Logical and Theoretical Foundations of Computer Science

LaTFoCS

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Kiel University
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Problems with Automated Checks

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automation is not possible in general!

Quantifier Equivalence Example

Consider $A = \{1, 2\}$, $P^{\mathscr{M}} = \{1\}$, and $Q^{\mathscr{M}} = \{2\}$

- is $\mathscr{M}$ a model of $\forall x P(x) \to \forall x Q(x)$?

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$$
\begin{aligned}
&\mathscr{M} \models \forall x P(x) \to \forall x Q(x)\\
\text{iff } &\mathscr{M} \models \forall x P(x) \text{ implies } \mathscr{M} \models \forall x Q(x)\\
\text{iff } &\text{for all } a \in A (\mathscr{M} \models_{\ell[x \mapsto a]} P(x)) \text{ implies}\\
&\text{for all } b \in A (\mathscr{M} \models_{\ell[x \mapsto b]} Q(x))
\end{aligned}
$$

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an implication is true if either the premise is false or both the premise and the conclusion are true

- premise only for $a = 1$ true but the conclusion isn't true for $b = 1$!

Undecidability of Predicate Logic

Decision Problem for Predicate Logic

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- Can we write a computer programme answering this question correctly?
- we cannot and we will prove this

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- and reduce it to our problem (each instance is mapped to one of our problem) …

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- can our problem be solvable?
- it cannot, using the rules $(\neg i)$ and $(\neg e)$

Post Correspondence Problem (PCP)

Definition

Given $(s_1, t_1), \ldots, (s_k, t_k) \in \{0,1\}^* \times \{0,1\}^*$ (binary strings), decide whether there exists a sequence of indices $i_1, \ldots, i_n$ such that

$$s_{i_1} \ldots s_{i_n} = t_{i_1} \ldots t_{i_n}.$$

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Notice that you are allowed to take a tuple more than once!

PCP and Domino

A visualisation of PCP is the game Domino:

- consider the tuples to be domino tiles $\begin{pmatrix} s_i \\ t_i \end{pmatrix}$
- you have each tile as often as you like
- put them next to each other such that the upper row and the lower row are identical

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we have a solution with

$$\begin{pmatrix} 1 \\ 101 \end{pmatrix}\begin{pmatrix} 011 \\ 11 \end{pmatrix}\begin{pmatrix} 10 \\ 00 \end{pmatrix}\begin{pmatrix} 011 \\ 11 \end{pmatrix} = \begin{pmatrix} 101110011 \\ 101110011 \end{pmatrix}$$

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$$\models \varphi \quad \text{iff} \quad C \text{ has solution}$$

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- idea for constructing φ: encode the bitstring in a formula
- we have constants, predicates, formulae

Proof Cont

- constant e for the empty string

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- function f_0 and f_1 (for the concatenation of 0 resp. 1)
- predicate P (for the existence of a sequence comparing s and t)
- $\varphi := \varphi_1 \wedge \varphi_2 \rightarrow \varphi_3$ with

$$\varphi_1 := \bigwedge_{i \in [k]} P(f_{s_i}(e), f_{t_i}(e))$$

$$\varphi_2 := \forall v \forall w (P(v, w) \rightarrow \bigwedge_{i \in [k]} P(f_{s_i}(v), f_{t_i}(w)))$$

$$\varphi_3 := \exists z P(z, z)$$

Proof Cont

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- $f_b^{\mathcal{M}}(s) = sb$ for $b \in \{0, 1\}$

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- assume $\models \varphi$
- $e^{\mathcal{M}} := \varepsilon$
- $f_b^{\mathcal{M}}(s) = sb$ for $b \in \{0, 1\}$
- $P^{\mathcal{M}} = \{(s, t) \mid \exists (i_1, \ldots, i_m) : \ s = s_1 \ldots s_m \wedge t = t_1 \ldots t_m\}$

