

Eigenvalues and Eigenvectors of Matrices

Determinant

A determinant is a scalar property of square matrices, denoted $\det(A)$ or $|A|$.

- Think of rows of an $n \times n$ matrix as n vectors in \mathbb{R}^n .
- The determinant represents the “space contained” by these vectors.

In this course, we will be working with 2×2 or 3×3 matrices.

Determinant (2x2 Matrix)

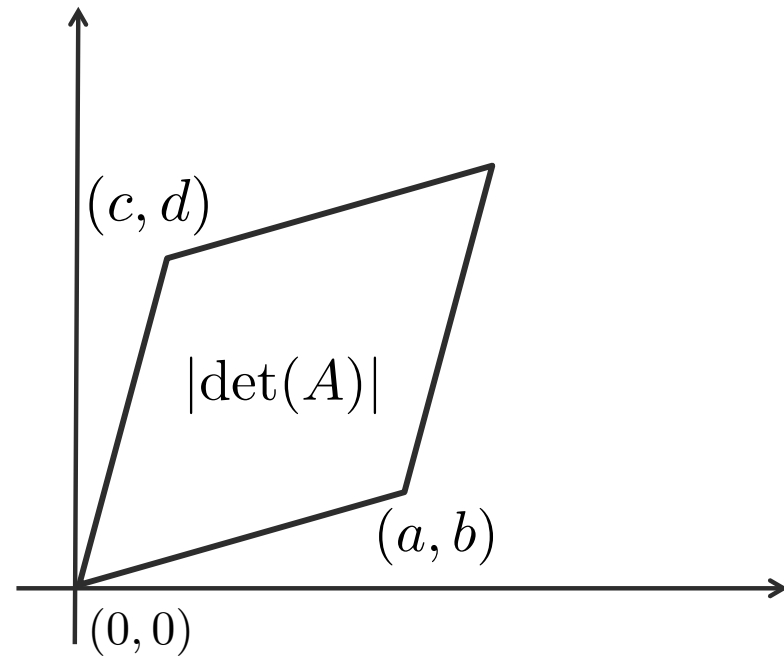
Consider:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

↑
Area of parallelogram
defined by the rows.



Example: Determinant (2x2 Matrix)

Consider:

$$\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = (1)(1) - (3)(4) = -11$$

Determinant (3x3 Matrix)

Consider:

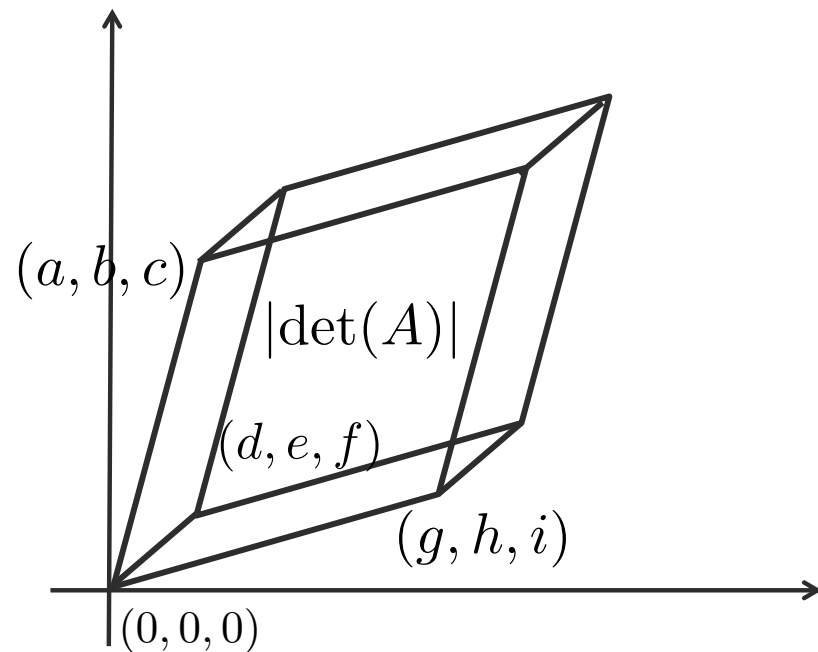
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh \\ - ceg - bdi - afh$$



Volume of parallelepiped
defined by the rows.



