LOGIC AND THEORETICAL FOUNDATION OF COMPUTER SCIENCE

LATFOCS

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Kiel University

Dependable Systems Group

SEIT 350 JAHREN GANZ WEIT OBEN

CAU

Natural Deduction

Infering Knowledge

- Black clouds are approaching. $\rightsquigarrow$ Computer: Please don't forget the umbrella.

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- Pain in the leg and normal concentration of white blood cells. $\rightsquigarrow$ Computer: Muscle or tendon problem.

How can we train a computer to deduce these things?

Proof Rules and Sequents

Definition

- S is a sequent iff it is of the form

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

for formulae $\varphi_1, \ldots, \varphi_n, \psi \in \Phi$

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- $\varphi \dashv\vdash \psi$ abbreviates $\varphi \vdash \psi$ and $\psi \vdash \varphi$ (φ and ψ are provable equivalent)

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... but what is a proof for a computer?

Notation for Proof Rules

- 1 premise

$$\frac{\text{premise}}{\text{conclusion}}$$

- n premises

$$\frac{\text{premise } 1 \qquad \dots \qquad \text{premise } n}{\text{conclusion}}$$

Rules for Natural Deduction - Rules for Conjunction

- and-introduction $(\wedge i)$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ (\wedge i)$$

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Our first proof

Foxes have fur and can walk. Foxes are mammals.

Conclusion: Foxes are walking mammals. (mammal and walk)

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Claim: $\varphi \wedge \psi, \vartheta \vdash \psi \wedge \vartheta$
Proof:

1.	$\varphi \wedge \psi$	(premise)
2.	ϑ	(premise)
3.	ψ	$(\wedge e_2)_1$
4.	$\psi \wedge \vartheta$	$(\wedge i)_{2,3}$

$$\dfrac{\dfrac{\varphi \wedge \psi}{\psi}\,(\wedge e_2) \qquad \vartheta}{\psi \wedge \vartheta}\,(\wedge i)$$

Rules for Natural Deduction - Rules for Double Negation

- double-negation-elimination

$$\frac{\neg\neg\varphi}{\varphi}\ (\neg\neg e)$$

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$$\frac{\neg\neg\varphi}{\varphi}\ (\neg\neg e)$$

- double-negation-introduction

$$\frac{\varphi}{\neg\neg\varphi}\ (\neg\neg i)$$

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In logic the double negation is well defined!

Rules for Natural Deduction - Rules for Implication

- modus-ponens (implies-elimination, arrow-eliminiation)

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- modus-tollens

$$\frac{\neg\psi \qquad \varphi \rightarrow \psi}{\neg\varphi}\ (\mathrm{mt})$$

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- if we buy tomatoes, onion, basil, olive oil, pepper, salt, then we have ingredients for a simple pasta sauce
- if additionally we had cheese, flour, yeast at home, we could prepare a simple pizza
- the latter one is an assumption
- Notice:
  - the pasta sauce does not depend on the assumption
  - the pizza depends on the assumption **and** the ingredients we tend to buy

Formalising Assumption based Proofs

- Everything based on an assumption is put in a box:

φ	assumption
$\vdots$	deduction steps
ψ	conclusion

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- thus we know: if φ holds, then ψ does so as well
- this is the colloquial description of $\varphi \to \psi$

Rules for Natural Deduction - Rules for Implication

- implication introduction

$$\dfrac{\boxed{\begin{array}{c} \varphi \\ \vdots \\ \psi \end{array}}}{\varphi \to \psi}\ (\to \mathrm{i})$$

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- implication introduction

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \rightarrow \psi}\ (\rightarrow \mathrm{i})$$

Notice: this rule also allows us the following

if we have a proof for ψ under the assumption φ,
we have also the implication $\varphi \rightarrow \psi$

Contraposition

Lemma

If $\varphi \to \psi$ holds, then also the **contraposition** $\neg\psi \to \neg\varphi$.

Contraposition

Lemma

If $\varphi \to \psi$ holds, then also the **contraposition** $\neg\psi \to \neg\varphi$.

