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A FILTER LAMBDA MODEL AND THE COMPLETENESS OF TYPE ASSIGNMENT[1]

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In [6, p. 317] Curry described a formal system assigning types to terms of the type-free λ-calculus. In [11] Scott gave a natural semantics for this type assignment and asked whether a completeness result holds.

Inspired by [4] and [5] we extend the syntax and semantics of the Curry types in such a way that filters in the resulting type structure form a domain in the sense of Scott [12]. We will show that it is possible to turn the domain of types into a λ-model, among other reasons because all λ-terms possess a type. This model gives the completeness result for the extended system. By a conservativity result the completeness for Curry's system follows.

Independently Hindley [8], [9] has proved both completeness results using term models. His method of proof is in some sense dual to ours.

For λ-calculus notation see [1].

§1. Curry type assignment.

1.1. DEFINITION. (i) The *Curry type schemes* form the smallest set T_C such that

1. $\varphi_0, \varphi_1, \ldots \in T_C$ (type variables).
2. $\sigma, \tau \in T_C \Rightarrow (\sigma \to \tau) \in T_C$.

(ii) A *Curry statement* is an expression of the form σM where $\sigma \in T_C$ and $M \in \Lambda$ (set of type free λ-terms). M is the *subject* and σ the *predicate* of σM.

A *basis* B is a set of Curry statements with only variables as subjects.

(iii) A *Curry type assignment* is defined by the following natural deduction system, see e.g. [10, Chapter I, §2A].

$$(\to\mathrm{I})\quad \dfrac{\begin{matrix}[\sigma\ x]\\ \vdots\\ \tau M\end{matrix}}{\sigma \to \tau\ \lambda x.\,M}\ (*) \qquad\qquad (\to\mathrm{E})\quad \dfrac{\sigma \to \tau M\ \ \sigma N}{\tau(MN)}$$

$$(\mathrm{EQ}_\beta)\quad \dfrac{\sigma M \quad M =_\beta N}{\sigma N}.$$

($*$) if x not free in assumptions on which τM depends other than σx.

(iv) If σM is derivable from a basis B, then we write $B \vdash_C \sigma M$. If D is a derivation showing this, then we write $D\colon B \vdash_C \sigma M$.

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We assume that the reader is familiar with the notion of λ-model (weakly extensional λ-algebra) and the interpretation of λ-terms in them. See [1] or [7].

1.2. DEFINITION. Let $\mathcal{M} = \langle D, \cdot, [\![\ ]\!] \rangle$ be a λ-model.

(i) If ξ is the valuation of variables of Λ in D, then $[\![M]\!]_\xi^{\mathcal{M}} \in D$ is the interpretation of M in $\mathcal{M}$ via ξ. Usually we omit the superscript $\mathcal{M}$.

(ii) Let $\mathcal{V}: \{\varphi_i \mid i \in \omega\} \to PD = \{X \mid X \subseteq D\}$. Then the interpretation of $\sigma \in T_C$ in $\mathcal{M}$ via $\mathcal{V}$, denoted $[\![\sigma]\!]_{\mathcal{V}}^{\mathcal{M}} \in PD$, is defined as follows.

1. $[\![\varphi_i]\!]_{\mathcal{V}}^{\mathcal{M}} = \mathcal{V}(\varphi_i)$.
2. $[\![\sigma \to \tau]\!]_{\mathcal{V}}^{\mathcal{M}} = \{d \in D \mid \forall e \in [\![\sigma]\!]_{\mathcal{V}}^{\mathcal{M}}\ d \cdot e \in [\![\tau]\!]_{\mathcal{V}}^{\mathcal{M}}\}$.

(iii) $\mathcal{M}, \xi, \mathcal{V} \models \sigma M$ iff $[\![M]\!]_\xi^{\mathcal{M}} \in [\![\sigma]\!]_{\mathcal{V}}^{\mathcal{M}}$,
$\mathcal{M}, \xi, \mathcal{V} \models B$ iff $\mathcal{M}, \xi, \mathcal{V} \models \sigma x$ for all $\sigma x \in B$,
$B \models \sigma M$ iff $\forall \mathcal{M}, \xi, \mathcal{V} \models B \quad \mathcal{M}, \xi, \mathcal{V} \models \sigma M$.

We will show the following completeness result: $B \vdash_C \sigma M \Leftrightarrow B \models \sigma M$. The soundness ($\Rightarrow$) has been proved in [2].

§2. Extended type assignment.

2.1. DEFINITION. (i) The set T of *extended types* is inductively defined by

1. $\varphi_0, \varphi_1, \ldots \in T$ type variables,
$\omega \in T$ type constant.
2. $\sigma, \tau \in T \Rightarrow (\sigma \to \tau) \in T, (\sigma \cap \tau) \in T$.

