematical manipulation of equations (66, 78):

$$\begin{cases} p_{1,\mathcal{B}_1,1} = \gamma_{1,\mathcal{B}_1}\, p_{1,\mathcal{R}_1} \left(q_{1,\mathrm{p}_1,2} - q_{1,\mathrm{p}_2,2}\right) \\ p_{1,\mathcal{B}_1,2} = \gamma_{1,\mathcal{B}_1}\, p_{1,\mathcal{R}_1} \left(q_{1,\mathrm{p}_2,1} - q_{1,\mathrm{p}_1,1}\right) \\ p_{1,\mathcal{B}_1,3} = p_{1,\mathcal{R}_1} \\ p_{1,\mathcal{B}_2,1} = q_{1,\mathrm{p}_1,2}\, p_{1,\mathcal{R}_1} + q_{1,\mathrm{p}_2,2} \left(\left(\gamma_{1,\mathcal{B}_2} - 1\right) p_{1,\mathcal{R}_1} + \gamma_{1,\mathcal{B}_2}\, p_{1,\mathcal{R}_2}\right) - \gamma_{1,\mathcal{B}_2} \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) q_{1,\mathrm{p}_3,2} \\ p_{1,\mathcal{B}_2,2} = -q_{1,\mathrm{p}_1,1}\, p_{1,\mathcal{R}_1} - q_{1,\mathrm{p}_2,1} \left(\left(\gamma_{1,\mathcal{B}_2} - 1\right) p_{1,\mathcal{R}_1} + \gamma_{1,\mathcal{B}_2}\, p_{1,\mathcal{R}_2}\right) + \gamma_{1,\mathcal{B}_2} \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) q_{1,\mathrm{p}_3,1} \\ p_{1,\mathcal{B}_2,3} = p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2} \\ p_{2,\mathcal{B}_1,1} = q_{1,\mathrm{p}_1,2}\, p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\, q_{1,\mathrm{p}_2,2} - \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) q_{1,\mathrm{p}_3,2} \\ p_{2,\mathcal{B}_1,2} = -q_{1,\mathrm{p}_1,1}\, p_{1,\mathcal{R}_1} - p_{1,\mathcal{R}_2}\, q_{1,\mathrm{p}_2,1} + \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) q_{1,\mathrm{p}_3,1} \\ p_{3,\mathcal{B}_1,1} = \gamma_{3,\mathcal{B}_1}\, p_{1,\mathcal{R}_1} \left(q_{1,\mathrm{p}_2,2} - q_{1,\mathrm{p}_1,2}\right) \\ p_{3,\mathcal{B}_1,2} = \gamma_{3,\mathcal{B}_1}\, p_{1,\mathcal{R}_1} \left(q_{1,\mathrm{p}_1,1} - q_{1,\mathrm{p}_2,1}\right) \\ p_{4,\mathcal{B}_1,1} = q_{1,\mathrm{p}_1,2}\, p_{1,\mathcal{R}_1} - q_{1,\mathrm{p}_2,2} \left(\left(\gamma_{4,\mathcal{B}_1} + 1\right) p_{1,\mathcal{R}_1} + \gamma_{4,\mathcal{B}_1}\, p_{1,\mathcal{R}_2}\right) + \gamma_{4,\mathcal{B}_1} \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) q_{1,\mathrm{p}_3,2} \\ p_{4,\mathcal{B}_1,2} = -q_{1,\mathrm{p}_1,1}\, p_{1,\mathcal{R}_1} + q_{1,\mathrm{p}_2,1} \left(\left(\gamma_{4,\mathcal{B}_1} + 1\right) p_{1,\mathcal{R}_1} + \gamma_{4,\mathcal{B}_1}\, p_{1,\mathcal{R}_2}\right) - \gamma_{4,\mathcal{B}_1} \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) q_{1,\mathrm{p}_3,1} \end{cases} \tag{80}$$

