LOGIC AND THEORETICAL FOUNDATION OF COMPUTER SCIENCE

LaTFoCS

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Dependable Systems Group

NORMAL FORMS

Motivation

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- but a decision procedure for satisfiability is essential!
- let's restrict ourselves to a subset of formula we can handle better

Conjunctive Normal Form (CNF)

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$$(p_1 \vee p_3 \vee \neg p_4) \wedge (p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2 \vee \neg p_4)$$

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$$(\overbrace{p_1}^{L_1} \lor \overbrace{p_3}^{L_2} \lor \overbrace{\neg p_4}^{L_3}) \land (\overbrace{p_2}^{L_4} \lor \overbrace{\neg p_3}^{L_2}) \land (\overbrace{\neg p_1}^{L_1} \lor \overbrace{p_2}^{L_4} \lor \overbrace{\neg p_4}^{L_3})$$

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- clauses are disjunction which valuate to `true` if at least one literal is `true`

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$$\underbrace{(\overbrace{p_1}^{L_1} \vee \overbrace{p_3}^{L_2} \vee \overbrace{\neg p_4}^{L_3})}_{C_1} \wedge \underbrace{(\overbrace{p_2}^{L_4} \vee \overbrace{\neg p_3}^{L_2})}_{C_2} \wedge \underbrace{(\overbrace{\neg p_1}^{L_1} \vee \overbrace{p_2}^{L_4} \vee \overbrace{\neg p_4}^{L_3})}_{C_3}$$

Valuation:

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5. for C_3 valuating to `true`, $\neg p_4$ has to valuate to `true`, thus $\beta(p_4) = \texttt{false}$

Example 2

$$\underbrace{(p_1 \vee p_2)}_{C_1} \wedge \underbrace{(\neg p_1 \vee p_2)}_{C_2} \wedge \underbrace{(p_1 \vee \neg p_2)}_{C_3} \wedge \underbrace{(\neg p_1 \vee \neg p_2)}_{C_4}$$

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4. but for C_4 valuating to `true`, p_2 needs to valuate to `false`
5. $\rightsquigarrow$ we have a problem if p_1 valuates to `true` (only assumption we made)

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4. but for C_3 valuating to `true`, p_2 needs to valuate to `false`
5. $\rightsquigarrow$ also problem if p_1 valuates to `false` (formula is contradiction)

Validity of Disjunctions

Lemma

Disjunction of literals $L_1 \vee \cdots \vee L_m$ *valid iff there exist* $i, j \in [m]$ *with* $L_i = \neg L_j$

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- thus disjuntion valuates to `false`

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- a CNF is valid iff all clauses are valid
- validity check of a CNF: test all clauses for validity

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- notice: high costs
- and how to get a CNF from an arbitrary formula?

Naïve algorithm for Transforming into CNF

Algorithm 1 Naïve algorithm for CNF

Input: propositional logic formula φ with atoms $p_1, \ldots, p_n$ and the truth table

```
for $\ell = 1$ to $2^n$ do
    $\varphi' = \texttt{true}$
    if $\varphi$ evaluates in $\ell^{th}$ line to false then
        $C_\ell = \texttt{false}$
        for $i = 1$ to $n$ do
            if $p_i$ evaluates to true in $\ell^{th}$ line then
                append $\vee\neg p_i$ to $C_\ell$
            else
                append $\vee p_i$ to $C_\ell$
    append $\wedge C_\ell$ to $\varphi'$
```

Example for naïve algorithm

Notice: the algorithm is based on the completeness proof of the propositional logic!

