STOCHASTIC PROCESS (D-K)

ANALYSIS OF ATM MACHINE QUEUE PROBLEM

GROUP 3

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CHAPTER I: INTRODUCTION

1.1 Background

Automatic Teller Machines (ATMs) play a critical role in providing 24/7 access to financial transactions. However, during peak usage hours, customer queuing and potential overcrowding can affect user satisfaction and system efficiency. This case study examines the dynamics of an ATM system where customers arrive randomly over time, with the arrival process following a Poisson distribution at a rate λ and some constraints as follows:

- The space in front of the ATM can accommodate at most 10 customers. Thus, if there are 10 customers already waiting and a new customer arrives, the customer walks away and is lost forever.
- The customers form a single line and use the ATM in a first-come, first-served fashion.
- The processing times at the ATM for the customers are independent and identically distributed (iid) exponential random variables with rate μ.
- Let X(t) be the number of customers at the ATM at time t.

This study uses historical ATM usage data from 2024, sourced from its.id/prosto2024, to analyze and model the queueing behavior. The goal is to answer critical questions regarding system dynamics, including transition rates, queue probabilities, idle times, and long-term behavior, as well as evaluating the system's economic performance. By understanding the dynamics of $X(t)$ as a CTMC, we aim to address operational challenges, such as minimizing customer loss and ensuring efficient ATM utilization, while also exploring profitability through various transaction fee scenarios.

1.2 Problem Statements

1. How can we define the states of the Markov chain representing the ATM queue and the transition rates between these states? How can the rate diagram be drawn?
2. How can the rate matrix and generator matrix for the CTMC describing the ATM queue system be derived?
3. How can the distribution of the inter-arrival times and service times of the ATM machine be visualized using historical data?
4. How can the average inter-arrival times and service times for the ATM system be calculated?
5. How can the parameters λ (arrival rate) and μ (service rate) of the ATM queue system be estimated using the Maximum Likelihood Estimation (MLE) method?
6. Estimate the parameters λ and μ using Maximum Likelihood Estimation method. If the ATM queue is empty at 5 AM, what is the probability of having k people in the queue at 7 AM for k = 0, 1, 2, …, 10, and what is the expected queue size?
7. If the ATM machine is idle at 8:00 AM, what is the expected amount of time the machine remains idle during the next hour?
8. What is the limiting distribution of the state of the ATM queue system?
9. Given the initial investment cost is I = 15,000 dollars, and the annual maintenance cost is M = 1,500 dollars with a total annual operating cost includes:
    - Electricity cost per year: Ce = 1,200 dollars per year, and

- Transaction processing cost: Ct = 0.25 dollars per transaction.

What is the annual profit and ROI if the transaction fee is set at:

- Cr = 0.5 dollars
- Cr = 1 dollars
- Cr = 2 dollars

for each transaction?

1.3 Objectives

1. Define the states of the Markov chain and the transition rates between them, and draw the rate diagram for the ATM queue system.
2. Derive the rate matrix and generator matrix for the CTMC representing the ATM queue system.
3. Visualize the distribution of the inter-arrival times and service times using historical data.
4. Calculate the average inter-arrival times and service times for the ATM system.
5. Estimate the parameters λ (arrival rate) and μ (service rate) using the Maximum Likelihood Estimation (MLE) method.
6. Calculate the probability that there are k people in the ATM queue at 7 AM, starting from an empty queue at 5 AM, for k = 0, 1, 2, …, 10, and compute its expected value.
7. Estimate the expected idle time of the ATM machine during the next hour, given it is idle at 8:00 AM.
8. Compute the limiting distribution of the states of the ATM queue system.
9. Calculate the annual profit and ROI for the ATM system, given the initial investment cost is I = 15,000 dollars, and the annual maintenance cost is M = 1,500 dollars with a total annual operating cost includes:
    - Electricity cost per year: Ce = 1,200 dollars per year, and
    - Transaction processing cost: Ct = 0.25 dollars per transaction.

