LOGIC AND THEORETICAL FOUNDATION OF COMPUTER SCIENCE

LATFOCS

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Kiel University Dependable Systems Group





 \bigcirc Black clouds are approaching. \rightsquigarrow Computer: Please don't forget the umbrella.



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- Braking lights of the preceding car on. ~ Computer:
 Please brake, too.



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- Braking lights of the preceding car on. → Computer: Please brake, too.
- Pain in the leg and high concentration of white blood cells.
 → Computer: Infection.



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How can we train a computer to deduce these things?



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$$\varphi_1,\ldots,\varphi_n\vdash\psi$$



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- $\circ \varphi \dashv \vdash \psi$ abbreviates $\varphi \vdash \psi$ and $\psi \vdash \varphi$ (φ and ψ are provable equivalent)



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... but what is a proof for a computer?

Notation for Proof Rules

○ 1 premise

premise conclusion

 \bigcirc *n* premises

 $\frac{\text{premise } 1 \qquad \dots \qquad \text{premise } n}{\text{conclusion}}$



Rules for Natural Deduction - Rules for Conjunction

 \bigcirc and-introduction ($\land i$)

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \stackrel{(\wedge i)}{}$$



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 (\lambde e_2)



Our first proof

Foxes have fur and can walk. Foxes are mammals. Conclusion: Foxes are walking mammals. (mammal and walk)



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Claim: $\varphi \land \psi, \vartheta \vdash \psi \land \vartheta$



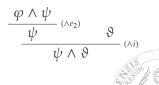
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Claim: $\varphi \land \psi, \vartheta \vdash \psi \land \vartheta$

Proof:

- 1. $\varphi \wedge \psi$ (premise)
- 2. ϑ (premise)
- 3. ψ $(\wedge e_2)_1$
- 4. $\psi \wedge \vartheta$ $(\wedge i)_{2,3}$



Rules for Natural Deduction - Rules for Double Negation

double-negation-elimination

$$\frac{\neg \neg \varphi}{\varphi}$$
 ($\neg \neg e$)



Rules for Natural Deduction - Rules for Double Negation

○ double-negation-elimination

$$\frac{\neg \neg \varphi}{\varphi}$$
 (¬¬e)

double-negation-introduction

$$\frac{\varphi}{\neg \neg \varphi}$$
 (¬¬i)



 \bigcirc It is not true that foxes cannot fly.



It is not true that foxes cannot fly. What does this mean?



- It is not true that foxes cannot fly. What does this mean?
 - Can all foxes fly?



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 - What do the answers *yes*, *no* mean?

In logic the double negation is well defined!



Rules for Natural Deduction - Rules for Implication

omodus-ponens (implies-elimination, arrow-eliminiation)

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \text{ (mp)}$$



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$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \text{ (mp)}$$

modus-tollens

$$\frac{\neg \psi \qquad \varphi \rightarrow \psi}{\neg \varphi} \ \ _{\text{(mt)}}$$



○ if we buy tomatoes, onion, basil, olive oil, pepper, salt, then we have ingredients for a simple pasta sauce



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- o if we buy tomatoes, onion, basil, olive oil, pepper, salt, then we have ingredients for a simple pasta sauce
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- the latter one is an assumption
- O Notice:
 - the pasta sauce does not depend on the assumption
 - the pizza depends on the assumption and the ingredients we tend to buy

Formalising Assumption based Proofs

○ Everything based on an assumption is put in a box:

 φ assumption \vdots deduction steps ψ conclusion



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Formalising Assumption based Proofs

O Everything based on an assumption is put in a box:

 φ assumption \vdots deduction steps ψ conclusion

- \bigcirc thus we know: if φ holds, then ψ does so as well
- \bigcirc this is the colloquial description of $\varphi \to \psi$



Rules for Natural Deduction - Rules for Implication

implication introduction

$$\begin{array}{c|c}
\varphi \\
\vdots \\
\psi
\end{array}$$

$$\varphi \to \psi \xrightarrow{(\to i)}$$



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implication introduction

$$\begin{array}{c|c}
\hline
\varphi \\
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\end{array}$$

$$\overline{\varphi \to \psi} \stackrel{(\to i)}{}$$

Notice: this rule also allows us the following

if we have a proof for ψ under the assumption $\varphi,$ we have also the implication $\varphi \to \psi$



Contraposition

Lemma

If $\varphi \to \psi$ *holds, then also the* **contraposition** $\neg \psi \to \neg \varphi$.



