Area-Sum Equality Between Graphs of Faulhaber-Bernoulli Polynomial & Euler-Riemann Zeta Function

A Self Conducted Research

Presented By

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To

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.... *Author's Note*

This paper is not a work of fiction or a child's scribble. The origin of what is done inside, dates back to my 9^{th} and 10^{th} grade (late 2017 to 2019) when I first saw *"Grandi's Series"* in the 11^{th} chapter of my *Higher Mathematics* textbook. At that neophyte time, I was not familiar with *Calculus* or basic *Number Theory* to make any sense of it. My being *"poor"* and other several factors contributed to having no access of internet and resources to acquire mathematical knowledge on infinite series. So, I came up with my own way of understanding them by *"Matrix Multiplication"* of finding area within a certain interval of plotted graph with its coordinates. However, that process is very tedious (ancient and lengthy) and will not be mentioned in the paper as it would not bear any fruit or useful connection with *"Bernoulli-Faulhaber Polynomial"*.

In this paper, I only deal with the *Negative Integer* inputs of *Euler-Riemann Zeta Function* and its connection with the area of the graph within a certain X-intercept interval to support my claim. I very well know that my theoretical knowledge has limits and the reader may find faults in my novice writing. I will delightfully accept critiques and suggestions along with mathematical theories to enrich this paper.

This is completely an investigation and result of my own with instructions and guidance of my math teacher Mr. Swapan Kumar Biswas, an Asst. Professor and Departmental Head of the faculty of Mathematics at St. Joseph Higher Secondary School. The materials used are described inside as per original source and the references are mentioned as I went on to the final result.

Do not use anything of this paper without my consent. I am a very friendly person and eager to collaborate if we are on the same page of this paper. If needed, please reach me via phone +8801633734006. Messaging is preferred.

Happy Reading ...

Mrenal Kanti Das
December 29, 2021

Chapter 00

Synopsis ....

The final result of this paper is;

$$\zeta(-s) = \frac{1}{s+1}\int\limits_a^b \left[\sum_{i=0}^{s} \binom{s+1}{i} \left\{\sum_{k=0}^{i} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k}{j} j^i\right\} x^{s+1-i}\right] dx$$

We show that the *Negative Integer* input of *Euler-Riemann Zeta Function* is equal to the integral at the right. It should be noted that the expression inside the curly bracket is *Bernoulli Numbers*, which is defined by;

$$B_i = \sum_{k=0}^{i} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k}{j} j^i \qquad ;\ \textit{for } i \geq 0$$

By which one has;

$$\zeta(-s) = \frac{1}{s+1}\int\limits_a^b \left[\sum_{i=0}^{s} \binom{s+1}{i} B_i\, x^{s+1-i}\right] dx$$

One will have a *Polynomial Function* via the expression inside the square bracket. And pictorially, one would have *"a"* and *"b"* as the intercept of *"x"* axis through the graph the polynomial would sketch. The integral represents the area within the interval $[a, b]$, where the graph touches or pierces the *"x"* axis. As a visual representation, 12 examples are given and one goes through the tedious process of finding the polynomials by evaluating the identity above.

We establish a relationship between *"The Graph of Faulhaber-Bernoulli Polynomial"* and the *"Negative Integer Inputs of Euler-Riemann Zeta Function"*, having a brilliant match of Areas between the intercepting or touching point of *"x"* axis.

Chapter 01

Background Materials ....

In this chapter, we will go through some of the foundations, from which one can derive the claim after which this paper is named. At section **1.1**, one will know the series of Triangular Numbers. **1.2** will cover the Partial Summation, whereas in **1.3**, we will have a brief historic recollection from Ramanujan's Letter to G.H. Hardy. Section **1.4** will present a simple proof of the sum of divergent series $1 + 2 + 3 + \cdots + \mathrm{n} + \cdots$ to be $-1/12$.

1.1 $\mathbf{1 + 2 + 3 + \cdots +} \boldsymbol{n} \mathbf{+ \cdots}$

One will start with the definition of *Divergent Series*, which are infinite series that do not converge, meaning it does not reach a specific value as we add or subtract more and more terms. One can calculate their partial sum, but the sum, does not have a limit to reach a certain value.

In this specific case, we have the sum of *Natural Numbers* $\mathbb{N}$. The sum will eventually explode to $+\infty$, *Positive Infinity* as we add more and more terms. If we keep adding terms, we get what we know as *Triangular Numbers* as sums. It could be written;

$$T_n = 1 + 2 + 3 + \cdots + n + \cdots \quad ; \quad n \in \mathbb{N}$$

The sequence of partial sums of the series is 1, 3, 6, 10, 15, 21, 28, 36, and so on. As can be seen, the sum only grows bigger and bigger as we keep adding further terms. So, the general definition and observation predicts that the sum should diverge to $+\infty$, and never tend to zero, or lesser.

1.2 *Partial Summation*

The idea of a sequence or series converging to a specific value comes from the notion of adding it term by term to see if it does. It is partial summation, where we take a certain section of series or sequence and try to find a pattern or limit in their summation. The *Partial Sum* of the series above, T_n can be calculated by a simple formula, given that $n \in \mathbb{N}$;

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

This formula outputs the partial summation to the "*n*"*-th* term, made famous by none other than *Carl Friedrich Gauss* when he was in his very early age. The formula tackles the problem by multiplying the "*n*"*-th* term to the sum of first and last term, and ultimately dividing by 2 as there are n number of $(n+1)$'s. The visual representation would be halving the area of a rectangle of n and $n+1$ sides, and then dividing it by 2 which will make it a triangle, hence the name *Triangular Sum*. Up until now, the definition of the series and its partial sum are within the general understanding and observation. Undoubtfully we cannot plug in zero or any negative number as the definition of this sum states itself that we should start from where the Natural Number, $\mathbb{N}$ begins.

