Rigid Body Transformations

Two distinct positions and orientations of the same rigid body

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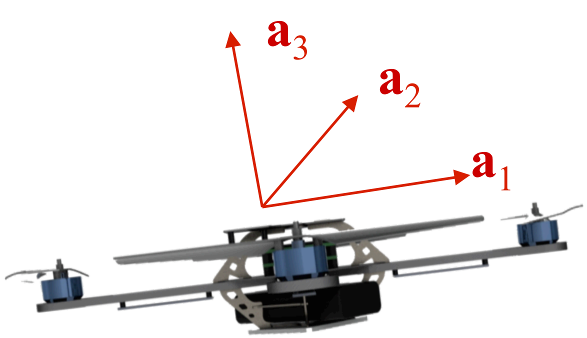
Reference Frames

We associate with any position and orientation a *reference frame*

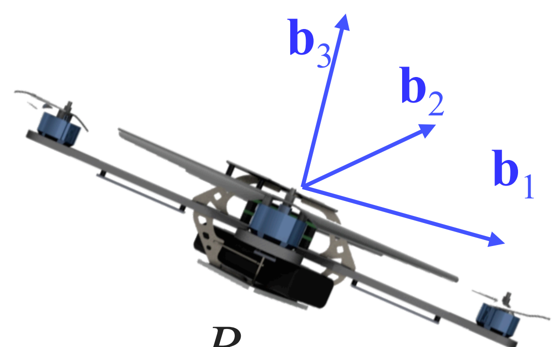
In reference frame $\{A\}$, we can find three linearly independent vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ that are basis vectors.

We can write any vector as a linear combination of the basis vectors in either frame.

$$\mathbf{v} = v_1\,\mathbf{a}_1 + v_2\,\mathbf{a}_2 + v_3\mathbf{a}_3$$



A



B

Notation

Vectors

- $\mathbf{x}$, $\mathbf{y}$, $\mathbf{a}$, …
- ${}^{A}\mathbf{x}$
- u, v, p, q, …

Potential for Confusion!

Reference Frames

- A, B, C, …
- a, b, c, …

Matrices

- $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, …

Transformations

- ${}^{A}\mathbf{A}_{B}$ ${}^{A}\mathbf{R}_{B}$ ${}^{A}\boldsymbol{\xi}_{B}$
- $\mathbf{A}_{ab}$ $\mathbf{R}_{ab}$
- g_{ab}, h_{ab}, …

