Time, Motion and Trajectories

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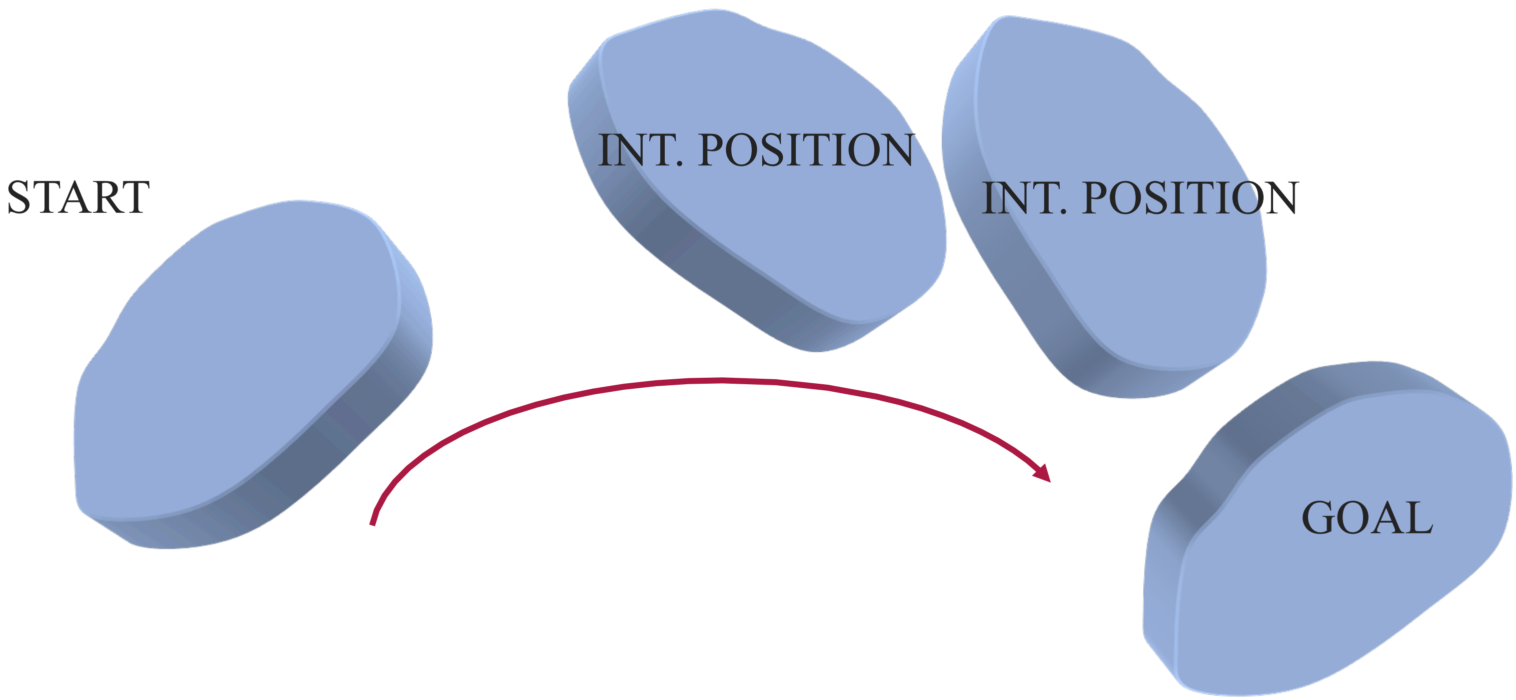
Smooth three dimensional trajectories



Applications

- Trajectory generation in robotics
- Planning trajectories for quad rotors

General Set up

- Start, goal positions (orientations)
- Waypoint positions (orientations)
- Smoothness criterion
  - Generally translates to minimizing rate of change of "input"
- Order of the system (n)
  - Order of the system determines the input
  - Boundary conditions on $(n-1)^{th}$ order and lower derivatives

Calculus of Variations

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dot{x}, x, t\right) dt$$

functional

function

Examples

- Shortest distance path (geometry) $\quad x^\star(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 dt$
- Fermat's principle (optics) $\quad x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T 1 dt$
- Principle of least action (mechanics) $\quad x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T T\left(\dot{x}, x, t\right) - V\left(x, t\right) dt$

