Logic and Theoretical Foundation of Computer Science

LaTFoCS

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Dependable Systems Group

Model Theory vs. Proof/Deduction Theory

Two ways - do the harmonise?

- we looked at truth tables
  - inductive definition when a formula is true or false; $\varphi_1, \ldots \varphi_n \models \psi$ if for every interpretation β in which $\varphi_1, \ldots, \varphi_n$ is true implies that $\hat{\beta}(\psi) = \mathtt{true}$
- we looked at natural deduction:
  - $\varphi_1, \ldots \varphi_n \vdash \psi$ valid sequent if ψ is deducable with the rules from $\varphi_1, \ldots, \varphi_n$
- **Suspicious question 1**: is it possible that $\varphi_1, \ldots, \varphi_n$ are all true, $\varphi, \ldots, \varphi_n \vdash \psi$ holds, but ψ is false?
- **Suspicious question 2**: Is it possible that the truth table tells us that ψ is always true if $\varphi_1, \ldots, \varphi_n$ and there does not exist a proof for that?

Soundness

Theorem

Given propositional logic formulae $\varphi_1, \ldots, \varphi_n, \psi$ *with* $\varphi_1, \ldots \varphi_n \vdash \psi$*, then* $\varphi_1, \ldots, \varphi_n \models \psi$ *holds.*

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Proof (Induction on the number of steps in the proof)

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Proof (Induction on the number of steps in the proof)

Base Case: $\varphi_1, \ldots, \varphi_n \vdash \psi$ has a proof in one step

- only premises are one-step-proofs
- thus: $n = 1, \varphi_1 = \psi$
- thus: $\psi \vdash \psi$
- ψ true iff ψ true
- thus $\psi \models \psi$

Soundness-Proof: Induction Hypothesis

Induction Hypothesis: Let k be in $\mathbb{N}$ arbitrary but fixed.
Assume that for all $n \in \mathbb{N}_0$ and for all proofs of length at most k
$\varphi_1, \ldots, \varphi_n \vdash \psi$ implies $\varphi_1, \ldots, \varphi_n \models \psi$

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$\varphi_1, \ldots, \varphi_n \vdash \psi$ implies $\varphi_1, \ldots, \varphi_n \models \psi$
Induction Step: Let n be in $\mathbb{N}$ and $\varphi_1. \ldots, \varphi_n \vdash \psi$ a proof of length $k + 1$.

Soundness-Proof: Induction Step

- proof of length $k + 1$ has structure

1.	φ_1	premise
2.	φ_2	premise
$\vdots$	$\vdots$	$\vdots$
n.	φ_n	premise
$\vdots$	$\vdots$	$\vdots$
(k+1).	ψ	conclusion

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- what do we know about this proof?
- there is a step k leading to ψ
- we **do not** know which rule was applied $\leadsto$ proof for all rules necessary

Soundness Proof: Case 1

last applied rule is $(\wedge i)$

- $\psi = \psi_1 \wedge \psi_2$
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- $\psi = \psi_1 \wedge \psi_2$
- ψ_1, ψ_2 occur up in the proof

  - ψ_1, ψ_2 above ψ
  - $\varphi_1, \dots, \varphi_n \vdash \psi_1$
  - $\varphi_1, \dots, \varphi_n \vdash \psi_2$
  - (IH) $\varphi_1, \dots, \varphi_n \models \psi_1$
  - (IH) $\varphi_1, \dots, \varphi_n \models \psi_2$
  - truth table:

    $\varphi_1, \dots, \varphi_n \models \psi_1 \wedge \psi_2 = \psi$

1.	φ_1	premise
$\vdots$	$\vdots$	$\vdots$
n.	φ_n	premise
$\vdots$	$\vdots$	$\vdots$
i_1	ψ_1	
$\vdots$	$\vdots$	$\vdots$
i_2	ψ_2	
$\vdots$	$\vdots$	$\vdots$
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Soundness Proof: Case 2

last applied rule is $(\vee e)$

$$\dfrac{\psi_1 \vee \psi_2 \qquad \boxed{\begin{array}{c}\psi_1\\ \vdots\\ \psi\end{array}} \qquad \boxed{\begin{array}{c}\psi_2\\ \vdots\\ \psi\end{array}}}{\psi}\;(\vee e)$$

last applied rule is $(\lor e)$

$$\dfrac{\psi_1 \lor \psi_2 \qquad \boxed{\begin{array}{c}\psi_1\\ \vdots\\ \psi\end{array}} \qquad \boxed{\begin{array}{c}\psi_2\\ \vdots\\ \psi\end{array}}}{\psi}\;(\lor e)$$

