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# Example solution for Series #3

## Exercise 1 — 10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

a) The conjunction is associative. ☒ true ◯ false

b) A formula is a tautology if it is true for an interpretation. ◯ true ☒ false

c) *Iff* is an abbreviation for two implications. ☒ true ◯ false

d) For proving a claim about all objects of a kind, it suffices to take three examples and to prove the claim for them. ◯ true ☒ false

e) $\mathrm{true}$ is the right-neutral element of $\rightarrow$. ☒ true ◯ false

f) $\varphi \leftrightarrow \psi \equiv (\neg\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$ ☒ true ◯ false

g) $(p \rightarrow q) \wedge (p \rightarrow \neg q)$ is a contradiction. ◯ true ☒ false

h) $p \wedge p$ is a tautology. ◯ true ☒ false

i) $\varphi_1 \equiv \varphi_2$ implies $\varphi_1 \leftrightarrow \varphi_2$. ☒ true ◯ false

j) For $M = \{0, 1\}$, consider $f, g : \mathbb{N} \rightarrow M$ with $f(n) = 0$ iff $g(n) = 0$ for all $n \in \mathbb{N}$. Then $f(n) = 1$ iff $g(n) = 1$. ☒ true ◯ false

## Exercise 2 — 4 Points

Give the following definitions and notations:

a) Logically equivalence of $\varphi, \psi \in \Phi$. (1P)
b) Substitution (1P)
c) Expressive power of $\Phi$. (1P)
d) Explain the notion *same formula up to logical equivalence*. (1P)

### Solution:

a) Two formula $\varphi_1, \varphi_2 \in \Phi$ are called logically equivalent ($\varphi_1 \equiv \varphi_2$) iff $\hat{\beta}(\varphi_1) = \hat{\beta}(\varphi_2)$ for all interpretations $\beta$.
b) A partial function $\sigma : \Phi \rightarrow \Phi$ which is the identity on all undefined formulae, is called a substitution.
c) The expressive power of $\Phi$ is the set of all formula $\varphi$ such there does not exist a $\psi \in \Phi$ with $\psi \equiv \phi$.
d) $\varphi$ is the same formula as $\psi$ up to logical equivalence if $\varphi \equiv \psi$ holds.

## Exercise 3 — 38.5 Points

a) Prove that
   1) $\neg$ is idempotent. (4P)
   2) $\rightarrow$ is not commutative. (8.5P)
b) Prove $\varphi \rightarrow \mathrm{false} \equiv \neg\varphi$. (4P)
c) Prove that $\varphi \rightarrow \varphi$ is a tautology. (3P)

d) Prove whether $\Phi_T$ for $T = \{\neg, \leftrightarrow\}$ has the same expressive power as $\Phi$. (19P)

**Solution:**

a) For proving that $\neg$ is idempotent, we have to prove $\neg\neg\varphi \equiv \varphi$ for all $\varphi \in \Phi$. (0.5P)
By definition, we have to prove $\hat{\beta}(\varphi) = \hat{\beta}(\neg\neg\varphi)$ for all interpretations $\beta$ and all formula $\varphi \in \Phi$. (0.5P)
Let $\varphi \in \Phi$ and $\beta$ an interpretation. (0.5+0.5P)
By definition we have that $\hat{\beta}(\neg\neg\varphi) = \text{true}$ iff $\hat{\beta}(\neg\varphi) = \text{false}$. (0.5P)
Again by definition we have $\hat{\beta}(\neg\varphi) = false$ iff $\hat{\beta}(\varphi) = \text{true}$. (0.5P)
Thus, all over we have $\hat{\beta}(\neg\neg\varphi) = \text{true}$ iff $\hat{\beta}(\varphi) = \text{true}$. (0.5P)
Since $|\mathcal{T}| = 2$, we have $\hat{\beta}(\varphi) = \hat{\beta}(\neg\neg\varphi)$. $\square$ (0.5P)