1.3 *Ramanujan's Letter to G.H. Hardy*

In section XI of the letter *(ref. Ramanujan: Letters and Commentary by Bruce C. Berndt and Robert A. Rankin, page* 29*)*, *Srinivasa Ramanujan* Claims to *G.H. Hardy* that he has values to some specific divergent series and the theorems by which he derived to such conclusion. Those specific divergent series include;

$$1-2+3-4+\cdots=1/4 \;\;;$$

$$1-1!+2!-3!+4!-\cdots=0.596\ldots;$$

$$1+2+3+\cdots=-1/12\;;$$

$$1^3+2^3+3^3+4^3+\cdots=1/120\;.$$

Other than the third one, none of these are of concern, at least for now. Here *Ramanujan* claims the summation to be $1/12$, despite the sum being a divergent and to make things weird, he claims the sum should be negative. This contradicts the properties of the series and the *Triangular Sum* that was discussed on 1.1 and 1.2. By general observation it can be stated clearly that the sum would be something positive. Even if it is a divergent series, the sum of positive numbers could never converge to something negative. There is a simple proof by which it can be shown the above 3^rd^ claim could make sense.

3.3 *A Simple Proof of* $1+2+3+4+5+\cdots+n+\cdots=-1/12$

So, we begin with three series S_1, S_2 , and S, by defining;

$$S_1=1-1+1-1+1-\cdots$$

$$S_2=1-2+3-4+5-\cdots$$

$$S=1+2+3+4+5+\cdots$$

Now one approaches to find the sums of these three series by simple algebra.
One has;

$$S_1=1-1+1-1+1-\cdots$$

Bracketing the numbers gives us;

$$S_1=1-(1-1+1-1+1-\cdots)$$

$$S_1=1-\;S_1$$

$$2S_1=1$$

And now;

$$S_1=1/2$$

So, $$S_1=1-1+1-1+1-\cdots=1/2$$

For S_2, we double the series by adding the series to itself. Now, here we rearrange the series by shifting the entries by one term right.

This arrangement will should be the double of the initial series and since we move the terms right, we will be able to add and subtract fitting to our needs.

One will have;

$$2S_2 = 1 - 2 + 3 - 4 + 5 - \cdots$$
$$ 1 - 2 + 3 - 4 + 5 - \cdots$$

The sum gives us; $2S_2 = 1 - 1 + 1 - 1 + 1 - \cdots$; which values to ½.

So, we substitute; $2S_2 = 1/2$

$$S_2 = 1/4$$

Now, one look at the third series, S = $1 + 2 + 3 + 4 + 5 + \cdots$ and subtract S_2 from S;

$$S - S_2 = 1 + 2 + 3 + 4 + 5 + \cdots$$
$$-[1 - 2 + 3 - 4 + 5 - \cdots]$$
$$S - S_2 = 4 + 8 + 12 + 16 + 20 + \cdots$$
$$S - S_2 = 4[1 + 2 + 3 + 4 + 5 + \cdots]$$

Now by substituting the values;

$$S - 1/4 = 4S$$
$$-1/4 = 3S$$
$$S = -1/12$$

So, by this adding and subtracting through rearrangement, the series gives a sum;

$$S = 1 + 2 + 3 + 4 + 5 + \cdots = -\frac{1}{12}$$

One might think it gives a solid argument, but it does not. It still lacks an argument. The main thing here is *Rearrangement* and *Partial Summing* through which we reach the conclusion. But it will need some mathematical definition or proof to believe so. It will be further discussed in the later chapters with detailed explanations.

Chapter 02

The Summability Of A Series ....

This chapter will give detailed explanations of *Partial Summing* and *Rearranging*, consecutively in section **2.1** with *Grandi's Series*, followed by method of *Cesàro Summation* in section **2.2**, and ending with *George Friedrich Bernhard Riemann*'s *Series Rearrangement Theorem* in section **2.3**.

2.1 *Grandi's Series*

One starts with the first series, which goes by the name *Grandi's Series.* It is a divergent series goes by definition;

$$G = \sum_{n=0}^{\infty} (-1)^n$$

To get the sum to converge to ½, it is needed to be defined how it reaches such a sum.

2.2 *Cesàro Summation*

We introduce another new term, *Cesàro Summing.* Traditional summing says;

$$\sum_{n=0}^{\infty} a_n = L, \qquad if \qquad \lim_{n \to \infty} S_n = L$$

A series "a_n" will converge to "L" if the limit of its *Partial Sum* "S_n" converges to "L". The *Cesàro Summation* takes the average of the partial summing. Meaning;

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k\,=\,0}^{n} S_k$$

So, the above definition states, by the traditional method one infinite series should converge to a specific value if its partial summation happens to have a limit. After which the *Cesàro Summation* defines something specific about it, bounding with a condition of arithmetic average of the sum. It says the arithmetic mean of the partial summation should converge to a limit. Now one starts;

$$S_1 = 1 - 1 + 1 - 1 + 1 - \cdots$$

$$S_n\ ; \qquad 1 \quad 0 \quad 1 \quad 0 \quad 1\ \ldots$$

Now, the averages of partial summations;

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \frac{4}{7}, \frac{4}{8}, \frac{5}{9}, \frac{5}{10}, \frac{6}{11}, \frac{6}{12}, \frac{7}{13}, \frac{7}{14}, \frac{8}{15}, \frac{8}{16}, \cdots \to \frac{1}{2}$$

As can be seen, the average of the partial sum (of the partial sum) approaches 1/2.

It has a common fraction ½ at every $2n$ or even terms but that is not the limit it converges to. The ½ is a common term. The $(2n+1)$ or odd term fractions converge to ½.

By same method, the average of the partial sums of the second series;

$$\frac{1}{1}, \frac{-1}{2}, \frac{2}{3}, \frac{-2}{4}, \frac{3}{5}, \frac{-3}{6}, \frac{4}{7}, \frac{-4}{8}, \frac{5}{9}, \frac{-5}{10}, \frac{6}{11}, \frac{-6}{12}, \frac{7}{13}, \frac{-7}{14}, \frac{8}{15}, \frac{8}{16}, \cdots \rightarrow \frac{1}{4}$$

Now comes the sense of what *S. Ramanujan* claims in section XI of his letter to *G.H. Hardy.*

2.3 *Riemann Rearrangement Or, Series Theorem*

There is a very important argument based on which one can prove the above claims, which is *Rearrangement* and *Partial Summing.* One might think these are out of *Classical Mathematics* or child's play. Which sometimes is, but the definition and properties of the partial summing is taken care of in the previous sections. A question may arise, does the final value change? If we shuffle the terms continuously, does it still converge to the exact value that is converged by the traditional summation? Here again, we introduce another theorem from *Georg Friedrich Bernhard Riemann.* Mathematically speaking, his claim states below;

