CH 1 Linear Equations

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July 30, 2020

0.1 Systems

Linear Equation (LE): $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where a are the coefficients.
Linear System (LS): system of linear equations.
Solution set (SS): set of all possible
solutions of an LS.
Equivalent: two LS are equivalent if SS are the same. LS: has no solution (inconsistent), one (consistent), or infinite (consistent). Coefficient Matrix (CM): matrix of coefficients of LS Augmented Matrix (AM): matrix of coefficients and constants
Size: $(m \times n)$

0.1.1 Solving Methods

1. Gaussian Elimination.

0.1.2 Elementary Row Operations

1. (Replacement) (Pivot)Replace one row with sum of itself and a multiple of another row.
$R_1 - 3R_2 \to R_1$
2. (Interchange) (Swap) Interchange two rows.
$R_1 \leftrightarrow R_2$
3. (Scaling) (Scale)multiply a row by a NZ constant.
$R_1 \to R_2$
Row Equivalence (RE): if an MX is transformed by ERO because reversible. If AM are RE then same SS

0.1.3   Existence and Uniqueness

1. Is system consistent? 2. Is there a unique solution?

0.2   Row Reduction, Echelon Forms

Nonzero Row (NZR): row or column containing at least one NZ entry.
Leading Entry (LE): leftmost NZ entry in a row.
Echelon Form (EF) has:
1. All NZR are above ZR
2. Each LE is in a col to the right of the LE of the row above it.
3. All entries in a col below LE are zero
Creates a triangular matrix.
Reduced Echelon Form (REF) also has:
4. The LE in each NZR is 1.
5. Each leading 1 is the only NZEntry in its col
Echelon Matrix (EFM, REFM) is a matrix in EF

0.2.1   Row Reduction

Any NZM may be RR (by ERO).

0.2.2   Uniqueness of the REF Theorem

Each matrix is RE to one and only one REFM.
LE in any EF of MX is the same bc Uniqueness of REF.

0.2.3   Pivot Position

PP is a location in MX that corresponds to a L1 in REF.
Pivot Column contains a pivot position.

0.2.4   Row Reduction Algorithm

1. Leftmost NZCol is a PC. PP at top.
2. Select NZ entry in PC as P. Possibly swap rows for PP
3. Use replacement to create zeros below the Pivot
4. Repeat 1-3 on submatrix below top row.
5. with rightmost P, and working up and left, create zeros above P. If pivot is not 1,

then scale.

Foward Phase (1-4). Backward Phase (5). Computers use partial pivoting instead.

0.2.5 Solutions

Variables in pivot columns are basic variables (BV).

Variables not in pivot columns are free variables (FV).

Consistent systems can have basic variables in terms of FV. (These are parametric descriptions (PD of SS))

These are general solutions (exactly the same as DE because it is just system of equations).

When a system has a FV in a PP then no parametrix description (inconsistent).

0.2.6 Back-Substitution

Computers use it...

A Flop is a float arithmetic operation

0.2.7 Existence and Uniqueness Theorem

LS is consistent IFF rightmost col of AM is not a PC, meaning IFF EF of AM has no $[0 \cdots 0 b]$.

Consistent has:

(i) a US (unique solution), w/ no FV.

(ii) infinite solutions, w/ at least 1 FV.

0.2.8 RR to Solve LS

1. AM
2. RRA -¿ EFM (if consistent then go next)
3. RR -¿ REFM
4. MX to SE
5. Parametric Description

0.3 Vector Equations

Vector (vec): Ordered list of numbers.

INSIGHT: think of vectors' components as scalars that scale the basis vectors and the vector is the vector sum of those.

0.3.1 Vectors in x$\mathbb{R}^2$

Column vector (CVec): vec with 1 col
Two vec are equal if their entries are equal.
Sum is by adding entries. Scalar is by distribution

0.3.2 Parallelogram Rule for Addition

$\vec{u} + \vec{v}$ represents the forth vertex of a parallelogram created by $\vec{u}$ and $\vec{v}$ (verticies are $\vec{0}, \vec{u}, \vec{v}$).

0.3.3 Vectors in x$\mathbb{R}^n$

is collection of all ordered n-tuples of n real numbers. Written by $n \times 1$ MX.
see algebraic properties of $\mathbb{R}^n$.