Consider also the following identities that can be easily obtained from the geometry of the RR mechanism:

$$\begin{cases} q_{1,\mathrm{p}_2,1} = q_{1,\mathrm{p}_1,1} + a_{1,1} \cos\left(q_{1,\mathcal{R}_1}\right) \\ q_{1,\mathrm{p}_2,2} = q_{1,\mathrm{p}_1,2} + a_{1,1} \sin\left(q_{1,\mathcal{R}_1}\right) \\ q_{1,\mathrm{p}_3,1} = q_{1,\mathrm{p}_1,1} + a_{1,1} \cos\left(q_{1,\mathcal{R}_1}\right) + a_{1,2} \cos\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right) \\ q_{1,\mathrm{p}_3,2} = q_{1,\mathrm{p}_1,2} + a_{1,1} \sin\left(q_{1,\mathcal{R}_1}\right) + a_{1,2} \sin\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right) \end{cases} \tag{81}$$

Making the following definitions:

$$\begin{aligned} &q_{\mathrm{RR\_},\mathcal{R}_1} = q_{1,\mathcal{R}_1} && p_{\mathrm{RR\_},\mathcal{R}_1} = p_{1,\mathcal{R}_1} \\ &q_{\mathrm{RR\_},\mathcal{R}_2} = q_{1,\mathcal{R}_2} && p_{\mathrm{RR\_},\mathcal{R}_2} = p_{1,\mathcal{R}_2} \\ &q_{\mathrm{RR\_},\mathrm{p}_1,1} = q_{1,\mathrm{p}_1,1} && u_{\mathrm{RR\_},1} = u_{1,1} \\ &q_{\mathrm{RR\_},\mathrm{p}_1,2} = q_{1,\mathrm{p}_1,2} && u_{\mathrm{RR\_},2} = u_{1,2} \\ &q_{\mathrm{RR\_},\mathrm{p}_3,1} = q_{1,\mathrm{p}_3,1} && \\ &q_{\mathrm{RR\_},\mathrm{p}_3,2} = q_{1,\mathrm{p}_3,2} && \end{aligned}$$

the new variables of the RR_ model are given by:

$$\mathbb{q}_{\mathrm{RR\_}} = \begin{bmatrix} q_{\mathrm{RR\_},\mathcal{R}_1} & q_{\mathrm{RR\_},\mathcal{R}_2} & q_{\mathrm{RR\_},\mathrm{p}_1,1} & q_{\mathrm{RR\_},\mathrm{p}_1,2} & q_{\mathrm{RR\_},\mathrm{p}_3,1} & q_{\mathrm{RR\_},\mathrm{p}_3,2} \end{bmatrix}^{\mathsf{T}} \tag{82}$$

$$\mathbb{p}_{\mathrm{RR\_}} = \begin{bmatrix} p_{\mathrm{RR\_},\mathcal{R}_1} & p_{\mathrm{RR\_},\mathcal{R}_2} \end{bmatrix}^{\mathsf{T}} \tag{83}$$

Define the following parameters:

$$\begin{cases} I_{\mathrm{RR\_U},1} = a_{1,2}^2 \left(m_{1,\mathcal{B}_2}\, \gamma_{1,\mathcal{B}_2}^2 + m_{2,\mathcal{B}_1} + m_{4,\mathcal{B}_1}\, \gamma_{4,\mathcal{B}_1}^2\right) + I_{1,\mathcal{B}_2} \\ I_{\mathrm{RR\_U},2} = a_{1,1}^2 \left(m_{1,\mathcal{B}_1}\, \gamma_{1,\mathcal{B}_1}^2 + m_{3,\mathcal{B}_1}\, \gamma_{3,\mathcal{B}_1}^2 + m_{1,\mathcal{B}_2} + m_{2,\mathcal{B}_1} + m_{4,\mathcal{B}_1}\right) + I_{1,\mathcal{B}_1} \\ I_{\mathrm{RR\_U},3} = a_{1,1}\, a_{1,2} \left(m_{1,\mathcal{B}_2}\, \gamma_{1,\mathcal{B}_2} - m_{4,\mathcal{B}_1}\, \gamma_{4,\mathcal{B}_1} + m_{2,\mathcal{B}_1}\right) \\ \mu_{\mathrm{RR\_U},1} = g\, a_{1,1} \left(m_{1,\mathcal{B}_1}\, \gamma_{1,\mathcal{B}_1} - m_{3,\mathcal{B}_1}\, \gamma_{3,\mathcal{B}_1} + m_{1,\mathcal{B}_2} + m_{2,\mathcal{B}_1} + m_{4,\mathcal{B}_1}\right) \\ \mu_{\mathrm{RR\_U},2} = g\, a_{1,2} \left(m_{1,\mathcal{B}_2}\, \gamma_{1,\mathcal{B}_2} - m_{4,\mathcal{B}_1}\, \gamma_{4,\mathcal{B}_1} + m_{2,\mathcal{B}_1}\right) \end{cases} \tag{84}$$

