Sums of Powers, Packings, Zeta, and All That

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Abstract: [to be written later when I finish]

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Ramanujan's Letter to Hardy

Srinivasa Ramanujan, a the then clerk in the Port Trust Office at Madras during the British Raj, sent a series of letters to *G.H. Hardy*; a Cambridge mathematician, beginning in January 1913. In his first letter, Ramanujan introduced himself to be a very poor and autodidact individual with no formal education in mathematics, but he attached around hundred mathematical results with the letter – many of which are novel and original, deep, and previously unknown to the Western continents. The results include many important infinite series, continued fractions, asymptotic formulas, expressions involving Gamma and Zeta functions, and prime numbers.

Going through the pages, the reaction of Hardy was one of skepticism mixed with intrigue. The letter, filled with many highly original mathematical results, but without proofs, seemed at first glance to be the work of a madman. Hardy was joined by *J. E. Littlewood*, and when they looked closely, they realized that the entities were not only correct, but many were extraordinarily deep and original. Hardy, upon realizing the depth of Ramanujan's results commented; "*A single look at them is enough to show that they could only be written down by a mathematician of the highest class*", further saying, "*They must be true, because, if they were not true, no one would have had the imagination to invent them*". This led Hardy to arrange for Ramanujan to come to Trinity College, Cambridge, beginning a legendary collaboration that would change the landscape of pure mathematics forever.

However, that is another topic for another paper, and one is requested to see "*The Man Who Knew Infinity*" by Kanigel[1] for further reading on Ramanujan's life[2]. For the time being, we focus on the section XI of Ramanujan's letter to Hardy on January 16, 1913. Ramanujan writes in his excellent cursive handwriting;

> **XI.** *I have got theorems on divergent series, theorems to calculate the convergent values corresponding to the divergent series, viz.*
>
> $$1 - 2 + 3 - 4 + \cdots = \frac{1}{4},$$
> $$1 - 1! + 2! - 3! + \cdots = 0.569\ldots,$$
> $$1 + 2 + 3 + 4 + \cdots = \frac{-1}{12},$$
> $$1^3 + 2^3 + 3^3 + 4^3 + \cdots = \frac{1}{120}.$$
>
> *Theorems to calculate such values for any given series (say* $1-1^1+2^2-3^3+4^4-5^5+\cdots$*), and the meaning of such values. I have also dealt with such questions: "When to use, where to use, and how to use such values? Where do they fail and where do they not?"*

We focus only on this part, and right at the heart of section XI, lies the famous and one of the most misunderstood expressions (referring to the memes and other cringe cultural phenomenons across the internet) in all mathematics;

$$1 + 2 + 3 + \cdots = -\frac{1}{12} \quad (\mathfrak{R})$$

where one notes that we write $\mathfrak{R}$, which denotes *Ramanujan Summation*, which in terms mean that there is a mathematically bona fide reason for why the sum of natural numbers add up to the negative fraction on the right, something we will be diving deep later as we proceed with our discussion.

[1] a biographical novel by Robert Kanigel on the life of Srinivasa Ramanujan.
[2] see Stephen Wolfram's blog Who Was Ramanujan?

Faulhaber's Formulae

It is very much expected that an average high school student will encounter, or will be introduced with the concept of *Partial Summation* around his 8th or 9th grade even if one does not take calculus. At the very least one will learn how to calculate which is very commonly known as *sum of natural numbers*, and consecutively the *sums of natural numbers raised to a certain integer power*. The means through which the author encountered the entity was through a table that looked like this;

Necessary Formulae:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2},$$
$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6},$$
$$1^3+2^3+3^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2,$$

N.B: $$1^3+2^3+3^3+\cdots+n^3=(1+2+3+\cdots+n)^2.$$

which, of course, as a high school textbook of 9th grade, had no explanations and derivations of where these actually came from. There were some numerical approaches to formulate the formulas which were to show how the series worked, but not any geometric ones that would paint an intuitive picture of the proof.

One recalls from *Ramanujan's letter* that at section **XI** Ramanujan had two of the above entities listed, but with astonishingly absurd values (to a high school student), which one can try to write side by side to compare just for the sake of curiosity;

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}, \quad \& \quad 1+2+3+\cdots=-\frac{1}{12} \quad (\mathfrak{R})$$
$$1^3+2^3+3^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2, \quad \& \quad 1^3+2^3+3^3+4^3+\cdots=\frac{1}{120} \quad (\mathfrak{R}).$$

The catch is not that hard to find, once someone finally notices that the left hand side; LHS for the terms from the table is finite, whereas Ramanujan's expressions are infinite. Hence the sums on the left are called *partial sums*. But even though two of these convey entirely different things, the Ramanujan sums being very alien, they seem eerily connected. Though one cannot see it right away. One takes the first one from Faulhaber's table, and tries to observe;

$$1+2+3+\cdots+n=\frac{n(n+1)}{2};$$

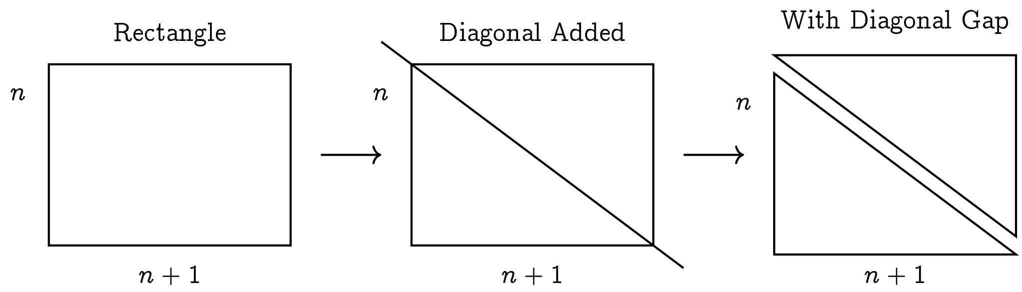
One sees that $n \in \mathbb{N}$, and the LHS represents the sums of all natural numbers, but to a finite extent. One has no doubt that such is an arithmetic series, and has a common difference of $d=1$. One can know it in the form of:

$$S_n=\frac{n}{2}(2a+(n-1)\cdot d);$$

However, the series $1+2+3+\cdots$ is a divergent series as $n\to\infty$. In classical sense, the series grows without bound as more and more terms are added. Such much should be obvious from what is in front of one, and when one looks at the RHS, one sees a shape that resembles a triangle, or the *area* of a general triangle.

$$\frac{n(n+1)}{2}=\frac{1}{2}\cdot n\cdot(n+1)$$

One sees that such is the shape of a rectangle of sides n & $n+1$, which one is halving to get an area through the product of its base and height.



Since we have a rectangle, and we are halving it to get an area that looks like a triangle. For the area, one can say that the shape does not matter, because it is just *area*, but for geometric intuition, it makes sense to go along the diagonal of the rectangle. Because of such, we have an idea that what we have on hands is a right angle triangle. Even if one tries, there are doubts that it can be anything else, because we have a rectangle, and rectangles have 4 right angles on their 4 edges.

Kepler's Conjecture & Circle Packing

It is very much curious to see a shape that very much resembles a triangle. One also asks the question that why there would be a very specific type of triangle; meaning a triangle that has a right angle and three unequal sides. After all, anyone can see such structure in the grocery market [3]. One will have;

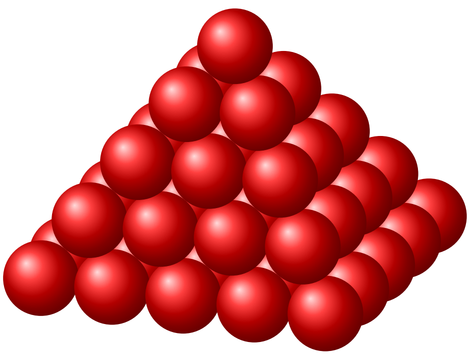


Fig: Recreation of figure E in *Strena Seu de Nive Sexangula*, illustrating the Kepler conjecture.

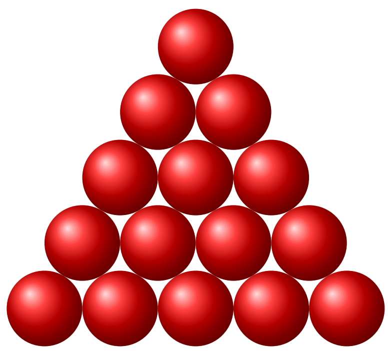


Fig: triangular side view of the square based pyramid.

One observes that what the orange stacks have is a square pyramid shape. Consecutively one notices that if one starts from the top, then;

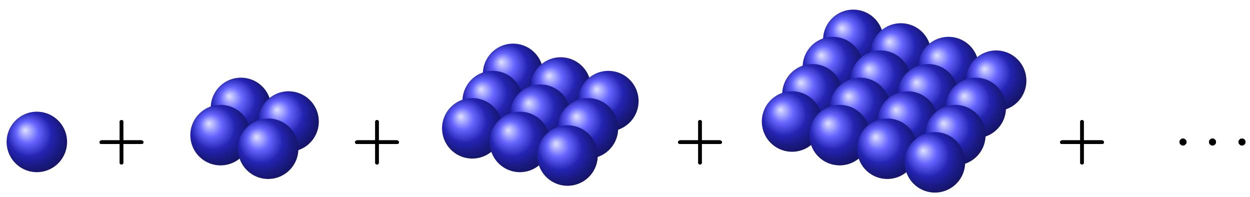


Fig: A Geometric construction of the Pyramid stack as sums spheres (1, 4, 9, 16, and so on).

is a geometric shape that looks like a pyramid that stacks up as 1 spheres + 4 spheres + 9 spheres + 16 spheres + $\cdots$, which can be written as $1^2+2^2+3^2+4^2+\cdots$. Let one observe what is happening here;

One has a pyramid like shape, and if, through our vivid imaginal and practical physical means, break it down each as a layer by layer summation of the spheres, one notices that it follows a square number pattern of the natural numbers $\mathbb{N} : \mathbb{N}\backslash 0$, which is mapped to $\mathbb{N}^+$, since it is just the squares of the whole set.

