Looking at Some Special Statistical Distributions

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Goals:

Explore more exotic statistical distributions.

Overview

Bernoulli Distribution

Poisson Distribution

Normal Distribution Revisited

Lognormal Distribution

Multinomial Distribution

Bernoulli Distribution

Patient treatment can either succeed or fail

$X = 0$ **if fails,** $X = 1$ **if successful**

Probability $p = P(X = 1)$

The collection of all distributions with $0 \leq p \leq 1$ **is the family of Bernoulli distributions**

Bernoulli Distribution

$P(X = 1) = p$, $P(X = 0) = 1 - p$

Probability function:

$$f(x) = \begin{cases} p^x(1-p)^{1-x}, & \text{for } x = 0,1 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$$

$E[X^2] = E[X] = p$ **(because** $x = x^2$**)**

$$V[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

$$\psi(t) = E[e^{tX}] = pe^t + (1-p)$$

Bernoulli Trials

Given a set of rvs $X_1, X_2, \ldots, X_n$ where $X_i \sim \text{Bernoulli}(p)$, we call $X_1 \ldots X_n$ Bernoulli *trials* with parameter p

An infinite sequence of Bernoulli trials is called a Bernoulli *process*

E.g., a repeated fair coin toss generates $X_1 \ldots X_n$ Bernoulli trials with $p = 1/2$

Binomial Distributions

A sum $X = X_1 + \cdots + X_n$ of n Bernoulli trials with parameter p has a Binomial distribution with parameters n, p

$$f(x,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & \text{for } x = 0,1,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

We can easily derive the mean/variance/mgf of a sum of Bernoulli trials:

$$E[X] = \sum_{i=1}^{n} E[X_i] = np$$

$$V[X] = \sum_{i=1}^{n} V[X_i] = np(1-p)$$

$$\psi(t) = E(e^{tX}) = \prod_{i=1}^{n} E(e^{tX_i}) = (pe^t + 1 - p)^n$$

Bernoulli Trial Example

Suppose a part has a 10% chance to be defective

You sample 10 parts from the production line

How many parts would you expect to be defective?

Bernoulli trials with $p = 0.1$ and $n = 10$

$E[X] = np = 0.1 \cdot 10 = 1$

Poisson Distribution

Measurements of number of occurrences within a time period

Customers at a store, number of calls, number of extreme weather events

Poisson distributions model the number of occurences within a fixed time period

Can be used to approximate binomial distributions for very small success probabilities

Poisson Distribution Example

Store owner wants to model the distribution X of customers arriving in a particular one-hour time period

Models arrivals in different time periods as being independent

Sees 5 customers an hour on average, so models arrival rate as $\frac{5}{3600} = 0.00138$ customers per second

During each second, either 0 or 1 customer arrives with $p = 0.00138$

Binomial distribution with $n = 3600$, $p = 0.00138$

Poisson Distribution Example

Calculations of probability function f too tedious

Successive values are closely related

$$\frac{f(x+1)}{f(x)} = \frac{\binom{n}{x+1} p^{x+1}(1-p)^{n-x-1}}{\binom{n}{x} p^{x}(1-p)^{n-x}} = \frac{(n-x)p}{(x+1)(1-p)}$$

Let's try to simplify:

- For the first few values, $n - x \approx n$
- Dividing by $1 - p$ has little effect because p is tiny

$$\frac{(n-x)p}{(x+1)(1-p)} \approx \frac{np}{x+1}$$

Thus, letting $\lambda = np$, we approximate

$$f(x+1) \approx \frac{f(x) \cdot \lambda}{x+1}$$

Poisson Distribution Example

Expanding out the recurrence relationships, we get:

$$f(1) = f(0)\lambda$$

$$f(2) = f(1)\frac{\lambda}{2} = f(0)\frac{\lambda^2}{2}$$

$$f(3) = f(2)\frac{\lambda}{3} = \cdots = f(0)\frac{\lambda^3}{6}$$

$$\vdots$$

Generalizing, $f(x) = f(0) \cdot \lambda^x / x!$

A valid pf requires that $\sum_{x=0}^{\infty} f(x) = 1$

Poisson Distribution Example

To guarantee the sum of 1, we set

$$f(0) = \frac{1}{\sum_{x=0}^{\infty} \lambda^x / x!} = \frac{1}{e^{\lambda}} = e^{-\lambda}$$

for all $\lambda > 0$

We have our probability function!

