Rotations and Angular Velocities

Time Derivatives of Rotations

Rotation matrix

$$R(t)$$

Orthogonality

$$R^T(t)R(t) = I \qquad \frac{d}{dt}(\,.\,) \qquad \dot{R}^T R + R^T \dot{R} = 0$$
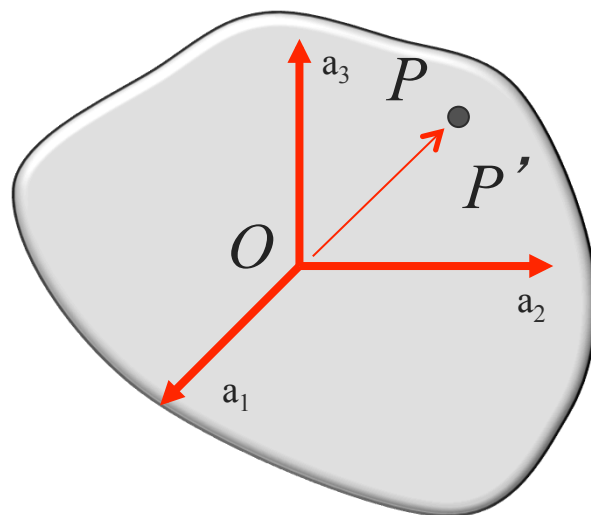
$$R(t)R^T(t) = I \qquad\qquad\qquad R\dot{R}^T + \dot{R}R^T = 0$$

$R^T\dot{R}$ and $\dot{R}R^T$ are skew symmetric

Rotation with O fixed

$$\overrightarrow{OP} = p_1 \mathbf{a}_1 + p_2 \mathbf{a}_2 + p_3 \mathbf{a}_3$$

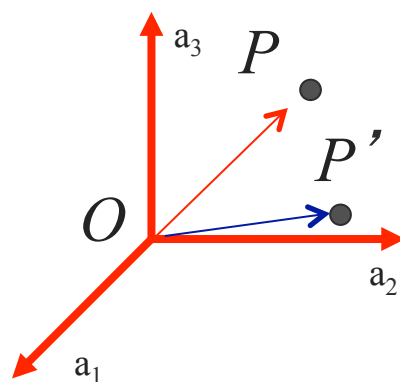$$\overrightarrow{OP'} = q_1 \mathbf{a}_1 + q_2 \mathbf{a}_2 + q_3 \mathbf{a}_3$$

$$q \quad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad p$$

constant coordinates of P in body-fixed frame

$$q(t) = R(t)p$$

changing coordinates of P as the rigid body rotates

Rotation with O fixed

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$q(t) = R(t)p$$

$$\dot{q} = \dot{R}p$$

velocity in inertial frame   *position in body-fixed frame*

$$R^T \dot{q} = R^T \dot{R} p \qquad \hat{\omega}^b$$

velocity in body-fixed frame   *encodes angular velocity in body-fixed frame*

$$\dot{q} = \dot{R} R^T q$$

velocity in inertial frame   *encodes angular velocity in inertial frame*

$$\hat{\omega}^s$$

Penn Engineering

Exercise

What is the angular velocity for a rotation about the *z* axis?

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dot{R} = \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

Angular velocity for a rotation about the *z*-axis

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T \dot{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$= \dot{R}R^T = \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \widehat{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \dot{\theta}$$

Two Rotations

$$R = R_z(\theta) R_x(\phi)$$

$$\hat{\omega}^b = R^T \dot{R} = (R_z R_x)^T (\dot{R}_z R_x + R_z \dot{R}_x)$$

$$= R_x^T R_z^T \dot{R}_z R_x + R_x^T \dot{R}_x$$

$$\hat{\omega}^s = \dot{R} R^T = (\dot{R}_z R_x + R_z \dot{R}_x)(R_z R_x)^T$$

$$= \dot{R}_z R_z^T + R_z \dot{R}_x R_x^T R_z^T$$