Force, Work, and Energy in Field-Particle Interactions

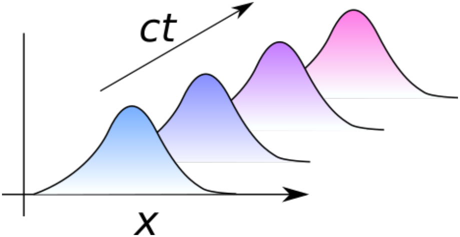
Mike Reppert

September 7, 2020

Last time on CHM676...

The Homogeneous Wave Equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{e}(\boldsymbol{x},t) = 0.$$



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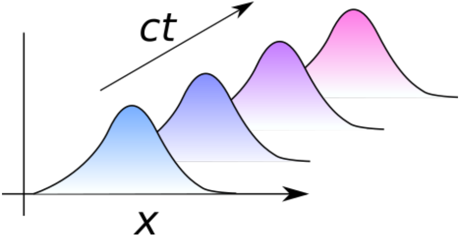

Today: Force, energy, and work in the EM field

Outline for Today:

1. Electromagnetic Work
2. The Poynting Vector and Energy Density
3. Detection of the EM Field

Electromagnetic Work

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$$\boldsymbol{F}_{EM} = q\left(\boldsymbol{e}(\boldsymbol{r},t) + \boldsymbol{v} \times \boldsymbol{b}(\boldsymbol{r},t)\right)$$

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Here Be Dragons! $\Rightarrow \approx -\int_{t_1}^{t_2} dt \int_V d\boldsymbol{x} \boldsymbol{j}(\boldsymbol{x}, t) \cdot \boldsymbol{e}(\boldsymbol{x}, t)$

Take-Home Points

The electric field does work on charged particles – the integral of the Lorentz force over the particle displacement.

The magnetic field does no work on charged particles.

For *finite particles*, the EM work can be written as an integral over the current density $\boldsymbol{j}(\boldsymbol{x}, t)$.

$$W_{\mathsf{el}} \approx -\int_{t_1}^{t_2} dt \int_V d\boldsymbol{x}\boldsymbol{j}(\boldsymbol{x}, t) \cdot \boldsymbol{e}(\boldsymbol{x}, t)$$

The Poynting Vector and Energy Density

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Solving the Maxwell-Faraday equation

$$\nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x}, t),$$

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The Poynting Vector and Energy Density

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$$\nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x}, t),$$

for $\boldsymbol{j}(\boldsymbol{x}, t)$, we get (after some cross-product magic...)

$$W_{\text{el}} = \int_{t_1}^{t_2} dt \int_V d\boldsymbol{x} \left(\nabla \cdot \boldsymbol{S} + \frac{\partial u}{\partial t} \right)$$

with the *Poynting vector*

$$\boldsymbol{S}(\boldsymbol{x}, t) \equiv \frac{c}{4\pi}\boldsymbol{e} \times \boldsymbol{b}$$

and the *electromagnetic energy density*

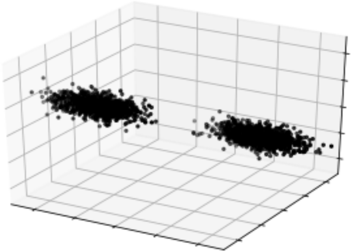
$$u(\boldsymbol{x}, t) = \frac{1}{8\pi}\left(\|\boldsymbol{e}\|^2 + \|\boldsymbol{b}\|^2\right).$$

The Poynting Vector and Energy Density

The **energy density** represents the "amount" of electromagnetic energy in a given region of space.

If the volume V contains the whole field:

$$W_{\text{el}} = \int_{t_1}^{t_2} dt \int_V d\boldsymbol{x}\, \frac{\partial u}{\partial t} = \int_V d\boldsymbol{x}\, u(x, t_2) - \int_V d\boldsymbol{x}\, u(x, t_1).$$

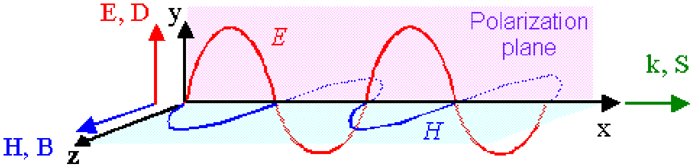


The Poynting Vector and Energy Density

The **Poynting vector** $\boldsymbol{S}$ represents the *magnitude and direction* of energy flow.