Consider $\varphi = p_1 \to (p_2 \wedge (p_3 \to \neg p_2)$ with the truth table

p_1	p_2	p_3	φ
`false`	`false`	`false`	`true`
`false`	`false`	`true`	`true`
`false`	`true`	`false`	`true`
`false`	`true`	`true`	`true`
`true`	`false`	`false`	`false`
`true`	`false`	`true`	`false`
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p_1	p_2	p_3	φ	
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`false`	`true`	`false`	`true`	
`false`	`true`	`true`	`true`	
`true`	`false`	`false`	`false`	$\neg p_1 \vee p_2 \vee p_3$
`true`	`false`	`true`	`false`	$\neg p_1 \vee p_2 \vee \neg p_3$
`true`	`true`	`false`	`true`	
`true`	`true`	`true`	`false`	$\neg p_1 \vee \neg p_2 \vee \neg p_3$

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`true`	`false`	`false`	`false`	$\neg p_1 \vee p_2 \vee p_3$
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`true`	`true`	`false`	`true`	
`true`	`true`	`true`	`false`	$\neg p_1 \vee \neg p_2 \vee \neg p_3$

thus: $\varphi \equiv (\neg p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee p_2 \vee \neg p_3)$
$\wedge(\neg p_1 \vee \neg p_2 \vee \neg p_3)$

Advantages of CNF

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- different measures: number of conjuncts, overall length, etc

Stepwise Approach to Build an Algorithm

- we will build a deterministic algorithm
- we will use the structure of formulae
- w.l.o.g. we will assume that the formulae are implication free (preprocessing)
- w.l.o.g. we will assume that the formulae are in negation normal form (preprocessing)

Preprocessing: implication freedom

With the equivalence $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$ we can calculate an implication free, equivalent formula.

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Algorithm 2 ImplFree

```
Input: propositional logic formula $\varphi$
if $\varphi$ is a literal then
    return $\varphi$
if $\varphi = \neg\psi$ then
    return $\neg\,\mathrm{ImplFree}(\psi)$
if $\varphi = \psi_1 \vee \psi_2$ then
    return $\mathrm{ImplFree}(\psi_1) \vee \mathrm{ImplFree}(\psi_2)$
if $\varphi = \psi_1 \wedge \psi_2$ then
    return $\mathrm{ImplFree}(\psi_1) \wedge \mathrm{ImplFree}(\psi_2)$
if $\varphi = \psi_1 \rightarrow \psi_2$ then
    return $\neg\,\mathrm{ImplFree}(\psi_1) \vee \mathrm{ImplFree}(\psi_2)$
```

Example

Notice that the algorithm ImplFree has to be applied recursively!

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$$
\begin{aligned}
&\mathrm{ImplFree}((p \lor (q \to r)) \to (s \to (p \to t)))\\
&= \neg\, \mathrm{ImplFree}(p \lor (q \to r)) \lor \mathrm{ImplFree}(s \to (p \to t))\\
&= \neg(p \lor \mathrm{ImplFree}(q \to r)) \lor (\neg s \lor \mathrm{ImplFree}(p \to t))\\
&= \neg(p \lor (\neg q \lor r)) \lor (\neg s \lor (\neg p \lor t))\\
&= (\neg p \land (\neg q \lor r)) \lor (\neg s \lor \neg p \lor t)
\end{aligned}
$$

NNF-Algorithm

Algorithm 3 NNF

```
Input: implication-free, propositional logic formula $\varphi$
if $\varphi$ is a literal then
    return $\varphi$
if $\varphi = \neg\neg\psi$ then
    return $\psi$
if $\varphi = \psi_1 \circ \psi_2$ for $\circ \in \{\wedge, \vee\}$ then
    return $\mathrm{NNF}(\psi_1) \circ \mathrm{NNF}(\psi_2)$
if $\varphi = \neg(\psi_1 \wedge \psi_2)$ then
    return $\mathrm{NNF}(\neg\psi_1) \vee \mathrm{NNF}(\neg\psi_2)$
if $\varphi = \psi_1 \vee \psi_2$ then
    return $\mathrm{NNF}(\neg\psi_1) \wedge \mathrm{NNF}(\neg\psi_2)$
```

Negation Normal Form

If we have a CNF for φ can we compute efficiently a CNF for $\neg\varphi$?

