Fundamenta Informaticae XXI (2012) 253–275
DOI 10.3233/FI-2012-778
IOS Press

Intersection Types from a proof-theoretic perspective

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Abstract. In this work we present a proof-theoretical justification for the intersection type assignment system (**IT**) by means of the logical system Intersection Synchronous Logic (**ISL**). **ISL** builds classes of equivalent deductions of the implicative and conjunctive fragment of the intuitionistic logic (**NJ**). **ISL** results from decomposing intuitionistic conjunction into two connectives: a *synchronous* conjunction, that can be used only among equivalent deductions of **NJ**, and an *asynchronous* one, that can be applied among any sets of deductions of **NJ**. A term decoration of **ISL** exists so that it matches both: the **IT** assignment system, when only the synchronous conjunction is used, and the simple types assignment with pairs and projections, when the asynchronous conjunction is used. Moreover, the proof of strong normalization property for **ISL** is a simple consequence of the same property in **NJ** and hence strong normalization for **IT** comes for free.

Keywords: Intersection types, λ-calculus, type assignment systems, structural proof-theory

*Work partially supported by CNPq and FAPEMIG.
†Work partially supported by PRIN '07 CONCERTO.

1. Introduction

The intersection type assignment system (**IT**) [5] is a deductive system that assigns formulae (built from the intuitionistic implication $\rightarrow$ and the intersection $\cap$) as types to the untyped λ-calculus. **IT** has been used as an investigation tool for a large variety of problems like, for example, characterizations of the strongly normalizing λ-terms [14].

The main goal of this work is to give a proof-theoretical justification to **IT**. This goal sounds very much alike, for example, to the one of giving a proof-theoretical characterization of linear functions. To that purpose, one could use λ-terms with exactly one single occurrence of every free and bound variable. However, proof-theoretically equivalently, the same set, under the "derivations-as-programs" analogy, is characterized by a deductive system of second order propositional logic without weakening and contraction.

To our aim, a basic step is to clarify, within a pure logical system, the difference between the connectives intersection ($\cap$) and intuitionistic conjunction ($\wedge$), by imposing constraints on the use of the logical and structural rules of intuitionistic logic. Recall that deductions of **IT** form a strict subset of deductions of the implicative and conjunctive fragment of Intuitionistic logic (which we will call **NJ**, by abusing the notation), in the sense that the λ-terms to which **IT** gives types to are used as meta-theoretical modalities. More specifically, for every $\Pi : \Gamma \vdash_{\mathrm{IT}} M : \sigma$ of **IT**, the term M records where $\rightarrow$-introductions and eliminations are used inside Π. Then the intersection can be introduced only between formulae typing the *same* term. Hence the rule for the introduction of the intersection $(\cap I)$ can be seen, roughly speaking, as a "mistaken decoration" of the rule for the introduction of the conjunction $(\wedge I)$ of **NJ**, where pairs are forgotten:

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash (M, N) : \sigma \wedge \tau}\ (\wedge I) \qquad\qquad \frac{\Gamma \vdash_{\mathbf{IT}} M : \sigma \quad \Gamma \vdash_{\mathbf{IT}} M : \tau}{\Gamma \vdash_{\mathbf{IT}} M : \sigma \cap \tau}\ (\cap I)$$

In order to evidence, at the level of λ-terms, the difference between the usual conjunction $\wedge$ of **NJ** and the intersection $\cap$ of **IT**, we start by defining a non standard decoration for **NJ** called **NJR** (**R**efinement of **NJ**) that has explicit structural rules and where the original conjunction $\wedge$ is split into two conjunctions $\cap$ and $\&$, whose introduction rules are the following:

$$\frac{\Gamma \vdash_{\mathrm{NJR}} M : \sigma \quad \Gamma \vdash_{\mathrm{NJR}} M : \tau}{\Gamma \vdash_{\mathrm{NJR}} M : \sigma \cap \tau}\ (\cap I) \qquad \frac{\Gamma \vdash_{\mathrm{NJR}} M : \sigma \quad \Gamma \vdash_{\mathrm{NJR}} N : \tau}{\Gamma \vdash_{\mathrm{NJR}} (M, N) : \sigma \& \tau}\ (\& I)$$

Note that the two rules are not mutually exclusive, so the system is not deterministic. Moreover the splitting of the conjunction cannot be expressed directly inside **NJ** without collapsing $\cap$ and $\&$.

From **NJR**, we build **ISL** (Intersection Synchronous Logic), a logical system that internalizes this splitting, maintaining explicit the structural rules. The rules of **ISL** inductively build in parallel a class of **NJ** deductions. Such class of deductions of **NJ** is formalized in **ISL** through the notion of *molecule*, as multiset of *atoms*, each one representing a **NJ** deduction. The atoms of a molecule are built in a *synchronous* way, in the sense that all rules are applied in the same time to all atoms of a molecule. In this way a new conjunction can be defined, $\cap$, which can be introduced only between two atoms of the same molecules. That conjunction is dubbed as *synchronous*, to recall that it can be applied only on formulae built in a synchronous way, and it corresponds to the intersection operator of **IT** in a very precise way. Namely if we restrict **ISL** by erasing the conjunction $\&$, then **IT** can be obtained through a decoration of it, according to the standard notion of decoration.

We shall conclude by saying that the present work gives a proof-theoretical justification for **IT** since **ISL**: (i) highlights the role of structural rules to delineate **IT** inside intuitionistic logic; (ii) reinterprets the intersection operator $\cap$ of **IT** in terms of an operator that can be used among sets of structurally equivalent deductions of intuitionistic logic; (iii) reformulates the tree structures that **IL** [18] required in order to characterize **IT**. The reformulation is in terms of (simultaneous) logical and structural operations on the equivalence classes. Finally, **ISL** is technically good, since it enjoys strong normalization and subformula properties.

The rest of the paper is organized as follows: Section 2 recalls the implicative and conjunctive fragment of Intuitionistic Logic (**NJ**) and introduces the system **NJR**, a refinement of the standard decoration for **NJ**. The Intersection Types Assignment System (**IT**), with explicit weakening and exchange rules, is then introduced as a subsystem of **NJR**. Section 3 introduces **ISL**, which embodies all our intuitions into a formal system. Section 4 formalizes how **ISL**, **NJ** and **IT** correspond. Also in this section, we give a technical justification for the conditions on the explicit structural rules, required to reformulate **IT** in terms of **ISL**. Section 5 proves that **ISL** is a good deductive system and describes the behavior of the two **ISL** conjunctions with respect to the implication. Finally, Section 6 describes the relationship between this and some related work and presents initial thoughts on some other theoretical aspects of **ISL** that can be further explored in a future work.

2. Splitting the conjunction

In this section we first recall the implicative and conjunctive fragment of Intuitionistic Logic (**NJ**) in natural deduction style. We then present **NJR**, a type assignment system based on the splitting of the standard conjunction into two connectives, each one catching a particular aspect of its behavior. Finally, the Intersection Types Assignment System (**IT**) will be presented as a subsystem of **NJR**.

Definition 2.1. ($\{\wedge \rightarrow\}$-fragment of NJ.)

i) The set $\mathcal{F}_{\mathbf{NJ}}$ of *formulae* of the implicative and conjunctive fragment of **NJ** are generated by the grammar: $\sigma ::= a \mid \sigma \rightarrow \sigma \mid \sigma \wedge \sigma$, where a belongs to a denumerable set of constants. As usual, $\rightarrow$ is right-associative while $\wedge$ is left-associative. Formulae of **NJ** will be denoted by Greek small letters.

ii) A *context* is a *finite sequence* $\sigma_1, \ldots, \sigma_m$ of formulae. Contexts are denoted by Γ and Δ.

iii) The implicative and conjunctive fragment of **NJ** proves statements of the shape $\Gamma \vdash_{\mathrm{NJ}} \sigma$, where Γ is a context and σ a formula. The rules are in Figure 1. $\Pi : \Gamma \vdash_{\mathrm{NJ}} \sigma$ means that the deduction Π proves $\Gamma \vdash_{\mathrm{NJ}} \sigma$. Finally, $\vdash_{\mathrm{NJ}} \sigma$ abbreviates $\emptyset \vdash_{\mathrm{NJ}} \sigma$.

Note that the contraction rule is derivable. By somewhat abusing the name, **NJ** will always name the implicative and conjunctive fragment of **NJ**.

NJR is a type assignment for λ-terms with pairs. It splits the original conjunction $\wedge$ of **NJ** into two conjunctions, depending on the form of the λ-terms M and N that could be typed by the premises $\Gamma \vdash_{\mathrm{NJ}} \sigma$ and $\Gamma \vdash_{\mathrm{NJ}} \tau$. If M and N are different, then the only possible refinement of $\wedge$ is the $(\&)$ that gives type to the pair (M, N). On the other hand, if the conclusion of the two premises is the type of the same λ-term, it is also possible to replace $\wedge$ by a different conjunction $(\cap)$. $\&$ and $\cap$ will be called respectively *asynchronous* and *synchronous* conjunction, for reasons that will be given in the following. Observe that **NJR** is *not* a standard decoration of **NJ**, since it is not even a function from proofs to λ-terms: the $(\wedge I)$ rule is decorated in two different ways.

$$\frac{}{\sigma \vdash_{\mathrm{NJ}} \sigma}\ (A) \qquad \frac{\Gamma \vdash_{\mathrm{NJ}} \sigma}{\Gamma, \tau \vdash_{\mathrm{NJ}} \sigma}\ (W) \qquad \frac{\Gamma_1, \sigma_1, \sigma_2, \Gamma_2 \vdash_{\mathrm{NJ}} \sigma}{\Gamma_1, \sigma_2, \sigma_1, \Gamma_2 \vdash_{\mathrm{NJ}} \sigma}\ (X)$$

$$\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \quad \Gamma \vdash_{\mathrm{NJ}} \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}\ (\wedge I) \qquad \frac{\Gamma \vdash_{\mathrm{NJ}} \sigma_l \wedge \sigma_r}{\Gamma \vdash_{\mathrm{NJ}} \sigma_i}\ (\wedge E^i)(i = l, r)$$

$$\frac{\Gamma, \sigma \vdash_{\mathrm{NJ}} \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma \to \tau}\ (\to I) \qquad \frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \to \tau \quad \Gamma \vdash_{\mathrm{NJ}} \sigma}{\Gamma \vdash_{\mathrm{NJ}} \tau}\ (\to E)$$

Figure 1. The system **NJ**.

