On Equivalence Classes of Interpolation Equations

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Abstract. An Interpolation Equation is an equation of the form $[(x)c_1 \dots c_n = b]^1$, where $c_1 \dots c_n$, b are simply typed terms containing no instantiable variable. A natural equivalence relation between two interpolation equations is the equality of their sets of solutions. We prove in this paper that given a typed variable x and a simply typed term b, the quotient by this relation of the set of all interpolation equations of the form $[(x)w_1 \dots w_p = b]$ contains only a finite number of classes, and relate this result to the general study of Higher Order Matching.

1 Introduction

Interpolation Equations are particular instances of the *Higher Order Matching* problem, which is the problem of determining, given two simply typed terms a and b, whether there exists a substitution σ such that $\sigma(a)$ and b normalize to the same term, or equivalently, the problem of solving the equation $a =_\beta b$ (written $[a = b]$) where b contains no instantiable variables.

The decidability of Higher Order Matching is still open. The Third Order Matching problem, or particular case of instantiable variables being of order at most three, has been proven decidable by Gilles Dowek in [4].

An interpolation equation is a matching problem of the form $[(x)c_1 \dots c_n = b]$, where $c_1 \dots c_n$, b are normal terms containing no instantiable variable, and b is of atomic type. The set of solutions of this equation is defined as the set of all terms t such that $(t)c_1 \dots c_n$ is well typed, and normalizes to b. A natural equivalence relation between two interpolation equations is the equality of their sets of solutions (in general, infinite). Write $\sim$ the relation thus defined. We prove in this paper the two following results:

1. Given a typed variable x and a term b, the quotient by the relation $\sim$ of the set of all interpolation equations of the form $[(x)w_1 \dots w_p = b]$ contains only a *finite* number of classes.

[1] we write $(u)v$ the application of u to v, and $(x)v_1 \dots v_n$ for $(\dots((x)v_1)v_2 \dots v_{n-1})v_n$

2. The decidability of the following problem implies the decidability of Higher Order Matching:
   "Given two finite sets of interpolation equations Φ and Ψ, determine whether there exists a term t such that for each $E \in \Phi$, t is a solution of E, for each $F \in \Psi$, t is not a solution of F."

We have proven in [5] and [6]: the decidability of this latter problem in two particular cases:

1. the case where all equations are at most fourth order. As a consequence, we get the decidability of Fourth Order Matching.
2. the case where all right members of the equations considered are first-order constants. As a consequence, we get the decidability of Atomic Matching (the problem of solving a finite set of equations whose right members are all first-order constants).

2 Terms

We assume that the reader is familiar with the notions of λ-term, β and η-reduction and type systems. These notions will not be redefined, and the reader is invited to refer to [3] or [1] for an introduction to these notions.

We first inductively define a set of types (starting from a finite set of type variables, the set of atomic types, and using the symbol $\rightarrow$ as a binary connective). Considering three kinds of typed terms variables - constant, local and instantiable - we build the set of Simply Typed Terms, following a given set of rules. The rules for λ-abstraction (third and fourth rules) are used in a special way, according to the following requirement: local variables are the only kind of variables that may be bound in the terms produced by these rules. In other words, a constant symbol or an instantiable variable, appearing in a considered term, is always free in this term.

2.1 Types

We first consider a language consisting of: a *finite* set of constants $\mathcal{O}$, and a binary connective $\rightarrow$. The set $\mathcal{T}$ of all formulas of this language is inductively defined as follows:

0) $\mathcal{O} \subset \mathcal{T}$.
1) $A, B \in \mathcal{T} \Rightarrow (A \rightarrow B) \in \mathcal{T}$.

We write $A_1 \ldots A_k \rightarrow A$ for $(A_1 \rightarrow (\ldots A_k \rightarrow A) \ldots)$. We call *order* of a formula the integer computed as follows:

0) $\text{Ord}\ (o) = 1 \quad \text{for } o \in \mathcal{O}$,
1) $\text{Ord}\ (A_1 \ldots A_k \rightarrow o) = \sup\ (\text{Ord}\ (A_1), \ldots, \text{Ord}\ (A_k)) + 1 \quad \text{for } o \in \mathcal{O}$.

2.2 Typed Variables

Given an infinite, countable set of variables $\mathcal{X} = \{x, y \ldots\}$, we consider an application from $\mathcal{X}$ to the set of formulas such that each element of $\mathcal{T}$ has an infinite number of antecedents. For each formula A in $\mathcal{T}$, we call set of *variables of type* A the (infinite) subset of $\mathcal{X}$ of all antecedents of A. We call *typed variables* all pairs of the form (x, B), written $x : B$, where B is the type of x.

From now on, we will deal with three particular sets of typed variables $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{I}$, called respectively set of *constants*, set of *local variables*, and set of *instantiable variables*, with the following properties:

$\mathcal{C}$, $\mathcal{L}$ and $\mathcal{I}$ are mutually disjoint;
$\mathcal{C}$ is finite;
$\mathcal{L}$ and $\mathcal{I}$ both contain an infinite number of variables of each type.

2.3 Simply Typed Terms

A *context* Γ is defined as a finite subset of the union of $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{I}$. Assuming that a typed variable $x : A$ is not already in Γ, we write $\Gamma, x : A$ for $\Gamma \cup \{x : A\}$.

Given a formula A, a context Γ, and a term t of pure λ-calculus (written with elements of $\mathcal{X}$ as variables), we define the notion "t is a simply typed (typable) term of type A, in the context Γ", written $\Gamma \vdash t : A$, by means of the following rules:

1) $x : A \vdash x : A$ for $x : A \in \mathcal{C}$, $\mathcal{L}$ or $\mathcal{I}$,
2) if $\Gamma \vdash u : A \to B$ and $\Gamma' \vdash v : A$ then $(\Gamma \cup \Gamma') \vdash (u)v : B$,
3) if $\Gamma, x : A \vdash u : B$ and $x : A \in \mathcal{L}$, then $\Gamma \vdash \lambda x.u : A \to B$,
4) if $\Gamma \vdash u : B$, $x : A \in \mathcal{L}$ and $x : A \notin \Gamma$, then $\Gamma \vdash \lambda x.u : A \to B$.

