ATM Machine Queue Problem

Customers arrive at a 24-hour automatic teller machine (ATM) according to a Poisson process with rate λ. The space in front of the ATM can accommodate at most 10 customers. Thus, if there are 10 customers already waiting and a new customer arrives, the customer walks away and is lost forever. The customers form a single line and use the ATM in a first-come, first-served fashion. The processing times at the ATM for the customers are independent and identically distributed (iid) exponential random variables with rate μ. Let $X(t)$ be the number of customers at the ATM at time t.

Given the historical data of ATM usage in year 2024 (its.id/prosto2024), model the number of customers $X(t)$ at the ATM as a Continuous-Time Markov Chain (CTMC) and answer the questions below.

1. Define the states of the Markov chain and the transition rates between the states (draw the rate diagram).

2. Derive the rate matrix and generator matrix for the CTMC.

3. Given the historical data, visualize the distribution of the inter-arrival times and the service times of the ATM machine.

4. Calculate the average inter-arrival times and the service times.

5. Estimate the parameter λ and μ using Maximum Likelihood Estimation method.

6. Suppose that one day at 5 AM the ATM queue is empty, calculate the probability that there are k people there at 7 AM ($k = 0, 1, 2, ..., 10$) [calculate for each k] and calculate its expected value.

7. Suppose that the ATM machine is idle at 8:00 AM. What is the expected amount of time the machine is idle during the next hour?

8. Compute the limiting distribution of the state of the ATM queue.

9. Let the initial investment in the ATM is $I = 15,000$ dollars, and the annual maintenance cost is $M = 1,500$ dollars. The total annual operating cost includes:

   - Electricity cost per year: $C_e = 1,200$ dollars per year.
   - Transaction processing cost: $C_t = 0.25$ dollars per transaction.

Calculate the annual profit and the ROI if we charge the customer the transaction fee with three different scenarios:

- $C_r = 0.5$ dollars
- $C_r = 1$ dollars
- $C_r = 2$ dollars

for each transaction.