Definition 2.2. (NJR.)

i) The set $\mathcal{F}_{\mathbf{NJR}}$ of *formulae* of **NJR** is generated by the grammar that we can obtain from Definition 2.1 by replacing $\sigma ::= \sigma \wedge \sigma$ by $\sigma ::= \sigma \cap \sigma \mid \sigma \& \sigma$, where $\cap$ and $\&$ are the *synchronous* and *asynchronous* conjunctions, respectively.

ii) The set Λ of terms of the λ-calculus is defined by the grammar $M ::= x \mid \lambda x.M \mid MM$, where x belongs to a countable set of variables.
The set Λ_p of terms of the λ-calculus with pairs is obtained by adding the clauses $M ::= (M, M) \mid \pi_l(M) \mid \pi_r(M)$ to the grammar for Λ. As usual, terms will be considered modulo α-conversion and application is left associative. $M \equiv N$ denotes that the λ-terms M and N are equal up to α-conversion.

iii) A **NJR**-context is a finite sequence of pairs $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ that assigns formulae to variables so that $i \neq j$ implies $x_i \not\equiv x_j$. By abusing the notation, **NJR**-contexts will be denoted by Γ. If $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, then $dom(\Gamma) = \{x_1, ..., x_n\}$.

iv) **NJR** derives judgments $\Gamma \vdash_{\mathrm{NJR}} M : \sigma$ where $M \in \Lambda_p$, Γ is an **NJR**-context, and σ is a formula. The rules of **NJR** are given in Figure 2. Deductions are ranged over by Π, Σ. $\Pi : \Gamma \vdash_{\mathrm{NJR}} M : \sigma$ means that the deduction Π proves $\Gamma \vdash_{\mathrm{NJR}} M : \sigma$.

Intuitively, in **NJR**, $\cap$ merges synchronous sub-deductions, i.e., subdeductions where $\to$ and $\&$ are introduced or eliminated at the "same points".

The erasing function $e : \mathcal{F}_{\mathbf{NJR}} \longrightarrow \mathcal{F}_{\mathbf{NJ}}$ here below, which obviously extends to contexts and deductions, allows to relate **NJR** and **NJ**:

$$e(a) = a \qquad e(\sigma \to \tau) = e(\sigma) \to e(\tau) \qquad e(\sigma \& \tau) = e(\sigma \cap \tau) = e(\sigma) \wedge e(\tau)$$

Theorem 2.3. (Relating NJR and NJ.)

i) If $\Pi : \Gamma \vdash_{\mathrm{NJR}} M : \sigma$ then $e(\Pi) : e(\Gamma) \vdash_{\mathrm{NJ}} e(\sigma)$.

ii) If $\Pi : \Gamma \vdash_{\mathrm{NJ}} \sigma$, then there is $\Pi' : \Gamma' \vdash_{\mathrm{NJR}} M : \sigma'$ such that $e(\Pi') = \Pi$.

Proof:
The proof for both items are straightforward. Note only that, for the part (ii) there is always an obvious Π' where the only connectives are $\to$ and $\&$. $\square$

$$\frac{}{x:\sigma \vdash_{\mathrm{NJR}} x:\sigma}\ (A)$$

$$\frac{\Gamma \vdash_{\mathrm{NJR}} M:\sigma \quad x \notin dom(\Gamma)}{\Gamma, x:\tau \vdash_{\mathrm{NJR}} M:\sigma}\ (W) \qquad \frac{\Gamma_1, x:\sigma_1, y:\sigma_2, \Gamma_2 \vdash_{\mathrm{NJR}} M:\sigma}{\Gamma_1, y:\sigma_2, x:\sigma_1, \Gamma_2 \vdash_{\mathrm{NJR}} M:\sigma}\ (X)$$

$$\frac{\Gamma \vdash_{\mathrm{NJR}} M:\sigma \quad \Gamma \vdash_{\mathrm{NJR}} M:\tau}{\Gamma \vdash_{\mathrm{NJR}} M:\sigma \cap \tau}\ (\cap I) \qquad \frac{\Gamma \vdash_{\mathrm{NJR}} M:\sigma_l \cap \sigma_r}{\Gamma \vdash_{\mathrm{NJR}} M:\sigma_i}\ (\cap E^i (i=l,r))$$

$$\frac{\Gamma \vdash_{\mathrm{NJR}} M:\sigma \quad \Gamma \vdash_{\mathrm{NJR}} N:\tau}{\Gamma \vdash_{\mathrm{NJR}} (M,N):\sigma \& \tau}\ (\& I) \qquad \frac{\Gamma \vdash_{\mathrm{NJR}} M:\sigma_l \& \sigma_r}{\Gamma \vdash_{\mathrm{NJR}} \pi_i(M):\sigma_i}\ (\& E^i (i=l,r))$$

$$\frac{\Gamma, x:\sigma \vdash_{\mathrm{NJR}} M:\tau}{\Gamma \vdash_{\mathrm{NJR}} \lambda x.M:\sigma \to \tau}\ (\to I) \qquad \frac{\Gamma \vdash_{\mathrm{NJR}} M:\sigma \to \tau \quad \Gamma \vdash_{\mathrm{NJR}} N:\sigma}{\Gamma \vdash_{\mathrm{NJR}} MN:\tau}\ (\to E)$$

Figure 2. The system **NJR**.

Now we can define the Intersection Type Assignment System **IT** as a subsystem of **NJR** where only synchronous conjunction is used.

Definition 2.4. (IT.)

i) The set $\mathcal{F}_{\mathbf{IT}}$ of *types* of **IT** is the subset of $\mathcal{F}_{\mathbf{NJR}}$ generated by the grammar: $\sigma ::= a \mid \sigma \to \sigma \mid \sigma \cap \sigma$, where a belongs to a denumerable set of constants.

ii) The Intersection Type Assignment System **IT** proves statements of the shape: $\Gamma \vdash_{\mathrm{IT}} M:\sigma$ where M is a λ-term in Λ, Γ is an **IT**-context, i.e., a finite sequence of pairs $x_1:\sigma_1,\ldots,x_n:\sigma_n$ that assigns formulae to variables so that $i \neq j$ implies $x_i \not\equiv x_j$, and σ is a type. The rules of the system are the rules of **NJR** but $(\& I)$, $(\& E^l)$ and $(\& E^r)$.
$\Pi : \Gamma \vdash_{\mathrm{IT}} M:\sigma$ means that the deduction Π proves $\Gamma \vdash_{\mathrm{IT}} M:\sigma$.

The difference between synchronous, and asynchronous conjunction cannot be expressed inside **NJ**, because it is related to a meta-condition on the form of the deductions. The following example can be useful for better understanding this.

Example 2.5. Let $\sigma = ((\alpha \to \alpha) \to \alpha \to \alpha)\&(\alpha \to \alpha)$ and let us consider the following deduction:

$$\Pi: \quad \frac{\Pi_2'' :\vdash_{\mathrm{NJR}} \lambda x.\pi_l(x)\pi_r(x):\sigma \to \alpha \to \alpha \quad \Pi_1'' :\vdash_{\mathrm{NJR}} (\lambda x.x, \lambda x.x):\sigma}{\vdash_{\mathrm{NJR}} (\lambda x.\pi_l(x)\pi_r(x))(\lambda x.x, \lambda x.x):\alpha \to \alpha}\ (\to E)$$

where Π_1'' is:

$$\frac{\dfrac{\dfrac{}{x:\alpha \to \alpha \vdash_{\mathrm{NJR}} x:\alpha \to \alpha}\ (A)}{\vdash_{\mathrm{NJR}} \lambda x.x:(\alpha \to \alpha) \to \alpha \to \alpha}\ (\to I) \quad \dfrac{\dfrac{}{x:\alpha \vdash_{\mathrm{NJR}} x:\alpha}\ (A)}{\vdash_{\mathrm{NJR}} \lambda x.x:\alpha \to \alpha}\ (\to I)}{\vdash_{\mathrm{NJR}} (\lambda x.x, \lambda x.x):((\alpha \to \alpha) \to \alpha \to \alpha)\&(\alpha \to \alpha)}\ (\& I)$$

and Π_2'' is:

$$\dfrac{\dfrac{\dfrac{}{x:\sigma \vdash_{\text{NJR}} x:\sigma}(A)}{x:\sigma \vdash_{\text{NJR}} \pi_l(x):(\alpha\to\alpha)\to\alpha\to\alpha}(\&E^l) \quad \dfrac{\dfrac{}{x:\sigma \vdash_{\text{NJR}} x:\sigma}(A)}{x:\sigma \vdash_{\text{NJR}} \pi_r(x):\alpha\to\alpha}(\&E^r)}{\dfrac{x:\sigma \vdash_{\text{NJR}} \pi_l(x)\pi_r(x):\alpha\to\alpha}{\vdash_{\text{NJR}} \lambda x.\pi_l(x)\pi_r(x):\sigma\to\alpha\to\alpha}(\to I)}(\to E)$$

