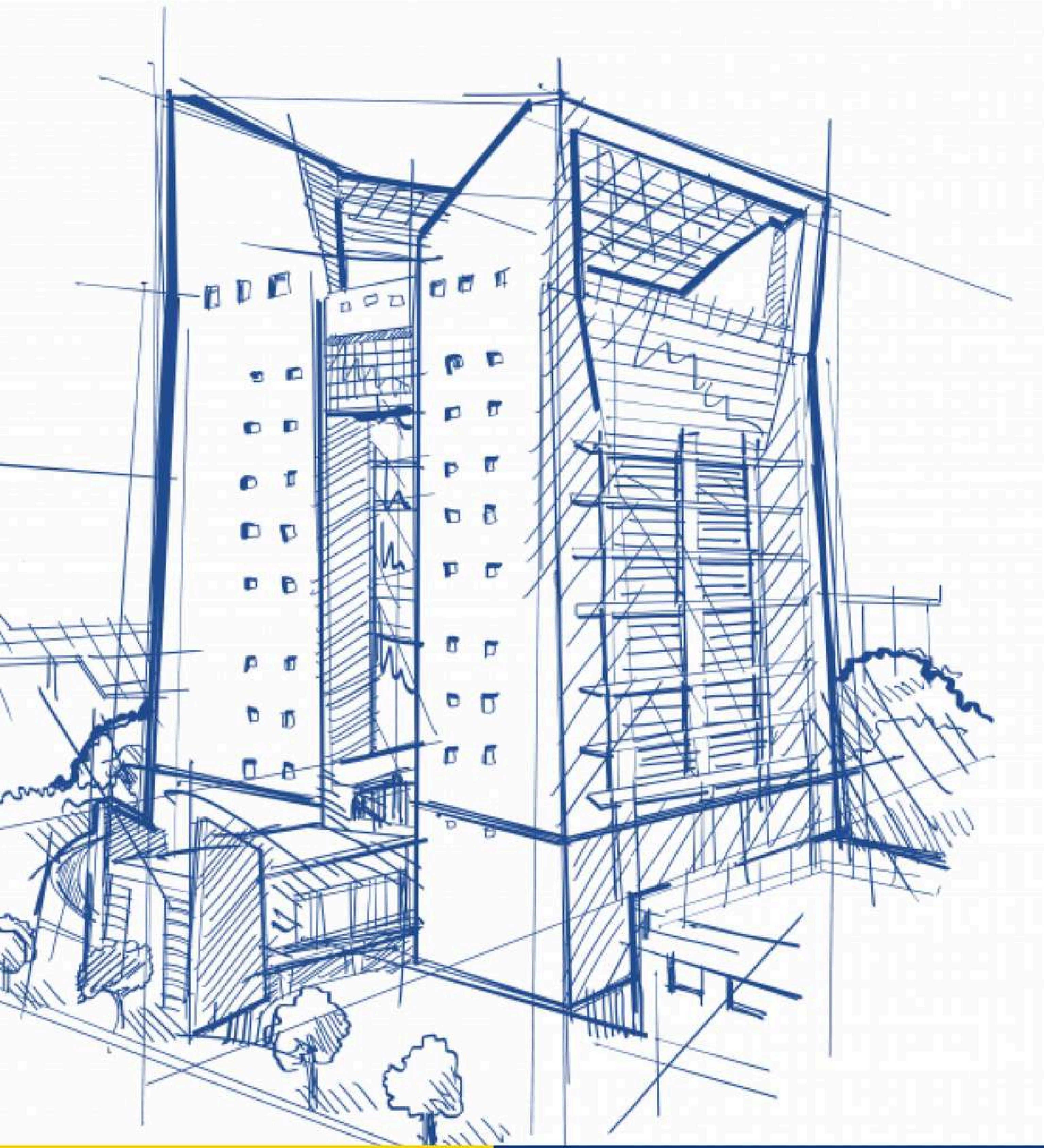
ANALYSIS OF ATM MACHINE QUEUE PROBLEM



Group 3 | STOCHASTIC PROCESS - K

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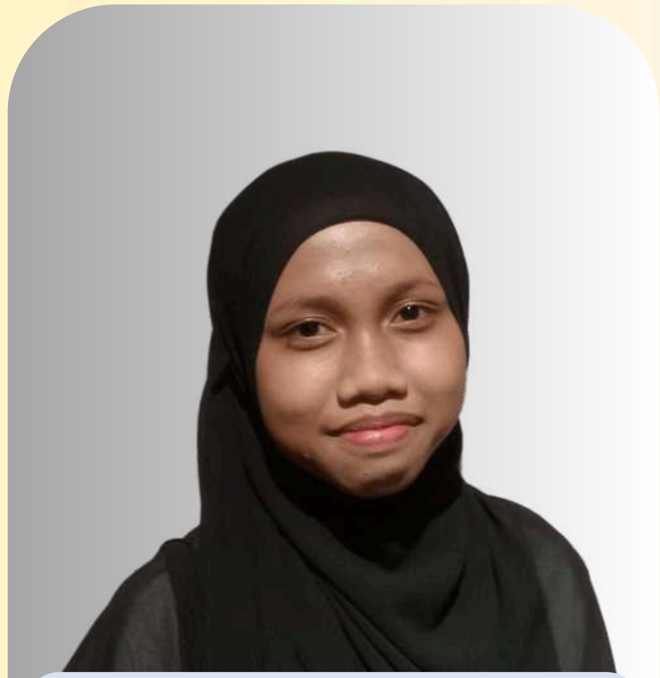

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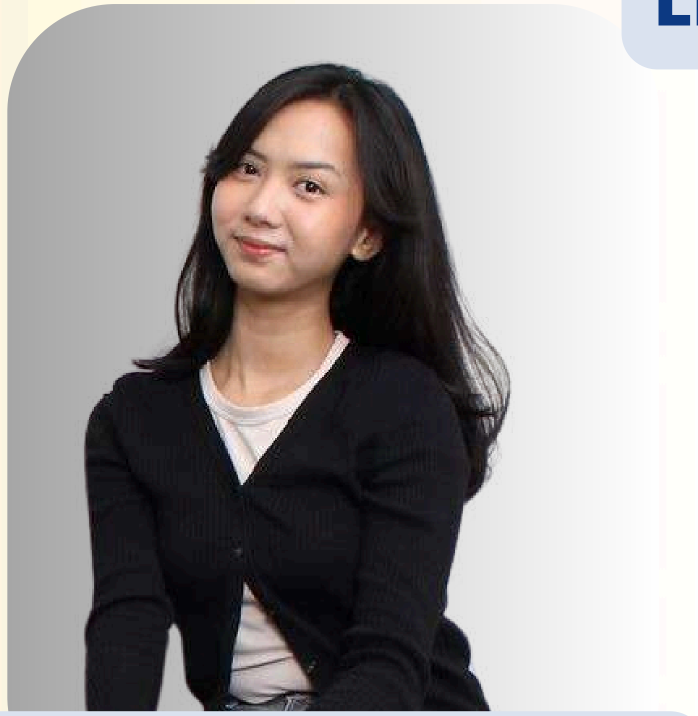

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CHAPTER I: INTRODUCTION

BACKGROUND

Automatic Teller Machines (ATMs) play a critical role in providing 24/7 access to financial transactions. However, during peak usage hours, customer queuing and potential overcrowding can affect user satisfaction and system efficiency. This case study examines the dynamics of an ATM system where customers arrive randomly over time, with the arrival process following a Poisson distribution at a rate λ and some constraints as follows:

- The space in front of the ATM can accommodate at most 10 customers. Thus, if there are 10 customers already waiting and a new customer arrives, the customer walks away and is lost forever.
- The customers form a single line and use the ATM in a first-come, first-served fashion.
- The processing times at the ATM for the customers are independent and identically distributed (iid) exponential random variables with rate . Let X(t) be the number of customers at the ATM at time t.