    Under three different transaction fee scenarios:
    - Cr = 0.5 dollars,
    - Cr = 1 dollars, and
    - Cr = 2 dollars.

CHAPTER II: RESULTS

2.1 States of the Markov Chain & the Transition Rates between the States

2.1.1 States of the Markov Chain

The system is modeled as a Continuous-Time Markov Chain (CTMC) with the state space:

$$S = \{0,1,2,3,4,5,6,7,8,9,10\}$$

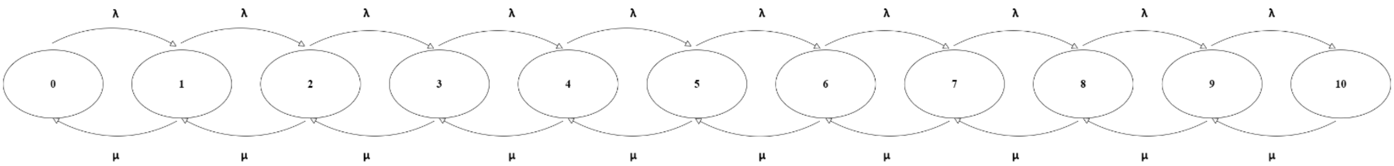
Each state $i \in$ represents the number of customers in the system, including those being served at the ATM and those waiting in the queue. The following conditions define the state transitions:

- State 0: The system is empty; no customers are present.
- State i: The system has i customers (where $1 \leq i \leq 9$).
- State 10: The system is at full capacity (maximum of 10 customers). Any additional arrivals are lost.

This discrete state space accurately models the queue dynamics, accounting for customer arrivals and service completions.

2.1.2 The Transition Rates between The States

The diagram below illustrates the transition rates between states in the Continuous-Time Markov Chain (CTMC) model. Each state represents the number of customers in the system, including those being served and those waiting in the queue. Transitions between states are determined by the arrival rate, denoted as λ, and the service rate, denoted as μ. When a customer arrives, the system transitions from state i to $i+1$, indicating an increase in the number of customers. Conversely, when a service is completed, the system transitions from state i to $i-1$, reducing the number of customers in the system.



This representation succinctly demonstrates how the system handles customer flow, capturing both arrivals and service completions within the defined capacity constraints.

2.2 The Rate Matrix & Generator Matrix for the CTMC

2.2.1 The Rate Matrix

As in the case of the ATM queue system, we can calculate the estimated of λ_i and μ_i first. The parameters of λ_i and μ_i are called, respectively, the arrival rate and service rate. The arrival rate is the parameter of Poisson distribution with a value of 0,0032 and the service rate is the parameter of Exponential distribution with a value of 0,0042. In the *rate matrix*, the diagonal entries R are always zero. The details can be described below.

$$
R = \begin{bmatrix}
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0
\end{bmatrix}
$$

$$
R = \begin{bmatrix}
0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0032 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0 & 0{,}0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{,}0042 & 0
\end{bmatrix}
$$

2.2.2 Generator Matrix for the CTMC

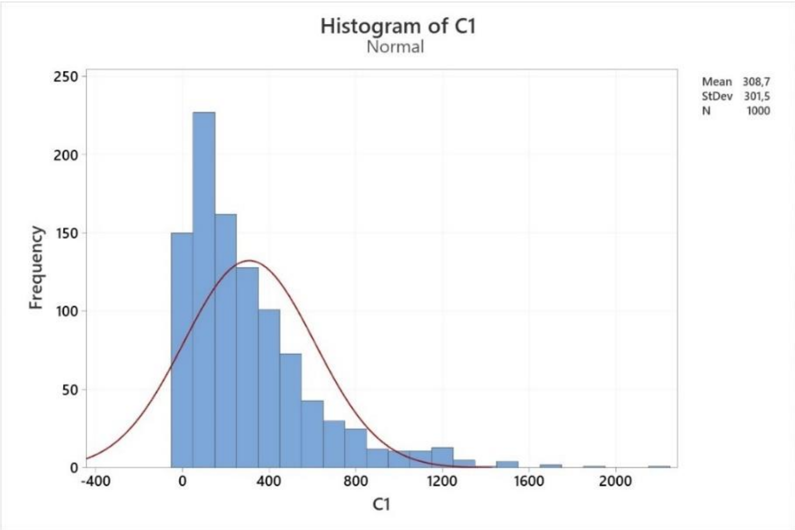
The generator matrix is the same as the rate matrix with the diagonal elements replaced by $-r_i$'s. It is common in the literature to describe a CTMC by the Q matrix that given by