b) For proving that $\to$ is not commutative, we have to prove $\varphi \to \psi \not\equiv \psi \to \varphi$ for some $\varphi, \psi \in \Phi$ and an interpretation $\beta$. (0.5P)
Thus, we have to find appropriate $\varphi, \psi \in \Phi$ and an interpretation $\beta$ with $\hat{\beta}(\varphi \to \psi) \neq \hat{\beta}(\psi \to \varphi)$. (0.5P)
Set $\varphi = p$, $\psi = q$ for $p, q \in A$, and define $\beta$ by $\beta(p) = 0$ and $\beta(q) = 1$. (1P)
By definition we get

$$\begin{aligned} & \hat{\beta}(\varphi \to \psi) = \text{false} \\ \text{iff } & \hat{\beta}(\varphi) = \text{true and } \hat{\beta}(\psi) = \text{false} \\ \text{iff } & \hat{\beta}(p) = \text{true and } \hat{\beta}(q) = \text{false} \\ \text{iff } & \beta(p) = \text{true and } \beta(q) = \text{false} \\ \text{iff } & 0 = 1 \text{ and } 1 = 0. \end{aligned}$$

(2.5P)

This implies $\hat{\beta}(\varphi \to \psi) = \text{true}$. (0.5P)
On the other hand we have

$$\begin{aligned} & \hat{\beta}(\psi \to \varphi) = \text{false} \\ \text{iff } & \hat{\beta}(\psi) = \text{true and } \hat{\beta}(\varphi) = \text{false} \\ \text{iff } & \hat{\beta}(q) = \text{true and } \hat{\beta}(p) = \text{false} \\ \text{iff } & \beta(q) = \text{true and } \beta(p) = \text{false} \\ \text{iff } & 1 = 1 \text{ and } 0 = 0. \end{aligned}$$

(2.5P)

This implies $\hat{\beta}(\psi \to \varphi) = \text{false}$. (0.5P)
By $\hat{\beta}(\varphi \to \psi) \neq \hat{\beta}(\psi \to \varphi)$ we have proven that $\to$ is not commutative. (0.5P)

c) For proving that $\varphi \to \text{false} \equiv \neg\varphi$ we have to prove $\hat{\beta}(\varphi \to \text{false}) = \hat{\beta}(\neg\varphi)$ for all interpretations $\beta$. (0.5P)
Let $\beta$ be an interpretation. (0.5P)
Then we get by definition

$$\begin{aligned} & \hat{\beta}(\varphi \to \text{false}) = \text{false} \\ \text{iff } & \hat{\beta}(\varphi) = \text{true and } \hat{\beta}(\text{false}) = \text{false} \\ \text{iff } & \hat{\beta}(\varphi) = \text{true} \\ \text{iff } & \hat{\beta}(\neg\varphi) = \text{false}. \end{aligned}$$

(2.5P)

By $\mathcal{T} = \{\text{true}, \text{false}\}$, we get $\hat{\beta}(\varphi \to \text{false}) = \hat{\beta}(\neg\varphi)$. (0.5P)

d) For proving that $\varphi \to \varphi$ is a tautology we have to prove that $\hat{\beta}(\varphi \to \varphi) = \text{true}$ for all interpretations $\beta$. (0.5P)
Let $\beta$ be an interpretation. (0.5P)

By definition we have

$$\begin{aligned} \hat{\beta}(\varphi \to \varphi) &= \text{false} \\ \text{iff } \hat{\beta}(\varphi) = \text{true} \text{ and } \hat{\beta}(\varphi) &= \text{false}\,. \end{aligned}$$

(0.5P)

