Logic and Theoretical Foundation of Computer Science

LaTFoCS

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Kiel University
Dependable Systems Group

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- Recall: we are only interested in decision problems!

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- number of $a \in \Sigma$ in a word $w \in \Sigma^*$: $|w|_a$

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- for a word $w \in \Sigma^n$ for $n \in \mathbb{N}_0$, $w[i]$ denotes the i$^{\text{th}}$ letter of w for all $i \in [n]_0$
- denote by $w[i \ldots j]$ the word which is obtained by concatenating the letters $w[i] \ldots w[j]$ for $0 \le i \le j \le |w|$

Properties of the Concatenation

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$\Rightarrow \Sigma^*$ is a free monoid

Be careful: concatenation is not commutative!
(washing-machine $\neq$ machine-washing)

Finite Automata and Regular Sets

Intuition for Automata

What is an automaton?

- Input

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- States (snapshot of time)

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- Transitions (changes of states)

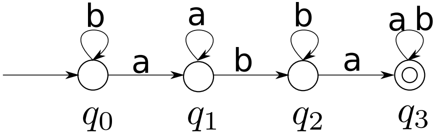
Finite Automata

Definition (Deterministic Finite Automata)

quintuple $\mathscr{A} = (Q, \Sigma, \delta, q_0, F)$ is DFA iff

- **finite** set of states Q
- input alphabet Σ
- initial/starting state $q_0 \in Q$
- transition **function** $\delta : Q \times \Sigma \to Q$
- final/accepting states $F \subseteq Q$

Visualisation



	a	b
q_0	q_1	q_0
q_1	q_1	q_2
q_2	q_3	q_2
q_3F	q_3	q_3

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$\hat{\delta} : Q \times \Sigma^* \to Q$ extension of δ:

- $\hat{\delta}(q, \varepsilon) = q$ for $q \in Q$
- $\hat{\delta}(q, \mathsf{a}x) = \hat{\delta}(\delta(q, \mathsf{a}), x)$ for $q \in Q, x \in \Sigma^*, a \in \Sigma$

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$\mathscr{A}$ accepts $w \in \Sigma^*$: $\hat{\delta}(q_0, w) \in F$ (otherwise reject)

Language of an Automaton and Regular Set of Words

Definition

$\mathscr{A}$'s language: $L(\mathscr{A}) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\}$

Definition

$A \subseteq \Sigma^*$ regular: $\exists \mathscr{A}$ DFA: $A = L(\mathscr{A})$

Regular Sets

Closure Properties

Lemma

A, B regular sets $\Longrightarrow$

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A, B *regular sets* $\Longrightarrow$

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- $A \cap B$ *regular*
- $\Sigma^* \backslash A$ *regular*

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- $A \cap B$ *regular*
- $\Sigma^* \setminus A$ *regular*
- AB *regular*
- A^* *regular*

Proof of the Regularity of $A \cap B$

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  - $F_\cap = F_A \times F_B$
  - $q_0^\cap = (q_0^A, q_0^B)$
  - $\delta_\cap((q, q'), a) = (\delta_A(q, a), \delta_B(q', a))$ (extension as usual)

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- $\Leftrightarrow x \in L(\mathscr{A})$ and $x \in L(\mathscr{B})$
- $\Leftrightarrow x \in L(\mathscr{A}) \cap L(\mathscr{B})$

$\square$

Regular Sets are closed under Morphisms

Definition

Σ, Γ alphabets:

- $h : \Sigma^* \longrightarrow \Gamma^*$ (homo)morphism:
  $\forall x, y \in \Sigma^* : h(xy) = h(x)h(y)$

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- *morphism h uniquely determined by values on Σ*

Morphisms and Regular Sets

$h : \Sigma^* \longrightarrow \Gamma^*$ morphism, $A \subseteq \Sigma^*$, $B \subseteq \Gamma^*$

Definition

- image of A under h: $h(A) := \{h(x) | \, x \in A\}$
- preimage of B under h: $h^{-1}(B) := \{x | \, h(x) \in B\}$

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Theorem

- $B \subseteq \Gamma^*$ *regular* $\Rightarrow h^{-1}(B)$ *regular*
- $A \subseteq \Sigma^*$ *regular* $\Rightarrow h(A)$ *regular*

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- Consider the problem: *Has a word w the same number of* a *and* b*?*
- We need something more powerful?