Since, for any variable x, there exists a unique type A such that $(x : A)$, an immediate induction on the length of proofs leads to the following result:

Proposition 1. *If a term t of pure λ-calculus is simply typable (in the sense defined above) then there exists a unique context Γ and a unique type A such that $\Gamma \vdash t : A$. If $\Gamma = \{x_1 : A_1, \ldots, x_k : A_k\}$ then $\{x_1, \ldots, x_k\}$ is the set of all variables free in t.*

Definition 2. For any simply typable term t, the set Γ and the formula A such that $\Gamma \vdash t : A$ will be called the *context of t* and the *type of t* respectively. We define the *order of t* as the order of its type.

Remark. The definition presented above is slightly different from the usual definition of simply typed terms (which can be found for instance in [4]), generally presented as follows:

i) $\Gamma, x : A \vdash^* x : A$ for $\Gamma, x : A$ included in the union of $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{I}$,

ii) if $\Gamma \vdash^* u : A \rightarrow B$ and $\Gamma \vdash^* v : A$ then $\Gamma \vdash^* (u)v : B$,

iii) if $\Gamma, y : A \vdash^* u : B$ and $y : A \in \mathcal{L}$, then $\Gamma \vdash^* \lambda y.u : A \rightarrow B$,

If $\Gamma \vdash^* t : A$, then for any $\Gamma^* \supset \Gamma$, $\Gamma^* \vdash^* t : A$, therefore the notion of "context of a typed term $t : A$" is not well defined in this system.

However, if $\Gamma \vdash t : A$, then $\Gamma \vdash^* t : A$ and conversely, if $\Gamma^* \vdash^* t : A$, then there exists a unique context $\Gamma \subset \Gamma^*$ such that $\Gamma \vdash t : A$. Thus, the well-known results of strong normalization of all typable terms and the stability of their typing under β-reduction hold for our presentation.

3 Reduction on Terms

We assume that the set of local variables $\mathcal{L}$ is split into two infinite subsets $\mathcal{A}$ and $\mathcal{P}$, each of these sets containing an infinite number of variables of each type. Elements of $\mathcal{A}$ will be called *active* variables, elements of $\mathcal{P}$ *passive* variables. In the following, these two kinds of variables will allow us to discern immediately in a given term the variables which cannot take part in the process of reduction of this term.

3.1 α-Equivalence, $\mathcal{S}$-Terms, Terms

We write $\equiv$ the α-equivalence on terms of λ-calculus. Renamings of bound variables in Simply Typed Terms are assumed to respect the kind (active or passive) and the type of the variables renamed.

Example 1. the variable $x \in \mathcal{A}$ of type B in $\lambda x.x : B \rightarrow B$ may only be renamed by an active variable of the same type. For any active variable $y : B$, $\lambda x.x : B \rightarrow B$ and $\lambda y.y : B \rightarrow B$ are α-equivalent. For $l \in \mathcal{P}$, $\lambda x.x$ and $\lambda l.l$ are not α-equivalent. For $z : D$ with $B \neq D$, $\lambda x.x : B \rightarrow B$ and $\lambda z.z : D \rightarrow D$ are not α-equivalent.

We let $\mathcal{S}$ be the set of Simply Typed Terms, and define $\overline{\mathcal{S}}$ as the quotient of this set by the α-equivalence ($\mathcal{S}/\equiv$). By convention, elements of $\overline{\mathcal{S}}$ and $\mathcal{S}$ will be called *terms* and *$\mathcal{S}$-terms* respectively. Greek letters shall be used to denote arbitrary $\mathcal{S}$-terms. An $\mathcal{S}$-term τ of the α-class (the term) t will be called *a representative* of t.

3.2 β-reduction

The definition of β-reduction used in this section is borrowed from [3]. The β-reduction on terms is the least binary relation β reflexive, transitive, and including the relation β_0 defined by the following rules:

0) if t is an element of $\mathcal{C}$, $\mathcal{L}$ or $\mathcal{I}$ then $t\,\beta_0\,t'$ is false for all t'.
1) if $t = \lambda z\, u$, then $t\,\beta_0\,t'$ if and only if $t' = \lambda z\, u'$ with $u\,\beta_0\,u'$.
2) if $t = (u)v$, then $t\,\beta_0\,t'$ if and only if: either $t' = (u')v$, with $u\,\beta_0\,u'$,
or $t' = (u)v'$, with $v\,\beta_0\,v'$,
or $u = \lambda x\, w$, $t' = w[u/x]$.

We let $\beta^* \subset \beta$ be the least binary relation reflexive, transitive, and including the relation β_0^* defined by:

i) β_0^* satisfies conditions (0) and (1).
ii) if $t = (u)v$, then $t\,\beta_0^*\,t'$ if and only if: either $t' = (u')v$, with $u\,\beta_0^*\,u'$,
or $t' = (u)v'$, with $v\,\beta_0^*\,v'$,
or $u = \lambda x\ w$, $x \in \mathcal{A}$, $t' = w[u/x]$.

3.3 β-Normal, η-Long Forms

Let $t = \lambda x_1 \ldots x_m.(x)u_1 \ldots u_p : A_1 \ldots A_n \to o$ (where $m \leq n$ and o is an atomic type i.e $o \in \mathcal{O}$) be a β-normal term. A *β-normal η-long form* of t is defined as a term of same type of the form

$$t' = \lambda x_1 \ldots x_m x_{m+1} \ldots x_n.(x)u'_1 \ldots u'_p x'_{m+1} \ldots x'_n$$

where u'_i is a β-normal η-long form of u_i, and x'_i is a β-normal η-long form of x_i. From now on, all normal terms will be supposed to be in β-normal η-long form.

Remark. By definition of α-equivalence, every β-normal term has only a finite number of η-long forms. Furthermore, if t, $v_1, \ldots, v_n$ are β-normal terms such that $(t)v_1 \ldots v_n$ is well-typed and first-order, $(t)v_1 \ldots v_n$ normalizes to b iff there exists t^*, $v_1^*, \ldots v_n^*$, b^*, η-long forms of t, $v_1, \ldots, v_n$, b respectively, such that $(t^*)v_1^* \ldots v_n^*$ normalizes to b^*. Therefore, we may restrict whitout loss of generality the set of normal terms to the set of β-normal η-long forms.

3.4 Restriction of the Set of Terms

For $t \in \overline{\mathcal{S}}$, the notation $t = \lambda y_1 \ldots y_n.u$ supposes: for every representative ν of u, $\lambda y_1 \ldots y_n.\nu$ is a representative of t; the variables $y_1, \ldots, y_n$ are distinct; the term u is first-order. For $\mathcal{Y} = (y_1, \ldots, y_n)$, we write $\lambda\mathcal{Y}.u$ for $\lambda y_1 \ldots y_n.u$.