1.	φ_1	premise
$\vdots$	$\vdots$	$\vdots$
n.	φ_n	premise
$\vdots$	$\vdots$	$\vdots$
i_1	$\psi_1 \lor \psi_2$	
$i_1 + 1$	ψ_1	assumption
$\vdots$	$\vdots$	$\vdots$
j_1	ψ	
$j_1 + 1$	ψ_2	assumption
$\vdots$	$\vdots$	$\vdots$
k	ψ	
$(k+1)$	ψ	conclusion

Soundness Proof: Case 2

last applied rule is $(\vee e)$

- ψ_1, ψ_2 above ψ
- $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \vee \psi_2$
- $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi$
- $\varphi_1, \ldots, \varphi_n, \psi_2 \vdash \psi$
- (IH) $\varphi_1, \ldots, \varphi_n \models \psi_1 \vee \psi_2$
- (IH) $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi$
- (IH) $\varphi_1, \ldots, \varphi_n, \psi_2 \vdash \psi$
- truth table: $\varphi_1, \ldots, \varphi_n \models \psi$

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$\vdots$	$\vdots$	$\vdots$
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Soundness Proof

the remaining cases are left as exercises

Value of Soundness - how to use it

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- if we find one valuation where $\varphi_1, \ldots \varphi_n$ is `true` but ψ is `false` under this valuation, we know...

Value of Soundness - how to use it

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- either it does not hold or
- we are not smart enough to find one
- if we find one valuation where $\varphi_1, \ldots \varphi_n$ is `true` but ψ is `false` under this valuation, we know...
- ... that we can't find a proof (contraposition to our Soundness-Theorem)

Model but no Proof?

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- does there exist a proof or not?
- it does

Completeness of Propositional Logic

Theorem

For an $n \in \mathbb{N}_0$ and $\varphi_1, \ldots \varphi_n \models \psi$, $\varphi_1, \ldots \varphi_n \vdash \psi$ also holds.

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1. $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi \ldots)))$ holds

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3. $\varphi_1, \ldots, \varphi_n \vdash \psi$

Tautology and Contradiction

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propositional formula φ tautology iff it evaluates to `true` under all its valuations, i.e. $\models \varphi$

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Example

- $p \vee \neg p$ is a tautology
- $p \wedge \neg p$ is a contradiction

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  - thus: φ_1 `true` and conclusion `false`

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  - inductively: all φ_i `true` and ψ `false`

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  - inductively: all φ_i `true` and ψ `false`
  - this would contradict $\varphi_1, \ldots \varphi_n \models \psi$
  - consequently $\varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi) \ldots))$ tautology

Proof: Step 2

- we want to show that if $\varphi_1 \ldots, \varphi_n \models \psi$ holds then $\vdash \varphi_1 \longrightarrow (\varphi_2 \longrightarrow (\ldots(\varphi_n \longrightarrow \psi \ldots)))$ is valid

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Theorem

If the formula η is a tautology that η is a theorem.

Proof: Step 2 Cont

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  - in the truth table all lines valuate to `true`

p_1	$\ldots$	p_n	η
`false`	...	`false`	`true`
	$\vdots$		
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- η theorem
  - we need a proof for η based on the 2^n lines

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Lemma

- *φ formula with exactly the propositional atoms $p_1, \ldots, p_n$*

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Proof by structural induction on the formulae.

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  1. p_1 in line ℓ `true` then $p_1 \vdash p_1$
  2. p_1 in line ℓ `false` then $\neg p_1 \vdash \neg p_1$
- there exists 1-line-proofs for $p_1 \vdash p_1$ and $\neg p_1 \vdash \neg p_1$

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- application of $\varphi = \neg\psi$: we have to prove
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- application of $\neg$`true` = `false`: we have to prove
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  2. $\neg\psi$ in line ℓ `false` then $\hat{p}_1, \dots, \hat{p}_n \vdash \psi$
- application of $\neg$`true` = `false`: we have to prove
  1. ψ in line ℓ `false` then $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\psi$
  2. ψ in line ℓ `true` then $\hat{p}_1, \dots, \hat{p}_n \vdash \psi$
- there exists proofs by IH
- first part done, for the second apply $(\neg\neg i)$

