

ADVANCED ABACUS

Japanese Theory and Practice

by

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FOREWORD

Mr. Kojima's second book on the abacus gives important information on the further practical use of the abacus and on the principles of its use in business. I believe that his complete explanation of operational methods and their theoretical basis will be of especial help to those foreign students who have no guide or instructor except books.

Aside from its immense utility in business and everyday calculation, the abacus is a far more effective instrument for teaching arithmetic in blind schools than is braille. Moreover, if introduced into ordinary schools, it will prove an excellent time-saver in arithmetic instruction. Half of the problems in arithmetic textbooks are calculation problems and the other half can be reduced to calculation problems by some mathematical reasoning. Consequently, those arithmetic hours allotted for the teaching of abacus operation, by improving the mental arithmetic of students, will enable them to calculate much faster than with pencil and paper, thus creating additional time for a more advanced study of arithmetic.

As Chairman of the Committee of the International Association of Abacus Operators of the Japan Chamber of Commerce and Industry, I have been most pleased to assist Mr. Kojima by making available the findings of recent technical and theoretical studies and by revising his manuscript in the light of all the latest information.

Yoemon Yamazaki
Professor of Economics, Nihon University
Vice-President, All-Japan Federation of Abacus Operators
Chairman, Committee of the Int. Assn. of Abacus Operators

AUTHOR'S PREFACE

This book has been written as a sequel to my earlier work *The Japanese Abacus: Its Use and Theory*. In this volume I shall both expand the explanation of some of the basic abacus operations given in the first volume and discuss new operations and new ways of doing the basic operations.

In the first chapter I shall discuss the Oriental history of the abacus in greater detail than was done in my first book. In the second chapter I shall deal with the new problem of negative numbers or negative answers resulting from the subtraction of a larger number from a smaller one. The third and fourth chapters will respectively consider other methods of multiplication and division than the ones explained in my first book. Chapter five is an expanded discussion of decimals in multiplication and division. Chapter six is a practical discussion of how to handle calculations involving more than one unit of measurement (such as inches and feet or pounds and ounces). Chapter seven introduces a method of extracting square roots on the abacus. Finally, chapter eight provides more exercises for the reader to practice on. Through it he will be able to measure his abacus ability by taking actual examinations given to Japanese applicants for proficiency grades eight to one.

I would like to include here some noteworthy statistics concerning the recent license examinations. Of the successful examinees for the third-grade license, about 70% were between thirteen and eighteen years old, 5% were nineteen and over, and 25 % were twelve and under (including 0.3 % who were nine and under). Of the successful examinees for the second-grade license, about 91 % were between thirteen and eighteen, 2 % were nineteen and over, and 7% were between ten and twelve. Of the successful examinees for the first-grade license, about 87% were between thirteen and eighteen, 12% were nineteen and over, and 1 % were under twelve. These figures do not completely tell the story because among those who pass the first-grade examination every year are some who have already passed it, but either want the practice or a higher score. This explains why the percentage of persons nineteen and over who passed the first-grade examination is larger than the percentage of the same group who passed the second-grade examination.

An interesting conclusion can be drawn from these statistics—it is rather difficult for persons over nineteen and under twelve to pass the first- and second-grade examinations.

The ratio of boys to girls who passed these examinations is also worthy of mention. Of the successful examinees for the third- and second-grade licenses, about 60% were girls and 40% were boys, while the reverse was true of the first-grade examination.

In Japan the abacus is definitely a practical skill. It has found its way into the curriculum of all Japanese grade schools as a fundamental part of arithmetic. Many senior commercial high schools require all students to pass at least the third-grade examination. There have also been many abacus schools established to meet the needs of those preparing to go into business. And, as I hope I am demonstrating in these two books, there is a good reason why the abacus can be found in practically every Japanese household.

I am greatly indebted to several authorities who kindly furnished me with valuable information and suggestions. My most grateful acknowledgments are due to Professor Yoemon Yamazaki of Nihon University. He is the ViceChairman of the Abacus Research Institute, and Advisor to the Central Committee of the Federation of Abacus Workers (hereafter referred to simply as the Abacus Committee). These organizations, being under the sponsorship of the Japan Chamber of Commerce and Industry, are far and away the largest and most important of all abacus organizations in Japan.

I also extend my most sincere gratitude to Professor Miyokichi Ban, of the above-mentioned Abacus Committee, who was kind enough not only to furnish this book with a great many exercises especially prepared and arranged for the sake of the foreign student but also to read the book in proof and give me many valuable suggestions.

I also must express my sincere thanks to Mr. Shinji Ishikawa, President of the Japan Association of Abacus Calculation, who spared himself no trouble in reading the manuscript and the proof and furnishing much invaluable up-to-date information.

Grateful acknowledgments are also due to Mr. Hisao Suzuki for his information on the history of the abacus and to Mr. Zenji Arai for valuable suggestions on the uses of the abacus.

I also wish to express my sincere thanks to Mr. Yataro Nagata, Chief of the Abacus Operators' License Examinations Section in the Japan Chamber of Commerce and Industry, for information on the national examinations for abacus operators' licenses and for permission to reprint in this book the problems presented in the 1959 National License examinations.

Last, but not least, I must thank Mr. William R. Whitney and the editorial staff of the Charles E. Tuttle Company for their valuable suggestions and improvements in both the manuscript and the proof stages.

TAKASHI KOJIMA

I - FURTHER REMARKS ON ABACUS HISTORY

The ancient Chinese books on mathematics which have been preserved furnish hardly any information on the abacus. Accordingly, nothing definite is known about its origin. The only reliable account of the origin of the Oriental abacus is in a book entitled *Mathematical Treatises by the Ancients* compiled by Hsu Yo toward the close of the Later Han dynasty (A.D. 25—220) at the beginning of the third century and annotated by Chen Luan in the sixth century. This book gives some information about various reckoning devices of those days and was one of the *Ten Books on Mathematics* (Suan-hwei-shi-chu) which were included among the textbooks to be read for government service examinations in China and Japan for many centuries.