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- assume $\models \varphi$
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- $P^{\mathscr{M}} = \{(s,t) | \exists (i_1, \ldots, i_m) : s = s_1 \ldots s_m \wedge t = t_1 \ldots t_m\}$
- $\models \varphi$ implies $\mathscr{M} \models \varphi$

Proof Cont

Claim: $\mathscr{M} \models \varphi$ implies $\mathscr{M} \models \varphi_1$ and $\mathscr{M} \models \varphi_2$

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Proof:

- $(s, t) \in \mathscr{P}^{\mathscr{M}}$ implies sequence $(i_1, \ldots, i_m)$ with $s = s_1 \ldots s_{i_m}$ and $t = t_1 \ldots t_{i_m}$

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- then $ss_i = s_{i_1} \ldots s_{i_m} s_i$ and $tt_i = t_{i_1} \ldots t_{i_m} t_i$

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- choose $(i_1, \ldots, i_m, i)$
- then $ss_i = s_{i_1} \ldots s_{i_m} s_i$ and $tt_i = t_{i_1} \ldots t_{i_m} t_i$
- thus $\mathscr{M} \models \varphi_2$

Proof Cont

- $\mathscr{M} \models \varphi_1 \wedge \varphi_2 \rightarrow \varphi_3$ and $\mathscr{M} \models \varphi_1 \wedge \varphi_2$ imply $\mathscr{M} \models \varphi_3$

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- $\mathscr{M} \models \varphi_1 \wedge \varphi_2 \rightarrow \varphi_3$ and $\mathscr{M} \models \varphi_1 \wedge \varphi_2$ imply $\mathscr{M} \models \varphi_3$
- thus there is a solution for C

Proof Cont

$\Leftarrow$

- $(i_1, \ldots, i_n)$ solution for C

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- to show: all models $\mathcal{M}'$ with a constant $e^{\mathcal{M}'}$, two binary functions $f_0^{\mathcal{M}'}, f_1^{\mathcal{M}'}$ and a binary predicate $P^{\mathcal{M}'}$ satisfy φ ($\mathcal{M}' \models \varphi$)

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- if $\mathscr{M}' \not\models \varphi_1$ or $\mathscr{M}' \not\models \varphi_2$, the claim is true due to the implication
- assume $\mathscr{M}' \models \varphi_1 \wedge \varphi_2$
- we need to verify $\mathscr{M} \models \varphi_3$

Proof Cont

- idea: interpreting finite, binary strings in the domain of values of $\mathscr{A}'$ (like interpreter do for one programming language in another)

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- idea: interpreting finite, binary strings in the domain of values of $\mathscr{A}'$ (like interpreter do for one programming language in another)
- define

$$\begin{aligned} \text{interpret}(\varepsilon) &= e^{\mathscr{M}'} \\ \text{interpret}(s0) &= f_0^{\mathscr{M}'}(\text{interpret}(s)) \\ \text{interpret}(s1) &= f_1^{\mathscr{M}'}(\text{interpret}(s)). \end{aligned}$$

Proof Cont

- by $\mathscr{M}' \models \varphi_1$ we have for all $i \in [k]$

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- by $\mathcal{M}' \models \varphi_1$ we have for all $i \in [k]$

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- recursive application leads to

$$(\text{interpret}(s_{i_1} \ldots s_{i_n}), \text{interpret}(t_{i_1} \ldots t_{i_n})) \in P^{\mathcal{M}'}$$

Proof Cont

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- $\rightsquigarrow$ interpret($s_{i_1} \dots s_{i_n}$), and interpret($t_{i_1} \dots t_{i_n}$) are equal
- $\mathscr{M}' \models \exists z P(z, z)$ $\square$

Undecidability of Satisfiability

- consequence of the definitional clause $\mathscr{M} \models_{\ell} \neg\varphi$:

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Theorem

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Corollary

Satisfiability is not decidable.

