Logical and Theoretical Foundations of Computer Science

LaTFoCS

Pamela Fleischmann

fpa@informatik.uni-kiel.de

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Dependable Systems Group

Substitutions

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- if x has not a free occurrence in φ: $\varphi[x/t] = \varphi$

Example

$\varphi = (\forall x\, x \wedge y) \vee (\exists y\, x \wedge z)$

- $\varphi[f(z,u)/x] \rightsquigarrow$

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We have to avoid that free occurrences are substituted by bounded!

Free Terms in Formulae and Renaming of Variables

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Given a variable x, and a formula φ, the term t is free for x in φ if no free x leaf in φ occurs in the scopes of $\forall y$ and $\exists y$ for all variables y in t.

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Substituting x by t in φ and t is not free for x in φ

1. for all variables y_i violating t's freedom for x in φ choose a fresh variable z_i(not occurring in neither t nor φ
2. perform $t' = t[z_i/y_i]$ (t' is free for x in φ
3. apply the substitution $\varphi[t'/x]$

Proof Theory of Predicate Logic

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Convention: $\varphi[t/x]$ implies implicitely that t is free for x in φ!

Rule for Equality

- for all terms t

$$\frac{}{t = t}\ (= i)$$

- for all terms t_1, t_2 and all formula φ

$$\frac{t_1 = t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]}\ (= e)$$

Proof of Commutativity of the Equality of Terms

$$\dfrac{t_1 = t_2 \text{ (Premise)} \qquad \dfrac{}{(t_1 = t_1)\hat{=}(x = t_1)[t_1/x]}\,(= i)}{(t_2 = t_1)\hat{=}(x)t_1)[t_2/x]}\,(= e)$$

Proof of Transitivity of the Equality of Terms

$$\dfrac{t_2 = t_3 \qquad \dfrac{t_1 = t_2 \qquad (t_1 = x)[t_1/x]}{(t_1 = x)[t_2/x]}\,(= e)}{(t_1 = x)[t_3/x]}\,(= e)$$

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With $(= i)$ we proved that $=$ is an equivalence relation.

Rules for Universal Quantification

- elimination of $\forall$

  (scheme of rules: for all free terms t one rule)

$$\frac{\forall x \varphi}{\varphi[t/x]}\ (\forall x e)$$

- introduction of $\forall$

$$\frac{\boxed{\begin{array}{c} x_0 \\ \vdots \\ \varphi[x_0/x] \end{array}}}{\forall x \varphi}\ (\forall x\ \mathrm{i})$$

  where x_0 is a variable that does not occur outside the box

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  - x_0 is arbitrary (not special, constraint-free) $\rightsquigarrow$ φ holds for all x

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Proof:

Premise	$\forall x\, P(x) \rightarrow Q(x)$	
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$\forall x e$ with $t = x_0$	$P(x_0) \rightarrow Q(x_0)$	
	fresh variable	x_0
	$\forall x e$ with $t = x_0$	$P(x_0)$
	mp	$Q(x_0)$
$\forall x i$	$\forall x\, Q(x)$	

$\forall$ and $\bigwedge$

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- if a conjunction is true, we know that each single conjunct is true
- if universally quantified formula is true, we know that the scope for each substitution is true
- if formulae are true, then also there conjunction
- if a formula is true for all substitutions for a variable, then the universally quantified formula is true as well

Rules for Existential Quantification

- exists introduction
  (scheme of rules: for all free terms t one rule)

$$\frac{\varphi[t/x]}{\exists x \varphi}\ (\exists x i)$$

- exists elimination

$$\frac{\exists x \varphi \qquad \boxed{\begin{array}{cc} x_0 & \varphi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ (\exists x\ \mathrm{e})$$

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- if one disjunct is true, the disjunction is true
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- if for one arbitrary value χ follows by $\varphi[x_0/x]$ and φ is true for one x, then χ holds always

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$$\dfrac{\dfrac{\forall x \varphi}{\varphi[x/x]}\,\forall x e}{\exists x \varphi}\,\exists x i$$

Example II

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- $R(x)$ - x is rich
- $D(x)$ - x is a doctor

$$\exists x(D(x) \land F(x)), \forall x(F(x) \rightarrow R(x)) \vdash \exists x(D(x) \land R(x))$$

Example Cont.

Claim: $\exists x(D(x) \land F(x)), \forall x(F(x) \to R(x)) \vdash \exists x(D(x) \land R(x))$

Proof:

1	premise		$\forall x(F(x) \to R(x))$
2	premise		$\exists x(D(x) \land F(x)$
3	assumption	x_0	$D(x_0) \land F(x_0)$
4	$(\forall x e)$ 1		$F(x_0) \to R(x_0)$
5	$(\land e_2)$ 3		$Q(x_0)$
6	$(\land e_1)$ 3		$D(x_0)$
7	(mp) 4		$R(x_0)$
8	$(\land i)$ 6,7		$D(x_0) \land R(x_0)$
9	$(\exists i)$ 8		$\exists x(D(x) \land R(x)$
10	$(\exists e)$ 1, box		$\exists x(D(x) \land R(x))$

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  - good for showing that φ is not a consequence of $\varphi_1, \ldots, \varphi_n$ (find model in which all φ_i are true and φ not)

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- Semantics:
  - good for showing that φ is not a consequence of $\varphi_1, \ldots, \varphi_n$ (find model in which all φ_i are true and φ not)
  - harder for showing that φ is a consequence of $\varphi_1, \ldots, \varphi_n$ (claim about all models)