Chen Luan in his note gives the following description of the calculating device:

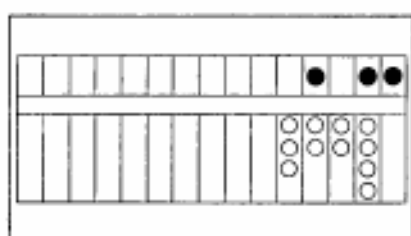


FIG. 1

“The abacus is divided into three sections. In the uppermost and lowest section, idle counters are kept. In the middle section designating the places of numbers, calculation is performed. Each column in the middle section may have five counters, one uppermost five-unit counter and four differently colored one-unit counters.”

The above figure represents the abacus as pictured in accordance with the foregoing description. The board represents the number 37 295.

The extent to which the counting board was used may be told by Hsu Yo's poetical description of the board. The verse, which is highly figurative and difficult to decipher, may read: “It controls the four seasons, and coordinates the three orders, heaven, earth, and man.” This means that it was used in astronomical or calendar calculations, in geodetic surveys, and in calculations concerning human affairs.

The reader will notice a close similarity between this original Oriental abacus and the Roman grooved abacus, except for the difference that counters were laid down in the former while they were moved along the grooves in the latter. Because of this and other evidence, many leading Japanese historians of mathematics and the abacus have advanced the theory that the above-mentioned prototype of the abacus was the result of the introduction into the East of the Roman grooved abacus.

The following corroborative pieces of evidence in favor of this theory are cited in the latest works by Prof. Yoemon Yamazaki and Prof. Hisao Suzuki of Nihon University.

(1) The original Chinese abacus has a striking resemblance in construction to the Roman grooved abacus, as is evident in the foregoing quotation from Hsu Yo's book, e.g., four one-unit counters and one five-unit counter in each column.

(2) The method of operation of the ancient Chinese abacus was remarkably similar to the ancient Roman method.

In ancient China, multiplication and division were performed by the repetition of addition and subtraction:

MULTIPLICATION:

Procedure A: $23 \times 5 = (23 \times 2) + (23 \times 2) + 23 = 115$ (Ans.)

Procedure B: $23 \times 5 = 23 + 23 + 23 + 23 + 23 = 115$ (Ans.)

DIVISION:

Procedure A: $115 \div 23 = 115 - 23 - 46 - 46 = 0$ (Ans. 5)

Procedure B: $115 \div 23 = 115 - 23 - 23 - 23 - 23 - 23 = 0$ (Ans. 5)

In the case of multiplication, each time 23 or 46 was added, 1 or 2 was added to the factor on the left of the board. In the case of division, each time 23 or 46 was subtracted, 1 or 2 was added to the quotient on the left of the board. It is obvious that anyone could easily learn and perform these simple primitive operations.

(3) Traces of reckoning by 5's may be found in the Chinese pictorial representation of reckoning-block calculation as in the Roman numerals, as:

six: VI (5 + 1)

seven: VII (5 + 2)

eight: VIII (5 + 3)

four: IV (5 - 1)

(4) Trade was carried on between China and Rome. Chinese historical documents written in the Han dynasty (206 B.C.-A.D. 220) furnish descriptions of two land routes, called silk roads, connecting the two great empires.

Inasmuch as even in olden days valuable products or devices made in one country were transmitted to others with astonishing rapidity, the above facts may well substantiate this theory.

Among the dozen other reckoning devices mentioned in this book are the reckoning boards pictured below. These boards are presumed to date back to the days of the Chou dynasty, which ended in 249 B.C.

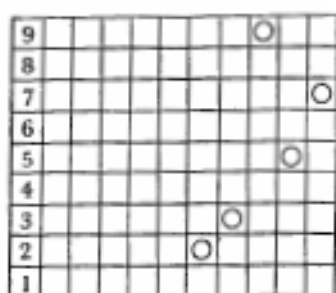


FIG. 2

(The number on the board is 23 957)



FIG. 3

(When yellow counters were used, the squares in each column represented 1, 2, 3, and 4 respectively. When blue ones were used, they represented 5, 6, 7, 8, and 9 respectively. The black balls in the figure stand for blue counters. The number on the board represents 3581.)

These and other reckoning devices are believed to have gone out of use as the previously mentioned abacus developed and gained popularity.

Now two questions present themselves. One is why this ancient abacus developed in the East to become such an efficient calculating machine. The other is why this development did not also take place in the West.

The reasons lay perhaps in the systems of calculation and the numerical nomenclature which were used in the East and West. They differ significantly. In ancient China and Japan numbers were named, written, and set on a calculating board from left to right, from the highest denomination to the lowest. Thus the introduction of the abacus to China provided the Chinese with an ideal tool in terms of their method of naming and using numbers. This compatibility and normal inventiveness caused the primitive abacus to be developed into its modern form during the long development of the Chinese civilization.

The chief calculating devices which are known to have been used in China from before 1 000 B.C. to the days when the abacus came into wide use are reckoning blocks called *ch'eu* in China and *sangi* in Japan and slender bamboo sticks called *chanchu* in China and *zeichiku* in Japan. The former device continued to be used in the East for calculation until not many years ago, and the latter device, which was more awkward, was largely replaced by the former for calculating purposes and is presently used only by fortunetellers for purposes of divination.

Until the introduction of Western mathematics, mathematicians in China and Japan utilized reckoning-block calculation, which had not only been developed to the point of performing basic arithmetic operations but was also used to solve quadratic, cubic, and even simultaneous equations. It is presumed that they did not think it worth while to concern themselves with the other reckoning devices, including the abacus, which was, in their eyes, an inferior calculator barely capable of performing multiplication and division by means of the primitive cumulative method of addition and subtraction. Probably another reason which alienated mathematicians from these reckoning devices was that these instruments gave only the result of calculation, and were incapable of showing either the process of calculation or the original problem.

In ancient times China was primarily a nomadic and agricultural country, and business in those days had little need of instruments of rapid calculation. Anyway a millennium after the Han dynasty there was no record of the abacus. During the dozen centuries beginning with its first mention in the Han dynasty until its development, this primitive calculator remained in the background.

