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Example solution for Series #2

Exercise 1 — 10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

a) By the induction axiom $\mathbb{N} = X$ is proven. ☒ true ○ false

b) $|\mathbb{N}_{>5}| > |\mathbb{N}|$. ○ true ☒ false

c) Natural induction works on every total ordered set. ○ true ☒ false

d) *The main building of Kiel university has three floors.* is an atomic proposition. ☒ true ○ false

e) Propositional formulae do not have an intrinsic semantic (meaning). ☒ true ○ false

f) $p \vee q$ is a subformula of $((q \wedge r) \vee (p \wedge q)) \rightarrow (r \leftrightarrow q)$ ○ true ☒ false

g) The truth table for a formula on $n \in \mathbb{N}$ atoms has 2^n lines. ☒ true ○ false

h) In a parse tree, atoms do not have children. ☒ true ○ false

i) $\hat{\beta}(\psi \rightarrow \chi)$ is true iff $\hat{\beta}(\psi) = \text{false}$ or $\hat{\beta}(\psi) = \hat{\beta}(\chi) = \text{true}$. ☒ true ○ false

j) A subformula of a propositional formula does not have to be a propositional formula itself. ○ true ☒ false

Exercise 2 — 14 Points

Give the following definitions and notations:

a) Induction Hypothesis. (1P)
b) Structural Induction Principle. (2P)
c) (Well-formed) propositional formulae. (5.5P)
d) Valuation. (1P)
e) Truth value of a formula. (4.5P)

Solution:

a) Assume that the property P holds for an arbitrary but fixed $n \in \mathbb{N}$.

b)
- If a property P holds for all $a \in A$ (base case) (0.5P)
- and if the fact that P holds for fixed but arbitrary $m_1, \ldots, m_\ell \in \mathcal{M}$ (0.5P)
- implies that P holds for $s(m_1, \ldots, m_k)$ for all $s \in S$ (induction step) (0.5P)
- then P holds for all elements of $\mathcal{M}$. (0.5P)

c) The set Φ of (well-formed) propositional formulae for a given set of $A = \{p_i | i \in \mathbb{N}\}$ is inductively defined by (1P)

a) each propositional atom is a propositional formula, (0.5P)
b) if $\phi \in \Phi$ then also $(\neg\phi)$ (negation) (0.5P)
c) if $\phi, \psi \in \Phi$ then also $(\phi \wedge \psi)$ (conjunction) (0.5P)
d) if $\phi, \psi \in \Phi$ then also $(\phi \vee \psi)$ (disjunction) (0.5P)
e) if $\phi, \psi \in \Phi$ then also $(\phi \rightarrow \psi)$ (implication) (0.5P)
f) if $\phi, \psi \in \Phi$ then also $(\phi \leftrightarrow \psi)$ (equivalence) (0.5P)

g) if $\phi, \psi \in \Phi$ then also $(\phi \dot{\vee} \psi)$ (exclusive or) (0.5P)
h) if $\phi, \psi \in \Phi$ then also $(\phi \downarrow \psi)$ (nor) (0.5P)
i) if $\phi, \psi \in \Phi$ then also $(\phi \uparrow \psi)$ (nand) (0.5P)

d) A valuation is a total mapping $\beta : A \to \mathcal{T}$. (0.5+0.5P)

e) Let $\varphi \in \Phi$ and β an interpretation. Then $\hat{\beta}(\varphi)$ is inductively defined by

- $\hat{\beta}(p) = \beta(p)$ (0.5P)
- $\hat{\beta}(\neg\psi) = \text{true}$ iff $\hat{\beta}(\psi) = \text{false}$ (0.5P)
- $\hat{\beta}(\psi \wedge \chi) = \begin{cases} \text{true} & \text{if } \hat{\beta}(\psi) = \text{true and } \hat{\beta}(\chi) = \text{true} \\ \text{false} & \text{otherwise} \end{cases}$ (0.5P)
- $\hat{\beta}(\psi \vee \chi) = \begin{cases} \text{false} & \text{if } \hat{\beta}(\psi) = \text{false and } \hat{\beta}(\chi) = \text{false} \\ \text{true} & \text{otherwise} \end{cases}$ (0.5P)
- $\hat{\beta}(\psi \to \chi) = \begin{cases} \text{false} & \text{if } \hat{\beta}(\psi) = \text{true and } \hat{\beta}(\chi) = \text{false} \\ \text{true} & \text{otherwise} \end{cases}$ (0.5P)
- $\hat{\beta}(\psi \leftrightarrow \chi) = \text{true}$ iff $\hat{\beta}(\psi) = \hat{\beta}(\chi)$ (0.5P)
- $\hat{\beta}(\psi \dot{\vee} \chi) = \text{true}$ iff $\hat{\beta}(\psi) \neq \hat{\beta}(\chi)$ (0.5P)
- $\hat{\beta}(\psi \downarrow \chi) = \text{true}$ iff $\hat{\beta}(\psi \vee \chi) = \text{false}$ (0.5P)
- $\hat{\beta}(\psi \uparrow \chi) = \text{true}$ iff $\hat{\beta}(\psi \wedge \chi) = \text{false}$ (0.5P)