(ii) *A statement* is of the form σM with $\sigma \in T$, $M \in \Lambda$. A *basis* is a set of statements with only variables as subjects.

The semantics for T_C is extended to T.

2.2. DEFINITION. (i) Let $\mathcal{V}: \{\varphi_i\} \to PD$, where D is the domain of a λ-model $\mathcal{M}$. Then for $\sigma \in T$ the set $[\![\sigma]\!]_{\mathcal{V}}^{\mathcal{M}} \subseteq D$ is defined by adding to 1.2(ii):

3. $[\![\omega]\!]_{\mathcal{V}}^{\mathcal{M}} = D$.
4. $[\![\sigma \cap \tau]\!]_{\mathcal{V}}^{\mathcal{M}} = [\![\sigma]\!]_{\mathcal{V}}^{\mathcal{M}} \cap [\![\tau]\!]_{\mathcal{V}}^{\mathcal{M}}$.

(ii) As before one defines $\mathcal{M}, \xi, \mathcal{V} \models \sigma M$; $\mathcal{M}, \xi, \mathcal{V} \models B$ and $B \models \sigma M$.

In order to introduce the formal system of extended type assignment one first defines a preorder $\leq$ on T. The intended interpretation of $\sigma \leq \tau$ is $\forall \mathcal{M}, \mathcal{V}\ [\![\sigma]\!]_{\mathcal{V}}^{\mathcal{M}} \subseteq [\![\tau]\!]_{\mathcal{V}}^{\mathcal{M}}$.

2.3. DEFINITION. (i) The relation $\leq$ on T is inductively defined by (i.e. is the smallest relation satisfying):

$\tau \leq \tau$,
$\tau \leq \omega$,
$\omega \leq \omega \to \omega$,
$\tau \leq \tau \cap \tau$,
$\sigma \cap \tau \leq \sigma, \sigma \cap \tau \leq \tau$,
$(\sigma \to \rho) \cap (\sigma \to \tau) \leq \sigma \to (\rho \cap \tau)$,
$\sigma \leq \tau \leq \rho \Rightarrow \sigma \leq \rho$,
$\sigma \leq \sigma', \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$,
$\sigma' \leq \sigma, \tau \leq \tau' \Rightarrow \sigma \to \tau \leq \sigma' \to \tau'$.

(ii) $\sigma \sim \tau \Leftrightarrow \sigma \leq \tau \leq \sigma$.

Note that, e.g., $(\sigma \to \rho) \cap (\sigma \to \tau) \sim \sigma \to (\rho \cap \tau)$; $\omega \sim \sigma \to \omega$; $\sigma \cap (\rho \cap \tau) \sim (\sigma \cap \rho) \cap \tau$. T may be considered modulo $\sim$; then $\leq$ becomes a partial order.

2.4. LEMMA. (i) $\sigma \to \tau \sim \omega \Leftrightarrow \tau \sim \omega$;

(ii) $(\mu_1 \to \nu_1) \cap \cdots \cap (\mu_n \to \nu_n) \leq \sigma \to \tau$ *and* $\tau \not\sim \omega$, *then there are* $i_1, \ldots, i_k \in \{1, \ldots, n\}$ *such that* $\mu_{i_1} \cap \cdots \cap \mu_{i_k} \geq \sigma$ *and* $\nu_{i_1} \cap \cdots \cap \nu_{i_k} \leq \tau$.

PROOF. (i) Define $\Omega \subseteq T$ inductively by: $\omega \in \Omega$; $\rho \in \Omega \Rightarrow \sigma \to \rho \in \Omega$; $\sigma, \rho \in \Omega \Rightarrow \sigma \cap \rho \in \Omega$. Note that $\sigma \in \Omega \Rightarrow \sigma \sim \omega$. By induction on the definition of $\leq$ one can show $\sigma \in \Omega$, $\sigma \leq \tau \Rightarrow \tau \in \Omega$. It follows that $\sigma \in \Omega \Leftrightarrow \sigma \sim \omega$. The rest is clear.