In the remaining, we will focus on a particular subset of $\overline{\mathcal{S}}$, the set $\overline{\mathcal{S}}_0$ defined by the following rules:

0) for every $x : o$ in $\mathcal{C}, \mathcal{A}, \mathcal{P}$ or $\mathcal{I}$ with o atomic, $x \in \overline{\mathcal{S}}_0$.
1) let $u : o \in \overline{\mathcal{S}}_0$ with o atomic. For every sequence of active variables $(y_1 : A_1, \ldots, y_n : A_n)$, $\lambda y_1 \ldots y_n.u : A_1 \ldots A_n \to o \in \overline{\mathcal{S}}_0$.
2) let $v_1 : A_1, \ldots, v_n : A_n \in \overline{\mathcal{S}}_0$. For every $x : A_1 \ldots A_n \to o$ in $\mathcal{A}$ or $\mathcal{I}$, $(x)v_1 \ldots v_n : o \in \overline{\mathcal{S}}_0$.

3) let $u_1 : \circ_1, \ldots, u_n : \circ_n \in \overline{\mathcal{S}}_0$ with $\circ_1, \ldots, \circ_n$ atomic. Let $\mathcal{K}_1, \ldots, \mathcal{K}_n$ be finite sequences of passive variables. Let $A_1, \ldots, A_n$ be the types of $\lambda\mathcal{K}_1.u_1, \ldots, \lambda\mathcal{K}_n.u_n$.
   For every $K : A_1 \ldots A_n \to \circ$ in $\mathcal{C}$ or $\mathcal{P}$, $(K)\lambda\mathcal{K}_1.u_1 \ldots \lambda\mathcal{K}_n.u_n \in \overline{\mathcal{S}}_0$.
4) let $w_0 : A_1 \ldots A_n \to \circ$, $w_1 : A_1 \ldots w_n : A_n \in \overline{\mathcal{S}}_0$.
   $(w_0)w_1 \ldots w_n : \circ \in \overline{\mathcal{S}}_0$.

Note that all normal terms in $\overline{\mathcal{S}}_0$ are in β-normal η-long form and conversely, for any $\mathcal{S}$-term τ in β-normal η-long form, there exists a unique renaming ρ of bound variables in τ (which may require to change the kind of the variables renamed) such that the class of $\rho(\tau)$ is an element of $\overline{\mathcal{S}}_0$.

Thus, we may assume without loss of generality that every term on β-normal η-long form is an element of $\overline{\mathcal{S}}_0$.

Remark. A term in $\overline{\mathcal{S}}_0$ of non-atomic type may only be obtained by application of rule 1, that is to say, if $t : B_1 \ldots B_n \to \circ \in \overline{\mathcal{S}}_0$, then there exists an $\mathcal{A}$-sequence $(x_1 : B_1, \ldots, x_n : B_n)$ and a first order $u : \circ \in \overline{\mathcal{S}}_0$ such that $t = \lambda x_1 \ldots x_n.u$.

Remark. The β^*-reduction on α-equivalence classes forbids the reduction of a redex of the form $(\lambda l.u)v$ where l is a passive variable. However, the following lemma proves that this relation is sufficient to reduce (in the usual sense) all non-normal elements of $\overline{\mathcal{S}}_0$.

Proposition 3. *Let $u{:}A$, $v{:}B \in \overline{\mathcal{S}}_0$. For any $x{:}B \in \mathcal{A}$ or $\mathcal{I}$, $u[v/x]{:}A \in \overline{\mathcal{S}}_0$.*

Proof. Straightforward induction on the number of rules used in the proof of $u : A \in \overline{\mathcal{S}}_0$. Note that the conclusion does not hold if we allow x to be in $\mathcal{C}$ or $\mathcal{P}$.

Lemma 4. *Let $t : A \in \overline{\mathcal{S}}_0$. If $t\,\beta_0\,t'$ then $t\,\beta_0^*\,t'$ and $t' : A \in \overline{\mathcal{S}}_0$.*

Proof. Induction on the number of rules used in the proof of $t : A \in \overline{\mathcal{S}}_0$. The cases of rules 1, 2 and 3 are immediate, so we only treat in details the case of rule 4, $t = (\lambda x \lambda y_1 \ldots y_n.u)v\, v_1 \ldots v_n : \circ$ with $t' = (\lambda y_1 \ldots y_n.u[v/x])v_1 \ldots v_n$ (we may assume that $y_1, \ldots, y_n$ are not free in v). Since $\lambda x \lambda y_1 \ldots y_n.u : A \to B$ is of higher order, x, $y_1, \ldots, y_n$ are necessarily active variables hence $t\,\beta_0^*\,t'$. By the preceding proposition, $\lambda y_1 \ldots y_n.u[v/x] : B \in \overline{\mathcal{S}}_0$. By rule 4, $t' : \circ \in \overline{\mathcal{S}}_0$. □

Lemma 5. *The Church-Rosser property holds for the β-reduction on simply typed terms (and in particular, for the β^*-reduction on $\overline{\mathcal{S}}_0$)*

Proof. See for instance [2]. □

Through sections 4 and 5, the set of terms will be restricted to $\overline{\mathcal{S}}_0$. We will call β-reduction the β^*-reduction on terms and β_0 the relation β_0^*. We will write $\simeq$ the β^*-equivalence.

4 Pattern Matching

Definition 6. A *matching problem* is by definition a finite set of equations of the form $[a = b]$, where a, b are normal terms and b contains no instantiable variable. A *solution* of a matching problem Ψ is a finite substitution σ on the set of instantiable variables free in Ψ to the set of normal terms, such that for each equation $[a = b] \in \Psi$, $\sigma(a)$ normalizes to b. We call *order of* Ψ the maximal order of an instantiable variable in this problem.

Note that we can assume without loss of generality that Ψ consists of a single equation of first order members: from $\{[a_1 = b_1], \ldots, [a_m = b_m]\}$, we construct the matching equation $[(K)a_1 \ldots a_m = (K)b_1 \ldots b_m]$, where K is a new constant of adequate type. Obviously, Ψ and this equation have the same set of solutions.

Definition 7. An *interpolation equation* E of *arguments* $(c_1, \ldots, c_n)$, of *result* b is by definition a matching equation of the form $[(x)c_1 \ldots c_n = b : \circ]$ where x is instantiable, b is of atomic type, $c_1, \ldots, c_n$ (and by definition, b) contain no instantiable variable. A *solution* of E is a normal term t such that $(t)c_1 \ldots c_n$ is well-typed and normalizes to b.