Example: Determinant (3x3 Matrix)

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$\begin{aligned} &= (1)(-4)(-1) + (2)(1)(0) + (3)(0)(3) \\ &\quad - (3)(-4)(0) - (2)(0)(-1) - (1)(1)(3) \\ &= 1 \end{aligned}$$

Eigenvalues and Eigenvectors

A matrix is a transformation.

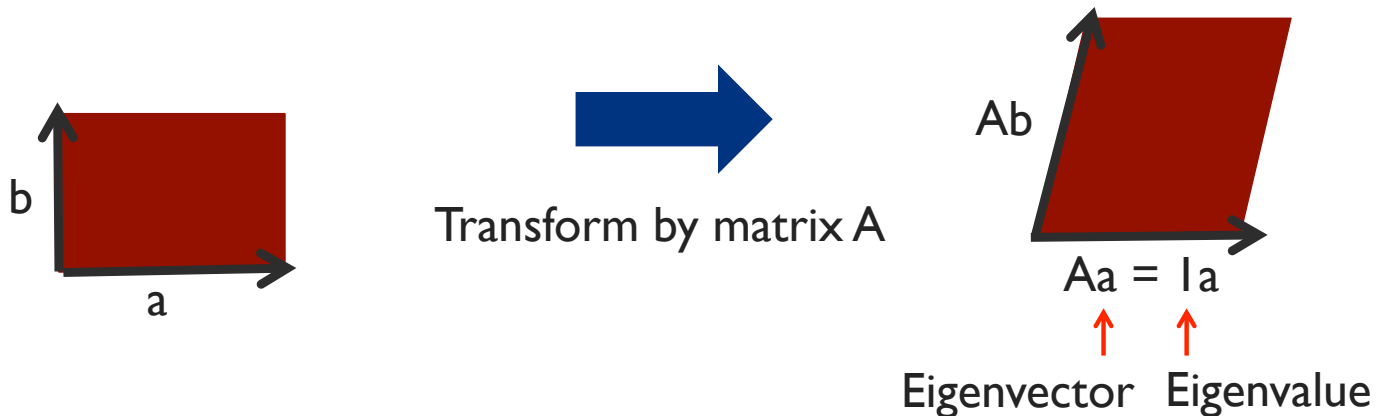
$$y = Ax$$

Eigenvalues and Eigenvectors

Eigenvectors are vectors associated by a square matrix that do not change in direction when multiplied by the matrix.

Eigenvalues are scalar values representing how much each eigenvector changes in length.

$$A\mathbf{v} = \lambda\mathbf{v}$$



Finding Eigenvalues

1. Calculate:

$$\det(A - \lambda \mathbf{I})$$

2. Find solutions to:

$$\det(A - \lambda \mathbf{I}) = 0$$

There will be n eigenvalues for an $n \times n$ matrix, but not all of them have to be distinct or real values.

Example: Finding Eigenvalues

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

I. Calculate $\det(A - \lambda \mathbf{I})$

$$\begin{aligned} \det(A - \lambda \mathbf{I}) &= \det \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \det \left(\begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} \right) \\ &= (2 - \lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \end{aligned}$$

Example: Finding Eigenvalues

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

2. Find solutions to $\det(A - \lambda \mathbf{I}) = 0$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_1 = 1, \lambda_2 = 3$$



2 eigenvalues for a 2 x 2 matrix

Finding Eigenvectors

I. For each eigenvalue, solve the equation:

$$A\mathbf{v} = \lambda\mathbf{v}$$

or:

$$(A - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

Notice that if \mathbf{v} is an eigenvector, then $\alpha\mathbf{v}$ is also an eigenvector, where α is any scalar.

Thus, we typically think about **linearly independent** eigenvectors.

Eigenvectors

n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are **linearly independent (LI)** if the only solution to the equation:

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_n \mathbf{v}_n = \mathbf{0}$$

is $a_1 = a_2 = \dots = a_n = 0$.

There will be at least one LI eigenvector for each eigenvalue.
If eigenvalues are repeated, there might be multiple LI eigenvectors for that eigenvalue.

Example: Finding Eigenvectors

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

I. For $\lambda_1 = 1$:

$$\begin{aligned} & (A - \lambda_1 \mathbf{I}) \mathbf{v}_1 \\ &= \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{v}_1 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{v}_1 \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$



$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \mathbf{v}_1 &= \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$