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Rigid Body Displacement

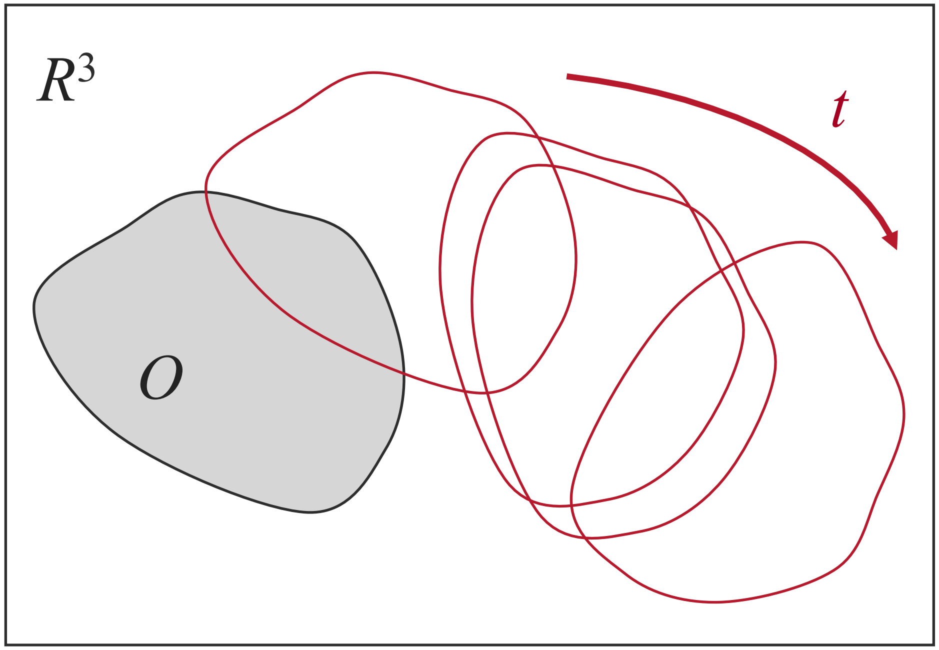
Object

$$O \subset R^3$$

Rigid Body Displacement

Map

$$g : O \to R^3$$

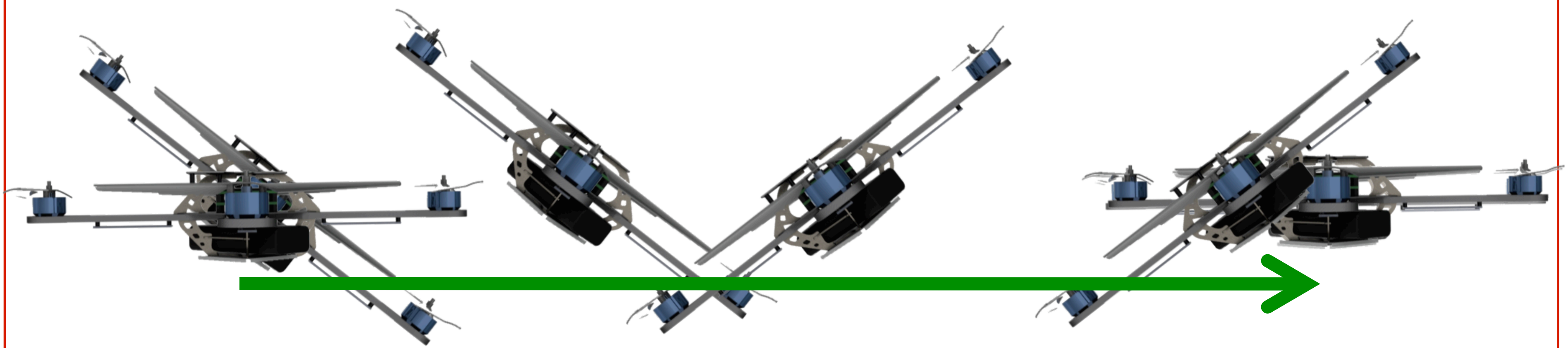


Rigid Body Motion

Continuous family of maps

$$g(t) : O \to R^3$$

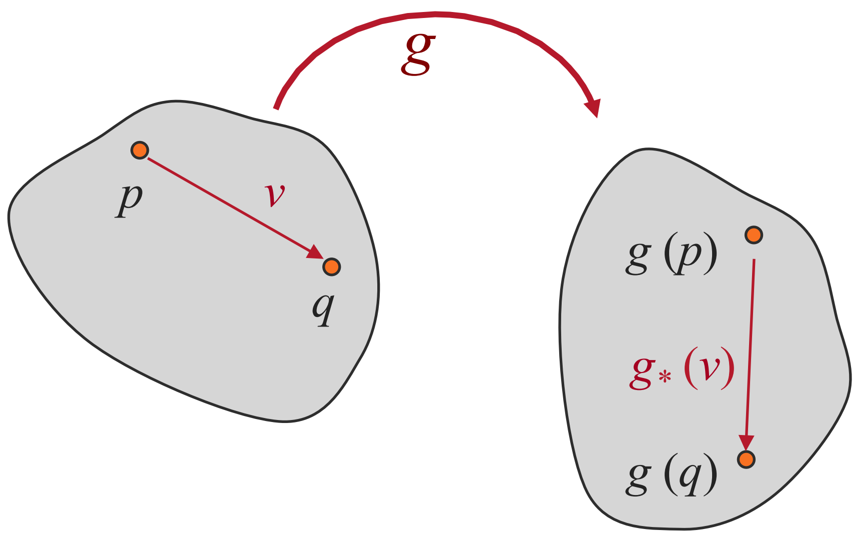
Example



Rigid Body Displacement