If the adjustable physical parameters $m_{3,\mathcal{B}_1}$, $\gamma_{3,\mathcal{B}_1}$, $m_{4,\mathcal{B}_1}$ and $\gamma_{4,\mathcal{B}_1}$ are chosen in such a way that at least one of the parameters among $I_{\mathrm{RR\_U},3}$, $\mu_{\mathrm{RR\_U},1}$ and $\mu_{\mathrm{RR\_U},2}$ are not null, model RR_ represents an unbalanced RR mechanism and will be denoted as RR_U. Its mathematical model is given by:

$$\mathbb{C}_{\text{RR\_U}} = \mathbb{1} \tag{85}$$

$$\mathbb{M}_{\text{RR\_U}} = \begin{bmatrix} I_{\text{RR\_U},1} + I_{\text{RR\_U},2} + 2\cos\left(q_{\text{RR\_U},\mathcal{R}_2}\right) I_{\text{RR\_U},3} & I_{\text{RR\_U},1} + \cos\left(q_{\text{RR\_U},\mathcal{R}_2}\right) I_{\text{RR\_U},3} \\ I_{\text{RR\_U},1} + \cos\left(q_{\text{RR\_U},\mathcal{R}_2}\right) I_{\text{RR\_U},3} & I_{\text{RR\_U},1} \end{bmatrix} \tag{86}$$

$$\mathbb{g}_{\text{RR\_U}} = \begin{bmatrix} I_{\text{RR\_U},3}\, p^2_{\text{RR\_U},\mathcal{R}_2} \sin\left(q_{\text{RR\_U},\mathcal{R}_2}\right) + 2\, I_{\text{RR\_U},3}\, p_{\text{RR\_U},\mathcal{R}_1}\, p_{\text{RR\_U},\mathcal{R}_2} \sin\left(q_{\text{RR\_U},\mathcal{R}_2}\right) \\ -I_{\text{RR\_U},3}\, p^2_{\text{RR\_U},\mathcal{R}_1} \sin\left(q_{\text{RR\_U},\mathcal{R}_2}\right) \end{bmatrix} \tag{87}$$

$$\mathbb{f}_{\text{RR\_U}} = \begin{bmatrix} u_{\text{RR\_U},1} - \left(\mu_{\text{RR\_U},1} \cos\left(q_{\text{RR\_U},\mathcal{R}_1}\right) + \mu_{\text{RR\_U},2} \cos\left(q_{\text{RR\_U},\mathcal{R}_1} + q_{\text{RR\_U},\mathcal{R}_2}\right)\right) \\ u_{\text{RR\_U},2} - \mu_{\text{RR\_U},2} \cos\left(q_{\text{RR\_U},\mathcal{R}_1} + q_{\text{RR\_U},\mathcal{R}_2}\right) \end{bmatrix} \tag{88}$$

On the other hand, if parameters $m_{3,\mathcal{B}_1}$, $\gamma_{3,\mathcal{B}_1}$, $m_{4,\mathcal{B}_1}$ and $\gamma_{4,\mathcal{B}_1}$ are chosen to satisfy the following relations:

$$\begin{cases} m_{1,\mathcal{B}_2}\, \gamma_{1,\mathcal{B}_2} - m_{4,\mathcal{B}_1}\, \gamma_{4,\mathcal{B}_1} + m_{2,\mathcal{B}_1} = 0 \\ m_{1,\mathcal{B}_1}\, \gamma_{1,\mathcal{B}_1} - m_{3,\mathcal{B}_1}\, \gamma_{3,\mathcal{B}_1} + m_{1,\mathcal{B}_2} + m_{2,\mathcal{B}_1} + m_{4,\mathcal{B}_1} = 0 \end{cases} \tag{89}$$