***Theorem* 01:**

*If $\sum_{n=0}^{\infty} a_n$ converges **absolutely**, then any arrangement or rearrangement of the series different from the original will converge to the same value.*

***Theorem* 02:**

*If $\sum_{n=0}^{\infty} a_n$ converges **conditionally**, then there exist numerous rearrangements that could converge the series to any value.*

Explanation;

If a series, specifically an infinite series of Real Numbers, $n \in \mathbb{R}$ is *Conditionally* convergent, then its terms can be arranged or rearranged in several permutations as though that the new series converges to an arbitrary real number, or it could diverge. Which also makes it clear that a series of real numbers will absolutely converge if and only if it is unconditionally convergent. We are, however, familiar with ***Theorem* 01** very nicely. It states what we already know that a series should have the exact same sum despite its arrangements or rearrangements. But ***Theorem* 02** states, if there is a condition by which it *Conditionally* converges, then we can rearrange it to our needs to reach a specific value. It could be π, e or anything withing the domain of real number, $\mathbb{R}$.

Thus, it can be concluded that $S = 1 + 2 + 3 + \cdots = -1/12$, proof is valid.

Chapter 03

Calculating The Polynomials ....

In this chapter, one will calculate the Bernoulli Numbers, Faulhaber-Bernoulli Polynomials as we will graph those polynomials in later chapters.

3.1 *Faulhaber-Bernoulli Formula*

So far it is made sense of that the series *Srinivasa Ramanujan* claimed in his letter. What about $1^3 + 2^3 + 3^3 + \cdots = 1/120$?

Let's define a function as a sum of infinite series;

$$a_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

If we input $n = 1$, we end up with another interesting series called *Harmonic Series.*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

The series is divergent and very slowly approaches to $+\infty$. And if we input negatives, we get something called the Power Sum. For example, the above case in *Ramanujan's* letter;

$$S_3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots = \frac{1}{120}$$

For that one introduces *Faulhaber's Formula* to the definition, named after the 17$^{\text{th}}$ century German Mathematician *Johann Faulhaber.*

$$\sum_{k}^{\infty} k^p = \frac{n^{P+1}}{p+1} + \frac{1}{2} n^p + \sum_{k=2}^{p} \frac{B_k}{k!} (p)_{k-1} \; n^{p-k+1}$$

$$where, \quad (p)_{k-1} = \frac{p!}{(p-k+1)!} \quad ; \quad is\ a\ Falling\ Factorial$$

The B_k term represents the *Bernoulli Numbers.* The *Faulhaber's Formula* is also called *Bernoulli's Formula* as *Johann Faulhaber* did not know the properties of the B_k terms during his time, later discovered by *Jacob Bernoulli.* In Jacob Bernoulli's 1713 published paper *Sammae Potestatum*, *Bernoulli* writes;

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{1}{2} n^p + \frac{1}{p+1} \sum_{j=2}^{p} \binom{p+1}{j} B_j \; n^{p+1-j}$$

Which happens to have two different variations if we plug in two first Bernoulli Numbers. For $B_1 = 1/2$ *and* $B_2 = \; -1/2$;

$$\frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j \; n^{\,p+1-j}$$

$$\frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \binom{p+1}{j} B_j \; n^{\,p+1-j}$$

These are the same Faulhaber's Formula but with a very definitive understandings of B_k or Bernoulli Numbers and its properties inside the *Polynomial Function* with much easier formation. Among the variations, for our purposes, we will continue to use B_1 as one would not want any trouble with negatives.

3.2 *Bernoulli Numbers*

Bernoulli Number is defined by;

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k}{j} j^{\,n}, \qquad n \geq 0$$

N	*fraction*
0	1
1	$\frac{\pm 1}{2}$
2	$\frac{1}{6}$
3	0
4	$\frac{-1}{30}$
5	0

Where couple of B_n are at the right of the page. We will be taking the *Positive* term of generating *Bernoulli Numbers* as one should not bother with the negatives inside the formula. It would be too much hassle. As per the list shown, only 5 are at the table. But we would need numbers up to 12$^{\text{th}}$ term for our purpose. After that one plugs in the numbers into the *Faulhaber-Bernoulli Equation* to find the polynomials.

3.3 *Calculating The Polynomial Functions*

We, define a polynomial function (by *Faulhaber's Formula*) by naming it $f(x)$;

$$f(x) = \frac{1}{x+1} \sum_{j=0}^{x} \binom{x+1}{j} B_j \, n^{\,x+1-j}$$

Where n is the polynomial terms. Now one goes through the tedious process of calculating a dozen of *Bernoulli-Faulhaber Polynomials;*

$$f(1) = \frac{1}{1+1} \sum_{j=0}^{1} \binom{1+1}{j} B_j \, n^{\,1+1-j} = \frac{1}{2}\left[\binom{2}{0} B_0 n^2 + \binom{2}{1} B_1 n\right] = \frac{n^2+n}{2}$$

As can be seen, this first output is the partial summation formula for the *Triangular Series.*

$$\sum_{k=1}^{n} n = \frac{n(n+1)}{2}$$

And the list will go on as we raise the power.

$$\begin{aligned} f(2) = \frac{1}{2+1} \sum_{j=0}^{2} \binom{2+1}{j} B_j \, n^{\,3-j} &= \frac{1}{3}\left[\binom{3}{0} B_0 n^3 + \binom{3}{1} B, n^2 + \binom{3}{2} B_i n\right] \\ &= \frac{2n^3+3n^2+n}{6} \end{aligned}$$

For the second input;

$$\sum_{k=1}^{n} n^2 = \frac{2n^3+3n^2+n}{6}$$

The second output is the partial summation of the series of natural numbers squared. It is fun to note that for every "n" input, the degree of the polynomial is "$n+1$".

$$\begin{aligned} f(3) &= \frac{1}{3+1} \sum_{j=0}^{3} \binom{3+1}{j} B_j \ n^{\,3+1-j} \\ &= \frac{1}{4}\left[\binom{4}{0} B_0 n^4 + \binom{4}{1} B_1 n^3 + \binom{4}{2} B_2 n^2 + \binom{4}{3} B_3 n\right] \\ &= \frac{n^4 + 2n^3 + n^2}{4} \end{aligned}$$

