

Notes on Higher-Order Matching

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1 Bounded Parallel Intersection Type System

We denote *intersection types* by A, B , finite sets of intersection types by σ, τ , and *type atoms* by a, b .

Definition 1 (Intersection Types).
$$\begin{aligned} A, B &::= a \mid \sigma \rightarrow A \\ \sigma, \tau &::= \{A_1, \dots, A_n\} \text{ where } n \geq 0 \end{aligned}$$

An *environment*, denoted by Γ , is a finite set of *type assumptions* having the shape $x : \sigma$ for distinct term variables.

Definition 2 (Environment). $\Gamma ::= \{x_1 : \sigma_1, \dots, x_n : \sigma_n\}$ where $x_i \neq x_j$ for $i \neq j$.

We may extend an environment Γ by an additional assumption $x : \sigma$, written $\Gamma, x : \sigma$, where x does not appear in any assumption in Γ .

We denote vectors of types or environments by \bar{A} , $\bar{\sigma}$, or $\bar{\Gamma}$ respectively. If $\bar{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\bar{A} = (A_1, \dots, A_n)$, then $\bar{\sigma} \Rightarrow \bar{A} = (\sigma_1 \rightarrow A_1, \dots, \sigma_n \rightarrow A_n)$.

Borrowing notation [DU21] we define type vector transformations as follows.

Definition 3 (Type Vector Transformation). Let $f : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ be a surjective function.

- $f(A_1, \dots, A_n) = (C_1, \dots, C_m)$ such that $C_i = \{A_j \mid f(j) = i\}$
- $f^{-1}(B_1, \dots, B_m) = (D_1, \dots, D_n)$ such that $D_i = B_{f(i)}$

We tacitly extend the definition to environments where set formation is taken pointwise.

Definition 4 (Parallel Intersection Type System).

$$\begin{aligned} &\frac{}{\{x : \bar{A}\} \vdash x : \bar{A}} \text{ (Ax)} & \frac{\bar{\Gamma}, x : \bar{\sigma} \vdash t : \bar{A}}{\bar{\Gamma} \vdash \lambda x. t : \bar{\sigma} \Rightarrow \bar{A}} (\Rightarrow \text{I}) \\ &\frac{\bar{\Gamma} \vdash t : f(\bar{A}) \Rightarrow \bar{B} \quad \bar{\Delta} \vdash u : \bar{A}}{\bar{\Gamma} \cup f(\bar{\Delta}) \vdash t u : \bar{B}} (\Rightarrow \text{E}) \end{aligned}$$

Definition 5 (Dimensional Restriction). $\bar{\Gamma} \vdash_k t : \bar{A}$ means that the length of vectors in some derivation is at most k .

Theorem 6. $\Gamma \vdash_{\text{BCD}} t : A$ iff for some k we have $(\Gamma) \vdash_k t : (A)$.

Theorem 7. Given Γ , A , and k it is decidable whether $(\Gamma) \vdash_k t : (A)$ holds.

References

- [DU21] Andrej Dudenhefner and Pawel Urzyczyn. “Kripke Semantics for Intersection Formulas”. In: *ACM Trans. Comput. Log.* 22.3 (2021), 15:1–15:16. DOI: 10.1145/3453481. URL: <https://doi.org/10.1145/3453481>.