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○ nobody knows the answer

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- how to transform a disjunction into a semantically equivalent conjunction

Transforming a DNF into a CNF

its similar to transform a product of sums into a sum of products:

$$(a + b) \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$$

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$$(a \wedge b) \vee (c \wedge d) = (a \vee c) \wedge (a \vee d) \wedge (b \vee c) \wedge (b \vee d)$$

resp.

$$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$$

DNF to CNF

Algorithm 4 DNF2CNF

```
Input: $\varphi = \psi_1 \lor \psi_2$ and $\psi_1, \psi_2$ in CNF
if $\psi_1$ is $\chi_1 \land \chi_2$ then
    return DNF2CNF($\chi_1 \lor \psi_2$) $\land$ DNF2CNF($\chi_2 \lor \psi_2$)
else if $\psi_2 = \chi_1 \land \chi_2$ then
    return DNF2CNF($\psi_1 \lor \chi_1$) $\land$ DNF2CNF($\psi_1 \lor \chi_2$)
else
    return $\psi_1 \lor \psi_2$
```

The CNF-Algorithm

Now we are able to present the algorithm:

Algorithm 5 CNF

```
Input: $\varphi$ implication-free and in NNF
if $\varphi$ literal then
    return $\varphi$
else if $\varphi = \psi_1 \wedge \psi_2$ then
    return CNF($\varphi_1$) $\wedge$ CNF($\varphi_2$)
else if $\varphi = \psi_1 \vee \psi_2$ then
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```

Conclusion for CNF in general

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- how are we thinking?
- often in implications!
- ... not all implications are easy to handle

Motivation for restricted implications

disjunctions in conclusions are often problematic in normal life:

- if Paula comes to the party Tom and Max are not coming both ($p \rightarrow \neg t \vee \neg m$)
- Tom is allergic to peanuts.
- Max will bring a barbecue.
- Paula cannot decide wether she comes before the very last minute.

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well... do we have to organise a barbecue on our own?
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- things are easier if we only have one positive atom as a conclusion

Motivation refined

If we now restrict the premise also to a conjunction of positive atoms we get something nice

$$p_1 \wedge \cdots \wedge p_n \longrightarrow q \equiv \neg p_1 \vee \cdots \vee \neg p_n \vee q$$

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Definition

○ Clause $C = (\ell_1 \vee \ldots \ell_n)$ of literals ℓ_i, $i \in [n]$, $n \in \mathbb{N}$ is a Horn Clause iff there exists exactly one $i \in [n]$ such that ℓ_i is a positive atom.

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- these clauses are very important in logic programming

Satisfiability of Horn Formulae

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- $\psi \rightarrow \bot$ means *if* ψ *holds we have a contradiction*

Satisfiability of Horn Formulae - Algorithm

Algorithm 6 Horn-SAT

