

THE SIMPLE SEMANTICS FOR
COPPO-DEZANI-SALLÉ TYPES

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ABSTRACT.

The Coppo-Dezani-Sallé type-language has " \cap " (intersection) and " ω " (universal type), besides the usual " \rightarrow " (exponentiation). Coppo, Dezani and Sallé have presented formal rules for assigning types to type-free λ -terms, and have proved metatheorems which give their system significance and interest. (Sallé 1978, Coppo et al. 1981.)

But no precise semantics has been given for their system yet, though the authors have made it fairly obvious informally what semantics they intended.

The present paper defines a precise semantics in an arbitrary model of type-free λ -calculus. The rules of Coppo, Dezani and Sallé turn out to be incomplete with respect to this semantics, but they become complete when three extra rules (two trivial, one not) are added. The completeness proof uses the term-model only.

INTRODUCTION.

In a series of manuscripts from 1977 onwards, Coppo, Dezani and Sallé have developed an extension of the usual Church-Curry type-language. They have added " \cap " (intersection) and " ω " (universal type) to the usual Church-Curry exponentiation connective (here denoted by " \rightarrow "). They have stated formal rules for assigning type-schemes to untyped λ -terms, which cover more terms than Curry's system, but still preserve the essential property of types; that the type should describe the term's functional behaviour in some significant way. (See for example Sallé 1978, Coppo et al. 1978, 1980, 1981.) A similar rule-system with " \cap " but not " ω " has been created independently by Pottinger, but not taken so far; Pottinger 1980.

The Coppo-Dezani-Sallé rules are motivated by a fairly obvious informal semantics. The present paper defines this semantics formally, in an arbitrary model of type-free λ -calculus; it is the same as the "simple semantics" of Hindley 1983.

The rules turn out to be incomplete with respect to this semantics; but they become complete when three extra rules (two trivial, one not) are added. Completeness is proved here using the term model, by the method of Hindley 1983.

A key lemma is the invariance of types under β -conversion; this will be proved via a translation into the system of Coppo et al. 1981, which will clarify the exact relation between that system and the present one.

The completeness problem has also been solved by Barendregt, Coppo and Dezani, independently of the present work and simultaneously with it (Spring 1980). Their version of the one non-trivial extra rule is a bit easier to handle than mine, so I

shall use it below. The model they use is more complicated than the term-model used here, though it is interesting in its own right. (Barendregt et al. 198-.)

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1. BASIC SYNTAX.

The reader is assumed to know at least the early parts of Coppo et al. 1980 or 1981. (1981 was written before 1980 and is the best reference for the basic ideas.) Also Hindley 1983 would be useful for motivation. The notation below is based on these papers, though with some changes.

Type-free λ -terms are built up as usual from *term-variables* (denoted by "x", "y", "z", ...), but no atomic constants. Capital letters will denote type-free λ -terms. Identity will be called " $=$ ". The usual β -convertibility and reducibility relations will be called " $=_\beta$ ", " $>_\beta$ ". (Definitions are in Barendregt 1977 §1.) $\beta\eta$ -reduction will not be considered here, nor will combinatory terms.

DEFINITION 1. Coppo-Dezani-Sallé type-schemes.

(i) An infinity of *type-variables* (a, b, c, d, \dots) are type-schemes. ω is a type-scheme.

(ii) If α and β are type-schemes, then so are $(\alpha \rightarrow \beta)$, $(\alpha \cap \beta)$. (Greek letters " α ", " β ", " γ ", " δ " will denote arbitrary type-schemes. Informally, ω represents the universe, $\alpha \rightarrow \beta$ the set of all functions from α into β , and \cap is intersection.)

Type-assignment statements are expressions αX (read as "assign α to X ", or " $X \in \alpha$ "). X is called the *subject* of the statement.

A basis \mathcal{B} is any finite or infinite set of statements whose subjects are variables (not necessarily all distinct; in fact one subject may occur infinitely often).

DEFINITION 2. The pre-order \leq is defined by the following axioms and rules, taken from Barendregt et al. 198-. It formalizes the subset relation, and will be used in the non-trivial extra type-assignment rule.

$$\begin{array}{ll} (A1) \quad \alpha \leq \alpha; & (A4) \quad \alpha \leq \alpha \cap \alpha; \\ (A2) \quad \alpha \leq \omega; & (A5) \quad \alpha_1 \cap \alpha_2 \leq \alpha_i \quad (i = 1, 2); \\ (A3) \quad \omega \leq \omega \rightarrow \omega; & (A6) \quad (\alpha \rightarrow \beta) \cap (\alpha \rightarrow \gamma) \leq \alpha \rightarrow (\beta \cap \gamma); \\ \\ (R1) \quad \alpha \leq \beta, \beta \leq \gamma \implies \alpha \leq \gamma; & \\ (R2) \quad \alpha \leq \alpha', \beta \leq \beta' \implies \alpha \cap \beta \leq \alpha' \cap \beta' & \\ (R3) \quad \alpha \leq \alpha', \beta \leq \beta' \implies \alpha' \rightarrow \beta \leq \alpha \rightarrow \beta'. & \end{array}$$

Note the reversal in (R3); this is justified by thinking of \leq as the subset relation. Note also (A3); (A3) says that every member of ω can be considered as a function on ω . This agrees with the semantics below. But there are other possible semantics, for example the F-semantics of Hindley 1983 §4, for which (A3) would have to be dropped.

DEFINITION 3. *The axioms and rules for type-assignment.* These are in the style of Gentzen's "Natural deduction"; cf. Hindley 1983 §1, or Coppo et al. 1981 §1.

