

Electromagnetic Waves in Vacuum

Mike Reppert

August 21, 2019

The Lorentz Force Law:

$$\mathbf{F}_{\text{EM}} \approx q\mathbf{e}(\mathbf{r}, t) + \frac{q}{c}\mathbf{v} \times \mathbf{b}(\mathbf{r}, t)$$

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$$\nabla \cdot \mathbf{e} = 4\pi\rho(\mathbf{x}, t)$$

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Today: How does the EM field propagate in vacuum?

Outline for Today:

- 1 Decoupling the Electric and Magnetic Fields
- 2 Propagating Waves
- 3 Oscillating Signals: The Fourier Basis
- 4 Plane Waves

Decoupling the Electric and Magnetic Fields

Decoupling the E&M Fields

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\Downarrow

$$\nabla \times (\nabla \times \mathbf{e}(\mathbf{x}, t)) + \frac{1}{c^2} \frac{\partial^2 \mathbf{e}(\mathbf{x}, t)}{\partial t^2} = 0.$$

A Dirty Trick

Use the vector identity:

$$\nabla \times (\nabla \times \mathbf{v}) = -\nabla^2 \mathbf{v} + \nabla(\nabla \cdot \mathbf{v})$$

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or

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{e}(\mathbf{x}, t) = 0.$$

This is the **homogeneous wave equation**.

Take-Home Point

In vacuum, Maxwell's equations imply that *each component* of the electric field obeys the **homogeneous wave equation** (HWE).

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So what?

Propagating Waves

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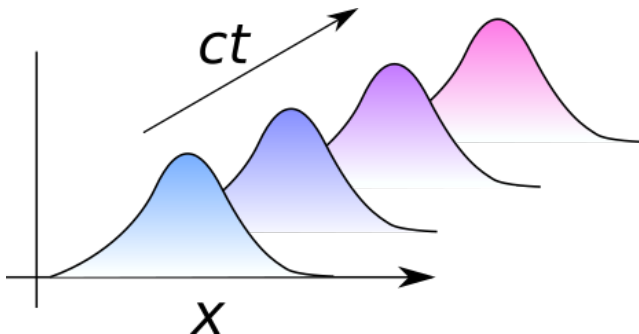
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Displacement along the unit vector $\hat{\mathbf{s}}$ is equivalent to a shift in time, i.e. the solution *propagates* at speed c .

Take-Home Point

Solutions to the HWE can take *any form* that propagates at the speed of light.



Oscillating Signals: The Fourier Basis



The Fourier Basis

The HWE is solved by *any* propagating function. So why do we usually think of “light waves” as oscillatory?

The Fourier Basis

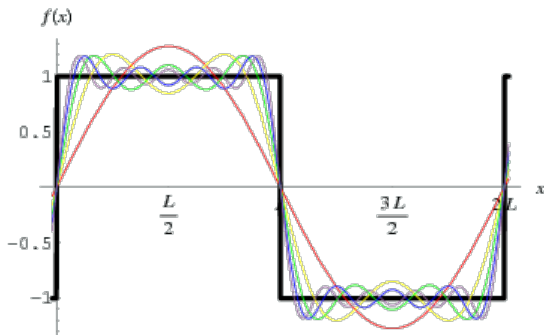
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The Fourier Basis

The HWE is solved by *any* propagating function. So why do we usually think of “light waves” as oscillatory?

- ① *Many* physical sources have well-defined frequencies
- ② *All* waves can be *represented* as a *sum* of oscillatory signals



Fourier decomposition

The Fourier Basis

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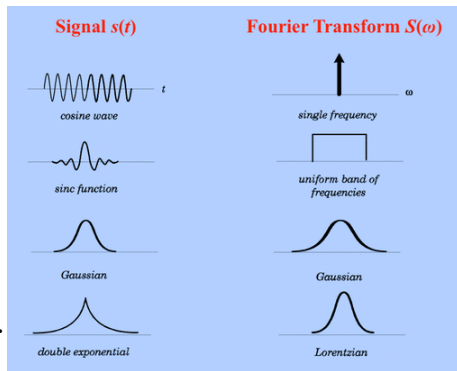
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<http://mriquestions.com/fourier-transform-ft.html>

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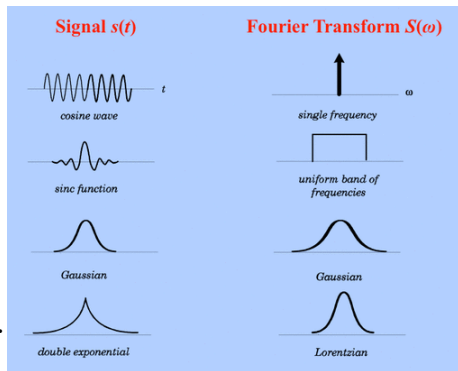
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Note: Widths are inversely related!

