CH 7 Symmetric Matricies and Quadratic Forms

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0.1 Diagonalization of Symmetric Matricies

Symmetric MX: a MX that $A^T = A$ (is square) maybe: row i = col i?

0.1.1 Orthogonal Evec Symmetry Theorem

if A is symmetric, then any two Evec from different ES are orthogonal.

0.1.2 Orthogonally Diagonalizable Theorem

a nxn is orthogonally diagonalizable if there are an orthogonal P ($P^{-1} = P^T$) and d such that:

$$A = PDP^T = PDP^{-1}$$

Theorem: nxn A is orthogonally diagonalizable IFF A is a symmetric matrix.

0.1.3 Spectral Theorem

set of Evec of A is sometimes called the spectreum of A.
Theorem: a nxn symmetric A has:
a. A has n real Evec, counting multiplicities.
b. dim of ES for each EV λ equals the multiplicity of λ as a root of the chara equation.
c. ES are mutually orthogonal: Evec corresponding to different EV are orthogonal.
d. A is orthogonally diagonalizable.

0.1.4 Spectral Decomposition

$$A = \cdots + \lambda_n \boldsymbol{u_n}\boldsymbol{u_n^T}$$

because it breaks up A into pieces determined by the spectrum of A.
$\boldsymbol{u_j}\boldsymbol{u_j}^T$ is a projection matrix:
for each x, the vector $()\boldsymbol{u_j}\boldsymbol{u_j}^T)\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x}$ onto subS spanned by $\boldsymbol{u_j}$

0.2 Quadratic Forms

a quadratic form on $\mathbb{R}^n$ is a fxn Q defined on $\mathbb{R}^n$ whose value at $\boldsymbol{x}$ is $\mathbb{R}^n$ can be computed by $Q(\boldsymbol{x})\boldsymbol{x}^T A\boldsymbol{x} where A is nxn symmetric.$
That A is the MX of the quadratic form.
quadratic forms are easier when no cross-product terms (when MX of quadratic form is diagonal)
cross-product term can be eliminated with change of variables

0.2.1 Change of Variable in a Quadratic Form

if $\boldsymbol{x}$ is a variable vec in rn, then change of variable is an equation of the form:

$$\boldsymbol{x} = P\boldsymbol{y} \text{ or } \boldsymbol{y} = P^{-1}\boldsymbol{x}$$

CHANGE OF VARIABLES (not change of coordinates)
*read 404 and check Theorem 2
SEE FIG1

0.2.2 The Principal Axes Theorem

A is nxn symmetric, then there is an orthogonal change of variable $\boldsymbol{x} = P\boldsymbol{y}$ that transforms the quadratic form $\boldsymbol{x}^T A\boldsymbol{x}$ into a quadratic form $\boldsymbol{y}^T D\boldsymbol{y}$ with no cross-product term (x_1x_2)
col of P in the theorem are the principle axes of xTAx.
y is the coordinate vec of x relative to the orthonormal basis of $\mathbb{R}^n$ given by these principle axes

0.2.3 Geometry of Pinciple Axes

if $\boldsymbol{x}^T A\boldsymbol{x} = c$ where A is an invertible 2x2 symm, then it is either an ellipse, hyperbola, intersecting lines, a point, or no points.
If A is a DMX, then graph is in standard position.
nonDMX is rotated out of standard position.
The principle axes (determined by Evec of A) means finding a new coord system wrt which the graph is in standard position.
SEE EX5

0.2.4 Classifying Quadratic Forms

quadratic form Q is:
a. positive definite if $Q(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$
b. negative definite if $Q(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \neq \mathbf{0}$
c. indefinite if $Q(\boldsymbol{x})$ assumes both positive and negative vals
Q is positive semidefinite if ≥ 0
Q is negative semidefinite if ≤ 0

0.2.5 QF and EV Theorem

let A be nxn symm. then QF $\boldsymbol{x}^T A\boldsymbol{x}$ is:
a. +def IFF EV of A are all positive
b. -def IFF EV of A are all negative
c. indefinite IFF A has both positive and negative EV
see diagram

0.2.6 QF MX

positive definite MX is a SMX for which the QF is +def
...

0.3 Constrained Optimization

the requirement that $\boldsymbol{x}$ is a unit vector:
mag = 1, mag squared = 1, $\boldsymbol{x}^T\boldsymbol{x} = 1$
See EX1 for min/max of Q

0.3.1 Optimization for SMX Theorem

let A be SMX, m is min, M is max EV of A.
value of $\boldsymbol{x}^T A\boldsymbol{x}$ is M when x is a unit Evec u corresponding to M.
the value of $\boldsymbol{x}^T A\boldsymbol{x}$ is m when x is a unit Evec corresponding to m.

0.3.2 Maximum for SMX Theorem(SEE NEXT THEOREM)

a,λ,$\boldsymbol{u}$ as in previous thm, max val of $\boldsymbol{x}^T A\boldsymbol{x}$ is subject to:

$$\boldsymbol{x}^T\boldsymbol{x} = 1 \ \ , \boldsymbol{x}^T\boldsymbol{u}_1 = 0$$

is the second greatest EV λ_2, and this max is attained when $\boldsymbol{x}$ is an Evec $\boldsymbol{u_2}$ corresponding to λ_2

0.3.3 General Maxmimum for SMX Theorem

let A be symm nxn w/ orthogonal diagonalization $A = PDP^{-1}$ where the entries on the diag of D are arranged so that each EV is less than the previous entry, where col of P are corresponding unit Evec u. Then for $k = 2, \ldots, n$, the maximum value of $\boldsymbol{x}^T Ax$ subject to:

$$\boldsymbol{x}^T\boldsymbol{x} = 1 \ \ , \boldsymbol{x}^T\boldsymbol{u}_1 = 0, \ldots \ \ , \boldsymbol{x}^T\boldsymbol{u}_{k-1} = 0$$

is the EV λ_k and this max is at $\boldsymbol{x} = \boldsymbol{u}_k$

0.4 APPLICATIONS to Image Processing and Stats

later in STATS and round2