Non-deterministic Finite Automata

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- How can we adopt this to finite automata?

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- previous automaton defined by transition-**function** $\Rightarrow$ determinism

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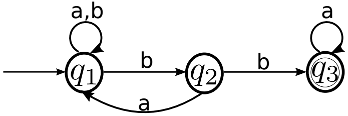
- starting/initial state $q_0 \in Q$
- set of final states $F \subseteq Q$

Example for a NFA



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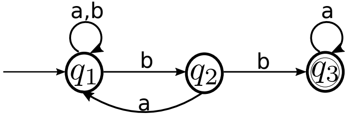

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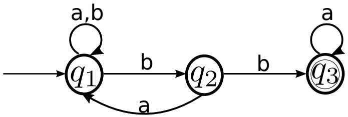

transitions:

as relation

$(q_1, a, q_1), (q_1, b, q_1), (q_1, b, q_2)$

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(q_3, a, q_3)

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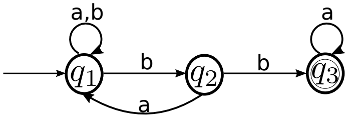

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(q_3, a, q_3)

as function $\Delta' : Q \times \Sigma \longrightarrow \mathscr{P}(Q)$

$(q_1, a) \mapsto \{q_1\}$

$(q_1, b) \mapsto \{q_1, q_2\}$

$(q_2, a) \mapsto \{q_1\}$

$(q_2, b) \mapsto \{q_3\}$

$(q_3, a) \mapsto \{q_3\}$

+ remaining to the empty set

Acceptance by an NFA

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$\hat{\Delta} : \mathscr{P}(Q) \times \Sigma^* \longrightarrow \mathscr{P}(Q)$ with

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- $\hat{\Delta}(A, \mathrm{a}x) = \bigcup_{q \in \Delta(A,\mathrm{a})} \hat{\Delta}(q, x)$

Acceptance by an NFA

Definition (Extending Δ)

$\hat{\Delta} : \mathscr{P}(Q) \times \Sigma^* \longrightarrow \mathscr{P}(Q)$ with

- $\hat{\Delta}(A, \varepsilon) = A$
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- each DFA is an NFA
- $\forall x, y \in \Sigma^* \forall A \subseteq Q : \hat{\Delta}(A, xy) = \hat{\Delta}(\hat{\Delta}(A, x), y)$
- $\hat{\Delta}$ commutes with $\cup$: $\forall n \in \mathbb{N} \forall A_1, \ldots, A_n \subseteq Q$:

$$\hat{\Delta}\left(\bigcup_{i \in [n]} A_i, x\right) = \bigcup_{i \in [n]} \hat{\Delta}(A_i, x)$$

The Subset Construction

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Did we build something more powerful?

- Only if there exists an NFA accepting a language that cannot be accepted by a DFA, we build something more powerful ...
- ... and we can have a look if NFAs are sufficient for the definition of *computation*
- we will show in the following that NFAs are not more powerful $\sim$ guessing does not bring any advantage in finite automata

Subset Construction (Power Set Construction)

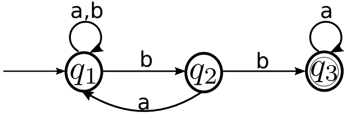
Input: NFA $\mathcal{N} = (Q, \Sigma, \Delta, q_0, F)$

Output: DFA $\mathcal{A} = (\mathbb{Q}, \Sigma, \delta, \{q_0\}, \mathcal{F})$

Construction:

- $\mathbb{Q} = \mathcal{P}(Q)$
- $\delta : \mathbb{Q} \times \Sigma \to \mathbb{Q}, (R, a) \mapsto \{q' \in Q \mid \exists q \in R : (q, a, q') \in \Delta\}$
- $\mathcal{F} = \{S \in \mathbb{Q} \mid S \cap F \neq \emptyset\}$

Example



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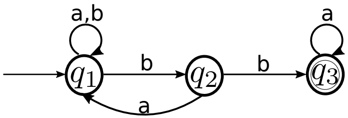

