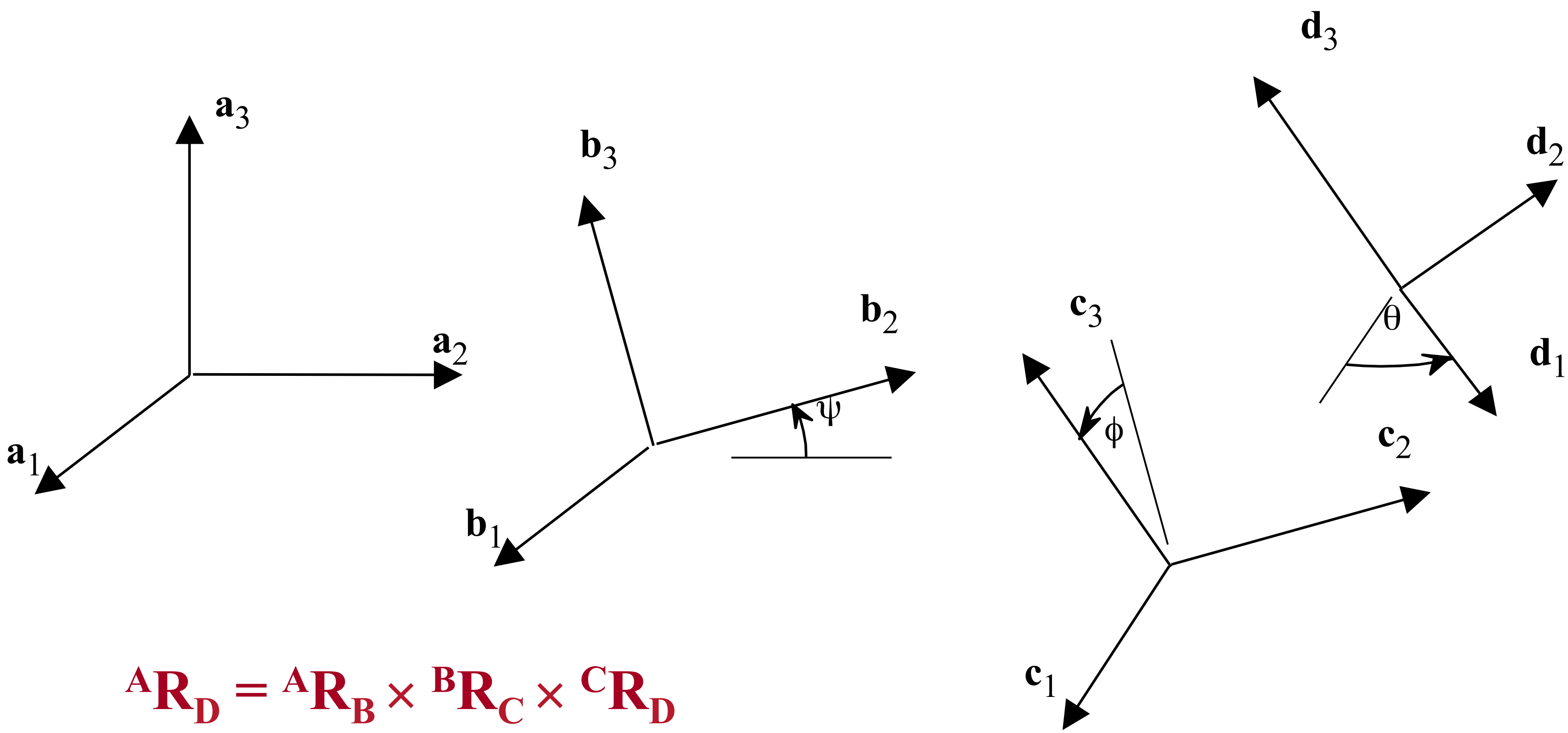
Euler Angles

Composition of Three Rotations



$$^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D}$$

$$^{A}\mathbf{R}_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta)$$