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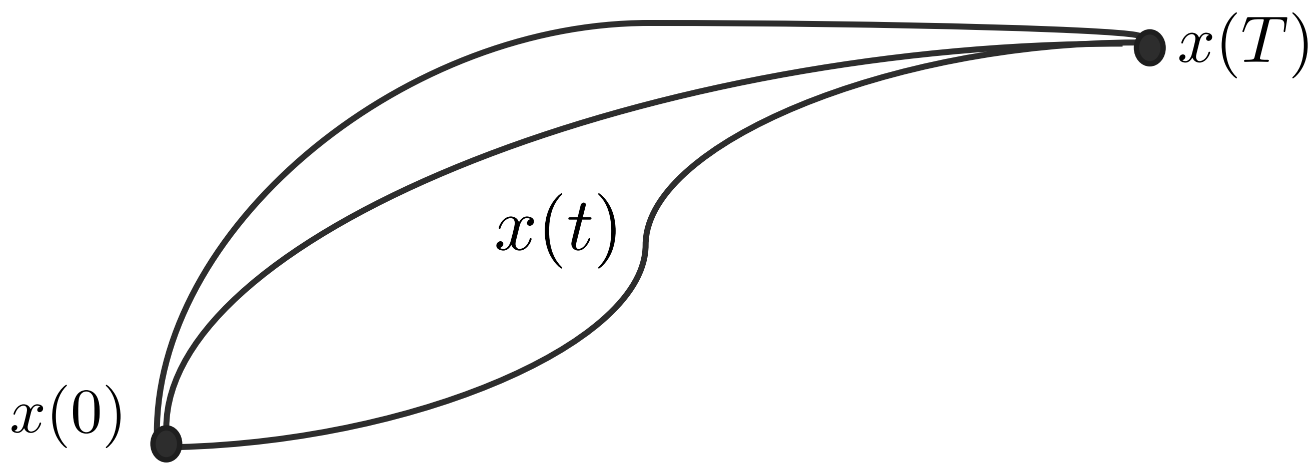
Calculus of Variations

$$x^\star(t) = \operatorname*{argmin}_{x(t)} \int_0^T \mathcal{L}\left(\dot{x}, x, t\right) dt$$

Consider the set of all differentiable curves, $x(t)$, with a given $x(0)$ and $x(T)$.



Calculus of Variations

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dot{x}, x, t\right) dt$$

Euler Lagrange Equation

Necessary condition satisfied by the "optimal" function $x(t)$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

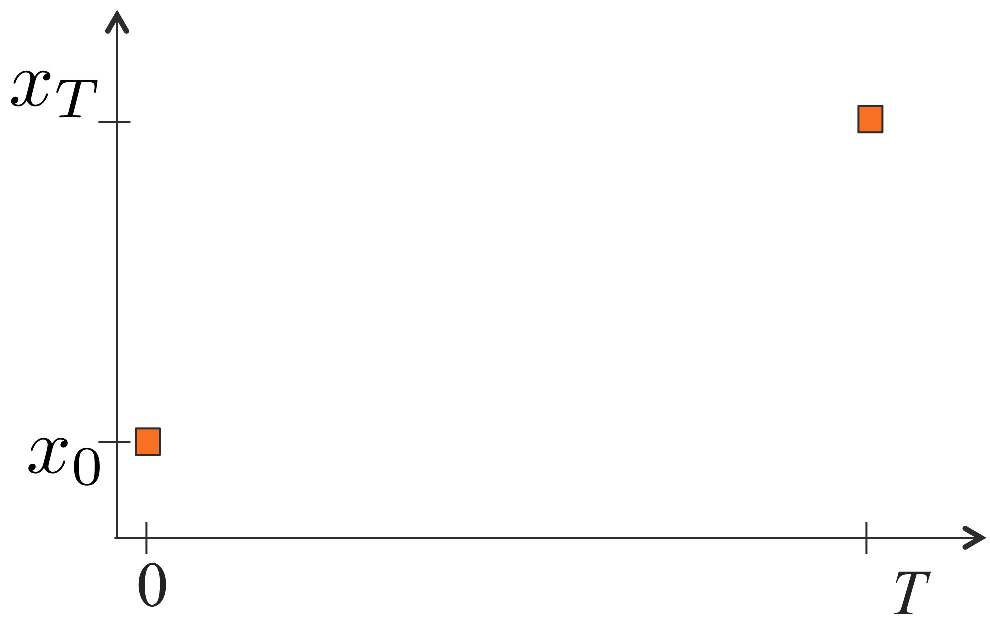
Smooth trajectories (n=1)

