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Example solution for Series #9

Exercise 1

10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

a) The Pumping Lemma can be used for proving the regularity of a language. ○ true ⊗ false

b) The Myhill-Nerode is relation is an equivalence relation. ⊗ true ○ false

c) There exists a regular set A such that $\Sigma^*/\equiv_A$ is infinite. ○ true ⊗ false

d) 2DFAs are more powerful than DFAs. ○ true ⊗ false

e) Context-free grammars are grammars. ⊗ true ○ false

f) In a grammar the alphabet and the set of variables have to be disjoint. ⊗ true ○ false

g) While using the Pumping Lemma to prove the unregularity of a language, p can be chosen as it suits best. ○ true ⊗ false

h) Grammars have a final state. ○ true ⊗ false

i) Balanced words can be produced by a context-free grammar. ⊗ true ○ false

j) The | in production rules is an abbreviation for multiple rules. ⊗ true ○ false

Exercise 2

12 Points

Give the following definitions and notations:

a) Pumping Lemma. (4.5P)
b) Myhill-Nerode-Relation. (2P)
c) Grammar. (3P)
d) $L(G)$ for a grammar G. (1P)
e) Reflexive-transitive closure of a relation R. (1.5P)

Solution:

a) $L \subseteq \Sigma^*$ regular set $\Rightarrow$ (0.5P)
$\exists p \in \mathbb{N}$ (0.5P)
$\forall w \in L^{\geq p}$ (0.5+0.5P)
$\exists x, y, z \in \Sigma^*$: (0.5P)
  a) $w = xyz$ (0.5P)
  b) $|y| \geq 1$ (0.5P)
  c) $|xy| \leq p$ (0.5P)
  d) $\forall i \in \mathbb{N}_0 : xy^iz \in L$ (0.5P)

b) $A \subseteq \Sigma^*$ regular, $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ with $L(\mathcal{A}) = A$: (0.5P)
$\equiv_{\mathcal{A}}$ Myhill-Nerode-Relation iff

$$\forall x, y \in \Sigma^* : x \equiv_{\mathcal{A}} y :\Leftrightarrow \hat{\delta}(q_0, x) = \hat{\delta}(q_0, y).$$

(1.5P)

c) $G = (V, \Sigma, S, P)$ grammar iff
   - V finite set of variables (0.5P)
   - Σ alphabet with $V \cap \Sigma = \emptyset$ (0.5+0.5P)
   - $S \in V$ start symbol (0.5P)
   - $P \subseteq (V \cup \Sigma)^* \times (V \cup \Sigma \cup \{\varepsilon\})^*$ production rules (1P)

d) $L(G) = \{w \in \Sigma^* \mid S \vdash^* w\}$ (1P)

e) $R^* = \{(x, y) | \, x = y \lor (x, y) \in R^+\}$ (1P)
   with $R^+ = \{(x, y) | \, \exists n \geq 0 \exists z_1, \ldots, z_n : (x, z_1), (z_1, z_2), \ldots, (z_{n-1}, z_n), (z_n, y) \in R\}$ (1.5P)

Exercise 3 — 24 Points

a) Prove that the language of all palindromes $L = \{w \in \Sigma^* | \, w = w^R\}$ over Σ is not regular. (11.5P)

b) Prove that the Myhill-Nerode relation is right-congruent. (3.5P)

c) Prove that $\{w \in \{\mathtt{a}, \mathtt{b}\}^* | \, |w|_{\mathtt{a}} = |w|_{\mathtt{b}}\}$ is not regular. (2.5P)

d) Prove that $\{w \in \{\mathtt{a}, \mathtt{b}\}^* | \, w[2] = \mathtt{a}\}$ is regular by the Theorem of Myhill-Nerode. (5P)

e) Construct a grammar for the language of all palindromes over Σ. (You don't have to prove the correctness.) (1.5P)