In the deduction Π_1'', the conjunction $\&$ has been introduced. But since the two subjects are syntactically the same, we could replace it by $\cap$, so obtaining the following deduction, where $\tau = ((\alpha\to\alpha)\to\alpha\to\alpha)\cap(\alpha\to\alpha)$:

$$\Pi' : \qquad \dfrac{\Pi_2' :\vdash_{\text{NJR}} (\lambda x.xx):\tau\to\alpha\to\alpha \quad \Pi_1' :\vdash_{\text{NJR}} \lambda x.x:\tau}{\vdash_{\text{NJR}} (\lambda x.xx)(\lambda x.x):\alpha\to\alpha}(\to E)$$

where Π_1' is:

$$\dfrac{\dfrac{\dfrac{}{x:\alpha\to\alpha \vdash_{\text{NJR}} x:\alpha\to\alpha}(A)}{\vdash_{\text{NJR}} \lambda x.x:(\alpha\to\alpha)\to\alpha\to\alpha}(\to I) \quad \dfrac{\dfrac{}{x:\alpha \vdash_{\text{NJR}} x:\alpha}(A)}{\vdash_{\text{NJR}} \lambda x.x:\alpha\to\alpha}(\to I)}{\vdash_{\text{NJR}} \lambda x.x:((\alpha\to\alpha)\to\alpha\to\alpha)\cap(\alpha\to\alpha)}(\cap I)$$

and Π_2' is:

$$\dfrac{\dfrac{\dfrac{}{x:\tau \vdash_{\text{NJR}} x:\tau}(A)}{x:\tau \vdash_{\text{NJR}} x:(\alpha\to\alpha)\to\alpha\to\alpha}(\cap E^l) \quad \dfrac{\dfrac{}{x:\tau \vdash_{\text{NJR}} x:\tau}(A)}{x:\tau \vdash_{\text{NJR}} x:\alpha\to\alpha}(\cap E^r)}{\dfrac{x:\tau \vdash_{\text{NJR}} xx:\alpha\to\alpha}{\vdash_{\text{NJR}} \lambda x.xx:\tau\to\alpha\to\alpha}(\to I)}(\to E)$$

Both the previous deductions correspond, in the sense of Theorem 2.3, to the following deduction in **NJ**:

$$\dfrac{\Pi_2 :\vdash_{\text{NJ}} \rho\to\alpha\to\alpha \quad \Pi_1 :\vdash_{\text{NJ}} \rho}{\vdash_{\text{NJ}} \alpha\to\alpha}(\to E)$$

where: $\rho = e(\sigma) = e(\tau) = ((\alpha\to\alpha)\to\alpha\to\alpha)\wedge(\alpha\to\alpha)$ and Π_1, Π_2 are, respectively:

$$\dfrac{\dfrac{\dfrac{}{\alpha\to\alpha \vdash_{\text{NJ}} \alpha\to\alpha}(A)}{\vdash_{\text{NJ}} (\alpha\to\alpha)\to\alpha\to\alpha}(\to I) \quad \dfrac{\dfrac{}{\alpha \vdash_{\text{NJ}} \alpha}(A)}{\vdash_{\text{NJ}} \alpha\to\alpha}(\to I)}{\vdash_{\text{NJ}} ((\alpha\to\alpha)\to\alpha\to\alpha)\wedge(\alpha\to\alpha)}(\wedge I)$$

and

$$\dfrac{\dfrac{\dfrac{}{\rho \vdash_{\text{NJ}} \rho}(A)}{\rho \vdash_{\text{NJ}} (\alpha\to\alpha)\to\alpha\to\alpha}(\wedge E^l) \quad \dfrac{\dfrac{}{\rho \vdash_{\text{NJ}} \rho}(A)}{\rho \vdash_{\text{NJ}} \alpha\to\alpha}(\wedge E^r)}{\dfrac{\rho \vdash_{\text{NJ}} \alpha\to\alpha}{\vdash_{\text{NJ}} \rho\to\alpha\to\alpha}(\to I)}(\to E)$$

We also notice that Curry's type assignment for Λ_p [6] can be seen as a sub-system of **NJR**, where only the asynchronous conjunction is used. In this system, the term $(\lambda x.xx)(\lambda x.x)$ is not typeable.

In order to better understand the problem we want to solve, it is necessary to formalize the notion of synchronous deductions, which has been introduced before in an informal way.

Definition 2.6. (Skeleton of a deduction.)
We call *skeleton of a deduction* Π a tree that contains the names of the rules that occur in Π, but the structural rules and the rules involving $\cap$. The skeleton of a deduction Π, denoted by $\mathcal{SK}(\Pi)$, is defined inductively as follows:

$$\mathcal{SK}\left(\frac{}{x:\alpha \vdash_{\mathrm{NJR}} x:\alpha}\ (A)\right) = \overline{(A)}$$

$$\mathcal{SK}\left(\frac{\Pi}{\Gamma \vdash_{\mathrm{NJR}} M:\sigma}\ (R)\right) = \frac{\mathcal{SK}(\Pi)}{(R)} \qquad \text{if } R \in \{(\to I), (\&E)\}$$

$$\mathcal{SK}\left(\frac{\Pi}{\Gamma \vdash_{\mathrm{NJR}} M:\sigma}\ (R)\right) = \mathcal{SK}(\Pi) \qquad \text{if } R \in \{(W), (X), (\cap E)\}$$

$$\mathcal{SK}\left(\frac{\Pi_1 \quad \Pi_2}{\Gamma \vdash_{\mathrm{NJR}} M:\sigma}\ (R)\right) = \frac{\mathcal{SK}(\Pi_1) \quad \mathcal{SK}(\Pi_2)}{(R)} \qquad \text{if } R \in \{(\&I), (\to E)\}$$

$$\mathcal{SK}\left(\frac{\Pi_1 \quad \Pi_2}{\Gamma \vdash_{\mathrm{NJR}} M:\sigma}\ (\cap I)\right) = \mathcal{SK}(\Pi_1)$$

Then, we say that Π_1 and Π_2 are *synchronous* if and only if $\mathcal{SK}(\Pi_1) = \mathcal{SK}(\Pi_2)$.

The term decoration in **NJR** allows to identify synchronous deductions.

Example 2.7. (Syncronous deductions in NJR.)
Let Π_1' and Π_1'' be:

$$\Pi_1' : \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\alpha_1 \vdash_{\mathrm{NJR}} x:\alpha_1}\ (A)}{x:\alpha_1, y:\delta \vdash_{\mathrm{NJR}} x:\alpha_1}\ (W)}{y:\delta, x:\alpha_1 \vdash_{\mathrm{NJR}} x:\alpha_1}\ (X)}{y:\delta \vdash_{\mathrm{NJR}} \lambda x.x:\alpha_1 \to \alpha_1}\ (\to I)}{y:\delta, z:\gamma \vdash_{\mathrm{NJR}} \lambda x.x:\alpha_1 \to \alpha_1}\ (W)
\qquad
\Pi_1'' : \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\alpha_2 \vdash_{\mathrm{NJR}} x:\alpha_2}\ (A)}{x:\alpha_2, y:\delta \vdash_{\mathrm{NJR}} x:\alpha_2}\ (W)}{x:\alpha_2, y:\delta, z:\gamma \vdash_{\mathrm{NJR}} x:\alpha_2}\ (W)}{y:\delta, x:\alpha_2, z:\gamma \vdash_{\mathrm{NJR}} x:\alpha_2}\ (X)}{y:\delta, z:\gamma, x:\alpha_2, \vdash_{\mathrm{NJR}} x:\alpha_2}\ (X)}{y:\delta, z:\gamma \vdash_{\mathrm{NJR}} \lambda x.x:\alpha_2 \to \alpha_2}\ (\to I)$$

Π_1' and Π_1'' are synchronous, and hence so are Π_1 and Π_2 where:

$$\Pi_1 : \frac{\Pi_1' : y:\delta, z:\gamma \vdash_{\mathrm{NJR}} \lambda x.x:\alpha_1 \to \alpha_1 \quad \Pi_1'' : y:\delta, z:\gamma \vdash_{\mathrm{NJR}} \lambda x.x:\alpha_2 \to \alpha_2}{y:\delta, z:\gamma \vdash_{\mathrm{NJR}} \lambda x.x:(\alpha_1 \to \alpha_1) \cap (\alpha_2 \to \alpha_2)}\ (\cap I)$$

$$\Pi_2 : \quad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{x : \alpha_3 \vdash_{\mathrm{NJR}} x : \alpha_3}\ (A)}{x : \alpha_3, y : \delta \vdash_{\mathrm{NJR}} x : \alpha_3}\ (W)}{y : \delta, x : \alpha_3 \vdash_{\mathrm{NJR}} x : \alpha_3}\ (X)}{y : \delta, x : \alpha_3, z : \gamma \vdash_{\mathrm{NJR}} x : \alpha_3}\ (W)}{y : \delta, z : \gamma, x : \alpha_3 \vdash_{\mathrm{NJR}} x : \alpha_3}\ (X)}{y : \delta, z : \gamma \vdash_{\mathrm{NJR}} \lambda x.x : \alpha_3 \to \alpha_3}\ (\to I)$$

The next Lemma shows that synchronous deductions have the same subject.

Lemma 2.8. (Equal terms, syncronous deductions.)
Let $\Pi_1 : \Gamma \vdash_{\mathrm{NJR}} M : \alpha$ and $\Pi_2 : \Delta \vdash_{\mathrm{NJR}} N : \beta$. If $M \equiv N$, then Π_1 and Π_2 are synchronous.

Proof:
All the names in a skeleton of Π_1 (Π_2) correspond to the term constructors of M (or N). Note that the converse of the theorem is trivially not true, since the skeleton of the following proofs are the same:

$$\cfrac{}{x : \sigma \vdash_{\mathrm{NJR}} x : \sigma}\ (A) \qquad \cfrac{}{y : \sigma \vdash_{\mathrm{NJR}} y : \sigma}\ (A)$$

and the terms are different for $x \not\equiv y$. □

Our goal is to define **ISL**, a deductive system that internalizes, by means of structural properties, the notion of “being synchronous”, encoded by λ-terms, as far as deductions of **NJR** are concerned.