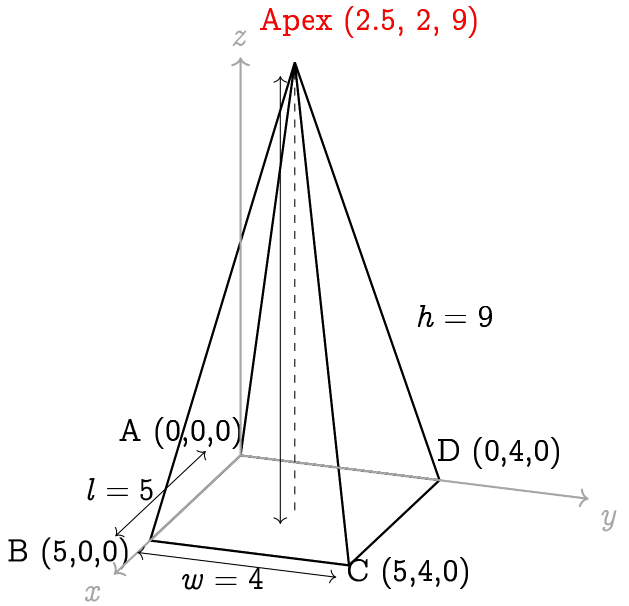
One recalls what one has seen from the previous section that;

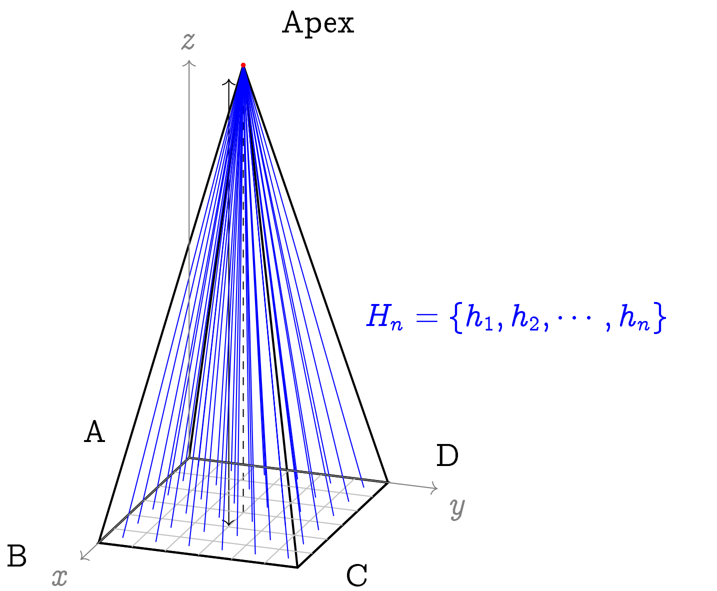
$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6};\ (\Sigma n^2 \text{ for finite terms})$$

[3] *personal experience of the author, see Mathematics, and Higher Mathematics at NCTB.*

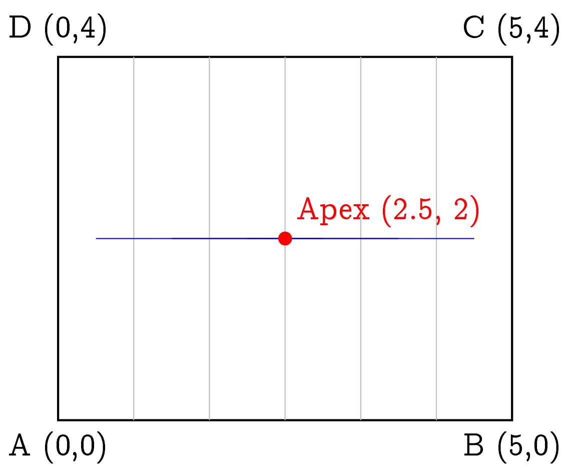
Now let us look at the exponents of the LHS, and then the dimensions of the RHS (keeping n as variable). One sees that exponent is 2 and the dimension on the right is 3. Just like the rectangular shape of the first series, one investigates the geometry of the RHS.

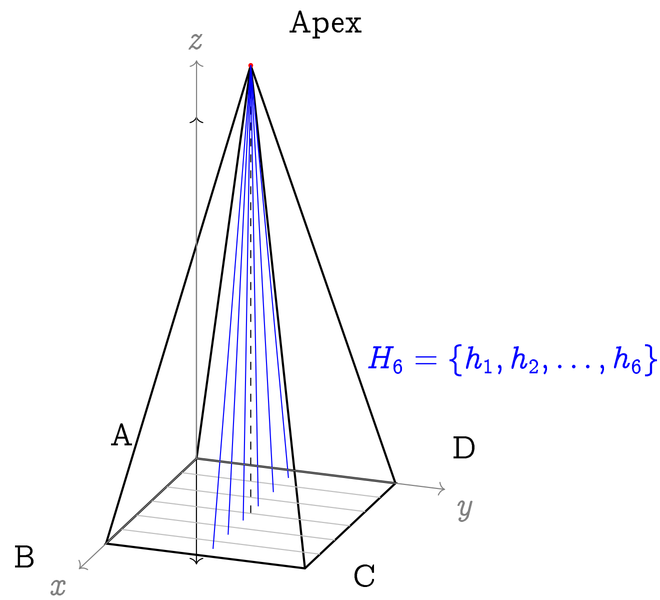
Meaning, what is this shape actually in the algebraic expression? If one were to look at it, one sees, or finds three dimensions. Such means that it is a 3D shape. We took a rectangle before, let us take such as a base, and height $(2x+1)$. Let us observed a scaled down version of the pyramid at $x=4$.





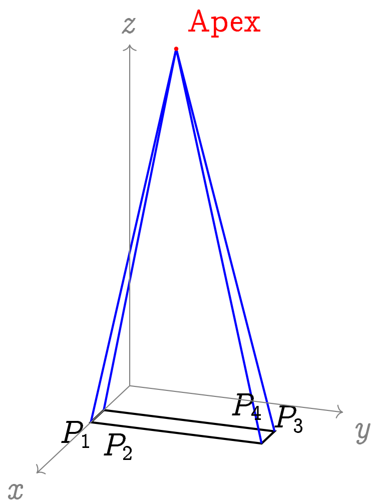
One looks at the one at the left, and then on the right. For the left one, we have a pyramid, and the coordinates are shown. For our purposes, meaning, trying to visuallize what the shape $\frac{x(x+1)(2x+1)}{6}$ looks like.





One constructed a pyramid from $x(x+1)(2x+1)$, and then divides it into 6 equal parts. To visualize, one has the right image, where one notices that for n equal divisions, one has a variable length of h along the *square* grid (as approximation, 36 rectangles making up the base). One also observes that, one cannot divide the base rectangle into equal squares. And thus, one takes the scaled down copy of the rectangle for square $n \times n$ grid. 6 is not a square number, so linearly, along the axis of height at a certain length direction, one slices the rectangle into equal areas, and then draws triangular height (2D) from the Apex. [4]

[4] One notices that, linearly the $V \to H$, and as one shrinks the volume, it approaches zero. Such makes an infinite collection of needle like shapes that eventually reaches 0 in volume, and the whole collection of that approaching zero volume makes up the pyramid.



This diagram shows the **3rd vertical slice** of a rectangular-based pyramid, which has a Base: 5×4 (length $\times$ width), Height: 9, Apex: $(2.5,\ 2,\ 9)$, Slice range: $x = \frac{10}{6}$ to $x = \frac{15}{6}$. Each corner of the slice is connected to the top apex, forming a trapezoidal wedge. Together, all 6 slices reconstruct the full pyramid.
This indeed, is the trapezoidal structure which we are after. It is a visual representation of $\frac{x(x+1)(2x+1)}{6}$. Now what does this tell us one asks, and sees that such shape, meaning the trapezoidal slice is not a pyramid of the size we thought we would have. Because we desired a square base pyramid, for which (which one could try actually, because this is a simple geometric proof), but ended up with a shape that does nor resemble our initial pattern. *WHY* such?

One goes back to the equilateral triangle, and the square based pyramid of the original Kepler's conjecture. Recalling that it was a square pyramid, one can know that the volume, at most occupied by the pyramid is;

$$V_P = \frac{a^3}{3\sqrt{2}}$$

where a is the side length of the square base (and also the side of the equilateral triangles).

One can also look at the *slice*, and calculate its area. Each vertical slice (like our diagram) is a trapezoidal wedge, formed between two vertical planes at $x = x_1$, and $x = x_2$ bounded above by the apex. To compute its volume, one can treat it as a frustum of a pyramid section (with trapezoidal base), and compute by subtracting the volume of two smaller pyramids or via integral. But since we already divided the full base into 6 equal vertical slices (in the x-direction), we can approximate.

