Dependable Systems Group
D. Nowotka, P. Fleischmann

Christian-Albrechts-Universität zu Kiel
Technische Fakultät

Example solution for Series #10

Exercise 1 — 10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

		true	false
a)	A PDA is deterministic.	○	⊗
b)	Context-free languages are closed under complement.	○	⊗
c)	PDAs and DPDAs have the same expressive power.	○	⊗
d)	Each grammar can be transformed into CNF (except the empty word).	⊗	○
e)	A grammar in GNF is context-free.	⊗	○
f)	The input alphabet and the stack alphabet of PDAs have to be disjoint.	○	⊗
g)	CFGs and PDAs have the same expressive power.	⊗	○
h)	A derivation in a grammar is biunique.	○	⊗
i)	Every PDA can be simulated by PDA with one state.	⊗	○
j)	The membership-problem for CFGs is efficiently decidable.	⊗	○

Exercise 2 — 7 Points

Give the following definitions and notations:

a) Grammar in Chomsky normal form. (1P)
b) Leftmost derivation. (0.5P)
c) PDA. (4P)
d) Acceptance by PDA by empty stack. (1.5P)

Solution:

a) Context-free grammar G in Chomsky normal form iff $P \subseteq (V \times V^2) \cup (V \times \Sigma)$. (1P)

b) $\alpha \rightarrow_G^L \beta$ is a leftmost derivation if β is derived from α be always replacing the leftmost variable. (0.5P)

c) $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, \bot, F)$ (0.5P)
PDA iff
- Q finite set of states, q_0 initial state (0.5P)
- Σ input alphabet, Γ stack alphabet (0.5P)
- $\bot$ inital stack symbol (0.5P)
- $F \subseteq Q$ final states (0.5P)
- $\Delta \subseteq (Q \times \Sigma \cup \{\varepsilon\} \cup \Gamma) \times (Q \times \Gamma^*)$ (1.5P)

d) $\mathcal{A}$ accepts w by empty stack iff $\exists q \in Q$ (0.5P)
with $(q_0, w, \bot) \rightarrow_{\mathcal{A}}^* (q, \varepsilon, \varepsilon)$ (0.5P)

Exercise 3 — 24 Points

a) Let be $\Sigma = \{0, \ldots, 9, (,), \cdot, +)\}$.

1) Define a grammar G that produces all well-formed arithmetical expression. (For instance $(3+4)\cdot 5$, $7\cdot(4\cdot 3)$, etc). (5P)
2) Prove the correctness of your grammar. (19P)
3) Transform your grammar into CNF. (3.5P)
4) Apply the CYK algorithm on $w = (3 + (\cdot$ and $u = 3 + (4 \cdot 5)$.

b) 1) Construct a PDA for $L = \{w \in \Sigma^* | \, \exists u \in \Sigma^+ : w = uu^R\}$.
2) Describe the way your PDA works. (You do not need to prove the correctness). (2P)

Solution:

a) 1) Define $G = (V, \Sigma, S, P)$ by $V = \{O, S\}$ and P contains
- $S \to 0|1|2|3|4|5|6|7|8|9|(S)|SOS$
- $O \to +|\cdot$

(5P)

2) Consider firstly $w \in L(G)$. (0.5P)
Then there exists a derivation $S \vdash_G^* w$. (0.5P)
We will prove that w is a well-formed arithmetical expression by induction on the length of the derivation. (0.5P)
For the induction base consider a derivation of length 1. (0.5P)
Then $w \in \{0, \dots, 9\}$ and thus a well-formed arithmetical expression. (0.5P)
Assume that the claim holds for derivations of length $k \leq n$ for an arbitrary but fixed $n \in \mathbb{N}$. (0.5P)
Consider now $S \vdash_G^{n+1} w$. (0.5P)
Thus the first applied production rule is either $S \to (S)$ or $S \to SOS$. (0.5P)
Case 1 $S \to (S)$ (0.5P)
This implies the derivation

$$S \vdash_G (S) \vdash_G^n w.$$

(1P)