A displacement is a transformations of points

- Transformation (g) of points induces an action (g_*) on vectors



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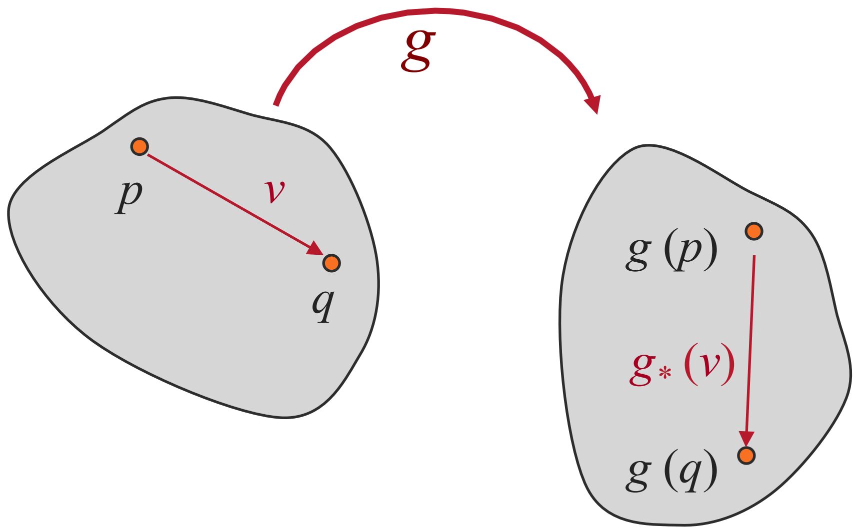
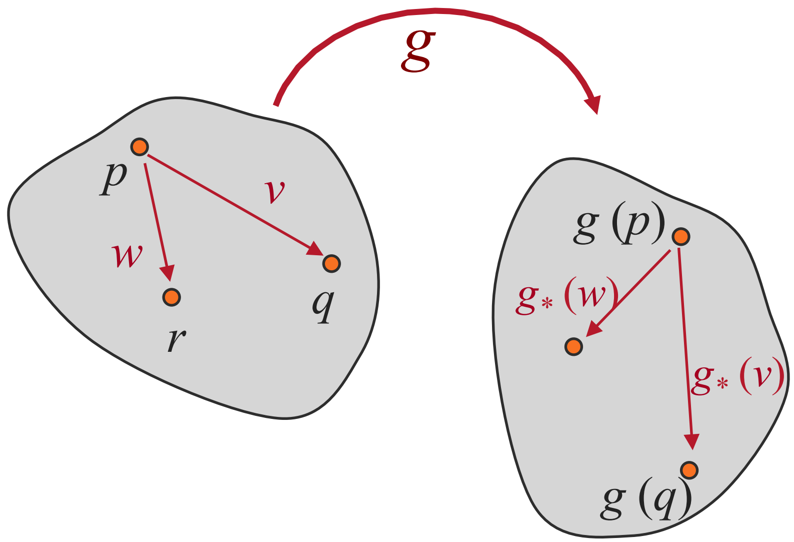
What makes *g* a *rigid body displacement?*



$$\|g(p) - g(q)\| = \|p - q\|$$

1. Lengths are preserved

What makes g a *rigid body displacement*?