In vacuum, $\boldsymbol{S}$ and u satisfy a *continuity equation*:

$$\nabla \cdot \boldsymbol{S} + \frac{\partial u}{\partial t} = 0.$$



https://www.tf.uni-kiel.de/matwis/amat/admat_en/kap_5/backbone/r5_1_4.html

Take-Home Points

Electromagnetic work can be written as a time and space integral over two quantities:

The *energy density* $u(\boldsymbol{x}, t)$ characterizes the "amount" of EM energy in a given region of space

The *Poynting vector* $S(\boldsymbol{x}, t)$ characterizes the magnitude and direction of EM energy flow

The two are related by the *continuity equation*

$$\nabla \cdot \boldsymbol{S} + \frac{\partial u}{\partial t} = 0.$$

Detection of the EM Field

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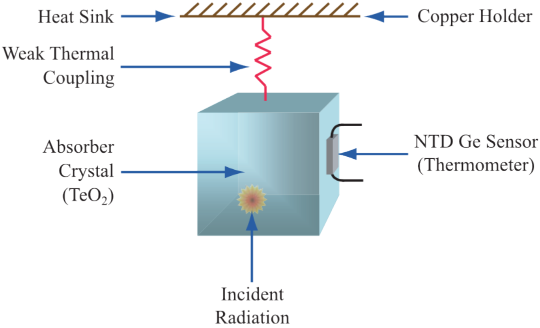
A: We measure the *energy absorbed* by a detector.



An example: the bolometer

https://cuore.lngs.infn.it/en/about/detectors

Energy Metrics for EM fields

Pulse Energy:

$$U_{\text{pulse}} = \int d\boldsymbol{x}\, u(\boldsymbol{x}, t).$$

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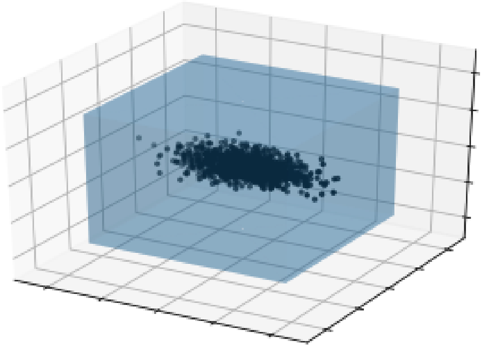

Energy Metrics for EM fields

Irradiance $\propto \boldsymbol{S}(\boldsymbol{x},t)\cdot\hat{\boldsymbol{n}}$:

$$\begin{aligned}\mathsf{Ir}_{\mathsf{det}} &= \frac{c\,(\hat{\mathbf{s}}\cdot\hat{\mathbf{n}})}{4\pi\tau_{\mathsf{det}}A_{\mathsf{det}}}\int_{t_o}^{t_o+\tau_{\mathsf{det}}} dt\int dA\,\|\boldsymbol{e}(\boldsymbol{x},t)\|^2 \\ &\approx \frac{c\,(\hat{\mathbf{s}}\cdot\hat{\mathbf{n}})}{8\pi^2\tau_{\mathsf{det}}A_{\mathsf{det}}}\int dA\int d\omega\,\|\breve{\boldsymbol{e}}(\boldsymbol{x},\omega)\|^2\,.\end{aligned}$$

Determined by $\hat{\mathrm{s}}\cdot\hat{\mathrm{n}}$ and the **intensity** $I(\boldsymbol{x},t)$ or $I(\boldsymbol{x},\omega)$.

