Christian-Albrechts-Universität zu Kiel
Technische Fakultät

Dependable Systems Group
D. Nowotka, P. Fleischmann

Example solution for Series #4

Exercise 1 — 10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

a) If $\Psi \subset \Phi$ is satisfiable and $\varphi \in \Phi$ is satisfiable then $\Psi \cup \{\varphi\}$ is satisfiable. ○ true ⊗ false

b) If $\Psi \models \varphi$ for $\Psi \subset \Phi$ and $\varphi \in \Phi$, and $\psi \in \Phi$ is satisfiable then $\Psi \backslash \{\psi\} \models \varphi$. ○ true ⊗ false

c) W.l.o.g. stands for *with loss of generality*. ○ true ⊗ false

d) For proving by contradiction, we suppose the opposite of our claim, and deduce a contradiction. ⊗ true ○ false

e) If $\mathcal{T}_\varphi$ is closed then φ is falsifiable. ⊗ true ○ false

f) For proving the correctness of an algorithm, it suffices to prove that the algorithm terminates. ○ true ⊗ false

g) A completed tableau is called closed, if at least one leaf is closed. ○ true ⊗ false

h) A formula is in negation normal form if $\neg$ is the only logical operator used. ○ true ⊗ false

i) A set of literals is satisfiable if it does not contain a complementary pair of literals. ⊗ true ○ false

j) $\{p, \neg q\}$ for $p, q \in A$ is a complementary pair of literals. ○ true ⊗ false

Exercise 2 — 5 Points

Give the following definitions and notations:

a) Satisfiability of $\varphi \in \Phi$. (1P)
b) Decision procedure for $\Psi \subseteq \Phi$. (1P)
c) $T \subseteq \Phi$ theory. (1P)
d) Axiomatisiable theory. (1P)
e) Literal. (1P)

Solution:

a) φ is satisfiable iff $\hat{\beta}(\varphi) = \mathrm{true}$ for some interpretation β (1P)
b) An algorithm A is a decision procedure for $\Psi \subseteq \Phi$ if for all $\varphi \in \Phi$ it returns true iff $\varphi \in \Psi$. (1P)
c) A theory is an under logical equivalence closed subset of Φ. (1P)
d) A theory T is axiomatisable iff there exists $A \subseteq \Phi$ with $T = \{\varphi | A \models \varphi\}$ (1P)
e) A literal is an atom or the negation of an atom. (1P)

Exercise 3 — 14.5 Points

a) Justify, why $\neg$ is necessary (implicit or explicit) in each set of operators S such that Φ_S has the same expressive power as Φ. (2P)
b) Let $\Psi \subset \Phi$, $\psi \in \Phi$. Prove that if Ψ is satisfiable and ψ is valid then $\Psi \cup \{\psi\}$ is satisfiable. (4P)
c) Let $\Psi \subset \Phi$ and $\psi, \varphi \in \Phi$. Prove that, if $\Psi \models \varphi$ then $\Psi \cup \{\psi\} \models \varphi$. (4P)
d) Apply the algorithm for constructing a semantic tableau on $(p \vee q) \to (s \wedge \neg p)$. (4.5P)

Solution:

a) $\neg$ is implicitely contained in $\uparrow$ and $\downarrow$ and false (as negation of true. (0.5P)
Thus the set of operators S need to be a subset of $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$. (0.5P)
This implies that we have to model $\neg p$ only with true and these four operators. (0.5P)
By the definition of $\hat{\beta}$ none of these operators is able to swap the truth value. (0.5P)

b) Let Ψ be satisfiable and ψ be valid. (1P)
Since Ψ is satisfiable there exists an interpretation $\beta : A \to \mathcal{T}$ such that $\hat{\beta}(\varphi) = \mathrm{true}$ for all $\varphi \in \Psi$. (0.5P)
Since ψ is valid we have $\hat{\beta_1}(\psi) = \mathrm{true}$ for all interpretations β. (0.5P)
For proving that $\Psi \cup \{\psi\}$ is satisfiable we have to find an interpretation β_2 with $\hat{\beta_2}(\varphi) = \hat{\beta_2}(\psi) = \mathrm{true}$ for all $\varphi \in \Psi$. (0.5P)
Set $\beta_2 = \beta$. (0.5P)
By definition $\hat{\beta}(\varphi) = \mathrm{true}$ for all $\varphi \in \Psi$. (0.5P)
Since ψ is evaluated to true under all interpretations, we get especially $\hat{\beta_2}(\psi) = \mathrm{true}$. (0.5P)
This concludes the proof. $\square$

c) Assume $\Psi \models \varphi$. (0.5P)
We have to prove $\Psi \cup \{\psi\} \models \varphi$. (0.5P)
This is by definition that we have to prove that each model β of $\Psi \cup \{\psi\}$ is also a model of φ. (0.5P)
Thus let β be a model of $\Psi \cup \{\psi\}$. (0.5P)
By the definition of model, we get $\hat{\beta}(\chi) = \mathrm{true}$ for all $\chi \in \Psi$ and $\hat{\beta}(\psi) = \mathrm{true}$. (0.5P)
By the first part we get that β is a model of π. (0.5P)
By the assumption we get that β is a model of φ. (0.5P)
This proves $\Psi \cup \{\psi\} \models \varphi$. $\square$ (0.5P)

d) Since the algorithm is only applicable to formulae in negation normal form, we have firstly to transform it: (0.5P)

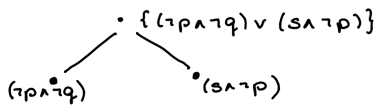
$$
\begin{aligned}
(p \vee q) \rightarrow (s \wedge \neg p) &\equiv \neg(p \vee q) \vee (s \wedge \neg p) \\
&\equiv (\neg p \wedge \neg q) \vee (s \wedge \neg p).
\end{aligned}
$$

(1P)

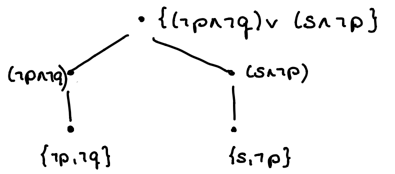
Step 1 • $\{(\neg p \wedge \neg q) \vee (s \wedge \neg p)\}$

Step 2 2.1 is not applicable

2.2.2 is applicable



Step 3 2.2.1 is applicable twice



Step 4: Since no complementary pairs of literals occur the formula is satisfiable.

(3P)