model RR_ represents a fully gyroscopic adaptive balanced RR mechanism and will be denoted as RR_G. Defining the following parameters:

$$\begin{cases} I_{\text{RR\_G},1} = a^2_{1,2}\left(m_{1,\mathcal{B}_2}\, \gamma^2_{1,\mathcal{B}_2} + m_{4,\mathcal{B}_1}\, \gamma^2_{4,\mathcal{B}_1} + m_{2,\mathcal{B}_1}\right) + I_{1,\mathcal{B}_2} \\ I_{\text{RR\_G},2} = a^2_{1,1}\left(m_{1,\mathcal{B}_1}\, \gamma^2_{1,\mathcal{B}_1} + m_{3,\mathcal{B}_1}\, \gamma^2_{3,\mathcal{B}_1} + m_{1,\mathcal{B}_2} + m_{2,\mathcal{B}_1} + m_{4,\mathcal{B}_1}\right) + I_{1,\mathcal{B}_1} \end{cases} \tag{90}$$

it can be stated that RR_G model is given by:

$$\mathbb{C}_{\text{RR\_G}} = \mathbb{1} \tag{91}$$

$$\mathbb{M}_{\text{RR\_G}} = \begin{bmatrix} I_{\text{RR\_G},1} + I_{\text{RR\_G},2} & I_{\text{RR\_G},1} \\ I_{\text{RR\_G},1} & I_{\text{RR\_G},1} \end{bmatrix} \tag{92}$$

$$\mathbb{g}_{\text{RR\_G}} = \mathbb{0} \tag{93}$$

$$\mathbb{f}_{\text{RR\_G}} = \begin{bmatrix} u_{\text{RR\_G},1} \\ u_{\text{RR\_G},2} \end{bmatrix} \tag{94}$$

4.3 Adaptive balancing of a 5R mechanism

Consider a mechanical system denoteb by 5R_ , that represents a 5-bar mechanism with revolute joints, transporting a payload (punctual mass) and with balancing masses attached to each of its movable links as shown in Figure 5.

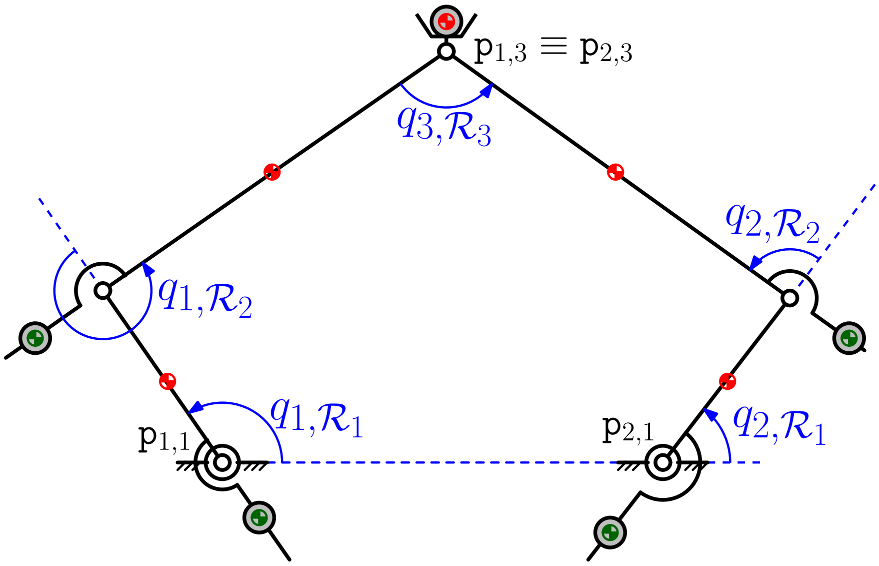

Figure 5: 5R mechanism

The strategy to model this mechanical system, synthesized in the diagram of Figure 6, is to consider that it can be conceived as two RR_ subsystems, denoted by the indexes 1 and 2, constrained by a revolute joint in their end effectors. These RR_ subsystems, however, do not have actuators in their second joints. However, this fact does not represent a limitation on the strategy, once these active joints can be modelled as passive by imposing an identically null torque input. Basically, it will be considered that $u_{1,2} = 0$ and $u_{2,2} = 0$. Also, consider the notation $u_{5R\_,1} = u_{1,1}$ and $u_{5R\_,2} = u_{2,1}$, such that $\mathbb{u}_{5R\_} = [\ u_{5R\_,1} \quad u_{5R\_,2}\ ]^{\mathsf{T}}$.