(ii) By induction on the definition of $\leq$ one can show for $n, n', m, m' \geq 0$ that for all $l \in \{1, \ldots, n'\}$ one has

$$\begin{aligned}
&[(\mu_1 \to \nu_1) \cap \cdots \cap (\mu_n \to \nu_n) \cap \varphi_{j_1} \cap \cdots \cap \varphi_{j_m} \\
&\quad \leq (\sigma_1 \to \tau_1) \cap \cdots \cap (\sigma_{n'} \to \tau_{n'}) \cap \varphi_{j'_1} \cap \cdots \cap \varphi_{j'_{m'}} \cap \omega \cap \cdots \cap \omega, \\
&\quad \text{and } \tau_l \not\sim \omega \Rightarrow \exists i_1, \ldots, i_k \in \{1, \ldots, n\}\ \mu_{i_1} \cap \cdots \cap \mu_{i_k} \geq \sigma_l \\
&\quad \text{and } \nu_{i_1} \cap \cdots \cap \nu_{i_k} \leq \tau_l].
\end{aligned}$$

Then the result follows. ■

2.5. DEFINITION (i) Extended type assignment is defined by the following natural deduction system.

$$(\to \mathrm{I}) \qquad \dfrac{\begin{array}{c}[\sigma x]\\ \vdots \\ \tau M\end{array}}{\sigma \to \tau\ \lambda x.M}\ (*) \qquad\qquad (\to \mathrm{E}) \qquad \dfrac{\sigma \to \tau M \quad \sigma N}{\tau\ MN}$$

$$(\cap \mathrm{I}) \qquad \dfrac{\sigma M \quad \tau M}{\sigma \cap \tau M} \qquad\qquad (\cap \mathrm{E}) \qquad \dfrac{\sigma \cap \tau M}{\sigma M} \quad \dfrac{\sigma \cap \tau M}{\tau M}$$

$$(\leq) \qquad \dfrac{\sigma M \quad \sigma \leq \tau}{\tau M}$$

$$(\omega) \qquad \overline{\omega M}\,.$$

$(*)$ if x not free in assumptions on which τM depends other than σx.

The rule $(\cap \mathrm{E})$ is superfluous, since it is directly derivable from rule $(\leq)$; the rule (EQ_β) is not included since it is also derivable (see 3.8).

(ii) If σM is derivable from a basis B in the extended system, then we write $B \vdash \sigma M$. Moreover D: $B \vdash \sigma M$ is as in 1.1(iv).

EXAMPLE. $\vdash ((\sigma \to \tau) \cap \sigma) \to \tau\ \lambda x.xx$.

2.6. DEFINITION. A *filter* is a subset $d \subseteq T$ such that:

(i) $\omega \in d$;

(ii) $\sigma, \tau \in d \Rightarrow \sigma \cap \tau \in d$;

(iii) $\sigma \geq \tau \in d \Rightarrow \sigma \in d$.

2.7. LEMMA. (i) $\{\sigma | B \vdash \sigma M\}$ *is a filter.*

(ii) $B \vdash \sigma x \Leftrightarrow \sigma$ *is in the filter generated by* $\{\tau | \tau x \in B\}$.

(iii) *If* τM *is derived from* $\sigma_1 M, \ldots, \sigma_n M$ *only by means of rules* $(\cap \mathrm{I})$, $(\cap \mathrm{E})$ *and* $(\leq)$, *then* $\tau \geq \sigma_1 \cap \cdots \cap \sigma_n$.

PROOF. (i) By rules (ω), $(\leq)$ and $(\cap \mathrm{I})$.

(ii) By induction on derivations.

(iii) From (ii) since, in the rules in question, M behaves like a variable. ■

2.8. LEMMA. (i) $B \vdash \tau MN \Rightarrow \exists\, \sigma \in T$: *[*$B \vdash \sigma \to \tau M$ *and* $B \vdash \sigma N$*].*

(ii) *Suppose* $\forall \sigma, \tau \in T$ $[B \cup \{\sigma x\} \vdash \tau M \Rightarrow B \cup \{\sigma x\} \vdash \tau N]$ *and x not in B; then* $\forall \rho \in T\, [B \vdash \rho\, \lambda x.M \Rightarrow B \vdash \rho\, \lambda x.N]$.

(iii) *If x is not in B then* $B \vdash \sigma \to \tau\, \lambda x.M \Leftrightarrow B \cup \{\sigma x\} \vdash \tau M$.

PROOF. (i) By induction on the derivation of τMN. The only interesting case is when the last applied rule is ($\bigcap$I), i.e. $\tau \equiv \tau_1 \cap \tau_2$. Then

$$\frac{\tau_1 MN \quad \tau_2\, MN}{\tau_1 \cap \tau_2\, MN}$$

is the last step.

By the induction hypothesis there are σ_1, σ_2 such that $B \vdash \sigma_i \to \tau_i M$, $B \vdash \sigma_i N$ for $i = 1, 2$. Then $B \vdash \sigma_1 \cap \sigma_2\, N$ and $B \vdash (\sigma_1 \to \tau_1) \cap (\sigma_2 \to \tau_2) M$. It is easy to verify that

$$(\sigma_1 \to \tau_1) \cap (\sigma_2 \to \tau_2) \le (\sigma_1 \cap \sigma_2) \to (\tau_1 \cap \tau_2),$$

so we can take $\sigma \equiv \sigma_1 \cap \sigma_2$.