However, with the gradual rise of commerce and industry, the need for rapid calculation grew. The modern, highly efficient abacus, which probably appeared late in the Sung dynasty (906—1279), came into common use in the fourteenth century. The great rise and prosperity of free commerce and industry during the Ming dynasty (1368 - 1636) are presumed to have promoted the use and development of the abacus. A number of books on mathematics brought out in those days give descriptions of the modern Chinese abacus and give accounts of the modern methods of abacus operation, including those of multiplication and division.

Bamboo, indigenous in the East, has furnished an abundant source of ideal material for an efficient and inexpensive abacus. Since the Ming period, on account of its remarkable efficiency, low price, and handiness, the abacus has been the favorite instrument of calculation in the East.

The Chinese abacus of the Ming period had two five-unit counters and five one-unit ones on each rod. The primitive abacus was changed into the present Chinese form to suit the convenience of figuring up the Chinese weights not based on the decimal system. The weights were also important for conversion of currency. Another cogent reason why the Chinese abacus has two five-unit counters on each rod is that a rod with two five-unit counters is more convenient to abacus operation by means of the Chinese method of multiplication and also by means of the older method of division which uses a special division table.

In Europe, the line abacus or counting board appeared first in France about the beginning of the thirteenth century and rapidly became popular. From the fourteenth to the seventeenth century the practice of this manual arithmetic was universal in business and in households, as well as in the departments of government. Its immense popularity may well be illustrated by the following pleasantly expressed stanza attributed to Brébeuf as it is quoted in Francis Pierrepont Barnard's *Casting Counter and the Counting-Board*.

*Les courtisans sont des jetons;
Leur valeur dépend de leur place;
Dans la faveur, des millions;
Et des zéros dans la disgrâce.*

The same book also quotes the phrase, "Faux comme un jeton," which arose from the practice of gilding or plating jettons and passing them as money, or creating a deceptive impression.

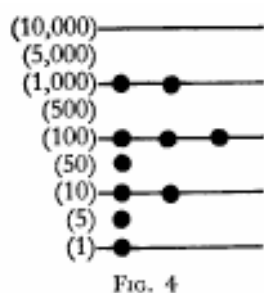


FIG. 4

The number 2 376 would be expressed by jettons on the line abacus or counting board as in Fig. 4.

(Each line upwards is ten times the value of that below it. Each space is five times as much as the line next below it. In addition, the process began at the units, and in subtraction at the higher digits.)

However, in Europe the line abacus failed to develop into the efficient rod abacus, and gradually gave way to the cipher system of greater efficiency, until it was given the *coup de grâce* by the French Revolution, which enforced the nation-wide ciphering system. One of the major causes for this result is presumed to be found in the fact that before the introduction of Arabic numerals European countries used diverse systems of numerical notation—duodecimal, binary, sexagesimal, etc. The division of daytime into twelve hours and that of one hour into sixty minutes, etc. may be mentioned as vestiges of these numerical systems. The rod abacus can never be worked with efficiency on these numerical scales. Another remote cause may be traced back to the way in which the Arabs, who introduced the cipher system into Europe, named their numbers. The Semites, including the Arabs, named their numbers beginning at the units, although they wrote

from right to left. Thus for instance, in Arabic, one hundred and twenty-five was called five and twenty and one hundred, the result appearing as 125, as Prof. Cargill G. Knott says in his treatise on the abacus. This is believed to be the primary reason why the Arabs, who achieved remarkable development in mathematics in the medieval ages, made use of their Arabic numerals without recourse to the less efficient lime abacus or other calculating devices. Nor could the rod abacus have been used with efficiency by a race which used such a system of naming their numbers. The early Indians, who are credited with the invention of the cipher, spoke like the Arabs although they wrote from left to right. The Chinese named their numbers beginning with the largest denomination, although they wrote from top to bottom, proceeding from right to left.

Now the second question before us is: What causes prevented the adoption of a cipher system in China and Japan? The Chinese and the Japanese write in vertical columns from top to bottom, while the cipher system is worked from left to right. However, this is not considered the primary cause, for in the remote past coeval with the origination of Chinese characters, the Chinese carried out their calculation by means of reckoning blocks working left to right. However, this reckoning-block calculation was cumbersome and was no more fit for rapid operation than the Western line abacus.

A couple of examples of the arrangement of reckoning blocks are given below. The numbers 123 and 5 078 are represented:

| | | | | |
|-------|-----|---|---|-----|
| 123 | | I | = | III |
| 5,078 | ≡≡≡ | ○ | ⏟ | ≡≡≡ |

In the units, hundreds, and other odd places, the numbers up to five are each represented by the corresponding number of vertical strokes, and the numbers from six to nine are each represented by the addition of the requisite number of strokes below a five-unit horizontal line. In the tens, thousands, and other even places, the numbers up to five are each represented by the corresponding number of horizontal strokes, and the numbers from six to nine are each represented by the addition of the requisite number of strokes above a five-unit horizontal line.

The Chinese numerical notation, which was probably the pictorial representation of reckoning-block calculation, was of far less practical use in calculation than reckoning blocks. Accordingly, mathematical calculation was generally performed with reckoning blocks and later also with the abacus. In remote antiquity, probably more than 2 000 years back, reckoning blocks were arranged differently for calculation. In those days the numbers in units and hundreds places were represented by horizontal blocks instead of vertical ones, and numbers in tens and thousands places were represented by vertical blocks. Thus the Chinese numerals, 一 (1), 二(2), 三 (3), and 百 (100), comprised of horizontal strokes, are pictographs, representing horizontally arranged blocks (一 = 〇), and the numerals, 十(10), 廿(20), 卅(30), and 千 (1 000), comprised of vertical strokes, are pictographic imitations of blocks arranged vertically.

What caused the change in the arrangement of reckoning blocks is a knotty problem, to which no satisfactory solution has been offered. However, some scholars conjecture that because of the great importance of divination in early China the

arrangement of reckoning blocks might probably have been influenced by the method of arranging reckoning sticks for divining purposes.