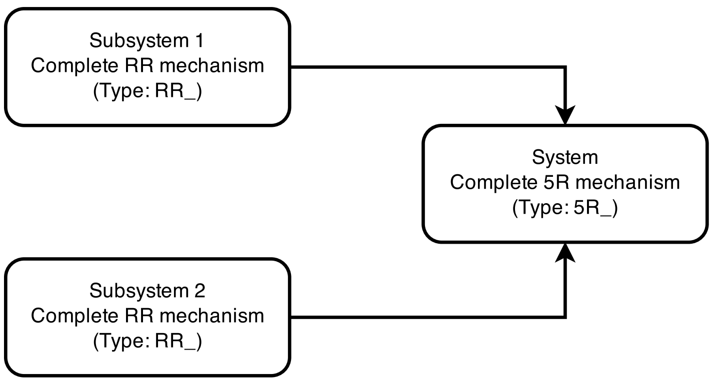

Figure 6: Dynamic modelling of the complete 5R mechanism

In addition to the generalized coordinates from each RR_ subsystem, define a third set of coordinates, represented by a column-matrix $\mathbb{q}_3$, with a single angular coordinate $q_{3,\mathcal{R}_3}$ that represents the angle between the longitudinal axes of bodies $\mathcal{B}_{1,2}$ and $\mathcal{B}_{2,2}$ (see Figure 5). Thus, the generalized coordinates of the model 5R_ are given by:

$$\begin{cases} \mathbb{q}_{5R\_} = [\ \mathbb{q}_1^{\mathsf{T}} \quad \mathbb{q}_2^{\mathsf{T}} \quad \mathbb{q}_3^{\mathsf{T}}\ ]^{\mathsf{T}} \\ \mathbb{q}_1 = [\ q_{1,\mathcal{R}_1} \quad q_{1,\mathcal{R}_2} \quad q_{1,\mathrm{p}_1,1} \quad q_{1,\mathrm{p}_1,2} \quad q_{1,\mathrm{p}_3,1} \quad q_{1,\mathrm{p}_3,2}\ ]^{\mathsf{T}} \\ \mathbb{q}_2 = [\ q_{2,\mathcal{R}_1} \quad q_{2,\mathcal{R}_2} \quad q_{2,\mathrm{p}_1,1} \quad q_{2,\mathrm{p}_1,2} \quad q_{2,\mathrm{p}_3,1} \quad q_{2,\mathrm{p}_3,2}\ ]^{\mathsf{T}} \\ \mathbb{q}_3 = [\ q_{3,\mathcal{R}_3}\ ]^{\mathsf{T}} \end{cases} \tag{95}$$

The (internal) holonomic constraints of subsystems 1 and 2 and the external holonomic constraints between them can be described by the following invariants:

$$\begin{cases} h_{1,1} = q_{1,\mathrm{p}_3,1} - \cos\left(q_{1,\mathcal{R}_1}\right) a_{1,1} - \cos\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right) a_{1,2} - q_{1,\mathrm{p}_1,1} = 0 \\ h_{1,2} = q_{1,\mathrm{p}_3,2} - \sin\left(q_{1,\mathcal{R}_1}\right) a_{1,1} - \sin\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right) a_{1,2} - q_{1,\mathrm{p}_1,2} = 0 \\ h_{2,1} = q_{2,\mathrm{p}_3,1} - \cos\left(q_{2,\mathcal{R}_1}\right) a_{2,1} - \cos\left(q_{2,\mathcal{R}_1} + q_{2,\mathcal{R}_2}\right) a_{2,2} - q_{2,\mathrm{p}_1,1} = 0 \\ h_{2,2} = q_{2,\mathrm{p}_3,2} - \sin\left(q_{2,\mathcal{R}_1}\right) a_{2,1} - \sin\left(q_{2,\mathcal{R}_1} + q_{2,\mathcal{R}_2}\right) a_{2,2} - q_{2,\mathrm{p}_1,2} = 0 \\ \widehat{h}_{5\mathrm{R}\_,1} = q_{1,\mathrm{p}_3,1} - q_{2,\mathrm{p}_3,1} = 0 \\ \widehat{h}_{5\mathrm{R}\_,2} = q_{1,\mathrm{p}_3,2} - q_{2,\mathrm{p}_3,2} = 0 \\ \widehat{h}_{5\mathrm{R}\_,3} = q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2} - q_{2,\mathcal{R}_1} - q_{2,\mathcal{R}_2} + q_{3,\mathcal{R}_3} - 2\,\pi = 0 \end{cases} \tag{96}$$