```
Input: $\varphi$ Horn formula
mark all occurrences of $\top$
while ($\exists p_1 \wedge \cdots \wedge p_{k_i} \longrightarrow q$ and $\forall j \in [k_i] : p_j$ marked and $q$ unmarked) do
    mark $q$
if ($\bot$ marked) then
    return UNSAT
return SAT
```

Correctness and Completeness of Horn-SAT

Theorem

The algorithm Horn-SAT is correct, complete and the while loop is executed at most $n + 1$ *times for the number of atoms* $n \in \mathbb{N}$ *in the formula.*

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- there are only n atoms to mark
- thus the while-conditions is checked at most $n + 1$ times
- $\rightsquigarrow$ termination (and roughly a run-time)

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Claim: At any time the marked variables need to evaluate to `true` in all valuations in which φ evaluated to `true`.
Proof by induction on the numbers of while-loop executions:

- Base Case: no execution of the while loop
- thus only $\bot$ is marked with `true`
- $\top$ needs to be `true` by definition

Poof of Soundness Cont

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- IH $\Rightarrow$ in any valuation in which φ is `true` all these p_j have to be `true`
- by truth table q has to be true in these valuation since otherwise the implication (conjunct of φ) would be `false` and thus also φ

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- thus φ is not satisfiable

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  - thus: conclusion cannot be `false`
- φ `true` $\square$

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- and even more generalised: in a conjunction each conjunct needs to be true
