Notes on Higher-Order Matching

Andrej Dudenhefner Aleksy Schubert

September 30, 2025

1 Bounded Parallel Intersection Type System

We denote *intersection types* by A, B, finite sets of intersection types by σ, τ, and *type atoms* by a, b.

Definition 1 (Intersection Types).
$$\begin{array}{rcl} A, B & ::= & a \mid \sigma \to A \\ \sigma, \tau & ::= & \{A_1, \ldots, A_n\} \textit{ where } n \geq 0 \end{array}$$

An *environment*, denoted by Γ, is a finite set of *type assumptions* having the shape $x : \sigma$ for distinct term variables.

Definition 2 (Environment). $\Gamma ::= \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ *where* $x_i \neq x_j$ *for* $i \neq j$.

We may extend an environment Γ by an additional assumption $x : \sigma$, written $\Gamma, x : \sigma$, where x does not appear in any assumption in Γ.

We denote vectors of types or environments by $\bar{A}$, $\bar{\sigma}$, or $\bar{\Gamma}$ respectively. If $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $\bar{A} = (A_1, \ldots, A_n)$, then $\sigma \Rightarrow A = (\sigma_1 \to A_1, \ldots, \sigma_n \to A_n)$.

Borrowing notation [DU21] we define type vector transformations as follows.

Definition 3 (Type Vector Transformation). *Let* $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ *be a surjective function.*

- $f(A_1, \ldots, A_n) = (C_1, \ldots, C_m)$ *such that* $C_i = \{A_j \mid f(j) = i\}$
- $f^{-1}(B_1, \ldots, B_m) = (D_1, \ldots, D_n)$ *such that* $D_i = B_{f(i)}$

We tacitly extend the definition to environments where set formation is taken pointwise.

Definition 4 (Parallel Intersection Type System).

$$\frac{}{\{x : \bar{A}\} \vdash x : \bar{A}}\ (\mathrm{Ax}) \qquad \frac{\bar{\Gamma}, x : \bar{\sigma} \vdash t : \vec{A}}{\bar{\Gamma} \vdash \lambda x.t : \bar{\sigma} \Rightarrow \bar{A}}\ (\Rightarrow\mathrm{I})$$

$$\frac{\bar{\Gamma} \vdash t : f(\bar{A}) \Rightarrow \bar{B} \qquad \bar{\Delta} \vdash u : \bar{A}}{\bar{\Gamma} \cup f(\bar{\Delta}) \vdash t\ u : \bar{B}}\ (\Rightarrow\mathrm{E})$$

Definition 5 (Dimensional Restriction). $\bar{\Gamma} \vdash_k t : \bar{A}$ *means that the length of vectors in some derivation is at most* k.

Theorem 6. $\Gamma \vdash_{\mathrm{BCD}} t : A$ *iff for some* k *we have* $(\Gamma) \vdash_k t : (A)$.

Theorem 7. *Given* Γ*,* A*, and* k *it is decidable whether* $(\Gamma) \vdash_k t : (A)$ *holds.*

References

[DU21] Andrej Dudenhefner and Paweł Urzyczyn. “Kripke Semantics for Intersection Formulas”. In: *ACM Trans. Comput. Log.* 22.3 (2021), 15:1–15:16. DOI: `10.1145/3453481`. URL: `https://doi.org/10.1145/3453481`.