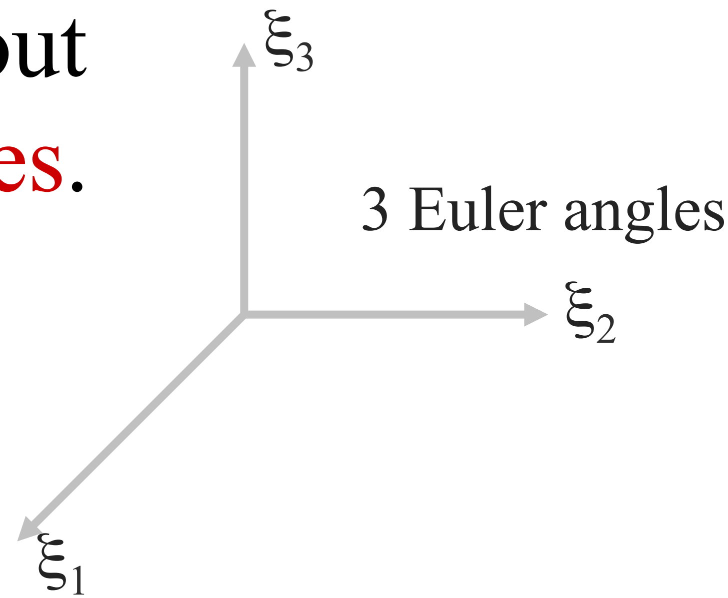
roll *pitch* *yaw*

Euler Angles

Image from wikipedia

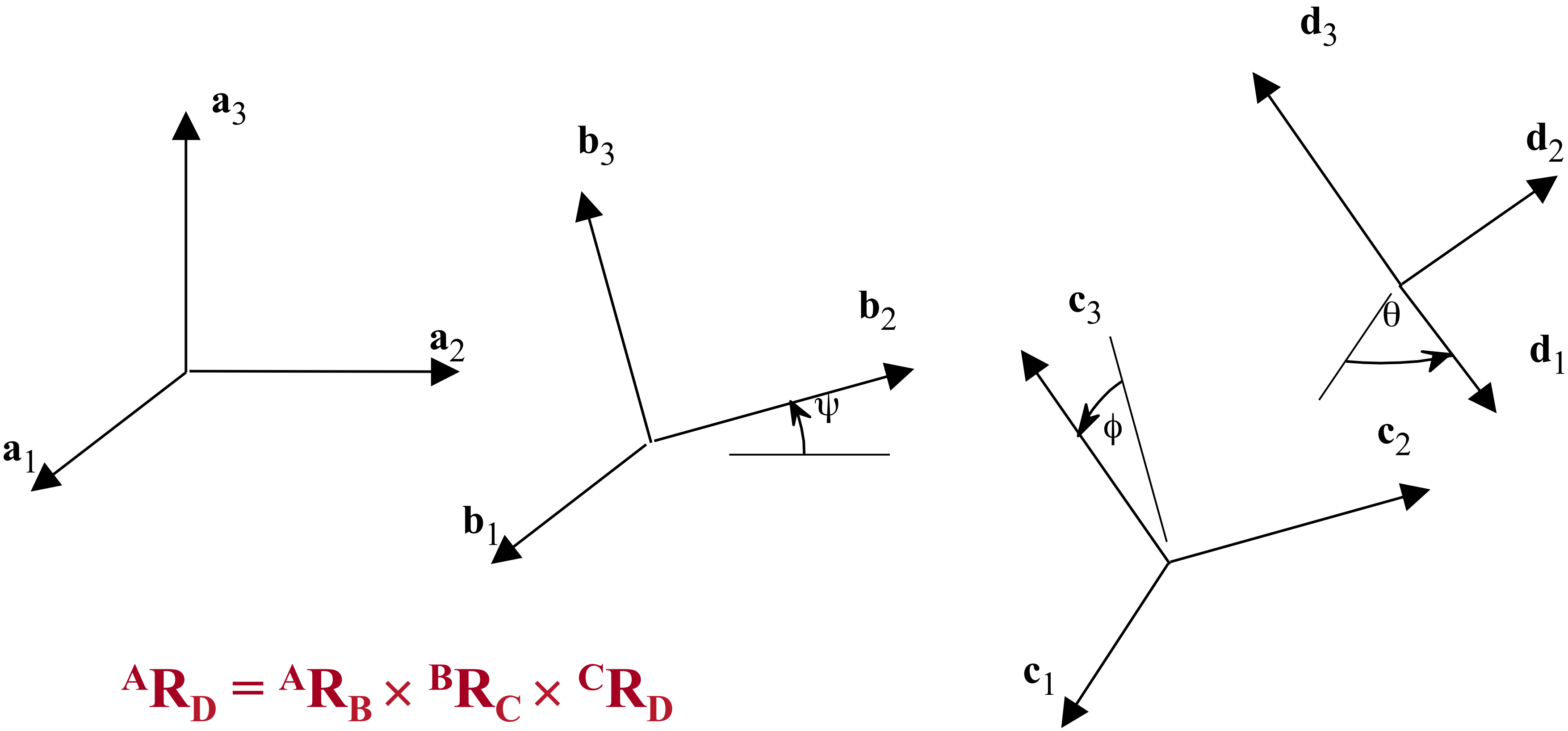
Any rotation can be described by three successive rotations about linearly independent axes.

