Discrete Mathematics: Propositional Logic

Formal Mathematical Notes

1 Continuous vs. Discrete Systems

Definition. Let $\mathcal{S}$ be a system with state space X. Then $\mathcal{S}$ is classified as:

- *Continuous* iff $X \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $|X| = \mathfrak{c}$ (cardinality of continuum)
- *Discrete* iff X is countable, i.e., $|X| \leq \aleph_0$

Examples. Let $\mathcal{S}_1, \mathcal{S}_2$ be systems with the following state transition functions:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(t) = \sin(t) \quad \text{(Continuous system)}$$

$$g : \mathbb{N} \to \{0, 1\}, \quad g(n) = n \bmod 2 \quad \text{(Discrete system)}$$

where $|\mathrm{dom}(f)| = \mathfrak{c}$ and $|\mathrm{dom}(g)| = \aleph_0$

Remark. The distinction between continuous and discrete systems forms a fundamental dichotomy in mathematics:

- Continuous domain: {calculus, analysis} $\subset$ continuous mathematics
- Discrete domain: {logic, graph theory, algorithms} $\subset$ discrete mathematics

2 Propositions and Truth Values

Definition. Let $\mathcal{L}$ be the set of all declarative sentences and let $\mathbb{B} = \{T, F\}$ be the Boolean domain. Define:

$$\mathrm{Val} : \mathcal{L} \rightharpoonup \mathbb{B}$$

where $\rightharpoonup$ denotes a partial function. Then:

- A sentence $p \in \mathcal{L}$ is a *proposition* iff $p \in \mathrm{dom}(\mathrm{Val})$
- The set of all propositions is $\mathcal{P} = \{p \in \mathcal{L} \mid p \in \mathrm{dom}(\mathrm{Val})\}$

Examples. Let $p, q \in \mathcal{P}$ where:

$$\begin{aligned} p &: 2 + 2 = 4 &&\implies \mathrm{Val}(p) = T \\ q &: 5 > 9 &&\implies \mathrm{Val}(q) = F \end{aligned}$$

3 Non-Propositions

Definition. A sentence $s \in \mathcal{L}$ is a *non-proposition* iff $s \notin \mathrm{dom}(\mathrm{Val})$.

The following classes of sentences are non-propositions:

- $\mathcal{Q} = \{s \in \mathcal{L} \mid s \text{ is a question}\}$
- $\mathcal{C} = \{s \in \mathcal{L} \mid s \text{ is a command}\}$
- $\mathcal{O}(X) = \{s(x) \in \mathcal{L} \mid x \in X\}$ where X is a set of variables

Examples.

$$\begin{aligned} &q \in \mathcal{Q} : \text{“What time is it?”} &&\implies \mathrm{Val}(q) \text{ undefined} \\ &c \in \mathcal{C} : \text{“Read this carefully.”} &&\implies \mathrm{Val}(c) \text{ undefined} \\ &o(x) \in \mathcal{O}(\mathbb{R}) : x + 1 = 2 &&\implies \mathrm{Val}(o(x)) \text{ depends on } x \end{aligned}$$

4 Proposition Variables

Definition. Let $\mathcal{V}$ be a countable set of symbols and $\mathcal{P}$ the set of propositions. Then:

- A *proposition variable* is a pair (v, I) where $v \in \mathcal{V}$ and $I : \mathcal{V} \to \mathcal{P}$
- The interpretation function I assigns each variable to a proposition: $\forall v \in \mathcal{V} : I(v) \in \mathcal{P}$

Examples. Let $\mathcal{V} = \{p, q, r, \ldots\}$ and define I as:

$$\begin{aligned} I(p) &= \text{“The sky is blue”} && \implies \mathrm{Val}(I(p)) \in \mathbb{B} \\ I(q) &= \text{“Dhaka is the capital of Bangladesh”} && \implies \mathrm{Val}(I(q)) \in \mathbb{B} \end{aligned}$$

5 Compound Propositions

Definition. The set $\mathcal{C}$ of compound propositions is defined inductively:

1. Base case: If $v \in \mathcal{V}$ then $v \in \mathcal{C}$
2. Inductive cases: If $\alpha, \beta \in \mathcal{C}$ then:

$$\begin{aligned} &\neg\alpha \in \mathcal{C} && \text{(negation)} \\ &(\alpha \wedge \beta) \in \mathcal{C} && \text{(conjunction)} \\ &(\alpha \vee \beta) \in \mathcal{C} && \text{(disjunction)} \\ &(\alpha \to \beta) \in \mathcal{C} && \text{(implication)} \\ &(\alpha \leftrightarrow \beta) \in \mathcal{C} && \text{(biconditional)} \\ &(\alpha \oplus \beta) \in \mathcal{C} && \text{(exclusive or)} \end{aligned}$$

We write this concisely in BNF notation as:

$$\mathcal{C} ::= v \;\Big|\; \neg\mathcal{C} \;\Big|\; (\mathcal{C} \wedge \mathcal{C}) \;\Big|\; (\mathcal{C} \vee \mathcal{C}) \;\Big|\; (\mathcal{C} \to \mathcal{C}) \;\Big|\; (\mathcal{C} \leftrightarrow \mathcal{C}) \;\Big|\; (\mathcal{C} \oplus \mathcal{C})$$

where $v \in \mathcal{V}$.