(ii) Induction on the derivation of $\rho\lambda x.M$. The only nontrivial case is ($\to$I). Then the result follows from the assumption.

(iii) ($\Leftarrow$) By rule ($\to$I). ($\Rightarrow$) We may suppose that $\tau \not\sim \omega$. Let D: $B \vdash \sigma \to \tau$ $\lambda x.\, M$. Let $\mu_i \to \nu_i \lambda x.M$ $(1 \le i \le n)$ be all the statements in D on which $\sigma \to \tau\lambda x.M$ depends and which are conclusions of ($\to$I):

$$\frac{\begin{array}{c}[\mu_i x]\\ \vdots\\ \nu_i M\end{array}}{\mu_i \to \nu_i \lambda x.M}$$

The statement $\sigma \to \tau\lambda x.M$ is derived from the $\mu_i \to \nu_i\lambda x.M$ using only rules ($\bigcap$I), ($\bigcap$E) and ($\le$). By 2.7(iii) it follows that $(\mu_1 \to \nu_1) \cap \cdots \cap (\mu_n \to \nu_n) \le \sigma \to \tau$ and hence, by Lemma 2.4(ii), there are $i_1, \ldots, i_p$ such that $\mu_{i_1} \cap \cdots \cap \mu_{i_p} \ge \sigma$ and $\nu_{i_1} \cap \cdots \cap \nu_{i_p} \le \tau$. Hence we can construct D': $B \cup \{\sigma x\} \vdash \tau M$ as follows:

$$\begin{array}{l}
\dfrac{\sigma x}{\mu_{i_1} \cap \cdots \cap \mu_{i_p} x}\,(\le)\\[2ex]
\dfrac{\mu_{i_1} \cap \cdots \cap \mu_{i_p} x}{\mu_{i_k} x \qquad 1 \le k \le p}\,(\bigcap\mathrm{E})\\
\qquad\vdots\\
\dfrac{\nu_{i_k} M \qquad 1 \le k \le p}{\nu_{i_1} \cap \cdots \cap \nu_{i_p} M}\,(\bigcap\mathrm{I})\\[2ex]
\dfrac{\nu_{i_1} \cap \cdots \cap \nu_{i_p} M}{\tau M}\,(\le)
\end{array}$$

■

2.9. LEMMA. *If* $B \vdash \sigma M$, *then* $B \restriction M \vdash \sigma M$, *where* $B \restriction M = \{\sigma x \in B | x \in FV(M)\}$.

PROOF. Induction on the derivations. ■

2.10. REMARK. If $M \twoheadrightarrow_{\beta\eta} M'$ and $B \vdash \tau M$, then $B \vdash \tau M'$ (subject reduction theorem). We do not need this fact, however.

§3. The filter model.

3.1. PROPOSITION. (i) $\sigma \leq \tau \Rightarrow \forall \mathscr{M}, \mathscr{V}\ [\![\sigma]\!]^{\mathscr{M}}_{\mathscr{V}} \subseteq [\![\tau]\!]^{\mathscr{M}}_{\mathscr{V}}$.

(ii) (*Soundness*). $B \vdash \sigma M \Rightarrow B \models \sigma M$.

PROOF. (i) Induction on the definition of $\leq$.

(ii) Induction on derivations, using (i). ■

3.2. DEFINITION. (i) $\mathscr{F} = \{d | d \text{ is a filter}\}$.

(ii) For $d_1, d_2 \in \mathscr{F}$ define

$$d_1 \cdot d_2 = \{\tau \in T | \exists\, \sigma \in d_2\ \sigma \to \tau \in d_1\}.$$

3.3. LEMMA. $d_1, d_2 \in \mathscr{F} \Rightarrow d_1 \cdot d_2 \in \mathscr{F}$.

PROOF. Easy. ■

It will be shown that $\langle \mathscr{F}, \cdot, [\![\ ]\!] \rangle$ is a λ-model. In order to do so we apply the method of Hindley and Longo by defining directly $[\![M]\!]_\xi$ and show that this satisfies conditions (i)–(vi) in [7].

3.4. DEFINITION. (i) Let ξ be a valuation in $\mathscr{F}$. Then $B_\xi = \{\sigma x | \sigma \in \xi(x)\}$.

(ii) For $M \in \Lambda$ define $[\![M]\!]_\xi = \{\sigma | B_\xi \vdash \sigma M\}$ ($\in \mathscr{F}$ by 2.7(i)).

