1 - Horses and Riders

We use addition and multiplication on Peano numbers to solve the puzzle.

```
% Adding two Peano numbers.
add(o,    Y, Y).
add(s(X), Y, s(Z)) :- add(X, Y, Z).

% Multiplying two  Peano numbers.
mult(o,    _, o).
mult(s(X), Y, Z) :- mult(X, Y, A), add(A, Y, Z).
```

We define a quaternary relation `rider`, which relates the number of heads and legs to the number of horses and riders.

```
rider(Heads, Feet, Humans, Horses) :-
```

The number of heads should then result from the sum of horses and riders.

```
  add(Humans, Horses, Heads),
```

The number of legs is the sum of the number of horses multiplied by 4 and the number of riders multiplied by 2.

```
  add(HuF, HoF, Feet),
  mult(s(s(o)), Humans, HuF),
  mult(s(s(s(s(o)))), Horses, HoF).
```

With this predicate we can solve the puzzle in Prolog.

```
?- rider(s(s(s(s(s(s(s(s(o)))))))),
         s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(o)))))))))))))))))))),
         Riders,
         Horses).
Riders = s(s(s(s(s(s(o)))))),
Horses = s(s(o));
false.
```

There are 6 riders and 2 horses on the farm.

2 - Substitution

$$\theta \circ \sigma = \{X_1 \mapsto \theta(t_1), \ldots, X_k \mapsto \theta(t_k)\} \cup \{Y_j \mapsto u_j \in \theta \mid Y_j \notin (\sigma)\}$$

Whereby we define so that only those variables are included that are changed by the substitution.

$$(\sigma) := \{X \mid \sigma(X) \neq X\}$$

$$(\theta \circ \sigma)(X) = \begin{cases} \theta(t_i) & \text{if } X \mapsto t_i \in \sigma, \\ u_i & \text{if } X \notin (\sigma) \text{ and } X \mapsto u_i \in \theta, \\ X & \text{otherwise.} \end{cases}$$

3 - Unification

$$
\begin{aligned}
t_0 &= f(a, g(X, f(Z)), h(Z)) \\
t_1 &= f(Y, g(Y, f(b)), h(Y))
\end{aligned}
$$

$$
\begin{aligned}
k &= 0 \\
\sigma_0 &= \varnothing \\
\sigma_0(t_0) &= f(a, g(X, f(Z)), h(Z)) \\
\sigma_0(t_1) &= f(Y, g(Y, f(b)), h(Y)) \\
\mathrm{ds}(\sigma_0(t_0), \sigma_0(t_1)) &= \{a, Y\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1 &= \{Y \mapsto a\} \\
k &= 1 \\
\sigma_1(t_0) &= f(a, g(X, f(Z)), h(Z)) \\
\sigma_1(t_1) &= f(a, g(a, f(b)), h(a)) \\
\mathrm{ds}(\sigma_1(t_0), \sigma_1(t_1)) &= \{X, a\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2 &= \{X \mapsto a, Y \mapsto a\} \\
k &= 2 \\
\sigma_2(t_0) &= f(a, g(a, f(Z)), h(Z)) \\
\sigma_2(t_1) &= f(a, g(a, f(b)), h(a)) \\
\mathrm{ds}(\sigma_2(t_0), \sigma_2(t_1)) &= \{Z, b\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_3 &= \{X \mapsto a, Y \mapsto a, Z \mapsto b\} \\
k &= 3 \\
\sigma_3(t_0) &= f(a, g(a, f(b)), h(b)) \\
\sigma_3(t_1) &= f(a, g(a, f(b)), h(a)) \\
\mathrm{ds}(\sigma_3(t_0), \sigma_3(t_1)) &= \{b, a\}
\end{aligned}
$$

$$\Longrightarrow \text{„fail“ (Clash)}$$

$$
\begin{aligned}
t_0 &= f(f(a, X), b) \\
t_1 &= f(Y, b)
\end{aligned}
$$

$$
\begin{aligned}
k &= 0 \\
\sigma_0 &= \varnothing \\
\sigma_0(t_0) &= f(f(a, X), b) \\
\sigma_0(t_1) &= f(Y, b) \\
\mathrm{ds}(\sigma_0(t_0), \sigma_0(t_1)) &= \{f(a, X), Y\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1 &= \{Y \mapsto f(a, X)\} \\
k &= 1 \\
\sigma_1(t_0) &= f(f(a, X), b) \\
\sigma_1(t_1) &= f(f(a, X), b)
\end{aligned}
$$

$$\Longrightarrow \sigma_1 \text{ is mgu}$$

$$
\begin{aligned}
t_0 &= f(g(X,Y),Z,h(Z))\\
t_1 &= f(Z,g(Y,X),h(g(a,b)))
\end{aligned}
$$