Solution:

a) Prove by Pumping-Lemma.
   Suppose: L is regular. (0.5P)
   Then there exists $p > 0$ (0.5P)
   such that for all $w \in L^{\geq p}$ (0.5P)
   there exists $x, y, z \in \Sigma^*$ (0.5P)
   with
   (1) $w = xyz$ (0.5P)
   (2) $|y| > 0$ (0.5P)
   (3) $|xy| \leq p$ (0.5P)
   (4) $\forall i \in \mathbb{N}_0 : xy^iz \in L$ (0.5P)
   Let $p \in \mathbb{N}$. (0.5P)
   Set $w = \mathtt{a}^p\mathtt{b}^p\mathtt{a}^p$. (0.5P)
   Then $|w| = 3p > p$ (0.5P)
   and $w^R = (\mathtt{a}^p\mathtt{b}^p\mathtt{a}^p)^R = \mathtt{a}^p\mathtt{b}^p\mathtt{a}^p$ and thus $w \in L$. (0.5P)
   Choose $x, y, z \in \Sigma^*$ with (1)-(4). (0.5P)
   By (3) we know that there exists $k_1, k_2, k_3 \in \mathbb{N}_0$ with $x = \mathtt{a}^{k_1}$ $y = \mathtt{a}^{k_2}$, $z = \mathtt{a}^{k_3}\mathtt{b}^p\mathtt{a}^p$, and $k_1 + k_2 + k_3 = p$. (0.5+0.5+0.5+0.5P)
   By (2) we have $k_2 > 0$. (0.5P)
   This implies $k_1 + k_3 < p$. (0.5P)
   Hence we get
   $$xy^0z = \mathtt{a}^{k_1}\mathtt{a}^{k_3}\mathtt{b}^p\mathtt{a}^p = \mathtt{a}^{k_1+k_3}\mathtt{b}^p\mathtt{a}^p.$$
   (0.5P)
   But $(xy^0z)^R = \mathtt{a}^p\mathtt{b}^p\mathtt{a}^{k_1+k_3} \neq xy^0z$. (0.5P)
   Thus we have a contradiction to (4). (0.5P)
   This implies that L is not regular. $\square$ (0.5P)

b) Let be $x, y \in \Sigma^*$ and $a \in \Sigma$ with $x \equiv_{\mathcal{A}} y$. (0.5P)
   Choose $q \in Q$ with $q = \hat{\delta}(q_0, x)$. (0.5P)
   By $x \equiv_{\mathcal{A}} y$ we have $q = \hat{\delta}(q_0, y)$. (0.5P)
   Since $\mathcal{A}$ is a DFA, $\delta(q, \mathtt{a})$ is unique (0.5P)
   and we get
   $$\hat{\delta}(q_0, x\mathtt{a}) = \hat{\delta}(q, \mathtt{a}) = \hat{\delta}(q_0, y\mathtt{a}).$$
   (1P)
   This is equivalent to $x\mathtt{a} \equiv_{\mathcal{A}} y\mathtt{a}$. $\square$ (0.5P)

c) Consider the words $\mathtt{a}^k$ for $k \in \mathbb{N}_0$. (0.5P)
For different k_1, k_2, $[\mathtt{a}^{k_1}]$ and $[\mathtt{a}^{k_2}]$ are different classes since either appending $\mathtt{b}^{k_1}$ or $\mathtt{b}^{k_2}$ ends in the language. (0.5P)
Since there exists infinitely many k, (0.5P)
we have infinitely many classes, (0.5P)
and the language is not regular. $\square$ (0.5P)

d) Consider $u, w \in \{\mathtt{a}, \mathtt{b}\}^*$ with $u[2] = \mathtt{a}$ and $w[2] = \mathtt{b}$. (0.5+0.5P)
Each $v \in \{\mathtt{a}, \mathtt{b}\}$ with $v[2] = \mathtt{a}$ is in $[u]$ (0.5P)
since appending $z \in \Sigma^*$ to u and v resp. does not change the second letter (0.5P)
and thus $vz, uz \in L$. (0.5P)
Each $v \in \{\mathtt{a}, \mathtt{b}\}$ with $v[2] = \mathtt{b}$ is in $[w]$ (0.5P)
since appending $z \in \Sigma^*$ to u and v resp. does not change the second letter (0.5P)
and thus $vz, wz \notin L$. (0.5P)
Consequently $|\Sigma^* / \equiv_L| = 2$ (0.5P)
and L is regular. (0.5P)

e) $S \to xSx | \varepsilon$ for all $x \in \Sigma$. (1.5P)