Let the pyramid have base side length: L (square base $L \times L$), height: H, then the volume of the full pyramid is:

$$V = \frac{1}{3} \cdot L^2 \cdot H$$

One divides the base into n vertical slices along the x-axis. Each slice spans:

$$\Delta x = \frac{L}{n}$$

The i-th slice spans:

$$x_i = \frac{(i-1)L}{n}, \quad x_{i+1} = \frac{iL}{n}$$

At a distance $x \in [0, L]$, the square cross-section's side shrinks linearly:

$$\text{Side length at } x = L\left(1 - \frac{x}{L}\right) = L - x$$

So one has area at x is:

$$A(x) = (L-x)^2 = L^2\left(1 - \frac{x}{L}\right)^2$$

One integrates to find the exact volume:

$$V_i = \int_{x_i}^{x_{i+1}} A(x) \cdot \frac{H}{L}\, dx = HL \int_{x_i}^{x_{i+1}} \left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right) dx$$

Evaluating:

$$V_i = HL\left[x - \frac{x^2}{L} + \frac{x^3}{3L^2}\right]_{x_i}^{x_{i+1}}$$

One has general form;

$$\int_a^b f(x)dx \approx \Delta x \sum_{i=1}^{n} f(x_i)$$

In this case, one assumes *cross-sectional areas* $A(x_i)$ multiplied by the width of the slice Δx to get approximate volume. One then has;

$$V \approx \Delta x \sum_{i=1}^{n} f(x_i)$$

One has $\Delta x = \frac{l}{n}$ which is the width of the slice, $A(x_i) = L^2(1 - \frac{i}{n})^2$ being the area of the cross section at x_i. We approximate the volume using a Riemann sum:

$$V \approx \Delta x \sum_{i=1}^{n} A(x_i) = \frac{L}{n} \sum_{i=1}^{n} L^2 \left(1 - \frac{i}{n}\right)^2$$

One takes the constants out:

$$V \approx \frac{L^3}{n} \sum_{i=1}^{n} \left(1 - \frac{i}{n}\right)^2$$

Expanding the square:

$$\left(1 - \frac{i}{n}\right)^2 = 1 - \frac{2i}{n} + \frac{i^2}{n^2}$$

Now:

$$V \approx \frac{L^3}{n} \left[\sum_{i=1}^{n} 1 - \frac{2}{n} \sum_{i=1}^{n} i + \frac{1}{n^2} \sum_{i=1}^{n} i^2\right]$$

Using summation formulas:

$$\sum_{i=1}^{n} 1 = n, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

One then substitutes:

$$V \approx \frac{L^3}{n} \left[n - \frac{2}{n} \cdot \frac{n(n+1)}{2} + \frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

Simplifying:

$$V \approx \frac{L^3}{n} \left[n - (n+1) + \frac{(n+1)(2n+1)}{6n}\right]$$

$$V \approx \frac{L^3}{n} \cdot \left[\frac{(n+1)(2n+1) - 6n}{6n}\right]$$

$$\begin{aligned} V &\approx \frac{L^3}{n} \cdot \frac{2n^2 - 3n + 1}{6n} \\ &= \frac{L^3(2n^2 - 3n + 1)}{6n^2} \end{aligned}$$

Taking the limit as $n \to \infty$:

$$\lim_{n \to \infty} \frac{L^3(2n^2 - 3n + 1)}{6n^2} = \frac{L^3}{3}$$

These are arbitrary ways of saying that it cam be written in these and those forms, but ultimately one notices that the dimension is $[L^3]$, and such stayed the same. But for our purposes; let $x_i = \frac{iL}{n}$, then:

$$A(x_i) = L^2 \left(\frac{n-i}{n}\right)^2$$

So:

$$V \approx \frac{L^3}{n^2} \sum_{j=0}^{n-1} j^2 = \frac{L^3}{n^2} \cdot \frac{(n-1)n(2n-1)}{6}$$

Now apply pyramid height factor:

$$V = \frac{HL^2}{n^2} \cdot \frac{(n-1)n(2n-1)}{6}$$

The volume of the first k vertical slices is:

$$V_k = \frac{HL^2}{n^2} \cdot \sum_{i=1}^{k} (n-i)^2$$

This gives:

$$V_k = \frac{HL^2}{n^2} \left[\frac{(n-1)n(2n-1)}{6} - \frac{(n-k-1)(n-k)(2n-2k-1)}{6} \right]$$

And each slice i has volume:

$$V_i = \frac{HL^2}{n} \left(1 - \frac{2i-1}{n} + \frac{(2i-1)^2}{4n^2} \right)$$

As example; given $L = 6, H = 9, n = 6, i = 3$, one computes

$$V_3 = \frac{9 \cdot 36}{6} \left(1 - \frac{5}{6} + \frac{25}{144} \right) = 54 \cdot \left(\frac{1}{6} + \frac{25}{144} \right) = 54 \cdot \frac{49}{144} = \frac{2646}{144} = \frac{49}{16} \approx 3.0625$$

The full volume of the wedge is just the same formula for $i = 1$ to 6, and the sum of all slice volumes. One has;

$$\sum_{i=1}^{6} V_i = \frac{1}{3} \cdot 36 \cdot 9 = 108$$

One can see that the pyramid can also be divided into a 2×3 rectangles, and since the area stays the same; no matter the divisions and the sizes of area of the base, the formula might be different to calculate the slices or divisions. And one also visually understands that the area stays the same, and what one has been doing thinking about pyramids being of such a similar scaled or sliced up shape. Such is not very hard to prove, even in terms of simple algebra and geometry. Calculus could be speeding up the process, but such is not a mandatory to such a mean.

One can see H as x, particularly since the pyramid shrinks linearly as $x \to L$. In the integral one can see;

$$dV = A(x) \cdot \frac{H}{L} \text{ with } A(x) = (L-x)^2$$

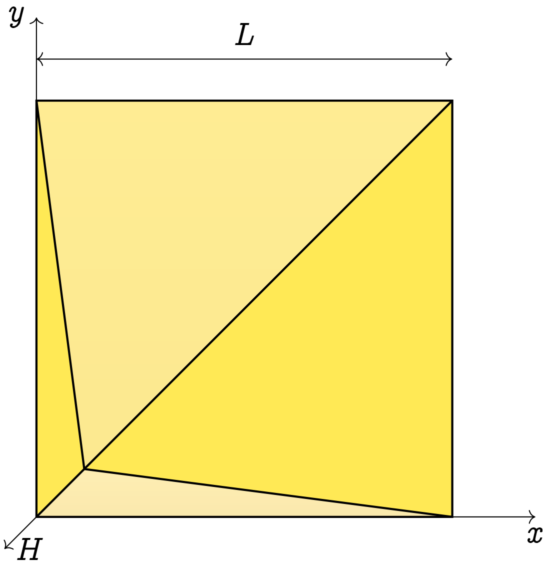
The factor $\frac{H}{L}$ appears because the vertical height is spread across the horizontal base from 0 to L. One has then;

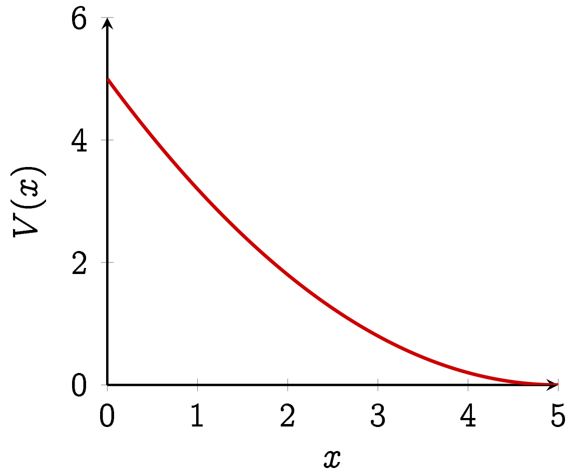
$$h(x) = \frac{H}{L} x \Rightarrow V(x) = \frac{Lh(x)}{nx} (L-x)^2$$

Geometrically, the shape;

$$\frac{H}{n} (L-x)^2 \text{ has dimension } [L] \cdot [L^2] = [L^3].$$

which interprets itself as a trapezoidal wedge, or frustum as x increases and side length shrinks. The entire pyramid is built stacking these varying square slabs from $x = 0$ to $x = L$. One sees a square pyramid, and each $V(x)$ is a some constant voluminous slab with side length $(L-x)$ and height $\frac{H}{L} \cdot \Delta x$.





The volume per slice decreases quadratically as one moves along x. The rectangular, or equal areas shrinks linearly if the base is taken into account. Now what of the differences of V_P, and the slice we have on hour hand?

One has;

$$V_P \,\Delta\, V(x)$$

meaning the difference (using the Δ sign to note the difference) of these two forms to observe. Is such the reason we are having a difference? [5] Such turns out to be;

$$V_P \,\Delta\, V(x) = \frac{x^3}{3\sqrt{2}} \,\Delta\, \frac{Lh(x)}{nx}(L-x)^2$$

One has L as a fixed length (e.g., the total length of a base), n is a positive integer (e.g., number of divisions or slices), $h(x)$ is a height function depending on x, and x is the position along the length L. We extract all constant terms (in terms of dimensional or scalar factors); and such gives one;

$$V_P \,\Delta\, V(x) \approx x^3 \,\Delta\, \frac{h(x)}{x}(L-x)^2$$

To examine the pure structural or geometric behavior, the function is reduced, and one notices a **Special Case:** If $h(x) = x$, then:

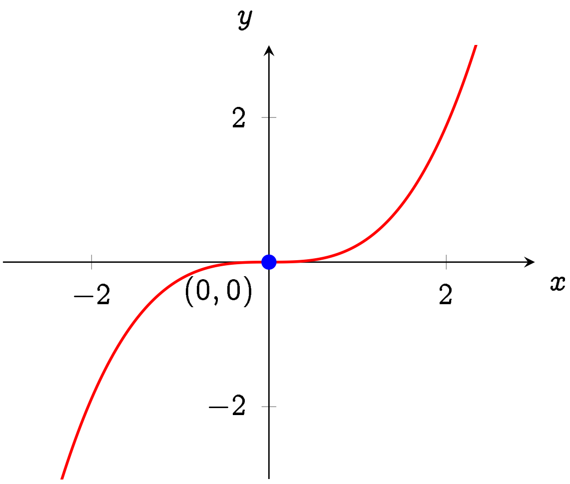
$$V_{\text{reduced}}(x) = \frac{x}{x}(L-x)^2 = (L-x)^2,$$

which represents a parabolic volume decay; e.g., a vertically tapering wedge. One also has dimensional consistency. Both functions, i.e. $\frac{x^3}{3\sqrt{2}}$ and $V_{\text{reduced}}(x)$ have dimension of volume $[L^3]$.

