CH 2 Matrix Algebra

mrevanisworking

August 3, 2020

0.1 Matrix Operations

columns are vectors in $\mathbb{R}^m$

$$A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_1} & \cdots & \boldsymbol{a_1} \end{bmatrix}$$

Diagonal entries form the main diagonal (MD).
Diagonal Matrix (DM) is $n \times n$ MX whose nondiags are zero.
Zero Matrix (ZM) is zero for all entries

0.1.1 Sums and Scalar Multiples

MX equal if same size and cols equal.
Sum $A + B$ is the sum of the columns. Only defined when A and B are the same size.
Scalar multiple of a matrix is scalar times cols.

0.1.2 Matrix Operations Theorem

see 95

0.1.3 Matrix Multiplication Definition

If A is $m \times n$ and B is $n \times p$ with $\boldsymbol{b_1}, \ldots, \boldsymbol{b_p}$ AB is the $m \times p$ whose columns are $A\boldsymbol{b_1}, ..., A\boldsymbol{b_p}$

$$AB = A \begin{bmatrix} \boldsymbol{b_1} & \boldsymbol{b_2} & \cdots & \boldsymbol{b_p} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{b_1} & A\boldsymbol{b_2} & \cdots & A\boldsymbol{b_p} \end{bmatrix}$$

In other words, it is the MX A times the cols of B

0.1.4 Row-Column Rule for Computing AB

If AB is defined then

$$(AB)_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$$

0.1.5 Properties of Matrix Multiplication Theorem

Associative, distributive, scalar, identity
left-multiplied (BA), right-multiplied (AB)
$AB \neq BA$
Cancelation laws do not apply $AB = AC \neq B = C$
if $AB = 0$ then cannot conclude that $A = 0, B = 0$ A^0 is the identity MX

0.1.6 Transpose of a MX properties Theorem

switching r and c where $m \times n- > n \times m$
T is not exponent
a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. $(rA)^T = rA^T$
d. $(AB)^T = B^T A^T$

0.2 The Inverse of a Matrix

A is invertible if there is $n \times n$ MX C that

$$CA = I, AC = I$$

where I is ID MX. C is the inverse of A and unique

$$A^{-1}A = I = AA^{-1}$$

Not invertible MX is singular matrix SMX, invertible is NSMX

0.2.1 Invertible Theorem

If the determinant $ad - bc \neq 0$ A invertible

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

0.2.2 Invertible Unique Solutions Theorem

If A is invertible $n \times n$ then each $\boldsymbol{b} \in \mathbb{R}^n$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

0.2.3 Properties of Invertible MX Theorem

b. if A,B are $n \times n$ invertible MX, then so it AB $(AB)^{-1} = B^{-1}A^{-1}$
c.$(A^T)^{-1} = (A^{-1})^T$

0.2.4 Elementary MX

obtained by performing one ERO on IDMX
if an ERO on an $m \times n$ A, then result is EA where $m \times m$ E is created by same row operation on I^m
All EMX are invertible. it transforms E back into I

0.2.5 Inveritble IFF Row Equivalent Theorem

A is invertible IFF A is RE to I_n
ERO sequence on A reduces it to I^n

0.2.6 Algorithm for Finding A^{-1}

RR the AM

$$\begin{bmatrix} A & I \end{bmatrix}$$

If A is RE to I, then

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

else A no inverse

0.3 Characterizations of Invertible MX

0.3.1 Invertible Matrix Theorem for Square MX

see proof
A is a square MX then all true or all false:
a. A is invertible

b. A is RE to square ID MX
c. A has n PP
d. $A\boldsymbol{x} = \mathbf{0}$ has only TS
e. cols of A form linearly independent set
f. lin trans $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is one-to-one
g. $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for each $\boldsymbol{b} \in \mathbb{R}^n$
h. the cols of A span $\mathbb{R}^n$
i. the lin trans above$\hat{m}$aps $R^n \mapsto R^n$
j. square MX C that $CA = I$
k. square MX D that $AD = I$
l. A^T is invertible
therefore if $AB = I$ then A and B are both invertible, with $B = A^{-1}, A = B^{-1}$

0.3.2 Invertible Linear Transformations Theorem

ILT if T is invertible.
Theorem: T is invertible IFF A is invertible MX. meaning the LT S is $S(\boldsymbol{x}) = A^{-1}\boldsymbol{x}$ is a unique fxn
ill-conditioned matrix: invMX that can become similar if some entries changed.
condition number: larger the num, closer MX is to being singular.