This study uses historical ATM usage data from 2024, sourced from its.id/prosto2024, to analyze and model the queueing behavior. The goal is to answer critical questions regarding system dynamics, including transition rates, queue probabilities, idle times, and long-term behavior, as well as evaluating the system's economic performance. By understanding the dynamics of $X(t)$ as a CTMC, we aim to address operational challenges, such as minimizing customer loss and ensuring efficient ATM utilization, while also exploring profitability through various transaction fee scenarios.

PROBLEM STATEMENTS

1. How can we define the states of the Markov chain representing the ATM queue and the transition rates between these states? How can the rate diagram be drawn?
2. How can the rate matrix and generator matrix for the CTMC describing the ATM queue system be derived?
3. How can the distribution of the inter-arrival times and service times of the ATM machine be visualized using historical data?
4. How can the average inter-arrival times and service times for the ATM system be calculated?
5. How can the parameters λ (arrival rate) and μ (service rate) of the ATM queue system be estimated using the Maximum Likelihood Estimation (MLE) method?
6. Estimate the parameters λ and μ using Maximum Likelihood Estimation method. If the ATM queue is empty at 5 AM, what is the probability of having k people in the queue at 7 AM for k = 0, 1, 2, ..., 10, and what is the expected queue size?
7. If the ATM machine is idle at 8:00 AM, what is the expected amount of time the machine remains idle during the next hour?
8. What is the limiting distribution of the state of the ATM queue system?
9. Given the initial investment cost is I = 15,000 dollars, and the annual maintenance cost is M = 1,500 dollars with a total annual operating cost includes:

- Electricity cost per year: Ce = 1,200 dollars per year, and
- Transaction processing cost: Ct = 0.25 dollars per transaction.

What is the annual profit and ROI if the transaction fee is set at:

- Cr = 0.5 dollars
- Cr = 1 dollars
- Cr = 2 dollars

for each transaction?

OBJECTIVES

1. Define the states of the Markov chain and the transition rates between them, and draw the rate diagram for the ATM queue system.
2. Derive the rate matrix and generator matrix for the CTMC representing the ATM queue system.
3. Visualize the distribution of the inter-arrival times and service times using historical data.
4. Calculate the average inter-arrival times and service times for the ATM system.
5. Estimate the parameters λ (arrival rate) and μ (service rate) using the Maximum Likelihood Estimation (MLE) method.
6. Calculate the probability that there are k people in the ATM queue at 7 AM, starting from an empty queue at 5 AM, for k = 0, 1, 2, ..., 10, and compute its expected value.
7. Estimate the expected idle time of the ATM machine during the next hour, given it is idle at 8:00 AM.
8. Compute the limiting distribution of the states of the ATM queue system.
9. Calculate the annual profit and ROI for the ATM system, given the initial investment cost is I = 15,000 dollars, and the annual maintenance cost is M = 1,500 dollars with a total annual operating cost includes:

- Electricity cost per year: Ce = 1,200 dollars per year, and
- Transaction processing cost: Ct = 0.25 dollars per transation.

Under three different transaction fee scenarios:

- Cr = 0.5 dollars,
- Cr = 1 dollars, and
- Cr = 2 dollars.

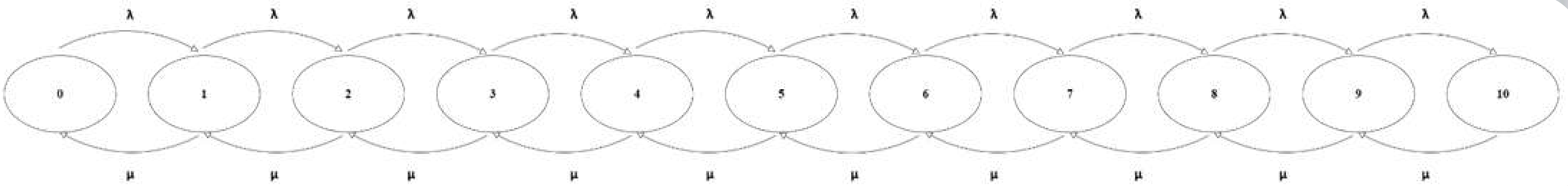
CHAPTER *II*: ANALYSIS RESULT

States of the Markov Chain & Rate Diagram

The system is modeled as a Continuous-Time Markov Chain (CTMC) with the state space:

$$S = \{0,1,2,3,4,5,6,7,8,9,10\}$$

The diagram below illustrates the transition rates between states in the Continuous-Time Markov Chain (CTMC) model



