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Example solution for Series #5

Exercise 1 — 10 Points

You get for each correct answer 1 point, but you will lose 1 point for an incorrect answer.

a) Semantic Tableau is a decision procedure for satisfiability in propositional logic. ☒ true ○ false

b) The or-elimination rule is assumption based. ☒ true ○ false

c) *Ex falso quodlibet* is a rule for implication. ☒ true ○ false

d) The contradiction rule is based on the fact that we only have two truth values. ☒ true ○ false

e) *Reductio ad absurdum* and ¬-introduction are two different rules. ○ true ☒ false

f) *Tertium non datur* is a theorem. ☒ true ○ false

g) Logically equivalence is not implied by provable equivalence. ○ true ☒ false

h) ∧-introduction has two premises. ☒ true ○ false

i) If one is not able to find a proof for a claim then the claim is wrong. ○ true ☒ false

j) If a claim is a logical consequence then there exists a proof. ☒ true ○ false

Exercise 2 — 5 Points

Give the following definitions and notations:

a) Sequent. (1P)
b) And-elimination rule ($\wedge e_2$) (1P)
c) Modus-Ponens. (1P)
d) Soundness-Theorem of Propositional Logic. (1P)
e) Completeness-Theorem of Propositional Logic. (1P)

Solution:

a) S is a sequent iff it is of the form $\varphi_1, \ldots, \varphi_n \vdash \psi$ for formulae $\varphi_1, \ldots, \varphi_n, \psi \in \Phi$, $n \in \mathbb{N}$. (1P)

b) $$\frac{\varphi \wedge \psi}{\psi}\ (\wedge e_2)$$ (1P)

c) $$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ (\mathrm{mp})$$ (1P)

d) For an $n \in \mathbb{N}_0$, $\varphi_1, \ldots, \varphi_n \vdash \psi$ implies $\varphi_1, \ldots, \varphi_n \models \psi$. (1P)

e) For an $n \in \mathbb{N}_0$, $\varphi_1, \ldots, \varphi_n \models \psi$ implies $\varphi_1, \ldots, \varphi_n \vdash \psi$. (1P)

Exercise 3 — 19.5 Points

a) Prove the transitivity of $\to$ by natural deduction. (4P)
b) Prove $\neg p \vee q \dashv\vdash p \to q$ by natural deduction. (12.5P)

c) Prove the case that the last applied rule is $(\neg\neg i)$ in the proof of the soundness theorem for propositional logic. (3P)

Solution:

a) Let $\varphi, \psi, \chi \in \Phi$. (0.5P)
We have the premise $\varphi \to \psi$. (1) (0.5P)
We have the premise $\psi \to \chi$. (2) (0.5P)
Since we want to prove $\varphi \to \chi$, last applied rule needs to be the implication introduction. (0.5P)
This rule has an assumption based premise and thus we assume φ. (3) (0.5P)
Modus ponens applied on (1) and (3) gives ψ. (4) (0.5P)
Modus ponens applied on (2) and (4) gives χ. (5) (0.5P)
Implication introduction leads to $\varphi \to \chi$. (0.5P)

b) We have to prove two parts, namely $\neg p \vee q \vdash p \to q$ and $p \to q \vdash \neg p \vee q$. (1P)
first part: $\neg p \vee q \vdash p \to q$.
We have the premise $\neg p \vee q$. (1) (0.5P)
Since we want to prove $p \to q$, last applied rule needs to be the implication introduction. (0.5P)
This rule has an assumption based premise and thus we assume p. (2) (0.5P)
By the or-elimination rule we have two assumptions. (3) (0.5P)
Assume $\neg p$. (3.1) (0.5P)
And-introduction on (3) and (3.1) leads to $p \wedge \neg p$. (0.5P)
This leads to $\perp$ by (cd). (0.5P)
By (efq) we get q. (3.1.1) (0.5P)
Assume q. (3.2) (0.5P)
Thus q holds. (3.2.1) (0.5P)
Since in both cases (3.1.1) and (3.2.1) we get q, by $\vee$-elimination we get q. (0.5P)
By implication introduction we get $p \to q$. (0.5P)

second part: $p \to q \vdash \neg p \vee q$
We have the premise $p \to q$. (1) (0.5P)
The last applied rule needs to be $\vee$-introduction. (0.5P)
Thus we have to prove that $\neg p$ or q holds. (0.5P)
By tertium non datur we have $p \vee \neg p$. (0.5P)
We apply $\vee$-elimination on that and get two cases. (0.5P)
case 1: $\neg p$ holds. (0.5P)
By $\vee$-introduction we get $\neg p \vee q$. (2) (0.5P)
case 2: p holds (0.5P)
By modus ponens and (1) we get q. (0.5P)
$\vee$-introduction on q gives us $\neg p \vee q$. (0.5P)
Since in both cases we get $\neg p \vee q$, the claim is proven. (0.5P)

c) If the last applied rule is $(\neg\neg i)$, we have $\psi = \neg\neg\psi_1$. (0.5P)
ψ_1 occurs up in the proof. (0.5P)
This implies $\varphi_1, \ldots, \varphi_n \models \psi_1$. (0.5P)
By (IH) we have $\varphi_1, \ldots, \varphi_n \models \psi_1$. (0.5P)
By a truth tabel we get $\varphi_1, \ldots, \varphi_n \models \neg\neg\psi_1 = \psi$. (0.5P)