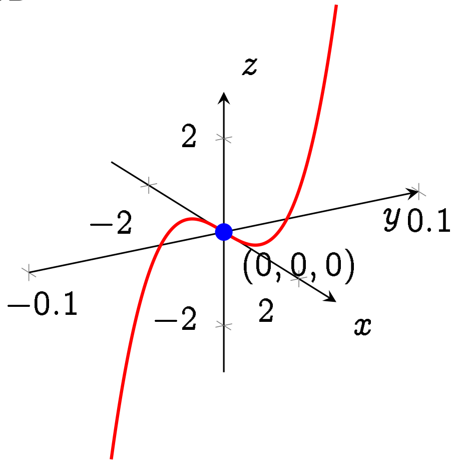
One now takes a basic algebraic approach, to see what it looks like in the Cartesian plane;

[5] One asks, "*what does the difference* $V_P \,\Delta\, V(x)$mean?". The author is armature, and does not know what such could mean mathematically. Just what does this shape represent? And what does it even mean for higher dimensions? One asks that one starts with a square based pyramid with equilateral sides that represent a cannonball structure, but we end up with a wedge (keeping the pyramid shape intact, but the area stays the same even if one does not preserve the shape.) Why such? Or does this difference means something in what shape we acquire from it? Because we do not see what the initial shape predicted in terms of similarity.

2D



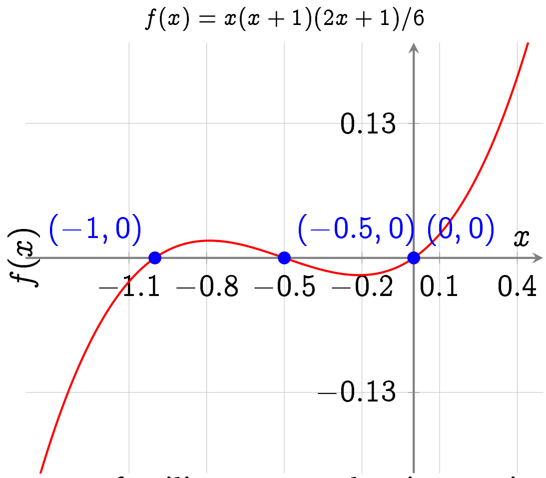
3D

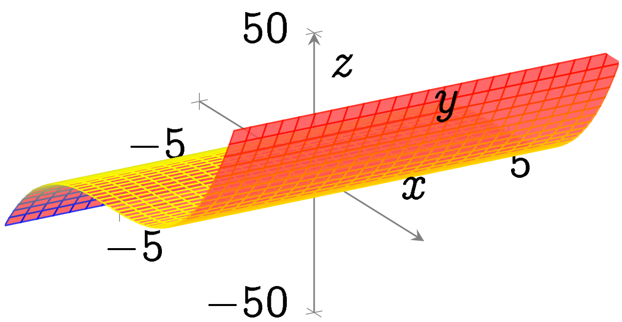


The function $f(x) = \frac{x^3}{3\sqrt{2}}$ is just a cubic function with a constant factor multiplied to it, and as expected of a simple cubic, it passes through the origin[6]. One then takes the latter function

$$f(x) = \frac{x(x+1)(2x+1)}{6}, \text{ for all } x \in \mathbb{R}$$

and observes the graph of it to see how it looks compared to the simple cubic one has. One has a constant attached to it, meaning the factor $1/6$, which in general terms might not matter, but we see it anyway.





One sees a familiar s-curve that is associated with cubics. One sees from the equation that it goes through $x = -1, -1/2$, and 0. But for the cubic term prior has none other than the origin to go through, now, not only it goes through three points, but also it is an s-curve that occupies some area under and up the axis line. The curve seems to have mirror symmetry at point $-1/2$, and the area on both sides of the points happen to be the same.

Here also one sees the difference in coordinates, showing the difference factor $V_P \,\Delta\, V(x)$. Why does only the square right pyramid correspond with the shape that is optimal for sphere packing in 3D? Why the other does not? And what does the factor mean? Could this be the reason the configuration is different? Or is it not optimally possible to pack it like such a shape which it corresponds to in terms of dimensions?

An Auxilary Observation: We somewhat see the geometric construction where a right rectangular pyramid is approximated by continuously subdividing into an $n \times n$, or rectangular grids, and connecting each subdivisions center to a fixed apex, forming sliced pyramid like shapes, and then ultimately which is reduced to needle like lines that approaches the height of the divisions itself.

One sees that although the sub-pyramids has non-zero volumes, *the total volume of their union is zero as* $n \to \infty$. Meanwhile, the actual pyramids volume stays arbitrarily constant, and

[6] The shape x^3 represents a cube, and the factor contributes in making it a pyramid in terms of volume, sometimes preserving the shape.

can be computed by integral. Such a discrepancy reveals a dimensional collapse and emphasizes the subtle difference between naive radial approximations and proper volumetric measures.
Let $\ell, w, H > 0$ be fixed and $\in \mathbb{R}$ representing the length, width, and height of a right rectangular pyramid $\mathcal{P} \in \mathbb{R}^3$. The base lies in the xy-plane:

$$B := [0, \ell] \times [0, w].$$

One has the apex is at the point

$$A := \left(\frac{\ell}{2}, \frac{w}{2}, H\right).$$

Then the pyramid is the convex hull of the base and apex:

$$\mathcal{P} := \mathrm{Conv}(B \times \{0\} \cup \{A\}).$$

One partitions B into $n \times n$ congruent squares, or rectangular shapes of $n \times (n+1)$ each of area;

$$\Delta A := \frac{\ell w}{n^2}.$$

For indices $i, j = 0, \ldots, n-1$, let the center of the (i, j)-th subrectangle be;

$$(x_i, y_j) := \left(\left(i + \frac{1}{2}\right)\frac{\ell}{n}, \left(j + \frac{1}{2}\right)\frac{w}{n}\right).$$

For each (x_i, y_j), define the *mini-pyramid*;

$$\mathcal{P}_{ij} := \mathrm{Conv}(R_{ij} \cup \{A\}),$$

where one has R_{ij} as the (i, j)-th subrectangle. The slant height of $\mathcal{P}_{ij}$ is the Euclidean distance from $(x_i, y_j, 0)$ to A;

$$h_{ij} \text{ is defined as: } \sqrt{(x_i - \frac{\ell}{2})^2 + (y_j - \frac{w}{2})^2 + H^2}.$$

The volume of each mini-pyramid is;

$$V_{ij} = \frac{1}{3}\Delta A \cdot h_{ij}.$$

The *approximate total volume* by summation is;

$$V_n := \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} V_{ij} = \frac{1}{3}\frac{\ell w}{n^2}\sum_{i,j} h_{ij}.$$

One defines the union of all mini-pyramids for fixed n;

$$\mathcal{P}_n := \bigcup_{i,j} \mathcal{P}_{ij}.$$

Definition: Define;

$$\mathcal{C} := \left\{(1-\lambda)(x, y, 0) + \lambda\left(\frac{\ell}{2}, \frac{w}{2}, H\right) : (x, y) \in B, \lambda \in [0, 1]\right\}.$$

This is the conical fan formed by rays connecting every base point to the apex. According to the definition, $B = [0, \ell] \times [0, w]$ is the rectangular base in the xy-plane and the apex is located at the point $(\frac{\ell}{2}, \frac{w}{2}, H)$. This set describes the union of all line segments from every point (x, y) on the base to the apex. Each fixed base point $(x, y) \in B$ defines a unique line segment to the apex. Varying $\lambda \in [0, 1]$ traces that segment;

$$\text{For fixed } (x, y): \quad \lambda \mapsto (1-\lambda)(x, y, 0) + \lambda\left(\frac{\ell}{2}, \frac{w}{2}, H\right).$$

Then, $\mathcal{C}$ is the union of all such segments over all $(x, y) \in B$. Such forms a "conical fan" or "skeleton" of the full pyramid. Although $\mathcal{C}$ spans the entire region of the pyramid visually, it only consists of 1-dimensional needle like very line segment like lines in $\mathbb{R}^3$. Such does *not* fill in any area or volume between the lines.

One has each to be a 1-Dimensional line segments, or very thin. The union of uncountable number of many 1D objects can remain measure-zero in higher-dimensional space if they do not cover any 2D or 3D regions, or contribute to the corresponding next dimension of the geometric object. Such is a somewhat visually obvious observation, but the process that happens during such is interesting. One has an example of such volume collapse, or in a special case, a **parabolic volume decay**. Also, $\mathcal{C}$ lies within the full pyramid but does not constitute its interior. Therefore, the set $\mathcal{C}$ has **Lebesgue measure zero** in $\mathbb{R}^3$. That is;

$$\mathrm{Vol}_3(\mathcal{C}) = 0.$$

Analogically, imagine stretching threads from every point of a flat square (the base) to a single point above it (the apex). The resulting web of strings gives you the outline or skeleton of a tent but not the fabric that encloses the space. The total "thread" structure spans the whole region, but it has no bulk; meaning no volume.

One concludes that $\mathcal{C}$ is a set of line segments from the base to the apex. Such shape is geometrically dense but contains no 3D interior: $\Rightarrow$ Its 3D Lebesgue measure is zero: $\mathrm{Vol}_3(\mathcal{C}) = 0$.

Theorem: Dimensional Collapse / Volume Degeneration: The volume of the union of mini-pyramids degenerates:

$$\lim_{n\to\infty} \mathrm{Vol}(\mathcal{P}_n) = 0.$$

Proof: Each V_{ij} has magnitude on the order of $1/n^2$ and there are n^2 such mini-pyramids. At first glance, one might expect V_n to remain bounded.