Axioms (ω): αX (one axiom for each term X);

Rules:

$$(\rightarrow i) \quad \frac{[\alpha x]}{\beta Y} \quad (\rightarrow e) \quad \frac{(\alpha \rightarrow \beta) U \quad \alpha V}{\beta(UV)}$$

$$(\wedge i) \quad \frac{\alpha X \quad \beta X}{(\alpha \wedge \beta) X} \quad (\wedge e) \quad \frac{(\alpha_1 \cap \alpha_2) X}{\alpha_i X} \quad (i = 1, 2)$$

$$(\leq) \quad \frac{\alpha X \quad \alpha \leq \beta}{\beta X}.$$

Note: each time rule $(\rightarrow i)$ is used, we 'cancel' (shown above by "[]") all occurrences of αx at the tops of branches above βY , that have not previously been cancelled. We are allowed to use $(\rightarrow i)$ even when there are no such occurrences of αx ("vacuous cancellation"). We must not use $(\rightarrow i)$ when the uncancelled premises above βY include a γx with $\gamma \neq \alpha$.

$\mathcal{B} \vdash \alpha X$ will mean that there is a deduction whose uncancelled premises are either axioms or members of \mathcal{B} , and whose conclusion is αX . (A deduction is a tree as usual.) If \mathcal{B} is empty, we shall say $\vdash \alpha X$.

- EXERCISE 1.**
- (i) $\vdash ((\alpha \wedge (\alpha \rightarrow \beta)) \rightarrow \beta)(\lambda x. xx).$
 - (ii) $\vdash (\alpha \rightarrow \omega)(\lambda x. x)$
 - (iii) $\vdash (\alpha \rightarrow (\omega \rightarrow \alpha))(\lambda xy. x).$

(Exercise 1 (i) shows how Coppo, Dezani and Sallé can give significant types to terms that Curry and Church could not. Exercise 1 (ii) and (iii) are examples of two different kinds of vacuous cancellation; (ii) can be done using (ω) and $(\rightarrow i)$.)

REMARK 1. The type-language in Definition 1 is the same as that in Barendregt et al. 198-; it is more expressive than in Coppo et al. 1980, 1981. Coppo and Dezani deliberately restricted the language in their 1980, 1981 in order to simplify the metatheory. I shall take advantage of this simpler metatheory in the proof of the β -invariance theorem below, which proceeds by translating into the restricted language. The translation will also give a quick decision procedure for the relation \leq .

REMARK 2. The rules in Coppo et al. 1980, 1981 are not complete, even for the more restricted type-language used there. Moreover, the rules in 1980 differed in detail from the two sets of rules given in 1981; for example vacuous cancellation was allowed in the first set of rules in 1981, but restricted to a special case in the second set, and in 1980.

Rule (ne) was omitted from both papers. The authors remarked that adding (ne) would not give any more provable statements αX (X closed); but adding (ne) does give more deductions $\emptyset \vdash \alpha X$ (X not closed), so it is included here.

Rule (\leq) was not included. But because of the restrictions on types, only a few special cases of (\leq) were relevant. The form of Rule (\leq) given above is due to Barendregt, Coppo and Dezani 198-; my original form was equivalent but clumsier.

DEFINITION 4. $\alpha \sim \beta$ iff $\alpha \leq \beta \leq \alpha$. (This is not the relation \sim of Coppo et al. 1980.)

LEMMA 1.

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|---|--|
| (i) $\alpha \sim \alpha;$ | (vii) $\alpha \cap \omega \sim \alpha;$ |
| (ii) $\alpha \sim \beta, \beta \sim \gamma \Rightarrow \alpha \sim \gamma;$ | (viii) $\alpha \dot{\times} \omega \sim \omega;$ |
| (iii) $\alpha \sim \beta \Rightarrow \beta \sim \alpha;$ | (ix) $\alpha \cap \alpha', \beta \sim \beta' \Rightarrow \alpha \cap \beta \sim \alpha' \cap \beta';$ |
| (iv) $\alpha \cap \beta \sim \beta \cap \alpha;$ | (x) $\alpha \sim \alpha', \beta \sim \beta' \Rightarrow \alpha \dot{\times} \beta \sim \alpha' \dot{\times} \beta';$ |
| (v) $(\alpha \cap \beta) \cap \gamma \sim \alpha \cap (\beta \cap \gamma);$ | (xi) $\alpha \leq \beta, \alpha \sim \alpha', \beta \sim \beta' \Rightarrow \alpha' \leq \beta';$ |
| (vi) $\alpha \sim \alpha \cap \alpha;$ | (xii) $(\alpha \dot{\times} \beta) \cap (\alpha \dot{\times} \gamma) \sim \alpha \dot{\times} (\beta \cap \gamma).$ |

2. SEMANTICS.

The semantics is based on the concept of "model of the type-free $\lambda\beta$ -calculus". The best way to define this concept is not yet agreed (some of the issues are discussed in Hindley and Longo 1980). But here only one model will be used, the term model, so the disagreements will not matter; everyone accepts the term model (I think). Type-schemes will be interpreted as subsets of models.

Models: Common to all definitions is that a model \mathfrak{D} has a non-empty set D (its domain), a map $\circ : D^2 \rightarrow D$ (called application), and a map $\llbracket \cdot \rrbracket$ which assigns a member $\llbracket X \rrbracket_\rho$ of D to each term X and each map ρ : term-variables $\rightarrow D$. These satisfy

- (i) $\llbracket x \rrbracket_\rho = \rho(x);$
- (ii) $\llbracket XY \rrbracket_\rho = \llbracket X \rrbracket_\rho \circ \llbracket Y \rrbracket_\rho;$
- (iii) if $\sigma(x) = \rho(x)$ for all x free in X , then $\llbracket X \rrbracket_\sigma = \llbracket X \rrbracket_\rho;$
- (iv) $X = \beta Y \Rightarrow (\forall \rho) \llbracket X \rrbracket_\rho = \llbracket Y \rrbracket_\rho.$

The term model $\mathcal{M}(\lambda\beta)$ has for its domain the set of all β -convertibility-classes of terms. (For all X let $[X] = \{Y : Y =_{\beta} X\}.$) Its map \circ is defined by

$$[X] \circ [Y] = [XY].$$

And $\llbracket \cdot \rrbracket$ is defined by

$$\llbracket X \rrbracket_{\rho} = [[Y_1, \dots, Y_n/x_1, \dots, x_n]x],$$

where x_1, \dots, x_n are the free variables of X , and $\rho(x_i) = [Y_i]$, and $[\dots / \dots]$ denotes simultaneous substitution.