Probably the major cause which prevented the replacement of the cumbersome numerical notation by the cipher system was the development of the abacus, which could meet everyday public needs in business and household calculation. Reckoning-block calculation, which was applied to the primitive abacus, made a remarkable development, and by the Ming dynasty the abacus had become a far more efficient computer than the cipher system.

In the days of feudal government, learning was mostly of classics and was the exclusive heritage of certain officials and limited circles of scholars. Mathematics was studied only by the few who were initiated into this mystery of learning, and many of them formed exclusive esoteric sects of hereditary transmission to preserve their patrimonial positions or living. Under these social conditions it was none of their concern to teach or popularize their secrets of mathematics. The enlightened scholars who were favored with exceptional opportunities to study Western science and mathematics may have been aware of the superiority of Arabic numerals to the cumbersome Oriental numerical notation. But these intellectuals must have been too few and far between and their outcry to initiate the reform too feeble to arouse public attention.

Among the other important causes may be mentioned the want of free international trade and communication, the virtual isolation of Eastern countries from the West, and the consequent lack of understanding of international situations and national prejudice against foreign culture, and among the rest, the conservatism of human nature. The Chinese officialdom was so prominently conservative that it would firmly have resisted any attempts at such reforms or improvements in the hoary customs or timehonored classics of national veneration, many of which had been included among textbooks for government service examinations during the long Chinese historical period extending over twenty centuries.

In Japan it was not until several years after the 1868 political revolution, which overthrew the shogunate (government by the supreme feudal ruler), that the progressive modern government, awakened to the progress of the world, enacted the compulsory education law, including in the curriculum the cipher system, without which the effective teaching of modern mathematics to the public is impossible.

Now the Japanese word for abacus, *soro ban*, is probably the Japanese rendering of the Chinese *suan-pan*, (*soo-pan* in the southern dialect or *sur-pan* in Manchuria). The soroban in Japan did not come into common use until the seventeenth century. However, the historical fact that beginning with the seventh century, there were at times as many as 2 000 Japanese students studying at the then Chinese capital in Chang-an, now called Si-an, furnishes us with reliable evidence that the abacus was introduced into Japan at a far earlier date, although the oldest documentary evidence of the Japanese abacus does not date further back than the sixteenth century.

In any case, once this convenient instrument of calculation gained popularity in Japan, it was studied extensively and intensively by many mathematicians including Seki Kowa (1640—1709), who discovered a native calculus independent of the Newtonian theory. As a result, the form and methods of operating the abacus have undergone one improvement after another. For a long time in Japan two kinds of abacus were used

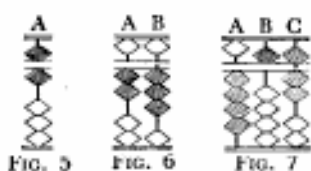
concurrently until the 1868 political revolution: the Chinese-style one with two five-unit counters and five one-unit counters and the older Japanese-style one with one five-unit counter and five one-unit counters. After the time of the revolution, the Chinese-style abacus went completely out of use. Finally since around 1940, the older-style Japanese abacus has largely been replaced by the present more advanced and efficient one with one five-unit counter and four one-unit counters.

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II - NEGATIVE ANSWERS FROM SUBTRACTION

Subtraction of larger numbers from smaller ones is performed by means of complementary numbers.



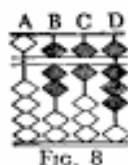
For this purpose it is necessary to be able to read complementary numbers on the board at a glance.

For example, the complementary digit of 6 with respect to 10 is 4. However, the board shows 3 in the position of the complementary digits (Fig. 5).

The complementary numbers of 23 and 457 are 77 and 543 respectively. However, on the board they appear as 76 and 542 (Figs. 6 and 7). Therefore, on the board, the true complementary number can be obtained by adding one to the complementary digit on the last rod.

The following problem involving complements can be solved without making any mental or written calculations.

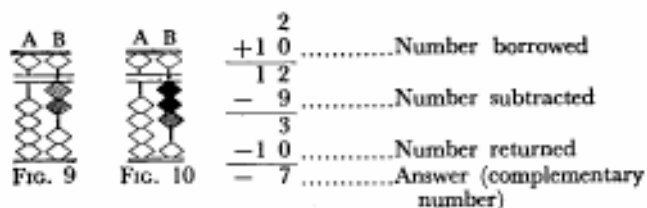
PROBLEM: A customer made a purchase of 7 dollars 68 cents and gave a clerk a ten-dollar bill. How much change should the customer receive?



If the clerk simply sets 768 on the abacus, the answer, 2 dollars 32 cents, will naturally appear on the board of its own accord in the form of the complementary number (Fig. 8).

Here are some examples of abacus calculation by means of complementary digits.

EXAMPLE 1: $2 - 9 = -7$



STEP 1: Set 2 on B (Fig. 9).

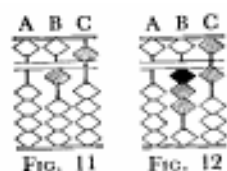
STEP 2: Since you cannot subtract 9 from 2, you must borrow 10 from the tens rod A, and subtract 9 from 12. This gives you 3 on B. However, since you borrowed

10 previously, you must return it. In other words, as 10 was added to the 3 on rod B, you must subtract 10 from it. You can do this very simply by setting minus 10 on rod B in your mind. Then the difference between the 3 and the minus 10 on rod B, i.e. minus 7, will mechanically appear in the form of the complementary number. This is the answer (Fig. 10). Note that the advantage of the abacus operation is to work out and change the difference between 2 and minus 9, i.e. minus 7, into that between 3 and minus 10 and to show the difference in the clearer and more obvious form.

NOTES: (a) In subtracting 9 from 2, do not make the mental calculation of subtracting 9 from 12 but add, to the 2 on B, the complementary number 1, with respect to 10, with the idea that you are subtracting 9 from 10. Nor should you take the trouble of setting 1 on rod A except for practice. (b) In calculation by complementary numbers, it is important for you to remember that the counters or beads which have been moved next

to the beam always indicate positive numbers, while their complements represent negative numbers.