$$\begin{pmatrix} 3 \times 3 \text{ rotation matrix} \end{pmatrix}$$



Almost 1-1 transformation

X-Y-Z Euler Angles



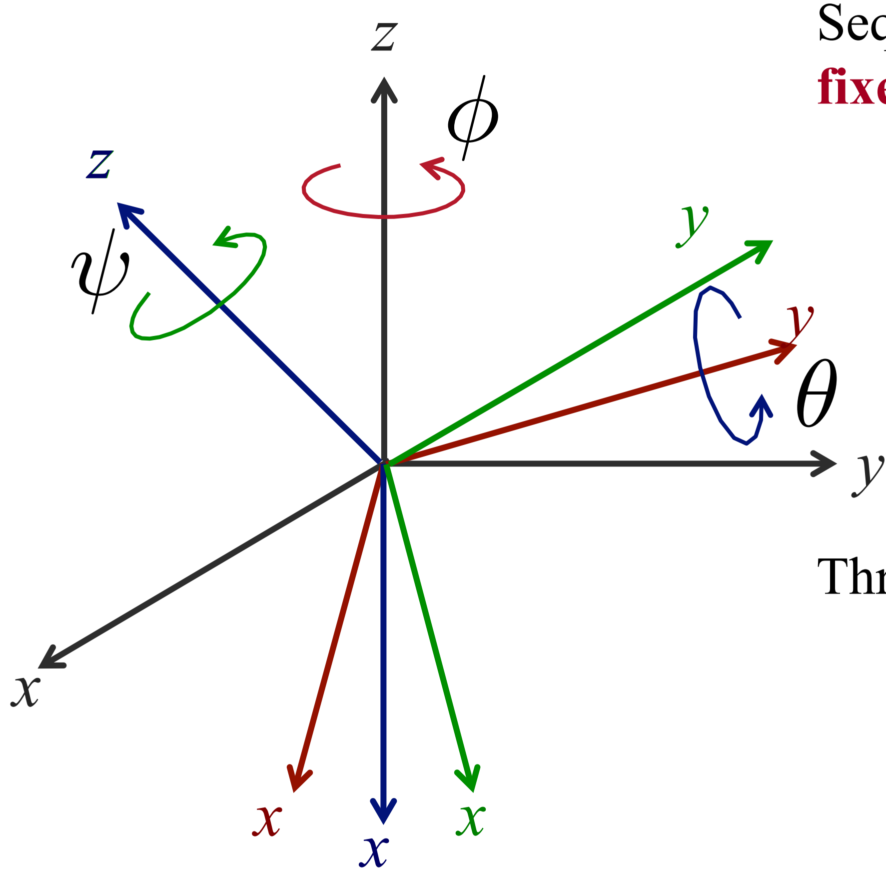
$$ {}^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D} $$

