Notes on Higher-Order Matching

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1 Bounded Parallel Intersection Type System

We denote *intersection types* by A, B, finite sets of intersection types by σ, τ, and *type atoms* by a, b.

Definition 1 (Intersection Types).
$$\begin{array}{rcl} A, B & ::= & a \mid \sigma \to A \\ \sigma, \tau & ::= & \{A_1, \ldots, A_n\} \textit{ where } n \geq 0 \end{array}$$

An *environment*, denoted by Γ, is a finite set of *type assumptions* having the shape $x : \sigma$ for distinct term variables.

Definition 2 (Environment). $\Gamma ::= \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ *where* $x_i \neq x_j$ *for* $i \neq j$.

We may extend an environment Γ by an additional assumption $x : \sigma$, written $\Gamma, x : \sigma$, where x does not appear in any assumption in Γ.

We denote vectors of types and sets of types by $\bar{A}$ and $\bar{\sigma}$ respectively. If $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $\bar{A} = (A_1, \ldots, A_n)$, then $\sigma \Rightarrow A = (\sigma_1 \to A_1, \ldots, \sigma_n \to A_n)$.

Extending notation [DU21] from surjective functions to binary relations we define type vector transformations as follows.

Definition 3 (Type Vector Transformation). *Let* $R \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}$ *be a left-total and right-total binary relation.*

- $R(A_1, \ldots, A_n) := (\tau_1, \ldots, \tau_m)$ *where* $\tau_i = \{A_j \mid j\, R\, i\}$
- $R(\sigma_1, \ldots, \sigma_n) := (\tau_1, \ldots, \tau_m)$ *where* $\tau_i = \bigcup\{\sigma_j \mid j\, R\, i\}$
- $R(\{x_1 : \sigma_1, \ldots, x_l : \sigma_l\}) := \{x_1 : R(\sigma_1), \ldots, x_l : R(\sigma_l)\}$

Example 4. *For* $R = \{(1,1), (1,3), (2,1), (3,2), (3,3)\}$ *we have*

$$R(a, b, c) = (\{a, b\}, \{c\}, \{a, c\})$$

Definition 5 (Parallel Intersection Type System).

$$\frac{}{\{x : \bar{A}\} \vdash x : \bar{A}}\ (\mathrm{Ax}) \qquad \frac{}{\emptyset \vdash t : ()}\ (\omega)$$

$$\frac{\Gamma, x : \bar{\sigma} \vdash t : \vec{A}}{\Gamma \vdash \lambda x.t : \bar{\sigma} \Rightarrow \bar{A}}\ (\Rightarrow\mathrm{I}) \qquad \frac{\Gamma \vdash t : R(\bar{A}) \Rightarrow \bar{B} \qquad \Delta \vdash u : \bar{A}}{\Gamma \cup R(\Delta) \vdash t\, u : \bar{B}}\ (\Rightarrow\mathrm{E})$$

Example 6.
Let

- $\Gamma_1 = \{x : (\{b, c\} \to a)\}$
- $\Gamma_2 = \{y : (a \to b, a \to c), z : (a, a)\}$
- $R_1 = \{(1, 1), (2, 1)\}$
- $R_2 = \{(1, 1), (1, 2)\}$

We have the following derivation

$$\dfrac{\Gamma_1 \vdash x : R_1(b, c) \Rightarrow (a) \qquad \dfrac{\{y : (a \to b, a \to c)\} \vdash y : R_2(a) \Rightarrow (b, c) \qquad \{z : (a)\} \vdash z : (a)}{\Gamma_2 \vdash y\, z : (b, c)}}{\Gamma_1 \cup R_1(\Gamma_2) \vdash x\, (y\, z) : (a)}$$

Definition 7 (Dimensional Restriction)**.** $\Gamma \vdash_k t : \bar{A}$ *means that the length of vectors in some derivation of* $\Gamma \vdash t : \bar{A}$ *is at most* k*.*

Theorem 8. $\Gamma \vdash_{\mathrm{CDV}} t : A$ *iff for some* k *we have* $\Gamma \vdash_k t : (A)$*.*

Proposition 9. *Given* Γ*,* $\bar{A}$*, and* k *it is decidable whether* $\Gamma \vdash_k t : \bar{A}$ *holds for some* t*.*

(Proof Idea). In order to decide whether $\Gamma \vdash_k t : \bar{A}$ holds for some t, it suffices to consider t in β-normal form. For an ASPACE decision algorithm we need to limit the effective number of variables in type environments. Assume $x : \bar{\sigma}$ occurs in some environment for an inhabitant in β-normal form. Then each type B which occurs in each set in each component of $\bar{\sigma}$ is a subformula of some type occurring in Γ or A. Therefore, there is finitely many distinct $\bar{\sigma}$. By the pigeonhole principle, if there are more variables in a type environment, some must have identical types and thus a redundant copy can be removed. Therefore, there are finitely many environements and types to consider throughout the cesition procedure. □

Remark 10. *For fourth order matching* k *is the product for the number of fragments of the right-hand sides. However, intersection types in* Γ *come from subject expansion of an intersection typing for an arbitrary candidate solution. Therefore, not a sufficient restriction (at first glance). However, one can try to relate types in* Γ *to given simple types in the given fourth-order matching problem, possibly obtaining a finite bound of occurring types.*

Definition 11 (Refinement)**.** *An intersection type* A *refines a simple type* φ*, written* $A \prec \varphi$ *if*

- A *is a constant and* $\varphi = \bullet$
- $A = \sigma \to B$ *and* $\varphi = \varphi_1 \to \varphi_2$ *and* $B \prec \varphi_2$ *and for all* $A_1 \in \sigma$ *we have* $A_1 \prec \varphi_1$

Definition 12 (Shift)**.** *We consider type atoms as words of natural numbers.*

- $\downarrow_i w = wi$
- $\downarrow_i(\{A_1, \ldots, A_n\} \to A) = (\{\downarrow_i A_1, \ldots, \downarrow_i A_n\} \to \downarrow_i A)$

For example $\downarrow_2(\{12, 3\} \to \varepsilon) = \{122, 32\} \to 2$.

Lemma 13. *If r is in β-normal form and $\Phi \vdash_{\mathrm{STLC}} r : \varphi$ is a η-long derivation, then there exists Γ, A such that*

1. $\Gamma \prec \Phi$ *and* $A \prec \varphi$
2. $\Gamma \vdash r : A$
3. *for all t such that $\Phi \vdash_{\mathrm{STLC}} t : \varphi$ is an η-long derivation (not necessarily β-normal) and $\Gamma \vdash t : A$, then $t \twoheadrightarrow_\beta r$*

(Proof Sketch). First we focus on β-normal forms. The crucial step application. Consider the example term $x\, r_1\, r_2$. By inversion we have $(x : \varphi_1 \to \varphi_2 \to \bullet) \in \Phi$ and $\Phi \vdash_{\mathrm{STLC}} r_i : \varphi_i$. By induction hypothesis there are $\Gamma_1, A_1, \Gamma_2, A_2$ satisfying the above conditions. We set $\Gamma = (x : \downarrow_1 A_1 \to \downarrow_2 A_2 \to \varepsilon) \cup \downarrow_1 \Gamma_1 \cup \downarrow_2 \Gamma_2$ and $A = \varepsilon$. $\square$

References

[DU21] Andrej Dudenhefner and Paweł Urzyczyn. "Kripke Semantics for Intersection Formulas". In: *ACM Trans. Comput. Log.* 22.3 (2021), 15:1–15:16. DOI: 10.1145/3453481. URL: https://doi.org/10.1145/3453481.