Contraposition

Lemma

If $\varphi \to \psi$ *holds, then also the* **contraposition** $\neg \psi \to \neg \varphi$.

Proof:

$$\begin{array}{c|c}
p \to q & \neg q \\
\hline
 \neg p & \\
\hline
 \neg q \to \neg p & (\rightarrow i)
\end{array}$$
(mt)



 \bigcirc we defined a sequent by $\varphi_1, \ldots, \varphi_n \vdash \psi$ and called it valid if there exists a proof inferring ψ from $\varphi_1, \ldots, \varphi_n$



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Definition

 φ theorem iff $\vdash \varphi$ is valid sequent



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Definition

 φ theorem iff $\vdash \varphi$ is valid sequent

we will see later why theorem in model-theory and theorem in decuction theory are the same

Example for a Theorem

Claim:
$$(q \to r) \to ((\neg q \to \neg p) \to (p \to r))$$
 is a theorem



Example for a Theorem

```
Claim: (q \to r) \to ((\neg q \to \neg p) \to (p \to r)) is a theorem
Proof:
Assumption: q \rightarrow r
                         Assumption: \neg q \rightarrow \neg p
                                                 Assumption: p
                                                              (\neg \neg i) \neg \neg p
                                                               (MT) \neg \neg q
                                                             (\neg \neg e) q
                                                            ((\rightarrow e) r
                                     (\rightarrow i) p \rightarrow r
             (\rightarrow i) (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)
(\rightarrow i) (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))
```



Connection: Proof - Theorem

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Every proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ is transformable into a theorem.



Connection: Proof - Theorem

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Every proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ is transformable into a theorem.

Proof:

 \bigcirc transform the proof by using $(\rightarrow i)$ to

$$\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\cdots \rightarrow (\varphi_n \rightarrow \psi) \dots))$$



 \bigcirc or-introduction ($\lor i_1$)

$$\frac{\varphi}{\varphi \vee \psi}$$
 ($\vee i_1$)



 \bigcirc or-introduction ($\lor i_1$)

$$\frac{\varphi}{\varphi \vee \psi}$$
 ($\vee i_1$)

 \bigcirc or-introduction ($\lor i_2$)

$$\frac{\psi}{\varphi \vee \psi}$$
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 \bigcirc or-introduction ($\vee i_1$)

$$\frac{\varphi}{\varphi \vee \psi}$$
 (vi₁)

 \bigcirc or-introduction ($\lor i_2$)

$$\frac{\psi}{\varphi \vee \psi}$$
 ($\vee i_2$)

Notice that in both rules the respectively other formulae is arbitrary! (If foxes are mammals then foxes are mammals or cows are birds.)

What can we deduce from a disjunction $\varphi \lor \psi$? We **do not** know if φ or ψ holds.



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I catch the bus or I take a taxi.

If I catch the bus, then I will arrive at the bus-stop.

If I arrived at the bus-stop, I will walk to the hotel.

If I take the taxi, it will bring me to the hotel.



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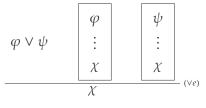
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 \bigcirc or-elimination ($\lor e$)





\land - \lor -Distributivity

Claim:
$$p \land (q \lor r) \vdash (p \land q) \lor (p \land r)$$



∧-∨-Distributivity

```
Claim: p \land (q \lor r) \vdash (p \land q) \lor (p \land r)
Proof:
       Premise: p \land (q \lor r)
             (\wedge e_1) p
             (\wedge e_2) q \vee r
                       Assumption: q
                                     (\wedge i) p \wedge q
                                    (\vee i_1) (p \wedge q) \vee (p \wedge r)
                       q \to (p \land q) \lor (p \land r)
                       Assumption: r
                                     (\wedge i) p \wedge r
                                    (\vee i_2) (p \wedge q) \vee (p \wedge r)
                       r \rightarrow (p \land q) \lor (p \land r)
              (\vee e) (p \wedge q) \vee (p \wedge r)
```