- we capture these marking generalisation in the following algorithm

A linear solver: Translating formula into adequate fragment

- $T(p) = p$
- $T(\neg\varphi) = \neg T(\varphi)$
- $T(\varphi_1 \wedge \varphi_2) = T(\varphi_1) \wedge T(\varphi_2)$
- $T(\varphi_1 \vee \varphi_2) = \neg(\neg T(\varphi_1) \wedge \neg T(\varphi_2))$
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after applying T a formula only contains atoms, $\neg$ and $\wedge$

A linear solver: Example

$$\varphi = p \wedge (q \longrightarrow \neg p)$$

A linear solver: Example

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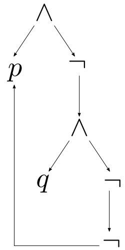
- we have to transform it

$$\begin{aligned} T(\varphi) &= T(p) \wedge T(q \longrightarrow \neg p) \\ &= p \wedge \neg(T(q) \wedge \neg T(\neg p)) \\ &= p \wedge \neg(q \wedge \neg(\neg T(p)) \\ &= p \wedge \neg(q \wedge \neg(\neg p)) \end{aligned}$$

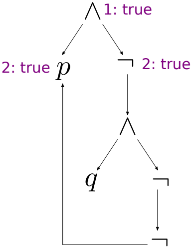
A linear solver: DAG

directed acyclic graph (DAG) for $p \land \neg(q \land \neg(\neg p))$



A linear solver: DAG

we now analyse the graph:



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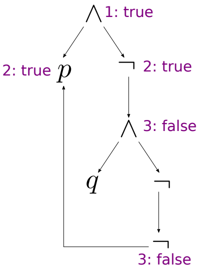

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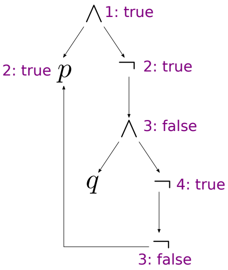

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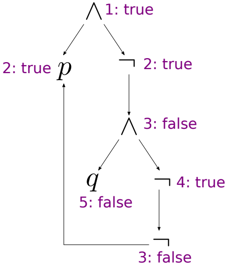

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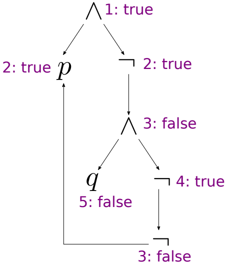
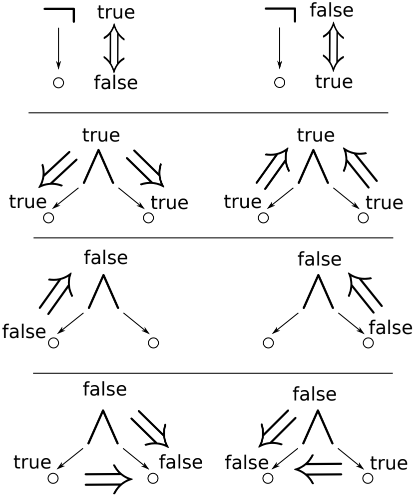


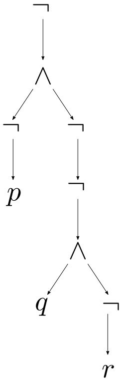
Thus the formula is satisfiable

A linear solver: Forcing Laws



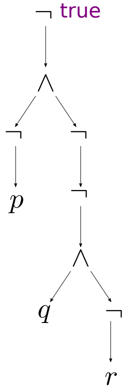
A linear solver: Counter-Example