3.5. THEOREM. $\langle \mathscr{F}, \cdot, [\![\ ]\!] \rangle$ *is a λ-model, i.e.*

(i) $[\![x]\!]_\xi = \xi(x)$;

(ii) $[\![MN]\!]_\xi = [\![M]\!]_\xi \cdot [\![N]\!]_\xi$:

(iii) $[\![\lambda x.M]\!]_\xi \cdot d = [\![M]\!]_{\xi(x/d)}$;

(iv) $(\forall x \in FV(M).\ [\![x]\!]_\xi = [\![x]\!]_{\xi'}) \Rightarrow [M]_\xi = [\![M]\!]_{\xi'}$;

(v) $[\![\lambda x.M]\!]_\xi = [\![\lambda y.\ M[x := y]]\!]_\xi$, *if y not in M*;

(vi) $[\forall d \in \mathscr{F}\ [\![M]\!]_{\xi(x/d)} = [\![N]\!]_{\xi(x/d)}] \Rightarrow [\![\lambda x.M]\!]_\xi = [\![\lambda x.N]\!]_\xi$.

PROOF. (i) If $\tau \in [\![x]\!]_\xi$, i.e., $B_\xi \vdash \tau x$, then by 2.7(ii) τ is in the filter (generated by) $\xi(x)$. The converse is trivial.

(ii) If $\tau \in [\![MN]\!]_\xi$, i.e. $B_\xi \vdash \tau MN$, then by 2.8(i) $\exists \sigma \in [\![N]\!]_\xi\ \sigma \to \tau \in [\![M]\!]_\xi$, i.e. $\tau \in [\![M]\!]_\xi \cdot [\![N]\!]_\xi$. The converse is trivial.

(iii)
$$\begin{aligned}
\tau \in [\![M]\!]_{\xi(x/d)} &\Leftrightarrow B_{\xi(x/d)} \vdash \tau M \\
&\Leftrightarrow B'_\xi \cup \{\sigma x | \sigma \in d\} \vdash \tau M, \quad \text{where } B'_\xi = B_\xi - \{\sigma x | \sigma \in \xi(x)\} \\
&\Leftrightarrow B'_\xi \cup \{\sigma x\} \vdash \tau M \quad \text{for some } \sigma \in d \text{ (use compactness and that } d \text{ is a filter)} \\
&\Leftrightarrow B'_\xi \vdash \sigma \to \tau\ \lambda x.M \quad \text{for some } \sigma \in d \text{ (use 2.8(iii))} \\
&\Leftrightarrow B_\xi \vdash \sigma \to \tau\ \lambda x.M \quad \text{(use 2.9)} \\
&\Leftrightarrow \sigma \to \tau \in [\![\lambda x.M]\!]_\xi \quad \text{for some } \sigma \in d \\
&\Leftrightarrow \tau \in [\![\lambda x.M]\!]_\xi \cdot d.
\end{aligned}$$

(iv) Trivial by 2.9.

(v) Trivial.

(vi) Assume the LHS and $\rho \in [\![\lambda x.M]\!]_\xi$. Then $\rho \in [\![\lambda x.N]\!]_\xi$ by 2.8(ii). By symmetry we are done. ■

3.6. DEFINITION. (i) $\mathscr{V}_0(\varphi_i) = \{d \in \mathscr{F} \mid \varphi_i \in d\}$.

(ii) Given a basis B, define

$$\xi_B(x) = \{\sigma \in T \mid B \vdash \sigma\, x\} \quad (\in \mathscr{F}).$$

3.7. LEMMA. (i) $\forall\, \sigma \in T\ [\![\sigma]\!]_{\mathscr{V}_0} = \{d \in \mathscr{F} \mid \sigma \in d\}$.

(ii) $B \vdash \sigma\, M \Leftrightarrow B_{(\xi_B)} \vdash \sigma\, M$.

(iii) $\mathscr{F}, \xi_B, \mathscr{V}_0 \models B$.

PROOF. (i) By induction on σ.

(ii), (iii). Easy. ■

3.8. COROLLARY. *The following is a derived rule for extended type assignment*

$$(\mathrm{EQ}_\beta) \qquad \frac{\sigma M \quad M =_\beta N}{\sigma N}.$$

PROOF. Suppose $M =_\beta N$ and $B \vdash \sigma M$. Then $B_{(\xi_B)} \vdash \sigma M$, hence $\sigma \in [\![M]\!]_{\xi_B} = [\![N]\!]_{\xi_B}$ since $\mathscr{F}$ is a λ-model. So $B \vdash \sigma N$. ■

3.9. COROLLARY. (i) $\sigma \le \tau \Leftrightarrow \forall \mathscr{M}, \mathscr{V}\ [\![\sigma]\!]^{\mathscr{M}}_{\mathscr{V}} \subseteq [\![\tau]\!]^{\mathscr{M}}_{\mathscr{V}}$.