For 3, we have *Triangular Number Squared*;

$$\sum_{k=1}^{n} k^3 = \frac{n^4 + 2n^3 + n^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

For the 4th;

$$\begin{aligned} f(4) &= \frac{1}{4+1} \sum_{j=0}^{4} \binom{4+1}{j} B_j \ n^{\,4+1-j} \\ &= \frac{1}{5}\left[\binom{5}{0} B_0 n^5 + \binom{5}{1} B_1 n^4 + \binom{5}{2} B_2 n^3 + \binom{5}{3} B_3 n^2 + \binom{5}{4} B_4 n\right] \\ &= \frac{6n^5 + 15n^4 + 10n^3 - n}{30} \end{aligned}$$

For the 5th;

$$\begin{aligned} f(5) &= \frac{1}{5+1} \sum_{j=0}^{5} \binom{5+1}{j} B_j n^{5+1-j} \\ &= \frac{1}{6}\left[\binom{6}{0} B_0 n^6 + \binom{6}{1} B_1 n^5 + \binom{6}{2} B_2 n^4 \ldots + \binom{6}{5} B_5 n\right] \\ &= \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12} \end{aligned}$$

For the 6th;

$$\begin{aligned} f(6) &= \frac{1}{6+1} \sum_{j=0}^{6} \binom{6+1}{0} B_j n^{6+1-j} \\ &= \frac{1}{7}\left[\binom{7}{0} B_0 n^7 + \binom{7}{1} B_1 n^6 + \binom{7}{2} B_2 n^5 + \cdots + \binom{7}{6} B_6 n\right] \\ &= \frac{6n^7 + 21n^6 + 21n^5 - 7n^3 + n}{42} \end{aligned}$$

For the 7th;

$$\begin{aligned} f(7) &= \frac{1}{7+1} \sum_{j=0}^{7} \binom{7+1}{j} B_j n^{7+1-j} \\ &= \frac{1}{8}\left[\binom{8}{0} B_0 n^8 + \binom{8}{1} B_1 n^7 + \binom{8}{2} B_2 n^6 + \cdots + \binom{8}{7} B_7 n\right] \\ &= \frac{3n^8 + 12n^7 + 14n^6 - 7n^4 + 2n^2}{24} \end{aligned}$$

For the 8$^{\text{th}}$;

$$
\begin{aligned}
f(8) = \frac{1}{8+1} \sum_{j=0}^{8} \binom{8+1}{j} B_j\, n^{8+1-j} \\
&= \frac{1}{9}\left[\binom{9}{0} B_0 n^9 + \binom{q}{.\,1} B_1 n^8 + \binom{9}{2} B_2 n^7 + \cdots + \binom{9}{8} B_8 n\right] \\
&= \frac{10n^9 + 45n^8 + 60n^7 - 42n^5 + 20n^3 - 3n}{90}
\end{aligned}
$$

For the 9$^{\text{th}}$;

$$
\begin{aligned}
f(9) = \frac{1}{9+1} \sum_{j=0}^{9} \binom{9+1}{j} B_j n^{9+1-j} \\
&= \frac{1}{10}\left[\binom{10}{0} B_0 n^{10} + \binom{10}{1} B_2 n^9 + \binom{10}{3} B_3 n^7 + \cdots + \binom{10}{9} B_9 n\right] \\
&= \frac{2n^{10} + 10n^9 + 15n^8 - 14n^6 + 10n^4 - 3n^2}{20}
\end{aligned}
$$

For 10$^{\text{th}}$ and 11$^{\text{th}}$;

$$
\begin{aligned}
f(10) = \frac{1}{10+1} \sum_{j=0}^{10} \binom{10+1}{j} B_j n^{10+1-j} \\
&= \frac{1}{10}\left[\binom{11}{0} B_0 n^{11} + \binom{11}{1} B_1 n^{10} + \binom{11}{2} B_2 n^9 + \cdots + \binom{11}{10} B_{10} n\right] \\
&= \frac{6n^{11} + 33n^{10} + 55n^9 - 66n^7 + 66n^5 - 33n^3 + 5n}{66}
\end{aligned}
$$

$$
\begin{aligned}
f(11) = \frac{1}{11+1} \sum_{j=0}^{11} \binom{11+1}{j} B_j n^{11+1-j} \\
&= \frac{1}{12}\left[\binom{12}{0} B_0 n^{12} + \binom{12}{1} B_1 n^{11} + \binom{12}{2} B_2 n^{10} + \cdots + \binom{12}{11} B_{11} n\right] \\
&= \frac{2n^{12} + 12n^{11} + 22n^{10} - 33n^8 + 44n^6 - 33n^4 + 10n^2}{24}
\end{aligned}
$$

And the last one for our observation, the 12$^{\text{th}}$ input

$$
\begin{aligned}
f(12) = \frac{1}{12+1} \sum_{j=0}^{12} \binom{12+1}{j} B_j n^{12+1-j} \\
&= \frac{1}{13}\left[\binom{13}{0} B_0 n^{13} + \binom{13}{1} B_1 n^{12} + \binom{13}{2} B_2 n^{11} + \cdots + \binom{13}{12} B_{12} n\right]
\end{aligned}
$$

$$
= \frac{210n^{13} + 1365n^{12} + 2730n^{11} - 5005n^9 + 8580n^7 - 9009n^5 + 4550n^3 - 691n}{2730}
$$

We have finished calculating our 12 samples or Polynomials that we will graph to prove our claim.

Chapter 04

The Zeta Function & The Graphs ....

In this chapter we will go to the heart of the claim, the Euler-Riemann Zeta Function. For the claim, we will work only with the negative integer inputs of the function. Our inputs will not contain any Complex Numbers ($\boldsymbol{i}$) and for observation, we will look at first 12 examples or inputs of the function.

4.1 *Defining The Function & The Negative Inputs*

The Zeta Function is denoted by the Greek Letter ζ (Zeta) which has definitions both in Real Number, $\mathbb{R}$ and Complex Number, $\mathbb{C}$ realms. It is a series function raised to the power of inputs. It goes by the definition;

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \quad ; \quad \textit{for } Re(s) > 1$$

As can be seen, the definition cannot go smaller than 1 in the number line. If one looks close enough, is very clear that this function relates to what were done in previous pages.