Two interpolation equations E and E' will be called *equivalent*, written $E \sim E'$, if and only if they have the same set of solutions.

Definition 8. We call *interpolation problem* any finite set of interpolation equations. A *dual problem* (Φ, Ψ) is by definition a pair of interpolation problems whose equations contain the same instantiable variable. A *solution* of (Φ, Ψ) is a normal term t such that for each $E \in \Phi$, t is a solution of E, and for each $F \in \Psi$, t is a not a solution of F.

4.1 Accessible Contexts

From now on, we allow the constants appearing in $\overline{\mathcal{S}}_0$-terms to be either in $\mathcal{C}$, or in a new separated set of constants, the set *Nil*: this set contains, for each atomic type $\circ$, a new element $nil_\circ$ of type $\circ$. As seen in the following, we do not need to explicitly differentiate these constants, i.e we will write *nil* all elements of *Nil*. For any set of variables $\mathcal{Z} = \{z_1 : A_1, \ldots, z_n : A_n\}$, and for any term u, we will write $u\,[\mathcal{Z} \leftarrow Nil]$ the term $u\,[\lambda\mathcal{X}_1.nil : A_1/z_1 \ldots \lambda\mathcal{X}_n.nil : A_n/z_n]$ (where all elements of $\mathcal{X}_1, \ldots, \mathcal{X}_n$ are active variables of expected type).

Lemma 9. *Let $u \in \overline{\mathcal{S}}_0$. For any set of variables $\mathcal{Z}$, there exists a term $v \in \overline{\mathcal{S}}_0$ such that $u' = u[\mathcal{Z} \leftarrow Nil]\,\beta\, v$.*

Proof. Straightforward induction on the number of rules in the proof of $u \in \overline{\mathcal{S}}_0$. The only non-trivial case is $u = (K)\lambda\mathcal{K}_1.u_1 \ldots \mathcal{K}_p.u_p$ with $K \in \mathcal{Z}$, $K \in \mathcal{P}$. In this case, $u' = (\lambda\mathcal{X}.nil)\lambda\mathcal{K}.u'_1 \ldots \lambda\mathcal{K}_p.u'_p \notin \overline{\mathcal{S}}_0$. As $\mathcal{X} \subset \mathcal{A}$, $u'\,\beta\, nil \in \overline{\mathcal{S}}_0$. □

Lemma 10. *Let Δ be any context disjoint from Nil. Call set of* accessible contexts *the set of all subsets of* $(\Delta \cup Nil)$.

i) Let b be a normal term such that $\Delta \vdash b : o$ with o atomic. Let E be any interpolation equation of result b. There exists an equation E' equivalent to E, whose arguments are of accessible context.

ii) Let $b_1, \ldots, b_m$ be normal terms of same atomic type o such that $\Delta \vdash b_1, \ldots, b_m$. Let $\Phi = \{[a_1 = b_1], \ldots, [a_m = b_m]\}$ be any interpolation problem. Let Ψ be any interpolation problem such that the result of each element of Ψ contains no free element of Nil. If (Φ, Ψ) has a solution, then (Φ, Ψ) has a solution of accessible context.

Proof. We write $\Gamma \vdash c_1, \ldots, c_n$ the relation "$c_1, \ldots, c_n$ are normal terms elements of $\overline{S}_0$, and the union of their contexts is included in Γ". Remark that $u \,\beta_0\, v$ implies $u[w/x] \,\beta_0\, v[w/x]$. If x is not free in v, then $u[w/x] \,\beta_0\, v$. Hence,

i) suppose E is of the form $[(x)c_1 \ldots c_n = b]$, with $\Gamma, \Delta \vdash c_1, \ldots, c_n$. For each i, let $c'_i = c_i[\Gamma \leftarrow Nil]$. For every t, $(t)c_1 \ldots c_n \,\beta\, b$ if and only if $(t)c'_1 \ldots c'_n \,\beta\, b$, that is to say, E and $E' = [(x)c'_1 \ldots c'_n = b]$ are equivalent.

ii) Let t be any solution of (Φ, Ψ). Suppose $\Gamma, \Delta_0 \vdash t : A$, where $\Delta_0 \subset \Delta$ and Γ contains no element of Δ. Let $t' = t[\Gamma \leftarrow Nil]$. Let $E = [(x)c_1 \ldots c_n = b_j]$ be any element of Φ. Since no element of Γ is free in b_j, t' is still a solution of E. Let $[(x)d_1 \ldots d_n = e]$ be any element of Ψ. Since e contains no element of *Nil*, t' is still not a solution of F. In other words, t' is still a solution of (Φ, Ψ). □

5 Equivalence Classes of Equations

The aim of this section is to prove the following result :

> Let b be a normal term such that $\mathcal{C} \vdash b : o$ with o atomic. Let A be any type. The quotient by $\sim$ of the set of all interpolation equations of the form $[(x)w_1 \ldots w_n = b]$ where x is of type A contains only a finite number of classes.

As a corollary of this result, we will prove that the decidability of Dual Interpolation implies the decidability of Pattern Matching.

5.1 Characterization Theorem

We give in this section a necessary and sufficient condition on t, t' in $[(x)t = b]$, $[(x)t' = b]$, so that these two equations are equivalent.

Definition 11. We assume the existence of a computable function *Rep* that given a term w, returns a representative ε of w such that for any z, z is not simultaneously free and bounded in ε, and "λz" appear at most once in ε.

Definition 12. For any normal term b such that $\mathcal{C} \vdash b : \circ$ with $\circ$ atomic, we write $RSub(b)$ the set of α-equivalence classes of all first order subterms of $Rep(b)$. Remark that as b is normal, first order and of constant context, all free variables in the elements of $RSub(b)$ are in the union of $\mathcal{C}$ and $\mathcal{P}$.

Definition 13. Let S be a finite set of terms, and let t, t' be two terms of same type $A = A_1 \dots A_n \to \circ$. We will say that t and t' are *parallel on* S (or S*-parallel*) if and only if $\forall v_1 : A_1, \dots, \forall v_n : A_n, \forall s \in S, (t)v_1 \dots v_n \,\beta\, s \Leftrightarrow (t')v_1 \dots v_n \,\beta\, s$.

Remark that S-parallelism is an equivalence relation. In the particular case of $n = 0$, t and t' are S-parallel if and only if either $t \notin S$ and $t' \notin S$, or t and t' are both in S and in that case, are equal terms.