The only ρ we shall need here is the simplest one, ρ_o , defined by

$$\rho_o(x) = [x].$$

It is easy to see that for all X ,

$$\llbracket X \rrbracket_{\rho_o} = [X].$$

The simple semantics (Hindley 1983 §2). Given any model \mathcal{D} , a valuation of the type-variables is any map V which assigns to each type-variable a subset of D . Any such V determines an interpretation $\llbracket \cdot \rrbracket_V$ of all the type-schemes as follows:

- (i) $\llbracket a \rrbracket_V = V(a);$
- (ii) $\llbracket \omega \rrbracket_V = D;$
- (iii) $\llbracket \beta \rightarrow \gamma \rrbracket_V = \{d \in D : (\forall e) e \in \llbracket \beta \rrbracket_V \Rightarrow d \circ e \in \llbracket \gamma \rrbracket_V\};$
- (iv) $\llbracket \beta \cap \gamma \rrbracket_V = \llbracket \beta \rrbracket_V \cap \llbracket \gamma \rrbracket_V.$

Satisfaction: A statement αX is satisfied by \mathcal{D}, ρ, V iff $\llbracket X \rrbracket_{\rho} \in \llbracket \alpha \rrbracket_V$. Iff every \mathcal{D}, ρ, V which satisfy all statements in β also satisfy αX , we say

$$\beta \models \alpha X.$$

EXERCISE 2. $\alpha \leq \beta \Rightarrow (\forall \mathcal{D})(\forall V) \llbracket \alpha \rrbracket_V \subseteq \llbracket \beta \rrbracket_V.$

(In particular, note that $\llbracket \omega \rightarrow \omega \rrbracket_V = D$. The converse of the above implication is also true, see later.)

3. MAIN RESULTS.

SOUNDNESS THEOREM. If $\mathcal{B} \vdash \alpha X$ then $\mathcal{B} \models \alpha X$.

Proof. Straightforward induction on \vdash , using the exercise at the end of §2. (By the way, this theorem holds even when one-member domains are counted as models; they were not excluded in §2.)

THE β -INVARIANCE THEOREM. If $\mathcal{B} \vdash \alpha X$ and $X =_{\beta} Y$, then $\mathcal{B} \vdash \alpha Y$.

Proof. By translating the present system into the simpler one of Coppo et al. 1981 and using their β -invariance proof, 1981 Theorem 1. (Details in §§4-5 below.)

REMARK 3. Contrast this theorem with Curry's type-assignment, e.g. in Hindley 1983. Curry's type-statements could only be made β -invariant by adjoining a rule

$$(\text{eq}) \quad \alpha X, X =_{\beta} Y \vdash \alpha Y.$$

This rule is non-recursive. In contrast, all the Coppo-Dezani rules, even (\leq) , are recursive. (That is, there is an algorithm which, when given a finite tree of statements, will decide whether that tree is a deduction.) For recursiveness of (\leq) , see the corollary to Lemma 5. Of course the relation $\mathcal{B} \vdash \alpha X$ is non-recursive in both systems.

THE η -LEMMA. If $\mathcal{B}, \beta z \vdash \gamma(Yz)$ and z is not in \mathcal{B} nor free in γ , then $\mathcal{B} \vdash (\beta \rightarrow \gamma)Y$.

Proof. See §5.

COMPLETENESS THEOREM. If $\mathcal{B} \models \alpha X$ then $\mathcal{B} \vdash \alpha X$.

Proof. (Cf. Hindley 1983 §3.) First extend \mathcal{B} to a set \mathcal{B}^+ by adding an infinity of statements

$$\partial y_{\partial, i} \quad (i = 1, 2, \dots)$$

for each type-scheme ∂ in the language. (The variables $y_{\partial, i}$ must all be distinct, and must not occur in \mathcal{B} or X .)

If this extension is not possible (i.e. if only a finite number of term-variables are not subjects of \mathcal{B}), then list all the term-variables as v_1, v_2, \dots , and replace each v_i by v_{2i} in \mathcal{B} and X . Let the results be \mathcal{B}' and X' . By routine calculations,

$$\begin{aligned} \mathcal{B} \models \alpha X &\iff \mathcal{B}' \models \alpha X', \\ \mathcal{B} \vdash \alpha X &\iff \mathcal{B}' \vdash \alpha X'. \end{aligned}$$

So we may assume the extension is possible.

Now take the term model $JM(\lambda\beta)$, and define V thus:

$$V(a) = \{[Y] : \mathcal{B}^+ \vdash aY\}.$$

Then for all ∂, Y we have, by a proof to be given later,

$$(1) \quad [Y] \in \llbracket \partial \rrbracket_Y \iff \mathcal{B}^+ \vdash \partial Y.$$

Now $\mathcal{B} \models \alpha X$, so in particular for the V above, $[X] \in \llbracket \alpha \rrbracket_V$. Hence by (1), $\mathcal{B}^+ \vdash \alpha X$. Now the subject of any statement of \mathcal{B}^+ that occurs in this deduction will appear as a free variable in X . But no variable in $\mathcal{B}^+ - \mathcal{B}$ is free in X , so only statements in \mathcal{B} are used in the above deduction. That is,

$$\mathcal{B} \vdash \alpha X,$$

as required for completeness. It only remains to prove (1).

Proof of (1). Induction on β . If β is a variable, use the definition of V . If $\beta = \omega$, use the ω -axioms. If $\beta = \beta_1 \cap \beta_2$, use the induction hypothesis and rules (ni), (ne).