The negative inputs of this *Euler-Riemann Zeta Function* outputs the power sums.

$$\zeta(-1) = 1 + 2 + 3 + \cdots$$
$$\zeta(-2) = 1^2 + 2^2 + 3^2 + \cdots$$
$$\zeta(-3) = 1^3 + 2^3 + 3^3 + \cdots$$

And so, due respect to the 12$^{\text{th}}$ input of the *Faulhaber-Bernoulli Formula*;

$$\zeta(-12) = 1^{12} + 2^{12} + 3^{12} + \cdots$$

In general, for all negative integers, one will have;

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$
$$\zeta(-2n) = 0$$

For all the negative inputs, we get outputs in terms of *Bernoulli Numbers*. The second one, is called *Ramanujan Summation*, which results to the *Trivial Zeroes* of the *Euler-Riemann Zeta Function*. That means, we will have all zeroes for the negative even number inputs. The first values of negative numbers to 12 are given below;

$$\zeta(-1) = \frac{-1}{12} \qquad \zeta(-5) = \frac{-1}{252}$$
$$\zeta(-2) = 0 \qquad \zeta(-6) = 0$$
$$\zeta(-3) = \frac{1}{120} \qquad \zeta(-7) = \frac{1}{240}$$
$$\zeta(-4) = 0 \qquad \zeta(-8) = 0$$

$$\zeta(-9) = \frac{-1}{132} \qquad \zeta(-11) = \frac{691}{32760}$$

$$\zeta(-10) = 0 \qquad \zeta(-12) = 0$$

Now one approaches to the final stage. These values of *Euler-Riemann Zeta Function* seem to give us some interesting values, including the first one we started with, the sum of all natural numbers, $\mathbb{N}$.

4.2 *"x" Intercept & Calculating The Area Within The Interval*

As per Ramanujan Summation, the even negatives give us *Trivial Zeroes*. If one graphs the *Polynomial Functions* found by the Faulhaber-Bernoulli Formula above, the area within the intercept of *x-axis* matches the values of *Euler-Riemann Zeta Function*'s negative input values. For that we graph those Polynomial Functions and calculate the area within the interval of x intercept;

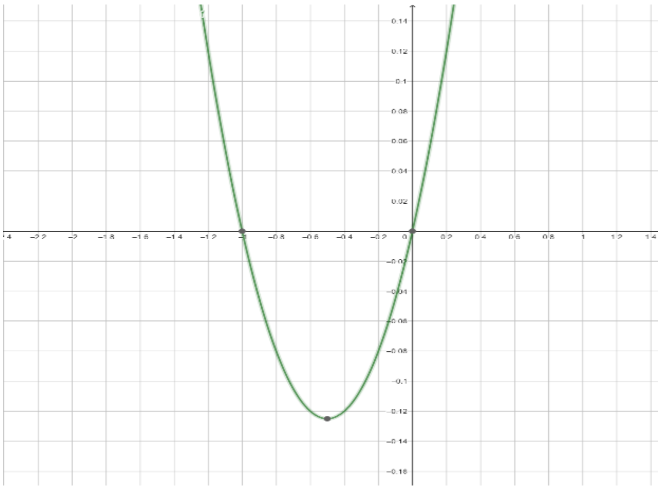


Fig 01: graph of $\frac{x(x+1)}{2}$

For the first graph, $f(x) = \frac{x(x+1)}{2}$ is a parabola that intercepts the x axis at $(-1,0)$ *and* $(0,0)$. It happens to have three roots with x intercepts. For visual convenience, one will take 12 squares and 1 square as units consecutively for x and y axis.

Now, one is to find the area within the intercept interval $[-1,0]$;

The integration gives us;

$$\int_{-1}^{0} \frac{x(x+1)}{2}\,dx = \frac{1}{2}\left[\int_{-1}^{0}(x^2+x)\,dx\right] = \frac{1}{2}\left[\int_{-1}^{0} x^2\,dx + \int_{-1}^{0} x\,dx\right] = -\frac{1}{12}$$

The calculated area under x axis of the graph corresponds with the sum of the infinite series of first negative integer input of Euler-Riemann Zeta Function. Since the tip of the parabola sits under the x axis, the area is negative.

For the graph of $f(x) = \frac{2n^3+3n^2+n}{6}$, one will take the same units in x and y axis consecutively. The cubic function shows three roots, and as x intercept, and it passes through $(-1,0)$ *and* $(0,0)$ having $(-0.5,0)$ in the middle.

Now;

$$\int_{-1}^{0} \frac{2x^3 + 3x^2 + x}{6}\, dx = \frac{1}{6}\left[\int_{-1}^{0} 2x^3\, dx + \int_{-1}^{0} 3x^2\, dx + \int_{-1}^{0} x\, dx\right] = 0$$

The cubic graph has two tips both up and above x axis, and is of same area. But, the area above x axis is positive, whereas the below is negative. Since they are of same area in unit, they cancel each other as a sum in whole. Which is the result one could find after integrating the function within [−1,0] interval. It is to be noted that the graph, like the previous graph of the first negative integer input, passes through points (−1,0), (0,0) within interval [−1,0]. This matches with the *Trivial Zero* of *Ramanujan's Summation*.