0.3.4 Linear Combinations

$$\vec{y} = c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p$$

is a linear combination (LC) of $\vec{v}_1, \ldots, \vec{v}_p$ with weights $c_1, \ldots, c_p$

0.3.5 LC and MX

a VE (vector equation)

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b}$$

has the same SS as the AM

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots \vec{a}_n & \vec{b} \end{bmatrix}$$

$\vec{b}$ can be generated by an LCof a_1... IFF there exists a solution to the LS corresponding to the MX

0.3.6 Span

If $...\vec{v}_p$ are in $\mathbb{R}^n$ then the set of all LC of $...\vec{v}_p$ is the Span$\langle \vec{v}_1, \ldots, \vec{v}_p \rangle$. and is called the subset of $\mathbb{R}^n$ spanned by $\vec{v}_1, \ldots, \vec{v}_p$
Span: collection of all vec as

$$c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p$$

Span: collection of all LC
   vec b is in Span if

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

has a solution. Span contains all scalars of v. Think of it as a change in coordinates.

0.4   The Matrix Equation $A\vec{x} = \vec{b}$

DEF: If A is an $m \times n$ MX with cols a_1..., and if $\vec{x}$ is in $\mathbb{R}_n$, then the product of A and $\vec{x}$ is the LC of the cols of A using corresponding x as weights

$$A\vec{x} = [\vec{a}_n][x_n] = x_n\vec{a}_n$$

Defined only if A cols == x len

0.4.1   Theorem

$$A\vec{x} = \vec{b}$$
$$+...x_n\vec{a}_n = \vec{b}$$
$$= [...\vec{a}_n]$$

All have the same solution set

0.4.2   Existence

$A\vec{x} = \vec{b}$ IFF $\vec{b}$ is an LC of A cols
Questions with: is b in Span a, is Ax=b consistent, and is Ax=b consistent for all b

0.4.3   CM Equivalence Theorem

Either all statements are true or all are false.
a. For each $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution
b. each $\vec{b} \in \mathbb{R}^m$ is an LC of cols of A
c. cols of A span $\mathbb{R}^m$
d. A has a PP in every row

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0.4.4   Computations of Ax

Row-Vector Rule for Ax 1. if Ax is defined, then the ith entry in Ax is the dot product of corresponding entries from row i of A and from x 2. repeat for each row
   Identity matrix is 1 on diagonals of CM

0.4.5   Properties of Ax Theorem

a. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
b. $A(c\vec{u}) = c(A\vec{u})$

0.5   The Solution Sets of Linear Systems

0.5.1   Homogeneous Linear Systems (HLS) SEE EXAMPLES

Homogeneous if $A\vec{x} = \vec{0}$ and always has at least one solution $\vec{x} = \vec{0}$
which is called the Trivial Solution (TS) Important question is if a NTS exists. FACT: the HE $A\vec{=}\vec{0}$ has an NTS IFF the equation as at least one FV.
ss of he can always be expressed as span

0.5.2   Parametric Vector Form

Parametric Vector Equation/Form is explicit

$$\vec{x} = s\vec{u} + t\vec{v}(s, t, \in \mathbb{R})$$

The oirginal equation of a plane is an implicit description.
(Think implicit and explicit solutions of DE)

0.5.3   Solutions of Nonhomogeneous Systems

Do Ax=b and solve system, then
put in parametric form (this is the particular solution)
   Vector Addition here can be thought of TRANSLATION. Where adding p to v moves v parallel to its previous position. translated by p to v + p.
   SS of Ax=b is a line through p parallel to SS of Ax=0.
see Figure 5.

0.5.4 SS for NHE, HE Theorem

$A\vec{x} = \vec{b}$ and $\vec{p}$ is a solution.
Then SS of $A\vec{x} = \vec{b}$ is the set of all vec as $\vec{w} = \vec{p} + \vec{v}_h$ where $\vec{v}_h$ is any solution of the HE $A\vec{x} = \vec{0}$
APPLIES WHEN $A\vec{x} = \vec{b}$ has ast least one NZ solution

0.5.5 Summary, SS of Consistent System in PVF

1. RR AM to REF
2. Express basic vars in terms of FV
3. Write typical solution $\vec{x}$ as vec depending on FV if any
4. Decompose $\vec{x}$ into LC of vec using FV as param.