EXAMPLE 2: $15 - 84 = -69$



STEP 1: Set 15 on BC (Fig. 11).

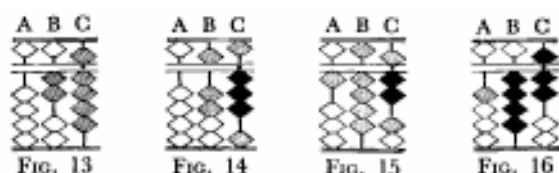
STEP 2: Borrow 100 from the hundreds rod A, and subtract 84 from 115.

This gives you 31 on BC, and the answer 69 mechanically appears on the board in the form of the complementary number of 31 for 100 (Fig. 12).

NOTE: The above process can be analyzed as follows:

$$\begin{array}{r}
 \text{Step 1:} \quad 15 \\
 \text{Step 2:} \quad +100 \dots\dots\dots \text{Number borrowed} \\
 \hline
 115 \\
 -84 \dots\dots\dots \text{Number subtracted} \\
 \hline
 31 \\
 -100 \dots\dots\dots \text{Number returned} \\
 \hline
 -69 \dots\dots\dots \text{Answer (complementary number)}
 \end{array}$$

EXAMPLE 3: $29 - 76 + 94 = 47$



STEP 1: Set 29 on BC (Fig. 13).

STEP 2: Borrow 100 from A, and subtract 76 from 129. This gives you 53 on BC (Fig. 14).

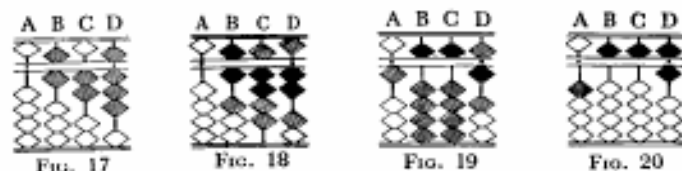
STEP 3: Add 94 to the 53 on BC. This gives you 147 on ABC (Fig. 15).

STEP 4: As you borrowed 100 previously, you must return it. So remove the 1 on A. The result is 47 on BC (Fig. 16).

NOTE: In step 3, the result 147 which you got is larger than the 100 which you borrowed. This shows that the result is positive. Accordingly, the result is 47 on BC.

$$\begin{array}{r}
 29 \\
 + (100) \dots\dots\dots \text{Number borrowed} \\
 \hline
 129 \\
 -76 \dots\dots\dots \text{Number subtracted} \\
 \hline
 (53) \dots\dots\dots \text{The real result is minus 47.} \\
 +94 \dots\dots\dots \text{Number added} \\
 \hline
 147 \\
 - (100) \dots\dots\dots \text{Number returned} \\
 \hline
 47 \dots\dots\dots \text{Answer}
 \end{array}$$

EXAMPLE 4: $628 - 936 + 864 = 556$



STEP 1: Set 628 on BCD (Fig. 17).

STEP 2: Borrow 1 000 from rod A, and subtract 936 from 1 628. This

gives you 692 on BCD (Fig. 18).

STEP 3: Add 864 to the 692 on BCD. This gives you 1 556 on ABCD (Fig. 19).

STEP 4: Remove the 1 on A which you borrowed in step 2. The answer is 556 (Fig. 20).

| | | |
|----------|-------------|------------------------------------|
| Step 1 : | 6 2 8 | |
| Step 2 : | + (1 0 0 0) |Number borrowed |
| | 1 6 2 8 | |
| | - 9 3 6 |Number subtracted |
| | 6 9 2 |The real result is minus 308. |
| Step 3 : | + 8 6 4 |Number added |
| | 1 5 5 6 | |
| Step 4 : | - (1 0 0 0) |Number returned |
| | 5 5 6 |Answer |

EXAMPLE 5: $60 - 84 - 96 = -120$

In making successive subtractions, if it is not sufficient to borrow 100 from the first rod to the left, borrow 1,000 from the second rod to the left at the start of your calculations. We shall call this Method A.

The alternative is to borrow 100 from the first rod to the left in making the first subtraction and later borrow 900 when you need another 100. We shall call this Method B.

Although Method A is more efficient in that you only have to borrow once, you may not be able to anticipate your later need for more. In case you haven't borrowed enough at the beginning you have to resort to Method B.

METHOD A:



FIG. 21



FIG. 22



FIG. 23

STEP 1: Set 60 on CD (Fig. 21).

STEP 2: As it will not be sufficient to borrow 100, borrow 1 000 from rod A, and subtract 84 from 1 060. This gives you 976 on BOD. In borrowing 1 000, set 9 on B and, leaving the 9

intact there, shift 100 to B and C and then subtract 84 from 160. In this way you will automatically get the answer, minus 24, on CD in the form of the complementary number of 976 with respect to 1 000 (Fig. 22).

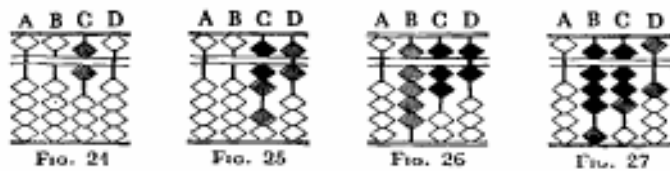
STEP 3: Next subtract 96 from the 976 on BCD. This leaves 880 on BCD. The answer is minus 120, which appears in the form of the complementary number on the board (Fig. 23).

| | | |
|----------|-------------|-----------------------------------|
| Step 1 : | 6 0 | |
| Step 2 : | + (1 0 0 0) |Number borrowed |
| | 1 0 6 0 | |
| | - 8 4 |First number subtracted |
| | 9 7 6 |The real result is minus 24. |
| Step 3 : | - 9 6 |Second number subtracted |
| | 8 8 0 | |
| | - (1 0 0 0) |Number returned |
| | 1 2 0 |Answer |

NOTE: This minus 120 is the difference between the 1 000 which you borrowed and the 880 which you have on the board.

METHOD B:

STEP 1: Set 60 on CD (Fig. 24).