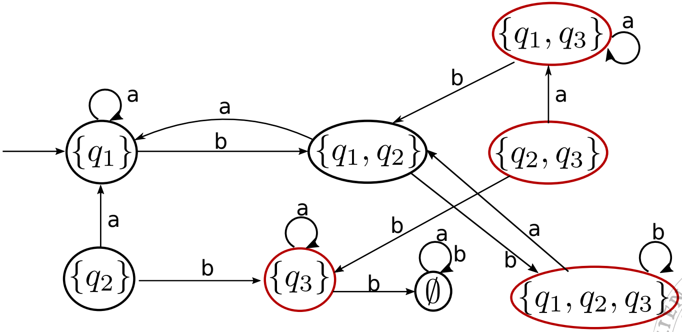

Connection: DFA and NFA

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The constructed DFA is equivalent to the given NFA.

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Theorem

NFA and DFA are equivalent!

Pattern Matching - Regular Expressions

Application for NFA/DFA

Problem: Given a text T and a set of words P, does T contain the words?

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In this tutorial we learn something about patterns.

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This does not match!

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Formal Approach to Regular Expressions

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Simplifying Regular Expressions

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The proof is left to the reader.

Connection between Regularity, RegExp, Finite Automata

Theorem

$A \subseteq \Sigma^* \Rightarrow$ *following statements are equivalent:*

1. *A is regular*
2. *exists DFA $\mathcal{A}$ such that $L(\mathcal{A}) = A$*
3. *exists NFA $\mathcal{N}$ such that $L(\mathcal{N}) = A$*
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This theorem is important!

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- it remains to show: 3. $\to$ 4. and 4. $\to$ 1.

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- induction step: $X \neq \emptyset$, $r \in X$
  - $\alpha^{X}_{qq'} = \alpha^{X \setminus \{r\}}_{qq'} + \alpha^{X \setminus \{r\}}_{qr} (\alpha^{X \setminus \{r\}}_{rr})^* \alpha^{X \setminus \{r\}}_{rq'}$

Regular Expression $\longrightarrow$ Regular Set

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Definition (NFA with ε-Transistion)

$\mathcal{N} = (Q, \Sigma, \Delta, Q_0, F)$ with Q, Σ, Q_0, F as in the *normal* NFA and

$$\Delta \subseteq Q \times \Sigma \cup \{\varepsilon\} \times Q$$

ε-transitions

Lemma

NFA with ε-transitions is ***not*** *more powerful than an NFA*

- M_ε accepts $x \in \Sigma^*$ if there exists $y \in (\Sigma \cup \{\varepsilon\})^*$ such that
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formally: deleting ε is morphism $h : (\Sigma \cup \{\varepsilon\})^* \to \Sigma$ with

- $h(a) = a$ for $a \in \Sigma$ and $h(\varepsilon) = \varepsilon$
- image and preimage of regular sets are regular

Regular Expression $\longrightarrow \varepsilon$-NFA

Thompson-Construction for regular expression r

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  with $\mathcal{A}$ ε-NFA for α, $\mathcal{B}$ ε-NFA for β

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- $r = \alpha^* \leadsto q \xrightarrow{\varepsilon} q_0^{\mathscr{A}} \longrightarrow q_f^{\mathscr{A}} \xrightarrow{\varepsilon} f \in F, f \xrightarrow{\varepsilon} q, q_f^{\mathscr{A}} \xrightarrow{\varepsilon} q_0^{\mathscr{A}}$

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- before we build a better automata model, let's try to handle the state explosion

DFA State Minimization

State Explosion and Handling it

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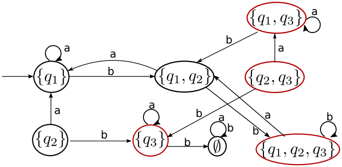
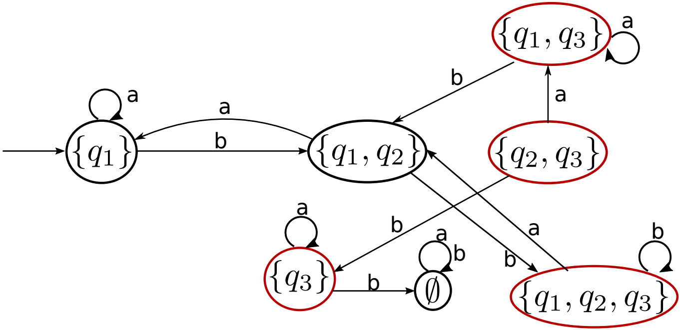
- Have a look at the constructed DFA discussed here:



- no incoming arrow $\Rightarrow$ state is not reachable
- here: $\{q_2\}, \{q_1, q_2, q_3\}$

Cont.