$$
\begin{aligned}
k &= 0\\
\sigma_0 &= \varnothing\\
\sigma_0(t_0) &= f(g(X,Y),Z,h(Z))\\
\sigma_0(t_1) &= f(Z,g(Y,X),h(g(a,b)))\\
\mathrm{ds}(\sigma_0(t_0),\sigma_0(t_1)) &= \{g(X,Y),Z\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1 &= \{Z \mapsto g(X,Y)\}\\
k &= 1\\
\sigma_1(t_0) &= f(g(X,Y),g(X,Y),h(g(X,Y)))\\
\sigma_1(t_1) &= f(g(X,Y),g(Y,X),h(g(a,b)))\\
\mathrm{ds}(\sigma_1(t_0),\sigma_1(t_1)) &= \{X,Y\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2 &= \{X \mapsto Y, Z \mapsto g(Y,Y)\}\\
k &= 2\\
\sigma_2(t_0) &= f(g(Y,Y),g(Y,Y),h(g(Y,Y)))\\
\sigma_2(t_1) &= f(g(Y,Y),g(Y,Y),h(g(a,b)))\\
\mathrm{ds}(\sigma_2(t_0),\sigma_2(t_1)) &= \{Y,a\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_3 &= \{X \mapsto a, Y \mapsto a, Z \mapsto g(a,a)\}\\
k &= 3\\
\sigma_3(t_0) &= f(g(a,a),g(a,a),h(g(a,a)))\\
\sigma_3(t_1) &= f(g(a,a),g(a,a),h(g(a,b)))\\
\mathrm{ds}(\sigma_3(t_0),\sigma_3(t_1)) &= \{a,b\}
\end{aligned}
$$

$$\Longrightarrow \text{„fail“ (Clash)}$$

$$
\begin{aligned}
t_0 &= f(X,g(X))\\
t_1 &= f(g(Y),Y)
\end{aligned}
$$

$$
\begin{aligned}
k &= 0\\
\sigma_0 &= \varnothing\\
\sigma_0(t_0) &= f(X,g(X))\\
\sigma_0(t_1) &= f(g(Y),Y)\\
\mathrm{ds}(\sigma_0(t_0),\sigma_0(t_1)) &= \{X,g(Y)\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1 &= \{X \mapsto g(Y)\}\\
k &= 1\\
\sigma_1(t_0) &= f(g(Y),g(g(Y)))\\
\sigma_1(t_1) &= f(g(Y),Y)\\
\mathrm{ds}(\sigma_1(t_0),\sigma_1(t_1)) &= \{g(g(Y)),Y\}
\end{aligned}
$$

$$\Longrightarrow \text{„fail“ (Occurs Check)}$$

The last example shows that the calculated unifier can become exponentially large.

$$
\begin{aligned}
t_0 &= f(B, C, D) \\
t_1 &= f(g(A, A), g(B, B), g(C, C))
\end{aligned}
$$

$$
\begin{aligned}
k &= 0 \\
\sigma_0 &= \varnothing \\
\sigma_0(t_0) &= f(B, C, D) \\
\sigma_0(t_1) &= f(g(A, A), g(B, B), g(C, C)) \\
\mathrm{ds}(\sigma_0(t_0), \sigma_0(t_1)) &= \{B, g(A, A)\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1 &= \{B \mapsto g(A, A)\} \\
k &= 1 \\
\sigma_1(t_0) &= f(g(A, A), C, D) \\
\sigma_1(t_1) &= f(g(A, A), g(g(A, A), g(A, A)), g(C, C)) \\
\mathrm{ds}(\sigma_1(t_0), \sigma_1(t_1)) &= \{C, g(g(A, A), g(A, A))\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2 &= \{B \mapsto g(A, A), C \mapsto g(g(A, A), g(A, A))\} \\
k &= 2 \\
\sigma_2(t_0) &= f(g(A, A), g(g(A, A), g(A, A)), D) \\
\sigma_2(t_1) &= f(g(A, A), g(g(A, A), g(A, A)), \\
&\qquad g(g(g(A, A), g(A, A)), g(g(A, A), g(A, A)))) \\
\mathrm{ds}(\sigma_2(t_0), \sigma_2(t_1)) &= \{D, g(g(g(A, A), g(A, A)), g(g(A, A), g(A, A)))\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_3 &= \{B \mapsto g(A, A), C \mapsto g(g(A, A), g(A, A)), \\
&\quad D \mapsto g(g(g(A, A), g(A, A)), g(g(A, A), g(A, A)))\} \\
k &= 3 \\
\sigma_3(t_0) &= f(g(A, A), g(g(A, A), g(A, A)), \\
&\qquad g(g(g(A, A), g(A, A)), g(g(A, A), g(A, A)))) \\
\sigma_3(t_1) &= f(g(A, A), g(g(A, A), g(A, A)), \\
&\qquad g(g(g(A, A), g(A, A)), g(g(A, A), g(A, A))))
\end{aligned}
$$

$$\Longrightarrow \sigma_3 \text{ is mgu}$$

4 - Occurs Check

For the first equation the following type is inferred.

```
f :: [a] -> [a]
```

The second equation yields the following type.

```
f :: b -> [b]
```

When unifying `[a] -> [a]` and `b -> [b]`, first `[a]` and `b` are unified with $\sigma = \{b \mapsto [a]\}$. After applying σ, `[a]` and `[[a]]` must then be unified and so must `a` and `[a]`. Since `a` now appears in the more complex type term `[a]` it comes to the above mentioned error message.

5 - House of Nicholas

```
edges([(1, 2), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (4, 5)]).
```

```
flip([],           []).
flip([(X, Y)|Xs], [(Y, X)|Ys]) :- flip(Xs, Ys).
flip([(X, Y)|Xs], [(X, Y)|Ys]) :- flip(Xs, Ys).

nicholas([]).
nicholas([_]).
nicholas([(_, X), (X, Y)|Xs]) :- nicholas([(X, Y)|Xs]).

houseOfNicholas(N) :- edges(Es),
                      permutation(Es, EsP),
                      flip(EsP, N),
                      nicholas(N).
```