$$g_\star(v) \times g_\star(w) = g_\star(v \times w)$$

2. Cross products are preserved

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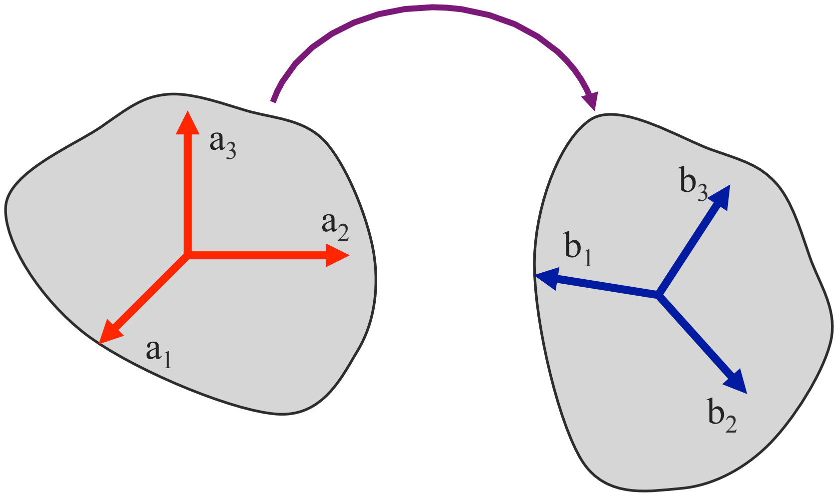
g is a *rigid body* displacement



mutually orthogonal unit vectors get mapped to mutually orthogonal unit vectors

You should be able to prove

- orthogonal vectors are mapped to orthogonal vectors

- g_* preserves inner products

$$g_\star(v) \cdot g_\star(w) = g_\star(v \cdot w)$$

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Summary

Rigid body displacements are transformations (maps) that satisfy two important properties

1. The map preserves lengths

2. Cross products are preserved by the induced map

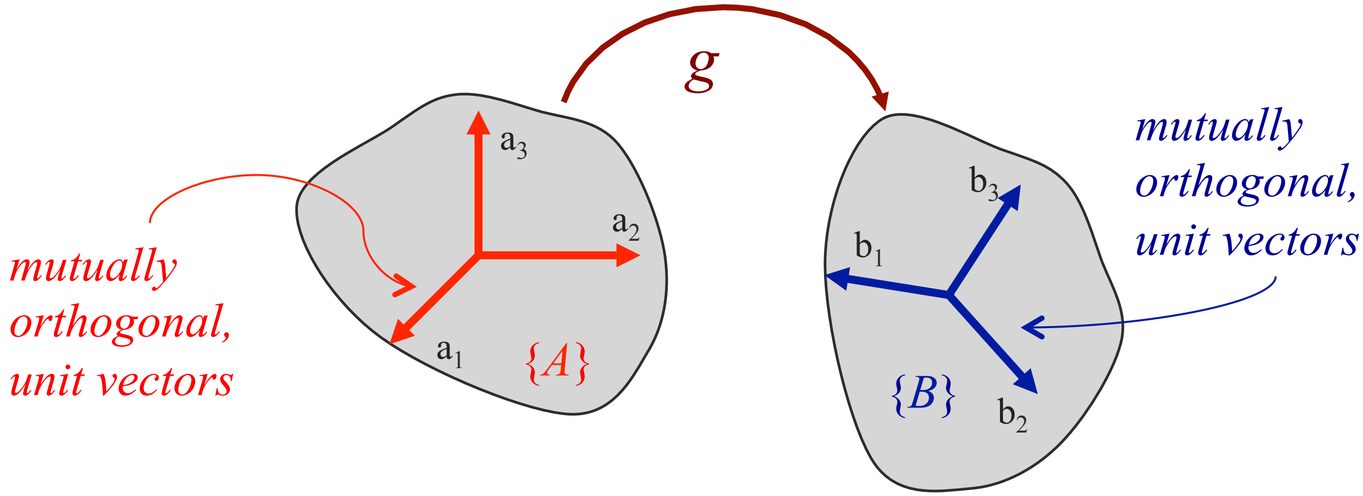
Note

Rigid body displacements and rigid body transformations are used interchangeably

1. Transformations generally used to describe relationship between reference frames attached to different rigid bodies.

2. Displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body

g is a *rigid body* displacement



$$\mathbf{b}_1 = R_{11}\mathbf{a}_1 + R_{12}\mathbf{a}_2 + R_{13}\mathbf{a}_3$$
$$\mathbf{b}_2 = R_{21}\mathbf{a}_1 + R_{22}\mathbf{a}_2 + R_{23}\mathbf{a}_3$$
$$\mathbf{b}_3 = R_{31}\mathbf{a}_1 + R_{32}\mathbf{a}_2 + R_{33}\mathbf{a}_3$$