Also, the quasi-velocities of the model 5R_ are given by:

$$\begin{cases} \mathbb{p}_{5\mathrm{R}\_} = \left[\begin{array}{cc} \mathbb{p}_1^{\mathsf{T}} & \mathbb{p}_2^{\mathsf{T}} \end{array}\right]^{\mathsf{T}} \\ \mathbb{p}_1 = \left[\begin{array}{cc} p_{1,\mathcal{R}_1} & p_{1,\mathcal{R}_2} \end{array}\right]^{\mathsf{T}} \\ \mathbb{p}_2 = \left[\begin{array}{cc} p_{2,\mathcal{R}_1} & p_{2,\mathcal{R}_2} \end{array}\right]^{\mathsf{T}} \end{cases} \tag{97}$$

Due to the constraints between subsystems 1 and 2 the following conditions must be satisfied:

$$\begin{cases} \widehat{\psi}_{5\mathrm{R}\_,1} = \sin\left(q_{1,\mathcal{R}_1}\right) a_{1,1}\, p_{1,\mathcal{R}_1} + \sin\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right) a_{1,2} \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) \\ \qquad\qquad - \sin\left(q_{2,\mathcal{R}_1}\right) a_{2,1}\, p_{2,\mathcal{R}_1} - \sin\left(q_{2,\mathcal{R}_1} + q_{2,\mathcal{R}_2}\right) a_{2,2} \left(p_{2,\mathcal{R}_1} + p_{2,\mathcal{R}_2}\right) \\ \widehat{\psi}_{5\mathrm{R}\_,2} = \cos\left(q_{1,\mathcal{R}_1}\right) a_{1,1}\, p_{1,\mathcal{R}_1} + \cos\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right) a_{1,2} \left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right) \\ \qquad\qquad - \cos\left(q_{2,\mathcal{R}_1}\right) a_{2,1}\, p_{2,\mathcal{R}_1} - \cos\left(q_{2,\mathcal{R}_1} + q_{2,\mathcal{R}_2}\right) a_{2,2} \left(p_{2,\mathcal{R}_1} + p_{2,\mathcal{R}_2}\right) \end{cases} \tag{98}$$

Again following the procedure described in Section 2.2, a matrix $\widehat{\mathbb{C}}_{5\mathrm{R}\_}$ satisfying the condition $\widehat{\mathbb{A}}_{5\mathrm{R}\_}\,\widehat{\mathbb{C}}_{5\mathrm{R}\_} = \mathbb{0}$ is given by:

$$\widehat{\mathbb{C}}_{5\mathrm{R}\_} = \left[\begin{array}{cc} 1 & 0 \\ -\dfrac{\sin\left(q_{1,\mathcal{R}_2} + q_{3,\mathcal{R}_3}\right) a_{1,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{1,2}} - 1 & \dfrac{\sin\left(q_{2,\mathcal{R}_2}\right) a_{2,1}}{\sin\left(q_{3,\mathcal{R}_3}\right) a_{1,2}} \\ 0 & 1 \\ -\dfrac{\sin\left(q_{1,\mathcal{R}_2}\right) a_{1,1}}{\sin\left(q_{3,\mathcal{R}_3}\right) a_{2,2}} & \dfrac{\sin\left(q_{2,\mathcal{R}_2} - q_{3,\mathcal{R}_3}\right) a_{2,1}}{\sin\left(q_{3,\mathcal{R}_3}\right) a_{2,2}} - 1 \end{array}\right] \tag{99}$$