Finally let $\beta = \beta \rightarrow \gamma$. Then

$$\begin{aligned} \beta^+ \vdash (\beta \rightarrow \gamma)Y &\Rightarrow (\forall Z)\{\beta^+ \vdash \beta Z \Rightarrow \beta^+ \vdash \gamma(YZ)\} \text{ by } (\rightarrow e) \\ &\Leftrightarrow (\forall Z)\{[Z] \in \llbracket \beta \rrbracket_V \Rightarrow [YZ] \in \llbracket \gamma \rrbracket_V\} \text{ by ind. hyp.} \\ &\Leftrightarrow [Y] \in \llbracket \beta \rightarrow \gamma \rrbracket_V \text{ by definition of } \llbracket \cdot \rrbracket_V. \end{aligned}$$

To prove the converse of the first implication, suppose

$$(\forall Z)\{\beta^+ \vdash \beta Z \Rightarrow \beta^+ \vdash \gamma(YZ)\}.$$

Choose Z to be a variable $z = y_{\beta, i}$ not in Y . Then $\beta^+ \vdash \beta z$, so

$$\beta^+ \vdash \gamma(Yz).$$

Hence by the η -reduction lemma,

$$\beta^+ \vdash (\beta \rightarrow \gamma)Y.$$

This proves (1) and the theorem.

NOTE. The β -invariance theorem was used implicitly throughout this proof. Because when selecting a member Y of a β -convertibility-class, it was assumed that all members had the same type.

COROLLARY (Barendregt, Coppo, Dezani).

- (i) $\alpha \leq \beta \Leftrightarrow (\forall \theta)(\forall V) \llbracket \alpha \rrbracket_V \subseteq \llbracket \beta \rrbracket_V$;
- (ii) $\alpha \sim \beta \Leftrightarrow (\forall \theta)(\forall V) \llbracket \alpha \rrbracket_V = \llbracket \beta \rrbracket_V$.

Proof. For " \Leftarrow " in (i): if the right side of (i) holds, then $\alpha x \models \beta x$, so by completeness, $\alpha x \vdash \beta x$. Hence by checking all possible deductions, $\alpha \leq \beta$.

4. THE TRANSLATION.

It remains to prove the β -invariance theorem and the η -lemma. This will be done via a translation into a system with a simpler metatheory, essentially the same as that of Coppo et al. 1981 §4.

DEFINITION 5. Normal type-schemes (the set NTS).

- (i) Type-variables and ω are in NTS;
- (ii) $(\sigma \sqcap \tau) \in \text{NTS}$ if $\sigma, \tau \in \text{NTS} - \{\omega\}$;
- (iii) $(\sigma \rightarrow \tau) \in \text{NTS}$ if $\sigma \in \text{NTS}$ and $\tau \in \text{NTS} - \{\omega\}$, intersections.

NTS is the set of all σ such that: either $\sigma = \omega$, or $\sigma = \sigma_1 \cap \dots \cap \sigma_n$ with some bracketing and with each σ_i having form

$$\sigma_{i1} \rightarrow (\dots (\sigma_{im_i} \rightarrow a_i) \dots), \quad m_i \geq 0, \quad n \geq 1.$$

NOTATION.

" α ", " β ", " γ " ' δ ' will denote arbitrary type-schemes of Def. 1;

" ρ ", " σ ", " τ " : members of NTS;

" $\alpha_1 \cap \dots \cap \alpha_n$ ", " $\sigma_1 \cap \dots \cap \sigma_n$ ", etc. : multiple intersections with parentheses

determined by context, and with the convention that none of α_i , σ_j is an intersection. (Then every α can be written uniquely as $\alpha = \alpha_1 \cap \dots \cap \alpha_n$, $n \geq 1$, with the parentheses determined by the structure of α . Note that $\alpha_i = \alpha_j$ is permitted.)

DEFINITION 6. \leq_N is the relation in $(NTS)^2$ obtained by restricting Definition 2 to NTS. (Later we shall see that

$$\sigma \leq_N \tau \iff \sigma \leq \tau;$$

but this is not obvious, a priori, because there may exist $\beta \notin NTS$ such that $\sigma \leq \beta \leq \tau$.)

LEMMA 3. \leq_N behaves very nicely, thus:

$$(i) (\sigma_1 \cap \dots \cap \sigma_m) \leq_N (\tau_1 \cap \dots \cap \tau_n) \iff (\forall \tau_j)(\exists \sigma_i) \sigma_i \leq_N \tau_j;$$

$$(ii) \omega \leq_N \tau \iff \tau = \omega;$$

$$(iii) (\sigma_1 \rightarrow (\dots (\sigma_m \rightarrow a) \dots)) \leq_N (\tau_1 \rightarrow (\dots (\tau_n \rightarrow b) \dots)) \iff a = b \text{ and } m = n \text{ and } (\forall i) \tau_i \leq_N \sigma_i;$$

$$(iv) (\sigma_1 \rightarrow (\dots (\sigma_m \rightarrow a) \dots)) \leq_N \tau \iff$$

either $\tau = \omega$, or $\tau = \tau_1 \cap \dots \cap \tau_k$ ($k \geq 1$) and

$(\forall i) \tau_i = \tau_{i,1} \rightarrow (\dots (\tau_{i,m} \rightarrow a) \dots)$ with $\tau_{i,j} \leq_N \sigma_j$.

(v) The relation \leq_N is recursive.

Proof. In each case " \Leftarrow " is trivial. For " \Rightarrow " in (i) - (iii), use induction on \leq_N . For (iv), use (i) and (iii). For (v), decide whether $\sigma \leq_N \tau$ by induction on σ , using (i) and (iv).

DEFINITION 7. The translation. Define α^* to be the NTS obtained by reducing α by the following replacement rules:

(i) $\beta \rightarrow (\gamma \cap \delta)$ may be replaced by $(\beta \rightarrow \gamma) \cap (\beta \rightarrow \delta)$;

(ii) $\beta \cap \omega$ may be replaced by β ;

(iii) $\omega \cap \beta$ may be replaced by β ;

(iv) $\beta \rightarrow \omega$ may be replaced by ω .