(ii) $\sigma \sim \tau \Leftrightarrow \forall \mathscr{M}, \mathscr{V}\ [\![\sigma]\!]^{\mathscr{M}}_{\mathscr{V}} = [\![\tau]\!]^{\mathscr{M}}_{\mathscr{V}}$.

PROOF. (i)($\Rightarrow$) 3.1(i). ($\Leftarrow$) Take $\mathscr{M} = \mathscr{F}$, $\mathscr{V} = \mathscr{V}_0$ and note that $\{\rho \mid \sigma \le \rho\}$ is a filter $\in [\![\sigma]\!]_{\mathscr{V}_0}$.

(ii) By (i). ■

3.10. COMPLETENESS THEOREM. $B \vdash \sigma M \Leftrightarrow B \models \sigma M$.

PROOF. ($\Rightarrow$) 3.1(ii). ($\Leftarrow$)

$$\begin{aligned} B \models \sigma M &\Rightarrow \mathscr{F}, \xi_B, \mathscr{V}_0 \models \sigma M \quad \text{by 3.7(iii)} \\ &\Rightarrow [\![M]\!]_{\xi_B} \in [\![\sigma]\!]_{\mathscr{V}_0} \\ &\Rightarrow \sigma \in [\![M]\!]_{\xi_B} \quad \text{by 3.7(i)} \\ &\Rightarrow B_{(\xi_B)} \vdash \sigma M \\ &\Rightarrow B \vdash \sigma M \quad \text{by 3.7(ii).} \ ■ \end{aligned}$$

It is interesting to compare Hindley's completeness proof with ours. He takes as a model a term model (cf. [1, 4.1.17]) and as valuations

$$\mathscr{V}_B(\varphi_i) = \{[M] \mid B^+ \vdash \varphi_i\, M\}, \qquad \xi_0(x) = [x],$$

where $[M] = \{N \in \Lambda \mid M =_\beta N\}$ and B^+ is a particular extension of B. Then he shows

$$[\![\sigma]\!]_{\mathscr{V}_B} = \{[M] \mid B^+ \vdash \sigma M\}, \qquad [\![M]\!]_{\xi_0} = [M].$$

REMARK. It is easy to prove that the filter model is a continuous λ-model; see [1, §19.3], ($\mathscr{F}$ is even an algebraic complete lattice). By an argument similar to the one in [3], we have Th $(\mathscr{F}) = \mathscr{B}$ (cf. [1, §16.4]). For the partial order $\subseteq$ in the model $\mathscr{F}$ one has $\boldsymbol{\Omega} \subseteq 1 \subseteq \mathsf{I}$. Therefore $(\mathscr{F}, \subseteq)$ is different from $(P\omega, \subseteq)$ and $(\mathscr{B}, \subseteq) \equiv (\boldsymbol{T}^\omega, \subseteq)$.

§4. Conservativity. Using a Prawitz normalization argument it will be shown that extended type assignment is conservative over that of Curry. Then the completeness for the latter theory follows from 3.10.

First we modify the extended type assignment theory.

4.1. DEFINITION. (i) A *large* basis is an arbitrary set of statements σM with $\sigma \in T$ and $M \in \Lambda$. To emphasize the difference, bases as in 1.1(i) and 2.1(ii) are called *small*.

(ii) Consider the type assignment system of 2.5 and replace rule ($\leq$) by

$$(\beta\eta) \qquad \frac{\sigma M \quad M \twoheadrightarrow_{\beta\eta} N}{\sigma N}.$$

$B \vdash^* \sigma M$ denotes derivability in the resulting system where we allow B to be large.

4.2. LEMMA. $B \vdash \sigma M \Rightarrow B \vdash^* \sigma M$.

PROOF. The only thing to show is that ($\leq$) is a derived rule in the $\vdash^*$ system: if $\sigma \leq \tau$, then $\sigma M \vdash^* \tau M$. This is done by induction on the definition of $\leq$ using rule ($\beta\eta$).

EXAMPLE. Let $\sigma \to \tau \leq \sigma' \to \tau'$ be a consequence of $\sigma' \leq \sigma$ and $\tau \leq \tau'$. Then one has the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\sigma' x}{\vdots}\ \text{ind. hyp.}}{\sigma x} \qquad \sigma \to \tau\ M}{\tau M x}\ (\to\mathrm{E})}{\vdots}\ \text{ind. hyp.}}{\tau' M x}}{\dfrac{\sigma' \to \tau'\ \lambda x.\, Mx}{\sigma' \to \tau'\ M}\ (\beta\eta).}\ (\to\mathrm{I}) \qquad \blacksquare$$

REMARK. By Remark 2.10 the converse of 4.2 is also true if B is a small basis. We do not need this result, however. For large bases the modified system is somewhat stronger than the system of §2 with large bases: $\varphi_0\ \lambda x.zx \vdash^* \varphi_0\ z$, $\varphi_0\ \lambda x.zx \nvdash \varphi_0\ z$.