STEP 2: Borrow 100 from rod B, and subtract 84 from the 60 on CD, treating the 60 as 160. This gives you 76 on CD. This number is really minus 24 (Fig. 25).

STEP 3: You cannot subtract 96 from minus 24. So set 9 on B (Fig. 26).

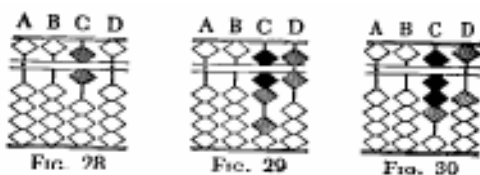
NOTE: Setting 9 on B now produces the same result as having borrowed 1 000 from rod A in the beginning since you borrowed 100 previously from rod B.

STEP 4: Now subtract 96 from the 976 on BCD. This leaves 880 on BCD. The answer is minus 120, which appears in the form of the complementary number (Fig. 27).

| | | |
|---------|----------|------------------------------------|
| Step 1: | 60 | |
| Step 2: | + (100) |Number borrowed |
| | 160 | |
| | - 84 |First number subtracted |
| | 76 |The real result is minus 24. |
| Step 3: | + (900) |Number borrowed again |
| | 976 | |
| Step 4: | - 96 |Second number subtracted |
| | 880 | |
| | - (1000) |Number returned |
| | - 120 |Answer (complementary number) |

NOTE: When you have to borrow again, after borrowing 100, never borrow 100 but 900. If you borrow 100 twice, you cannot get the complementary number for 200 on the board, which slows down your operation.

You will find this in the following steps, in which the above Example 5 is worked by this wrong method.



STEP 1: Set 60 on CD (Fig. 28).

STEP 2: Borrow 100 from rod B, and subtract 84 from the 60 on CD, treating the 60 as 160. This gives you 76 on CD (Fig. 29).

STEP 3: Borrow 100 again from B and subtract 96 from 176, and you get 80 on CD (Fig. 30).

Since you borrowed 200, now you must return 200. But as the board does not show the complementary number of 80 for 200, you must work out the answer by subtracting 80 from 200. Thus this method is rather awkward.

EXAMPLE 6: $55 - 67 - 4\,297 = -4\,309$

STEP 1: Set 55 on DE (Fig. 31).

STEP 2: Borrow 100 from C, and subtract 67 from 55, treating the 55 as 155. This gives you 88 on DE. The result is minus 12 (Fig. 32).

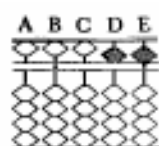


FIG. 31



FIG. 32

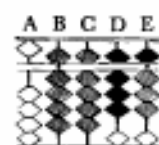


FIG. 33



FIG. 34

$$\begin{array}{rcl}
 \text{Step 1:} & & 55 \\
 \text{Step 2:} & + & (100) \text{.....Number borrowed} \\
 & - & 67 \text{.....First number subtracted} \\
 & = & 88 \text{.....The real result is minus 12.} \\
 \text{Step 3:} & + & (9900) \text{.....Number borrowed again} \\
 & = & 9988 \\
 \text{Step 4:} & - & 4297 \text{.....Second number subtracted} \\
 & = & 5691 \\
 & - & (10000) \text{.....Number returned} \\
 & = & -4309 \text{.....Answer (complementary number)}
 \end{array}$$

STEP 3: As you cannot subtract 4,297 from the 88 on DE, you must borrow 10 000 from rod A. So set 9 on both C and B. You must note that setting 9 on both C and B produces the same result as borrowing 10 000 from A, for the reason that you have previously borrowed 100 from C (Fig. 33).

STEP 4: Now subtract 4 297 from the 9 988 on BCDE. This gives you 5 691 on BCDE. The answer is the complementary number of 5 691 with respect to minus 10 000, i.e. minus 4309, which appears on the board (Fig. 34).

Exercises

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|------|------|-------|------|------|------|-------|-------|---------|---------|
| 167 | 283 | 309 | 434 | 512 | 608 | 274 | 713 | 82 | 903 |
| 280 | 406 | 472 | 159 | 296 | 152 | 460 | 264 | 39 | -75 |
| 349 | 137 | 563 | 618 | -947 | 591 | -958 | 380 | 65 | -829 |
| 265 | 459 | 128 | -703 | 613 | 235 | -215 | 521 | -721 | -6473 |
| 420 | 261 | -796 | -841 | 308 | -764 | 309 | -849 | -9604 | 146 |
| 593 | -825 | 204 | 397 | 247 | 318 | 132 | -706 | 76 | 2704 |
| 178 | -570 | -485 | 204 | 105 | -973 | 341 | -693 | 148 | 92 |
| -837 | -968 | -601 | -972 | 430 | -896 | -604 | -942 | -85 | -98650 |
| -672 | 309 | 815 | 120 | -867 | 409 | -897 | -874 | -629 | -3518 |
| -905 | 147 | 937 | 584 | -698 | 240 | -786 | 150 | -4053 | 637 |
| -159 | -361 | 1 546 | 0 | -1 | -80 | 1 944 | 2 036 | -14 682 | 105 063 |

III - OTHER METHODS OF MULTIPLICATION

There are several methods of multiplication that can be used on the abacus. A comparative study of these methods may be interesting. I hope that it will lead the reader to the conclusion that the standard method of multiplication as described in my first book is the best. In addition to discussing other basic methods of multiplication I shall introduce some methods of simplified multiplication.

1. The Variant of the Standard Multiplication

There is a variant method of the standard multiplication, which was popular in Japan around 1930, but which was largely replaced in less than ten years by the standard method of multiplication. The variant is still favored by quite a few experts including entrants in abacus contests, because it is a little faster than the standard method. Here is an example.