Poisson Distribution

A random variable X has a Poisson distribution with mean $\lambda > 0$ iff it has a probability function

$$f(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!}, & \text{for } x \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

Poisson Distribution

If $X \sim \text{Poisson}(\lambda)$,

$E[X] = \lambda$

$V[X] = \lambda$

$\psi(t) = e^{\lambda(e^t - 1)}$

Demo

Normal Distribution Revisited

A rv X is normally distributed $X \sim N(\mu, \sigma^2)$ if it has a pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[X] = \mu$$

$$V[X] = \sigma^2$$

$$\psi(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Linear Transformation

If X has a normal distribution with mean μ and variance σ^2,

and $Y = aX + b$ where a and b are constants s.t. $a \neq 0$,

Y has a normal distribution with mean $a\mu + b$ and variance $a^2\sigma^2$

Example:

- Suppose $X \sim N(0,1)$
- Let $Y = 2X + 3$
- Then $Y \sim N(a\mu + b, a^2\sigma^2) = N(3,4)$

Standard Normal Distribution

Normal distribution with $\mu = 0, \sigma = 1$

Has pdf

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and cdf

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)\, du$$

Cdf does not have an analytic solution

Due to symmetry, for all x and $0 < p < 1$,

$$\Phi(-x) = 1 - \Phi(x)$$

Conversion to Standard Normal Distribution

If $X \sim N(\mu, \sigma^2)$ and F is the cdf of X

Then $Z = \frac{X-\mu}{\sigma}$ has standard normal distribution

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Conversion to Standard Normal Distribution Example

Suppose X has $\mu = 5$ and $\sigma = 2$

Need to calculate $P(1 < X < 8)$

Let $Z = (X - 5)/2$, then $Z \sim N(0,1)$

$$P(1 < X < 8) = P(-2 < Z < 1.5)$$

$$\begin{aligned} P(-2 < Z < 1.5) &= P(Z < 1.5) - P(Z \leq -2) \\ &= \Phi(1.5) - \Phi(-2) \\ &= \Phi(1.5) - [1 - \Phi(2)] \end{aligned}$$

Values of Φ can be looked up to give us

$$P(1 < X < 8) = 0.91$$

Lognormal Distribution

Suppose we have rvs s.t. $Y = \log(X)$

If Y has a normal distribution with mean μ and variance σ^2, X is said to have a *lognormal* distribution

From the definition of normal distribution, we have the mgf of Y as

$$\psi(t) = \exp(\mu t + 0.5\sigma^2 t^2)$$

On the other hand,

$$\psi(t) = E[e^{tY}] = E\left[e^{t \cdot \log X}\right] = E[X^t]$$

$$E[X] = E[X^1] = \psi(1) = e^{\mu + 0.5\sigma^2}$$

$$V[X] = E[X^2] - (E[X])^2 = \psi(2) - \psi(1)$$
$$= \exp(2\mu + \sigma^2)\,(\exp(\sigma^2) - 1)$$

Multinomial Distribution

Binomial distribution models 2 possible outcomes

Multinomial coefficient allows >2 possible outcomes (e.g., blood types)

Given x_i items of type i $(i = 1, \ldots, k)$,

$$\binom{n}{x_1, \ldots, x_k} = \frac{n!}{x_1!\, x_2! \ldots x_k!}$$

If $X = (X_1, \ldots, X_k)$ is a vector of counts and $x = (x_1, \ldots, x_k)$ is a possible value of the random vector, the joint pmf of X is

$$\binom{n}{x_1, \ldots, x_k} p_1^{x_1} \cdots p_k^{x_k}$$

where p is a vector of probabilities

Multinomial Distribution Example

Suppose 23% of people watch TV for 0 to 10 mins a day, 59% between 10 mins and 1 hour, 18% for more than an hour

Given a sample of 20 people, what's the probability that

- 7 watch 10mins or less *and*
- 8 watch between 10mins and 1hr

$$P = \frac{20!}{7!8!5!} 0.23^7 \cdot 0.59^8 \cdot 0.18^5 = 0.00942$$

Summary

Bernoulli distribution

- One of two possible values with probability p
- A sequence is called Bernoulli trials
- Sum of Bernoulli trials has binomial distribution

Poisson distribution: approximation of the binomial distribution for small values

Normal distribution: linear transformation; lognormal distribution

Multinomial distribution: like binomial when >2 outcomes are possible

Course Summary

Probability: probability, experiments, events, set theory, sample spaces, counting methods, combinatorics

Conditional probability: event independence, Bayes theorem, Gambler's Ruin

Random variables and distributions: discrete vs. continuous, probability (mass) function, cdf, pdf; covered key distributions

Expectation: mean, variance, moments and the moment generating function

Special distributions