Proof:

$$\dfrac{\dfrac{p \to q \qquad \boxed{\neg q}}{\boxed{\neg p}}\,(\mathrm{mt})}{\neg q \to \neg p}\,(\to i)$$

Premiseless-Derivations (Theorems)

- we defined a sequent by $\varphi_1, \dots, \varphi_n \vdash \psi$ and called it valid if there exists a proof inferring ψ from $\varphi_1, \dots, \varphi_n$

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- we defined a sequent by $\varphi_1, \ldots, \varphi_n \vdash \psi$ and called it valid if there exists a proof inferring ψ from $\varphi_1, \ldots, \varphi_n$
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Definition

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we will see later why theorem in model-theory and theorem in decuction theory are the same

Example for a Theorem

Claim: $(q \to r) \to ((\neg q \to \neg p) \to (p \to r))$ is a theorem

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Proof:

Assumption: $q \to r$

Assumption: $\neg q \to \neg p$

Assumption: p

$(\neg\neg i)$ $\neg\neg p$

(MT) $\neg\neg q$

$(\neg\neg e)$ q

$((\to e)$ r

$(\to i)$ $p \to r$

$(\to i)$ $(\neg q \to \neg p) \to (p \to r)$

$(\to i)$ $(q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

Connection: Proof - Theorem

Lemma

Every proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ *is transformable into a theorem.*

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Proof:

- transform the proof by using $(\rightarrow i)$ to

$$\vdash \varphi_1 \longrightarrow (\varphi_2 \longrightarrow (\cdots \longrightarrow (\varphi_n \longrightarrow \psi) \ldots))$$

Rules for Natural Deduction - Rules for Disjunction

- or-introduction ($\vee i_1$)

$$\frac{\varphi}{\varphi \vee \psi}\ (\vee i_1)$$

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- or-introduction ($\vee i_2$)

$$\frac{\psi}{\varphi \vee \psi} \; (\vee i_2)$$

Rules for Natural Deduction - Rules for Disjunction

- or-introduction ($\vee i_1$)

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- or-introduction ($\vee i_2$)

$$\frac{\psi}{\varphi \vee \psi}\ (\vee i_2)$$

Notice that in both rules the respectively other formulae is arbitrary! (If foxes are mammals then foxes are mammals or cows are birds.)

Rules for Natural Deduction - Rules for Disjunction (Cont.)

What can we deduce from a disjunction $\varphi \vee \psi$? We **do not** know if φ or ψ holds.

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I catch the bus or I take a taxi.
If I catch the bus, then I will arrive at the bus-stop.
If I arrived at the bus-stop, I will walk to the hotel.
If I take the taxi, it will bring me to the hotel.

Rules for Natural Deduction - Rules for Disjunction (Cont.)

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- or-elimination ($\vee e$)

$$\dfrac{\varphi \vee \psi \qquad \boxed{\begin{array}{c}\varphi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi}\ (\vee e)$$

$\wedge$-$\vee$-Distributivity

Claim: $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$

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Claim: $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$
Proof:

Premise: $p \wedge (q \vee r)$

$(\wedge e_1)$ p

$(\wedge e_2)$ $q \vee r$

Assumption: q

$(\wedge i)$ $p \wedge q$

$(\vee i_1)$ $(p \wedge q) \vee (p \wedge r)$

$q \rightarrow (p \wedge q) \vee (p \wedge r)$

Assumption: r

$(\wedge i)$ $p \wedge r$

$(\vee i_2)$ $(p \wedge q) \vee (p \wedge r)$

$r \rightarrow (p \wedge q) \vee (p \wedge r)$

$(\vee e)$ $(p \wedge q) \vee (p \wedge r)$

Contradiction

- so far we have seen for negation: $(\neg\neg i)$ and $(\neg\neg e)$

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- how to introduce or eliminate just one negation

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- how to introduce or eliminate just one negation

Definition

for all formulae φ, $\varphi \wedge \neg\varphi$ is a contradiction

Ex falso quodlibet

What do you think? True or false?

