CH 4 Continuous Variables and Their Probability Distributions

mrevanisworking

August 11, 2020

0.1 Intro

Continuous Random Variable: a RV that can take on any value in an interval.

0.2 The Probability Distribution for a CRV

0.2.1 Definition of CDF (distribution function)

let Y denote any RV. The DF of Y denoted by $F(y)$, is such that $F(y) = P(Y \leq y)$ for $-\infty < y < \infty$
CDF for DRV are always step functions because the CDF increases only at the finite or countable num of points w/ positive probabilities

0.2.2 Properties of a Distribution Function(CDF)

If $F(y)$ is a DF, then
1. $F(-\infty) = \lim_{y\to-\infty} F(y) = 0$
2. $F(\infty) = \lim_{y\to\infty} F(y) = 1$
3. $F(y)$ is a nondecreasing fxn of y. $(y_1 < y_2, F(y_1) < F(y_2)$

0.2.3 Definition of a Continuous Random Varaible

a RV Y w/ DF F(y) is said to be continuous if $F(y)$ is continuous for $-\infty < y < \infty$

0.2.4 Derivative of CDF is PDF

let F(Y) be the DF for a CRV Y. Then $f(y)$, given by

$$f(y) = \frac{dF(y)}{dy} = F'(y)$$

whenever the derivative exists, is called the probability density fxn for the RV Y.

$$F(y) = \int_{-\infty}^{y} f(t)dt$$

where $f(\cdot)$ is the PDF
PDF is a theoretical model

0.2.5 Properties of a Density Function (PDF)

if f(y) is a PDF for a CRV then
1. $f(y) \geq 0$ for all $y, -\infty < y < \infty$
2.$\int_{-\infty}^{\infty} f(y)dy = 1$
$F(y_0)$ gives the P that $Y \leq y_0$, it's of interest to determine y of RV Y such that $P(Y \leq y) \geq$ some value

0.2.6 Definition of Quantile, Percentile HELP

Let Y denote any RV. If $0 < p < 1$, the pth quantile of Y denoted byϕ_p is the smallest val such that $P(Y \leq \pi_q) = F(\pi_p) \geq p$. If Y is cont.,$\pi_p$ is the smallest val such that $F(\phi_p) = P(Y \leq \phi_p) = p$
ϕ_p is referred to the 100pth percentile of Y

0.2.7 PDF Interval Theorem

If the RV Y has a DF $f(y)$ and$a < b$ then the P that Y falls in the interval $[a, b]$is

$$P(a \leq Y \leq b) = \int_{a}^{b} f(y)dy$$

NOT TRUE FOR DRV

0.3 Expected Values for CRV

0.3.1 Definition of EV of CRV

EV of CRV Y is

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy$$

exists if converges

0.3.2 EV of Function of Y Theorem

let g(Y) be a fxn of Y; then the EV of g(Y) is

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y)f(y)dy$$

0.3.3 EV of g(y) properties

let g1tok(Y) be fxns of CRV Y: 1.$E(c) = c$
2.$E[cgY)] = cE[g(Y)]$
3.$E[g1 + tok(Y)] = E[g_1(Y)] + tok$
some bonuses:
$g(Y) = (Y - \mu)^2$
$V(Y) = E(Y - \mu)^2$
$V(Y) = E(Y^2) - \mu^2$

0.4 The Uniform Probability Distribution

0.4.1 Definition of Uniform PD

If $\theta_1 < \theta_2$ a RV Y is said to have a cont. UPD on the interval (θ_1, θ_2) IFF the density fxn of Y is

$$f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0, & \text{elsewhere} \end{cases}$$

0.4.2 Parameters of Density Fxn

the constants that determine form of a density fxn are called parameters of the density fxn.

0.4.3 Mean and Variance of UPD

If $\theta_1 < \theta_2$ and Y is a RV uniformly distributed on the interval θ_1, θ_2, then

$$\mu = E(Y) = \frac{\theta_1 + \theta_2}{2} \text{ and } \sigma^2 = V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}$$

0.5 The Normal Probability Distribution

The most widely used cont PD is the normal distribution

0.5.1 Definition of Normal Probability Distribution

a RV Y is said to have a NPD IFF for $\sigma > 0$ and $-\infty < \mu < \infty$ the density fxn of Y is:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$$

for $-\infty < y < \infty$
area is the integral of this (which doesn't exist)

0.5.2 Mean and Variance of NPD

if Y is normally distributed RV w/ param μ, σ then

$$E(Y) = \mu \text{ and } V(Y) = \sigma^2$$

NPDF is symmetric around mean

0.5.3 Z and z score

z is the distance from the mean of a normal distribution expressed in units of standard deviation

$$z = \frac{y - \mu}{\sigma}$$

Can transform a normal RV Y to a standard normal RV Z by using

$$Z = \frac{Y - \mu}{\sigma}$$

SEE TABLE 4!!!