Undecidability of Provability

Theorem (Not proven here)

$\models \varphi$ *iff* $\vdash \varphi$

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Corollary

Provability is undecidable.

Expressiveness of Predicate Logic

Propositional Logic vs. Predicate Logic

- predicate logic much more expressive than propositional logic

Propositional Logic vs. Predicate Logic

- predicate logic much more expressive than propositional logic
- neither validity, nor satisfiability, nor provability are decidable

Interpretation as Directed Graphs

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- in reality we often have only binary relations: software models, design standards, execution models for hardware and software
- elements of A can be visualised as nodes, the ones of $R^{\mathscr{M}}$ as edges
- a question in software models for the automotive industry is for instance: is the node for braking always reachable?

Example

Consider

- $A = \{s_0, s_1, s_2, s_3\}$ and
- $R^{\mathscr{M}} = \{(s_0, s_1), (s_1, s_0), (s_1, s_1), (s_1, s_2), (s_2, s_0), (s_3, s_0), (s_3, s_2)\}$

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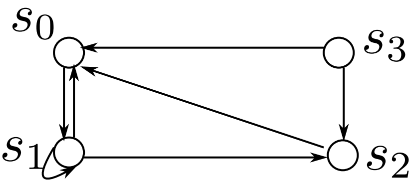

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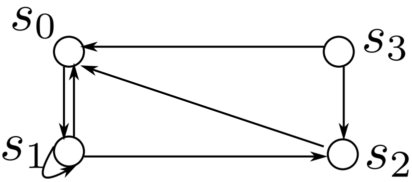

If s_3 is the *braking state*, we are not able to brake if we are in before in the states s_0, s_1, or s_2!

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- idea: constructing a formula φ for G, u, v such that φ is satisfiable iff there exists a path from u to v
- unfortunately we are not (φ would be infinitely large)
- we need two other important results before we prove this

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Theorem

If all finite subsets of a set of predicat logic sentences Γ *are satisfiable, then* Γ *is as well.*

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- completeness $\rightsquigarrow \Gamma \vdash \bot$ valid

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- proofs are finite $\rightsquigarrow$ only finitely many premises Δ from Γ are used
- $\rightsquigarrow \Delta \vdash \perp$
- $\rightsquigarrow \Delta \models \perp$ (soundness)

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- $\rightsquigarrow$ this sequent has a proof in natural deduction
- proofs are finite $\rightsquigarrow$ only finitely many premises Δ from Γ are used
- $\rightsquigarrow \Delta \vdash \bot$
- $\rightsquigarrow \Delta \models \bot$ (soundness)
- not all finite subsets are consistent

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Proof:

- $\varphi_n = \exists x_1 \ldots \exists x_n (\bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j))$ specifies the existence of at least n elements
- $\Gamma := \{\psi\} \cup \{\varphi_n \mid n \in \mathbb{N}\}$, Δ finite subset of Γ

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Let ψ be a predicate logic sentence such that for all $n \in \mathbb{N}$ there exists a model of ψ with at least n element. Then ψ has a model with infinitely many elements.

Proof:

- $\varphi_n = \exists x_1 \ldots \exists x_n (\bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j))$ specifies the existence of at least n elements
- $\Gamma := \{\psi\} \cup \{\varphi_n | \, n \in \mathbb{N}\}$, Δ finite subset of Γ
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Reachability Problem and Predicate Logic

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- Compactness Theorem $\rightsquigarrow$ contradiction

Second-Order Logic - An Outlook

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- description of first-order predicate logic:
    - propositional logic enriched by
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- we don't want to develop a complete new language
- how can we extend the first-oder predicate logic?

Second-Order Logic

Definition

A formula φ is in existential second-order predicate logic

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In universal Second-Order Logic describes the reachability problem is describable.

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- higher order logic: quantifying relations over relation over relation . . .
- keeping soundness and completness is hard (see Russel's Antimony)