In the following we prove the existence of a canonical form for every deduction of **NJR**, which will be used as technical tool for proving the correctness of **ISL** with respect to **NJR**.

Definition 2.9. (Canonical deductions of NJR.)
Let Π be a deduction of **NJR**. A *canonical sequence* in Π may contain an arbitrary number of instances of (W), immediately followed by an arbitrary number of instances of (X).

Π is *canonical* if every occurrence of (W) and (X) in Π belongs to a *canonical sequence* which is just below an occurrence of (A).

Lemma 2.10. (Existence of canonical deductions in NJR.)
Let Π be a deduction in **NJR**. There is a canonical deduction Π' with the same conclusion of Π.

Proof:
The essential observation is that structural and logical rules commute. The procedure of transforming a given deduction in its canonical counterpart consists of two steps:

1. the one that moves upward all the instances of (W) to form a first part of a canonical sequence, and

2. the one that moves upward all the instances of (X) to conclude the construction of a canonical sequence.

The effectiveness of 1 here above can be proved by induction on Π. The only nontrivial case is the one that requires the introduction of a new instance of (X):

$$\dfrac{\dfrac{\dfrac{\Gamma, x:\alpha \vdash_{\mathrm{NJR}} M:\beta}{\Gamma \vdash_{\mathrm{NJR}} \lambda x.M:\alpha\to\beta}\,(\to I)}{\Gamma, y:\gamma \vdash_{\mathrm{NJR}} \lambda x.M:\alpha\to\beta}\,(W) \quad \Gamma, y:\gamma \vdash_{\mathrm{NJR}} N:\alpha}{\Gamma, y:\gamma \vdash_{\mathrm{NJR}} (\lambda x.M)N:\beta}\,(\to E)$$

Such a deduction must be re-written as:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, x:\alpha \vdash_{\mathrm{NJR}} M:\beta}{\Gamma, x:\alpha, y:\gamma \vdash_{\mathrm{NJR}} M:\beta}\,(W)}{\Gamma, y:\gamma, x:\alpha \vdash_{\mathrm{NJR}} M:\beta}\,(X)}{\Gamma, y:\gamma \vdash_{\mathrm{NJR}} \lambda x.M:\alpha\to\beta}\,(\to I) \quad \Gamma, y:\gamma \vdash_{\mathrm{NJR}} N:\alpha}{\Gamma, y:\gamma \vdash_{\mathrm{NJR}} (\lambda x.M)N:\beta}\,(\to E)$$

Observe that the introduction of a new instance of (X) follows an application of (W), hence it does not contradict 2.

The effectiveness of 2 can then be proved by induction on the deduction that results from 1. □

In Example 2.7, the only derivation in canonical form is Π_1''.

3. The logical system ISL

The coming logical system **ISL** internalizes at the logical level the different behaviors of the two conjunctions of **NJR**. In **NJR** the difference between & and $\cap$ is the fact the the latter can be applied only to synchronous deductions. In **ISL** this is formalized through the notion of *molecule*: roughly speaking, a molecule is a multiset of deductions of **NJ**, where all rules, but ones dealing with $\cap$, are applied in parallel to all its elements.

Definition 3.1. (The system ISL.)

i) The set of formulae of **ISL** coincides with the set of formulae of **NJR**. Contexts are finite sequences of such formulae, ranged over by Δ, Γ.

ii) An *atom* is a pair $(\Gamma; \alpha)$, where the context Γ is a finite sequence of formulae. $\mathcal{A}, \mathcal{B}$ will range over atoms.

iii) A finite multiset of atoms, such that the contexts in all atoms have the same cardinality is called a *molecule*. $[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ denotes a molecule consisting of the atoms $\mathcal{A}_1, \ldots, \mathcal{A}_n$. $\mathcal{M}, \mathcal{N}$ will range over molecules. $\cup$ is multiset union.

iv) **ISL** derives molecules by the rules given in Figure 3.

v) $\vdash_{\mathbf{ISL}} \mathcal{M}$ denotes the existence of an **ISL** deduction rooted at $\mathcal{M}$.

$$\frac{}{[(\alpha_i;\alpha_i)\mid 1\leq i\leq r]}\ (A) \qquad \frac{\mathcal{M}\cup\mathcal{N}}{\mathcal{M}}\ (P)$$

$$\frac{[(\Gamma_i;\beta_i)\mid 1\leq i\leq r]}{[(\Gamma_i,\alpha_i;\beta_i)\mid 1\leq i\leq r]}\ (W) \qquad \frac{[(\Gamma_1^i,\beta_i,\alpha_i,\Gamma_2^i;\sigma_i)\mid 1\leq i\leq r]}{[(\Gamma_1^i,\alpha_i,\beta_i,\Gamma_2^i;\sigma_i)\mid 1\leq i\leq r]}\ (X)$$

$$\frac{[(\Gamma_i,\alpha_i;\beta_i)\mid 1\leq i\leq r]}{[(\Gamma_i;\alpha_i\to\beta_i)\mid 1\leq i\leq r]}\ (\to I) \qquad \frac{[(\Gamma_i;\alpha_i\to\beta_i)\mid 1\leq i\leq r]\quad [(\Gamma_i;\alpha_i)\mid 1\leq i\leq r]}{[(\Gamma_i;\beta_i)\mid 1\leq i\leq r]}\ (\to E)$$

$$\frac{[(\Gamma_i;\alpha_i)\mid 1\leq i\leq r]\quad [(\Gamma_i;\beta_i)\mid 1\leq i\leq r]}{[(\Gamma_i;\alpha_i\&\beta_i)\mid 1\leq i\leq r]}\ (\&I) \qquad \frac{[(\Gamma_i;\alpha_i^l\&\alpha_i^r)\mid 1\leq i\leq r]}{[(\Gamma_i;\alpha_i^k)\mid 1\leq i\leq r]}\ (\&E^k(k=l,r))$$

$$\frac{\mathcal{M}\cup[(\Gamma;\alpha),(\Gamma;\beta)]}{\mathcal{M}\cup[(\Gamma;\alpha\cap\beta)]}\ (\cap I) \qquad \frac{\mathcal{M}\cup[(\Gamma;\alpha_l\cap\alpha_r)]}{\mathcal{M}\cup[(\Gamma;\alpha_k)]}\ (\cap E^k(k=l,r))$$

Figure 3. The rules of **ISL**.

Some comments are in order. All rules of **ISL**, but (P), have the same name than rules in **NJR**, and a similar behavior, but sometimes they work in parallel on all the atoms of a molecule. In particular, structural rules, (A), $(\to I)$ and $(\&E^k)(k=1,2)$ modify at the same time all the atoms of a molecule. $(\to E)$ and $(\&E^k)(k=1,2)$ merge two different molecules, connecting pairwise their atoms. The behavior of $(\cap)$ is local, in the sense that both its introduction and elimination can be made inside a single atom of a molecule.

Making a parallel with the so called *hypersequents*[2], this means that the intersection is an *internal* connective, while the conjunction and implication are *external*. More about the relationship between molecules and hypersequents (in fact, with hyperformulae since **ISL** is in natural deduction style) can be seen in Section 6.

The rule (P) is in some sense a structural rule. We will see in the following that this rule is redundant in a deduction; it allows to split a molecule into two parts, and it is useful for formalizing the normalization proof.

Example 3.2. (Two deductions of ISL.)
Let τ denote $\alpha\to\alpha$, ρ denote $(\tau\to\tau)\&\tau$, and θ denote $(\tau\to\tau)\cap\tau$. First one:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[(\tau;\tau)]}\ (A)}{[(\emptyset;\tau\to\tau)]}\ (\to I)\quad \dfrac{\dfrac{}{[(\alpha;\alpha)]}\ (A)}{[(\emptyset;\tau)]}\ (\to I)}{[(\emptyset;\rho)]}\ (\&I) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{[(\rho;\rho)]}\ (A)}{[(\rho;\tau\to\tau)]}\ (\&E^l)\quad \dfrac{\dfrac{}{[(\rho;\rho)]}\ (A)}{[(\rho;\tau)]}\ (\&E^r)}{[(\rho;\tau)]}\ (\to E)}{[(\emptyset;\rho\to\tau)]}\ (\to I)}{[(\emptyset;\tau)]}\ (\to E)$$

and second one:

$$
\dfrac{
 \dfrac{\dfrac{\dfrac{}{[(\tau;\tau),(\alpha;\alpha)]}\,(A)}{[(\emptyset;\tau\to\tau),(\emptyset;\tau)]}\,(\to I)}{[(\emptyset;\theta)]}\,(\cap I)
 \quad
 \dfrac{\dfrac{\dfrac{\dfrac{}{[(\theta;\theta)]}\,(A)}{[(\theta;\tau\to\tau)]}\,(\cap E^l)\quad \dfrac{\dfrac{}{[(\theta;\theta)]}\,(A)}{[(\theta;\tau)]}\,(\cap E^r)}{[(\theta;\tau)]}\,(\to E)}{[(\emptyset;\theta\to\tau)]}\,(\to I)
}{[(\emptyset;\tau)]}\,(\to E)
$$

These derivations correspond to derivations Π and Π' in Example 2.5, as we will see in the following.

4. ISL, NJ and IT

We state the formal correspondence among **ISL**, **NJR**, **NJ** and **IT**. This is done by decorating deductions of **ISL** by means of λ-terms. The decoration is similar to the one described in [18] and it is inspired by the so called "Curry-Howard isomorphism": every deduction Π of **ISL** is associated to a λ-term. The decorating λ-term is *untyped*, so it does not encode the *whole* structure of Π, but only the order of instances of implicative and asynchronous conjunctive rules.