When subsystems 1 and 2 are fully gyroscopic adaptive balanced RR mechanisms (model RR_G), the 5R_ model obtained corresponds to a gyroscopic adaptive balanced 5R mechanism, and will be denoted by 5R_G. The dynamical equations of model 5R_G can be written in the following matrix form:

$$\mathbb{u}_{5\mathrm{R\_G}} = \mathbb{M}'_{5\mathrm{R\_G}}\,\dot{\mathbb{p}}_{5\mathrm{R\_G}} \tag{100}$$

with matrix $\mathbb{M}'_{5\mathrm{R\_G}}$ given by:

$$\mathbb{M}'_{5\mathrm{R\_G}} = \left[\begin{array}{cc} I_{1,2} - \dfrac{\sin\left(q_{1,\mathcal{R}_2} + q_{3,\mathcal{R}_3}\right) a_{1,1}\, I_{1,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{1,2}} & \dfrac{\sin\left(q_{2,\mathcal{R}_2}\right) a_{2,1}\, I_{1,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{1,2}} \\ -\dfrac{\sin\left(q_{1,\mathcal{R}_2} + q_{3,\mathcal{R}_3}\right) a_{1,1}\, I_{1,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{1,2}} & \dfrac{\sin\left(q_{2,\mathcal{R}_2}\right) a_{2,1}\, I_{1,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{1,2}} \\ -\dfrac{\sin\left(q_{1,\mathcal{R}_2}\right) a_{1,1}\, I_{2,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{2,2}} & \dfrac{\sin\left(q_{2,\mathcal{R}_2} - q_{3,\mathcal{R}_3}\right) a_{2,1}\, I_{2,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{2,2}} + I_{2,2} \\ -\dfrac{\sin\left(q_{1,\mathcal{R}_2}\right) a_{1,1}\, I_{2,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{2,2}} & \dfrac{\sin\left(q_{2,\mathcal{R}_2} - q_{3,\mathcal{R}_3}\right) a_{2,1}\, I_{2,1}}{\sin\left(q_{3,\mathcal{R}_3}\right)\, a_{2,2}} \end{array}\right]^{\mathsf{T}} \tag{101}$$

On the other hand, when subsystems 1 and 2 are unbalanced RR mechanisms (model RR_U), so will be the corresponding 5R mechanism, whose model will be denoted by 5R_U. In this case, the dynamical equations can be written in the following form:

$$
\begin{aligned}
u_{5\mathrm{R\_U},1} = & \left(\frac{a_{1,2}\left(I_{1,2} + \cos\left(q_{1,\mathcal{R}_2}\right) I_{1,3}\right) - \csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{1,\mathcal{R}_2} + q_{3,\mathcal{R}_3}\right) a_{1,1}\left(I_{1,1} + \cos\left(q_{1,\mathcal{R}_2}\right) I_{1,3}\right)}{a_{1,2}} \right) \dot{p}_{1,\mathcal{R}_1} \\
& + \left(\cos\left(q_{1,\mathcal{R}_2}\right) I_{1,3} - \frac{\csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{1,\mathcal{R}_2} + q_{3,\mathcal{R}_3}\right) a_{1,1} I_{1,1}}{a_{1,2}} \right) \dot{p}_{1,\mathcal{R}_2} \\
& + \left(- \frac{\csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{1,\mathcal{R}_2}\right) a_{1,1}\left(I_{2,1} + \cos\left(q_{2,\mathcal{R}_2}\right) I_{2,3}\right)}{a_{2,2}} \right) \dot{p}_{2,\mathcal{R}_1} \\
& + \left(- \frac{\csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{1,\mathcal{R}_2}\right) a_{1,1} I_{2,1}}{a_{2,2}} \right) \dot{p}_{2,\mathcal{R}_2} \\
& + \frac{1}{a_{1,2}\, a_{2,2}} \Big(a_{1,2}\big(a_{2,2}\big(\mu_{1,1} \cos\left(q_{1,\mathcal{R}_1}\right) - I_{1,3}\left(p_{1,\mathcal{R}_1} + p_{1,\mathcal{R}_2}\right)^2 \sin\left(q_{1,\mathcal{R}_2}\right)\big) \\
& - a_{1,1} \sin\left(q_{1,\mathcal{R}_2}\right) \csc\left(q_{3,\mathcal{R}_3}\right) \left(I_{2,3} p_{2,\mathcal{R}_1}^2 \sin\left(q_{2,\mathcal{R}_2}\right) + \mu_{2,2} \cos\left(q_{2,\mathcal{R}_1} + q_{2,\mathcal{R}_2}\right)\right)\big) \\
& - a_{1,1} a_{2,2} \sin\left(q_{3,\mathcal{R}_3} + q_{1,\mathcal{R}_2}\right) \csc\left(q_{3,\mathcal{R}_3}\right) \big(I_{1,3} p_{1,\mathcal{R}_1}^2 \sin\left(q_{1,\mathcal{R}_2}\right) \\
& + \mu_{1,2} \cos\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right)\big)\Big)
\end{aligned} \tag{102}
$$