Preparations for the remaining Cases

- consider $\varphi = \varphi_1 \circ \varphi_2$ for $\circ \in \{\wedge, \vee, \rightarrow\}$
- propositional atoms of φ_1: $q_1, \ldots, q_\ell$
- propositional atoms of φ_2: $r_1, \ldots, r_k$
- $\{q_1, \ldots, q_\ell, r_1, \ldots, r_k\} = \{p_1, \ldots, p_n\}$
- $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \psi_1$ and $\hat{r}_1, \ldots, \hat{r}_k \vdash \psi_2$ then $\hat{p}_1, \ldots, \hat{p}_n \vdash \psi_1 \wedge \psi_2$ (with rule ($\wedge i$))

Proof for Implication

consider $\varphi = \varphi_1 \rightarrow \varphi_2$

Proof for Implication

consider $\varphi = \varphi_1 \rightarrow \varphi_2$

- we have to prove
  1. φ in line ℓ $\texttt{true}$ then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$
  2. φ in line ℓ $\texttt{false}$ then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$

Proof for Implication

consider $\varphi = \varphi_1 \rightarrow \varphi_2$

- we have to prove
  1. φ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$
  2. φ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$
- application of $\varphi = \varphi_1 \rightarrow \varphi_2$: we have to prove
  1. $\varphi_1 \rightarrow \varphi_2$ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi_1 \rightarrow \varphi_2$
  2. $\varphi_1 \rightarrow \varphi_2$ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg(\varphi_1 \rightarrow \varphi_2)$

Proof for Implication: Part 1

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`

Proof for Implication: Part 1

$\varphi_1 \to \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$

Proof for Implication: Part 1

$\varphi_1 \to \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \lor \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$

Proof for Implication: Part 1

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$

Proof for Implication: Part 1

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \wedge \varphi'$

Proof for Implication: Part 1

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \lor \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \land \varphi'$
  - remains to show: $\neg\varphi_1 \land \varphi' \vdash \varphi_1 \rightarrow \varphi_2$ (Exercise)

Proof for Implication: Part 1

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \wedge \varphi'$
  - remains to show: $\neg\varphi_1 \wedge \varphi' \vdash \varphi_1 \rightarrow \varphi_2$ (Exercise)
- case 2: φ_1 in line ℓ `true`

Proof for Implication: Part 1

$\varphi_1 \to \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \wedge \varphi'$
  - remains to show: $\neg\varphi_1 \wedge \varphi' \vdash \varphi_1 \to \varphi_2$ (Exercise)
- case 2: φ_1 in line ℓ `true`
  - then φ_2 in line ℓ `true`

Proof for Implication: Part 1

$\varphi_1 \longrightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \wedge \varphi'$
  - remains to show: $\neg\varphi_1 \wedge \varphi' \vdash \varphi_1 \longrightarrow \varphi_2$ (Exercise)
- case 2: φ_1 in line ℓ `true`
  - then φ_2 in line ℓ `true`
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$

Proof for Implication: Part 1

$\varphi_1 \longrightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \wedge \varphi'$
  - remains to show: $\neg\varphi_1 \wedge \varphi' \vdash \varphi_1 \longrightarrow \varphi_2$ (Exercise)
- case 2: φ_1 in line ℓ `true`
  - then φ_2 in line ℓ `true`
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi_2$

Proof for Implication: Part 1

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \wedge \varphi'$
  - remains to show: $\neg\varphi_1 \wedge \varphi' \vdash \varphi_1 \rightarrow \varphi_2$ (Exercise)
- case 2: φ_1 in line ℓ `true`
  - then φ_2 in line ℓ `true`
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi_2$
  - $\hat{p}_1, \ldots, \hat{p}_k \vdash \varphi_1 \wedge \varphi_2$

Proof for Implication: Part 1

$\varphi_1 \longrightarrow \varphi_2$ in line ℓ `true`

- case 1: φ_1 in line ℓ `false`
  - set $\varphi' = \varphi_2 \vee \neg\varphi_2$
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi'$
  - $\hat{p}_1 \ldots \hat{p}_n \vdash \neg\varphi_1 \wedge \varphi'$
  - remains to show: $\neg\varphi_1 \wedge \varphi' \vdash \varphi_1 \longrightarrow \varphi_2$ (Exercise)
- case 2: φ_1 in line ℓ `true`
  - then φ_2 in line ℓ `true`
  - IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
  - IH: $\hat{r}_1, \ldots, \hat{r}_k \vdash \varphi_2$
  - $\hat{p}_1, \ldots, \hat{p}_k \vdash \varphi_1 \wedge \varphi_2$
  - remains to show: $\varphi_1 \wedge \varphi_2 \vdash \varphi_1 \longrightarrow \varphi_2$ (Exercise)

