Logic and Theoretical Foundation of Computer Science

LaTFoCS

Pamela Fleischmann

fpa@informatik.uni-kiel.de

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Kiel University

Dependable Systems Group

Limitations of Finite Automata

Stupidity of Finite Automata

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$$L = \{a^n b^n \mid n \in \mathbb{N}_0\} \subseteq \{a, b\}^*$$

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- if an NFA/DFA could recognize L it would have infinitely many states (see whiteboard)

Formalisation of Non-Regularity

Theorem (Pumping Lemma for Regular Sets)

$L \subseteq \Sigma^*$ *regular set* $\Rightarrow \exists p \in \mathbb{N} \forall w \in L^{\geq p} \exists x, y, z \in \Sigma^*$:

1. $w = xyz$
2. $|y| \geq 1$
3. $|xy| \leq p$
4. $\forall i \in \mathbb{N}_0 : xy^iz \in L$

Explanations: Pumping Lemma

- DO NOT (NEVER EVER) use the Pumping Lemma for proving regularity!
- Contradiction for proving that a language is NOT REGULAR
- idea: word longer than number of states $\Rightarrow$ loop

Using the Pumping Lemma

The language $L = \{a^n b^n | n \in \mathbb{N}_0\}$

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  - $|w| = 2p > p$ since $p > 0$

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- take $x, y, z \in \Sigma^*$ with

$$w = a^p b^p = xyz,\ |y| \geq 1, |xy| \leq p, \forall i \in \mathbb{N}_0 :\ xy^i z \in L$$

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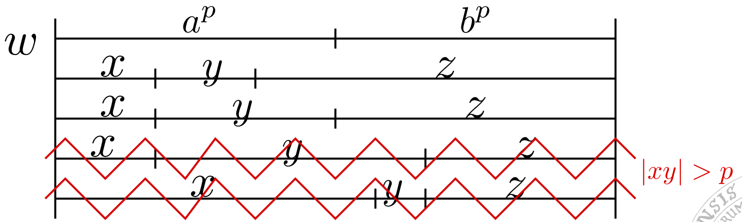

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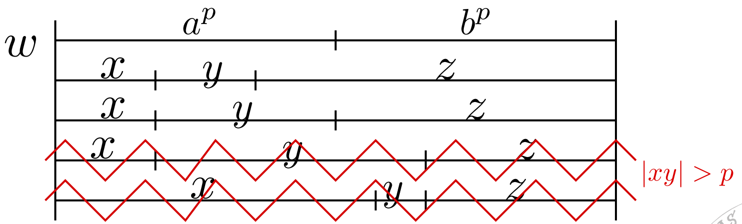

(1)-(3) fulfilled in the two cases $\Rightarrow$ we need contradiction to (4)

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Case 2: $|xy| = p$ analogously

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- we only know that w can be decomposed in xyz - we do not know anything how x, y, z look specifically!
- the difficulty is to find a *nice* w such that the cases for x, y, z are not so horrible

More non-regular languages

Every language in which data have to be stored (e.g. for counting) are not regular, e.g.

- $L_1 = \{\mathtt{a}^n\mathtt{b}^n + k \subseteq \{\mathtt{a},\mathtt{b}\}^* | n \in \mathbb{N}_0\}$ for a fixed $k \in \mathbb{N}_0$

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What do the last three languages tell us? We cannot build automata for calculating the factorial, the prime numbers, or the Fibonacci-numbers.

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Myhill-Nerode offer another option to prove not only unregularity but also regularity.

Myhill-Nerode-Relations

Definition

$A \subseteq \Sigma^*$ regular, $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ with $L(\mathcal{A}) = A$:
$\equiv_{\mathcal{A}}$ Myhill-Nerode-Relation iff

$$\forall x, y \in \Sigma^* : \; x \equiv_{\mathcal{A}} y :\Leftrightarrow \hat{\delta}(q_0, x) = \hat{\delta}(q_0, y).$$

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- $|A/\equiv_{\mathcal{A}}| < \infty$

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Theorem

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- L is regular $\square$

Two-Way Finite Automata

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- Idea: We reach in $a^n b^n$ somehow the middle and read afterwards from left and right

Ideas for 2DFA/2NFA

- reading head that moves over the word
- start marker $\vdash$ and end marker $\dashv$
- automaton can accept or reject (special states)

Example

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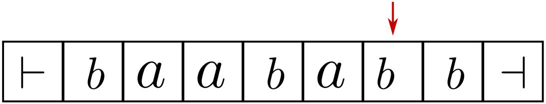

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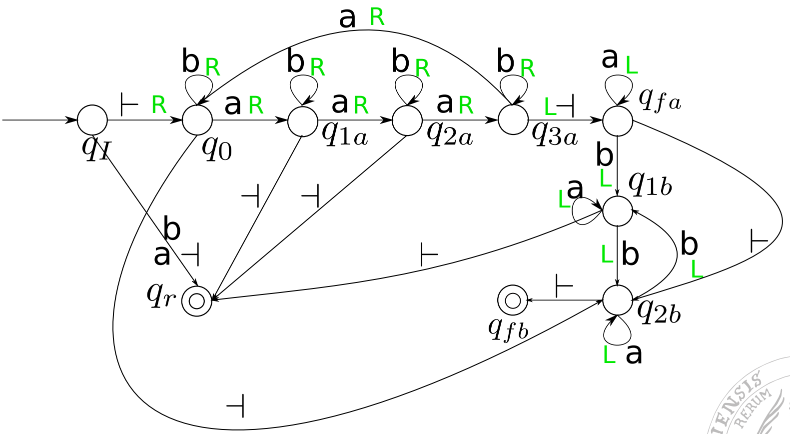

Formal Definition

Definition (2DFA)

$\mathcal{A} = (Q, \Sigma, \vdash, \dashv, \delta, q_0, q_a, q_r)$ 2DFA with

- Q, Σ, q_0 as usual

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Input: $a_0 \ldots a_{n+1} = \vdash x \dashv$

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- next configuration:

$$(p, i) \xrightarrow[x]{i} \begin{cases} (q, i-1) & \text{if } \delta(p, a_i) = (q, L), \\ (q, i+1) & \text{if } \delta(p, a_i) = (q, R). \end{cases}$$

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- $\mathscr{A}$ loops on x: $\forall i \in \mathbb{N} : (q_0, 0) \xrightarrow[x]{*} (q, i) \Rightarrow q \notin \{q_a, q_f\}$

2DFAs and Regular Sets

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- this following state is independent from z resp. from x!

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- infinitely many words $\Rightarrow$ words do have the same mapping

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  - only finitely many mappings $\Rightarrow$ index is finite

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- every DFA is a special 2DFA
- $\mathscr{A}$ 2DFA $\Rightarrow$
  - $T_x = T_y \Leftrightarrow (\forall z \in \Sigma^* : xz \in L(\mathscr{A}) \Leftrightarrow yz \in L(\mathscr{A}))$
    (x, y are indistinguishable)
  - define $x \equiv_{L(\mathscr{A})} y$ by $T_x = T_y$
  - only finitely many mappings $\Rightarrow$ index is finite
  - $L(\mathscr{A})$ is regular