However, the geometric shape of each $\mathcal{P}_{ij}$ collapses to a needle-like shape: their cross-sectional area shrinks as n increases since only the centers of the subrectangles connect to the apex, leaving gaps. Therefore, the union $\mathcal{P}_n$ does not fill $\mathcal{P}$. Instead,

$$\mathcal{P}_n \xrightarrow{n\to\infty} \mathcal{C},$$

which has measure zero. Hence,

$$\lim_{n\to\infty} \mathrm{Vol}(\mathcal{P}_n) = \mathrm{Vol}(\mathcal{C}) = 0.$$

This illustrates a *dimensional collapse* where 3D volumetric solids converge to a 1D set in $\mathbb{R}^3$, causing null-ness of volume. One can compute the volume of $\mathcal{P}$ by integrating the vertical height function over the base. One needs a function that linearly interpolates the height from the apex at $(\frac{\ell}{2}, \frac{w}{2})$ to zero on the edges of the base. The volume is given by the double integral;

$$V = \iint_B f(x, y)\, dx\, dy.$$

In the case of a right rectangular pyramid with base dimensions $\ell \times w$ and apex at height H located directly above the center of the base, one might ask whether the volume may be computed as a double integral over the base, using a height function $f(x, y)$.

One sees for a solid defined by a height function $f(x, y)$ over a planar domain $D \subset \mathbb{R}^2$, the volume is given by;

$$V = \iint_D f(x, y)\, dx\, dy$$

Thus, to compute the volume of the pyramid in this way, one must define a function $f(x, y)$ that correctly captures the geometry of the upper surface of the pyramid.

Attempt 1; Tent Function over the Center: One has the function;

$$f(x, y) := H\left(1 - \frac{|x - \ell/2|}{\ell/2}\right)\left(1 - \frac{|y - w/2|}{w/2}\right)$$

This function reaches maximum value H at the center $\left(\frac{\ell}{2}, \frac{w}{2}\right)$, and decays linearly to zero along all four edges. It is symmetric and continuous, but it does not represent the actual geometry of a pyramid we are looking for; the surface formed is a bilinear tent-like cap, not four planar triangular faces. The integral yields;

$$V = \iint_{[0,\ell]\times[0,w]} f(x, y)\, dx\, dy = \frac{1}{4}\ell w H$$

which is incorrect for a pyramid, but correct for the volume under this particular tent function.

Attempt 2: Slanted Surface from a Corner

$$f(x, y) := H\left(1 - \frac{x}{\ell}\right)\left(1 - \frac{y}{w}\right)$$

This function peaks at $(0, 0)$ and decreases linearly toward $(x = \ell, y = w)$. It describes a planar surface slanted from one corner, and again fails to match the true geometry of the pyramid whose apex lies at the center.

The computed volume is again:

$$V = \iint_{[0,\ell]\times[0,w]} f(x, y)\, dx\, dy = \frac{1}{4}\ell w H$$

matching the previous result, but still incorrect for a centered apex pyramid.

Obstruction: In both cases above, the height functions are separable and smooth, but they do not correspond to the true planar surfaces of a right rectangular pyramid with center-apex. A true pyramid has four distinct planar triangular faces, each defined over a subregion of the base, and cannot be captured by a single-valued function $z = f(x, y)$ over the entire base domain.

This is a geometric obstruction: the pyramid is not a graph over the base rectangle in the sense required by a single-valued function $f(x, y)$.

Resolution via Vertical Slicing: To circumvent this issue, one proceeds differently: by slicing the solid horizontally. At each height $z \in [0, H]$, the cross-sectional area is a rectangle whose dimensions scale linearly with z:

$$\ell(z) = \ell\left(1 - \frac{z}{H}\right), \qquad w(z) = w\left(1 - \frac{z}{H}\right)$$

$$A(z) = \ell(z) \cdot w(z) = \ell w\left(1 - \frac{z}{H}\right)^2$$

Thus, the volume is computed via the single integral:

$$V = \int_0^H A(z)\, dz = \ell w \int_0^H \left(1 - \frac{z}{H}\right)^2 dz$$

Using substitution $u = \frac{z}{H}$, $dz = H\, du$, we obtain:

$$V = \ell w H \int_0^1 (1 - u)^2\, du = \ell w H \cdot \frac{1}{3} = \boxed{\frac{1}{3}\ell w H}$$

□

Through the failure of the double integral approach, one knows such doesn not work, and the reason lies in the fact that a right rectangular pyramid with a center-apex cannot be represented by a single-valued function over the rectangular base. The vertical slicing method succeeds because it reduces the 3D geometry to a 1D integral over the height, where the cross-sectional area varies quadratically and captures the pyramid's geometry exactly. One is preferred over the other in this case, and A pyramid with planar triangular sides cannot be described by a single-valued function $f(x, y)$ over a rectangle, unless the apex lies directly above one corner of the base. We did manage to acquire another result, a trivially interesting one, that $V \to H$, which ultimately reaches null.

Parabolic Volume Decay and Dimensional Collapse: One lets $A(x) = (L-x)^2$ be the cross-sectional area function of a pyramid, which decreases quadratically as $x \to L$. We interpret this as a *parabolic decay* of volume contributions near the apex. The volume of a horizontal slab of thickness Δx at position $x \in [0, L]$ is given by;

$$V(x) = A(x) \cdot h(x) = (L-x)^2 \cdot \frac{H}{L}\Delta x.$$

As $x \to L$, we have;

$$V(x) \sim \frac{H}{L}(L-x)^2 \Delta x.$$

This decay is quadratic in the distance to the apex, hence the term *parabolic*. In particular, for a uniform partition with $x_k = k\frac{L}{n}$, the slab volume becomes:

$$V_k = \frac{H}{L}(L-x_k)^2 \Delta x = \frac{HL}{n}\left(1 - \frac{k}{n}\right)^2.$$

Asymptotic Decay Near Tips: One can consider the tips to be the last point of the geometric object, meaning it just has cartesian coordinates. Such is just a point in $\mathbb{R}^3$ space. It does not have any volume. For $k = n-1$, the final slab satisfies;

$$V_{n-1} = \frac{HL}{n^3},$$

demonstrating that the volume decays on the order of $O(n^{-3})$ near the apex. The parabolic decay ensures, despite summing n such slabs, the union of mini-pyramids (formed by radial connections to the apex) converges to a degenerate conical fan in the limit $n \to \infty$, with total volume zero.

Another Auxiliary Observation: One lets $M \subset \mathbb{R}^n$ be a smooth k-dimensional embedded submanifold. Then, for every point $p \in M$, there exists a smooth function $F : U \to \mathbb{R}^{n-k}$, defined on an open neighborhood $U \subset \mathbb{R}^n$, such that;

$$M \cap U = F^{-1}(0) \quad \text{and} \quad \operatorname{rank}(DF(x)) = n-k \quad \forall x \in M \cap U.$$

Conversely, if $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a smooth function and $0 \in \mathbb{R}^{n-k}$ is a **regular value**, then the level set $M = F^{-1}(0) \subset \mathbb{R}^n$ is a smooth embedded submanifold of dimension k.

One lets $M \subset \mathbb{R}^n$ be a smooth k-dimensional embedded submanifold. Then for every $p \in M$, there exists; an open neighborhood $U \subset \mathbb{R}^n$ of p, a smooth coordinate chart $\psi : U \to \mathbb{R}^n$ with $\psi(p) = 0$, such that $\psi(M \cap U) = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$. One defines the projection $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ by;

$$\pi(x, y) = y.$$

Then one defines the function;

$$F := \pi \circ \psi : U \to \mathbb{R}^{n-k}.$$

One has;

$$F(x) = 0 \iff \psi(x) \in \mathbb{R}^k \times \{0\} \iff x \in M \cap U.$$

So, $M \cap U = F^{-1}(0)$, and since ψ is a diffeomorphism and π has constant full rank, the Jacobian matrix $DF(x)$ has rank $n-k$. One then supposes $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is smooth and $0 \in \mathbb{R}^{n-k}$ is a **regular value**. This means;

$$\forall x \in F^{-1}(0), \quad \operatorname{rank}(DF(x)) = n-k.$$

Then by the **Regular Level Set Theorem**, the set;

$$M = F^{-1}(0)$$

is a smooth embedded submanifold of dimension $n - (n-k) = k$. Thus, the geometric object is locally defined by the zero set of a smooth function with full-rank differential.

Regularity Condition (Key Hypothesis)

Definition (Regular Value): A point $y \in \mathbb{R}^{n-k}$ is a *regular value* of a smooth function $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ if for all $x \in F^{-1}(y)$, the Jacobian matrix $DF(x)$ has full rank $n - k$. One has regularity requirement:

$$\boxed{\text{For all } x \in F^{-1}(0), \quad \operatorname{rank} DF(x) = n - k.}$$

Under this condition, the level set $F^{-1}(0)$ is a smooth k-dimensional submanifold of $\mathbb{R}^n$.

Example; The Circle as a Level Set: Let, $F(x, y) = x^2 + y^2 - 1$. Then the zero set is the unit circle;

$$F^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

This is a 1-dimensional smooth submanifold of $\mathbb{R}^2$. The gradient is:

$$\nabla F(x, y) = (2x, 2y),$$

which is non-zero on the entire circle, hence the Jacobian DF has full rank 1 everywhere on the manifold. This two-way theorem proves that; Every smooth geometric object (submanifold) can locally be represented as the level set of a smooth function, and every regular level set of a smooth function is a smooth geometric object. Thus, there exists a fundamental duality between function forms and geometric objects, provided the function satisfies a regularity condition (full rank Jacobian at every point on the level set).

Theorem (Equivalence/Duality):

Let $M \subset \mathbb{R}^n$ be a smooth embedded k-dimensional submanifold.

smooth geometric object (manifold):

$$M \subset \mathbb{R}^n \text{ of dimension } k$$

$$\Updownarrow$$

function from:

$$\exists \text{ open } U \subset \mathbb{R}^n, \text{ and a smooth function } F : U \to \mathbb{R}^{n-k}$$

$$\text{such that } M \cap U = F^{-1}(0), \quad \text{and } \operatorname{rank} DF(x) = n - k \quad \forall x \in M \cap U$$

This equivalence holds under the **regularity condition** that the Jacobian $DF(x)$ has full rank $n - k$ everywhere on $M \cap U$.