Proof for Implication: Part 2

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `false`

- φ_1 in line ℓ `true` and φ_2 in line ℓ `false`

Proof for Implication: Part 2

$\varphi_1 \to \varphi_2$ in line ℓ `false`

- φ_1 in line ℓ `true` and φ_2 in line ℓ `false`
- $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$

Proof for Implication: Part 2

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `false`

- φ_1 in line ℓ `true` and φ_2 in line ℓ `false`
- $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg\varphi_2$

Proof for Implication: Part 2

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `false`

- φ_1 in line ℓ `true` and φ_2 in line ℓ `false`
- $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg\varphi_2$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \varphi_1 \wedge \neg\varphi_2$

Proof for Implication: Part 2

$\varphi_1 \rightarrow \varphi_2$ in line ℓ `false`

- φ_1 in line ℓ `true` and φ_2 in line ℓ `false`
- $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg\varphi_2$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \varphi_1 \wedge \neg\varphi_2$
- remains to show: $\varphi_1 \wedge \neg\varphi_2 \vdash \neg(\varphi_1 \rightarrow \varphi_2)$ (Exercise)

Proof for Conjunction

consider $\varphi = \varphi_1 \wedge \varphi_2$

Proof for Conjunction

consider $\varphi = \varphi_1 \land \varphi_2$

- we have to prove
  1. φ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$
  2. φ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$

Proof for Conjunction

consider $\varphi = \varphi_1 \wedge \varphi_2$

- we have to prove
  1. φ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$
  2. φ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$
- application of $\varphi = \varphi_1 \wedge \varphi_2$: we have to prove
  1. $\varphi_1 \wedge \varphi_2$ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi_1 \wedge \varphi_2$
  2. $\varphi_1 \wedge \varphi_2$ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg(\varphi_1 \wedge \varphi_2)$

Proof for Conjunction: Part 1

$\varphi_1 \land \varphi_2$ in line ℓ `true`

- φ_1 and φ_2 in line ℓ `true`

Proof for Conjunction: Part 1

$\varphi_1 \wedge \varphi_2$ in line ℓ `true`

- φ_1 and φ_2 in line ℓ `true`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$

Proof for Conjunction: Part 1

$\varphi_1 \wedge \varphi_2$ in line ℓ `true`

- φ_1 and φ_2 in line ℓ `true`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi_2$

Proof for Conjunction: Part 1

$\varphi_1 \wedge \varphi_2$ in line ℓ `true`

- φ_1 and φ_2 in line ℓ `true`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi_2$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \varphi_1 \wedge \varphi_2$

Proof for Conjunction: Part 2

$\varphi_1 \land \varphi_2$ in line ℓ `false`

- set $\varphi' = \varphi_2 \lor \neg\varphi_2$

Proof for Conjunction: Part 2

$\varphi_1 \wedge \varphi_2$ in line ℓ `false`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume that φ_1 evaluates to `false`

Proof for Conjunction: Part 2

$\varphi_1 \wedge \varphi_2$ in line ℓ `false`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume that φ_1 evaluates to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$

Proof for Conjunction: Part 2

$\varphi_1 \wedge \varphi_2$ in line ℓ `false`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume that φ_1 evaluates to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi'$

Proof for Conjunction: Part 2

$\varphi_1 \wedge \varphi_2$ in line ℓ `false`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume that φ_1 evaluates to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi'$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \neg\varphi_1 \wedge \varphi'$

Proof for Conjunction: Part 2

$\varphi_1 \wedge \varphi_2$ in line ℓ `false`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume that φ_1 evaluates to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi'$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \neg\varphi_1 \wedge \varphi'$
- remains to show $\neg\varphi_1 \wedge (\varphi_2 \vee \neg\varphi_2) \vdash \neg(\varphi_1 \wedge \varphi_2)$

Proof of Disjunction

consider $\varphi = \varphi_1 \vee \varphi_2$

Proof of Disjunction

consider $\varphi = \varphi_1 \vee \varphi_2$

- we have to prove
  1. φ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$
  2. φ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$

Proof of Disjunction

consider $\varphi = \varphi_1 \vee \varphi_2$

- we have to prove
  1. φ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$
  2. φ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$
- application of $\varphi = \varphi_1 \vee \varphi_2$: we have to prove
  1. $\varphi_1 \vee \varphi_2$ in line ℓ `true` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi_1 \vee \varphi_2$
  2. $\varphi_1 \vee \varphi_2$ in line ℓ `false` then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg(\varphi_1 \vee \varphi_2)$

