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Example solution for Series #8

Exercise 1

10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

a)	ε is the neutral element w.r.t. concatenation.	☒ true	○ false
b)	Concatenation is commutative.	○ true	☒ false
c)	Let $h : \Sigma^* \to \Gamma^*$ be a morphism and $A \subseteq \Sigma^*$ regular. Then $h(A)$ is regular.	☒ true	○ false
d)	An NFA $\mathcal{N}$ accepts a word $x \in \Sigma^*$ if there exists $q \in F$ with $q \in \hat{\Delta}(Q_0, x)$.	☒ true	○ false
e)	An NFA is more powerful than a DFA.	○ true	☒ false
f)	A regular expression is more powerful than an NFA with ε-transitions.	○ true	☒ false
g)	The minimisation algorithm has a runtime in $\mathcal{O}(n^2)$.	☒ true	○ false
h)	The minimisation algorithm is applicable to automata with unreachable states.	○ true	☒ false
i)	The equivalence of states is an equivalence relations.	☒ true	○ false
j)	The subset construction is in PSPACE.	○ true	☒ false

Exercise 2

12.5 Points

Give the following definitions and notations:

a) DFA (3P)
b) Transition function for words (1.5P)
c) NFA (3P)
d) Regular Expression (pattern). (2.5P)
e) Quotient Automaton. (2.5P)

Solution:

a) Quintuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ is DFA (0.5P)
iff
- finite set of states Q (0.5P)
- input alphabet Σ (0.5P)
- initial/starting state $q_0 \in Q$ (0.5P)
- transition function $\delta : Q \times \Sigma \to Q$ (0.5P)
- final/accepting states $F \subseteq Q$ (0.5P)

b) $\hat{\delta} : Q \times \Sigma^* \to Q$ (0.5P)
extension of δ:
- $\hat{\delta}(q, \varepsilon) = q$ for $q \in Q$ (0.5P)
- $\hat{\delta}(q, \mathtt{a}x) = \hat{\delta}(\delta(q, \mathtt{a}), x)$ for $q \in Q, x \in \Sigma^*, a \in \Sigma$ (0.5P)

c) Quintuple $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ is NFA (0.5P)
iff
- finite set of states Q (0.5P)

- finite alphabet Σ (0.5P)
- transition-relation $\Delta \subseteq Q \times \Sigma \times Q$ (0.5P)
- starting/initial state $q_0 \in Q$ (0.5P)
- set of final states $F \subseteq Q$ (0.5P)

d) r is a regular expression iff
- new symbol $\emptyset$ for *the nothing* (0.5P)
- ε is the empty word/string (0.5P)
- atomic pattern: every element from $\Sigma \cup \{\varepsilon, \emptyset\}$ (0.5P)
- operations $+, \cdot, {}^*$ (0.5P)
- compound patterns for given patterns α, β: $\alpha + \beta$, $\alpha \cdot \beta$, α^* (0.5P)

e) Quotient automaton $\mathcal{A}/\approx = (Q', \Sigma, \delta', q_0', F')$ for DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$: (0.5P)
- $Q' = \{[q] | \, q \in Q\}$ (0.5P)
- $\delta' : Q' \times \Sigma \to Q'$ with $\delta'([q], a) = [\delta(q, a)]$ (0.5P)
- $q_0' = [q_0]$ (0.5P)
- $F' = \{[q] | \, q \in F\}$ (0.5P)

Exercise 3 — 44 Points

a) Prove that AB is regular, if $A, B \subseteq \Sigma^*$ are regular sets. (18P)

b) 1) Construct an NFA $\mathcal{N}$ for the language $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^{\geq 2} | w[|w| - 1] = \mathtt{a}\}$. (5P)

2) Prove $L(\mathcal{N}) = L$. (9P)

3) Apply the subset construction on $\mathcal{N}$. (7.5P)

4) Decide whether the constructed DFA $\mathcal{A}$ is minimal. (3P)

c) Prove that $+$ in regular expressions is commutative. (2.5P)

Solution:

a) Let $A, B \subseteq \Sigma^*$ be regular sets. (0.5P)
Then there exists an NFA $\mathcal{A} = (Q_A, \Sigma, q_0^A, \Delta_A, F_A)$ with $L(\mathcal{A}) = A$ (0.5P)
and an NFA $\mathcal{B} = (Q_B, \Sigma, q_0^B, \Delta_B, F_B)$ with $L(\mathcal{B}) = B$ (0.5P)
W.l.o.g. (e.g. by renaming) we may assume that Q_A, Q_B are disjoint. (0.5P)
Define $\mathcal{C} = (Q_C, \Sigma, q_0^A, \Delta_C, F_B)$ (2P)
with $Q_C = Q_A \cup Q_B$ (0.5P)

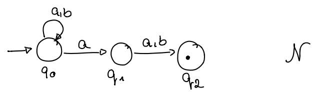
$$\Delta_C = \Delta_A \cup \Delta_B \cup \{(q, \varepsilon, q_0^B) | \, q \in F_A\}.$$

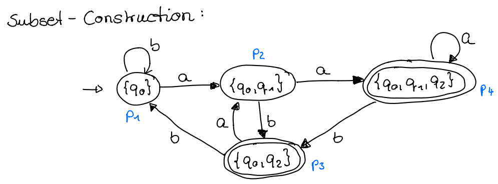
(1.5P)