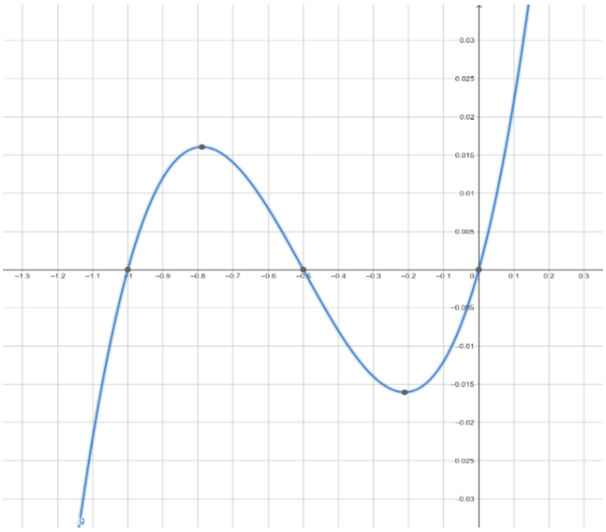


Fig 02: graph of $\frac{2x^3+3x^2+x}{6}$

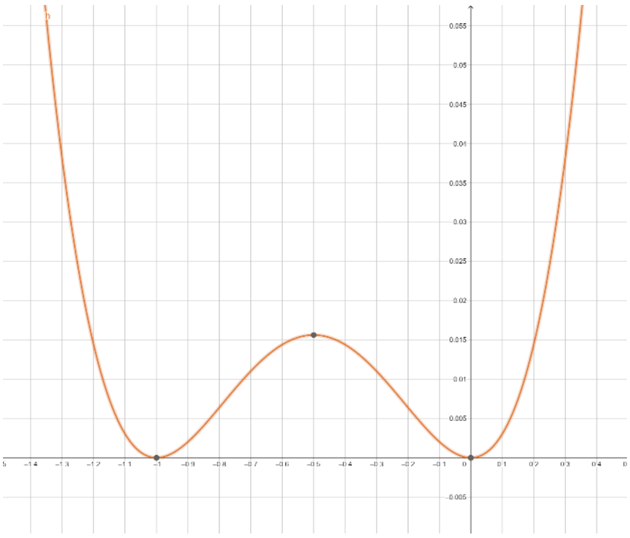


Fig 03: graph of $\frac{x^4+2x^3+x^2}{4}$

For $f(x) = \frac{x^4+2x^3+x^2}{4}$ we have two zeroes from the graph as the polynomial function is the square of the previous $\frac{x(x+1)}{2}$. So, the squares take the other two values away. The graph makes a "*w*" *curve* which touches the x axis at (−1,0) *and* (0,0). The area will be positive as the graph is above the axis.

This time too, the graph has an x intercept interval of [−1,0].

Integrating the function between [−1,0] gives us the value of $\zeta(-3)$;

$$\int_{-1}^{0} \frac{x^4 + 2x^3 + x^2}{4}\, dx = \frac{1}{4}\left[\int_{-1}^{0} x^4\, dx + \int_{-1}^{0} 2x^3\, dx + \int_{-1}^{0} x^2\, dn\right] = \frac{1}{120}$$

For $f(x) = \frac{6x^5+15x^4+10x-x}{30}$ one has 5 zeroes of the polynomial function respectively adding two more two points to the interval; $(-1.263762615826, 0), (0.263762615826, 0)$. However, the (−1,0), (0,0) points stay the same as the graph passes through them. The area will be zero, as the bumps of the graph are equal to their opposite sides of x axis.

The integral gives us 0;

$$\int_{-1.263762615826}^{0.263762615826} \left[\frac{6x^5+15x^4+10x^3-x}{30}\right] dx = 0$$

So far so good, the area of the graph within the interval matches the integral perfectly. It also passes through the initial points (-1,0) and (0,0) with which we started.

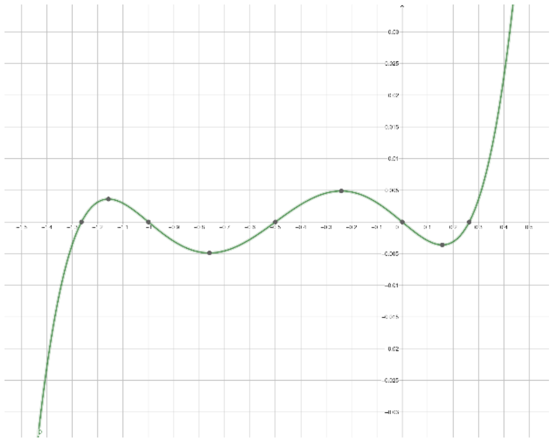


Fig 04: graph of $\frac{6x^5+15x^4+10x^3-x}{30}$

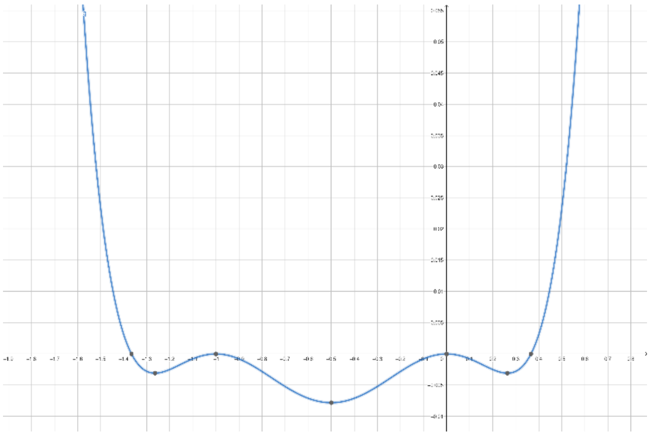


Fig 05: graph of $\frac{2x^6+6x^5+5x^4-x^2}{12}$

The graph of $f(x) = \frac{2x^6+6x^5+5x^4-x^2}{12}$ has 4 zeroes passing through $(-1.366025403844{,}0)$, $(-1{,}0)$, $(0{,}0)$, $(0.366025403844{,}0)$. Since the tips of the graph are pointed below the x axis, the area will be negative.

Within $[-1.366025403844, 0.366025403844]$,

One has;

$$\int_{-1.366025403844}^{0.366025403844} \frac{2x^6+6x^5+5x^4-x^2}{12}\, dx = -\frac{1}{252}$$

The graph of $f(x) = \frac{6x^7+21x^6+21x^5-7x^3+x}{42}$ passes through $(-1{,}0), (-0.5{,}0)$ *and* $(0{,}0)$ which is between the same $[-1{,}0]$ interval.

Now;

$$\int_{-1}^{0} \frac{6x^7+21x^6+21x^5-7x^3+x}{42} dx = 0$$

The integral gives us zero like the previous $\zeta(-2n)$.