Proposition 14. *Let b be a normal term such that $\mathcal{C} \vdash b : \circ$ with $\circ$ atomic. let $S = RSub(b)$. Let $\diamond$ be any atomic type.*

Let $(w : B_1 \dots B_p \to \diamond)$, $(e_1 : B_1), \dots, (e_p : B_m)$ be arbitrary terms. Let $t : A$ and $t' : A$ be two S-parallel terms of constant context. Let $z : A$ be a fresh active variable. Then $(w[t/z])e_1 \dots e_p$ and $(w[t'/z])e_1 \dots e_p$ are S-parallel terms.

Proof. We fix t and t', and prove the result by induction on P, and for each P, by induction on N, where P is the sum of the length of all normalizations of $(w[t/z])e_1 \dots e_p$, N is the number of rules used in the proof of $w \in \overline{\mathcal{S}}_0$. Since all terms in $\overline{\mathcal{S}}_0$ are strongly normalizing, this induction is well founded. We consider the last rule used in the proof of $w \in \overline{\mathcal{S}}_0$.

If it is rule 0, $w = x$ is a first order variable. Either x and z are distinct and $w[t/z] = w[t'/z] = x$, or $x = z$ and $w[t/z] = t$, S-parallel to $w[t'/z] = t'$.

If it is rule 1, w is of the form $\lambda x_1 \dots x_p.v$ with $p \neq 0$. We assume that z and $x_1, \dots, x_p$ are distinct. Since z is not free in $e_1 \dots e_p$, $(w[t/z])e_1 \dots e_p$ β-reduces to $v[t/z][e_1/x_1 \dots e_p/x_p] = v[t/z][\mathbf{e}/\mathbf{x}] = v[\mathbf{e}/\mathbf{x}][t/z]$. The sum of the length of all normalizations of this latter term is at most $(P - p)$ hence, by induction hypothesis, $(w[t/z])e_1 \dots e_n \,\beta\, v[\mathbf{e}/\mathbf{x}][t/z]$ and $v[\mathbf{e}/\mathbf{x}][t'/z] \simeq (w[t'/z])e_1 \dots e_p$ (recall that the symbol $\simeq$ stands for the β-equivalence) are S-parallel terms.

If it is rule 2, w is of the form $(x)v_1 \dots v_m$, with $m \neq 0$ and $x \in \mathcal{A}$. Note that, as x is an active variable, if x is not equal to z then $w[t/z]$ and $w[t'/z]$ cannot normalize in S. Suppose w of the form $(z)v_1 \dots v_m$ and t of the form $\lambda y_1 \dots y_m.t_0$. Then $w[t/z] = (t)v_1[t/z] \dots v_m[t/z] \;\beta\; t_0[v_1[t/z]/y_1 \dots v_m[t/z]/y_m] = t_0[v_1/y_1 \dots v_m/y_m][t/z] = t_0[\mathbf{v}/\mathbf{y}][t/z]$. The sum of the length of all normalizations of this latter term is at most $(P - m)$ hence, by induction hypothesis, $(t)v_1[t/z] \dots v_m[t/z] \simeq t_0[\mathbf{v}/\mathbf{y}][t/z]$ and $t_0[\mathbf{v}/\mathbf{y}][t'/z] \simeq (t)v_1[t'/z] \dots v_m[t'/z]$ are S-parallel terms. Furthermore, as t and t' are S-parallel, $(t)v_1[t'/z] \dots v_m[t'/z]$ and $(t')v_1[t'/z] \dots v_m[t'/z] = w[t'/z]$ are S-parallel terms.

If it is rule 3, w is of the form $(K)\lambda\mathcal{K}_1.u_1 \dots \lambda\mathcal{K}_m.u_m$, with $K \in \mathcal{C}$ or $\mathcal{P}$. Suppose for instance that $w[t/z]$ normalizes to $s \in S$. In this case, we may assume that $\mathcal{K}_1, \dots, \mathcal{K}_m$ are such that $s = (K)\lambda\mathcal{K}_1.s_1 \dots \lambda\mathcal{K}_m.s_m$ with $s_1, \dots, s_m \in S$.

Then $w[t/z] = (K)\lambda\mathcal{K}_1.u_1[t/z]\ldots\lambda\mathcal{K}_m.u_m[t/z] \simeq (K)\lambda\mathcal{K}_1.s_1\ldots\lambda\mathcal{K}_m.s_m$. By induction hypothesis on N, for each j, $u_j[t/z] \simeq u_j[t'/z] \simeq s_j$ therefore $w[t/z] \simeq (K)\lambda\mathcal{K}_1.s_1\ldots\lambda\mathcal{K}_m.s_m \simeq w[t'/z]$. The converse hypothesis ($w[t'/z] \simeq s' \in S$) leads to a similar conclusion.

If it is rule 4, w is of the of the form $(\lambda x_1...x_n.u)v_1...v_n$ with $n \neq 0$. We assume that $x_1, ..., x_n$, z are distinct. Then $w[t/z]\ \beta\ u[t/z][v_1[t/z]/x_1...v_n[t/z]/x_n] = u[v_1/x_1\ldots v_n/x_n][t/z] = u[\mathbf{v}/\mathbf{x}][t/z]$. The sum of the length of all normalizations of this latter term is at most $(P-n)$ hence, by induction hypothesis, $w[t/z] \simeq u[\mathbf{v}/\mathbf{x}][t/z]$ and $u[\mathbf{v}/\mathbf{x}][t'/z] \simeq w[t'/z]$ are S-parallel terms. □

Theorem 15. *Let b be a normal term such that $C \vdash b : \circ$ with $\circ$ atomic. Let t, t' be two terms of constant context. The equations $[(x)t = b]$ and $[(x)t' = b]$ are equivalent if and only if t and t' are parallel on $RSub(b)$.*