$$
Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda+\mu) & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -\mu
\end{bmatrix}
$$

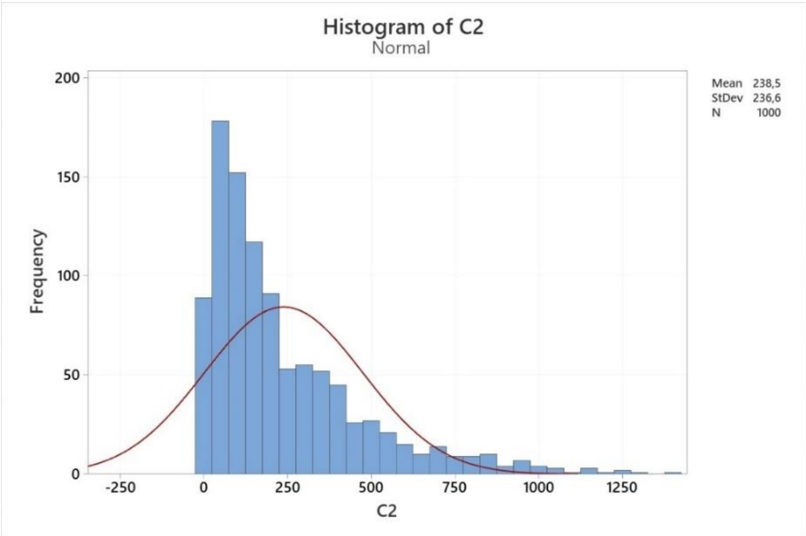
$$
Q = \begin{bmatrix}
0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_1 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_2 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_3 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_4 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_5 & 0 & \lambda_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_6 & 0 & \lambda_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_7 & 0 & \lambda_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_8 & 0 & \lambda_8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_9 & 0 & \lambda_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_{10} & 0
\end{bmatrix}
$$

2.3 The Distribution of the Inter-Arrival Times & the Service Times of the ATM Machine

Inter-arrival time determine the time between the arrivals of the consecutive customers to the system, in this case ATM queue. It represents how much time passes between one customer arriving and the next customer arriving. With the dataset given, the following figure is the visualization of inter-arrival times of the ATM Machine.



On other hand, Service Time is the time it takes for the server in the system to serve a customer until the service is completed. It represents the time a customer spends being processed or served by the system. Using the dataset given, service time is determined by subtracting the service start time from the service end time. With that calculation, the plot of the visualization of service times of the ATM Machine is found, as follows.



2.4 Calculation of the Averge Inter-Arrival Times & the Service Times

2.4.1 The Average of the Inter-Arrival Times

The equation for the **Average Inter-Arrival Times** depends on the process used to model arrivals time. When we have observed data of arrival times, then the Average Inter-Arrival time is computed as :

$$\bar{T} = \frac{\sum_{i=1}^{n-1} ⟦(T_i - T_{i-1}⟧)}{N}$$

Where T_i is in seconds. So, the average value of inter arrival is as follows.

$$\bar{T} = \frac{(173 + 398 + \cdots + 82 + 251)}{1000} = 308{,}74$$

From this calculation, we obtained the Average of Inter-Arrival Times value is 308,74.

