Limits of Context-Free Grammars/Languages

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5. is everything producible by context-free grammars?
6. what is with $\{a^n b^n c^n \mid n \in \mathbb{N}\}$?

The Pumping Lemma for CFLs

Reminder: How to prove non-regularity?

- no DFA/NFA/regexp exists for the language
- contradict the supposition that it is regular using the Pumping Lemma
- prove that the index of the Myhill-Nerode-relation is infinite

Pumping Lemma for Context-Free Languages

Lemma

L *context-free language* $\Rightarrow \exists p \in \mathbb{N} \forall z \in L^{\geq p} \exists u, v, w, x, y \in \Sigma^* :$

1. $z = uvwxy$
2. $vx \neq \varepsilon$
3. $|vwx| \leq p$
4. $\forall i \in \mathbb{N}_0 : uv^i wx^i y \in L$

Cont.

Definition

Parse Tree for a word z producible by a grammar in CNF:

- nodes $N = V \cup \Sigma$
- start with S
- if the rule $L \longrightarrow \ell$ is applied: ℓ is a child of L
- if the rule $L \longrightarrow R_1 R_2$ is applied: R_1 and R_2 are children

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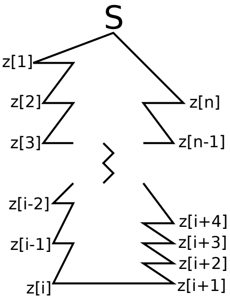
- any parse tree for z has depth at least $n + 1$
- $\Rightarrow$ the longest path has length at least $n + 1$
- only n variables $\Rightarrow$ one occurs at least twice

Cont.

read longest path from bottom to top and take the first two variables X occurring twice



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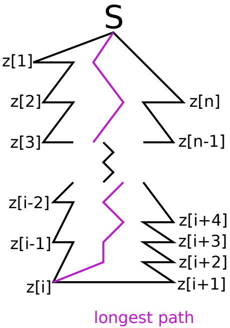

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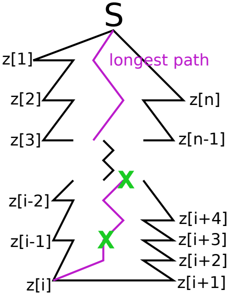

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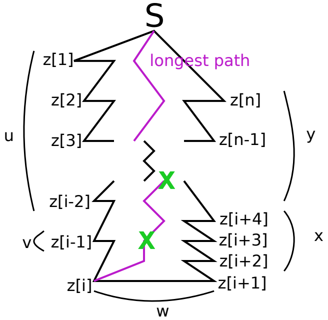

Choose u, v, w, x, y at the bottom of the tree with

- w is generated by lower occurrence of X
- v is generated by left child of upper occurrence of X
- x is generated by right child of upper occurrence of X
- u is part before v
- y is part after x

Cont.

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- for $i = 0$: exchange the upper one by the lower one
- v, x are gone!

Nasty Language II

Context-Freedom is not the jack of all trades device.

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- PL $\Rightarrow$ exists $p \in \mathbb{N}$ (take it)
- set $z = a^p b^p a^p$
- $\Rightarrow w \in L, |z| \geq p$

Cont.

$$\underbrace{a^n}\;\underbrace{b^n}\;\underbrace{a^n}$$

Cont.

$$\overset{a^n}{\vdash\!\!\text{———}\!\!\dashv}\ \overset{b^n}{\vdash\!\!\text{———}\!\!\dashv}\ \overset{a^n}{\vdash\!\!\text{———}\!\!\dashv}$$

we have to check all decompositions into $uvwxy$

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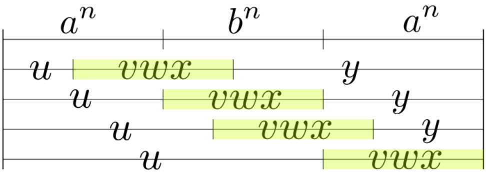

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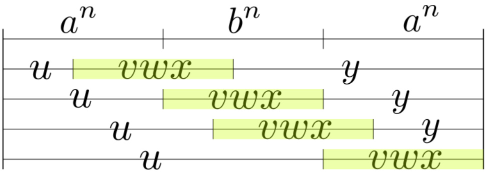

- case 1: no change in third part

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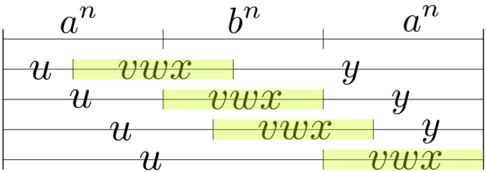

- case 1: no change in third part
- case 2: no change in third part

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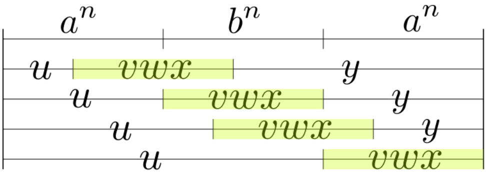

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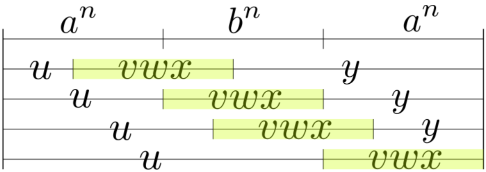

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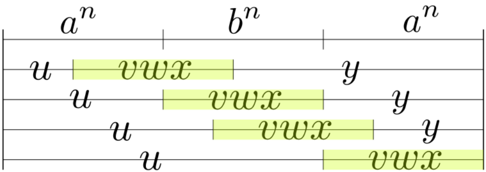

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contradiction to (4)! $\square$