Proof. Suppose t and t' $RSub(b)$-parallel. By the proposition 14, for every u, $u[t/z] \simeq b \Leftrightarrow u[t'/z] \simeq b$, thus $(\lambda z.u)t \simeq b \Leftrightarrow (\lambda z.u)t' \simeq b$ i.e. $[(x)t = b]$ and $[(x)t' = b]$ are equivalent. Conversely, suppose for instance that for $v_1\ldots v_n$, $(t)v_1\ldots v_n \simeq s \in RSub(b)$ and $(t')v_1\ldots v_n \simeq s' \neq s$. Let $\mathcal{X}_1\ldots\mathcal{X}_p$ be the sequence of symbols ("λ", "(", ")" or a typed variable) equal to $\varepsilon = Rep(b)$. Let j, k be such that $\mathcal{X}_j\ldots\mathcal{X}_{j+k} = \varepsilon_0$, representative of s. Replace this subsequence in ε by a representative of $(z)v_1\ldots v_n$. Call b^* the α-class of the sequence of symbols thus defined. As t is of constant context, no bounded variable in ε is free in t hence, there exists a representative of $b^*[t/z]$ of the form $\mathcal{X}_1\ldots\mathcal{X}_{j-1}\,\kappa\,\mathcal{X}_{j+k+1}\ldots\mathcal{X}_p$ where κ is a representative of $(t)v_1\ldots v_n$. The normal form of $b^*[t/z]$ has a representative of the form $\mathcal{X}_1\ldots\mathcal{X}_{j-1}\,\varepsilon_0\,\mathcal{X}_{j+k+1}\ldots\mathcal{X}_p$ i.e. is equal to b. As t' is of constant context, no bounded variable in ε is free in t' hence, there exists a representative of $b^*[t'/z]$ of the form $\mathcal{X}_1\ldots\mathcal{X}_{j-1}\,\kappa'\,\mathcal{X}_{j+k+1}\ldots\mathcal{X}_p$ where κ' is a representative of $(t')v_1\ldots v_n$. The normal form of $b^*[t'/z]$ has a representative of the form $\mathcal{X}_1\ldots\mathcal{X}_{j-1}\,\varepsilon_0'\,\mathcal{X}_{j+k+1}\ldots\mathcal{X}_p$ where ε_0' is a representative of $s' \neq s$ i.e. is distinct from b. Thus, $\lambda z.b^*$ is a solution of $[(x)t = b]$ and $[(x)t' \neq b]$, i.e. these equations are not equivalent. □

5.2 Specifying the Context of Solutions

We may add some new equations to a dual problem (Φ, Ψ) in order to forbid a particular set of variables from appearing in every solution of accessible context of the new problem. Consider, for instance, $E = [(x)A = A]$, where A is a first order constant. This interpolation equation has only two solutions, $\lambda y.y$ and $\lambda y.A$. Let $F = [(x)B = B]$, with $B \neq A$. The only solution of $\{E, F\}$, $\lambda y.y$, does not contain A. The following proposition generalizes this simple example.

Definition 16. We will say that two ordered sets of variables $\mathcal{Z}$, $\mathcal{Z}'$ are in one to one corespondance if and only if they are of the form $\mathcal{Z} = (z_1 : A_1, \ldots, z_n : A_n)$ and $\mathcal{Z}' = (z_1' : A_1, \ldots, z_n' : A_n)$. In this case, we write $[\mathcal{Z}'/\mathcal{Z}]$ the substitution $[z_1'/z_1\ldots z_n'/z_n]$.

Lemma 17. *Let s be a normal term such that $\mathcal{C}_0, \mathcal{K} \vdash s : \diamond$ with $\mathcal{C}_0 \subset \mathcal{C}$, $\mathcal{K} \subset \mathcal{P}$ and $\diamond$ atomic. Let $\mathcal{K}^*$ be a new subset of $\mathcal{P}$ in one to one correspondance with $\mathcal{K}$. Let $E = [(x)v_1 \ldots v_n = s]$ be an interpolation equation. Let $E^* = E[\mathcal{K}^*/\mathcal{K}] = [(x)v_1^* \ldots v_n^* = s^*]$. Then,*

i) $\forall w$, w is a solution of $\Phi = \{E, E^*\} \Leftrightarrow w[\mathcal{K}\mathcal{K}^* \leftarrow Nil]$ is a solution of Φ;

ii) for any t of constant context, t is a solution of $E \Leftrightarrow t$ is a solution of E^*.

Proof. i) As s and s^* contain no element of Nil, if $w[\mathcal{K}\mathcal{K}^* \leftarrow Nil]$ is a solution of Φ then w is a solution of Φ. Conversely, let w be any solution of Φ. Suppose for instance that $w[\mathcal{K} \leftarrow Nil]$ is not a solution of Φ. Let $\mathcal{Z}$ be a new subset of $\mathcal{P}$ in one to one correspondance with $\mathcal{K}$. Let $w_0 = w[\mathcal{Z}/\mathcal{K}]$. let s_0, s_0^* be the normal forms of $(w_0)v_1 \ldots v_n$ and $(w_0)v_1^* \ldots v_n^*$ respectively. Then at least one element of $\mathcal{Z}$ is free in s_0 or s_0^*. Otherwise,
$(w[\mathcal{K} \leftarrow Nil])v_1 \ldots v_n = (w_0)v_1 \ldots v_n[\mathcal{Z} \leftarrow Nil] \ \beta \ s_0[\mathcal{Z} \leftarrow Nil] = s_0$,
$s_0 = s_0[\mathcal{K}/\mathcal{Z}] \simeq (w_0)v_1 \ldots v_n[\mathcal{K}/\mathcal{Z}] = (w)v_1 \ldots v_n \ \beta \ s$, and
$(w[\mathcal{K} \leftarrow Nil])v_1^* \ldots v_n^* = (w_0)v_1^* \ldots v_n^*[\mathcal{Z} \leftarrow Nil] \ \beta \ s_0^*[\mathcal{Z} \leftarrow Nil] = s_0^*$,
$s_0^* = s_0^*[\mathcal{K}/\mathcal{Z}] \simeq (w_0)v_1^* \ldots v_n^*[\mathcal{K}/\mathcal{Z}] = (w)v_1^* \ldots v_n^* \ \beta \ s^*$, a contradiction.
Since $(w[\mathcal{Z}/\mathcal{K}])v_1 \ldots v_n \simeq s_0 \Leftrightarrow (w[\mathcal{Z}/\mathcal{K}])v_1^* \ldots v_n^* \simeq s_0[\mathcal{K}^*/\mathcal{K}]$, $s_0^* = s_0[\mathcal{K}^*/\mathcal{K}]$. Hence for any $z \in \mathcal{Z}$, z is free in s_0 iff z is free in s_0^*. As $s^* = s_0^*[\mathcal{K}/\mathcal{Z}]$, we conclude that s^* contains a $\mathcal{K}$-occurrence, a contradiction. The proof of "$w[\mathcal{K}^* \leftarrow Nil]$ is a solution of Φ" is symmetrical.