For any $\beta = \{\alpha_1 x_1, \alpha_2 x_2, \dots\}$, define $\beta^* = \{\alpha_1^* x_1, \alpha_2^* x_2, \dots\}$.

LEMMA 4. α^* exists and is unique. Also $(\beta \cap \gamma)^* = (\beta^* \cap \gamma^*)^*$ and $(\beta \rightarrow \gamma)^* = (\beta^* \rightarrow \gamma^*)^*$.

Proof. Call a replacement by one of (i) - (iv) a contraction ($\#_1$), and a sequence a reduction ($\#$). An ordinal number degree (α) will be assigned to each α , so that contraction strictly reduces degree. First define

$$|\alpha| = \text{number of occurrences of symbols in } \alpha.$$

(Contractions (ii) - (iv) reduce $|\alpha|$, but (i) does not.) Then define

$$\mathcal{I}(e) = 0 \quad (e = \omega \text{ or a variable});$$

$$\mathcal{I}(\alpha_1 \rightarrow (\dots (\alpha_m \rightarrow e) \dots)) = \mathcal{I}(\alpha_1) + \dots + \mathcal{I}(\alpha_m) \quad (m \geq 0);$$

$$\mathcal{I}(\alpha_1 \rightarrow (\dots (\alpha_m \rightarrow (\beta \cap \gamma)) \dots)) = m + \mathcal{I}(\alpha_1 \rightarrow (\dots (\alpha_m \rightarrow \beta) \dots)) + \mathcal{I}(\alpha_1 \rightarrow (\dots (\alpha_m \rightarrow \gamma) \dots)).$$

It is easy to prove by induction on $|\alpha|$ that contractions (i) reduce $\mathcal{I}(\alpha)$; also that if α' is part of α , then $\mathcal{I}(\alpha') \leq \mathcal{I}(\alpha)$, so (ii) - (iv) do not increase $\mathcal{I}(\alpha)$.

Finally, take the ordinal ω and define

$$\text{degree}(\alpha) = \omega \cdot \mathcal{I}(\alpha) + |\alpha|.$$

Contractions reduce degree, so all reductions must terminate. And when no further contraction is possible, α is obviously in NTS.

The uniqueness of α^* is proved by a Church-Rosser argument. First, by induction on $\text{degree}(\alpha)$, uniqueness follows from

$$(2) \quad \alpha \#_1 \beta_1 \text{ and } \alpha \#_1 \beta_2 \implies (\exists \gamma) \beta_1 \# \gamma \text{ and } \beta_2 \# \gamma.$$

Then (2) is proved by easy case-checking. For example, if

$$\alpha = (\beta \rightarrow (\omega \cap \theta)),$$

and α contracts to $((\beta \rightarrow \omega) \cap (\beta \rightarrow \theta))$ by (i) and to $(\beta \rightarrow \omega)$ by (iii), then we can choose

$$\gamma = \beta \rightarrow \omega.$$

LEMMA 5.

$$(i) \quad \alpha \sim \alpha^*;$$

$$(ii) \quad \alpha \leq \beta \iff \alpha^* \leq_N \beta^*;$$

$$(iii) \quad \alpha \sim \omega \iff \alpha^* = \omega.$$

Proof. For (i), use Lemma 1. For (iii), use (ii) and Lemma 3(ii) and note that $\omega^* = \omega$.

For (ii) " \Rightarrow ", use induction on Definition 2. (To prove $\alpha^* \leq \beta^*$ is easy, but to get \leq_N we must check that all type-schemes involved are in NTS.) The only non-trivial case is rule (R3). Let $\gamma \leq \gamma'$, $\theta \leq \theta'$, and

$$\alpha = \gamma' \rightarrow \theta, \quad \beta = \gamma \rightarrow \theta'.$$

By the induction-hypothesis,

$$\gamma^* \leq_N \gamma'^*, \quad \theta^* \leq_N \theta'^*.$$

And by Lemma 4,

$$\alpha^* = (\gamma'^* \rightarrow \theta'^*)^*, \quad \beta^* = (\gamma^* \rightarrow \theta'^*)^*.$$

Consider θ^* : if $\theta^* = \omega$, then by Lemma 3(ii) $\theta'^* = \omega$, and so $\beta^* = \omega$; then $\alpha^* \leq_N \beta^*$ by (A2). If $\theta^* \neq \omega$, then

$$\partial^* = \sigma_1 \cap \dots \cap \sigma_n \quad (n \geq 1),$$

$$\sigma_i = (\sigma_{i,1} \rightarrow \dots (\sigma_{i,m_i} \rightarrow a_i) \dots) \quad (1 \leq i \leq n).$$

If $\partial'^* = \omega$, then $\alpha^* \leq_N \beta^*$ as above. If not, then by Lemma 3,

$$\partial'^* = \tau_1 \cap \dots \cap \tau_p \quad (p \geq 1),$$

$$(\forall \tau_j)(\exists \sigma_i) \tau_j = (\tau_{j,1} \rightarrow \dots (\tau_{j,m_i} \rightarrow a_i) \dots) \text{ and}$$

$$(\forall k \leq m_i) \tau_{j,k} \leq_N \sigma_{i,k}.$$

Then

$$\alpha^* = (\gamma'^* \rightarrow \sigma_1) \cap \dots \cap (\gamma'^* \rightarrow \sigma_n),$$

$$\beta^* = (\gamma^* \rightarrow \tau_1) \cap \dots \cap (\gamma^* \rightarrow \tau_n).$$

So $\alpha^* \leq_N \beta^*$ by an easy calculation.

To prove (ii)" \Leftarrow ", use induction on Definition 2. The only nontrivial cases are (R2) and (R3).

