Logical and Theoretical Foundations of Computer Science

LaTFoCS

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Kiel University
Dependable Systems Group

Predicate Logic as a Formal Language - Syntax

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Notice: constants are 0-arity (nullary) functions (no argument, no output)

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3. for terms $t_1, \ldots, t_n$, $n \in \mathbb{N}$, a function symbol $f \in \mathcal{F}$ with arity n

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4. nothing else is a term

Examples for Terms

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- y

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- owner(car with plate KI KI 1234)

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- owner(car with plate KI KI 1234)
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- $\mathrm{sum}(1, x, 2, y)$ if sum is 4-ary

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5. nothing else is a formula

Binding Priorities

- $\neg$, $\forall$, $\exists$ bind most tightly
- $\wedge$, $\vee$ are next in the hierarchy
- $\rightarrow$ has loosest connectivity and is right-associative

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Is my mother now my aunt?

Parse Tree of Predicate Logic Formulae

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The parse tree of $\varphi \in \Phi_{\text{FO}}$ is build according to the rules for propositional logic formulae with three additional rules

- $\varphi = (\forall x \psi(x))$: $\forall x$ is a node with child $\psi(x)$
- $\varphi = (\exists x \psi(x))$: $\exists x$ is a node with child $\psi(x)$
- $\varphi = P(t_1, \ldots, t_n)$: P is a node with children $t_1, \ldots, t_n$

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ψ is called the scope of x

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- If $\{x_1, \ldots, x_n\} = \mathrm{Free}(\varphi)$ then $\exists x_1 \ldots \exists x_n(\varphi)$ is the existential closure of φ.

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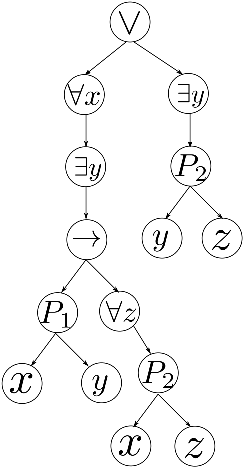
$$(\forall x(\exists y P_1(x,y) \rightarrow \forall z P_2(x,z)) \vee (\exists y P_2(y,z)))$$



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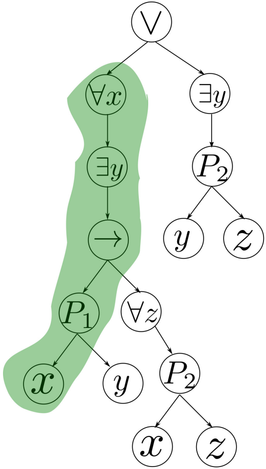

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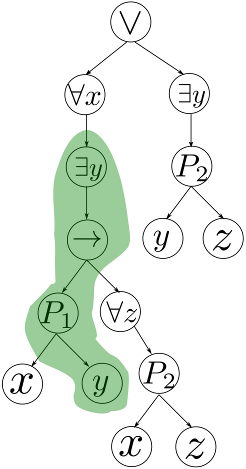

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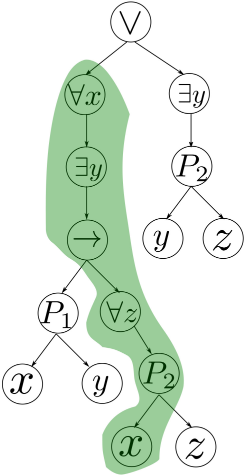

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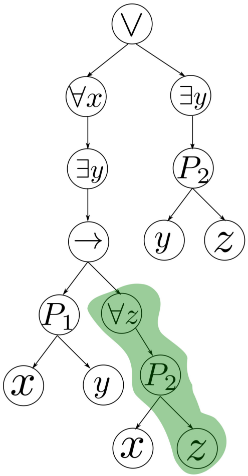

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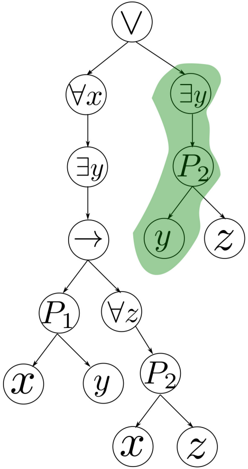

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