Logical and Theoretical Foundations of Computer Science

LaTFoCS

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Kiel University

Dependable Systems Group

Propositional Logic

Proposition/ Declarative Sentence

What do have all the sentences in common?

- A tree is an animal.

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Proposition/ Declarative Sentence

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They are all either true or false.

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- to no strict part of *A fox is a mammal.* the logical/truth values are assignable.
- Contrary: *A fox is a mammal and can fly.* is dividable into

  A fox is a mammal. A fox can fly.

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- ○ A child has a biological mother.

Semantics and Syntax

If we are about to define a language we need (as usual) syntax and semantics:

Syntax
: Which strings are a valid atomic proposition or formula?

Semantics
: What is the meaning of an atomic proposition or formula?

Formal Definition of the Propositional Formulae's Syntax

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9. if $\phi, \psi \in \Phi$ then also $(\phi \uparrow \psi)$ (nand)

Representations of propositional formulae

- By definition a propostional formula is just a string over the alphabet $A \cup \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \dot{\vee}, \downarrow, \uparrow, (,)\}$.
- Φ can also be generated by a grammar (validity is easy to check by a computer)

$$\begin{aligned} \varphi &\rightarrow (\neg\varphi)|(\varphi \wedge \varphi)|(\varphi \vee \varphi)|\varphi \rightarrow \varphi) \\ \forall i \in \mathbb{N}: \varphi &\rightarrow p_i \end{aligned}$$

Formulae as Trees

Definition

A tree $T = (V, E)$ is a parse-tree for a formula $\varphi \in \Phi$ iff

1. $\varphi \in A$: φ is a node
2. $\varphi = (\neg\psi)$: ψ is the only child of $\neg$
3. $\varphi = (\psi \circ \chi)$: ψ is left-child of $\circ$ and χ is right-child of $\circ$ for all $\circ \in \{\wedge, \vee, \rightarrow, , , \leftrightarrow, \dot{\vee}, \downarrow, \uparrow\}$

Ambiguity

What does

If a fox is a mammal then it can fly or it can walk.

mean?

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mean?

- (If a fox is a mammal then it can fly) or it can walk.
- If a fox is a mammal then (it can fly or it can walk).

Precedence Conventions

From tightest to weakest binding

- $\neg$
- $\wedge, \vee, \uparrow, \downarrow, \dot{\vee}$
- $\rightarrow$
- $\leftrightarrow$

All binary operators are assumed to be right-associative.

Subformulae

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Notice that each subformula ψ of φ is associated with a subtree of φ's parse tree.

Height of a Formula

Definition

The height of a well-formed formula is the length of the longest path from the root to a leaf in its parse-tree plus one.