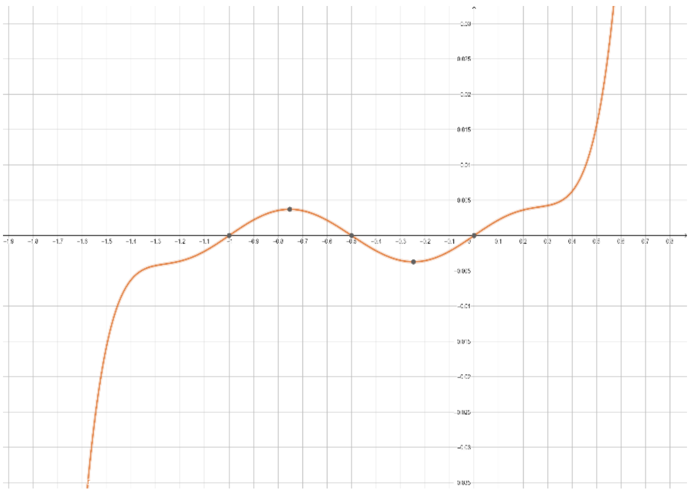


Fig 06: graph of $\frac{6x^7+21x^6+21x^5-7x^3+x}{42}$

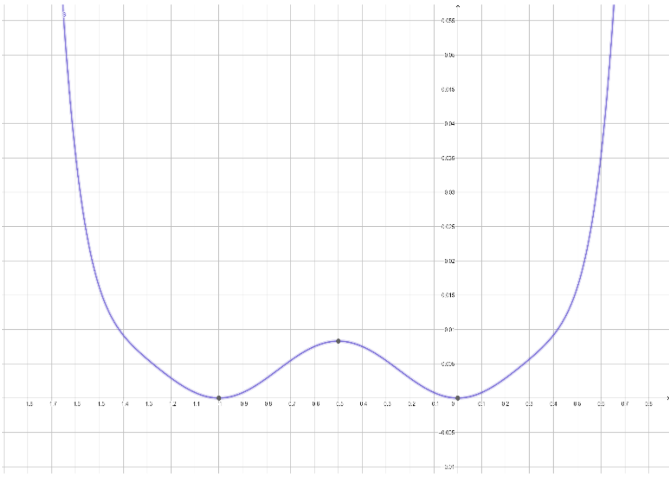


Fig 07: graph of $\frac{3x^8+12x^7+14x^6-7x^4+2x^2}{24}$

$f(x) = \frac{3x^8+12x^7+14x^6-7x^4+2x^2}{24}$ gives us another "w" curve having zeroes at (−1,0) *and* (0,0). Having the curvature tip above the x axis will make the area positive, and like the previous graphs, the curve has an x intercept interval of [−1,0].

As integral, one has;

$$\int_{-1}^{0} \frac{3x^8+12x^7+14x^6-7x^4+2x^2}{24}\,dx = \frac{1}{240}$$

The next graph gives us 5 zeroes, including (−1,0), $(-0.5, 0)$, (0,0), (−1.449106003964,0) and, (0.449106003964,0). As the previous $\zeta(-2n)$ inputs, the area of the graph within the interval is zero.

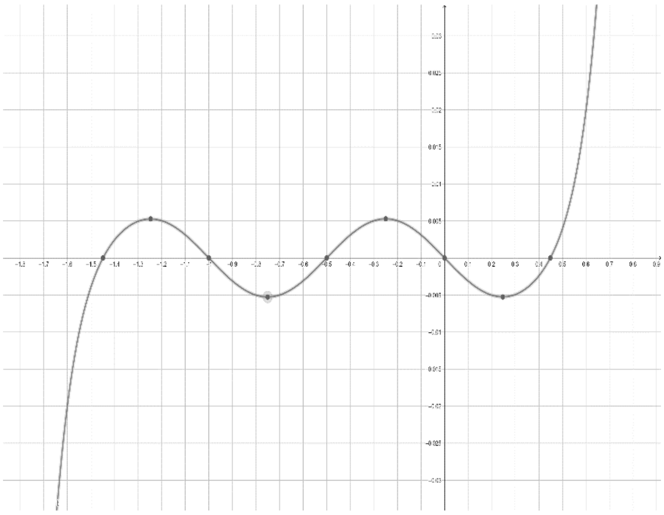


Fig 08: graph of $\frac{10x^9+45x^8+60x^7-42x^5+20x^3-3x}{90}$

$$\int_{-1.449106003964}^{0.449106003964} \frac{10x^9+45x^8+60x^7-42x^5+20x^3-3x}{90}\,dx = 0$$

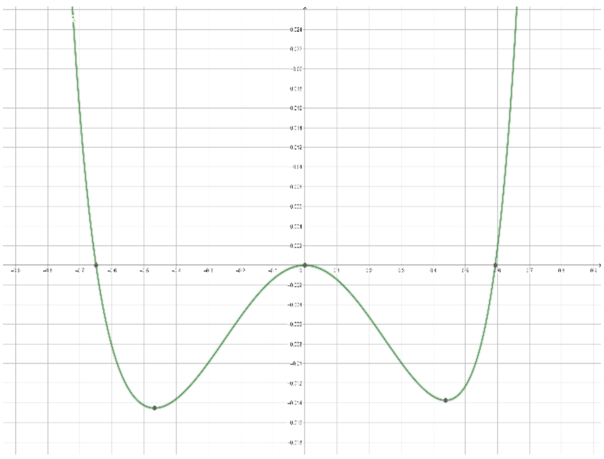


Fig 09: graph of $\frac{20x^{10}+10x^9+15x^8-14x^6+10x^4-3x^2}{20}$

$f(x) = \frac{20x^{10}+10x^9+15x^8-14x^6+10x^4-3x^2}{20}$ has only three zeroes. In order, (−0.6495984198519,0), (0,0) and (0.5929117944181,0). The area within the closed interval $(-0.6495984198519, 0.5929117944181)$ would be negative, as two tips of the graph is pointed downwords the x axis.

Integrating will give one;

$$\int_{-0.6495984198519}^{0.5929117944181} \frac{20x^{10}+10x^9+15x^8-14x^6+10x^4-3x^2}{20}\,dx = -\frac{1}{132}$$

For the next polynomial function;

$$\int_{-1.6180339887499}^{0.6180339887499} \frac{6x^{11}+33x^{10}+55x^9-66x^7+66x^5-33x^3+5x}{66}\,dx = 0$$

But, here one finds something very interesting. The graph has 7 zeroes, but all the points are in the interval of $[-1.6180339887499, 0.6180339887499]$. The other passing points match the previous graphs, but if one looks very closely, the interval shows the values of *Golden Ratio*.

