CH 5 Eigenvectors and Eigenvalues

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0.1 Inner Product, Length, Orthogonality

0.1.1 The Inner Product (Dot Product), Length of a vector

***dot product is a TYPE of Inner Product see calculus CH12
process of creating unit vector sometimes called normalizing v
distance formula

0.1.2 Dot Product Theorem, Pythagorean Theorem

see calculus: two vectors are orthogonal if DP is 0
two vec are orthogonal IFF pythagorean theorem

0.1.3 Orthogonal Complements

Orthogonal Component: z is orthogonal to W if the set of all z are orthogonal to W denoted by $W^{\perp}$
1. $\boldsymbol{x}$ is in $W^{\perp}$ IFF $\boldsymbol{x}$ is orthogonal to every vec in a set that spans W
2. $W^{\perp}$ is a subS of $\mathbb{R}^n$

0.1.4 Perpindicular Subspace Theorem

let A mxn. The orthogonal complement of the RS is the NS of A. and the orthogonal complement of CS is the NS of A^T :

$$(\text{Row}A)^{\perp} = \text{Nul}A \text{ and } (\text{Col}A)^{\perp} = \text{Nul}A^T$$

see FIG8, proof

0.1.5 Angles in $\mathbb{R}^2 or 3$

Dot Product Theorem corollary solve for angle.

0.2 Orthogonal Sets

a set of vectors u in $\mathbb{R}^n$ is an orthogonal set if each pair of distinct vectors from the set is orthogonal: $\boldsymbol{u_i} \cdot \boldsymbol{u_j} = 0$ when $i \neq j$

0.2.1 Linearly Independent Orthogonal Set Theorem

if S is an OS of NZ vec, then S is linearly indepdent and therefore is a basis for subS spanned by S.

0.2.2 Orthogonal Basis

a subS W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set

0.2.3 Orthogonal Decomposition Theorem

u vectors be an OB for subS W of $\mathbb{R}^n$. for each $\boldsymbol{y} inW$ the weights in the LC:

$$\boldsymbol{y} = c_1\boldsymbol{u_1} + \cdots + c_p\boldsymbol{u_p}$$

are given by

$$c_j = \frac{\boldsymbol{y} \cdot \boldsymbol{u_j}}{u_j \cdot u_j} \ \ (j = 1 \ldots p)$$

0.2.4 Orthogonal Projection HELP

see 342/FIG2
think adjacent/base of a right triangle
Vector Orthogonal Projection of y onto L:

$$\hat{\boldsymbol{y}} = \mathrm{proj}_L \boldsymbol{y} = \frac{\boldsymbol{y} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}} \boldsymbol{u}$$

0.2.5 Geometric Interpretation of Orthogonal Decomp Theorem

see 3blue1brown comments:
think of $\hat{u}$ as the basis vec of a 1D subS of W, then all the LC/projections of $\hat{i}, \hat{j}$ is mapped onto that subS.
Sum of projections.

0.2.6 Decomposing Force into Component Forces APPLICATION

later

0.2.7 Orthonormal Sets

orthonormal set if orthogonal set of UNIT vec.
orthonormal basis if subS spanned by orthonormal set.

0.2.8 Orthonormal Columns Identity Theorem

mxn U has orthonormal col IFF $U^T U = I$
see proof

0.2.9 Properties of Orthonormal Col Theorem

a. $||U||_{\boldsymbol{x}} = ||\boldsymbol{x}||$
b. $(U\boldsymbol{x}) \cdot (U\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$
c. $(U\boldsymbol{x}) \cdot (U\boldsymbol{y}) = 0$ IFF $\boldsymbol{x} \cdot \boldsymbol{y} = 0$
(a) and (c) say that linear mapping $\boldsymbol{x} \mapsto U\boldsymbol{x}$ preserves lengths and orthogonality.
Orthogonal Matrix:
square invertible matrix U such that

$$U^{-1} = U^T$$

and has orthonormal col and rows.