EXAMPLE 1:

$$56 \times 49 = 2\,744$$



Fig. 35

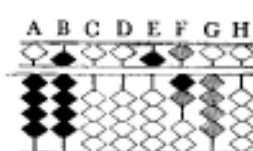


Fig. 36

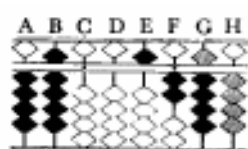


Fig. 37



Fig. 38

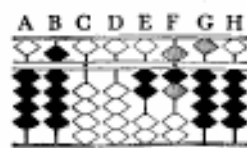


Fig. 39

| A B C D E F G H | | | | | | | | | |
|-----------------|---|---|---|---|---|---|---|--|--|
| 4 | 9 | 0 | 0 | 5 | 6 | 0 | 0 | | |
| | | | | 2 | 4 | | | | |
| | | | | | 5 | 4 | | | |
| 4 | 9 | 0 | 0 | 5 | 2 | 9 | 4 | | |
| | | | | 4 | 5 | | | | |
| 4 | 9 | 0 | 0 | 2 | 7 | 4 | 4 | | |

Step 1
Step 2
Step 3
Step 4
Step 5

STEP 1: Set the multiplicand 56 on EF and the multiplier 49 on AB (Fig. 35).

STEP 2: Multiplying the 4 on A by the 6 on F, set the product 24 on FG after clearing F of the 6. This gives you 24 on FG (Fig. 36).

STEP 3: Multiplying the 9 on B by the same 6 which you remember was on F, set the product 54 on GH. Since you had 24 on FG, you get a total product of 294 on FGH (Fig. 37).

STEP 4: Multiplying the 4 on A by the 5 on E, set the product 20 on EF after clearing E of the 5. This gives you a total of 2940 on EFGH (Fig. 38).

STEP 5: Multiplying the 9 on B by the same 5 which you remember was on E,

set the product 45 on FG. Since you had 2940 on EFGH, you get, on EFGH, a total of 2744, which is the answer (Fig. 39).

Advantages and Disadvantages

ADVANTAGE:

This method is a little faster than the standard method, because the distance between the multiplier and the product has been reduced by one rod.

DISADVANTAGES:

(1) The product of this multiplication does not form in the position of the dividend of the standard division, as is the case with the standard multiplication and division.

It should be noted that this method produces its product in the position of the dividend of the older method of division. In other words, it forms the counterpart of the older method of division, the performance of which requires the use of its own special division table. With the decay of the older method of division, this multiplication method has also given way to the standard method of multiplication.

(2) The beginner may find this method a little less easy to follow than the standard one, as he has to remember each digit in the multiplicand after it is removed.

2. Multiplication Starting with the Final Digits of the Multiplier and the Multiplicand

The order of multiplication in this method is identical with that of written multiplication. Accordingly, the student of the abacus will find it interesting and useful to compare this method with the standard method. This arithmetic method is the oldest multiplication method here introduced. It was used extensively in Japan until around 1930, when it was largely replaced by the variant of the standard multiplication method later developed, and which, in turn, was replaced by the standard method itself.

This method may be broken down into two variants. One of them, which formed the counterpart of the older method of division, used to be popular in Japan. We shall call this Variant B. But today, when the older method of division has fallen out of favor, the variant of this method which I shall call Variant A makes a counterpart of the standard method of division and is free from the incidental details of operation that complicate Variant B.

Variant A

EXAMPLE 1: $78 \times 89 = 6\,942$



Fig. 40

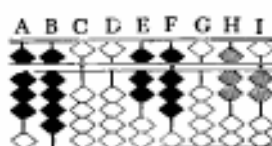


Fig. 41



Fig. 42

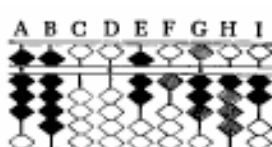


Fig. 43



Fig. 44

| A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|
| 8 | 9 | 0 | 0 | 7 | 8 | 0 | 0 | 0 |
| | | | | 7 | 2 | | | |
| | | | | 6 | 4 | | | |
| 8 | 9 | 0 | 0 | 7 | 0 | 7 | 1 | 2 |
| | | | | 6 | 3 | | | |
| | | | | 5 | 6 | | | |
| 8 | 9 | 0 | 0 | 0 | 6 | 9 | 4 | 2 |

STEP 1: Set 78 on EF and 89 on AB (Fig. 40).

STEP 2: Multiplying the 9 on B by the 8 on F, set the product 72 on HL (Fig. 41).

STEP 3: Multiplying the 8 on A by the same 8 on F, set the product 64 on GH, and clear F of the 8. This gives you a total product of 712 on GHI (Fig. 42).

STEP 4: Multiplying the 9 on B by the 7 on E, add the product 63 to the 71 on GH. This makes a total product of 1 342 on FGHI (Fig. 43).

STEP 5: Multiplying the 8 on A by the same 7 on E, add the product 56 to the 13 on FC, and clear E of its 7. This gives you, on FGHI, a total product of 6 942, which is the answer (Fig. 44).

NOTE: The procedure of this Variant A forms the unit rod of the product on the rod to the right of the unit rod of the multiplicand by as many rods plus one as there are digits in the multiplier.

Variant B

The disadvantages of Variant B may be illustrated best by the same example, since this method can become complicated when both multiplier and multiplicand have large digits. But the complications offer no serious obstacles to experienced operators. Hence, until recently, Variant B has been preferred to Variant A primarily because it is faster and also because it is the counterpart of the older method of division.

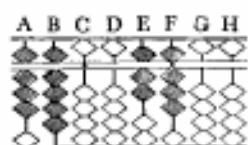


Fig. 45



Fig. 46

EXAMPLE 2: $78 \times 89 = 6\,942$

STEP 1: Set 78 on EF and 89 on AB (Fig. 45).

STEP 2: Multiplying the 9 on B by the 8 on F, set the product 72 on GH (Fig. 46).

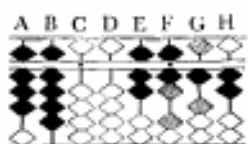


Fig. 47

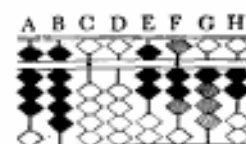


Fig. 48

NOTE: The reader can see that Variant B sets the product one rod closer to the multiplicand and thus is slightly speedier than Variant A.