Claim: $L(\mathcal{C}) = AB$. (0.5P)
Proof: "$\subseteq$" Let be $w \in L(\mathcal{C})$. (0.5P)
By the definition of the acceptance we have $\hat{\Delta}_C(q_0^A, w) \cap F_B \neq \emptyset$. (0.5P)
Choose $q_f \in \hat{\Delta}_C(q_0^A, w) \cap F$. (0.5P)
By the definition of $\hat{\Delta}$ there exists a state sequence $(q_0^A, q_1, \ldots, q_k, q_f)$ for a $k \in \mathbb{N}_0$. (0.5P)
Since $Q_A \cap Q_B = \emptyset$ we have $\Delta_A \cap \Delta_B = \emptyset$. (0.5P)
Notice that both sets are disjoint to $\{(q, \varepsilon, q_0^B) | \, q \in F_A\}$ as well. (0.5P)
Thus there exists $i \in [k]_0$ with $q_i \in F_A$ and $q_{i+1} = q_0^B$. (1P)
Set u as the word produced by the state sequence $(q_0^A, q_1, \ldots, q_i)$ (0.5P)
and v as the word produced by the state sequence $(q_{i+1}, \ldots, q_k, q_f)$. (0.5P)
By $q_i \in F_A$ and the definition of $\hat{\Delta}$ we have $u \in L(\mathcal{A}) = A$ (0.5P)
and by $q_{i+1} = q_0^B$ and the definition of $\hat{\Delta}$ we have $v \in L(\mathcal{B}) = B$. (0.5P)
Thus we have $w = u\varepsilon v = uv \in AB$. (0.5P)
"$\supseteq$" Let $w \in AB$. (0.5P)
Then there exist $u \in A$ and $v \in B$ with $w = uv$. (0.5P)
By $L(\mathcal{A}) = A$ we get $\hat{\Delta}(q_0^A, u) \cap F_A \neq \emptyset$. (0.5P)
Thus after the automaton $\mathcal{C}$ read u it is in a state belonging to F_A. (0.5P)
Hence after the automaton $\mathcal{C}$ read $u\varepsilon$ is is in the state q_0^B by using a transition from

$\{(q, \varepsilon, q_0^B) \mid q \in F_A\}$. (0.5P)
By $L(\mathcal{B}) = B$ we get $\hat{\Delta}(q_0^B, v) \cap F_B \neq \emptyset$. (0.5P)
Thus after the automaton $\mathcal{C}$ read $u\varepsilon v$ it is in a state belonging to F_B. (0.5P)
Since F_B are the final states of $\mathcal{C}$, q_0^A is $\mathcal{C}$'s initial state, we get $uv = u\varepsilon v \in L(\mathcal{C})$. $\square$ (1.5P)

b) The automata for 1) and 3) are given by





(5+7.5P)

For the proof of $\mathcal{L}(N) = L$ we have two prove two subset relations. (0.5P)
"$\subseteq$" Let be $w \in \mathcal{L}(N)$. (0.5P)
Suppose $w[|w| - 1] = \mathtt{b}$. (0.5P)
By the definition of Δ we have $\{(q_0, \mathtt{b}, q_0), (q_1, \mathtt{b}, q_2)\} = \{(q, \mathtt{b}, q') \in \Delta \mid q, q' \in Q\}$. (0.5P)
Choose $u \in \Sigma^*$ and $x \in \Sigma$ with $w = u\mathtt{b}x$. (0.5P)
case 1 After reading u, $\mathcal{N}$ is in state q_0. (0.5P)
Thus after reading $u\mathtt{b}$, $\mathcal{N}$ is in state q_0. (0.5P)
Consequently after reading x, $\mathcal{N}$ is in state q_0 or in state q_1. (0.5P)
Since q_2 is the only final state, $\mathcal{N}$ does not accept w - a contradiction. (0.5P)
case 2 After reading u, $\mathcal{N}$ is in state q_1. (0.5P)
Thus after reading $u\mathtt{b}$, $\mathcal{N}$ is in state q_2. (0.5P)
By the definition of Δ, $\mathcal{N}$ cannot proceed with reading x - a contradiction. (0.5P)
"$\supseteq$"Let w be in L. (0.5P)
Thus there exist $u \in \Sigma^*$ and $x \in \Sigma$ with $w = u\mathtt{a}x$. (0.5P)
By the definition of $\mathcal{N}$, u can be read by $\mathcal{N}$ while staying in q_0. (0.5P)
By reading $\mathtt{a}$, $\mathcal{N}$ moves to state q_1, (0.5P)
and x can be read with the transition $(q_1, \mathtt{a}/\mathtt{b}, q_2)$. (0.5P)
Since q_2 is a final state, $\mathcal{N}$ accepts w. $\square$ (0.5P)
Claim: The given DFA is minimal. (0.5P)

Proof: The states p_1 and p_2 are not equivalent since $\hat{\delta}(p_1, \mathtt{b}) = p_1 \notin F$ whereas $\hat{\delta}(p_2, \mathtt{b}) = p_3 \in F$. (0.5P)
The states p_1 and p_4 (resp. p_3) are not equivalent since p_1 is not a final state whereas p_4 and p_3 are and thus with $x = \varepsilon$ the property for equivalence is not fulfilled. (1P)
By the same reason p_2 is not equivalent to neither p_3 nor p_4. (1P)
Finally p_3 and p_4 are not equivalent because $\delta(p_4, \mathtt{a}) \in F$ and $\delta(p_3, a) \notin F$. (0.5P)
Since no two states are equivalent, the DFA is minimal. $\square$

c) Let α and β be regular expression. (0.5P)
By the semantic's definition we have $L(\alpha + \beta) = L(\alpha) \cup L(\beta)$. (0.5P)
Since the union is commutative, we have $L(\alpha) \cup L(\beta) = L(\beta) \cup L(\alpha)$ (0.5P)
which is equal to $L(\beta + \alpha)$. (0.5P)
Since $\alpha + \beta$ and $\beta + \alpha$ have the same semantic, $+$ commutes. (0.5P)