(ii) Indeed, $(t)v_1 \ldots v_n \, \beta \, s \Leftrightarrow ((t)v_1 \ldots v_n)[\mathcal{K}^*/\mathcal{K}] = (t)v_1^* \ldots v_n^* \, \beta \, s[\mathcal{K}^*/\mathcal{K}] = s^*$.
□

5.3 Finiteness Lemma

Preliminaries

Definition 18. For any context Δ disjoint from Nil, and for any type A, we write $Terms(A, \Delta, Nil)$ the (in general infinite) subset of $\overline{\mathcal{S}}_0$ of all normal terms of type A, of context included in the union of Δ and Nil.

For any normal term s such that $\Delta \vdash s : \diamond$ where $\diamond$ is atomic, we write $Equ(A, s)$ the set of all interpolation equations of the form $[(x)c_1 \ldots c_n = s]$, where x is an instantiable variable of type A and for $A = A_1 \ldots A_n \rightarrow \diamond$, each c_i is an element of $Terms(A_i, \Delta, Nil)$.

Lemma 19. *Let Δ be any context disjoint from Nil. Let s be any normal term such that $\Delta \vdash s : \diamond$, and let A be any type. The cardinal of $(Equ(A, s)/ \sim)$ is equal to the cardinal of the quotient by $\sim$ of the set of all interpolation equations of the form $[(x)w_1 \ldots w_n = s]$, where x is of type A.*

Proof. Clear, by lemma 10. For any interpolation equation $E = [(x)w_1 ... w_n = s]$, there exists in $Equ(A, s)$ an equation equivalent to E.

Proposition 20. *For $E = [(x)c_1 \ldots c_n = b]$, $F = [(x)d_1 \ldots d_n = b]$, E and F are equivalent if and only if for each i, $[(z)c_i = b]$ and $[(z)d_i = b]$ are equivalent.*

Proof. By induction on n. Suppose E and F equivalent, $[(z)c = b]$ and $[(z)d = b]$ equivalent. For $\lambda y y_1 \ldots y_n.u = \lambda y \mathcal{Y}.u$, assume that $y, y_1, \ldots, y_n$ are not free in $c, d, c_1 \ldots c_n, d_1 \ldots d_n$. Then $(\lambda y \mathcal{Y}.u) c\, c_1 \ldots c_n\, \beta\, b \Leftrightarrow (\lambda \mathcal{Y}.u[c/y]) c_1 \ldots c_n\, \beta\, b \Leftrightarrow (\lambda \mathcal{Y}.u[c/y]) d_1 \ldots d_n\, \beta\, b$ (as E and F are equivalent) $\Leftrightarrow (\lambda y.u[d_1/y_1 \ldots d_n/y_n]) c\, \beta\, b$ $\Leftrightarrow (\lambda y.u[d_1/y_1 \ldots d_n/y_n]) d\, \beta\, b$ (as $[(z)c = b]$ and $[(z)d = b]$ are equivalent) $\Leftrightarrow (t) d\, d_1 \ldots d_n\, \beta\, b$ □

Key Lemma From now on, we fix an enumeration of the set of all terms, the set of all variables, vand the set of all interpolation equations.

Lemma 21. *Let N be an arbitrary order.*

1) Let b be any normal term such that $\mathcal{C} \vdash b : \circ$ with $\circ$ atomic. For any type $A = A_1 \ldots A_n \to \circ$ of order at most N, the quotient by the relation $\sim$ of the set

$Equ(A, b) = \{ [(x)c_1 \ldots c_n = b] \mid x : A \text{ and } \forall i\, c_i \in Terms(A_i, \mathcal{C}, Nil)\}$

contains a finite number of classes

2) There exists a function Ω_N satisfying the two following properties

> *i) For any type of A of order at most N, and for any normal $s : \diamond$ of context included in the union of $\mathcal{C}$ and $\mathcal{P}$ with $\diamond$ atomic, $\Omega_N(A, s)$ contains a unique representative of each class in $(\text{Equ}(A, s)/\sim)$.*
>
> *ii) If Dual Interpolation of order $(N-1)$ is decidable, then Ω_N is computable.*

Proof. By induction on N. The case $N = 1$ is immediate, since for any $x : \circ \in \mathcal{I}$, the set $Equ(\circ, b)$ contains a unique equation of instantiable variable x, $[x = b]$. Suppose $N > 1$.

1) Let $S = RSub(b)$. Let $\mathcal{K}$ be the set of all passive variables free in the elements of S. Let $\mathcal{K}^*$ be a new subset of $\mathcal{P}$ in one to one correspondance with $\mathcal{K}$. For any type D of order at most $(N-1)$, we define the finite set of dual problems $Car(D, b)$ as follows:

> Let z be the first instantiable variable of type D. For each $s \in S$, for each $E \in \Omega_{N-1}(D, s)$, let $E^* = E[\mathcal{K}^*/\mathcal{K}]$. Define $P = Car(D, b)$ as the finite set of all dual problems (Φ, Ψ) of instantiable variable z satisfying
>
> - $\Phi \cup \Psi = \{E \mid \exists s \in S, E \in \Omega_{N-1}(D, s)\} \cup \{E^* \mid \exists s \in S, E \in \Omega_{N-1}(D, s)\}$,
>
> - for any $s \in S$, for any $E \in \Omega_{N-1}(D, s)$, $\{E, E^*\} \subset \Phi$ or $\{E, E^*\} \subset \Psi$.

Remark that for any $s \in S$ and for $\mathcal{K}_0$ defined as a new set of constants in one to one correspondance with $\mathcal{K}$, $s_0 = s[\mathcal{K}_0/\mathcal{K}]$ is of constant context, $(Equ(D, s)/\sim)$

and $(Equ(D, s_0)/\sim)$ have same cardinal. Therefore, by induction hypothesis, for each s in S, $(Equ(D, s)/\sim)$ and thereby $\Omega_{N-1}(D, s)$ are finite sets. Hence, $Car(D, b)$ is a finite set.