Definition 4.1. (Decorating deductions of ISL.)

1. Let $\Gamma = \beta_1, ..., \beta_n$ be a context. A decoration of Γ, with respect to a sequence of different variables $x_1, ..., x_n$, is $(\Gamma)^{x_1,...,x_n} = x_1 : \beta_1, ..., x_n : \beta_n$. The symbol s denotes a sequence of pairwise distinct variables.

2. Every Π that proves the molecule $\mathcal{M} = [(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]$ can be decorated so that the result is a type assignment that proves $M_s(\Pi) : (\mathcal{M})^s$, where $(\mathcal{M})^s \equiv [((\Gamma_i)^s; \beta_i) \mid 1 \leq i \leq r]$, and $M_s(\Pi)$ is a λ-term whose free variables FV$(M_s(\Pi))$ occur in the set corresponding to s. The decoration procedure is inductively defined in Figure 4. From now on, we will call $M_s(\Pi)$ by M_s.

3. $\vdash^*_{\mathbf{ISL}} M_s : (\mathcal{M})^s$ denotes the existence of a decorated deduction of **ISL** rooted at $M_s : (\mathcal{M})^s$.

The following theorem shows that **ISL** is as powerful as **NJR**, namely as well as **NJ**. In particular it proves that a molecule represents a set of synchronous deductions of **NJR**.

Theorem 4.2. (ISL and NJR.)
Let $\mathcal{M} = [(\Gamma_1; \alpha_1), \ldots, (\Gamma_m; \alpha_m)]$. Then $\vdash^*_{\mathbf{ISL}} M_s : (\mathcal{M})^s$ if and only if $(\Gamma_i)^s \vdash_{\mathrm{NJR}} M_s : \alpha_i$, for all $1 \leq i \leq m$.

Proof:
Let Π be the deduction of $\mathcal{M}$ and let $\Pi_i : (\Gamma_i)^s \vdash_{\mathrm{NJR}} M_s : \alpha_i$. The "only if" direction holds by induction on Π.

For the "if" direction, observe that, by Lemma 2.8, $\Pi_1, \ldots, \Pi_m$ are pairwise synchronous. Also, by Lemma 2.10, without loss of generality, we may assume that such derivations are canonical. The proof then proceeds by induction on M_s.
For the base case, let M_s be some variable x. Let $\Pi_i : (\Gamma'_i)^s \vdash_{\mathrm{NJR}} M_s : \alpha_i$ $(1 \leq i \leq m)$. This means that every Π_i consists of $p_i > 0$ instances of (A), proving $x : \beta_i \vdash_{\mathrm{NJR}} x : \beta_i$, possibly followed by

- $\Pi : \dfrac{}{[(\alpha_i;\alpha_i) \mid 1 \leq i \leq m]}\,(A) \Rightarrow \dfrac{}{M_x(\Pi) \equiv x : [(x : \alpha_i;\alpha_i) \mid 1 \leq i \leq m]}\,(A^*)$;
- $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i, \alpha_i;\beta_i) \mid 1 \leq i \leq r]}\,(W) \Rightarrow$
  $\dfrac{M_s(\Pi_1) : [((\Gamma_i)^s;\beta_i) \mid 1 \leq i \leq r] \quad x \notin dom(\Gamma)^s}{M_{s,x}(\Pi) \equiv M_s(\Pi_1) : [((\Gamma_i)^s, x : \alpha_i;\beta_i) \mid 1 \leq i \leq r]}\,(W^*)$;
- $\Pi : \dfrac{\Pi_1 : [(\Gamma^i_1, \beta_i, \alpha_i, \Gamma^i_2;\sigma_i) \mid 1 \leq i \leq r]}{[(\Gamma^i_1, \alpha_i, \beta_i, \Gamma^i_2;\sigma_i) \mid 1 \leq i \leq r]}\,(X) \Rightarrow$
  $\dfrac{M_{s_1,y,x,s_2}(\Pi_1) : [((\Gamma^i_1)^{s_1}, y : \beta_i, x : \alpha_i, (\Gamma^i_2)^{s_2};\sigma_i) \mid 1 \leq i \leq r]}{M_{s_1,x,y,s_2}(\Pi) \equiv M_{s_1,y,x,s_2}(\Pi_1) : [((\Gamma^i_1)^{s_1}, x : \alpha_i, y : \beta_i, (\Gamma^i_2)^{s_2};\sigma_i) \mid 1 \leq i \leq r]}\,(X^*)$;
- $\Pi : \dfrac{\Pi_1 : [(\Gamma_i, \alpha_i;\beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\alpha_i \to \beta_i) \mid 1 \leq i \leq r]}\,(\to I) \Rightarrow$
  $\dfrac{M_{s,x}(\Pi_1) : [((\Gamma_i)^s, x : \alpha_i;\beta_i) \mid 1 \leq i \leq r]}{M_s(\Pi) \equiv \lambda x.M_{s,x}(\Pi_1) : [((\Gamma_i)^s;\alpha_i \to \beta_i) \mid 1 \leq i \leq r]}\,(\to I^*)$;
- $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\alpha_i \to \beta_i) \mid 1 \leq i \leq r] \quad \Pi_2 : [(\Gamma_i;\alpha_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\beta_i) \mid 1 \leq i \leq r]}(\to E) \Rightarrow$
  $\dfrac{M_1 : [((\Gamma_i)^s;\alpha_i \to \beta_i) \mid 1 \leq i \leq r] \quad M_2 : [((\Gamma_i)^s;\alpha_i) \mid 1 \leq i \leq r]}{M_s(\Pi) \equiv M_1M_2 : [((\Gamma_i)^s;\beta_i) \mid 1 \leq i \leq r]}(\to E^*)$,
  where $M_1 \equiv M_s(\Pi_1), M_2 \equiv M_s(\Pi_2)$;
- $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\alpha_i) \mid 1 \leq i \leq r] \quad \Pi_2 : [(\Gamma_i;\beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\alpha_i\&\beta_i) \mid 1 \leq i \leq r]}\,(\&I) \Rightarrow$
  $\dfrac{M_s(\Pi_1) : [((\Gamma_i)^s;\alpha_i) \mid 1 \leq i \leq r] \quad M_s(\Pi_2) : [((\Gamma_i)^s;\beta_i) \mid 1 \leq i \leq r]}{M_s(\Pi) \equiv (M_s(\Pi_1), M_s(\Pi_2)) : [((\Gamma_i)^s;\alpha_i\&\beta_i) \mid 1 \leq i \leq r]}\,(\&I^*)$;
- $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\alpha^l_i\&\alpha^r_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\alpha_i) \mid 1 \leq i \leq r]}\,(\&E_k) \Rightarrow$
  $\dfrac{M_s(\Pi_1) : [((\Gamma_i)^s;\alpha^l_i\&\alpha^r_i) \mid 1 \leq i \leq r]}{M_s(\Pi) = \pi_t(M_s(\Pi_1)) : [((\Gamma_i)^s;\alpha_i) \mid 1 \leq i \leq r]}\,(\&E^*_k)$
  where $k \in \{L, R\}$, and, if $X = L$ then $t = l$ else $t = r$;
- $\Pi : \dfrac{\Pi_1 : \mathcal{M}_1}{\mathcal{M}_2}\,(R) \Rightarrow \dfrac{M_s(\Pi_1) : (\mathcal{M}_1)^s}{M_s(\Pi) = M_s(\Pi_1) : (\mathcal{M}_2)^s}\,(R^*)$
  where $R \in \{(\cap I), (\cap E_L), (\cap E_R), (P)\}$.

Figure 4. The decoration of ISL.

a canonical sequence $W_i^j X_i^j$ such that: W_i^j contains instances of (W) that prove $(\Gamma_i^j)^{s_j^i} \vdash_{\mathrm{NJR}} x : \beta_i$, and X_i^j contains instances of (X) that generate deductions Π_i^j that prove $(\Gamma_i')^s \vdash_{\mathrm{NJR}} x : \beta_i$ from $(\Gamma_i^j)^{s_j^i} \vdash_{\mathrm{NJR}} x : \beta_i$ $(1 \leq i \leq m, 1 \leq j \leq p_i)$. Finally, a sequence of applications of rules $(\cap I)$ and $(\cap E)$ are applied, connecting all Π_i^j in $\Pi_i : (\Gamma_i)^s \vdash_{\mathrm{NJR}} x : \alpha_i$ $(1 \leq i \leq m, 1 \leq j \leq p_i)$. Notice that every s_j^i are permutations of s, so the sequences W_i^j have the same length for all i, j and introduce the same variables. Moreover, for every i, all Π_i^j share the same context $(\Gamma_i)^s$ for all $1 \leq j \leq p_i$, since the rule $(\cap I)$ treats contexts in an additive way. But the order variables are added via applications of rule (W) may vary, as well as the type assigned to each of them in different Π_i. Hence the lengths of $X_i^1, \ldots, X_i^{p_i}$ may not be the same. But we can easily modify all deductions in such a way that variables are inserted in the same order and hence exchanges can be done in the same way for all i, j. Thus the applications of (A), (W) and (X) can be thought as being done in *parallel*. So we can build the desired decorated molecule in the following way:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x : [\underbrace{(x : \beta_1; \beta_1), \ldots, (x : \beta_1; \beta_1)}_{p_1}, \ldots\ldots, \underbrace{(x : \beta_m; \beta_m), \ldots, (x : \beta_m; \beta_m)}_{p_m}]}\,(A)}{x : [((\Gamma_1)^{s'}; \beta_1), \ldots, ((\Gamma_1)^{s'}; \beta_1), \ldots\ldots, ((\Gamma_m)^{s'}; \beta_m), \ldots, ((\Gamma_m)^{s'}; \beta_m)]}\,W^*}{x : [((\Gamma_1')^s; \beta_1), \ldots, ((\Gamma_1')^s; \beta_1), \ldots\ldots, ((\Gamma_m')^s; \beta_m), \ldots, ((\Gamma_m')^s; \beta_m)]}\,X^*}{x : [((\Gamma_1')^s; \alpha_1), ..., ((\Gamma_m')^s; \alpha_m)]}\,\delta$$

where W^* and X^* denote respectively the applications of the rule W introducing the same variable on all atoms and the applications of the rule X on the same position on all atoms, $(\Gamma_i)^{s'}$ is a permutation of $(\Gamma_i')^s$ and δ is a sequence, in any order, of all the applications of rules $(\cap I)$ and $(\cap E)$ that have been applied in Π_i, for each $1 \leq i \leq m$.