4.3. DEFINITION. Let D: $B \vdash^* \sigma M$.

(i) An $\to$*-cut* in D is a statement occurrence ρZ in D which is the major premise of ($\to$ E) and is obtained by ($\to$ I) and immediately followed by $k \geq 0$ applications of ($\beta\eta$). The *length* of this cut is $k + 1$.

(ii) An $\bigcap$*-cut* and its length are defined similarly.

(iii) The *degree* of a cut ρZ is $|\rho|$, the number of symbols in ρ.

(iv) The *ordinal* of D is $O(D) = \omega \cdot |\rho| + m$, where $|\rho|$ is the highest degree of a cut in D and m is the sum of the lengths of cuts with degree $|\rho|$; $O(D) = 0$ if D does not contain a cut.

(v) D is *normal* if $O(D) = 0$.

4.4. EXAMPLE. The following two derivations are not normal:

D_1:

$$\dfrac{\dfrac{\dfrac{\sigma_1 M \qquad \sigma_2 M}{\sigma_1 \cap \sigma_2 M}\,(\cap\mathrm{I})}{\sigma_1 \cap \sigma_2 N}\,(\beta\eta)}{\sigma_1 N}\,(\cap\mathrm{E}) \qquad \begin{array}{l} \cap\text{-cut}: \sigma_1 \cap \sigma_2 M \\ \text{degree } |\sigma_1 \cap \sigma_2| \\ \text{length } 2 \end{array}$$

D_2:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[\sigma x]\\ \vdots \\ \tau M\end{array}}{\sigma \to \tau\, \lambda x.M}\,(\to\mathrm{I})}{\sigma \to \tau P}\,(\beta\eta) \qquad \sigma N}{\tau\, PN}\,(\to\mathrm{E}) \qquad \begin{array}{l} \to\text{-cut}: \sigma \to \tau\lambda x.M \\ \text{degree } |\sigma \to \tau| \\ \text{length } 2. \end{array}$$

4.5. Lemma (Subformula Principle). *Let D: $B \vdash^* \sigma M$ be normal. Then each predicate in D is subtype of σ or of a predicate in B.*

Proof. Well known; see e.g. [10, pp. 41, 42]. ■

4.6. Lemma *Let D: $B \vdash^* \sigma M$. Then there is a deduction $D[x := L]$: $B[x := L] \vdash^* \sigma M[x := L]$ by replacing all free occurrences of x by L. Moreover $D[x := L]$ has the same tree structure and same ordinal as D (e.g. D is normal iff $D[x := L]$ is normal).*

Proof. Obvious. ■

4.7. Lemma.

$$\begin{array}{ccc} (\lambda x.M)\,N & \xrightarrow{\quad}\!\!\!\!\rightarrow_{\beta\eta} & PN \\ \Big\downarrow{\scriptstyle\beta} & & \vdots{\scriptstyle\beta} \\ M[x := N] & \twoheadrightarrow_{\beta\eta} & L \end{array}$$

Proof. Since $\twoheadrightarrow_\beta$ commutes with $\twoheadrightarrow_{\beta\eta}$, see [1, §3.3.]. ■

4.8. Lemma. *Let D: $B \vdash^* \sigma M$ have ordinal $O(D) \neq 0$. Then there is an M' and D' with*

1. $M \twoheadrightarrow_\beta M'$,
2. D': $B \vdash^* \sigma M'$,
3. $O(D') < O(D)$.

Proof. Since $O(D) \neq 0$, there is a cut in D. Let $|\rho|$ be the highest cut degree in D and consider an innermost cut ρZ with this degree (i.e. in the subderivation of this cut there are only cuts of lower degree).

If the length of ρZ is > 2, then one can perform two consecutive applications of $(\beta\eta)$ at once obtaining a derivation D': $B \vdash^* \sigma M$ with lower ordinal.

If the length of ρZ is $= 2$, then D has as subderivation D_1 or D_2 as in 4.4 with the cut ρZ being $\sigma_1 \cap \sigma_2 M$ or $\sigma \to \tau\lambda x.M$.

If ρZ is the $\cap$-cut $\sigma_1 \cap \sigma_2.M$, then replace D_1 by

$$\dfrac{\sigma_1 M}{\sigma_1 N}\,(\beta\eta)$$

and one obtains a derivation D': $B \vdash^* \sigma M$ with lower ordinal (the only possibly created cut has degree $|\sigma_1| < |\sigma_1 \cap \sigma_2| = |\rho|$).