2.4.2 The Average of the Service Times

The average service time serves as a critical measure of the system's efficiency, reflecting how long, on average, it takes to complete a single service. When we have observed data of Start Time and Departure Time, then the Average Service Time is computed as :

$$Service\ Time = \frac{\sum_{i=1}^{n} ⟦(Departure\ Time - Start\ Time⟧)}{N}$$

Where Departure Time and Start Time is in seconds. So, the average value of inter arrival is as follows.

$$Service\ Time = \frac{(53 + 28 + \cdots + 112 + 74)}{1000} = 238{,}47$$

From this calculation, we obtained the Average of Service Times value is 238,47.

2.5 Parameter Estimation

2.5.1 Estimation of Parameter λ

To estimate λ in Distribution Poisson for customer arrive, **Maximum Likelihood Estimation (MLE)** can be used. The probability mass function (PMF) for a Poisson random variable X with parameter λ > 0 is given by:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2, \ldots$$

Maximum Likelihood Estimation for λ :

- Given n observations $x_1, x_2, \ldots, x_n$, the Likelihood Function is given by :

$$L(\lambda) = \prod_{i=1}^{n} P(X = x_i) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

- From the likelihood function, the Log-Likelihood Function is given by :

$$l(\lambda) = \ln L(\lambda) = \sum_{i=1}^{n} \ln \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum_{i=1}^{n} (x_i \ln \lambda - \lambda - \ln x_i!)$$

- So, we can obtain the Estimation of Maximum Likelihood for λ :

$$\Leftrightarrow \frac{dl(\lambda)}{d\lambda} = 0$$

$$\Leftrightarrow \frac{\mathrm{d}}{d\lambda}(\ln \lambda \sum_{i=1}^{n} x_i) - \frac{\mathrm{d}}{d\lambda}(n\lambda) - \frac{d}{d\lambda}(\ln x_i!) = 0$$

$$\Leftrightarrow \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n = 0$$

$$\Leftrightarrow \frac{1}{\lambda}\sum_{i=1}^{n} x_i = n$$

$$\Leftrightarrow \hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\lambda} = \bar{x}$$

We can conclude that the estimation of $\hat{\lambda}$ is the average of the number of observed events.

2.5.2 Estimation of Parameter μ

To estimate μ in Distribution Exponential for service time, Maximum Likelihood Estimation (MLE) can be used. The probability mass function (PMF) for a Exponential random variable X with parameter μ > 0 is given by:

$$P(X = x) = \frac{1}{\mu} e^{-\frac{1}{\mu}x}, x \geq 0, \mu \geq 0$$

Maximum Likelihood Estimation for μ :

- Given n observations $x_1, x_2, \dots, x_n$, the Likelihood Function is given by :

$$L(\mu) = \prod_{i=1}^{n} P(X = x_i) = \prod_{i=1}^{n} \frac{1}{\mu} e^{-\frac{1}{\mu}x_i} = \left(\frac{1}{\mu}\right)^n e^{-\frac{1}{\mu}\sum_{i=1}^{n} x_i}$$

- From the likelihood function, the Log-Likelihood Function is given by :

$$l(\mu) = \ln L(\mu) = \ln(\left(\frac{1}{\mu}\right)^n e^{-\frac{1}{\mu}\sum_{i=1}^{n} x_i})$$

$$l(\mu) = -\mathrm{n} \ln \mu - \frac{1}{\mu}\sum_{i=1}^{n} x_i$$

- So, we can obtain the Estimation of Maximum Likelihood for μ :

$$\Leftrightarrow \frac{dl(\mu)}{d\mu} = 0$$

$$\Leftrightarrow \frac{\mathrm{d}}{d\mu}(-\mathrm{n}\ln\mu) - \frac{\mathrm{d}}{d\mu}\left(\frac{1}{\mu}\sum_{i=1}^{n} x_i\right) = 0$$

$$\Leftrightarrow -\frac{n}{\mu} + \frac{1}{\mu^2}\sum_{i=1}^{n} x_i = 0$$

$$\Leftrightarrow \frac{1}{\mu^2}\sum_{i=1}^{n} x_i = \frac{n}{\mu}$$

$$\Leftrightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\mu} = \bar{x}$$

We can conclude that the estimation of $\hat{\mu}$ is the average of the number of observed events.