Now let $M_s \equiv PQ$. The shape of the term says that all the Π_i consist of $p_i \geq 1$ applications of rule $(\rightarrow E)$, connected together by applications of rules dealing with $\cap$. Note that we are sure that there are not applications of structural rules, since the deductions are in canonical form. So we can apply the inductive hypothesis directly to all the left premises of $(\rightarrow E)$, deriving the existence of a molecule collecting all of them, and to all their right premises, deriving the existence of a second molecule. To both these molecules the **ISL** rule $(\rightarrow E)$ can be applied, and then we can mimic, in any order, all the applications of $\cap$ on the conclusion.

The remaining cases are similar, but simpler. □

Remark 4.3. **ISL** introduces a stronger notion of synchronicity between **NJR** deductions than that one in Definition 2.6. In fact, in **ISL** also the structural rules are applied in parallel. This is not a restriction, since Theorem 4.2 assures that there is a complete correspondence between the judgments of the two systems. But it is worth noticing that it is a correspondence between judgments, not between deductions, as evident from the proof of Theorem 4.2.

As corollary of the previous theorem, we obtain a formal correspondence between **ISL** and **NJ**.

Theorem 4.4. (ISL and NJ.)
Let $\vdash_{\mathbf{ISL}} [(\Gamma_1; \alpha_1), \ldots, (\Gamma_m; \alpha_m)]$. Then, $e(\Gamma_i) \vdash_{\mathrm{NJ}} e(\alpha_i)$, for every $1 \leq i \leq m$.

Proof:
From Theorems 4.2 and 2.3. □

ISL can be proposed as the logic for **IT**, thanks to the following theorem, whose proof follows directly from Theorem 4.2, since **IT** is a subsystem of **NJR**.

Theorem 4.5. (ISL and IT.)

1. Let $\mathcal{M} = [(\Gamma_1; \alpha_1), \ldots, (\Gamma_m; \alpha_m)]$, where α_i and all types in Γ_i belong to $\mathcal{F}_{\mathbf{IT}}$, and let $\vdash^*_{\mathbf{ISL}} M : (\mathcal{M})^s$ with $M \in \Lambda$, for some s. Then $(\Gamma_i)^s \vdash_{\mathrm{IT}} M : \alpha_i$.

2. If $(\Gamma_i)^{x_1,\ldots,x_n} \vdash_{\mathrm{IT}} M : \alpha_i$ $(1 \leq i \leq m)$, then $\vdash_{\mathbf{ISL}} [(\Gamma_i; \alpha_i) \mid 1 \leq i \leq m]$.

Example 4.6. • (**ISL** and **NJR**.) Theorem 4.2 says that the two synchronous deductions of **NJR** in Example 2.7, Π_1 and Π_2 correspond to the following single decorated **ISL** deduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : [(x : \alpha_i; \alpha_i) \mid 1 \leq i \leq 3]}\,(A)}{x : [(x : \alpha_i, y : \delta; \alpha_i) \mid 1 \leq i \leq 3]}\,(W)}{x : [(x : \alpha_i, y : \delta, z : \gamma; \alpha_i) \mid 1 \leq i \leq 3]}\,(W)}{x : [(y : \delta, x : \alpha_i, z : \gamma; \alpha_i) \mid 1 \leq i \leq 3]}\,(X)}{x : [(y : \delta, z : \gamma, x : \alpha_i; \alpha_i) \mid 1 \leq i \leq 3]}\,(X)}{\lambda x.x : [(y : \delta, z : \gamma; (\alpha_i \to \alpha_i)) \mid 1 \leq i \leq 3]}\,(\to I)}{\lambda x.x : [(y : \delta, z : \gamma; (\alpha_1 \to \alpha_1) \cap (\alpha_2 \to \alpha_2)), (y : \delta, z : \gamma; (\alpha_3 \to \alpha_3))]}\,(\cap I)$$

• (**ISL** and **IT**.) Let $\delta = ((\alpha \to \alpha) \to (\alpha \to \alpha)) \cap (\alpha \to \alpha)$. Let us consider the following deductions Π_1, Π_2, and Π in **IT**. Π_1 is:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : \delta \vdash_{\mathrm{IT}} x : \delta}\,(A)}{x : \delta \vdash_{\mathrm{IT}} x : (\alpha \to \alpha) \to (\alpha \to \alpha)}\,(\cap E^l) \quad \dfrac{\dfrac{}{x : \delta \vdash_{\mathrm{IT}} x : \delta}\,(A)}{x : \delta \vdash_{\mathrm{IT}} x : \alpha \to \alpha}\,(\cap E^r)}{x : \delta \vdash_{\mathrm{IT}} xx : \alpha \to \alpha}\,(\to E)}{\emptyset \vdash_{\mathrm{IT}} \lambda x.xx : \delta \to (\alpha \to \alpha)}\,(\to I)$$

Π_2 is:

$$\dfrac{\dfrac{\dfrac{}{y : \alpha \to \alpha \vdash_{\mathrm{IT}} y : \alpha \to \alpha}\,(A)}{\emptyset \vdash_{\mathrm{IT}} \lambda y.y : (\alpha \to \alpha) \to (\alpha \to \alpha)}\,(\to I) \quad \dfrac{\dfrac{}{y : \alpha \vdash_{\mathrm{IT}} y : \alpha}\,(A)}{\emptyset \vdash_{\mathrm{IT}} \lambda y.y : \alpha \to \alpha}\,(\to I)}{\emptyset \vdash_{\mathrm{IT}} \lambda y.y : \delta}\,\cap I$$

Π is:

$$\dfrac{\Pi_1 \qquad \Pi_2}{\emptyset \vdash_{\mathrm{IT}} (\lambda x.xx)(\lambda y.y) : \alpha \to \alpha}\,(\to E)$$

Theorem 4.5.2 implies that Π corresponds to the following deduction in **ISL**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[(\delta;\delta)]}\,(A)}{[(\delta;(\alpha\to\alpha)\to(\alpha\to\alpha))]}\,(\cap E_L) \quad \dfrac{\dfrac{}{[(\delta;\delta)]}\,(A)}{[(\delta;\alpha\to\alpha)]}\,(\cap E_R)}{[(\delta;\alpha\to\alpha)]}\,(\to E)}{[(\emptyset;\delta\to(\alpha\to\alpha))]}\,(\to I) \quad [(\emptyset;\delta)]}{[(\emptyset;\alpha\to\alpha)]}\,(\to E)$$

where the molecule $[(\emptyset;\delta)]$ has been built by the following deduction:

$$\dfrac{\dfrac{\dfrac{}{[(\alpha\to\alpha;\alpha\to\alpha),(\alpha;\alpha)]}\,(A)}{[(\emptyset;(\alpha\to\alpha)\to(\alpha\to\alpha)),(\emptyset;\alpha\to\alpha)]}\,(\to I)}{[(\emptyset;\delta)]}\,(\cap I)$$

5. Properties of ISL

We prove that **ISL** enjoys properties expected for logical systems, like strong normalization and subformula property. Both proofs follow the method described in [18] showing that, in fact, these properties are inherited from **NJ**. We also discuss the behavior of the implication with respect to the two conjunctions.

Strong normalization. The strong normalization property will be proved by reducing it to the strong normalization of **NJ**.

First of all, we will prove that rule (P) is redundant, and that deductions in **ISL** can assume a canonical form, similarly to those in **NJR**.

Definition 5.1. (Canonical deductions of ISL.)
Let Π a deduction of **ISL**. Let us assume to define the concept of *canonical sequence* in **ISL**, in analogy to the namesake concept in Definition 2.9.

1. Π is *pre-canonical* if it does not contain any occurrences of the rule (P).
2. Π is *canonical* if it is pre-canonical and every occurrence of (W) and (X) in Π belongs to a *canonical sequence* which is just below an occurrence of (A).

Lemma 5.2. (Existence of canonical deductions of ISL.)
Let Π be a deduction in **ISL**. There is a canonical deduction Π' with the same conclusion of Π.

Proof:
First we will prove that an **ISL** deduction can be transformed into an equivalent one in pre-canonical form, i.e., that every application of rule (P) can be eliminated. In case the application of (P) rule follows an axiom rule, i.e.:

$$\dfrac{\dfrac{}{[(\alpha_i;\alpha_i)\mid 1\le i\le r]\cup[(\alpha_j;\alpha_j)\mid r<j\le s]}\,(A)}{[(\alpha_i;\alpha_i)\mid 1\le i\le r]}\,(P)$$

just replace this sequence by:

$$\frac{}{[(\alpha_i;\alpha_i) \mid 1 \le i \le r]}\ (A)$$

Otherwise, it is easy to prove, by induction on the shape of the deduction, that an application of (P) commutes with an application of every other rule. The proof that a **ISL** pre-canonical deduction can be transformed into an equivalent canonical one is analogous to the strategy of proof we used in Lemma 2.10. □

Definition 5.3. (Redexes in ISL.)
Let Π a pre-canonical deduction of **ISL**.