0.4 Partitioned Matricies

PMX: subMX in MX

0.4.1 PMX Operations

if A, B same sizes then block by block addition/scalar mult
A and B are conformable for block multiplication if can separate blocks into a defined MX multiplication.

0.4.2 Column-Row Expansion of AB Theorem

if $m \times n, n \times p$

$$AB = [col_1(A) \cdots col_n(A)][\vdots row_n(B)]$$
$$AB = col_1(A) row_1(B) \cdots$$

0.4.3 Inverse of PMX

SEE EXAMPLES
A block diagonal MX (BDMX) is a PMX with no blocks off the main diagonal of blocks
BDMX is invertible IFF each block on diagonal is invertible.

0.5 Matrix Factorizations

0.5.1 LU Factorization

L is mxm lower triangular MX with 1's on diag
U is mxn echelon form of A.
L is invertible and the unit lower triagnular matrix (ULTM)
$A = LU$ where $A\boldsymbol{x} = \boldsymbol{b}$ and
$L\boldsymbol{y} = \boldsymbol{b}$
$U\boldsymbol{x} = \boldsymbol{y}$ to solve for L and U

0.5.2 LU Factorization Algorithm

1. Reduce A to EFM by RRO
2. Place entries L such that same sequence of RO reduces L to I
SEE EXAMPLE 2
RO that create zeros in first col of A create them in L as well.

0.6 The Leontirf Input-Output Model

APPLICATION SO SKIP

0.7 Applications to Computer Graphics

APPLICATION SO LATER (THIS ONE IS IMPORTANT)

0.8 Subspaces of $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties
a. The $\mathbf{0}$ is in H
b. For each $\boldsymbol{u}$ in H and scalar c, the vec $c\boldsymbol{u}$ is in H
Meaning: a subSpace is closed under addition and scalar mult.

note: line L not through the origin is not a subspace (doesn't contain origin)
$\mathbf{0}$ is zero subspace

0.8.1 Column Space and Null Space

DEF: col space of A is the set Col A of all LC of the cols of A
DEF: null space of A is the set Nul A of all solutions of $A\boldsymbol{x} = \mathbf{0}$

0.8.2 Null space in $\mathbb{R}^n$ Theorem

null space of mxn A is SBS of $\mathbb{R}^n$.
SS of $A\boldsymbol{x} = \mathbf{0}$ of m HLE in n unknowns is a SBS of $\mathbb{R}^n$
$\boldsymbol{v}$ is Nul A if $A\boldsymbol{v}$ is $\mathbf{0}$
Nul is defined implicitly.
Col is defined explicitly.

0.8.3 Basis for a SBS

DEF: a basis for SBS H of $\mathbb{R}^n$ is a LI set in H that spans H
Standard Basis Vectors is the set of vectors with a 1 entry and the rest zeros (Think $\hat{\imath}, \hat{\jmath}, \hat{k}$)
SEE EXAMPLES
Theorem: The pivot cols of A form a basis for col space of A

0.9 Dimension and Rank

0.9.1 Coordinate Vector Definition

set $\mathcal{B} = \{\boldsymbol{b_1}, \ldots, \boldsymbol{b_p}\}$ is a basis for SBS H. For each $\boldsymbol{x}$ in H, the coordinates of x relative to the basis $\mathcal{B}$ are weights $\ldots c_p$ such that $\boldsymbol{x} = \ldots c_p\boldsymbol{b_p}$ and vec in $\mathbb{R}^p$

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the coordinate vector of x relative to B or
$\mathcal{B}$-coordinate vector of $\boldsymbol{x}$
H is isomorphic to $\mathbb{R}^2$ when it is 1-1 (preserves LC)

0.9.2 DEF:The Dimension of a Subspace

dimension of NZ SBS H, dim H, is num of vec in any basis for H
dimension of ZSBS is zero.
$\mathbb{R}^n$ has dimension n.

0.9.3 Rank

rank of A, rank A, is the dimension of col space of A
pivot col of A form basis for Col A -¿ rank of A is num of pivot col in A

0.9.4 Rank Theorem

If A has n cols, then $\text{rank}A + \dim \text{Nul}A = n$

0.9.5 Basis Theorem

Let H be p-dim SBS of $\mathbb{R}^n$ Any LI set of p ele in H is a basis for H.
Any set of p ele of H that spans H is a basis for H

0.9.6 Invertible Matrix Theorem for Rank!!!!!

A is nxn then all true if invertible
m. the cols form a basis of $\mathbb{R}^n$
n. Col $A = \mathbb{R}^n$
o. dim Col $A = n$
p. rank $A = n$
q. Nul $A = \{\mathbf{0}\}$
r. dim Nul $A = 0$