Logical and Theoretical Foundations of Computer Science

LaTFoCS

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CAU

Semantic Entailment

Reminder for Propositional Logic

for propositional logic formulae $\varphi_1, \ldots, \varphi_n, \psi$ we defined

$\varphi_1, \ldots, \varphi_n \models \psi$ iff whenever φ evaluates to `true`, ψ does as well

Semantic Entailment in Predicate Logic

Γ set of predicate logic formulae, ψ predicate logic formula

Definition

1. $\Gamma \models_\ell \psi$ iff for all $\mathscr{M}$ and all ℓ and all $\varphi \in \Gamma$ we have that $\mathscr{M} \models_\ell \varphi$ implies $\mathscr{M} \models_\ell \psi$

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3. ψ is valid iff for all models $\mathcal{M}$ and all appropriate environments ℓ, $\mathcal{M} \models_\ell \psi$ holds

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3. ψ is valid iff for all models $\mathcal{M}$ and all appropriate environments ℓ, $\mathcal{M} \models_\ell \psi$ holds
4. Γ is consistent/satisfiable iff there exists a model $\mathcal{M}$ and there exists an environment ℓ such that for all $\varphi \in \Gamma$, $\mathcal{M} \models_\ell \varphi$ holds

Unsatisfiable and Falsifiable

The negations are defined like in propositional logic

- φ is unsatisfiable iff φ is not satisfiable,
- φ is falsifiable iff φ is not valid.

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- Abbreviation: $\models \varphi$

Equivalence of Formulae

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Two formulae φ and ψ are semantically equivalent ($\varphi \equiv \psi$) iff for all models $\mathcal{M}$ we have $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi$.

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Theorem

For formulae φ and ψ we have

$$\varphi \equiv \psi \textit{ iff } \varphi \leftrightarrow \psi.$$

Duality and Commutativity

Theorem

For predicate logic formulae φ and ψ and $Q \in \{\forall, \exists\}$ we have

1. $\neg(\forall x \varphi) \equiv \exists x \neg \varphi$
2. $\neg(\exists x \varphi) \equiv \forall x \neg \varphi$
3. $Q x Q y \varphi \equiv Q y Q x \varphi$

Quantifiers and Logical Connectives

Theorem

- *φ, ψ predicate logic formulae*
- *x not free in ψ*
- *$Q, Q_1, Q_2 \in \{\forall, \exists\}$ with $Q_1 \neq Q_2$*
- $\circ \in \{\wedge, \vee\}$

$$(Qx\,\varphi) \circ \psi \equiv Qx(\varphi \circ \psi)$$

$$Qx(\psi \to \varphi \equiv \psi \to Qx\varphi$$

$$Q_1x(\varphi \to \psi) \equiv Q_2x(\varphi \to \psi)$$

$$\forall x\varphi \wedge \forall x\psi \equiv \forall x(\varphi \wedge \psi)$$

$$\exists x\varphi \vee \exists x\psi \equiv \exists x(\varphi \vee \psi)$$

Some Implications

Theorem

- *φ, ψ predicate logic formulae*
- *x not free in ψ*
- $Q, Q_1, Q_2 \in \{\forall, \exists\}$ *with* $Q_1 \neq Q_2$
- $\circ \in \{\wedge, \vee\}$

$$\models \exists x \forall y \varphi(x, y) \to \forall y \exists x \varphi(x, y),$$
$$\models (\forall x \varphi(x)) \vee (\forall x \psi(x)) \to (\forall x(\varphi(x) \vee \psi(x))),$$
$$\models (\exists x \varphi(x)) \wedge (\forall x \psi(x)) \to (\exists x(\varphi(x) \wedge \psi(x))).$$