Rate Matrix for CTMC

$$
R = \begin{bmatrix}
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0
\end{bmatrix}
$$

$$
R = \begin{bmatrix}
0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0
\end{bmatrix}
$$

In the rate matrix, the diagonal entries R are always zero. The arrival rate is the parameter of Poisson distribution with a value of 0,0032 and the service rate is the parameter of Exponential distribution with a value of 0,0042.

Generator Matrix for CTMC

The generator matrix is the same as the rate matrix with the diagonal elements replaced by −ri's. It is common in the literature to describe a CTMC by the Q matrix that given by

$$Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -\mu
\end{bmatrix}$$

$$Q = \begin{bmatrix}
-0{,}0032 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0{,}0042 & -0{,}0074 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0{,}0042 & -0{,}0074 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0{,}0042 & -0{,}0074 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0{,}0042 & -0{,}0074 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0{,}0042 & -0{,}0074 & 0{,}0032 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0{,}0042 & -0{,}0074 & 0{,}0032 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & -0{,}0074 & 0{,}0032 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & -0{,}0074 & 0{,}0032 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & -0{,}0074 & 0{,}0032 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & -0{,}0042
\end{bmatrix}$$

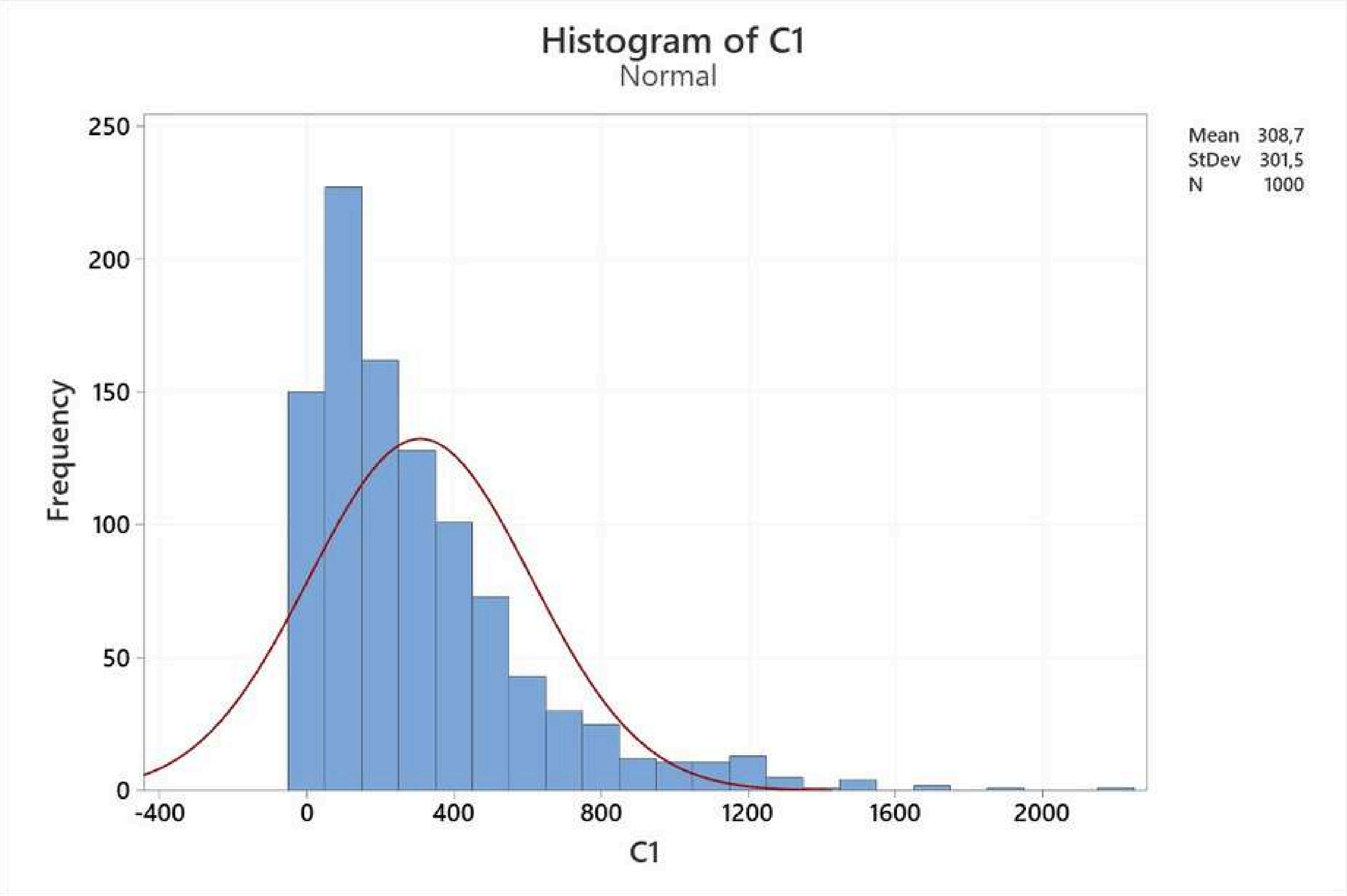
Inter-Arrival Time & its Visualization

Inter-Arrival Time determine the time between the arrivals of the consecutive customers to the system, in this case ATM queue. It represents how much time passes between one customer arriving and the next customer arriving.

Mathematically, if the arrival times of customers are denoted by t1, t2 , t3, …, the inter-arrival time between the i-th and (i+1)-th customer is:

Inter-Arrival Time = t(i+1) - t(i)