Thus there exists a $v \in \Sigma^*$ with $w = (v)$ (0.5P)
and $S \vdash_G^n v$. (0.5P)
By (IH) we know that v is a well-formed arithmetical expression. (0.5P)
With $w = (v)$, w is also a well-formed arithmetical expression. (0.5P)
Case 2 $S \to SOS$ (0.5P)
This implies the derivation

$$S \vdash_G SOS \vdash_G^n w.$$

(1P)

Thus there exists $x \in \{+, \cdot\}$ and $u, v \in \Sigma^*$ with $w = uxv$. (1.5P)
By (IH) we know that u, v are well-formed arithmetical expressions. (0.5P)
With $w = uxv$, w is also a well-formed arithmetical expression. (0.5P)
Thus for derivations of all length, G produces only well-formed arithmetical expression. (0.5P)

Consider now w to be a well-formed arithmetical expression. (0.5P)
We prove $w \in L(G)$ by induction on $|w|$. (0.5P)
If $w = 1$, we have $w \in [9]_0$ (0.5P)
and one of the first 10 rules produces w. (0.5P)
Assume that the claim holds for all w with $|w| = k \leq n$ for an arbitrary but fixed $n \in \mathbb{N}$. (0.5P)
Let be $w \in \Sigma^{n+1}$. (0.5P)
We have to distinguish the cases $w = (v)$ and $w = uxv$ for $u, v \in \Sigma^*$ and $x \in \{+, \cdot\}$. (0.5P)
Case 1 $w = (v)$. (0.5P)
By (IH) we know $S \vdash_G^* v$. (0.5P)
With the 11th rule we get $S \vdash_G^* w$. (0.5P)
Case 2 $w = uxv$. (0.5P)
By (IH) we know $S \vdash_G^* v$ and $S \vdash_G^* u$. (1P)
With the 12th and the O-rule we get $S \vdash_G^* uxv = w$. (0.5P)

3) Introduce the variable P_1, P_2 and the rules
   - $P_1 \to ($ (0.5P)
   - $P_2 \to)$ (0.5P)
   - $S \to P_1 S P_2$ (0.5P)

   For transforming $P_1 S P_2$ introduce a fresh variable A with
   - $S \to AP_2$ (0.5P)
   - $A \to P_1 S$ (0.5P)

   For transforming SOS introduce a fresh variable B with
   - $S \to BS$ (0.5P)
   - $B \to SO$

   Since S is not allowed on the right-hand side, define S' as new starting variable and introduce the rule $S' \to S$. (0.5P)
4) For $w = (3 + (\cdot$ we get

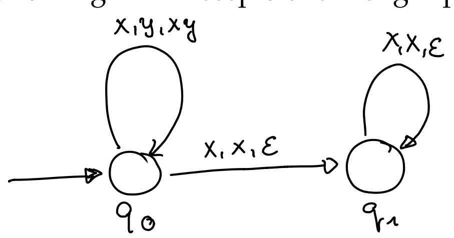
(	3	+	(	$\cdot$
P_1	S	O	P_1	O
A	B			

   Thus $w \notin L(G)$. (5P)
   For $u = 3 + (4 \cdot 5)$ we get

3	+	(	4	$\cdot$	5	)
S	O	P_1	S	O	S	P_2
B		S	B			
		B	S			
S		A, S				
		S				
S						
S						

   Thus $u \in L(G)$. (8P)

b) 1) The following PDA accepts even length palindromes:



acceptance by empty stack

2) The PDA writes the letter, that it read, on the stack while being in q_0. (0.5P)
   At some point it switches from putting on the stack to deleting them, i.e. the transition from q_0 to q_1. (0.5P)
   In q_1 the letters are removed from the stack if they correspond to the read letter. (0.5P)
   The PDA works as intended since for every palindrome there exists the point of switching - the middle. (0.5P)