

Quaternions

Quaternion Definition

Quaternion:

$$q = (q_0, q_1, q_2, q_3)$$

This can be interpreted as a constant + vector:

$$q = (q_0, \mathbf{q})$$

Operations with Quaternions

Quaternion addition/subtraction:

$$p \pm q = (p_0 \pm q_0, \mathbf{p} \pm \mathbf{q})$$

Quaternion multiplication:

$$pq = (p_0q_0 - \mathbf{p}^T \mathbf{q}, p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q})$$

Quaternion inverse:

$$q^{-1} = (q_0, -\mathbf{q})$$

Axis-Angle Representation to Quaternion

Quaternions can be used to represent rigid-body rotations.

Recall the axis-angle representation of rotations:

Angle of rotation: ϕ

Axis of rotation: \mathbf{u}

The equivalent quaternion is:

$$q = \left(\cos\left(\frac{\phi}{2}\right), u_1 \sin\left(\frac{\phi}{2}\right), u_2 \sin\left(\frac{\phi}{2}\right), u_3 \sin\left(\frac{\phi}{2}\right) \right)$$

Quaternions to Axis-Angle Representation

Given a quaternion:

$$q = (q_0, q_1, q_2, q_3)$$

The equivalent axis-angle representation is:

Angle of rotation: $2 \cos^{-1}(q_0)$

Axis of rotation: $\mathbf{u}_2 = \begin{bmatrix} \frac{q_1}{\sqrt{1-q_0^2}} \\ \frac{q_2}{\sqrt{1-q_0^2}} \\ \frac{q_3}{\sqrt{1-q_0^2}} \end{bmatrix}$

Vector Rotation with Quaternions

To rotate a vector \mathbf{p} in \mathbb{R}^3 by the quaternion q :

1. Define quaternion:

$$p = (0, \mathbf{p})$$

2. The result after rotation is:

$$p' = qpq^{-1} = (0, \mathbf{p}')$$

We can easily compose two rotations:

$$q = q_2q_1$$

Properties of Quaternions

- $q = (q_0, q_1, q_2, q_3)$ and $-q = (-q_0, -q_1, -q_2, -q_3)$ represent the same rotation.
- Compact representation of rotations, with only 4 parameters.
- No singularities
- Quaternion product is more numerically stable than matrix multiplication.