Since each formula is either true or false, (0.5P)
we know that the right-hand side of the equivalence is false. (0.5P)
Consequently we have $\hat{\beta}(\varphi \to \varphi) = \text{true}$. (0.5P)

e) We will prove that $\Phi_T$ for $T = \{\leftrightarrow, \neg\}$ has not the same expressive power as $\Phi$ by proving that $p \wedge q$ is not expressible just by $\leftrightarrow$ and $\neg$ for atoms $p, q \in A$. (1P)
Notice first that $p \wedge q$ is only true any interpretation $\beta$ if $\beta(p) = 1 = \beta(q)$. (0.5P)
We will prove by induction in $\Phi_T$ that each $\varphi \in \Phi_T$ having exactly two atoms, has always exactly two interpretations such that $\varphi$ is true. (2P)
The induction base is given by $\varphi = p \leftrightarrow q$ since this is the *smallest* formula in $\Phi_T$ having two atoms. (1P)
By definition we get that $\varphi$ is true under the interpretations

- $\beta_1(p) = 1 = \beta_1(q)$ **and**
- $\beta_2(p) = 0 = \beta_2(q)$

and false otherwise. (1P)
This proves the claim for the basis. (0.5P)
For the induction hypothesis assume that the claim holds for all formula of length $\leq n$ for an abitrary but fixed $n \in \mathbb{N}$. (0.5P)
For the induction step we have to prove the claim for $\varphi = \neg\psi$ and $\varphi = \psi \leftrightarrow \chi$. (0.5P)
Consider firstly $\varphi = \neg\psi$. (0.5P)
By the induction hypothesis $\psi$ has exactly two interpretations evaluating $\psi$ to true. (0.5P)
Choose $\beta_1, \beta_2$ as these interpretations. (0.5P)
Since $\varphi$ has exactly two atoms, $\psi$ has exactly two atoms. (0.5P)
Consequently there exists all over four interpretations for $\psi$. (0.5P)
Denote the remaining two valuations by $\beta_3$ and $\beta_4$. (0.5P)
Thus we get

- $\beta_1(\varphi) = \beta_1(\neg\psi) = 0$
- $\beta_2(\varphi) = \beta_2(\neg\psi) = 0$
- $\beta_3(\varphi) = \beta_3(\neg\psi) = 1$
- $\beta_4(\varphi) = \beta_4(\neg\psi) = 1$

(2P)

This proves that there exists exactly two valuations evaluating $\varphi$ to true. (0.5P)
Consider now $\varphi = \psi_1 \leftrightarrow \psi_2$. (0.5P)
By the induction hypothesis $\psi_1, \psi_2$ have each exactly two interpretations evaluating them to true. (0.5P)

- $\beta_1(\psi_1) = \beta_2(\psi_1) = 1$ and $\beta_3(\psi_1) = \beta_4(\psi_1) = 0$
- $\beta_1'(\psi_2) = \beta_2'(\psi_2) = 1$ and $\beta_3'(\psi_2) = \beta_4'(\psi_2) = 0$

(1P)

Again there exists overall four interpretations. (0.5P)
Define $\gamma_1, \ldots, \gamma_4$ by

$$\begin{aligned} \gamma_1(\psi_1) = \beta_1(\psi_1) = 1 &\text{ and } \gamma_1(\psi_2) = \beta_1'(\psi_2) = 1, \\ \gamma_2(\psi_1) = \beta_2(\psi_1) = 1 &\text{ and } \gamma_2(\psi_2) = \beta_3'(\psi_2) = 0, \\ \gamma_3(\psi_1) = \beta_3(\psi_1) = 0 &\text{ and } \gamma_3(\psi_2) = \beta_2'(\psi_2) = 1, \\ \gamma_4(\psi_1) = \beta_4(\psi_1) = 0 &\text{ and } \gamma_4(\psi_2) = \beta_4'(\psi_2) = 0. \end{aligned}$$

(1P)

By definition we get

$$\gamma_1(\varphi) = 1 = \gamma_4(\varphi) \quad \text{and} \gamma_2(\varphi) = 0 = \gamma_3(\varphi).$$

(1P)

This proves that we have exactly two interpretations that evaluate $\varphi$ to $\mathrm{true}$. (0.5P)
Thus we proved that each formula in $\Phi_T$ has exactly two interpretations evaluating the formula to true. (0.5P)
For expressing $p \wedge q$ we need amongst the four possible interpretations exactly one evaluating to true and this is not possible. $\square$ (1P)