$$
\begin{aligned}
u_{5\mathrm{R\_U},2} = & \left(\frac{\csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{2,\mathcal{R}_2}\right) a_{2,1}\left(I_{1,1} + \cos\left(q_{1,\mathcal{R}_2}\right) I_{1,3}\right)}{a_{1,2}} \right) \dot{p}_{1,\mathcal{R}_1} \\
& + \left(\frac{\csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{2,\mathcal{R}_2}\right) a_{2,1} I_{1,1}}{a_{1,2}} \right) \dot{p}_{1,\mathcal{R}_2} \\
& + \left(\frac{\csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{2,\mathcal{R}_2} - q_{3,\mathcal{R}_3}\right) a_{2,1}\left(I_{2,1} + \cos\left(q_{2,\mathcal{R}_2}\right) I_{2,3}\right) + a_{2,2}\left(I_{2,2} + \cos\left(q_{2,\mathcal{R}_2}\right) I_{2,3}\right)}{a_{2,2}} \right) \dot{p}_{2,\mathcal{R}_1} \\
& + \left(\frac{\csc\left(q_{3,\mathcal{R}_3}\right) \sin\left(q_{2,\mathcal{R}_2} - q_{3,\mathcal{R}_3}\right) a_{2,1} I_{2,1}}{a_{2,2}} + \cos\left(q_{2,\mathcal{R}_2}\right) I_{2,3} \right) \dot{p}_{2,\mathcal{R}_2} \\
& + \frac{1}{a_{1,2} a_{2,2}} \Big(a_{2,1} \csc\left(q_{3,\mathcal{R}_3}\right) \big(a_{1,2} \sin\left(q_{2,\mathcal{R}_2} - q_{3,\mathcal{R}_3}\right) \big(I_{2,3} p_{2,\mathcal{R}_1}^2 \sin\left(q_{2,\mathcal{R}_2}\right) \\
& + \mu_{2,2} \cos\left(q_{2,\mathcal{R}_1} + q_{2,\mathcal{R}_2}\right)\big) + a_{2,2} \sin\left(q_{2,\mathcal{R}_2}\right) \big(I_{1,3} p_{1,\mathcal{R}_1}^2 \sin\left(q_{1,\mathcal{R}_2}\right) \\
& + a_{1,2} a_{2,2} \big(\mu_{2,1} \cos\left(q_{2,\mathcal{R}_1}\right) - I_{2,3}\left(p_{2,\mathcal{R}_1} + p_{2,\mathcal{R}_2}\right)^2 \sin\left(q_{2,\mathcal{R}_2}\right)\big) \\
& + \mu_{1,2} \cos\left(q_{1,\mathcal{R}_1} + q_{1,\mathcal{R}_2}\right)\big)\big)\Big)
\end{aligned} \tag{103}
$$

Comparing the equations of models 5R_G and 5R_U it is remarkable the algebraic simplicity of the first one for calculating the control torques based on the current generalized coordinates, quasi-velocities and quasi-accelerations. Indeed, the following subsections will present a discussion, based on simulations, on the advantages of controling a giroscopic adaptive balanced 5R mechanism (over its unbalanced counterpart).