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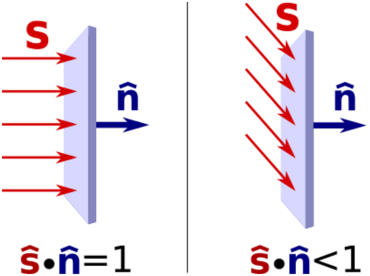

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$$\approx \frac{c\left(\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}\right)}{8\pi^2\tau_{\text{det}} A_{\text{det}}} \int dA \int d\omega \left\|\breve{\boldsymbol{e}}(\boldsymbol{x}, \omega)\right\|^2 .$$

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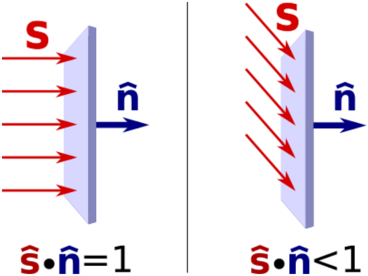

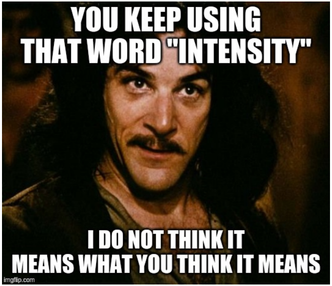

"Intensity"

Lots of different things get called "intensity"!