2.6 Probability of k-Length of Queue & Expected Value

2.6.1 Probability of k-Length of Queue

The probability of a queue having k-length is an essential parameter in evaluating system performance, as it indicates the likelihood of exactly k customers being present in the queue. This Transition Probability Matrix is modeled using the formula below:

$$P(t) = \sum_{k=0}^{\infty} e^{-rt} \frac{(rt)^k}{k!} p^k$$

The approximation of P(2) using Uniformization Algorithm, gives the result for the Transition Probability Matrix. Continuing with this value **in Transisition Probability Matrix , the probabilities for all values of k are computed and presented as follows:**

$$k = 0: P(X(2) = 0) = 0{,}993647$$
$$k = 1 : P(X(2) = 1) = 0{,}006333$$
$$k = 2: P(X(2) = 2) = 0{,}000024$$
$$k = 3: P(X(2) = 3) = 0{,}00000004$$
$$k = 4: P(X(2) = 4) = 6{,}899381 \times 10^{-11}$$
$$k = 5: P(X(2) = 5) = 8{,}81638 \times 10^{-14}$$
$$k = 6: P(X(2) = 6) = 0$$
$$k = 7: P(X(2) = 7) = 0$$
$$k = 8: P(X(2) = 8) = 0$$
$$k = 9: P(X(2) = 9) = 0$$
$$k = 10: P(X(2) = 10) = 0$$

From these calculations, it is evident that the probability decreases exponentially as k increases, showcasing the system's ability to maintain short queue lengths and prevent congestion.

2.6.2 Expected Value of k-Length of Queue

The expected value of the k -length of the queue, $E[X(t)]$, represents the average number of people in the queue over a given period. It is calculated using the formula:

$$\begin{aligned} E\big(X(2)\big) = {} & 0 \times (0{,}993647) + 1 \times (0{,}006333) + 2 \times (0{,}000024) + 3 \times (0{,}0000004) \\ & + 4 \times (6{,}899381 \times 10^{-11}) + 5 \times (8{,}81638 \times 10^{-14}) + 6 \times (0) + 7 \times (0) \\ & + 8 \times (0) + 9 \times (0) + 10 \times (0) = 0{,}0063822 \end{aligned}$$

In this calculation, $t = 2$ represents the time interval between the initial observation at 5 AM, when the queue is empty ($t = 0$), and the target time of 7 AM. This 2 hour interval allows us to analyze the system's behavior during this period.

Thus, the expected value of the queue length at 7 AM is 0,0063822**.** This indicates that, on average, less than one person is expected to be in the queue at this time, reflecting the system's efficiency in handling arrivals and departures.

2.7 Expected Duration of Time The Machine is Idle

In an M/M/1 queueing system, the customers arrive according to Poisson distribution with the paramater λ and the services times are Exponential distribution with the parameter μ random variables. There is a single server with the M/M/1 queue theorem define as follows.

$$\rho = \frac{\lambda}{\mu}$$

$$\rho = \frac{0{,}0032}{0{,}0042}$$

$$\rho = 0{,}7724$$

Beside of that, the probability that the server is idle is given by

$$\rho_0 = 1 - \rho$$

$$\rho_0 = 1 - 0{,}7724$$

$$\rho_0 = 0{,}2276$$

As a result, the expected of idle time of the ATM machine during the next hour is 0,2276 hours or 13,65 minutes if the ATM machine is idle at 8:00 AM.