For (R2), let $\alpha^* \leq_N \alpha'^*$, $\beta^* \leq_N \beta'^*$. To show $(\alpha^* \cap \beta^*)^* \leq_N (\alpha'^* \cap \beta'^*)^*$. If the right-hand side is ω , the result is obvious. If the right side is α'^* or β'^* , the result comes from (A5) and (R1). Otherwise, none of α^* , β^* , α'^* , β'^* is ω , and the result comes by (R2).

For (R3), let $\alpha^* \leq_N \alpha'^*$, $\beta^* \leq_N \beta'^*$. To show

$$(3) \quad (\alpha'^* \rightarrow \beta^*)^* \leq_N (\alpha^* \rightarrow \beta'^*)^*.$$

If the right side is ω , the result is obvious. Otherwise, neither β^* nor β'^* is ω , and

$$\beta^* = \sigma_1 \cap \dots \cap \sigma_m, \quad \beta'^* = \tau_1 \cap \dots \cap \tau_n \quad (m, n \geq 1; \sigma_i, \tau_j \neq \omega).$$

Also the left side of (3) is

$$(4) \quad (\alpha'^* \rightarrow \sigma_1) \cap \dots \cap (\alpha'^* \rightarrow \sigma_m)$$

and the right side is

$$(5) \quad (\alpha^* \rightarrow \tau_1) \cap \dots \cap (\alpha^* \rightarrow \tau_n).$$

Now $\beta^* \leq_N \beta'^*$, so by Lemma 3(i), $(\forall j \leq n)(\exists i \leq m) \sigma_i \leq_N \tau_j$. From this we get

(4) \leq_N (5). This proves Lemma 5.

COROLLARY. The relation \leq is recursive.

Proof. By (ii) above and Lemma 3(v).

LEMMA 6. $\sigma \leq \tau \iff \sigma \leq_N \tau$ for $\sigma, \tau \in \text{NTS}$.

Proof. By Lemma 5(ii), since $\sigma^* = \sigma$ and $\tau^* = \tau$.

NOTATION. From now on, in NTS " \leq " will be used for \leq_N .

DEFINITION 7. A deduction is called moderately restricted (\vdash_{MR}) when all its type schemes are in NTS.

LEMMA 7. If $\mathcal{B} \vdash \alpha X$, then $\mathcal{B}^* \vdash_{MR} \alpha^* X$.

Proof.

By induction on \vdash . The basis ($\alpha X \in \mathcal{B}$ or $\alpha = \omega$) is trivial. For rule (\leq) or $(\wedge e)$, use Lemma 5(ii).

Rule $(\wedge i)$: easy, because $(\alpha \wedge \beta)^*$ is either $\alpha^* \wedge \beta^*$ or α^* or β^* .

Rule $(\rightarrow i)$: let $\mathcal{B}^*, \alpha^* \vdash_{MR} \beta^* Y$. To show $\mathcal{B}^* \vdash_{MR} (\alpha^* \rightarrow \beta^*)^* (\lambda x. Y)$. If $\beta^* = \omega$, this is trivial by the ω -axiom. Otherwise

$$\beta^* = (\sigma_{1,1} \rightarrow (\dots (\sigma_{1,m_1} \rightarrow a_1) \dots)) \wedge \dots \wedge (\sigma_{n,1} \rightarrow (\dots (\sigma_{n,m_n} \rightarrow a_n) \dots))$$

where $n \geq 1$ and $m_i \geq 0$. Hence

$$(\alpha^* \rightarrow \beta^*)^* = (\alpha^* \rightarrow (\sigma_{1,1} \rightarrow \dots)) \wedge \dots \wedge (\alpha^* \rightarrow (\sigma_{n,1} \rightarrow \dots)).$$

Take the given deduction of $\beta^* Y$ and apply $(\wedge e)$ $n-1$ times, then $(\rightarrow i)$, then $(\wedge i)$ $n-1$ times.

Rule $(\rightarrow e)$: similar to $(\rightarrow i)$.

DEFINITION 8. A deduction is called restricted (\vdash_R) when all its type-schemes are in NTS and

(i) rule $(\wedge e)$ never immediately follows $(\wedge i)$,

(ii) $(\wedge e)$ and (\leq) are only used with atomic subjects.

TRANSLATION THEOREM. $\mathcal{B} \vdash \alpha X \iff \mathcal{B}^* \vdash_R \alpha^* X$.

Proof. For " \Leftarrow ": use Lemma 5(i) and rule (\leq) .

For " \Rightarrow ": by Lemma 7, it is enough to prove that $\mathcal{B} \vdash_{MR} \alpha X$ implies $\mathcal{B} \vdash_R \alpha X$. To do this, it is enough to prove

$$(6) \quad \mathcal{B} \vdash_R \tau X \text{ and } \tau \leq \tau' \iff \mathcal{B} \vdash_R \tau' X.$$

(The elimination of $(\wedge i)$ - $(\wedge e)$ pairs is easy, and shortens deductions.)

Proof of (6).

Induction on \vdash_R . If τX is in \mathcal{B} , then X is an atom and rule (\leq) is permitted.

Case (ω) : If $\tau = \omega$, then by Lemma 3(ii), $\tau' = \omega = \tau$.

Case $(\rightarrow i)$: If τX comes from $(\rightarrow i)$, then $X = \lambda x. Y$, $\tau = \rho \rightarrow \sigma$, and

$$(7) \quad \mathcal{B}, \rho x \vdash_R \sigma Y.$$

By Lemma 3(iv) applied to $\tau \leq \tau'$, either $\tau' = \omega$, or

$$\tau' = \tau_1 \cap \dots \cap \tau_k, k \geq 1, \tau_i = \rho_i \rightarrow \sigma_i, \rho_i \leq \rho, \sigma_i \leq \sigma.$$

If $\tau' = \omega$, then $\vdash_R \tau' (\lambda x. Y)$ by (ω). Now let $\tau' \neq \omega$. By the induction hypothesis applied to (7),

$$\mathcal{B}, \rho x \vdash_R \sigma_i Y \quad (\text{for } i = 1, \dots, k).$$

And by rule (\leq) with atomic subject,

$$\rho_i x \vdash_R \rho x.$$

Hence $\mathcal{B}, \rho_i x \vdash_R \sigma_i Y$; so by $(\rightarrow i)$,

$$\mathcal{B} \vdash_R \tau_i (\lambda x. Y) \quad (i = 1, \dots, k).$$

Then by $(\cap i)$,

$$\mathcal{B} \vdash_R \tau' (\lambda x. Y).$$

Case $(\rightarrow e)$: let $X = UV$ and $\mathcal{B} \vdash_R (\rho \rightarrow \tau) U$ and $\mathcal{B} \vdash_R \rho V$.