1. A $\cap$-redex of Π is a sequence of the following shape:

$$\frac{\dfrac{\mathcal{M} \cup [(\Gamma;\alpha_l),(\Gamma;\alpha_r)]}{\mathcal{M} \cup [(\Gamma;\alpha_l \cap \alpha_r)]}\ (\cap I)}{\mathcal{M} \cup [(\Gamma;\alpha_k)]}\ (\cap E^k)$$

   where $k \in \{l, r\}$.

2. A $\&$-redex of Π is the sequence:

$$\frac{\dfrac{[(\Gamma_i;\alpha_i^l) \mid 1 \le i \le p] \qquad [(\Gamma_i;\alpha_i^r) \mid 1 \le i \le p]}{[(\Gamma_i;\alpha_i^l \& \alpha_i^r) \mid 1 \le i \le p]}\ (\& I)}{[(\Gamma_i;\alpha_i^k) \mid 1 \le i \le p]}\ (\& E^k)$$

   where $k \in \{l, r\}$.

3. A $\rightarrow$-redex of Π is the sequence:

$$\frac{\dfrac{[(\Gamma_i,\alpha_i;\beta_i) \mid 1 \le i \le r]}{[(\Gamma_i;\alpha_i \rightarrow \beta_i) \mid 1 \le i \le r]}\ (\rightarrow I) \qquad [(\Gamma_i;\alpha_i) \mid 1 \le i \le r]}{[(\Gamma_i;\beta_i) \mid 1 \le i \le r]}\ (\rightarrow E)$$

As usual, the strong normalization proof needs a Substitution Lemma.

Lemma 5.4. (Substitution lemma.)
Let $\Pi_0 :\vdash_{\mathbf{ISL}} [(\Gamma_i,\alpha_i;\beta_i) \mid 1 \le i \le r]$ and $\Pi_1 :\vdash_{\mathbf{ISL}} [(\Gamma_i;\alpha_i) \mid 1 \le i \le r]$. Then there is a deduction $S(\Pi_0,\Pi_1)$ proving $[(\Gamma_i;\beta_i) \mid 1 \le i \le r]$.

Proof:
By induction on the structure of Π_0. We remark that the substitution, in general, does not preserve the canonical deductions. For example, let us consider the case Π_0 is an instance of (A), followed by a sequence s of instances of structural rules. Then, $S(\Pi_0,\Pi_1)$ coincides to Π_1, followed by s, which is not canonical. □

Definition 5.5. (Rewriting steps on ISL.)
Let Π be a canonical deduction of **ISL**.

1. A $\cap$-rewriting step on Π is:

$$\dfrac{\dfrac{\mathcal{M}\cup[(\Gamma;\alpha),(\Gamma;\beta)]}{\mathcal{M}\cup[(\Gamma;\alpha\cap\beta)]}\,(\cap I)}{\mathcal{M}\cup[(\Gamma;\alpha)]}\,(\cap E^l) \quad\hookrightarrow\quad \dfrac{\mathcal{M}\cup[(\Gamma;\alpha),(\Gamma;\beta)]}{\mathcal{M}\cup[(\Gamma;\alpha)]}\,(P)$$

The $(\cap E^r)$ case is analogous.

2. A &-rewriting step on Π is:

$$\dfrac{\dfrac{\Pi_1:\ [(\Gamma_i;\alpha_i)\mid 1\le i\le r] \qquad \Pi_2:\ [(\Gamma_i;\beta_i)\mid 1\le i\le r]}{[(\Gamma_i;\alpha_i\&\beta_i)\mid 1\le i\le r]}\,(\& I)}{[(\Gamma_i;\alpha_i)\mid 1\le i\le r]}\,(\& E^l) \quad\hookrightarrow \Pi_1:\ [(\Gamma_i;\alpha_i)\mid 1\le i\le r]$$

The $(\& E^r)$ case is analogous.

3. A $\rightarrow$-rewriting step on Π is:

$$\dfrac{\dfrac{\Pi_0:\ [(\Gamma_i,\alpha_i;\beta_i)\mid 1\le i\le r]}{[(\Gamma_i;\alpha_i\rightarrow\beta_i)\mid 1\le i\le r]}\,(\rightarrow I) \qquad \Pi_1:\ [(\Gamma_i;\alpha_i)\mid 1\le i\le r]}{[(\Gamma_i;\beta_i)\mid 1\le i\le r]}\,(\rightarrow E) \hookrightarrow S(\Pi_0,\Pi_1)$$

which exists by Lemma 5.4.

In order to prove the strong normalization property, we need to refine Theorem 4.4, stating the correspondence between **ISL** and **NJ**. We assume that the reader knows the notion of redex in **NJ**.

Lemma 5.6. Let Π be a deduction of $\mathcal{M}=[(\Gamma_1;\alpha_1),\ldots,(\Gamma_m;\alpha_m)]$ in **ISL**. There is a set $e(\Pi)=\{\Pi_i : e(\Gamma_i)\vdash_{\mathrm{NJ}} e(\alpha_i)\mid 1\le i\le m\}$ such that for every redex of Π there is a redex in at least one of the deductions of $e(\Pi)$.

Proof:
By structural induction on Π. We develop explicitly the cases when the last rule of Π is (A), $(\cap I)$, and $(\& I)$, all the remaining cases being analogous and simpler. In particular, we observe that every occurrence of (P) in Π is completely transparent in the construction of $e(\Pi)$.

Let Π be $\dfrac{}{[(\alpha_i;\alpha_i)\mid 1\le i\le r]}\,(A)$. Then $e(\Pi)$ is $\left\{\dfrac{}{e(\alpha_i)\vdash_{\mathrm{NJ}} e(\alpha_i)}\,(A)\mid 1\le i\le r\right\}$.

Let Π be $\dfrac{\mathcal{M}'\cup[(\Gamma;\alpha),(\Gamma;\beta)]}{\mathcal{M}'\cup[(\Gamma;\alpha\cap\beta)]}\,(\cap I)$ with $\mathcal{M}'=[(\Gamma'_1;\alpha'_1),\ldots,(\Gamma'_{m'};\alpha'_{m'})]$. By inductive hypothesis we can build the three sets $\{\Pi'_i : e(\Gamma'_i)\vdash_{\mathrm{NJ}} e(\alpha'_i)\mid 1\le i\le m'\}$, $\{\Pi'_l : e(\Gamma)\vdash_{\mathrm{NJ}} e(\alpha)\}$, and $\{\Pi'_r : e(\Gamma)\vdash_{\mathrm{NJ}} e(\beta)\}$. So, we can build

$$\{\Pi'_i : e(\Gamma'_i)\vdash_{\mathrm{NJ}} e(\alpha'_i)\mid 1\le i\le m'\}\cup\left\{\dfrac{\Pi'_l : e(\Gamma)\vdash_{\mathrm{NJ}} e(\alpha)\quad \Pi'_r : e(\Gamma)\vdash_{\mathrm{NJ}} e(\beta)}{e(\Gamma)\vdash_{\mathrm{NJ}} e(\alpha\cap\beta)}\,(\wedge I)\right\}$$

Let Π be $\dfrac{[(\Gamma_i;\alpha_i) \mid 1 \leq i \leq r] \quad [(\Gamma_i;\beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i;\alpha_i\&\beta_i) \mid 1 \leq i \leq r]}\ (\&I)$. By inductive hypothesis we can build the two sets $\{\Pi'_i : e(\Gamma_i) \vdash_{\mathrm{NJ}} e(\alpha_i) \mid 1 \leq i \leq r\}$, and $\{\Pi''_i : e(\Gamma_i) \vdash_{\mathrm{NJ}} e(\beta_i) \mid 1 \leq i \leq r\}$. So, using the pairwise corresponding deductions in the two sets we can build

$$\left\{ \dfrac{\Pi'_i : e(\Gamma_i) \vdash_{\mathrm{NJ}} e(\alpha_i) \quad \Pi''_i : e(\Gamma_i) \vdash_{\mathrm{NJ}} e(\beta_i)}{e(\Gamma_i) \vdash_{\mathrm{NJ}} e(\alpha_i\&\beta_i)}\ (\wedge I) \mid 1 \leq i \leq r \right\}$$

Then the result follows by concatenating two simple observations. The translation just developed here above maps every introduction and elimination of synchronous and asynchronous conjunctions of Π into an introduction and an elimination, respectively, of at least one deductions of $e(\Pi)$. Of course the same holds for implication introductions and eliminations. So, every introduction/elimination sequence inside Π which is a redex translates into a redex of at least one of the deductions of inside $e(\Pi)$. □

Theorem 5.7. (Strong normalization of ISL.)
ISL is strongly normalizable.

Proof:
Lemma 5.2 shows that (P) can be eliminated and that the other structural rules can be moved up from redexes, so they do not play a significant role in the normalization process. We can imagine that every normalization step consists of commuting the rules to get a canonical deduction, on which to apply a rewriting-step. Consider a sequence $\Pi_1 \hookrightarrow \ldots \hookrightarrow \Pi_n$ of normalization steps in **ISL**. Lemma 5.6 implies that every step $\Pi_i \hookrightarrow \Pi_{i+1}$ corresponds to a rewriting-step inside at least one element of $e(\Pi_i)$. Since every element of $e(\Pi_i)$ normalizes, we can bound the number of redexes we need to reduce Π_i to its normal form with the number of redexes we need to reduce all the deductions of $e(\Pi_i)$ to their normal forms. Since every of them is strongly normalizing, then Π_i is strongly normalizing as well. □

From the this result, and Theorem 4.5, we can obtain as corollary the well known property of strong normalization for **IT**.