2.8 The Limiting Distribution of the ATM Queue

The limiting distribution of each state of the ATM queue can be determined only if the queueing system is stable. If the queuing system is stable, its limiting distribution is given by the equation as follows.

$$p_i = (1 - \rho)\rho^i, \qquad i \geq 0,$$

where

$$\rho = \frac{\lambda}{\mu} \ .$$

Parameter ρ, referred to as the *traffic intensity*, represents the average server utilization within the system, specifically the $M/M/1$ queue in this study case. If the traffic intensity ρ is less than one, it indicates that the queuing system is stable. By applying the equation above, the traffic intensity of the $M/M/1$ queue is calculated to be 0,772, indicating that the system is stable.

$$\rho = \frac{308{,}74}{238{,}47} = 0{,}772$$

Consequently, the limiting distribution can be determined, as demonstrated in the calculation for state 0 below.

$$p_0 = (1 - 0{,}772)0{,}772^0 = 0{,}228$$

The calculation above provides the limiting distributions for all states with results as follows.

$$p^T = [0{,}228 \quad 0{,}176 \quad 0{,}136 \quad 0{,}105 \quad 0{,}081 \quad 0{,}063 \quad 0{,}048 \quad 0{,}037 \quad 0{,}029 \quad 0{,}022 \quad 0{,}017]$$

In the long run, the limiting distribution provides insight into the steady-state behavior of the ATM queue. The probabilities reflect how often the system is expected to be in each possible state over an extended period of time. In general, the limiting distribution shows that the ATM system is more likely to have fewer customers in the queue, with the probability of having a full queue (10 customers) being very low (1.7%). This suggests that the ATM generally operates with relatively low traffic, ensuring that customers are served in a timely manner and that the queue does not often reach its maximum capacity.

2.9 Annual Profit & the Return of Investment (ROI)

Annual profit refers to the total amount of money the ATM earns over the course of a year after accounting for all associated costs and expenses. To calculate the annual profit, we can use the following equation.

$$Annual\ Profit = Total\ Revenue - Total\ Expenses$$

$$Annual\ Profit = (N \times C_r) - [(N \times C_t) - M - C_e]$$

Where N represents the total number of customers per year, C_r is the customer transaction fee, C_t is transaction processing cost ($0,25 per transaction), M is the annual maintenance cost ($1.500), and C_e is the electricity cost per year ($1.200).

ROI or Return of Investment is a financial metric used to measure the profitability or efficiency of an investment, relative to its cost. ROI is commonly used to evaluate the potential returns from different investments or projects. The formula for calculating ROI is given by the following equation.

$$ROI = \frac{Net\ Profit}{Cost\ of\ Investment} \times 100\%$$

In this case net profit refers to the annual profit, while the cost of investment is the initial invesment in the ATM ($15.000). We will calculate the annual profit and ROI for thee different scenarios, based on how much we charge the customer transaction fee.

- **Transaction Fee $\boldsymbol{C_r = \$0,5}$**

  First, we calculate the annual profit for the first scenario, assuming a transaction fee of $0,50 per transaction. Based on the provided historical data, the ATM processes approximately 200 customers per day, which results in an annual total of approximately 73.000 customers. The annual profit can then be calculated as follows.

  $$Annual\ Profit = (73.000 \times 0{,}5) - (73.000 \times 0{,}25) - 1.500 - 1.200$$

  $$Annual\ Profit = 15.550$$

  For the first scenario, the annual profit is found to be $15.550. Then we can calculate the ROI as follows.

  $$ROI = \frac{15.550}{15.000} \times 100\% = 103{,}67\%$$

  An ROI of 103,67% implies that the return on investment is slightly more than the original investment amount. For every dollar invested, the profit earned is approximately $1,04. This ROI indicates that the investment successfully returned more than the initial amount, generating additional profit.

