Christian-Albrechts-Universität zu Kiel
Technische Fakultät

Dependable Systems Group
D. Nowotka, P. Fleischmann

Example solution for Series #7

Exercise 1 — 10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

a) The empty clause is a clause without literals. ☒ true ◯ false

b) A clause is called trivial if it is empty. ◯ true ☒ false

c) $\square$ is falsifiable. ☒ true ◯ false

d) The resolvant of $(p \vee \neg q \vee s)$ and $(\neg p \vee q \vee r)$ is $(s \vee r)$ ◯ true ☒ false

e) The resolvant is satisfiable iff the parent clauses are both satisfiable. ☒ true ◯ false

f) The empty string is not a word. ◯ true ☒ false

g) For a word $w \in \Sigma^*$ we have $\sum_{a \in \Sigma} |w|_a = |w|$. ☒ true ◯ false

h) For words $u, v \in \Sigma^*$ we have $|uv| > |u| + |v|$. ◯ true ☒ false

i) For a word $w \in \Sigma^n$ we have $w = w[1]w[2] \ldots w[n-1]w[n]$. ☒ true ◯ false

j) $|\varepsilon\varepsilon| > |\varepsilon|$. ◯ true ☒ false

Exercise 2 — 8 Points

Give the following definitions and notations:

a) Unit clause. (1P)
b) Clausal form of a formula $\varphi \in \Phi$. (2P)
c) C_1, C_2 clashing clauses. (2P)
d) Word. (1P)
e) Repetition. (2P)

Solution:

a) A clause with exactly one literal. (1P)
b) If φ has the clauses $C_1, \ldots, C_m$ then $C_\varphi = \{C_1, \ldots, C_m\}$ is the clausal form of φ. (2P)
c) There exists a literal ℓ with $\ell \in C_1$ and $\neg\ell \in C_2$. (2P)
d) Finite sequence of elements of an alphabet Σ. (1P)
e) Repetitions are defined inductively: $u^0 = \varepsilon$, $u^{i+1} = uu^i$. (2P)

Exercise 3 — 23 Points

a) Consider the formula $\varphi = (p \vee q) \wedge (p \to (\neg(r \vee s) \wedge q))$.
   1) Construct the DAG for φ. (8P)
   2) Determine with the DAG-algorithm whether the formula is satisfiable or not and if it is satisfiable, determine a satisfying interpretation. (5P)

b) 1) Prove for a clausal form S and a trivial clause $C \in S$ that $S \backslash C \equiv S$ holds. (6.5P)

2) Decide with the resolution procedure whether

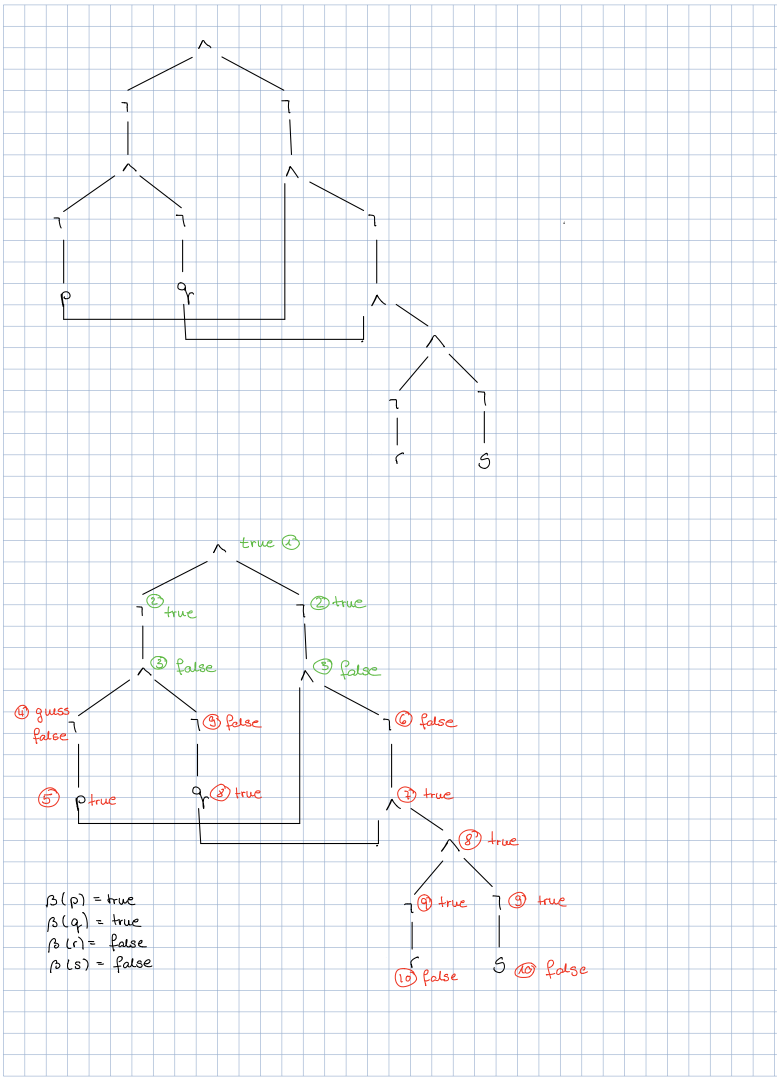
$$\varphi = (p \vee q \vee \neg r) \wedge (\neg q \vee r \vee s) \wedge (p \vee \neg s) \wedge (q \vee r \vee s \vee \neg q) \wedge (\neg p \vee s).$$