If ρZ is the $\to$-cut $\sigma \to \tau\ \lambda x.M$, then replace D_2 by

$$\frac{\begin{array}{c}\vdots \\ \sigma N \\ \vdots \\ \tau M\,[x := N]\end{array}}{\tau L}\ (\beta\eta)$$

using 4.6 and 4.7. Since $PN \twoheadrightarrow_\beta L$ the subjects in part of the rest of D have to be reduced (by $(\beta\eta)$) in order to match τL. In this way one obtains a derivation D': $B \vdash^* \sigma\, M'$ with $M \twoheadrightarrow_\beta M'$ and $O(D') < O(D)$. Duplicated cuts have degree $<|\rho|$ (since ρZ is innermost); possibly created cuts have degree $|\sigma|$ or $|\tau| < |\sigma \to \tau| = |\rho|$. Some extra applications of $(\beta\eta)$ may be needed in D' but only if τPN is followed by $(\bigcap\mathrm{I})$. But then no cut will be longer.

If the length of ρZ is 1, then the argument is slightly simpler. ∎

4.9. Corollary (Normalization Theorem). *If $B \vdash^* \sigma M$, then there is a normal D and M' such that $M \twoheadrightarrow_\beta M'$ and D: $B \vdash^* \sigma M'$.*

Proof. Immediate by 4.8. ∎

4.10. Corollary (Conservativity). *Let B, σM be Curry statements. Then*

$$B \vdash \sigma M \Rightarrow B \vdash_C \sigma M.$$

Proof. By 4.2 and 4.9 there is a normal D: $B \vdash^* \sigma M'$ with $M \twoheadrightarrow_\beta M'$. By 4.5 D is good as a Curry derivation. Hence $B \vdash_C \sigma M'$. But then $B \vdash_C \sigma M$ by (EQ_β). ∎

4.11. Theorem (Completeness for Curry type assignment). *Let B, σM be Curry statements. Then*

$$B \vdash_C \sigma M \Leftrightarrow B \models \sigma M.$$

Proof.

$$\begin{aligned} B \vdash_C \sigma M &\Leftrightarrow B \vdash \sigma M && \text{by 3.8, 4.10} \\ &\Leftrightarrow B \models \sigma M && \text{by 3.10. } \blacksquare \end{aligned}$$

The extended types allow us to characterize terms having a normal form or head normal form; the proof follows [5].

4.12. Lemma. *Let D: $B \vdash^* \tau M$ with D normal, $\tau \not\sim \omega$ and τM not obtained by $(\to \mathrm{I})$ or $(\bigcap \mathrm{I})$ immediately followed by $k \geq 0$ applications of $(\beta\eta)$. Then M is of the form $xM_1 \cdots M_n$.*

Proof. Induction on D. The only interesting cases are when the last steps in D are $(\to \mathrm{E})$ or $(\bigcap \mathrm{E})$ followed by $k \geq 0$ applications of $(\beta\eta)$. If it is $(\to \mathrm{E})$ then τM comes from $\sigma \to \tau P$, σQ, with $\sigma \to \tau P$ either an assumption in B (then P is a variable); or the induction hypothesis applies ($\sigma \to \tau P$ is not obtained by $(\to \mathrm{I})$ since D is normal, nor by $(\bigcap \mathrm{I})$). The case $(\bigcap\mathrm{E})$ is treated similarly. ∎

4.13. Theorem. (i) *$\exists B \exists \tau \not\sim \omega\ B \vdash \tau M \Leftrightarrow M$ has a head normal form.*

(ii) *$\exists B, \tau\{B \vdash \tau M$ and ω not in $B, \tau\} \Leftrightarrow M$ has a normal form.*

Proof. (i) $(\Leftarrow)$ Induction on M.

$(\Rightarrow)$ Let $B \vdash \tau M$, $\tau \not\sim \omega$. By 4.2 and 4.3 there is a normal D: $B \vdash^* \tau M'$. Induction on D. If $\tau M'$ is obtained in D by $(\to \mathrm{E})$, then 4.12 applies. If it is obtained by $(\to \mathrm{I})$, $(\bigcap \mathrm{I})$ or $(\bigcap \mathrm{E})$, then the induction hypothesis applies.

(ii) Similarly. ■

REMARKS. (i) A semantical proof of 4.13(ii) using soundness and the model $P\omega$ is also possible.

(ii) It is easy to show that $\vdash \varphi_0 \to \varphi_0 M \Leftrightarrow M =_\beta \lambda x.x$. Therefore type assignment is a recursively enumerable but not recursive theory (in fact Σ_1^0-complete).

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