Such, though somewhat trivial, is an interesting observation, because if such, then one can do all possible calculations on the shape one has. Functions, as a polynomial, or in terms of their zeros have invaluable information about them. Especially when we notice the rates of changes, and other needed tools, even the Jacobian Matrix. And if that function form can be projected, or represented in terms of any coordinate system, or one can have them in a space that one can use to visualize it, then such becomes a useful duality.

Volume Difference Factor

With such and auxiliaries put out of the way, one gets back to where we started from. In the case of our pyramid, one now notices the difference factor $V_P \,\Delta\, V(x)$ to be;

$$\begin{aligned} V_P \,\Delta\, V(x) &= \frac{x^3}{3\sqrt{2}} \,\Delta\, \frac{x(x+1)(2x+1)}{6} = \frac{x^3}{3\sqrt{2}} \,\Delta\, \frac{(x^2+x)(2x+1)}{6} \\ &= \frac{x^3}{3\sqrt{2}} \,\Delta\, \frac{2x^3+2x^2+x^2+x}{6} = \frac{x^3}{3\sqrt{2}} \,\Delta\, \frac{2x^3+3x^2+x}{6} \end{aligned}$$

One reduces the constants, and then it becomes;

$$V_P \;\Delta\; V(x) = x^3 \;\Delta\; x^3 + x^2 + x$$

which is very interesting. Visually such means the cube on the left, and on the right one has a cube, a square, and another line. What does such mean, or what is this difference contributing to? Again one has the factor $V_P \;\Delta\; V(x)$. Just what does this mean?

In terms; one will know that clearly;

$$x^3 + x^2 + x \geq x^3, \text{ and, } x^3 \propto x^3 + x^2 + x$$

so there is a clear difference. One observes the factor;

$$V_P \;\Delta\; V(x) = x^2 + x = x(x+1),$$

which goes through $(-1, 0)$, and $(0, 0)$ in Euclidean space $\mathbb{R}^2$. But why must we find a difference? What does such mean? One even notices something striking from our previous terms;

$$\Sigma n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \text{ has also a form; } x^2 + x;$$

if we decide to melt the constants. Why would it have a form of a parabola that corresponds to something of the *supposedly* of previous or lower dimension as a geometric object? Is is possible that a projection mapping occurs and such difference factor contributes, for which there is a discrepancy? What does though occur during the process?

Proposal, Another Observation, Another Question: One lets $\Sigma_F x^i$ denote the Faulhaber's formulaes for partial summation (sum of powers up to degree i) *without constants*. Then let $i \in \mathbb{R}_{\geq 1}$ represent the dimension or exponent. One supposes V_P to denote the volume of a i-dimensional pyramid (or similar simplex). Actually, volume might not be the term one is looking for, because in higher dimensions, such becomes arbitraty. One denotes the term to be F_i, and then one proposes that;

$$F_i \propto \Sigma_F x^i - \Sigma_F x^{i-1}$$

where $\propto$ denotes proportionality, absorbing constants like $\frac{1}{i}$, and assuming the forms are dimensionally consistent. Perhaps this is a good time to introduce Faulhaber's formula;

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j \, n^{p+1-j}$$

Here the term B_j means the **Bernoulli Numbers**. One sees that This expression yields a polynomial in n of degree $p+1$. Using the bernoulli polynomials $B_{p+1}(x)$, the formula becomes;

$$\sum_{k=0}^{n-1} k^p = \frac{B_{p+1}(n) - B_{p+1}(0)}{p+1}$$

this form is expressed in terms of Bernoulli polynomials, which generalizes the Bernoulli numbers $B_{p+1}(0) = B_{p+1}$. One also has;

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \, x^{n-k}$$

Anyway, one has an example: For the case $i = 3$, corresponding to a 3D pyramid or volume:

$$F_i \propto x^3 = x^3 + (x^2 + x) - (x^2 + x) = \Sigma_F x^2 - \Sigma_F x$$

Recursive Volume Functional via Faulhaber's Summations

One lets $\Sigma_F x^k$ denote the Faulhaber's summation formula of degree k, considered *up to proportionality*, i.e., omitting constant coefficients and lower-order terms, so that;

$$\boxed{\Sigma_F x^k \propto x^{k+1}.}$$

Here $k \in \mathbb{N}_0$. For each integer $i \geq 1$, define the functional F_i by the alternating sum;

$$\boxed{F_i := \sum_{k=0}^{i} (-1)^{i-k} \Sigma_F x^k.}$$

Under the proportionality assumption, such becomes;

$$F_i \propto \sum_{k=0}^{i} (-1)^{i-k} x^{k+1}.$$

Starting from the recursive form;

$$F_i \propto \Sigma_F x^i - \Sigma_F x^{i-1},$$

one expands recursively;

$$\begin{aligned} F_i &\propto \Sigma_F x^i - \left(\Sigma_F x^{i-1} - \Sigma_F x^{i-2}\right) \\ &= \Sigma_F x^i - \Sigma_F x^{i-1} + \Sigma_F x^{i-2} \\ &= \Sigma_F x^i - \Sigma_F x^{i-1} + \Sigma_F x^{i-2} - \Sigma_F x^{i-3} + \cdots + (-1)^i \Sigma_F x^0, \end{aligned}$$

which can be compactly written as;

$$\boxed{F_i \propto \sum_{k=0}^{i} (-1)^{i-k} \Sigma_F x^k.}$$

One re-indexes with $j = k + 1$, then one has;

$$F_i \propto \sum_{j=1}^{i+1} (-1)^{i+1-j} x^j,$$

yielding the explicit closed form;

$$\boxed{F_i \propto x^{i+1} - x^i + x^{i-1} - \cdots + (-1)^i x.}$$

Values;

$$F_i = \begin{cases} F_1 \propto x^2 - x, \\ F_2 \propto x^3 - x^2 + x, \\ F_3 \propto x^4 - x^3 + x^2 - x, \\ \quad \vdots \\ F_i \propto x^{i+1} - x^i + x^{i-1} - \cdots + (-1)^i x. \end{cases}$$

demonstrating the alternating pattern of powers. Such structure mirrors the classical Faulhaber's formula;

$$\sum_{n=1}^{N} n^k = \frac{1}{k+1}\left(N^{k+1} + \text{lower-order terms}\right),$$

where the leading term x^{k+1} is a proxy for the $(k+1)$-dimensional measure. The functional F_i isolates the pure i-dimensional contribution by recursively subtracting lower-dimensional parts;

$$F_i \propto \Sigma_F x^i - F_{i-1},$$

with the base case $F_0 \propto \Sigma_F x^0 \propto x$. Geometrically, this corresponds to;

$$F_i = \begin{cases} F_1 : \text{a line segment} - \text{a point}; \ (|) - (\cdot) \\ F_2 : \text{a square area} - \text{edges} + \text{vertices}; \ (\square) - (|) + (\cdot) \\ F_3 : \text{a cubic volume} - \text{square base} + \text{edges} - \text{vertices}; \ (\boxed{\square}) - (\square) + (|) - (\cdot) \\ \quad \vdots \\ F_i : x^{i+1} - x^i + x^{i-1} - \cdots + (-1)^i x; \ \text{who knows what this is...} \end{cases}$$

reflecting the alternating inclusion-exclusion principle. With constants attaches, such just makes a shape factor; meaning the counts of the constants which says how many shapes are involved. One has;

$$F_i := c_i \sum_{k=1}^{n} k^i - c_{i-1} \sum_{k=1}^{n} k^{i-1},$$

or using proportionality, which is if not fully equal. Also recursively;

$$F_i := F_{i-1} + \left[c_i \sum_{k=1}^{n} k^i - c_{i-1} \sum_{k=1}^{n} k^{i-1} \right].$$

One also sees that the constant $c_1, c_2, c_3, \cdots$ and so on[7], is native to the dimension. If one defines the constants geometrically;

$$c_i = \frac{1}{i+1},$$

which matches the geometric volume scaling, then one gets the geometric evolution. One finds the factor which is needed to make a **dimensional leap**;

$$F_i = \frac{1}{i+1} \sum_{k=1}^{n} k^i - \frac{1}{i} \sum_{k=1}^{n} k^{i-1}.$$

This represents the net quantity added when transitioning from dimension $i-1$ to dimension i, discretely approximating integral growth of the form;

$$\int x^i dx = \frac{x^{i+1}}{i+1}\ .$$

As example: $i = 2$;

$$\begin{aligned} F_2(n) = \frac{1}{3} \sum_{k=1}^{n} k^2 - \frac{1}{2} \sum_{k=1}^{n} k &= \frac{1}{3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{2} \cdot \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{36} \cdot (2n+1-9) = \frac{n(n+1)(2n-8)}{36}. \end{aligned}$$

Each $F_i(n)$ is a polynomial of degree $i+1$, constructed as a discrete analogue of the geometric volume difference between dimensions i and $i-1$. One notices the alternating pattern, and observes that such is analogous to the Euler characteristic of a cell complex;

$$\chi = \sum_{k=0}^{n} (-1)^k c_k,$$

where c_k counts k-dimensional cells. One sees, here, the $\Sigma_F x^k$ act as continuous analogues of k-dimensional measures, and the alternating signs remove boundary contributions to isolate the pure i-dimensional volume. Thus, F_i serves as a dimensionally stratified continuous analogue of Euler characteristic decomposition, and such can be viewed as a symbolic discrete differential of accumulated dimensional content;

$$F_i \sim \Delta \Sigma_F x^i,$$

[7] constants correspond to the number of shapes, and is very specific to the dimensions. one asks if this shape is responsible for the packings, and the shape it represents, i.e., a square based right triangular faced pyramid, and so on...

where Δ is an alternating difference operator that *"peels off"* boundary layers. Such recursion, and dimensional transition is, and will be useful because after 3D, we sort of lose the visibility, and imaginable landscape into abstraction. But it does not hurt to look at what happens in 4D, or try to understand what happens, or what can be the evolution. For $i = 4$, using what we already know;