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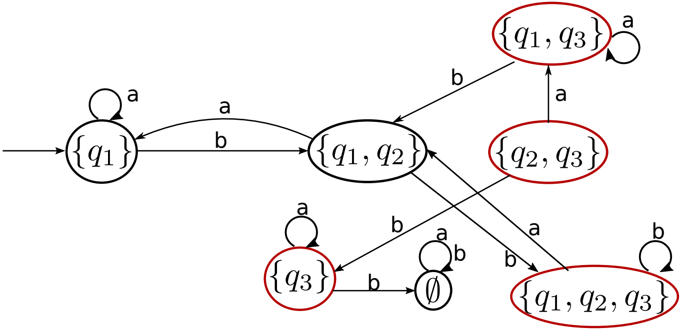

Can we reduce more? Exists something like a similar "behaviour" of states?

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Assume: $q_1, q_2 \in Q$ and whatever you read starting in them you either reach with both a final state or with none

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- $\approx$ equivalence relation $\Rightarrow$ factorisation possible

Quotient Automaton

Definition

quotient automaton $\mathcal{A}/{\approx} = (Q', \Sigma, \delta', q'_0, F')$ for DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$:

- $Q' = \{[q] \mid q \in Q\}$
- $\delta' : Q' \times \Sigma \rightarrow Q'$ with $\delta'([q], a) = [\delta(q, a)]$
- $q'_0 = [q_0]$
- $F' = \{[q] \mid q \in F\}$

Some Properties

$p, q \in Q, x \in \Sigma^*$

- $p \approx q \Rightarrow \delta(p, a) \approx \delta(q, a)$

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Theorem

For every DFA $\mathcal{A}$: $L(\mathcal{A}) = L(\mathcal{A}/\approx)$

Minimized DFA

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there exists a nice algorithm for detecting the equivalent states

Minimization Algorithm

DFA $\mathscr{A}$ with no unreachable states

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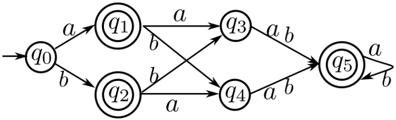
DFA $\mathscr{A}$ with no unreachable states

1. write down a table with all pairs $\{p, q\}$
2. **mark** $\{p, q\}$ if $p \in F$ and $q \notin F$ and vice versa
3. **Repeat Until** no changes
   - $\exists \{p, q\}$ unmarked $\exists a \in \Sigma$:
     $\{\delta(p, a), \delta(q, a)\}$ marked $\Rightarrow$ **mark** $\{p, q\}$

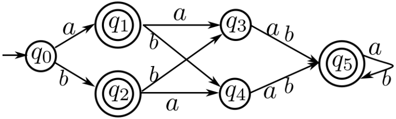
Remarks

- $\{p, q\}$ marked $\Rightarrow$ not equivalent
- every pair may be checked several times
- only stop if really nothing changes
- ≥ 1 mark in each iteration $\Rightarrow$ at most $\binom{|Q|}{2}$ iterations

Example

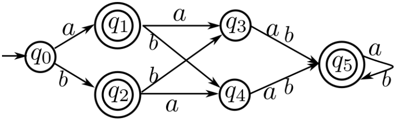


Example



	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1		-	-	-	-	-
q_2			-	-	-	-
q_3				-	-	-
q_4					-	-
q_5						-

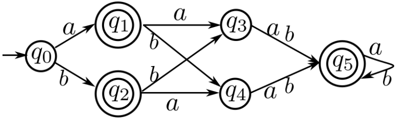
Example



Step 2: 1 final state, 1 non-final state

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3		×	×	-	-	-
q_4		×	×		-	-
q_5	×			×	×	-