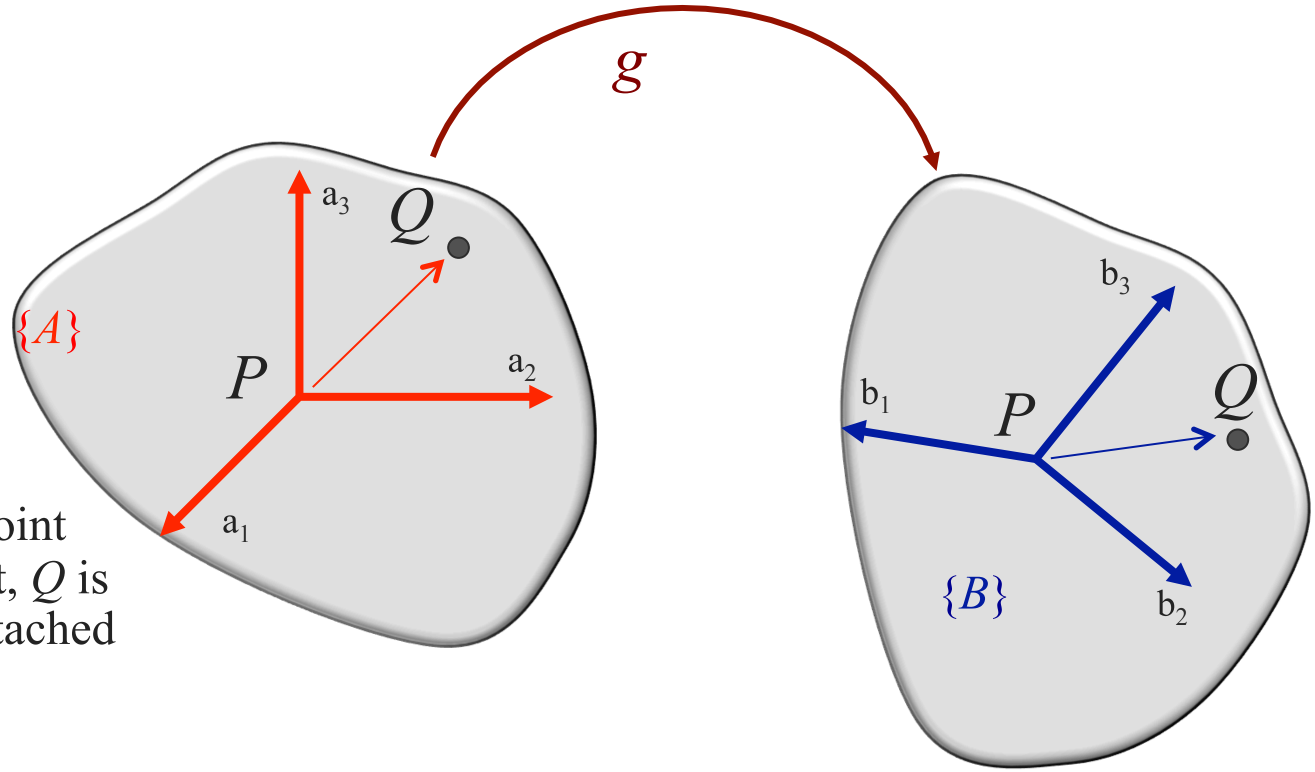
$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

rotation matrix

Properties of a Rotation Matrix

- Orthogonal
  - Matrix times its transpose equals the identity
- Special orthogonal
  - Determinant is +1
- Closed under multiplication
  - The product of any two rotation matrices is another rotation matrix
- The inverse of a rotation matrix is also a rotation matrix