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

one obtains;

$$\begin{aligned} F_4(n) &= \frac{1}{5} \cdot \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} - \frac{1}{4} \cdot \left(\frac{n(n+1)}{2}\right)^2 \\ &= \frac{n(n+1)}{150}(2n+1)(3n^2+3n-1) - \frac{n^2(n+1)^2}{16}. \end{aligned}$$

This is a degree-5 polynomial, representing the pure 4D volume content after subtracting 3D boundary contributions. Analogously;

$$F_i = \sum_{k=0}^{i} (-1)^{i-k} \Sigma_F x^k \sim \chi_{\text{measure}},$$

acting as a continuous analog of the Euler characteristic by subtracting out lower-dimensional layers of accumulated content. One knows the 4-dimensional Faulhaber-based alternating volume functional as;

$$F_4(x) = x^5 - x^4 + x^3 - x^2 + x$$

This expression can be interpreted as a recursive stratification of geometric content, where each power of x corresponds to a shape of a given dimension (with degree = dimension + 1), and the signs alternate according to inclusion-exclusion;

$$F_i(x) = \sum_{k=0}^{i} (-1)^{i-k} x^{k+1}$$

Each term corresponds to a geometric element in the structure of a 4D hypercube (tesseract):

$$\begin{array}{rcll} x^5 & : & \text{Tesseract (4D body)} & (+) \\ -x^4 & : & \text{Cubic 3D volumes (3D cells)} & (-) \\ +x^3 & : & \text{Square faces (2D)} & (+) \\ -x^2 & : & \text{Edges (1D)} & (-) \\ +x & : & \text{Vertices (0D)} & (+) \end{array}$$

One takes a table view;

Dimension	**Term**	**Count in Tesseract**	**Shape Type**
4	x^5	1	Tesseract (4D)
3	$-x^4$	8	Cubic cells (3D)
2	$+x^3$	24	Square faces (2D)
1	$-x^2$	32	Edges (1D)
0	$+x$	16	Vertices (0D)

Each alternating term subtracts the contribution of lower-dimensional boundaries, similar to a discrete version of the divergence theorem or Gauss-Bonnet, where excess boundary content is successively canceled.

Generalizing: For any $i \in \mathbb{N}_{\geq 1}$, the pattern generalizes as;

$$F_i(x) \propto \sum_{k=0}^{i} (-1)^{i-k} x^{k+1} \quad \text{or equivalently,} \quad F_i(x) = \Sigma_F x^i - \Sigma_F x^{i-1}$$

Each term represents a combinatorial-geometric component of an $(i+1)$-dimensional cube, expressed symbolically through partial summation polynomials (Faulhaber-type);

$$\Sigma_F x^i := \sum_{k=1}^{n} k^i \quad \text{(without constants)}$$

We have acquired some results that are very interesting. One now can understand, and state to mind that each shape has to fit some very specific, if not one specifically and most certainly so, configuration, for which we have a dimension factor which is attached to our form like a constant. Because we observed, that for double integral method, some attempts failed because it cannot represent the geometry of the pyramid. We had to slice vertically to make sense of it, and find a resulting resolution[8]. We also observed null-ness of volume, and dimensional collapse when one shrinks the volume to null. But what does this 4D shape tell us? One sees;

$$\begin{aligned} F_4(x) \sim \underbrace{\text{Tesseract (4D)}}_{corresponding\ x^5} - \underbrace{\text{Cubic volumes (3D)}}_{corresponding\ x^4} + \underbrace{\text{Square areas (2D)}}_{corresponding\ x^3} \\ - \underbrace{\text{Edges}}_{corresponding\ x^2} \quad (1D) + \underbrace{\text{Vertices}}_{corresponding\ x} \quad (0D), \end{aligned}$$

something interesting. One notices that each form has a term that is one dimension higher. One again considers the functional $F_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by the alternating sum of Faulhaber-type partial sums;

$$F_i(x) := \sum_{k=0}^{i} (-1)^{i-k} \Sigma_F x^k,$$

where;

$$\Sigma_F x^k := \sum_{m=1}^{n} m^k,$$

denotes the partial sum of k-th powers up to n, omitting constants for simplicity. One recalls Faulhaber's formula for sums of powers;

$$\sum_{m=1}^{n} m^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j},$$

where B_j are Bernoulli numbers. The leading term of this polynomial is

$$\sum_{m=1}^{n} m^p = \frac{n^{p+1}}{p+1} + \text{lower order terms in } n.$$

This implies that the degree of the polynomial in n corresponding to the partial sum of p-th powers is $p+1$. The appearance of x^{p+1} instead of x^p for the p-dimensional geometric objects is a direct consequence of the summation process: summing p-dimensional measures over an index m from 1 to n effectively adds an integration dimension, raising the degree by one.

Rigorously: Let $p \in \mathbb{N}_0$ denote the geometric dimension of a cell in the complex. Then the partial sum $\Sigma_F x^p$ behaves like a polynomial of degree $p+1$ in x. Formally;

$$\deg\left(\Sigma_F x^p\right) = p + 1.$$

Therefore, the term x^{p+1} in the expression for $F_i(x)$ corresponds to the accumulated content (volume, area, length, count, etc.) of p-dimensional cells up to scale x. This explains why, in dimension $i = 4$;

$$F_4(x) \sim x^5 - x^4 + x^3 - x^2 + x,$$

with x^5 corresponding to the full 4D hypervolume, x^4 to its cubic 3D faces, and so on down to vertices[9].

[8] why such very specific configuration of shape? Is it because those shapes represent packings in dimensions?

[9] can one say that each i-dimensional shape is the projection of $i+1$ dimension? is such the reason why sphere packings happen in certain configurations, and is a projection of its one dimension higher?

Another, Another Observation: Let $S \subset \mathbb{R}^i$ be a geometric object (e.g., a polytope, manifold, or compact set). Then there exists a compact set $T \subset \mathbb{R}^{i+1}$ and a linear projection;

$$\pi : \mathbb{R}^{i+1} \to \mathbb{R}^i$$

such that $\pi(T) = S$. Many optimal or symmetric configurations in low-dimensional sphere packings (e.g., the densest 2D or 3D packings) correspond to projections of lattices in higher dimensions, such as; the E_8 lattice (in $\mathbb{R}^8$), the Leech lattice (in $\mathbb{R}^{24}$). Thus, the structure of optimal sphere packings may be understood as *shadows* or *projections* from higher-dimensional order. While not all geometric configurations in $\mathbb{R}^i$ naturally arise as projections, many of the most symmetric and efficient structures do. This includes some of the best-known sphere packings. Therefore, viewing lower-dimensional shapes as projections of higher-dimensional forms is a fruitful heuristic.

Theorem: One lets $S \subset \mathbb{R}^i$ be any compact subset. Then there exists a compact set $T \subset \mathbb{R}^{i+1}$ and a linear projection;

$$\pi : \mathbb{R}^{i+1} \to \mathbb{R}^i \text{ such that} \pi(T) = S.$$

Proof. Define the projection $\pi(x_1, \ldots, x_i, x_{i+1}) = (x_1, \ldots, x_i)$. Then set;

$$T := S \times \{0\} \subset \mathbb{R}^i \times \mathbb{R} = \mathbb{R}^{i+1}.$$

Such set is compact since S is compact and $\{0\}$ is closed. Then;

$$\pi(T) = \pi(S \times \{0\}) = S,$$

is as desired. □

Stronger Result (Polytopes): Every convex polytope $P \subset \mathbb{R}^i$ is the image of a higher-dimensional simplex $Q \subset \mathbb{R}^{i+1}$ under a linear projection. This follows from the Minkowski-Weyl Theorem.

Algebraic Topology Analogy:

The alternating sum resembles the structure of a chain complex:

$$\partial_k \circ \partial_{k+1} = 0$$

and mimics cancellation of overcounted boundary structures, producing a pure measure of $(i+1)$-volume, analogous to an Euler characteristic:

$$\chi = \sum_{k=0}^{i} (-1)^k \cdot \#(k\text{-cells})$$

$$F_4(x) \sim \underbrace{\text{Tesseract}}_{x^5} - \underbrace{\text{Cubic Faces}}_{x^4} + \underbrace{\text{Square Faces}}_{x^3} - \underbrace{\text{Edges}}_{x^2} + \underbrace{\text{Vertices}}_{x}$$

Circle Packing

One now steps one dimension down to observe what has been going on in 2D. In fact, one can look at the equilateral sides of the square right pyramid. Corresponding to such, one has;

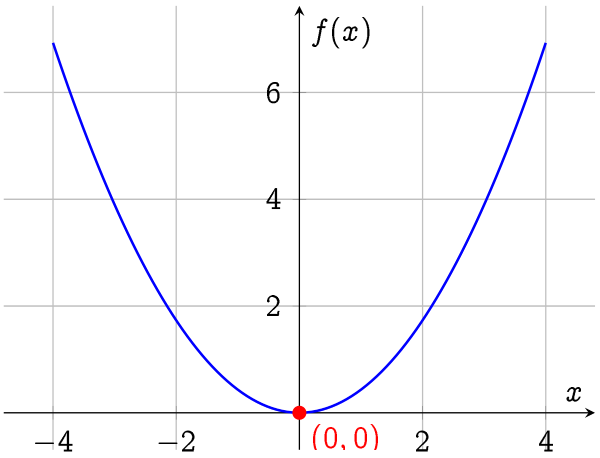
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad \& \quad 1 + 2 + 3 + \cdots = \frac{-1}{12} \ (\mathfrak{R})$$

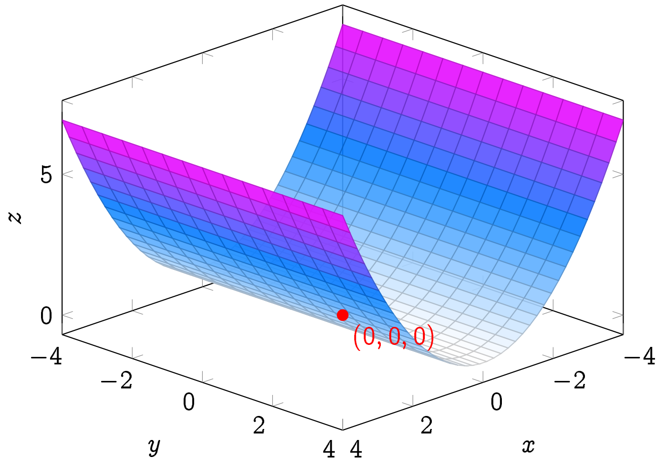
One knows that the triangle is equilateral, along with the sides of the pyramid. One has an area of;

$$A_\Delta(x) = \frac{\sqrt{3}}{4}x^2, \text{ where } x \text{ is the side length of the triangle.}$$

One knows that it is an equilateral triangle, and as like before, we graph it to see what is going on;

One can further say, a geometric shape such as an equilateral triangle can represent a simple parabola in cartesian plane. The function is smooth, defined for all $x \in \mathbb{R}$, and since $f(x) =$





0, $\iff x = 0$, the graph passes through origin only, shown in both 2D and 3D plots. One can see such algebraic expressions to be structurally identical to the area formulas[10]. Geometrically, one can analyze that the term $f(x) = \sqrt{3}/4 \cdot x^2$, though a parabola, it has a constant multiplied to a shape that looks like a square.