All elements of $Car(D, b)$ are then dual problems of order at most $(N-1)$. We let $SDual(D, b)$ be the least finite set containing, for each element (Φ, Ψ) in $Car(D, b)$ *which has a solution*[2], the first solution of this problem of minimal context. Then,

- Let (Φ, Ψ) be any element of $Car(D, b)$. By lemma 10 (ii), all solutions of (Φ, Ψ) of minimal context are of context included in the union of $\mathcal{C}$, $\mathcal{K}$, $\mathcal{K}^*$ and *Nil*. By lemma 17 (i) and by definition of *Car*, there is no solution of (Φ, Ψ) which is at once of minimal context and containing a free element of $(\mathcal{K} \cup \mathcal{K}^*)$. Therefore, all elements of $SDual(D, b)$ are of context included in the union of $\mathcal{C}$ and *Nil*.
- If (Φ, Ψ), (Φ', Ψ') are two distinct elements of $Car(D, b)$, then there exists at least one pair E, E^* in Φ which is in Ψ', or (if Φ is empty) at least one pair in Φ' which is in Ψ. By lemma 17 (ii), for any term c of constant context, c is a solution of E if and only if c is a solution of E^*, hence c cannot be at once a solution of (Φ, Ψ) and (Φ', Ψ'). Therefore, for any term c of constant context, of type D, there exists, a unique dual problem in $Car(D, b)$ of which c is a solution.
- For any $s \in RSub(b)$, for any $F = [(z)w_1 \dots w_n = s]$ in $Equ(D, s)$, there exists in $\Omega_{N-1}(D, s)$ an equation $[(z)v_1 \dots v_n = s]$ equivalent to F. Therefore, for any terms t, t' of constant context and of type D,

$$
\begin{aligned}
& t,\ t' \text{ are solutions of the same problem } (\Phi, \Psi) \in Car(D, b),\\
\Leftrightarrow\ & \text{for every } s \in S, \text{ for every } F \in Equ(D, s),\\
& t \text{ is a solution of } F \text{ if and only if } t' \text{ is a solution of } F,\\
\Leftrightarrow\ & t \text{ and } t' \text{ are } S\text{-parallel terms (by definition of parall ``}\\
\Leftrightarrow\ & [(x)t = b] \text{ and } [(x)t' = b] \text{ are equivalent (by theorem 15).}
\end{aligned}
$$

Hence, the finite set $\{[(x)t = b] \mid t \in SDual(D, b)\}$ contains a unique representative of each class in $(Equ(D \to \circ, b)/\sim)$ i.e. this latter set contains a finite number of classes. By proposition 20, for $A = D_1 \dots D_n \to \circ$ the finite set $\{[(x)t_1 \dots t_n = b] \mid \forall i, t_i \in SDual(D_i, b)\}$ contains for each class in $(Equ(A, b)/\sim)$, a unique representative of this class i.e. this latter set contains only a finite number of classes.

2) i) We may extend the function Ω_{N-1} to Ω_N by the following definition:

> Let s be a normal term such that $\mathcal{C}_0, \mathcal{K} \vdash s : \diamond$ with $\mathcal{C}_0 \subset \mathcal{C}$, $\mathcal{K} \subset \mathcal{P}$ and $\diamond$ atomic. Let $\mathcal{K}_0$ be a new set of constants in one to one correspondance with $\mathcal{K}$. For any type $A = D_1 \dots D_n \to \diamond$ of order N,
>
> let $\Omega_N(A, s) = \{[(x)t_1 \dots t_n = s] \mid \forall i, t_i[\mathcal{K}_0/\mathcal{K}] \in SDual(D_i, s[\mathcal{K}_0/\mathcal{K}])\}$

[2] As we don't know whether Dual Interpolation of order $N-1$ is decidable, the function *SDual* may be not computable

ii) The functions *RSub* and Ω_1 are computable. For any $1 < P < N$, if Ω_P is computable, then the restriction of *Car* to types of order at most P is computable. The decidability of Dual Interpolation of order P implies that the function *SDual* restricted to types of order P is computable and thereby, implies that the function Ω_{P+1} is computable. □

6 Main Results

Theorem 22. *Let $b : o$ be a normal term of atomic type. Let $A = A_1 \ldots A_n \to o$ be an arbitrary type. The quotient by the relation $\sim$ of set of interpolations equations*

$$EQU(A, b) = \{ [(x)c_1 \ldots c_n = b] \mid x : A,\ c_1 : A_1,\ \ldots, c_n : A_n \}$$

contains only a finite number of classes.

Proof. We may assume that b is of constant context, by substituting new constants for all free variables in this term. We may also assume that every β-normal term is on η-long form. At last, we may assume that every term on β-normal η-long form (in particular, b) is an element of $\overline{\mathcal{S}}_0$ (by adjusting the kind of bound variables in all terms considered) and that the β-reduction is restricted to β^*. The conclusion follows then from the preceding lemma and lemma 19. □

Theorem 23. *Let N be an arbitrary order. The decidability of Dual Interpolation of order N implies the decidability of Pattern Matching of order N.*

Proof. Indeed, by lemma 21, the decidability of Dual Interpolation of order N implies that the function Ω_{N+1} is a computable function. Let $z_1 : A_1 \ldots z_n : A_n$ be instantiable variables of order at most N. Let $A = A_1 \ldots A_n \to o$. For any normal term b such that $\mathcal{C} \vdash b : o$, let Σ be the finite set containing, for each $[(x)t_1 \ldots t_n = b]$ in $\Omega_{N+1}(A, b)$, the substitution $[t_1/z_1 \ldots t_n/z_n]$. (since A is of order at most $N + 1$, by hypothesis, the set Σ is computable). Let $F = [u[z_1 \ldots z_n] = b]$ be a matching equation. Let $[t_1^*/z_1 \ldots t_n^*/z_n]$ be an arbitrary solution of F. Then $\lambda y_1 \ldots y_n . u[y_1/z_1 \ldots y_n/z_n]$ $(y_1 \ldots y_n \in \mathcal{A})$ is a solution of $[(x)t_1^* \ldots t_n^* = b]$; there exists in $\Omega_{N+1}(A, b)$ an equation $[(x)t_1 \ldots t_n = b]$ equivalent to $[(x)t_1^* \ldots t_n^* = b]$; $u[t_1/z_1 \ldots t_n/z_n]$ still normalizes to b.

In other words, the set Σ contains a solution of F. □

7 Conclusion

So far, the results presented in this paper leave open the issue of the decidability of Pattern Matching. Since we do not consider the problem of solving simultaneously equations *and* inequations between simply typed terms (i.e. we do not consider inequations of the form $[a \neq b]$, where b contains no instantiable variables), Pattern Matching could be decidable without Dual Interpolation being decidable for all orders. The methods used in [5] and [6] in order to prove the decidability of Fourth Order Matching and Atomic Matching are quite different, and both rely on properties specific to these particular cases.

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