is satisfiable or not. (3.5P)

Solution:

a) Firstly we have to apply T on φ.

$$\begin{aligned} T(\varphi) &= T((p \vee q) \wedge (p \rightarrow (\neg(r \vee s) \wedge q))) \\ &= T(p \vee q) \wedge T(p \rightarrow (\neg(r \vee s) \wedge q)) \\ &= \neg(\neg T(p) \wedge \neg T(q)) \wedge \neg(T(p) \wedge \neg T(\neg(r \vee s) \wedge q)) \\ &= \neg(\neg p \wedge \neg q) \wedge \neg(p \wedge \neg(\neg T(r \vee s) \wedge T(q))) \\ &= \neg(\neg p \wedge \neg q) \wedge \neg(p \wedge \neg(\neg\neg(\neg T(r) \wedge \neg T(s)) \wedge q)) \\ &= \neg(\neg p \wedge \neg q) \wedge \neg(p \wedge \neg((\neg r \wedge \neg s) \wedge q)) \end{aligned}$$



b) 1) Let S be a clausal form and $C \in S$ trivial. (0.5P)
Then there exists clauses $C_1, \dots, C_m$ with $S = \{C_1, \dots, C_m, C\}$. (0.5P)
Moreover there exists a $p \in A$ with $p, \neg p \in C$. (0.5P)
We have to prove that for all interpretations β we have $\hat{\beta}(S \backslash C) = \hat{\beta}(S)$. (0.5P)
Let β be an interpretation. (0.5P)
case 1: $\hat{\beta}(S) = \text{true}$ (0.5P)
Since S is a conjunction of $C_1, \dots, C_m$, we have $\hat{\beta}(C_i) = \text{true}$ and $\hat{\beta}(C) = \text{true}$. (0.5P)

Thus, especially $\hat{\beta}(C_i) = \text{true}$ and consequently $\hat{\beta}(S \backslash C) = \text{true}$. (0.5P)
case 2: $\hat{\beta}(S) = \text{false}$ (0.5P)
Thus $\hat{\beta}(C) = \text{false}$ or there exists $i \in [m]$ with $\hat{\beta}(C_i) = \text{false}$. (0.5P)
Since C is trivial, i.e. contains p and $\neg p$, C is a tautology and we have $\hat{\beta}(C) = \text{true}$. (0.5P)
This implies that $\hat{\beta}(C_i) = \text{false}$ for some $i \in [m]$. (0.5P)
Consequently $\hat{\beta}(S \backslash C) = \hat{\beta}(\{C_1, \ldots, C_m\}) = \text{false}$. $\square$ (0.5P)

2) Since the second last clause is trivial, we can delete it from the clausal form. (0.5P)
Since the first two clause clash on q, we determine the resolvent $p \vee \neg r \vee r \vee s$. (0.5P)
Since this clause is trivial, we delete it. (0.5P)
Thus only the third clause and the last clause are remaining and and clash, we apply resolution on s. (0.5P)
This leads to $p \vee \neg p$. (0.5P)
Since this is a trivial clause, we delete it. (0.5P)
Since S is now empty, the formula is satisfiable. (0.5P)