Radiant intensity	$I_{e,\Omega}$[nb 5]	watt per steradian	W/sr	$\mathsf{M{\cdot}L^2{\cdot}T^{-3}}$	Radiant flux emitted, reflected, transmitted or received, per unit solid angle. This is
Spectral intensity	$I_{e,\Omega,\nu}$[nb 3] *or* $I_{e,\Omega,\lambda}$[nb 4]	watt per steradian per hertz *or* watt per steradian per metre	$\mathrm{W{\cdot}sr^{-1}{\cdot}Hz^{-1}}$ *or* $\mathrm{W{\cdot}sr^{-1}{\cdot}m^{-1}}$	$\mathsf{M{\cdot}L^2{\cdot}T^{-2}}$ *or* $\mathsf{M{\cdot}L{\cdot}T^{-3}}$	Radiant intensity per unit frequency or wavelength. The latter is commonly measured in $\mathrm{W{\cdot}sr^{-1}{\cdot}nm^{-1}}$. This is a *directional* quantity.
Radiance	$L_{e,\Omega}$[nb 5]	watt per steradian per square metre	$\mathrm{W{\cdot}sr^{-1}{\cdot}m^{-2}}$	$\mathsf{M{\cdot}T^{-3}}$	Radiant flux emitted, reflected, transmitted or received by a *surface*, per unit solid angle per unit projected area. This is a *directional* quantity. This is sometimes also confusingly called "intensity".
Spectral radiance	$L_{e,\Omega,\nu}$[nb 3] *or* $L_{e,\Omega,\lambda}$[nb 4]	watt per steradian per square metre per hertz *or* watt per steradian per square metre, per metre	$\mathrm{W{\cdot}sr^{-1}{\cdot}m^{-2}{\cdot}Hz^{-1}}$ *or* $\mathrm{W{\cdot}sr^{-1}{\cdot}m^{-3}}$	$\mathsf{M{\cdot}T^{-2}}$ *or* $\mathsf{M{\cdot}L^{-1}{\cdot}T^{-3}}$	Radiance of a *surface* per unit frequency or wavelength. The latter is commonly measured in $\mathrm{W{\cdot}sr^{-1}{\cdot}m^{-2}{\cdot}nm^{-1}}$. This is a *directional* quantity. This is sometimes also confusingly called "spectral intensity".
Irradiance Flux density	E_e[nb 2]	watt per square metre	$\mathrm{W/m^2}$	$\mathsf{M{\cdot}T^{-3}}$	Radiant flux *received* by a *surface* per unit area. This is sometimes also confusingly called "intensity".
Spectral irradiance Spectral flux density	$E_{e,\nu}$[nb 3] *or* $E_{e,\lambda}$[nb 4]	watt per square metre per hertz *or* watt per square metre, per metre	$\mathrm{W{\cdot}m^{-2}{\cdot}Hz^{-1}}$ *or* $\mathrm{W/m^3}$	$\mathsf{M{\cdot}T^{-2}}$ *or* $\mathsf{M{\cdot}L^{-1}{\cdot}T^{-3}}$	Irradiance of a *surface* per unit frequency or wavelength. This is sometimes also confusingly called "spectral intensity". Non-SI units of spectral flux density include jansky ($1\ \mathrm{Jy} = 10^{-26}\ \mathrm{W{\cdot}m^{-2}{\cdot}Hz^{-1}}$) and solar flux unit ($1\ \mathrm{sfu} = 10^{-22}\ \mathrm{W{\cdot}m^{-2}{\cdot}Hz^{-1}} = 10^4\ \mathrm{Jy}$).
Radiosity	J_e[nb 2]	watt per square metre	$\mathrm{W/m^2}$	$\mathsf{M{\cdot}T^{-3}}$	Radiant flux *leaving* (emitted, reflected and transmitted by) a *surface* per unit area. This is sometimes also confusingly called "intensity".
Spectral radiosity	$J_{e,\nu}$[nb 3] *or* $J_{e,\lambda}$[nb 4]	watt per square metre per hertz *or* watt per square metre, per metre	$\mathrm{W{\cdot}m^{-2}{\cdot}Hz^{-1}}$ *or* $\mathrm{W/m^3}$	$\mathsf{M{\cdot}T^{-2}}$ *or* $\mathsf{M{\cdot}L^{-1}{\cdot}T^{-3}}$	Radiosity of a *surface* per unit frequency or wavelength. The latter is commonly measured in $\mathrm{W{\cdot}m^{-2}{\cdot}nm^{-1}}$. This is sometimes also confusingly called "spectral intensity".
Radiant exitance	M_e[nb 2]	watt per square metre	$\mathrm{W/m^2}$	$\mathsf{M{\cdot}T^{-3}}$	Radiant flux *emitted* by a *surface* per unit area. This is the emitted component of radiosity. "Radiant emittance" is an old term for this quantity. This is sometimes also confusingly called "intensity".
Spectral exitance	$M_{e,\nu}$[nb 3] *or* $M_{e,\lambda}$[nb 4]	watt per square metre per hertz *or* watt per square metre, per metre	$\mathrm{W{\cdot}m^{-2}{\cdot}Hz^{-1}}$ *or* $\mathrm{W/m^3}$	$\mathsf{M{\cdot}T^{-2}}$ *or* $\mathsf{M{\cdot}L^{-1}{\cdot}T^{-3}}$	Radiant exitance of a *surface* per unit frequency or wavelength. The latter is commonly measured in $\mathrm{W{\cdot}m^{-2}{\cdot}nm^{-1}}$. "Spectral emittance" is an old term for this quantity. This is sometimes also confusingly called "spectral intensity".
Radiant exposure	H_e	joule per square metre	$\mathrm{J/m^2}$	$\mathsf{M{\cdot}T^{-2}}$	Radiant energy received by a *surface* per unit area, or equivalently irradiance of a *surface* integrated over time of irradiation. This is sometimes also called "radiant fluence".

`https://en.wikipedia.org/wiki/Intensity_(physics)`

Take-Home Points

Optical fields are characterized experimentally by the energy they carry.

Detectors monitor EM fields by measuring the *energy absorbed* by optically dense materials.

The *pulse energy* refers to the total EM energy carried by an ultrafast pulse, i.e. $\int d\boldsymbol{x}\, u(\boldsymbol{x}, t)$

Irradiance refers to the rate at which a beam transmits energy in a given direction, i.e. $\boldsymbol{S} \cdot \hat{\boldsymbol{n}}$

Informally, we use the term "intensity" for either $I(\boldsymbol{x}, t) = \|\boldsymbol{e}(\boldsymbol{x}, t)\|^2$ or $I(\boldsymbol{x}, \omega) = \|\breve{\boldsymbol{e}}(\boldsymbol{x}, \omega)\|^2$.