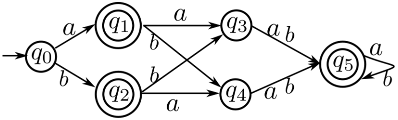
Example



Step 3: for marking $\{q_0, q_3\}$,
$\{\delta(q_0, a/b), \delta(q_3, a/b)\} = \{q_1/q_2, q_5\}$ needs to be marked $\Rightarrow$ nothing

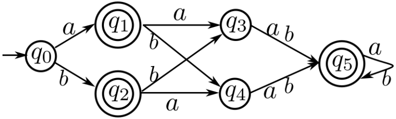
	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3		×	×	-	-	-
q_4		×	×		-	-
q_5	×			×	×	-

Example



Step 3: for marking $\{q_0, q_4\}$,
$\{\delta(q_0, a/b), \delta(q_4, a/b)\} = \{q_1/q_2, q_5\}$ needs to $\Rightarrow$ nothing

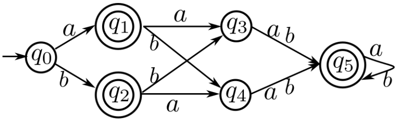
	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3		×	×	-	-	-
q_4		×	×		-	-
q_5	×			×	×	-



Step 3: for marking $\{q_1, q_2\}$,
$\{\delta(q_1, a/b), \delta(q_2, a/b)\} = \{q_3/q_4, q_4/q_3\}$ needs to be marked $\Rightarrow$ nothing

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3		×	×	-	-	-
q_4		×	×		-	-
q_5	×			×	×	-

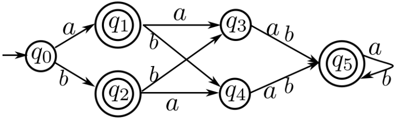
Example



Step 3: for marking $\{q_1, q_5\}$,
$\{\delta(q_1, a/b), \delta(q_5, a/b)\} = \{q_3/q_4, q_5\}$ needs to be marked $\Rightarrow$ mark

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	$\times$	-	-	-	-	-
q_2	$\times$		-	-	-	-
q_3		$\times$	$\times$	-	-	-
q_4		$\times$	$\times$		-	-
q_5	$\times$	$\times$		$\times$	$\times$	-

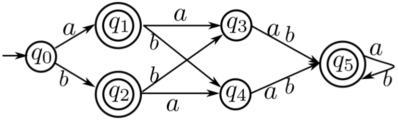
Example



Step 3: for marking $\{q_2, q_5\}$,
$\{\delta(q_2, a/b), \delta(q_5, a/b)\} = \{q_4/q_3, q_5\}$ needs to be marked $\Rightarrow$ mark

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3		×	×	-	-	-
q_4		×	×		-	-
q_5	×	×	×	×	×	-

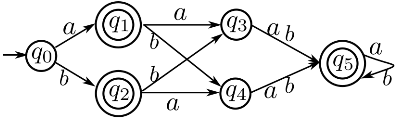
Example



Step 3: for marking $\{q_3, q_4\}$, $\{\delta(q_3, a/b), \delta(q_4, a/b)\} = \{q_5, q_5\}$ needs to be marked $\Rightarrow$ nothing

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3		×	×	-	-	-
q_4		×	×		-	-
q_5	×	×	×	×	×	-

Example



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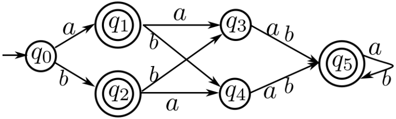
$\{\delta(q_0, a/b), \delta(q_3, a/b)\} = \{q_1/q_2, q_5\}$ needs to be marked $\Rightarrow$ mark

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3	×	×	×	-	-	-
q_4		×	×		-	-
q_5	×	×	×	×	×	-

Example



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	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3	×	×	×	-	-	-
q_4	×	×	×		-	-
q_5	×	×	×	×	×	-

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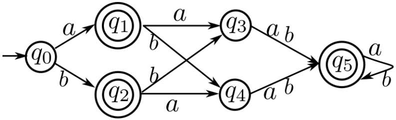

no more changes $\Rightarrow$ q_1, q_2 and q_3, q_4 are equivalent

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	-	-	-	-	-	-
q_1	×	-	-	-	-	-
q_2	×		-	-	-	-
q_3	×	×	×	-	-	-
q_4	×	×	×		-	-
q_5	×	×	×	×	×	-

Example