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Ex falso quodlibet

What do you think? True or false?

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- If a fox is a fish, then all trees are anemones.

Ex falso quodlibet

What do you think? True or false?

- ○ If a fox is a fish, then -350°C exists.
- ○ If a fox is a fish, then forests are under water.
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- ○ If a fox is a fish, then it can fly.

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Everything follows from the false. (*Deductive explosion*. First proven by William of Soissons in 12th century)

Rules for Natural Deduction - Rules for Negation

- bottom-eliminiation, false-eliminiation, ex falso quodlibet (efq)

$$\frac{\bot}{\varphi}\ \text{(efq)}$$

Rules for Natural Deduction - Rules for Negation

- bottom-eliminiation, false-eliminiation, ex falso quodlibet (efq)

$$\frac{\bot}{\varphi}\ \text{(efq)}$$

- not-eliminiation, contradiction (cd)

$$\frac{\varphi \wedge \neg\varphi}{\bot}\ \text{(cd)}$$

Reductio ad absurdum

Definition (Absurd (Oxford Dictionary))

1. Adjective:
   - Wildly unreasonable, illogical, or inappropriate.
   - Arousing amusement or derision; ridiculous.
2. Noun: An absurd state of affairs.

Mid 16th century: from Latin absurdus 'out of tune', hence 'irrational'; related to surdus 'deaf, dull'.

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- Not tossing the garbage implies that it rottens.
- Rotten garbage attracts fruit flies.
- Getting rid of fruit flies takes more than 10 min.
- This implies that not tossing the garbage is absurd.

Rules for Natural Deduction - Rules for Negation

- $\neg$-introduction, reductio ad absurdum

$$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}}{\neg\varphi}\ (\mathrm{raa})$$

Rules for Natural Deduction - Tertium non datur

- tertium non datur (law of excluded middle)

$$\frac{}{\varphi \vee \neg\varphi}\,(\text{tnd})$$

Rules for Natural Deduction - Tertium non datur

- tertium non datur (law of excluded middle)

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as long as we *are* binary, either the one or the opposite holds

- either a number is zero or greater than zero
- either I am eating pizza or not
- either the fox can fly or not

Not only for proving proof-rules

Lemma

Let $a \in \mathbb{N}_0$ *and* $b \in \mathbb{N}$ *with* $a|b$. *Then* $|a| \leq |b|$.

Not only for proving proof-rules

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Let $a \in \mathbb{N}_0$ *and* $b \in \mathbb{N}$ *with* $a|b$. *Then* $|a| \leq |b|$.

Proof.

Not only for proving proof-rules

Lemma

Let $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$ with $a|b$. Then $|a| \leq |b|$.

Proof. How to start?

Proof - Scrap Paper 1

Let's first collect what we have:

1. $a, b \in \mathbb{N}$: a, b natural numbers, $a \geq 0$, $b > 0$

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2. $a|b$: $\exists c \in \mathbb{N} : ac = b$
3. $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{otherwise} \end{cases}$

Proof - Scrap Paper 2

Reformulation of the lemma:

$$\underbrace{a \text{ natural number}}_{\varphi} \wedge \underbrace{b \text{ positive natural number}}_{\psi} \wedge$$

$$\underbrace{\text{exists natural number } c \text{ with } ac = b}_{\chi}$$

$$\vdash \underbrace{\text{absolute value of } a \text{ is at most absolute value of } b}_{\xi}$$

Proof

- χ allows us to take a c with $ac = b$
- tnd: either c is zero or not
- case 1: $c = 0$
  - multiplication with zero is zero
  - thus b is zero
  - $(\wedge i)$ $b = 0$ and $b > 0$
  - (cd) $\perp$
- case 2: $c \neq 0$
  - definition of absolute value: $|c| > 0$
  - total order on $\mathbb{N}$: $|a| = |a| \cdot 1 \leq |a| \cdot |c| = |ac| = |b|$ $\square$