Exercise 3 — 20 Points

a) Prove that $n^2 - n \equiv_2 0$ for all $n \in \mathbb{N}$. (8.5P)

b) Consider $A = \{p_1, p_2, p_3\}$. Determine all subformulae of

$$\varphi = ((p_1 \vee p_2) \to ((\neg p_1) \wedge p_3))$$

(3.5P)

c) 1) Prove that $\mathrm{Sub}(\psi) \subseteq \mathrm{Sub}(\varphi)$ for $\psi, \varphi \in \Phi$ and $\psi \in \mathrm{Sub}(\varphi)$. (2.5P)
2) What is the name of this property? (1P)

d) Prove that $\varphi = (\neg(\neg(\ldots(\neg p_1))))$ with n occurrences of $\neg$, has $n+1$ subformulae. (4.5P)

Solution:

a) Define $X = \{n \in \mathbb{N} | \, n^2 - n \equiv_2 0\}$. (0.5P)
For the induction base we have to prove $1 \in X$. (0.5P)
By $1^2 - 1 = 1 - 1 = 0 \equiv_2 0$ we get $1 \in X$. (0.5P)
Assume $n \in X$ for an arbitrary but fixed $n \in \mathbb{N}$, i.e. $n^2 - n \equiv_2 0$. (0.5+0.5+0.5P)
Consider $n + 1$. (0.5P)
We have

$$(n+1)^2 - (n+1) = n^2 + 2n + 1 - n - 1 = n^2 + n.$$

(1P)

By the induction hypothesis we know that $n^2 - n$ is even and by definition of being even there exists a $c \in \mathbb{N}$ with $n^2 - n = 2c$. (1P)
Moreover we have $n^2 + n = n^2 - n + 2n$. (1P)
Combined with the induction hypothesis we get

$$(n+1)^2 - (n+1) = n^2 - n + 2n = 2c + 2n = 2(c+n).$$

(1P)

By the definition of even, we have $(n+1)^2 - (n+1)$ is even and thus in X. (0.5P)
By the induction principle we get $\mathbb{N} \subset X$ and the claim is proven. (0.5P)

b) subformulae: $((p_1 \vee p_2) \to ((\neg p_1) \wedge p_3))$, $(p_1 \vee p_2)$, $((\neg p_1) \wedge p_3)$, p_1, p_2, $(\neg p_1)$, p_3

c) 1) For proving $\mathrm{Sub}(\psi) \subseteq \mathrm{Sub}(\varphi)$ we have to show that $\chi \in \mathrm{Sub}(\psi)$ implies $\chi \in \mathrm{Sub}(\varphi)$. (0.5P)
Let $\chi \in \mathrm{Sub}(\psi)$. (0.5P)
Thus χ is a subtree of ψ's parse tree. (0.5P).

By $\psi \in \mathrm{Sub}(\varphi)$, ψ is a subtree of φ's parse tree. (0.5P)
Since a subtree of a subtree is a subtree of the parse tree, χ is a subformula of φ. (0.5P)

2) Transitivity. (1P)

d) Proof by induction on $n \in \mathbb{N}_0$. (0.5P)
For $n = 0$ we have $\varphi = p_1$ and thus p_1 is the only subformula and the cardinality is $n + 1 = 0 + 1 = 1$. (0.5P)
Assume now that the claim holds for an arbitrary but fixed $n \in \mathbb{N}_0$. (0.5P)
Consider a formula φ of the given form with $n + 1$ occurrences of $\neg$. (0.5P)
Then there exists a $\psi \in \Phi$ with $\varphi = (\neg\psi)$. (0.5P)
The formula ψ has n occurrences of $\neg$. (0.5P)
By induction hypothesis, ψ has $n + 1$ subformulae. (0.5P)
In addition φ has $(\neg\psi)$ as a subformula. (0.5P)
This sums up to $n + 2$ subformulae of φ. (0.5P)