directed acyclic graph (DAG) for $\neg(\neg p \wedge \neg(\neg(q \wedge \neg r)))$



A linear solver: Counter-Example

we now analyse the graph:



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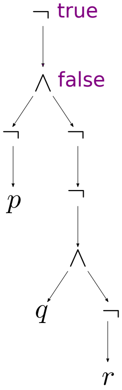

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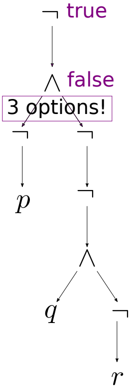

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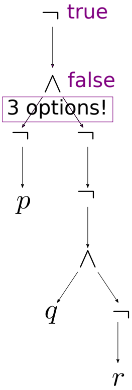

The linearity comes to the costs of not being able to solve this problem!

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- Problem: not all nodes marked (as seen in the example)

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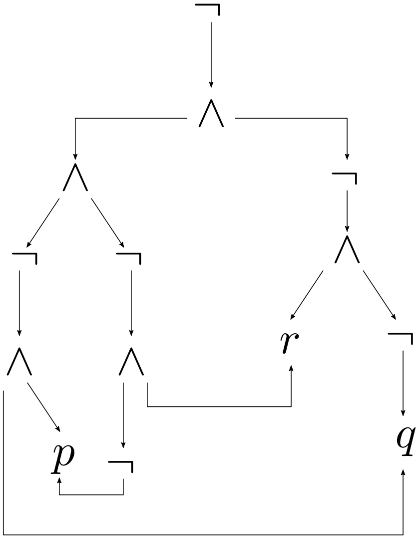
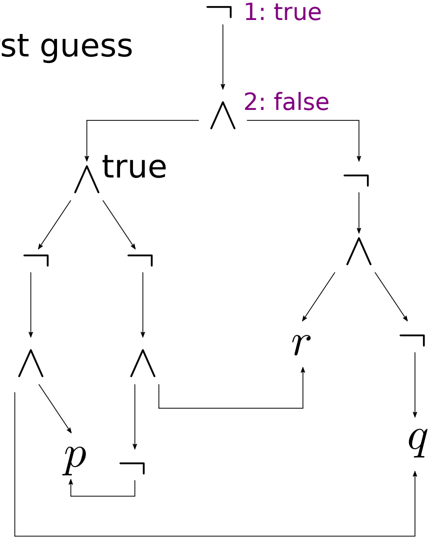
    - if not search for nodes with the same temporary marking in the `true` and the `false` case and turn them to permanent markings
  - delete the temporary markings

Example

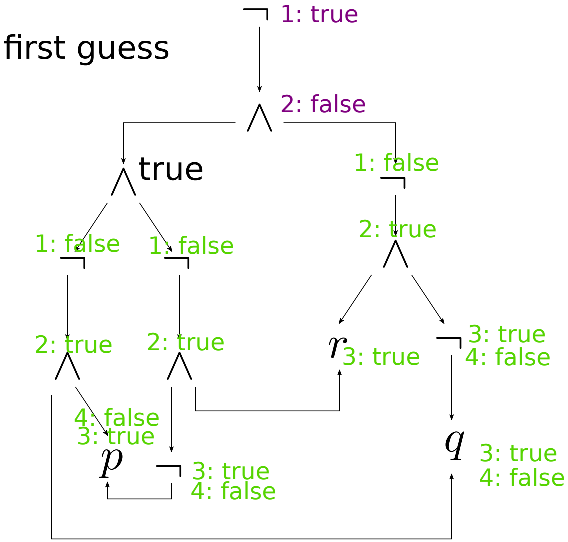


Example

first guess



Example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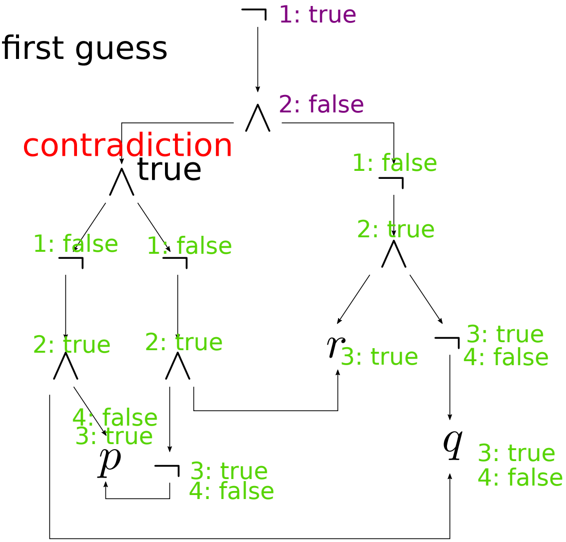

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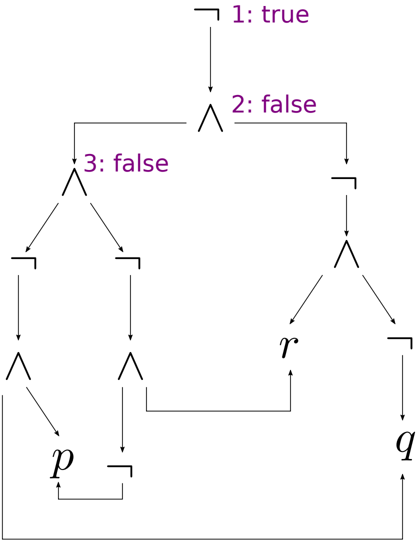

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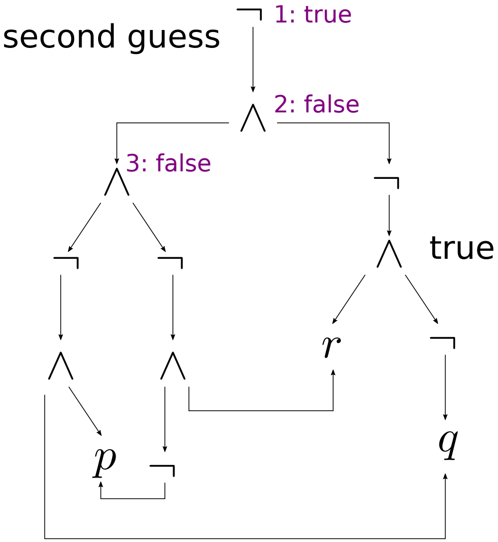

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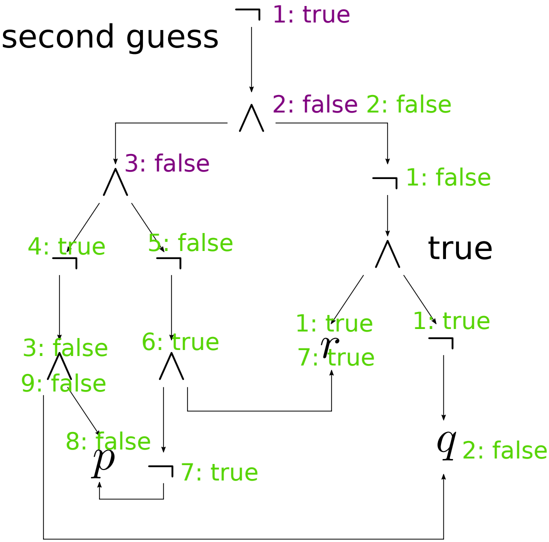
second guess
¬ 1: true
∧ 2: false 2: false
∧ 3: false
¬ 1: false
4: true ¬
5: false ¬
∧ true
3: false ∧ 9: false
6: true ∧
1: true r 7: true
1: true ¬
8: false p
¬ 7: true
q 2: false

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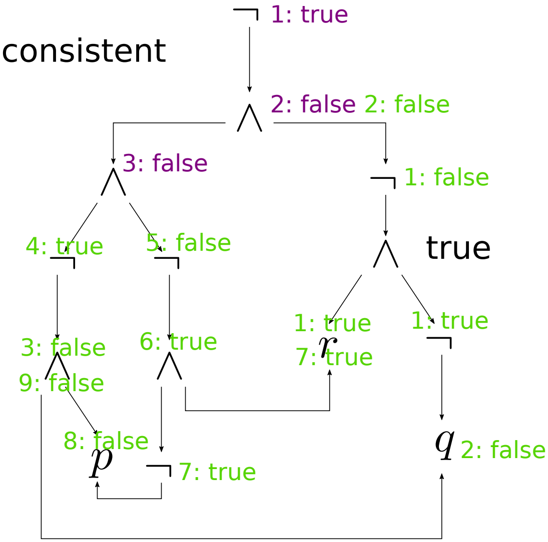
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