g is a *rigid body* displacement

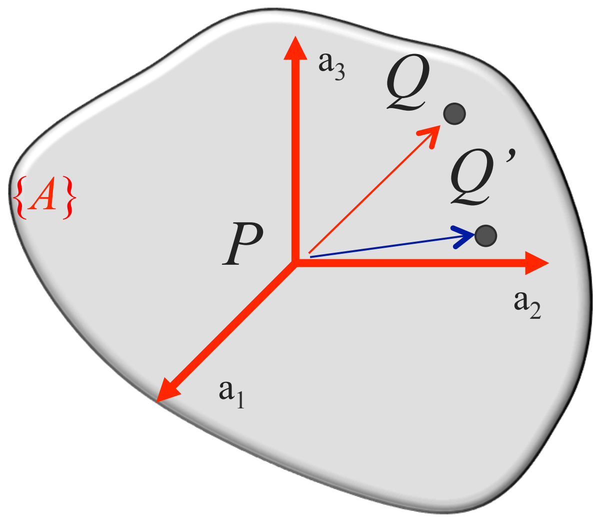


P is a reference point fixed to the object, Q is a generic point attached to the object.

$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\overrightarrow{PQ} = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$

g is a *rigid body* displacement

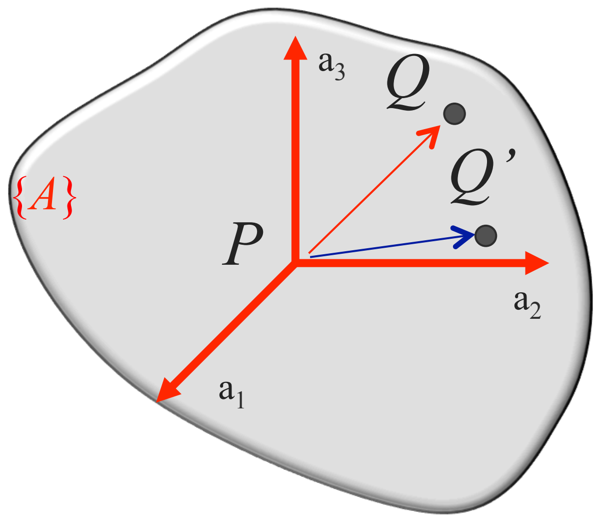


$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\overrightarrow{PQ'} = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \text{ or } \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix}$$

g is a *rigid body* displacement



$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\overrightarrow{PQ'} = q'_1\mathbf{a}_1 + q'_2\mathbf{a}_2 + q'_3\mathbf{a}_3$$