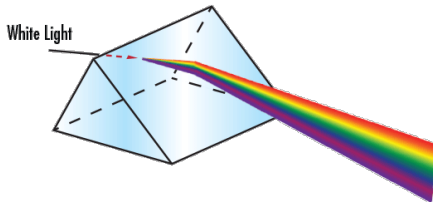
4D Fourier Transform

In electrodynamics, we use a 4D transform:

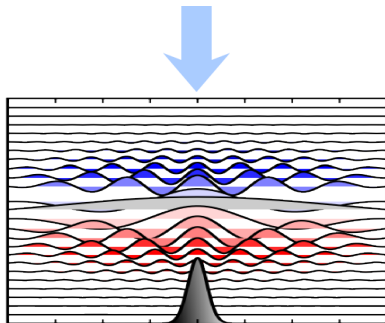
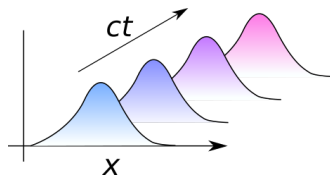
$$\tilde{e}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} d\mathbf{x} \int_{-\infty}^{\infty} dt e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} e(\mathbf{x}, t)$$

$$e(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\mathbf{k} \int_{-\infty}^{\infty} d\omega e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} \tilde{e}(\mathbf{k}, \omega).$$

The individual frequency/wavevector components in $\tilde{e}(\mathbf{k}, \omega)$ can be physically separated using a prism!

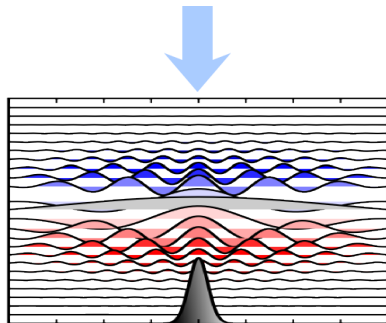
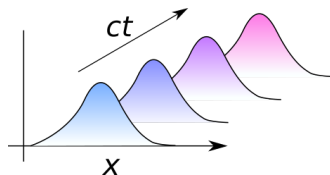


NB: The FT is completely general! *Any field* can be decomposed as an integral of Fourier components.



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What are the characteristic features of HWE solutions in Fourier space?



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For the HWE, this implies

$$\left(-\frac{\omega^2}{c^2} + k^2 \right) \tilde{e}(\mathbf{k}, \omega) = 0$$

$$\Downarrow$$

$$\omega = ck.$$

This is the **vacuum dispersion relation** connecting frequency and wavelength ($1/k$).

Take-Home Points

The Fourier Transform splits signals into *frequency components*.

Using a 4D FT, we can split the field into frequency components in *both time and space*.

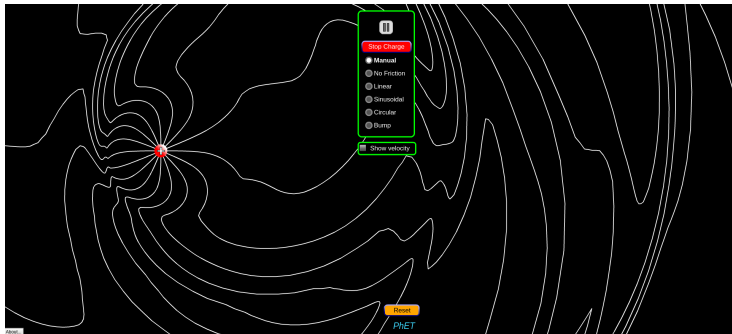
The FT converts differential equations into algebraic equations. In vacuum, the HWE implies the dispersion relation $\Rightarrow \omega = ck$.