- **Transaction Fee $\boldsymbol{C_r = \$1}$**

  The annual profit for the second scenario is calculated as follows.

  $$Annual\ Profit = (73.000 \times 1) - (73.000 \times 0{,}25) - 1.500 - 1.200$$

$$Annual\ Profit = 52.050$$

For the second scenario, the annual profit is found to be \$52.050. Then we can calculate the ROI as follows.

$$ROI = \frac{52.050}{15.000} \times 100\% = 347\%$$

An ROI of 347% indicates a very high return on investment. For every dollar invested, the profit earned is \$3,47. This means the investment has generated a return that is more than three times the original investment amount.

- **Transaction Fee $\boldsymbol{C_r} = \$\mathbf{2}$**

The annual profit for the third scenario is calculated as follows.

$$Annual\ Profit = (73.000 \times 2) - (73.000 \times 0{,}25) - 1.500 - 1.200$$

$$Annual\ Profit = 125.050$$

For the third scenario, the annual profit is found to be \$125.050. Then we can calculate the ROI as follows.

$$ROI = \frac{125.050}{15.000} \times 100\% = 833{,}67\%$$

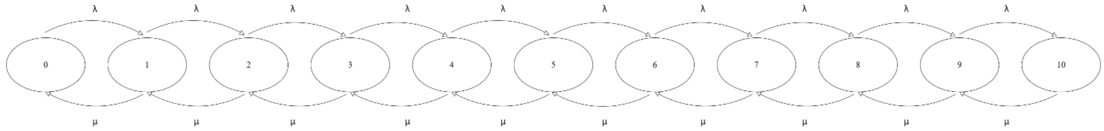
An ROI of 833.67% indicates an exceptionally high return on investment. For every dollar invested, the profit earned is approximately \$8.34. This means that the return is more than eight times the original investment amount.

CHAPTER III: CONCLUSION

1. The system is modeled as a Continuous-Time Markov Chain (CTMC) with the state space:

   S = {0,1,2,3,4,5,6,7,8,9,10}

   State 0: The system is empty; no customers are present.
   State i: The system has i customers (where $1 \leq i \leq 9$).
   State 10: The system is at full capacity (maximum of 10 customers). Any additional arrivals are lost.
   The diagram below illustrates the transition rates between states in the Continuous-Time Markov Chain (CTMC) model.

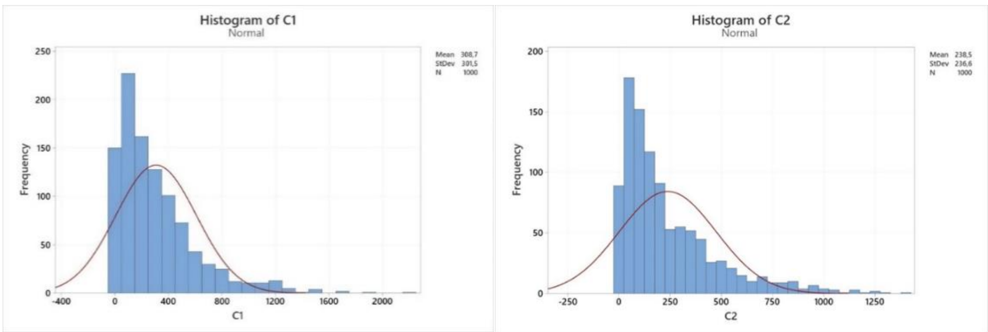


2. The rate matrix and generator matrix for the CTMC representing the ATM queue system is as follows:

$$R = \begin{bmatrix} 0 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0,0042 & 0 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0,0042 & 0 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0,0042 & 0 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0,0042 & 0 & 0,0032 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0,0042 & 0 & 0,0032 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0,0042 & 0 & 0,0032 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & 0 & 0,0032 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & 0 & 0,0032 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & 0 & 0,00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0,0032 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0,0042 & -0,0074 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0,0042 & -0,0074 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0,0042 & -0,0074 & 0,0032 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0,0042 & -0,0074 & 0,0032 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0,0042 & -0,0074 & 0,0032 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0,0042 & -0,0074 & 0,0032 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & -0,0074 & 0,0032 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & -0,0074 & 0,0032 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & -0,0074 & 0,0032 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,0042 & -0,0042 \end{bmatrix}$$