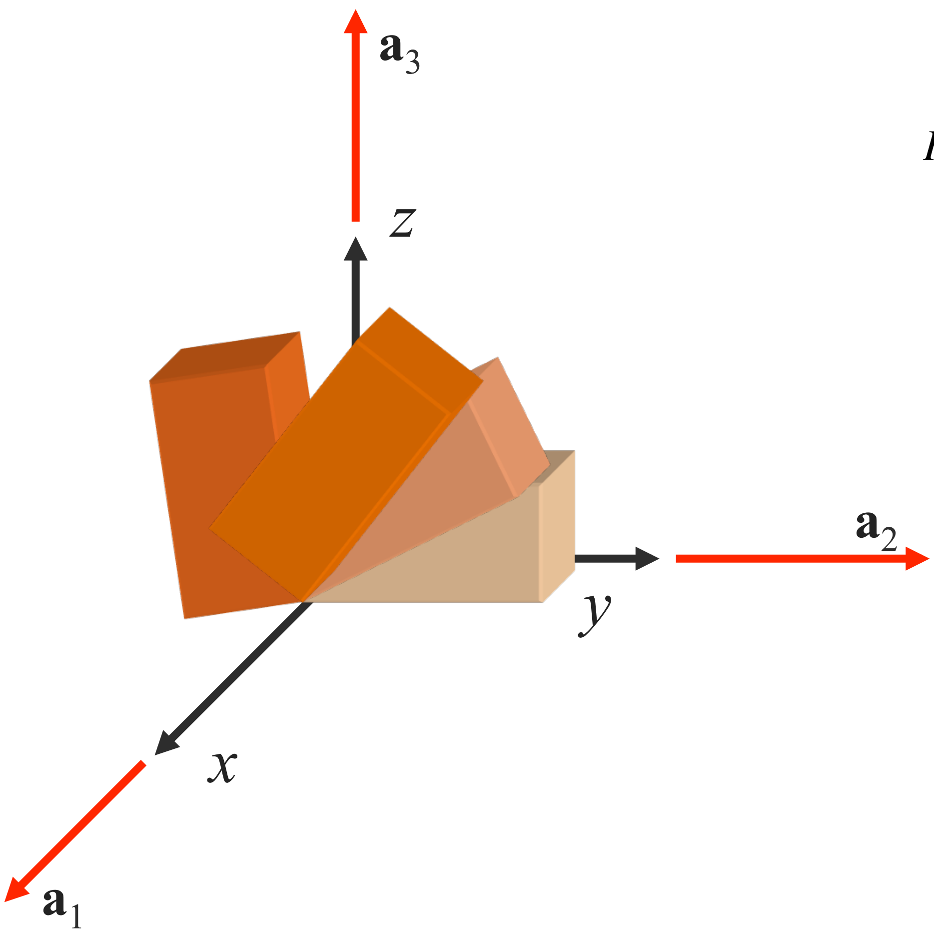
Verify

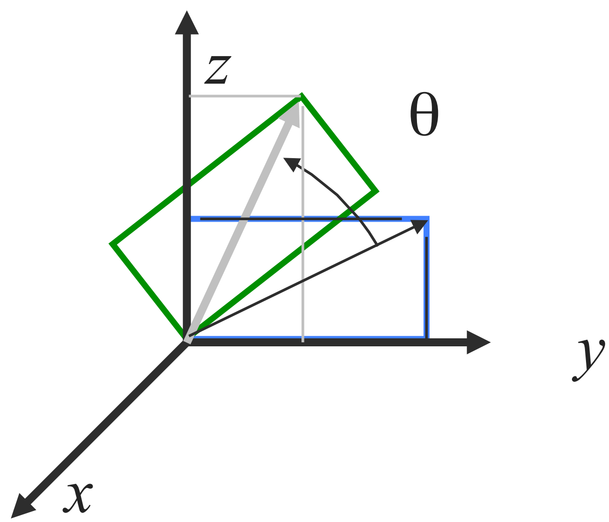
$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} q'_1 \\ q'_2 \\ q'_3 \end{bmatrix}$$

Example: Rotation

- Rotation about the x-axis through θ

$$Rot(x,\theta)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

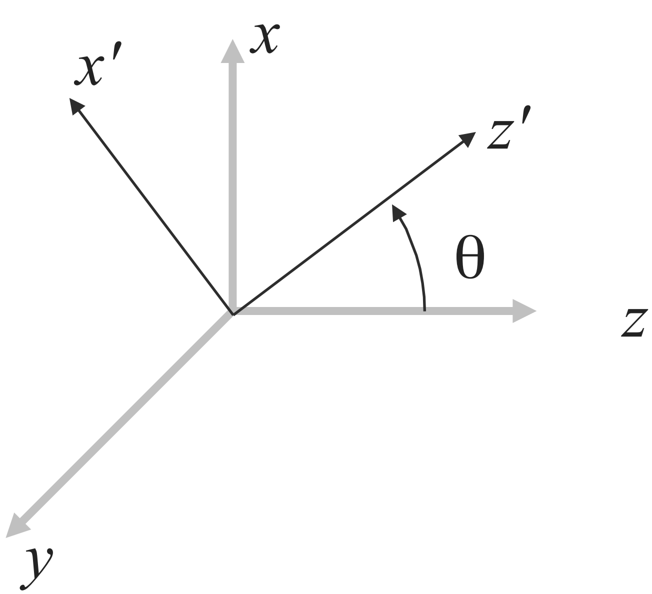




Example: Rotation

Rotation about the y-axis through θ

$$Rot(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$



Rotation about the z-axis through θ

$$Rot(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$