Let a square have side length a. Its area is:

$$A_{\square}(x) = x^2.$$

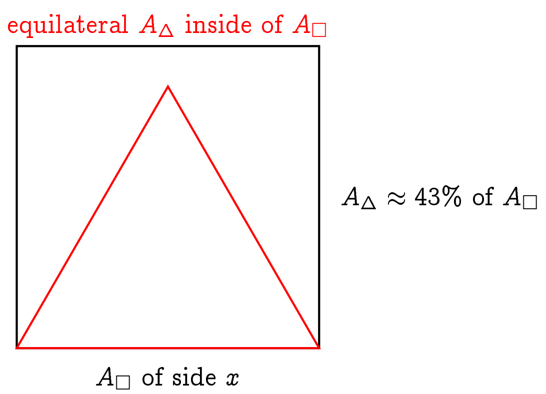
Now one considers an equilateral triangle with side length x. Its area is;

$$A_{\triangle}(x) = \frac{\sqrt{3}}{4} x^2.$$

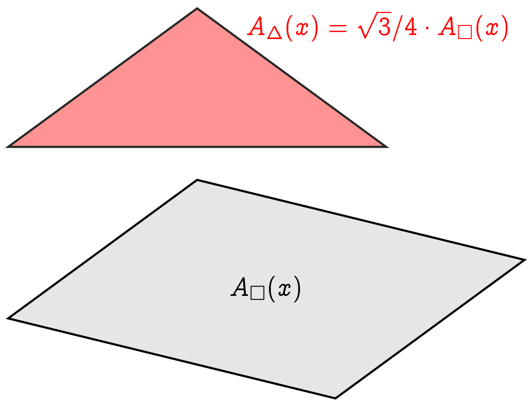
Hence,

$$A_{\triangle}(x) = \frac{\sqrt{3}}{4} \cdot A_{\square}(x).$$

The factor $\sqrt{3}/4$ is 0.4330127018922 and so on, and such means that an equilateral triangle of side x occupies approximately 43.3% of the area of a square with the same side.



2D: Square and Equilateral Triangle



3D: Triangle Sheet on Top of Square

The height of the equilateral triangle is $\frac{\sqrt{3}}{2}x$, so $A_{\triangle} \in A_{\square} \iff \frac{\sqrt{3}}{2}x \leq x$, i.e., $\sqrt{3} \leq 2$, which is true.

One considers a unit square of side length $a = 1$, with an equilateral triangle of the same side length positioned above it. To formulate an analogue of the Kepler packing density in this simplified 3D context, we introduce a uniform thickness $\varepsilon > 0$ for both the square base and the triangle layer. Assigning a thickness ε to both the square and the triangle sheet, we define their

[10] Can one say; A form $f(x) \xrightarrow{equivalance}$ A geometric area $A(x)$? Can; A geometric area $A(x) \xrightarrow{equivalance}$ A form $f(x)$, be said too, so that $f(x) \iff A(x)$? And what would be the case if one does not care what the dimensions are? Can such an equivalence be generalized so that we can observe its behavior in other dimensions?

volumes:

$$V_{\square} = A_{\square} \cdot \varepsilon = \varepsilon, \quad V_{\triangle} = A_{\triangle} \cdot \varepsilon = \frac{\sqrt{3}}{4}\varepsilon.$$

The total volume of the unit cell (i.e., the bounding box containing both layers) is:

$$V_{\text{cell}} = A_{\square} \cdot (2\varepsilon) = 2\varepsilon.$$

One now defines a density function $\delta_{\triangle}$, analogous to Kepler's density δ, as the ratio of the triangle volume to the total unit cell volume:

$$\delta_{\triangle} = \frac{V_{\triangle}}{V_{\text{cell}}} = \frac{\frac{\sqrt{3}}{4}\varepsilon}{2\varepsilon} = \frac{\sqrt{3}}{8} \approx 0.2165.$$

Kepler's conjecture (proven by Hales[11]) states that the maximum density achievable by packing congruent spheres in 3D Euclidean space is:

$$\delta_{\text{Kepler}} = \frac{\pi}{3\sqrt{2}} \approx 0.74048.$$

Thus, our triangle-over-square configuration yields a lower density:

$$\delta_{\triangle} < \delta_{\text{Kepler}}.$$

Such is a simplified planar analogue to demonstrate how relative area and volume distributions can be interpreted through a packing lens.

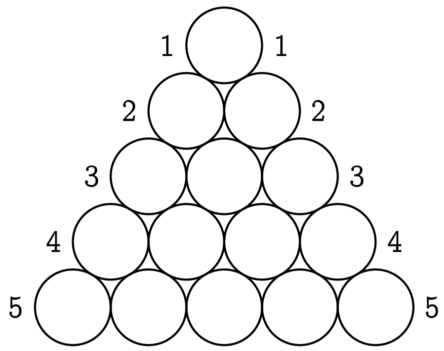


Fig: An equilateral arrangement of circle packing.

Talking about packing, one then looks at the side of the pyramid, to notice the exact same shape. One has the image on the left, and from top to bottom, if one continues on forever, would not one end up with a triangle T_n, where;

$$T_{n\to\infty} = 1 + 2 + 3 + 4 + 5 + \cdots \ (?)$$

At this point, it is better to use some formal notations. We denote:

$$T_{n\to\infty} = 1 + 2 + 3 + 4 + 5 + \cdots = \Sigma n; \quad \textit{where } n \in \mathbb{N}$$

Tis the reason for which these numbers are known as the *Triangular Numbers* (T_n), and they arrange themselves as an equilateral triangle.

This equilateral arrangement of circle packing, how many circles are in finite layers from the base to the top corresponds to the sums of natural numbers which we get from Faulhaber's formula. Formally, one can write:

$$T_n = 1 + 2 + 3 + \cdots + n = \sum_{i=1}^{k} n = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} = \binom{n+1}{2} \quad \text{(for finite terms)}$$

But wait... how can that be possible when our own geometric intuition gave us a right angle triangle? From the Faulhaber's formula; $x(x+1)/2$ we have a a geometric figure which is a right angle triangle that we get from halving the rectangle through the diagonal. And such keeps the intuition of T_n intact. Even if we consider the argument that area is just some constant number, such is not sufficient to make sense of what one has an equilateral arrangement in circle packing.

One looks at Kepler's original formulation in 3D, and observes. One wonders what is actually happening in the 3D Euclidean space, and what is the optimal packing in it, and then one looks at the 2D analog of the arrangement, and the shape represented by $x(x+1)/2$. One has;

[11] see the result by Thomas C. Hales.

Formal Mathematical Statement: Let $\mathcal{P} \subset \mathbb{R}^3$ be a packing of non-overlapping closed balls of radius $r > 0$, such that for any two distinct centers $x_i, x_j \in \mathcal{P}$, we have $\|x_i - x_j\| \geq 2r$. Define the upper density of $\mathcal{P}$ as:

$$\delta(\mathcal{P}) := \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^3} \frac{\mathrm{Vol}(\mathcal{P} \cap B(x,R))}{\mathrm{Vol}(B(x,R))}.$$

Then,

$$\sup_{\mathcal{P}} \delta(\mathcal{P}) = \frac{\pi}{3\sqrt{2}} \approx 0.74048\ldots,$$

and the supremum is achieved if and only if $\mathcal{P}$ is congruent (up to isometry) to the FCC or HCP packing.

One has above the statement for Kepler's Conjecture, which was formulated and stated by the German mathematician Johannes Kepler in 1611 on his paper titled "*On the six-cornered snowflake*". He had started to observe the arrangements of spheres as a result of his correspondence with the English mathematician and astronomer Thomas Harriot in 1606. Harriot was a friend and assistant of Sir Walter Raleigh, who, had asked Harriot to find formulas for counting stacked cannonballs on the deck of a ship, an assignment which in turn led Raleigh's mathematician acquaintance into wondering about what the best way to stack cannonballs was. Harriot published a study of various stacking patterns in 1591, and went on to develop an early version of lattice and packing based atomic theory.

Kepler was unable to provide any proof of his conjecture, and the next step was undertaken by Carl Friedrich Gauss in 1831, who provided a proof of the Kepler's conjecture being true if the spheres are arranged in a regular lattice. Which, in technical terms meant that any packing to disproof Kepler's statement has to be an irregular packing. But eliminating all possible irregular arrangements was proven very difficult, and this is what made the Kepler conjecture so hard to prove. In fact, there are irregular arrangements that are denser than the cubic close packing arrangement over a small enough volume, but any attempt to extend these arrangements to fill a larger volume is now known to always reduce their density. After Gauss, no further progress was made towards proving the Kepler conjecture in the nineteenth century. In 1900 David Hilbert included it in his list of twenty three unsolved problems of mathematics, it forms part of Hilbert's eighteenth problem.