$$x^\star(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 \, dt$$

$$x(0) = x_0, \;\; x(T) = x_T$$

input

$u = \dot{x}$



Smooth trajectories (n=1)

$$x^{\star}(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 dt$$

Euler Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

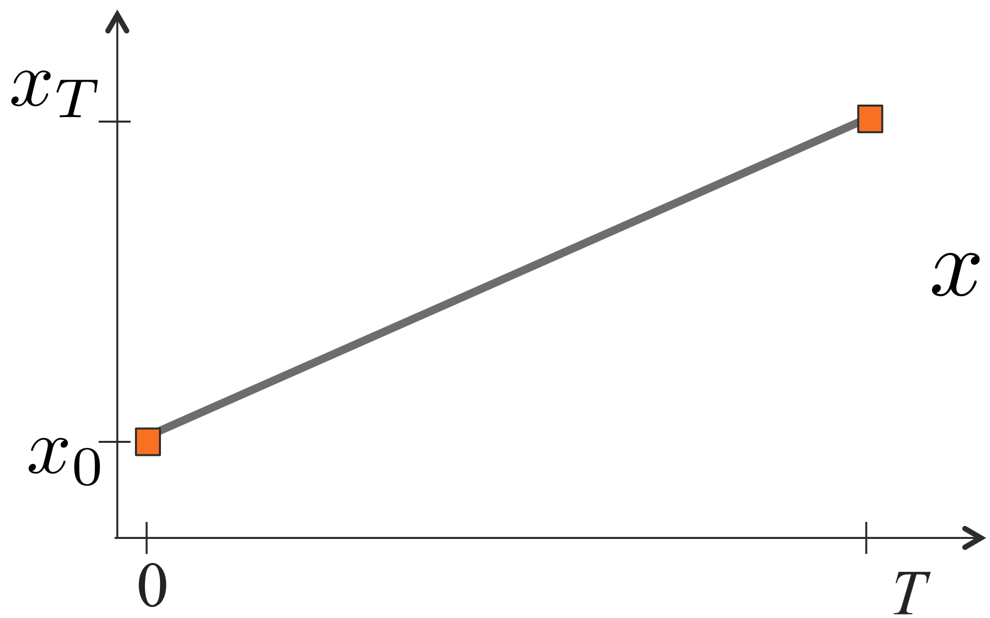
$$\mathcal{L}(\dot{x}, x, t) = (\dot{x})^2 \quad \Rightarrow \quad \ddot{x} = 0$$

$$x = c_1 t + c_0$$

Smooth trajectories (n=1)

$$x^\star(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 dt$$

$$x(0) = x_0, \;\; x(T) = x_T$$



$$x = c_1 t + c_0$$

Smooth trajectories (general *n*)

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \left(x^{(n)}\right)^2 dt$$

input

$u = x^{(n)}$

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Euler-Lagrange Equation

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(x^{(n)}, x^{(n-1)}, \ldots, \dot{x}, x, t\right) dt$$

Euler Lagrange Equation

Necessary condition satisfied by the "optimal" function

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{x}}\right) + \ldots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial \mathcal{L}}{\partial x^{(n)}}\right) = 0$$

Smooth Trajectories

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \left(x^{(n)}\right)^2 dt$$

- *n*=1, shortest distance $\boxed{\text{velocity}}$
- *n*=2, minimum $\boxed{\text{acceleration}}$
- *n*=3, minimum $\boxed{\text{jerk}}$
- *n*=4, minimum $\boxed{\text{snap}}$

n – order of system

n^{th} derivative is input

Smooth Trajectories

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \left(x^{(n)}\right)^2 dt$$

- n=1, shortest distance $\boxed{\text{velocity}}$
- n=2, minimum $\boxed{\text{acceleration}}$
- n=3, minimum $\boxed{\text{jerk}}$
- n=4, minimum $\boxed{\text{snap}}$

Why is the minimum velocity curve also the shortest distance curve?