0.6 The Gamma Probability Distribution

some PDF are skewed right where most values are near origin

0.6.1 Definition of GPD

a RV Y is said to have a gamma distribution w/ param $\alpha > 0$ and $\beta > 0$ IFF the density fxn of Y is:

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} & 0 \leq y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

where the gamma function is

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y}dy$$

$\alpha$is called the shape parameter associated w/ a gamma distribution.
β is called the scale parameter
Special case: $\alpha = 1$ can be expressed as a sum of certain Poisson probabilities

0.6.2 Mean and Varaince of GPD HELP magic

if Y has a gamma distribution w/ α, β then

$$\mu = E(Y) = \alpha\beta \text{ and } \sigma^2 = V(Y) = \alpha\beta^2$$

0.6.3 Definition of chi-square distribution

Let v be a +int. A RV Y is said to have a chi-square distribution with v degrees of freedom IFF Y is a gamma distributed RV w/ parameteres $\alpha = \frac{v}{2}$,$\beta = 2$
A RV w/ chi-square distribution is a χ^2 RV.

0.6.4 Mean and Variance of chi-square distribution

if Y is a χ^2 RV w/ v DoF, then:

$$\mu = E(Y) = v \text{ and } \sigma^2 = V(Y) = 2v$$

SEE TABLE 6
gamma density function in which $\alpha = 1$ si called the exponential density function

0.6.5 Definition Exponential Distribution

a RV Y is said to have an exponential distribution w/ parameter $\beta > 0$ IFF the density fxn of Y is:

$$f(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta}, & 0 \leq y < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

useful for modeling length of life of electronic components

0.6.6 Mean and Variance of Exponential Distribution

If Y is an exponential RV w/ param β then:

$$\mu = E(Y) = \beta \text{ and } \sigma^2 = V(Y) = \beta^2$$

see example 4.10:
exponential distribution has a property called memoryless property (geometric distribution also has)

0.7 The Beta Probability Distribution

the beta density fxn is a two param density fxn defined over $0 \leq y \leq 1$. Often used as a model for proportions

0.7.1 Definition of Beta Probability Distribution

a RV Y has a BPD w/ param $\alpha > 0, \beta > 0$ IFF desnity fxn of Y:

$$f(y) = \begin{cases} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}. & 0 \leq y \leq 1 \\ 0, & \text{elsewhere,} \end{cases}$$

where the beta function is

$$B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

if $c \leq y \leq d$, then $y* = \frac{y-c}{d-c}$ defines a new var such that $0 \leq y* \leq 1$, thus BDF can be applied to a RV defined on int $c \leq y \leq d$ by translation
CDF for beta RV is the incomplete beta function:

$$F(y) = \int_0^y \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha, \beta)} dt = I_y(\alpha, \beta)$$

see Tables of the Incomplete Beta Function
related to binomial function: when integration by parts

$$F(y) = \sum_{i=\alpha}^{n} \binom{n}{i} y^i (1-y)^{n-i}$$

where $n = \alpha + \beta - 1$

0.7.2 Mean and Varaince of Beta Probability Distribution

if Y is a beta-distributed RV w/ param $\alpha > 0, \beta$

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta} \text{ and } \sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

0.8 General Comments

a good model is one that yields good inferences about the population of interest. Selecting a reasonable models is a matter of acting on theoretical considerations. EX: Poisson RV is appropriate when the it is characterised by random behavior of events in time.
Another way to form model is from a frequency histogram and choose a density fxn similar to the frequency curve.
Can also calculate goodness of fit.
see Weibull distribution

0.9 Other Expected Values

Moments for CRV have definitions analogous to DRV

0.9.1 Definition Moments CRV

If Y is a CRV, then the kth moment about the origin is:

$$\mu_k' = E(Y^k)$$

k1toinf.
The kth moment about the mean, or the kth central moment:

$$\mu_k = E[(Y-\mu)^k],$$

k1toinf

0.9.2 Definition of Moment Generationg Function CRV

if Y is CRV, then the MGF of Y is:

$$m(t) = E(e^{tY})$$

MGF exists if there exists a constant $b > 0$such that$m(t)$is finite for $|t| \leq b$
binomial expansion can be used to solve

0.9.3 MGF for g(Y)

let Y be a RV with density fxn f(y) and g(Y) be a fxn of Y. then MGF for g(Y)

$$E[e^{tg(Y)}] = \int_{-\infty}^{\infty} e^{tg(y)} f(y)dy$$

0.10 Tchebysheff's Theorem CRV

0.10.1 Theorem

let Y be a RC w/ finite mean μ and variance σ^2 then for any $k > 0$

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} \text{ or } P(|Y - \mu| \leq k\sigma) \leq \frac{1}{k^2}$$

Allows to find bounds for P easier
Can find mean and variances of RV without knowledge of distribution

0.11 Expectations of DiscontFxn and Mixed PD (HELP R2)

a RV t that has some P at discrete points and the remainder spread over intervals is a mixed distribution. EX: insurance

0.11.1 Definition of a Mixed Distribution Function

$$F(y) = c_1F_1(y) + c_2F_2(y)$$

where F1 is a step distribution fxn and F2 is a continuous distribution fxn
suppose that X_1 is a DRV w/ dist fxn $F_1(y)$
suppose that X_2 is a CRV w/ dist fxn $F_2(y)$
let $g(Y)$ denote a fxn of :

$$E[g(Y)] = c_1E[g(X_1)] + c_2E[g(X_2)]$$

0.12 Summary

PDF, CDF, Normal, Exponential, Gamma, χ^2, Beta
Moments,MGF, Tchebysheff
IMPORTANT:
first moment is mean
second moment is variance
third moment is skewness(about mean)
fourth moment about mean is kurtosis