0.3 Orthogonal Projections

0.3.1 Orthogonal Decomposition Theorem see previous sec

0.3.2 Properties of Orthogonal Projections

If $\boldsymbol{y}$ is in W = Span $\{\ldots u_p\}$, then $\text{proj}_W \boldsymbol{y} = \boldsymbol{y}$

0.3.3 Best Approximation Theorem

let W be subS of $\mathbb{R}^n$, let $\boldsymbol{y}$ be any vec in $\mathbb{R}^n$, let $\hat{\boldsymbol{y}}$ be the orthogonal projection of $\boldsymbol{y}$ onto W. Then $\hat{\boldsymbol{y}}$ is the closest point in W to $\boldsymbol{y}$

$$||\boldsymbol{y} - \hat{y}|| \leq ||\boldsymbol{y} - \boldsymbol{v}||$$

for all $\boldsymbol{v}$ in W distinct from $\boldsymbol{y}$. (Think right triangle, or this looks like stats (error))

0.3.4 Orthonormal Basis Projection Theorem

u vec is orthonormal basis for a subS W of $\mathbb{R}^n$, then

$$\text{proj}_W \boldsymbol{y} =) \cdots + (\boldsymbol{y} \cdot \boldsymbol{u_p})\boldsymbol{u_p}$$

If U is the matrix of u vec, then

$$\text{proj}_W \boldsymbol{y} = UU^T\boldsymbol{y} \text{ for all } \boldsymbol{y} \in \mathbb{R}^n$$

0.4 The Gram-Schmidt Process, QR Factor

0.4.1 Gram-Schmidt Process Theorem

Given basis x for a NZ subS W of $\mathbb{R}^n$ define

$$\boldsymbol{v}_p = \boldsymbol{x}_p - \frac{\boldsymbol{x_p} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} \cdots - \frac{\boldsymbol{x_p} \cdot \boldsymbol{v_{p-1}}}{\boldsymbol{v_{p-1}} \cdot \boldsymbol{v_{p-1}}}\boldsymbol{v_{p-1}}$$

then set v is an orthogonal basis for W
span v = span x for $1 \leq k \leq p$
Process: read EX2

0.4.2 Orthonormal Bases

just normalize all v in orthogonal basis

0.4.3 QR Factorization Theorem

If A mxn with LI col, then A can be factored $A = QR$ where Q is an mxn whose col form orthogonal basis for Col A
and R is an nxn upper triangular invertible MX w/ positive entries on its diagonal.

0.5 Least-Squares Problems APPLICATIONS

later in stats!!!

0.6 APPLICATIONS to Linear Models

later in stats!!!

0.7 Inner Product Spaces

0.7.1 Definition

associates a real num ¡u, v¿ and and satisfies: 1. (Commutativity)$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
2. (Distribution)$\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u} + \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$
3. (Scalar Mult)$\langle \boldsymbol{cu}, \boldsymbol{v} \rangle = c\langle \boldsymbol{u}, \boldsymbol{v} \rangle$
4. (Orthogonality)$\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0$ and $= 0$ IFF $\boldsymbol{u} = \boldsymbol{0}$
VS w/ Inner Product is Inner Product Space (IPS)
(Inner Product can be any function bruh)

0.7.2 Lengths, Distances, Orthogonality HELP

length or norm is

$$||\boldsymbol{v}|| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$$

Distance between u and v is abs(u-v)
u, v orthogonal if ¡u, v¿=0

0.7.3 Gram-Schmidt Process for IPS, Best Approximations

HELP! use external resources, later round2

0.7.4 Pytagorean Orthogonal Inner Product

See FIG2

$$||\boldsymbol{v}||^2 = ||\text{proj}_W \boldsymbol{v}||^2 + ||\boldsymbol{v} - \text{proj}_W \boldsymbol{v}||^2$$

0.7.5 The Cauchy-Schwarz Inequality

For all $\boldsymbol{u}, \boldsymbol{v}$ in V,

$$|\langle \boldsymbol{u}, \boldsymbol{u} \rangle| \leq ||\boldsymbol{u}||||\boldsymbol{v}||$$

0.7.6 The Triangle Inequality

For all $\boldsymbol{u}, \boldsymbol{v}$ in V,

$$||\boldsymbol{u} + \boldsymbol{v}|| \leq ||\boldsymbol{u}|| + ||\boldsymbol{v}||$$

see proof

0.7.7 Inner Product Calculus

reimann sum see explanation
Example of inner product:
$\left\langle f, g \right\rangle = \int_a^b f(t)g(t)dt$ is an inner product that has axioms 1-4

0.8 APPLICATIONS of Linear Product Spaces

LATER in stats!!!