Visualization Plot of Inter-Arrival Times



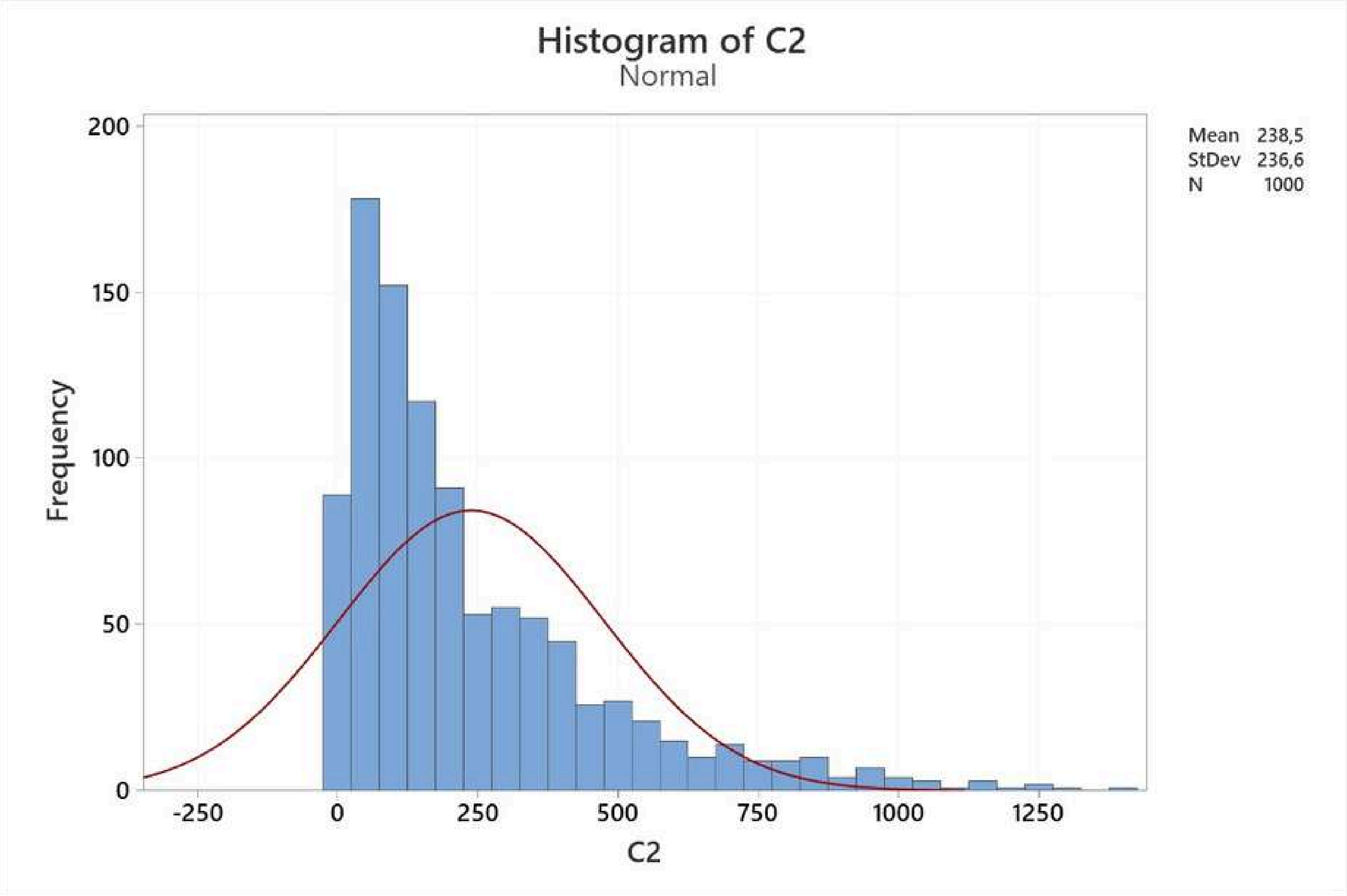
Service Time & its Visualization

Service Time is the time it takes for the server in the system to serve a customer until the service is completed. It represents the time a customer spends being processed by the system, Using the dataset given, service time is determined by subtracting the service start time from the service end time.

Mathematically, the Service Time is:

Service Time = Service End Time - Service Start Time

Visualization Plot of Service Times



Average Inter-Arrival Times

When we have observed data of arrival times, then the Average Inter-Arrival time is computed as :

$$\bar{T} = \frac{\sum_{i=1}^{n-1} [\![(T_i - T_{i-1}]\!])}{N}$$

Where T is in seconds. So, the average value of inter arrival is as follows.

$$\bar{T} = \frac{(173 + 398 + \cdots + 82 + 251)}{1000} = 308{,}74$$

From this calculation, we obtained the Average of Inter-Arrival Times value is 308,74.

Average Service Times

When we have observed data of Start Time and Departure Time, then the Average Service Time is computed as :

$$Service\,Time = \frac{\sum_{i=1}^{n} [\![(Departure\,Time - Start\,Time]\!])}{N}$$

Where Departure Time and Start Time is in seconds. So, the average value of Service Time is as follows.