Proof of Disjunction: Part 1

$\varphi_1 \vee \varphi_2$ in line ℓ `true`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$

Proof of Disjunction: Part 1

$\varphi_1 \vee \varphi_2$ in line ℓ `true`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume w.l.o.g. φ_1 to be `true`

Proof of Disjunction: Part 1

$\varphi_1 \vee \varphi_2$ in line ℓ `true`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume w.l.o.g. φ_1 to be `true`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$

Proof of Disjunction: Part 1

$\varphi_1 \vee \varphi_2$ in line ℓ `true`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume w.l.o.g. φ_1 to be `true`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi'$

Proof of Disjunction: Part 1

$\varphi_1 \vee \varphi_2$ in line ℓ $\texttt{true}$

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume w.l.o.g. φ_1 to be $\texttt{true}$
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi'$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \neg\varphi_1 \wedge \varphi'$

Proof of Disjunction: Part 1

$\varphi_1 \vee \varphi_2$ in line ℓ `true`

- set $\varphi' = \varphi_2 \vee \neg\varphi_2$
- assume w.l.o.g. φ_1 to be `true`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \varphi'$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \neg\varphi_1 \wedge \varphi'$
- remains to show $\varphi_1 \wedge (\varphi_2 \vee \neg\varphi_2) \vdash \varphi_1 \vee \varphi_2$

Proof of Disjunction: Part 2

$\varphi_1 \vee \varphi_2$ in line ℓ `false`

- φ_1 and φ_2 evaluate to `false`

Proof of Disjunction: Part 2

$\varphi_1 \vee \varphi_2$ in line ℓ `false`

- φ_1 and φ_2 evaluate to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$

Proof of Disjunction: Part 2

$\varphi_1 \lor \varphi_2$ in line ℓ `false`

- φ_1 and φ_2 evaluate to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg\varphi_2$

Proof of Disjunction: Part 2

$\varphi_1 \vee \varphi_2$ in line ℓ `false`

- φ_1 and φ_2 evaluate to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg\varphi_2$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \neg\varphi_1 \wedge \neg\varphi_2$

Proof of Disjunction: Part 2

$\varphi_1 \vee \varphi_2$ in line ℓ `false`

- φ_1 and φ_2 evaluate to `false`
- IH: $\hat{q}_1, \ldots, \hat{q}_\ell \vdash \neg\varphi_1$
- IH: $\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg\varphi_2$
- $\hat{p}_1, \ldots, \hat{p}_\ell \vdash \neg\varphi_1 \wedge \neg\varphi_2$
- we already proved: $\neg\varphi_1 \wedge \neg\varphi_2 \vdash \neg(\varphi_1 \vee \varphi_2)$ $\square$

Proof of Step 2

Step 2 $\varphi_1, \ldots, \varphi_n \vdash \psi$

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- we have a proof for $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$

Proof of Step 2

Step 2 $\varphi_1, \ldots, \varphi_n \vdash \psi$

- we have a proof for $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi)\ldots))$
- augment this proof by introducing $\varphi_1, \ldots, \varphi_n$ as premises

Proof of Step 2

Step 2 $\varphi_1, \ldots, \varphi_n \vdash \psi$

- we have a proof for $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$
- augment this proof by introducing $\varphi_1, \ldots, \varphi_n$ as premises
- applying n times $(\to e)$

Proof of Step 2

Step 2 $\varphi_1, \ldots, \varphi_n \vdash \psi$

- we have a proof for $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi)\ldots))$
- augment this proof by introducing $\varphi_1, \ldots, \varphi_n$ as premises
- applying n times $(\rightarrow e)$
- we get ψ as conclusion

Soundness and Completeness

Theorem

For propositional formulae $\varphi_1, \ldots, \varphi_n, \psi$ we have

$$\varphi_1, \ldots, \varphi_n \models \psi \quad \textit{iff} \quad \varphi_1, \ldots, \varphi_n \vdash \psi \textit{ valid}$$

Soundness and Completeness

Theorem

For propositional formulae $\varphi_1, \ldots, \varphi_n, \psi$ we have

$$\varphi_1, \ldots, \varphi_n \models \psi \quad \textit{iff} \quad \varphi_1, \ldots, \varphi_n \vdash \psi \textit{ valid}$$

we have now:

- everything provable is `true`
- everything `true` is provable