0.6 Applications...

0.7 Linear Independence

DEF: set $v_1, \ldots, v_p \in \mathbb{R}^n is$ linearly independent (LI) if the VE

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution, and is linearly dependent (LD) if there exist weights $c_1...$ not all zero, such that

$$c_1\vec{v}_1 + c_2 v_2 + \cdots + c_p\vec{p} = \vec{0}$$

0.7.1 LI of MC (MX cols)

$A\vec{x} = \vec{0} == ...x_n a_n = \vec{0}$
So cols of A are LI IFF $A\vec{x} = \vec{0}$ has only a trivial solution

0.7.2 Sets of One or Two Vec

set of one vec is LI IFF v is not $\vec{0}$
set of two vec is LD if one vec is a multiple of other

0.7.3 Characterization of LD Sets Theorem

An indexed set $S = \{\vec{v}_1, ..., \vec{v}_p\}$ of two or more vec is LD IFF at least one vec in S is a LC of the others.
if S is LD and $v_1 \neq \vec{0}$ then some v_j with j¿1 is a LC of the preceding vecs $v_1, ..., \vec{v}_{j-1}$

0.7.4 Vec ¿ Entries LD Theorem

If a set contains more vec than entries in each vec, then the set is LD. (if p ¿ n)

0.7.5 Zero vector LD Theorem

If a set S in $\mathbb{R}^n$ contains the zero vec, then set is LD

0.8 Intro to Linear Transformations

We can think of a the ME $A\vec{x} = \vec{b}$ equivalent to the VE notation, but we should think of the matrix notation as a transformation of the vector $\vec{x}$ to $A\vec{x}$

0.8.1 Definition of a Transformation

A transformation (or function, or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ assigns each $\vec{x} \in \mathbb{R}^n$ to a vec $T(\vec{x}) \in \mathbb{R}^n$.
The set $\mathbb{R}^n$ is the domain of T, $\mathbb{R}^m$ is called the codomain of T. The notation:
$T : \mathbb{R}^n \to \mathbb{R}^m$
means that the domain of $T is \mathbb{R}^n$ and the codomain is $\mathbb{R}^m$
n is columns, m is rows
For $\vec{x} \in \mathbb{R}^n$, the vec $T(\vec{x}) \in \mathbb{R}^m$ is called the IMAGE of $\vec{x}$ (under the action of T)
Range: set of all images $T(\vec{x})$
SEE FIGURE 2
VISUALS ARE VERY IMPORTANT FOR LIN ALG

0.8.2 Matrix Transformations HELP

denote a matrix transformation by $\vec{x} \mapsto A\vec{x}$

0.8.3 Linear Transformations HELP

DEF: a transformation (mapping) T is linear if:
(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in domain of T
(ii) $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in domain of T
So LT preserves operations of vec add and scalar mult therefore,
$T(\vec{0}) = \vec{0}$ and
$T(c_1\vec{v}_1 + ...c_p\vec{v}_p) = c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p)$
If a transformation satisfies the one above, must be linear.
The above is reffered to as superposition principle (same as DE)

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0.9   The Matrix of a Linear Transformation

T is completely determined by what it does to the cols of $n \times n IMX$
see Example 1

0.9.1   Transformation Matrix Theorem

$T : \mathbb{R}^n \to \mathbb{R}^m$ then there is a unique MX
such that

$$T(\vec{x}) = A(\vec{x}) \text{for all} \vec{x} \in \mathbb{R}^n$$

A is the $m \times n$ MX whose jth column is the vec $T(\vec{e}_j)$, where $\vec{e}_j$ is the jth col of the
IMX in R^n

$$A = \begin{bmatrix} A = T(\vec{e}_1) & \cdots & T(\vec{e}_n)) \end{bmatrix}$$

A is the standard matrix for the linear transformation T.
Matrix transformations focus on how it's implemented,
Linear Transformations focus on a property of mapping.

0.9.2   Geometric Linear Transformations of $\mathbb{R}^2$

Unit Square transformations (SEE TABLES 1-4 and FIG2-3)

0.9.3   Existence and Uniqueness

DEF EXISTENCE: a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$
is image of at least one $\vec{x} \in \mathbb{R}^n$
Meaning: $T(\vec{x}) = \vec{b}$, See FIG3
DEF UNIQUENESS: a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each $\vec{b} \in \mathbb{R}^m$ is the
image of at most one $\vec{x} \in \mathbb{R}^n$
Meaning: The $T(\vec{x}) = \vec{b}$ has either a unique solution or none at all.

0.9.4   1-1 IFF trivial solution Theorem

$T : \mathbb{R}^n \to \mathbb{R}^m$, then T is 1-1 IFF the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution

0.9.5

let T be an LT...
a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ IFF the cols of A span $\mathbb{R}^m$
b. T is 1-1 IFF the cols of A are LI

0.10 Linear Models (APPLICAIONS...)

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