$$Service\,Time = \frac{(53 + 28 + \cdots + 112 + 74)}{1000} = 238{,}47$$

From this calculation, we obtained the Average of Service Times value is 238,47.

Estimation of Parameter λ

The probability mass function (PMF) for a Poisson random variable X :

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2, \dots$$

The Estimation Parameter :

- Given n observations $x_1, x_2, \dots, x_n$, the Likelihood Function is given by :

$$L(\lambda) = \prod_{i=1}^{n} P(X = x_i) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

- From the likelihood function, the Log-Likelihood Function is given by :

$$l(\lambda) = \ln L(\lambda) = \sum_{i=1}^{n} \ln \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum_{i=1}^{n} (x_i \ln \lambda - \lambda - \ln x_i!)$$

- So, we can obtain the Estimation of Maximum Likelihood for λ :

$$\Leftrightarrow \frac{dl(\lambda)}{d\lambda} = 0$$

$$\Leftrightarrow \frac{\mathrm{d}}{d\lambda}(\ln \lambda \sum_{i=1}^{n} x_i) - \frac{\mathrm{d}}{d\lambda}(n\lambda) - \frac{d}{d\lambda}(\ln x_i!) = 0$$

$$\Leftrightarrow \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n = 0$$

$$\Leftrightarrow \frac{1}{\lambda}\sum_{i=1}^{n} x_i = n$$

$$\Leftrightarrow \hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\lambda} = \bar{x}$$

We can conclude that the estimation is the average of the number of observed events.