$$ {}^{A}\mathbf{R}_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta) $$

roll *pitch* *yaw*

Penn Engineering

Z-Y-Z Euler Angles



Sequence of three rotations about **body-fixed** axes

- Rot(z, ϕ)
- Rot(y, θ)
- Rot(z, ψ)

Are these linearly independent?

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

Note

- $\theta = 0$ is a special (singular) case

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

Determination of Euler Angles

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \boxed{\cos\phi\sin\theta} \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \boxed{\cos\theta} \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} & \boxed{R_{13}} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & \boxed{R_{33}} \end{bmatrix}$$

$R_{31} = -\sin\theta\cos\psi$

$R_{32} = \sin\theta\sin\psi$

$R_{33} = \cos\theta$

$R_{13} = \sin\theta\cos\phi$

$R_{23} = \sin\theta\sin\phi$

known rotation matrix

Determination of Euler Angles

If $|R_{33}| < 1$,

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

$$\theta = \sigma \arccos(R_{33}), \quad \sigma = \pm 1$$

$$\psi = a\tan 2\left(\frac{R_{32}}{\sin\theta}, \frac{-R_{31}}{\sin\theta}\right)$$

$$\phi = a\tan 2\left(\frac{R_{23}}{\sin\theta}, \frac{R_{13}}{\sin\theta}\right)$$

Two sets of Euler angles for every **R** for almost all **R**'s!

If $R_{33} = 1$,

$$f(\phi + \psi)$$

$$R = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi + \sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If $R_{33} = -1$,

Infinite set of Euler Angles!

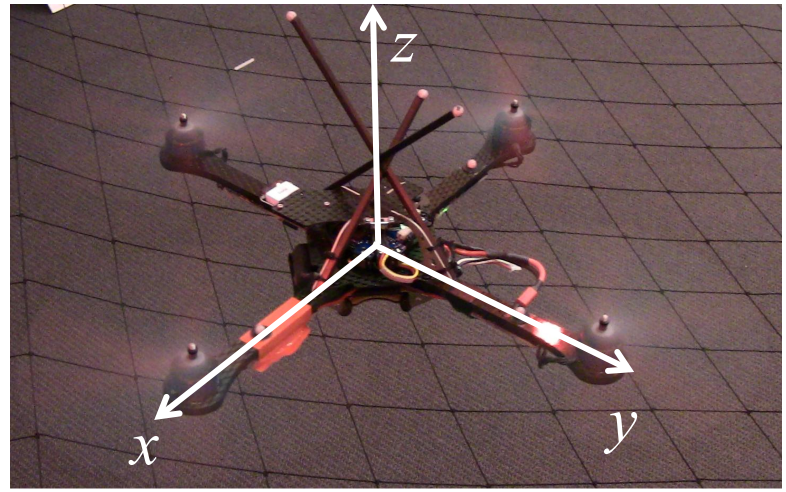
$$R = \begin{bmatrix} -\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$f(\phi + \psi)$$

Z-X-Y Euler Angles

Sequence of three rotations about **body-fixed** axes

- Rot(z, ψ)
- Rot(x, ϕ)
- Rot(y, θ)



Verify

$$R = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\theta s\phi c\psi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

N. Michael, D. Mellinger, Q. Lindsey, V. Kumar, *The GRASP Multiple Micro-UAV Testbed*, IEEE Robotics & Automation Magazine, vol.17, no.3, pp.56-65, Sept. 2010

What is the minimum number of sets of Euler angles you need to cover $SO(3)$?

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = R R^T = I\}$$