Since $\tau \leq \tau'$, we have $\rho \rightarrow \tau \leq \rho \rightarrow \tau'$. If $(\rho \rightarrow \tau') \in \text{NTS}$, then apply the induction hypothesis to $(\rho \rightarrow \tau)U$, giving

$$\mathcal{B} \vdash_R (\rho \rightarrow \tau') U.$$

Hence by $(\rightarrow e)$,

$$\mathcal{B} \vdash_R \tau' (UV).$$

This assumed $\rho \rightarrow \tau' \in \text{NTS}$. The only way this could fail would be $\tau' = \omega$ or an intersection.

If $\tau' = \omega$, then $\mathcal{B} \vdash_R \tau' (UV)$ by axiom (ω) .

If $\tau' = (\tau'_1 \cap \dots \cap \tau'_p)$ ($p \geq 2$, $\tau_i \neq \omega$): since $\rho \rightarrow \tau \in \text{NTS}$, τ is not an intersection, so by Lemma 3(iv), $(\forall i) \tau \leq \tau'_i$. Hence $(\rho \rightarrow \tau) \leq (\rho \rightarrow \tau'_1)$, and we get

$$\mathcal{B} \vdash_R \tau'_1 (UV). \quad \text{Then } \mathcal{B} \vdash_R \tau' (UV) \text{ by } (\cap i).$$

Case $(\cap i)$: let $\tau = \tau_1 \cap \dots \cap \tau_n$, $n \geq 2$, and let

$$\mathcal{B} \vdash_R (\tau_1 \cap \dots \cap \tau_k) X, \quad \mathcal{B} \vdash_R (\tau_{k+1} \cap \dots \cap \tau_n) X.$$

Now τ' has form

$$\tau' = \tau'_1 \cap \dots \cap \tau'_p \quad (p \geq 1).$$

And since $\tau \leq \tau'$, by Lemma 3(i) we have

$$(\forall \tau_j)(\exists \tau_i) \tau_i \leq \tau'_j.$$

The result then follows by induction hypothesis.

Case $(\rightarrow e)$ or (\leq) : use the induction hypothesis.

This proves the theorem.

COROLLARY. If all type-schemes in \mathcal{B} are in NTS, then

$$\mathcal{B} \vdash \sigma x \iff \mathcal{B} \vdash_R \sigma x.$$

5. PROOFS OF THE CONVERSION LEMMAS.

LEMMA 8. If $\mathcal{B} \vdash \alpha X$ and X converts to Y by changing bound variables, then $\mathcal{B} \vdash \alpha Y$. Same for \vdash_R .

THE β -INVARIANCE THEOREM. If $\mathcal{B} \vdash \alpha X$ and $X =_{\beta} Y$, then $\mathcal{B} \vdash \alpha Y$.

Proof. (Based on Coppo et al. 1981 Lemma 1 and Theorem 1.)

It is enough to prove

$$\mathcal{B} \vdash_R \alpha X, X =_{\beta} Y \Rightarrow \mathcal{B} \vdash \alpha Y.$$

(We only need to prove \vdash , because \vdash_R will follow by the corollary to the translation theorem.) It is also enough to consider only the case that $=_{\beta}$ is one step, contraction or expansion.

Case 1: $X = (\lambda y.P)Q$, $Y = [Q/y]P$. We are given

$$(8) \quad \mathcal{B} \vdash_R \sigma((\lambda y.P)Q),$$

and we want to prove

$$(9) \quad \mathcal{B} \vdash \sigma([Q/y]P).$$

Subcase 1a: the last step in (8) is (ω) . Then $\sigma = \omega$ and (9) follows by (ω) .

Subcase 1b: the last step in (8) is $(\rightarrow e)$:

$$\frac{(\rho \rightarrow \sigma)(\lambda y.P) \quad \rho Q}{\sigma((\lambda y.P)Q)}.$$

The step above $(\rho \rightarrow \sigma)(\lambda y.P)$ must be $(\rightarrow i)$. (It cannot be (\leq) or (\neq) because $\lambda y.P$ is composite.) So we have

$$\begin{array}{c} [\rho y] \\ \vdots \\ \sigma P \\ \hline (\rho \rightarrow \sigma)(\lambda y.P) \end{array}.$$

And $\mathcal{B} \vdash \rho Q$. Substitute the deduction $\mathcal{B} \vdash \rho Q$ for every premise ρy above σP . This will give a deduction (9).

Subcase 1c: the last step in (8) is (ηi) . As we move up the deduction (8), there will be a finite number of (ηi) -steps, and above these the only possibility is $(\rightarrow e)$. (We cannot have (ω) because $\sigma \in \text{NTS}$.) Hence

$$\sigma = \sigma_1 \cap \dots \cap \sigma_n \quad (n \geq 2),$$

and

$$\mathcal{B} \vdash_R \sigma_i ((\lambda y.P)Q) \quad (i = 1, \dots, n),$$

by deductions ending in $(\rightarrow e)$. Apply Subcase 1b to these deductions, giving $\mathcal{B} \vdash \sigma_i ([Q/y]P)$, and then use (ηi) to get (9).