6 Logical Connectives

Let $p, q \in \{T, F\}$. Define:

$$\neg p = \begin{cases} T & \text{if } p = F, \\ F & \text{if } p = T, \end{cases}$$

$$p \wedge q = \begin{cases} T & \text{if } p = T \text{ and } q = T, \\ F & \text{otherwise}, \end{cases}$$

$$p \vee q = \begin{cases} F & \text{if } p = F \text{ and } q = F, \\ T & \text{otherwise}, \end{cases}$$

$$p \oplus q = \begin{cases} T & \text{if exactly one of } p, q \text{ is } T, \\ F & \text{otherwise}, \end{cases}$$

$$p \to q = \begin{cases} F & \text{if } p = T, q = F, \\ T & \text{otherwise}, \end{cases}$$

$$p \leftrightarrow q = \begin{cases} T & \text{if } p = q, \\ F & \text{otherwise}. \end{cases}$$

7 Negation

Definition. $\neg p$ inverts the truth value of p.

p	$\neg p$
T	F
F	T

8 Logical Operations

9 Conjunction

Definition. The conjunction operator $\wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is defined by:

$$\forall p, q \in \mathbb{B} : p \wedge q = \begin{cases} T & \text{if } p = T \text{ and } q = T \\ F & \text{otherwise} \end{cases}$$

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Properties. For all $p, q, r \in \mathbb{B}$:

- Commutativity: $p \wedge q = q \wedge p$
- Associativity: $(p \wedge q) \wedge r = p \wedge (q \wedge r)$
- Distributivity: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
- Identity: $p \wedge T = p$
- Annihilator: $p \wedge F = F$

10 Disjunction

Definition. The disjunction operator $\vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is defined by:

$$\forall p, q \in \mathbb{B} : p \vee q = \begin{cases} F & \text{if } p = F \text{ and } q = F \\ T & \text{otherwise} \end{cases}$$

Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

11 Exclusive Or (XOR)

Definition. The exclusive or operator $\oplus : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is defined by:

$$\forall p, q \in \mathbb{B} : p \oplus q = \begin{cases} T & \text{if } |\{i \in \{p, q\} : i = T\}| = 1 \\ F & \text{otherwise} \end{cases}$$

Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Equivalence. $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$

12 Implication

Definition. The implication operator $\rightarrow: \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$ is defined by:

$$\forall p, q \in \mathbb{B} : p \rightarrow q = \begin{cases} F & \text{if } (p, q) = (T, F) \\ T & \text{otherwise} \end{cases}$$

Truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalence. $p \rightarrow q \equiv \neg p \vee q$

13 Biconditional

Definition. The biconditional operator $\leftrightarrow: \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$ is defined by:

$$\forall p, q \in \mathbb{B} : p \leftrightarrow q = \begin{cases} T & \text{if } p = q \\ F & \text{otherwise} \end{cases}$$

Truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Equivalence. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

14 Propositional Equivalences

Definition. For any compound proposition p:

- p is a *tautology* iff $\forall v : \mathrm{Val}_v(p) = T$
- p is a *contradiction* iff $\forall v : \mathrm{Val}_v(p) = F$
- p is a *contingency* iff $\exists v_1, v_2 : \mathrm{Val}_{v_1}(p) \neq \mathrm{Val}_{v_2}(p)$

Examples.

$$p \vee \neg p \text{ (tautology)}, \quad p \wedge \neg p \text{ (contradiction)}$$

15 Logical Equivalence

Definition. Propositions p, q are *logically equivalent*, denoted $p \equiv q$, iff:

$$\forall v : \mathrm{Val}_v(p) = \mathrm{Val}_v(q)$$

Examples. Laws of propositional logic:

$$\begin{aligned}
\text{De Morgan:} \quad & \neg(p \vee q) \equiv (\neg p \wedge \neg q) \\
\text{Distribution:} \quad & p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \\
\text{Implication:} \quad & p \to q \equiv \neg p \vee q
\end{aligned}$$

One learns about the reflexive relation, in a certain sense, that;

$$\forall a \in 1, 2, 3. \text{ the relation is;}$$

one then learns about symmetric relation, and transitive ones. One also knows about the ordered pairs. The set examples are better than the most, and gives one the basic ideas of what these things really are. Transitive relations are interesting, and they come in ordered pairs.