3. The histograms for the distribution of interarrival and service time are as follows:



4. The average value of inter arrival is as follows.

$$\bar{T} = \frac{(173 + 398 + \cdots + 82 + 251)}{1000} = 308,74$$

   While the average value of inter arrival is as follows.

$$Service\ Time = \frac{(53 + 28 + \cdots + 112 + 74)}{1000} = 238{,}47$$

5. The obtained Estimation of Maximum Likelihood for λ is as follows:

$$\hat{\lambda} = \bar{x}$$

We can conclude that the estimation of $\hat{\lambda}$ is the average of the number of observed events. While the obtained Estimation of Maximum Likelihood for μ is as follows:

$$\hat{\mu} = \bar{x}$$

We can conclude that the estimation of μ̂ is the average of the number of observed events.

6. The probabilities for all values of k are computed and presented as follows:

$$k = 0: P(X(2) = 0) = 0{,}993647$$
$$k = 1 : P(X(2) = 1) = 0{,}006333$$
$$k = 2: P(X(2) = 2) = 0{,}000024$$
$$k = 3: P(X(2) = 3) = 0{,}00000004$$
$$k = 4: P(X(2) = 4) = 6{,}899381 \times 10^{-11}$$
$$k = 5: P(X(2) = 5) = 8{,}81638 \times 10^{-14}$$
$$k = 6: P(X(2) = 6) = 0$$
$$k = 7: P(X(2) = 7) = 0$$
$$k = 8: P(X(2) = 8) = 0$$
$$k = 9: P(X(2) = 9) = 0$$
$$k = 10: P(X(2) = 10) = 0$$

The expected value of the queue length at 7 AM is as follows:

$$E[X(2)] = 6336.83$$

This indicates that, on average, less than one person is expected to be in the queue at this time, reflecting the system's efficiency in handling arrivals and departures.

7. There is a single server with the M/M/1 queue theorem define as follows.

$$\rho = \frac{\lambda}{\mu}$$
$$\rho = \frac{0{,}0032}{0{,}0042}$$
$$\rho = 0{,}7724$$

Beside of that, the probability that the server is idle is given by

$$\rho_0 = 1 - \rho$$
$$\rho_0 = 1 - 0{,}7724$$
$$\rho_0 = 0{,}2276$$

As a result, the expected of idle time of the ATM machine during the next hour is 13,65 minutes if the ATM machine is idle at 8:00 AM.

8. The matrix below provides the limiting distributions for all states.

$$p^T = [0{,}228 \quad 0{,}176 \quad 0{,}136 \quad 0{,}105 \quad 0{,}081 \quad 0{,}063 \quad 0{,}048 \quad 0{,}037 \quad 0{,}029 \quad 0{,}022 \quad 0{,}017]$$

9. The calculated annual profit and ROI for the first scenario ($C_r = \$0{,}5$) is as follows.

$$Annual\ Profit = (73.000 \times 0,5) - (73.000 \times 0,25) - 1.500 - 1.200$$
$$Annual\ Profit = 15.550$$
$$ROI = \frac{15.550}{15.000} \times 100\% = 103,67\%$$
The calculated annual profit and ROI for the second scenario ($C_r = \$1$) is as follows.
$$Annual\ Profit = (73.000 \times 1) - (73.000 \times 0,25) - 1.500 - 1.200$$
$$Annual\ Profit = 52.050$$
$$ROI = \frac{52.050}{15.000} \times 100\% = 347\%$$
The calculated annual profit and ROI for the third scenario ($C_r = \$2$) is as follows.
$$Annual\ Profit = (73.000 \times 2) - (73.000 \times 0,25) - 1.500 - 1.200$$
$$Annual\ Profit = 125.050$$
$$ROI = \frac{125.050}{15.000} \times 100\% = 833,67\%$$