Case 2: $X = [Q/y]P$, $Y = (\lambda y.P)Q$

By Lemma 8, we may assume no variable free in yQ is bound in P . We have

$$(10) \quad \mathcal{B} \vdash_R \sigma([Q/y]P).$$

If y does not occur in P , then $[Q/y]P = P$, so we have $\mathcal{B} \vdash \sigma P$. Hence by

($\rightarrow i$) with vacuous cancellation,

$$\mathcal{B} \vdash (\omega \rightarrow \sigma)(\lambda y.P).$$

But $\vdash \omega Q$ by (ω), so by ($\rightarrow e$), $\mathcal{B} \vdash \sigma((\lambda y.P)Q)$.

Now let y occur in P , and let y_1, \dots, y_n be its occurrences. In $[Q/y]P$ there will be corresponding occurrences Q_1, \dots, Q_n of Q . In the deduction (10), for each i , either Q_i will be part of a component $Z' = [Q/y]Z$ in $[Q/y]P$ such that $\omega Z'$ occurs in (10), or a sub-deduction $\mathcal{B} \vdash \rho_i Q_i$ occurs in (10) for some $\rho_i \neq \omega$. Say the former happens for Q_1, \dots, Q_k , and the latter happens for Q_{k+1}, \dots, Q_n .

Take the deduction (10), and eliminate all the sub-deductions $\mathcal{B} \vdash \rho_i Q_i$ ($i \geq k+1$), except for their last statements $\rho_i Q_i$. Then replace each Q_i by y , in these statements and in all statements below them. (By the nature of the rules in Definition 3, every statement below $\rho_i Q_i$ will contain a 'corresponding' occurrence of Q , in an obvious sense.)

For $i \leq k$, replace Q_i by y in the components Z' .

The result will be a deduction

$$\mathcal{B}, \rho_{k+1} y, \dots, \rho_n y \vdash \sigma P.$$

From this, by adding ($\wedge e$) at the top and ($\rightarrow i$) at the bottom, we get

$$\mathcal{B} \vdash ((\rho_{k+1} \wedge \dots \wedge \rho_n) \rightarrow \sigma)(\lambda y.P).$$

We had $\mathcal{B} \vdash \rho_i Q_i$ for $i \geq k+1$, that is $\mathcal{B} \vdash \rho_i Q$, so by ($\wedge i$) and ($\rightarrow e$), $\mathcal{B} \vdash \sigma((\lambda y.P)Q)$.

Case 3: Y comes from X by an arbitrary contraction or expansion. Use induction on X , with Cases 1 and 2 in the basis. This proves the β -invariance theorem.

THE η -LEMMA. If $\mathcal{B}, \beta z \vdash \gamma(Yz)$ and z is not in \mathcal{B} nor free in Y , then $\mathcal{B} \vdash (\beta \rightarrow \gamma)Y$.

Proof. Let $\mathcal{B}, \beta z \vdash \gamma(Yz)$. Then by the translation theorem,

$$(11) \quad \mathcal{B}^*, \beta^* z \vdash_R \gamma^*(Yz).$$

The last rule in (11) may be (ω) or ($\rightarrow e$) or ($\wedge i$).

Case (ω): $\gamma^* = \omega$. By Lemma 5(iii), $\omega \leq \gamma$. Hence

$$\omega \leq \omega \rightarrow \omega \leq \beta \rightarrow \omega \leq \beta \rightarrow \gamma.$$

So $\mathcal{B} \vdash (\beta \rightarrow \gamma)Y$ by (ω) and (\leq).

Case ($\rightarrow e$): we have, for some σ ,

$$\mathcal{B}^*, \beta^* z \vdash_R (\sigma \rightarrow \gamma^*)Y; \quad \mathcal{B}^*, \beta^* z \vdash_R \sigma z.$$

In the first of these deductions, $\beta^* z$ cannot occur, because z is not free in Y . In the second, no member of \mathcal{B}^* can occur, and the only rules possible are (ω), ($\wedge e$), (\leq). Hence we have

$$\mathcal{B}^* \vdash_R (\sigma \rightarrow \gamma^*)Y, \quad \beta^* \leq \sigma.$$

Therefore by (\leq) ,

$$\mathcal{B}^* \vdash (\beta^* \rightarrow \gamma^*)Y.$$

Now $\beta \sim \beta^*$ and $\gamma \sim \gamma^*$, so again by (\leq) , $\mathcal{B} \vdash (\beta \rightarrow \gamma)Y$.

Case (n1): This case can be reduced to Case $(\rightarrow e)$ just like Subcase 1c of the β -invariance theorem.

REFERENCES

- BARENDREGT, H. (1977): "The type-free λ -calculus", in "Handbook of Mathematical Logic", edited by J. Barwise, North-Holland Co., 1977.
- BARENDREGT, H., COPPO, M., DEZANI, M. (198-): "A filter lambda model and the completeness of type assignment", to appear, J. Symbolic Logic.
- COPPO, M., DEZANI, M. (1978): "A new type-assignment for λ -terms", Archiv. Math. Logik 19, pp.139-156 (1978).
- COPPO, M., DEZANI, M., VENNERI, B. (1980): "Principal type-schemes and λ -calculus semantics", in "To H.B. Curry", edited by J.P. Seldin and J.R. Hindley, Academic Press 1980.
- COPPO, M., DEZANI, M., VENNERI, B. (1981): "Functional characters of solvable terms", Z. Math. Logik 27, pp.45-58 (1981).
- HINDLEY, J.R. (1983): "The completeness theorem for typing λ -terms", to appear, Theoretical Computer Science 22 (1983).
- HINDLEY, J.R., LONGO, G. (1980): "Lambda-calculus models and extensionality", Z. Math. Logik 26, pp.289-310 (1980).
- POTTINGER, G. (1980): "A type assignment for the strongly normalizable λ -terms", in "To H.B. Curry", edited by J.P. Seldin and J.R. Hindley, Academic Press 1980.
- SALLÉ, P. (1978): "Une extension de la théorie des types", Lecture Notes in Computer Science 62, pp.398-410, Springer-Verlag 1978.