Contradiction

 \bigcirc so far we have seen for negation: $(\neg \neg i)$ and $(\neg \neg e)$



Contradiction

- \bigcirc so far we have seen for negation: $(\neg \neg i)$ and $(\neg \neg e)$
- how to introduce or eliminate just one negation



Contradiction

- \bigcirc so far we have seen for negation: $(\neg \neg i)$ and $(\neg \neg e)$
- $\, \bigcirc \,$ how to introduce or eliminate just one negation

Definition

for all formulae φ , $\varphi \land \neg \varphi$ is a contradiction





What do you think? True or false?

 $\, \bigcirc \,$ If a fox is a fish, then -350°C exists.



- \bigcirc If a fox is a fish, then -350°C exists.
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- If a fox is a fish, then forests are under water.
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- If a fox is a fish, then it can fly.

Everything follows from the false. (*Deductive explosion*. First proven by William of Soissons in 12th century)

Rules for Natural Deduction - Rules for Negation

bottom-eliminiation, false-eliminiation, ex falso quodlibet (efq)

$$\frac{\bot}{\varphi}$$
 (efq



Rules for Natural Deduction - Rules for Negation

 bottom-eliminiation, false-eliminiation, ex falso quodlibet (efq)

$$\frac{\bot}{\varphi}$$
 (efq)

not-eliminiation, contradiction (cd)

$$\frac{\varphi \land \neg \varphi}{\bot}$$
 (cd



Definition (Absurd (Oxford Dictionary))

- 1. Adjective:
 - Wildly unreasonable, illogical, or inappropriate.
 - Arousing amusement or derision; ridiculous.
- 2. Noun: An absurd state of affairs.



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Mid 16th century: from Latin absurdus 'out of tune', hence 'irrational'; related to surdus 'deaf, dull'.

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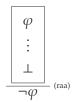
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- O Getting rid of fruit flies takes more than 10 min.
- O This implies that not tossing the garbage is absurd.



Rules for Natural Deduction - Rules for Negation

○ ¬-introduction, reductio ad absurdum





Rules for Natural Deduction - Tertium non datur

○ tertium non datur (law of excluded middle)

$$\varphi \lor \neg \varphi$$
 (tnd)



Rules for Natural Deduction - Tertium non datur

tertium non datur (law of excluded middle)

$$\varphi \lor \neg \varphi$$
 (tnd)

as long as we are binary, either the one or the opposite holds

- either a number is zero or greater than zero
- either I am eating pizza or not
- either the fox can fly or not



Not only for proving proof-rules

Lemma

Let $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$ with a|b. Then $|a| \leq |b|$.



Not only for proving proof-rules

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Proof.



Not only for proving proof-rules

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Let $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$ with a|b. Then $|a| \leq |b|$.

Proof. How to start?



Let's first collect what we have:

1. $a, b \in \mathbb{N}$: a, b natural numbers, $a \ge 0$, b > 0



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- 1. $a, b \in \mathbb{N}$: a, b natural numbers, $a \ge 0$, b > 0
- 2. $a|b: \exists c \in \mathbb{N} : ac = b$



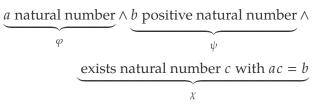
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3.
$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{otherwise} \end{cases}$$



Reformulation of the lemma:



 \vdash absolute value of a is at most absolute value of b

ć



Proof

- \bigcirc χ allows us to take a c with ac = b
- \bigcirc tnd: either *c* is zero or not
- \circ case 1: c = 0
 - o multiplication with zero is zero
 - thus *b* is zero
 - \circ ($\wedge i$) b = 0 and b > 0
 - o (cd) ⊥
- \bigcirc case 2: $c \neq 0$
 - definition of absolute value: |c| > 0
 - total order on \mathbb{N} : $|a| = |a| \cdot 1 \le |a| \cdot |c| = |ac| = |b|$