Estimation of Parameter μ

The probability mass function (PMF) for a Exponential random variable X :

$$P(X = x) = \frac{1}{\mu} e^{-\frac{1}{\mu}x}, x \geq 0, \mu \geq 0$$

The Estimation Parameter :

- Given n observations $x_1, x_2, \ldots, x_n$, the Likelihood Function is given by :

$$L(\mu) = \prod_{i=1}^{n} P(X = x_i) = \prod_{i=1}^{n} \frac{1}{\mu} e^{-\frac{1}{\mu}x_i} = \left(\frac{1}{\mu}\right)^n e^{-\frac{1}{\mu}\sum_{i=1}^{n} x_i}$$

- From the likelihood function, the Log-Likelihood Function is given by :

$$l(\mu) = \ln L(\mu) = \ln(\left(\frac{1}{\mu}\right)^n e^{-\frac{1}{\mu}\sum_{i=1}^{n} x_i})$$

$$l(\mu) = -\mathrm{n} \ln \mu - \frac{1}{\mu}\sum_{i=1}^{n} x_i$$

- So, we can obtain the Estimation of Maximum Likelihood for μ :

$$\Leftrightarrow \frac{dl(\mu)}{d\mu} = 0$$

$$\Leftrightarrow \frac{\mathrm{d}}{d\mu}(-\mathrm{n} \ln \mu) - \frac{\mathrm{d}}{d\mu}\left(\frac{1}{\mu}\sum_{i=1}^{n} x_i\right) = 0$$

$$\Leftrightarrow -\frac{n}{\mu} + \frac{1}{\mu^2}\sum_{i=1}^{n} x_i = 0$$

$$\Leftrightarrow \frac{1}{\mu^2}\sum_{i=1}^{n} x_i = \frac{n}{\mu}$$

$$\Leftrightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\mu} = \bar{x}$$

We can conclude that the estimation is the average of the number of observed events.

Expected Duration of Time The Machine is Idle

In an M/M/1 queueing system, the customers arrive according to Poisson distribution with the paramater and the services times are Exponential distribution with the parameter random variables. There is a single server with the M/M/1 queue theorem define as follows.

$$\rho = \frac{\lambda}{\mu}$$

$$\rho = \frac{0{,}0032}{0{,}0042}$$

$$\rho = 0{,}7724$$

Beside of that, the probability that the server is idle is given by

$$\rho_0 = 1 - \rho$$

$$\rho_0 = 1 - 0{,}7724$$

$$\rho_0 = 0{,}2276$$

As a result, the expected of idle time of the ATM machine during the next hour is 0,2276 hours or 13,65 minutes if the ATM machine is idle at 8:00 AM.

Probability of *k*-Length of Queue

The probability of a queue having k-length is an essential parameter in evaluating system performance, as it indicates the likelihood of exactly k customers being present in the queue. This Transition Probability Matrix is modeled using the formula below:

$$P(t) = \sum_{k=0}^{\infty} e^{-rt} \frac{(rt)^k}{k!} p^k$$

The approximation of P(2) using Uniformization Algorithm, gives the result for the Transition Probability Matrix. Continuing with this value in Transisition Probability Matrix , the probabilities for all values of k are computed and presented as follows:

$$k = 0: P(X(2) = 0) = 0{,}993647$$
$$k = 1: P(X(2) = 1) = 0{,}006333$$
$$k = 2: P(X(2) = 2) = 0{,}000024$$
$$k = 3: P(X(2) = 3) = 0{,}00000004$$
$$k = 4: P(X(2) = 4) = 6{,}899381 \times 10^{-11}$$
$$k = 5: P(X(2) = 5) = 8{,}81638 \times 10^{-14}$$
$$k = 6: P(X(2) = 6) = 0$$
$$k = 7: P(X(2) = 7) = 0$$
$$k = 8: P(X(2) = 8) = 0$$
$$k = 9: P(X(2) = 9) = 0$$
$$k = 10: P(X(2) = 10) = 0$$

