Logic and Theoretical Foundation of Computer Science

LaTFoCS

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Kiel University
Dependable Systems Group

Semantics of Propositional Logic

Motivation

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  - if I just don't like rain at all, the formula is true
- we have to formalise this!

Truth values and Models

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other notations for `true` and `false`: 1 and 0, `t` and `f`

Truth value of a Formula

Definition

Let $\varphi \in \Phi$ and β an interpretation. Then $\hat{\beta}(\varphi)$ is inductively defined by

$$\hat{\beta}(p) = \beta(p) \text{ for atoms } p \in A$$

$$\hat{\beta}(\neg\psi) = \texttt{true} \text{ iff } \hat{\beta}(\psi) = \texttt{false}$$

$$\hat{\beta}(\psi \wedge \chi) = \begin{cases} \texttt{true} & \text{if } \hat{\beta}(\psi) = \texttt{true} \text{ and } \hat{\beta}(\chi) = \texttt{true} \\ \texttt{false} & \text{otherwise} \end{cases}$$

$$\hat{\beta}(\psi \vee \chi) = \begin{cases} \texttt{false} & \text{if } \hat{\beta}(\psi) = \texttt{false} \text{ and } \hat{\beta}(\chi) = \texttt{false} \\ \texttt{true} & \text{otherwise} \end{cases}$$

$$\hat{\beta}(\psi \rightarrow \chi) = \begin{cases} \texttt{false} & \text{if } \hat{\beta}(\psi) = \texttt{true} \text{ and } \hat{\beta}(\chi) = \texttt{false} \\ \texttt{true} & \text{otherwise} \end{cases}$$

Truth value of a formula (Cont.)

Definition

$$
\begin{aligned}
\hat{\beta}(\psi \leftrightarrow \chi) &= \texttt{true} \text{ iff } \hat{\beta}(\psi) = \hat{\beta}(\chi) \\
\hat{\beta}(\psi \dot{\vee} \chi) &= \texttt{true} \text{ iff } \hat{\beta}(\psi) \neq \hat{\beta}(\chi) \\
\hat{\beta}(\psi \downarrow \chi) &= \texttt{true} \text{ iff } \hat{\beta}(\psi \vee \chi) = \texttt{false} \\
\hat{\beta}(\psi \uparrow \chi) &= \texttt{true} \text{ iff } \hat{\beta}(\psi \wedge \chi) = \texttt{false}
\end{aligned}
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Partial Interpretation

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Let $\varphi \in \Phi$ be a propositional formula with the atoms $A_\varphi \subseteq A$. A function $\overline{\beta}$ is a partional interpretation of φ if its domain is a subset of A_φ.

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For $\varphi = p \vee q$ $(p, q \in A)$ we have $\hat{\beta}(\varphi) = \texttt{true}$ if the interpretation of at least one atom is $\texttt{true}$. Thus with $\overline{\beta}(p) = \texttt{true}$ we have $\overline{\beta}(\varphi) = \texttt{true}$ although $\overline{\beta}$ is not total on A_φ.

Truth Tables - a convenient way to deal with truth

Definition

A truth table for $\varphi \in \Phi$ with the atoms $A_\varphi \subseteq A$ and $|A_\varphi| = n \in \mathbb{N}$ is a $2^n \times (n+1)$-matrix where the first n columns indicate one atom (assuming an (unimportant) order) and the last column indicates φ. Each row is associated with an interpretation of the atoms; the interpretation of the formula is stated in the last entry of a row.

Some truth tables

p	$\neg p$
0	1
1	0

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

p	q	$p \dot{\vee} q$
0	0	0
0	1	1
1	0	1
1	1	0

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Example

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
0	0				
0	1				
1	0				
1	1				

Example

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- Each row stands for a different interpretation.
- Is the procedure for determining the interpretation of a formula efficient?

Let's think about the operators: Disjunction

I buy milk or I buy bread.

- Do I lie if I buy only milk or only bread?
- Do I lie if I buy both?
- Do I lie if I buy nothing?

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I buy either milk or I buy bread.

- Do I lie if I buy only milk or only bread? No!
- Do I lie if I buy both? Yes!
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Let’s think about the operators: Implication

If I buy milk, then I buy bread.

- Do I lie if I buy only milk?
- Do I lie if I buy only bread?
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We defined **material implication**:
the premise/antecedent (LHS of $\longrightarrow$) does not cause necessarily the conclusion (RHS of $\longrightarrow$)

Logical Equivalence

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 f(a,b){
d=1
for i=2 to min{a,b}
  if i|a and i|b
    then d=i
return d
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 g(a,b){
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Revisiting an Example

What can we deduce from this truth table?

p	q	$p \longrightarrow q$	$\neg p$	$\neg p \vee q$	$(p \longrightarrow q) \leftrightarrow (\neg p \vee q)$
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0	1	1	1	1	1
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Logical Equivalence

Definition

Two formula $\varphi_1, \varphi_2 \in \Phi$ are called logically equivalent ($\varphi_1 \equiv \varphi_2$) iff $\hat{\beta}(\varphi_1) = \hat{\beta}(\varphi_2)$ for all interpretations β.

How to prove logical equivalence?

For checking if φ_1, φ_2 are logical equivalent, we have to check $\hat{\beta}(\varphi_1) = \hat{\beta}(\varphi_2)$ for all interpretations β.

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- Do we have to take every single interpretation and check the property? (exponentially many!)
- What if we take an arbitrary one, having only the properties each interpretation has, and checking the equality?

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- by the name representatives represent something
- notice: politicians are a bad example!
- better: among a group of objects or living beings find out all properties they have in common
- then define a new object or living being (representative, prototype) having only these properties
- whatever statement you are claiming about the group, if the new object has it, the group will have it

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Lemma

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- let $\varphi, \psi \in \Phi$, β an arbitrary interpretation
- by definition $\hat{\beta}(\varphi \vee \psi) = \texttt{false}$ iff $\hat{\beta}(\varphi) = \texttt{false}$ and $\hat{\beta}(\psi) = \texttt{false}$

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- by definition this is equivalent to $\hat{\beta}(\psi \vee \varphi) = \texttt{false}$
- since all steps are equivalent we have proven $\hat{\beta}(\varphi \vee \psi) = \hat{\beta}(\psi \vee \varphi)$ $\square$

The confusion with metalanguages

- in the last proof we used the word *equivalent* for stating that two claims imply each other

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- equivalence in the metalanguage is $\equiv$ (talking about Φ)

Why do we use twice the same word?

Theorem

Let $\varphi_1, \varphi_2 \in \Phi$. Then $\varphi_1 \equiv \varphi_2$ iff $\varphi_1 \leftrightarrow \varphi_2$ is true for all interpretations β.

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Definition

A formula φ that is `true` for all interpretations β is called a tautology; if it is `false` for all interpreations β it is called a contradiction.

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- I buy milk if I buy bread, and I do so exactly in the cases when I by bread - never else!
- thus I buy milk if I buy bread, and if I buy milk, I buy bread

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- we do not have to consider the case when the premise is `false`!
- we can assume that the premise is `true` and then we prove that the conclusion is also `true`

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Let $\varphi_1, \varphi_2 \in \Phi$. Then $\varphi_1 \equiv \varphi_2$ iff $\varphi_1 \leftrightarrow \varphi_2$ is true for all interpretations β.

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- we have to prove the statements
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