Corollary 5.8. (Strong normalization of IT.)
IT is strongly normalizing.

This is remarkable since we get the strong normalization for **IT** for free, while most of the known proofs use very complex techniques, like reducibility predicates [10, 20].

Sub-formula property. Sub-formulae in **ISL** are defined as follows:

Definition 5.9. (Sub-formula.)
Let α be a formula of **ISL**. Then:

i. α is a sub-formula of α.

ii. If $\beta \diamond \gamma$ is a sub-formula of α, then so are β and γ for $\diamond \in \{\&, \cap, \rightarrow\}$.

Definition 5.10. (Sub-formula property.)
Let Π be a **ISL** deduction of the molecule $[(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]$. Π enjoys the *sub-formula property*, written **sf**(Π), if every formula appearing in Π is a sub-formula of one of those occurring in $\Gamma_i \cup \{\alpha_i\}$.

Theorem 5.11. (Sub-formula property.)
Let Π be a **ISL** deduction in normal form. Then **sf**(Π).

Proof:
The proof is an easy extension of the same property for **NJ**, given the relationship between **NJ** and **ISL** described by Lemma 5.6. □

The adjoint property. In **NJ**, the conjunction $(\wedge)$ is the adjoint of the implication, that is, the formulae:

$$\alpha \wedge \beta \rightarrow \gamma \text{ and } \alpha \rightarrow \beta \rightarrow \gamma$$

are equivalent.We recall that two formulas α and β are said to be equivalent if $\alpha \vdash_{\mathrm{NJ}} \beta$ and $\beta \vdash_{\mathrm{NJ}} \alpha$. The question that arises then is if the conjunctions of **ISL** $(\&, \cap)$ also have this property.

The answer is *positive* for the asynchronous conjunction. The molecules

$$[(\emptyset; \alpha\&\beta \rightarrow \gamma)] \text{ and } [(\emptyset; \alpha \rightarrow \beta \rightarrow \gamma)]$$

are provable equivalent in **ISL**.

However, the answer is *negative* for the synchronous conjunction $(\cap)$. Indeed, $\alpha \cap \beta \rightarrow \gamma$ implicitly says that α and β depend one from the other as explained in Section 4. Namely, α and β must correspond to the same λ-term, while this is not required by the occurrences of α and β in $\alpha \rightarrow \beta \rightarrow \gamma$.

This same kind of behavior is observed, for example, in Linear Logic where the additive conjunction $(\&)$ is the adjoint of the linear implication, while the multiplicative one $(\otimes)$ is not.

The existence of a synchronous implication of which $\cap$ is left adjoint is open, and we conjecture the non existence of a natural solution.

5.1. The role of the structural rules

In the literature there are many different styles to present intersection types assignment systems. Here we want to consider a "minimal" version, in the sense that only the rules dealing with the two connectives $\rightarrow$ and $\cap$ occur (while there are systems with various kinds of subtyping and eta-rules) and also there is no universal type. The reason for this choice is clear, being this a foundational investigation, and being these extra features not motivated from a logical point of view. But also in this minimal version **IT** is usually presented in a different style, i.e. contexts are *sets* of pairs $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$, and the three rules (A),(W),(X) are replaced by:

$$(A)\frac{x : \sigma \in \Gamma}{\Gamma \vdash_{\mathrm{IT}} x : \sigma}$$

The two formulations are equivalent. But the design of **ISL**, and consequently a logical account of **IT**, needs explicit structural rules.

Indeed, let us assume for a while that **ISL**$'$ be defined from **ISL** by considering contexts as sets and by replacing the rules (A) and (W) by the axiom:

$$\frac{}{[(\Gamma_i \cup \{\alpha_i\}; \alpha_i) \mid 1 \leq i \leq r]}\ (A')$$

Then the following molecules could be proved:

$$[(\{(\alpha \cap \beta) \to \gamma\}; \alpha \to (\beta \to \gamma))] \text{ and } [(\{\alpha \to (\beta \to \gamma)\}; (\alpha \cap \beta) \to \gamma)]$$

hence collapsing $\cap$ to $\&$ (due to the uniqueness of the adjoint). This shows that implicit weakening cannot be used in the definition of **ISL**.

Also, we could pretend to have **ISL**$''$ defined from **ISL** by using contexts as sets (instead of sequences) but maintaining the explicit weakening rule (thus still having a linear axiom). Then it would be possible to derive:

$$\cfrac{\cfrac{[(\{\alpha\}; \alpha), (\{\beta\}; \beta)]}{[(\{\alpha, \beta\}; \alpha), (\{\alpha, \beta\}; \beta)]}\ (W)}{[(\{\alpha, \beta\}; \alpha \cap \beta)]}\ (\cap I)$$

The deduction above does not correspond to any deduction of **IT**. Indeed, let us assume the two atoms $(\{\alpha, \beta\}; \alpha)$ and $(\{\alpha, \beta\}; \beta)$ represent the two judgments $x : \alpha, y : \beta \vdash_{\text{IT}} x : \alpha$ and $x : \alpha, y : \beta \vdash_{\text{IT}} y : \beta$. They have the same context, being, however, labelled by different terms. So $\cap$ cannot be introduced.

Hence, in order to capture correctly the behavior of the intersection connective, we need *both* contexts as sequences and explicit structural rules.

6. Related and future work

The idea of studying the relationship between the intersection and intuitionistic conjunction connectives is not new. In fact, this kind of discussion started with Pottinger's observation [14] that $\cap$ does not correspond to the traditional conjunction (this was later formally proved by Hindley [11]). This subject was further motivated in [1, 3]. But still, the study of the behavior of these two connectives were always restricted to type assignment systems.

The first attempt of giving a logical foundation for **IT** appears in [21], where a new type inference system equivalent to **IT** was defined. This system, called $TA^*_\wedge$ avoids the traditional introduction rule for the intersection, and the logic $L_\wedge$ in a Hilbert-style axiom based formulation was proposed in such a way that combinators in the type assignment system can be associated to logical deductions. This approach is indeed very interesting, and it follows in many ways the ideas already in [14]. Still, the intersection type inference is investigated in the context of combinatory logic instead of λ-calculus and the presentation of the resultant logic is axiomatic. This work was further extended in order to support also union types [7]. The key observation in these two papers is that a logic for intersection types needs to be relevant, i.e., only relevant dependencies between axioms and conclusions need to be taken into account. This aspect of relevance appears also in the axioms of **ISL**, although **ISL** contains a weakening rule, which is absent in the previous cited papers.

In [4], hyperformulae were used in order to obtain the logic HL presented in standard natural deduction style, hence abandoning the axiomatic framework. Molecules are very much alike hyperformulae, the differences consisting in the fact that a context inside an atom (sequent) is a list of formulae (and hence the ordering is crucial), the existence in HL of a distinguished formula ε (the empty formula) and explicit substitutions. This makes the syntax of HL more complicated than the one presented here, but still easier to handle than *kits* appearing in [18] (see comment below).

Another approach on the logical foundation for **IT** is given in [18], where **IL** has been introduced. Roughly speaking, **ISL** can be viewed as **IL** enriched with conjunction. But, although inspired in this former work, the notation designed for **ISL** is completely different from that presented for **IL**, where kits (i.e., trees labeled by formulae) where used in order to keep track of the structure of deductions. The presence of trees introduces a beautiful geometry within the logical system, but at the same time it makes the definition of deductions harder to manipulate, in the sense that it is necessary the introduction of classes of equivalence between deductions in order to define valid deductions. It turns out that kits aren't really necessary: controlling the order of the leaves is enough in this case. That made it possible to choose a much simpler approach based on molecules, where we don't record the shape of deductions, but only group the equivalent ones, step by step. An investigation about the relation between **ISL** and **IL** is in [19].

In any case, the logical systems proposed so far admit the presence of only one between intersection and conjunction, giving the idea that it was impossible to mix them in the same setting. The main contribution of this work is to present a logical system in natural deduction style in which conjunction and intersection can be represented and hence making it possible to characterize, at the proof-theoretical level, the behavior of these two connectives. In this way, the intersection $\cap$ leaves the stigma of being a truly proof-functional connective (as described in [13]) in order to become a connective with synchronous behavior, contrasting with the asynchronous nature of the conjunction.

The present work can be extended in a number of ways.

The first and more natural one is to propose an adequate decoration of **ISL** so that the resulting language supports *discrete polymorphism*, given by the **ISL**'s synchronous conjunction. This result would be comparable with the known fact that second order λ-calculus is the language for *universal polymorphism* [8, 15]. In fact, intersection types provide type polymorphism by listing types instances, differing from universal quantifiers that provide type polymorphism by instantiating quantified type variables for types. It is worthy to note that a logic for **IT** always gives, as sub-product, a typed version of Λ with intersection types, through a complete decoration of deductions [17]. But typed versions of **IT** can be obviously defined following a non logical approach: examples are in [12, 16, 22].

Another interesting problem would be to investigate better the existence or not of a new logical connective $\hookrightarrow$ such that the synchronous conjunction $(\cap)$ would be its adjoint. That is, such that $\vdash_{\mathbf{ISL}}$ $[(\alpha \cap \beta; \gamma)]$ if and only if $\vdash_{\mathbf{ISL}}$ $[(\alpha; \beta \hookrightarrow \gamma)]$. The problem is that, in the formula $\alpha \cap \beta$, it is implicit that α and β are dependent in the sense that they are labeled by the same λ-term. Hence such an arrow $\hookrightarrow$ would have to internalize, within the logical system, this meta-condition. Being more specific, the connective $\hookrightarrow$ needs to be a new arrow that deals with hypothesis which are intersections one piece at a time (as expected from an arrow), but "remembering" partially discharged hypothesis (as needed in intersections).

Acknowledgements. The authors would like to thank the anonymous referees for their very useful comments and remarks.

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