Expected Value of *k*-Length of Queue

The expected value of the k-length of the queue, $E[X(t)]$, represents the average number of people in the queue over a given period. It is calculated using the formula:

$$\begin{aligned} E(X(2)) &= 0 \times (0{,}993647) + 1 \times (0{,}006333) + 2 \times (0{,}000024) + 3 \times (0{,}0000004) \\ &+ 4 \times (6{,}899381 \times 10^{-11}) + 5 \times (8{,}81638 \times 10^{-14}) + 6 \times (0) + 7 \times (0) \\ &+ 8 \times (0) + 9 \times (0) + 10 \times (0) = 0{,}0063822 \end{aligned}$$

In this calculation, $t=2$ represents the time interval between the initial observation at 5 AM, when the queue is empty ($t=0$), and the target time of 7 AM. This 2 hour interval allows us to analyze the system's behavior during this period.

Thus, the expected value of the queue length at 7 AM is **0.0063822.** This indicates that, on average, less than one person is expected to be in the queue at this time, reflecting the system's efficiency in handling arrivals and departures.

Limiting Distribution

The limiting distribution of each state state of M/M/1 model can be determined only if the queueing system is stable (rho < 1). Parameter rho represents the average server utilization within the system, specifically the queue in this study case. If the queuing system is stable, its limiting distribution is given by the equation as follows.

EQUATION

$$p_i = (1-\rho)\rho^i, \qquad i \geq 0,$$

where

$$\rho = \frac{\lambda}{\mu}.$$

Lambda : Average of Inter-Arrival

Mu : Average of Service

CALCULATION

For example, for state 0:

$$\rho = \frac{308{,}74}{238{,}47} = 0{,}772$$

$$p_0 = (1 - 0{,}772)0{,}772^0 = 0{,}228$$

MATRIX RESULTS

$$p^T = [0{,}228 \quad 0{,}176 \quad 0{,}136 \quad 0{,}105 \quad 0{,}081 \quad 0{,}063 \quad 0{,}048 \quad 0{,}037 \quad 0{,}029 \quad 0{,}022 \quad 0{,}017]$$

Annual Profit & ROI

Annual profit refers to the total amount of money the ATM earns over the course of a year after accounting for all associated costs and expenses.

$$Annual\ Profit = Total\ Revenue - Total\ Expenses$$
$$Annual\ Profit = (N \times C_r) - [(N \times C_t) - M - C_e]$$

ROI or Return of Investment is a financial metric used to measure the profitability or efficiency of an investment, relative to its cost.

$$ROI = \frac{Net\ Profit}{Cost\ of\ Investment} \times 100\%$$

Annual Profit & ROI

Transaction Fee Cr=$0,5

$$Annual\ Profit = (73.000 \times 0{,}5) - (73.000 \times 0{,}25) - 1.500 - 1.200$$
$$Annual\ Profit = 15.550$$

$$ROI = \frac{15.550}{15.000} \times 100\% = 103{,}67\%$$

Transaction Fee Cr=$1

$$Annual\ Profit = (73.000 \times 1) - (73.000 \times 0{,}25) - 1.500 - 1.200$$
$$Annual\ Profit = 52.050$$

$$ROI = \frac{52.050}{15.000} \times 100\% = 347\%$$

Annual Profit & ROI

Transaction Fee Cr=$2

$$Annual\ Profit = (73.000 \times 2) - (73.000 \times 0{,}25) - 1.500 - 1.200$$
$$Annual\ Profit = 125.050$$

$$ROI = \frac{125.050}{15.000} \times 100\% = 833{,}67\%$$

THANK YOU!