Noether's Theorem and Energy Conservation

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Date: April 27, to 28, 2025

Introduction

Conservation Laws are one of the crown jewels of fundamental physics and one of the core weapons to attack one of the hardest problems that plagues the landscape. Even though conservation laws are one of the key results, until the early 1920, the intellectual community didn't know the origin of them. They did know how the laws work, through something known to the mechanics community to be *Action*, but they could not find the core lying principle of where the laws came from. Many titans of mathematics and physics have tackled this problem, they did all their best, including providing their own formulations and certain bounds that describe the properties of such fundamental laws, until Emmy Noether came into the picture, and found the origin of all the conservation laws. This literature, very small, tries to collect the results in a very much fitting manner.

Setup: Actions and Symmetries

One considers a mechanical system that has configuration at any time $t \in \mathbb{R}$, which is described by generalized coordinates $q_i(t) \in \mathbb{R}$, with $i = 1, \ldots, n$. The space of all possible configurations forms the **configuration space** $\mathcal{Q} \cong \mathbb{R}^n$. The dynamics of the system are determined by a Lagrangian function:

$$L : T\mathcal{Q} \times \mathbb{R} \to \mathbb{R}, \quad (q_i, \dot{q}_i, t) \mapsto L(q_i, \dot{q}_i, t)$$

where $T\mathcal{Q}$ denotes the tangent bundle (the space of positions and velocities). The **action functional** is defined as:

$$S[q] = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)\, dt$$

and the principle of least action states that the physical trajectories $q_i(t)$ extremize $S[q]$.

An **infinitesimal transformation** of the variables is characterized by:

$$t \mapsto t' = t + \epsilon\ \delta t(q, \dot{q}, t),$$
$$q_i(t) \mapsto q_i'(t') = q_i(t) + \epsilon\ \delta q_i(q, \dot{q}, t)$$

where ϵ is an infinitesimally small quantity or parameter.

To first order in ϵ, the variation in the coordinates is:

$$\delta q_i = q_i'(t) - q_i(t) + \dot{q}_i \delta t$$

accounting for both the direct variation and the implicit time shift.

A transformation is called a **symmetry** of the action if the action is invariant under the transformation up to a boundary term:

$$\delta S = \int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \frac{dF}{dt}\, dt$$

for some function $F(q_i, t)$. This condition ensures that the equations of motion derived from the action remain unchanged under the symmetry transformation. Thus, in precise terms:

- The *configuration space* describes the possible states q_i of the system.
- The *action* $S[q]$ assigns a number to each trajectory based on the integral form of the Lagrangian.
- An *infinitesimal symmetry* leaves $S[q]$ invariant up to a total derivative.

Infinitesimal Transformations: Detailed View

An **infinitesimal transformation** of the system is a very small deformation of time and coordinates, expressed as:

$$t \mapsto t' = t + \epsilon\ \delta t(q, \dot{q}, t)$$

$$q_i(t) \mapsto q_i'(t') = q_i(t) + \epsilon\ \delta q_i(q, \dot{q}, t)$$

where ϵ is an infinitesimal parameter, and δt, δq_i describe how time and configuration change. Since q_i' is evaluated at the shifted time t', to express everything in terms of the original time t, we expand using Taylor's theorem:

$$q_i'(t) = q_i'(t') - \epsilon\ \dot{q}_i(t)\delta t$$

where $\dot{q}_i = dq_i/dt$. Thus, the total variation of q_i at fixed time t is:

$$\delta q_i(t) = q_i'(t) - q_i(t) = (q_i'(t') - q_i(t)) - \epsilon \dot{q}_i(t)\delta t$$

In other words, the total variation consists of two contributions:

- A *direct variation* $q_i'(t') - q_i(t)$, corresponding to how q_i is explicitly transformed,
- A *time-shift correction* $-\epsilon \dot{q}_i \delta t$, due to the small shift in time.

One has the correct formula for the infinitesimal variation at fixed t:

$$\boxed{\delta q_i = \delta q_i|_{\text{direct}} - \dot{q}_i \delta t}$$

This correction is essential when deriving conserved quantities through Noether's theorem.

Variation of the Action

One begins by considering the action $S[q]$ of the system, defined as:

$$S[q] = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)\, dt$$

where $L(q_i, \dot{q}_i, t)$ is the Lagrangian. To obtain the equations of motion, we seek the extremum of $S[q]$ under small variations of the path $q_i(t)$.

Step 1: Variation of the Lagrangian. We consider a small variation $\delta q_i(t)$ in the generalized coordinates:

$$q_i(t) \to q_i(t) + \delta q_i(t)$$

which induces a variation in the Lagrangian:

$$L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) \approx L(q_i, \dot{q}_i, t) + \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

Step 2: Variation of the Action. The variation of the action is:

$$\delta S = \int_{t_1}^{t_2} \left(\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

Step 3: Integration by Parts. The second term involves $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$. We use integration by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \, dt = \frac{\partial L}{\partial \dot{q}_i} \delta q_i \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt$$

Thus, the variation of the action becomes:

$$\delta S = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt$$

Step 4: Euler-Lagrange Equations. For arbitrary variations $\delta q_i(t)$, the action $S[q]$ will be stationary if:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

These are the *Euler-Lagrange equations*, which govern the motion of the system.

Noether Current:

The Noether current is defined as:

$$J^\mu = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i + L \delta t$$

The time component of the current j^0 is conserved:

$$\frac{d}{dt} j^0 = 0$$

This implies that the quantity associated with the symmetry is conserved over time.

Special Case: Energy Conservation

For infinitesimal time translation:

$$\delta t = 1, \quad \delta q_i = \dot{q}_i$$

thus:

$$J = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + L$$

but with a sign correction (since δt acts on the action integrand):

$$J = -\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right)$$

Therefore, the **conserved energy** E is:

$$\boxed{E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}$$

and:

$$\frac{dE}{dt} = 0$$

And such means, energy remains constant, for which if one differentiates, one gets zero.

Summary

Symmetry	Conserved Quantity
Spatial translation $x \mapsto x + \epsilon$	Linear momentum p
Rotation $\theta \mapsto \theta + \epsilon$	Angular momentum L
Time translation $t \mapsto t + \epsilon$	Energy E

References

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