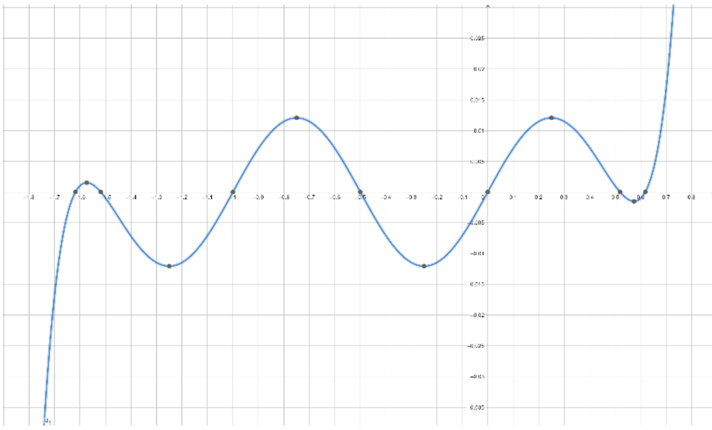


Fig 10: graph of $\frac{6x^{11}+33x^{10}+55x^9-66x^7+66x^5-33x^3+5x}{66}$

The Golden Ratio is defined by;

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887499\ldots$$

On the left side of the graph of $-x$ axis, the curve passes through (−1.6180339887499,0) which is just the negative of the *Golden Ratio*. The expression above is the solution of the quadratic equation $x^2 = x+1$ which has a conjugate solution of $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. The first gives the value 1.6180339887499 and the later −0.6180339887499. One should not be afraid that the graph passes through the polar opposite of the golden ratio numbers within the interval $(-1.6180339887499, 0.6180339887499)$, but it is very unexpected encounter as it passes through those points, though they are neagtive.

The 11^{th} polynomial gives us a very weird looking w curve, touching the x axis at (−1,0) and (0,0). It surprisingly, only has two zeroes. One is back to the point he started with. This time, the polynomial function is huge and the w curve's bell-ish part is above the x axis. So, the area within the interval [−1,0] will be positive.

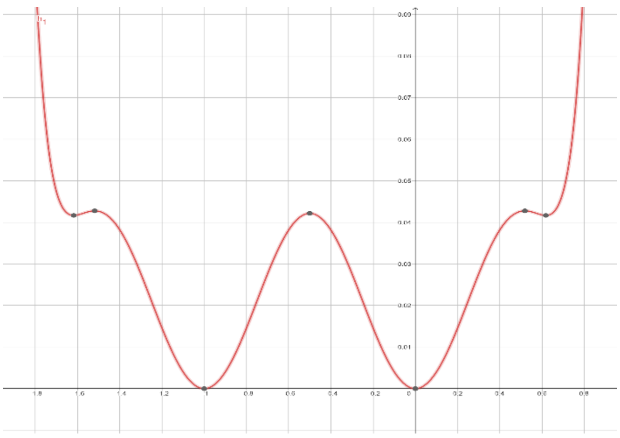
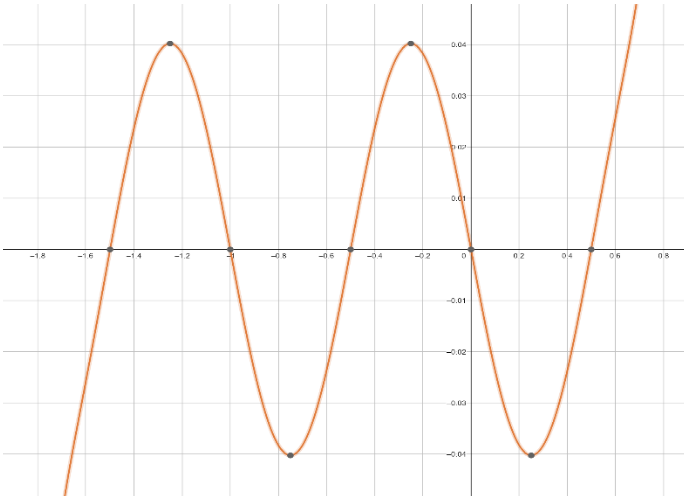


Fig 11: graph of $\frac{2x^{12}+12x^{11}22x^{10}-33x^8+44x^6-33x^4+10x^2}{24}$

Integrating the polynomial function from -1 *to* 0 gives;

$$\int_{-1}^{0} \frac{2x^{12} + 12x^{11} 22x^{10} - 33x^{8} + 44x^{6} - 33x^{4} + 10x^{2}}{24} = \frac{691}{32760}$$



For our last function, we have another zero, as it is a $\zeta(-2n)$ variant. It passes through the respective points;

$(-1.4990565978821{,}0), (-1{,}0), (-0.5{,}0), (0{,}0),$ *and* $(0.4990565978821{,}0)$ with an interval of

$[-1.4990565978821, 0.4990565978821]$ and integrating gives us a zero.

Fig 12: graph of $\frac{201x^{13}+1365x^{12}+2730x^{11}-5005x^{9}+8580x^{7}-9009x^{5}+4550x^{3}-691x}{2730}$

$$\int_{-1.4990565978821}^{0.4990565978821} \frac{201x^{13} + 1365x^{12} + 2730x^{11} - 5005x^{9} + 8580x^{7} - 9009x^{5} + 4550x^{3} - 691x}{2730} dx$$

$$= 0$$

***Chapter* 05**

***Conclusion* ....**

Compiling all the pages prior;

$$\zeta(-s) = \sum_{n=1}^{\infty} n^s = \frac{1}{s+1} \int_a^b \left[\sum_{i=0}^{s} \binom{s+1}{i} \left\{ \sum_{k=0}^{i} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k}{j} j^i \right\} x^{s+1-i} \right] dx$$

The notation at the left is the "Euler-Riemann Zeta Function", the middle is the summation form of the function. The integral at the right is the area within the X intercept interval of the graph produced by the expression inside the square bracket. The summation inside the curly bracket is the formula of finding the *Bernoulli Number*, and with the expression as a whole, it produces a curver that intersects X axis at *"a"* and *"b"*. The input "s" being an integer, in terms of Bernoulli Numbers and visual representation of graphs, one concludes.

.... fin ....