Plane Waves

Plane Waves

In general, electromagnetic fields can be very complex!

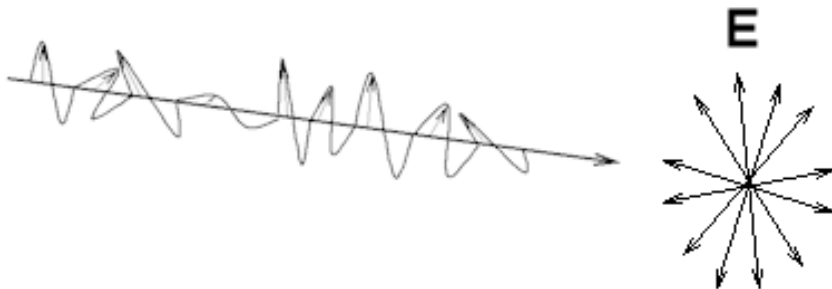
https://phet.colorado.edu/sims/radiating-charge/radiating-charge_en.html



Usually, we'll consider simplified forms
 \Rightarrow plane waves.

Ideal Beams

A *plane wave* is an electromagnetic field propagating with a fixed \hat{s} -vector.



<http://labman.phys.utk.edu/phys222core/modules/m6/polarization.html>

Plane Waves

General

$$e(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\mathbf{k} \int_{-\infty}^{\infty} d\omega e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} \tilde{e}(\mathbf{k}, \omega)$$

\Downarrow

Sum of Plane Waves

$$e(\mathbf{x}, t) = \frac{1}{2\pi} \sum_i \int_0^{\infty} d\omega \tilde{A}^{(i)}(\omega) e^{-i\frac{\omega}{c}(ct - \hat{\mathbf{s}}^{(i)} \cdot \mathbf{x})} + \text{c. c.}$$

\Downarrow

Plane Wave

$$e(\mathbf{x}, t) = \frac{1}{2\pi} \int_0^{\infty} d\omega \tilde{A}(\omega) e^{-i\frac{\omega}{c}(ct - \hat{\mathbf{s}} \cdot \mathbf{x})} + \text{c. c.}$$

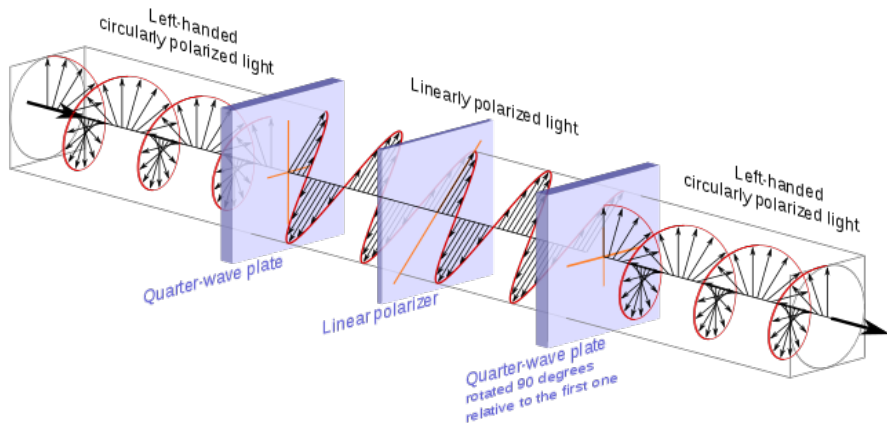
\Downarrow

Polarized Plane Wave

$$e(\mathbf{x}, t) = \frac{\hat{\mathbf{e}}}{2\pi} \int_0^{\infty} d\omega \tilde{A}(\omega) e^{-i\frac{\omega}{c}(ct - \hat{\mathbf{s}} \cdot \mathbf{x})} + \text{c. c.}$$

Plane Waves

Polarized plane waves have both a propagation axis \hat{s} and a polarization vector \hat{e}



<https://en.wikipedia.org/wiki/Polarizer>

Take-Home Points

In general: Electromagnetic fields are complicated!

A *plane wave* is an EM field with a well-defined propagation axis \hat{s}

A *polarized plane wave* has both a propagation axis \hat{s} and a polarization vector $\hat{\epsilon}$

Polarization comes in several flavors: Circular, elliptical, linear.