STEP 3: Multiplying the 8 on A by the 8 on F, set the product 64 on FG after clearing F of its 8. This gives you a total product of 712 on FGH (Fig. 47).



Fig. 49

| A B C D E F G H | | | | | | | | | |
|-----------------|---|---|---|---|---|---|---|--|--|
| 8 | 9 | 0 | 0 | 7 | 8 | 0 | 0 | | |
| | | | | | 7 | 2 | | | |
| | | | | | 6 | 4 | | | |
| 8 | 9 | 0 | 0 | 7 | 7 | 1 | 2 | | |
| | | | | | 6 | 3 | | | |
| | | | | | 5 | 6 | | | |
| 8 | 9 | 0 | 0 | 6 | 9 | 4 | 2 | | |

STEP 4: Multiplying the 9 on B by the 7 on E, set the product 63 on FC. This makes a total of 7 342 on EFGH. In this procedure do not add 1 to the 7 on E, but remember to add it in the next step (Fig. 48).

STEP 5: Multiplying the 8 on A by the 7 on E, set the product 56 on EF after removing the 7 on E. In this step do not forget the 1 which must be added to E. This leaves you with a total of 6 942 on EFCH, which is the answer (Fig. 49).

NOTE: The 1 to be added in step 4 must be remembered till step 5. This kind of situation occurs especially when both the multiplier and the multiplicand are large numbers.

EXAMPLE 3: $78 \times 456 = 35\,568$



Fig. 50

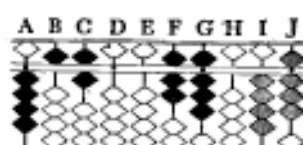


Fig. 51

STEP 1: Set the multiplicand 78 on FG and the multiplier 456 on ABC (Fig. 50).

STEP 2: Multiplying the 6 on C by the 8 on G set the product 48 on IJ

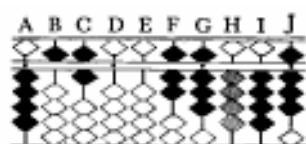


Fig. 52



Fig. 53

(Fig. 51).

STEP 3: Multiplying the 5 on B by the same 8 on set the product 40 on HL. This makes a total of 448 on HIJ (Fig. 52).

STEP 4: Multiplying the 4 on A by the same 8 on G, set the product 32 on GH, and clear G of its 8. This makes a total of 3 648 on GHIJ (Fig. 53).



Fig. 54



Fig. 55

STEP 5: Multiplying the 6 on C by the 7 on F, set the product 42 on HI. This makes a total of 4 068 on GHIJ (Fig. 54).

STEP 6: Multiplying the 5 on B by the same 7 on F, set the product 35 on GH. This makes a total of 7 568 on

| A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 6 | 0 | 0 | 7 | 8 | 0 | 0 | 0 |
| | | | | | | | 4 | 8 | |
| | | | | | | | 4 | 0 | |
| | | | | | | | 3 | 2 | |
| 4 | 5 | 6 | 0 | 0 | 7 | 3 | 6 | 4 | 8 |
| | | | | | | | 4 | 2 | |
| | | | | | | | 3 | 5 | |
| | | | | | | | 2 | 8 | |
| 4 | 5 | 6 | 0 | 0 | 3 | 5 | 5 | 6 | 8 |

Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7

GHIJ (Fig. 55).

STEP 7: Multiplying the 4 on A by the same 7 on F, set the product 28 on FG, and clear F of the 7. This gives you, on FGHIJ, a total of 35 568, which is the answer (Fig. 56).

Advantages and Disadvantages

ADVANTAGES:

(1) Variant A of this method forms the counterpart of the standard method of division, that is, it forms its product in the position of the dividend of the latter method.

(2) Since the order of multiplication is identical with that of written multiplication, the beginner may find it easier to learn.

DISADVANTAGES:

This method is not commonly used for the following reasons.

(1) The operator has to take the trouble of counting the digits in the multiplier. As was previously explained, Variant A requires that the unit rod of the first product be separated from the unit rod of the multiplicand by as many rods plus one as there are digits in the multiplier, while Variant B requires that the unit rod of the first product be separated from the last rod of the multiplicand by as many rods as there are digits in the multiplier. The greater the distance, the greater the inconvenience.

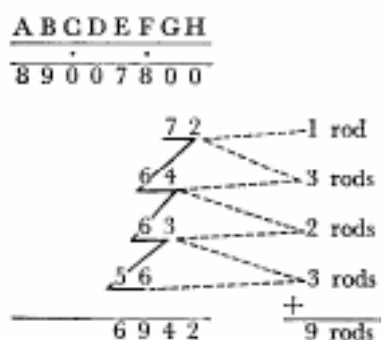
(2) Right-to-left operation makes this method a little slower than the other methods.

(3) Variant B of this method is often complicated by the kind of inconvenience which arose in Example 2 of this section.

Why Is Right-to-Left Operation Slower than Left-to-Right Operation?

The reasons for the slowness of right-to-left operation of this method may be seen through the analytical comparison of this method with the standard one.

The procedure of multiplication illustrated in Example 2 sets the products as given below.



Step 1

Step 2

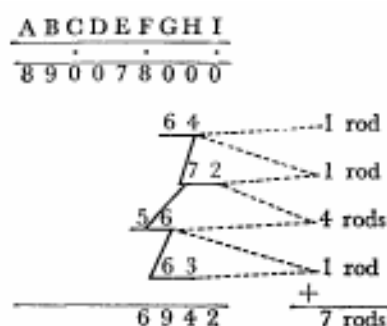
Step 3

Step 4

Step 5

The above shows that in this operation the finger has to travel a total of 9 rods. In step 2, the finger travels 1 rod from G to H. In step 3, it travels 3 rods from H back to F and forward to G. In step 4, it travels 2 rods from G back to F and forward again to G. In step 5, it travels 3 rods from G back to E and forward to F.

If the same problem, 78 x 89, is performed by the standard method of multiplication, the finger has to travel no more than 7 rods, as indicated in the figure below.



Step 1

Step 2

Step 3

Step 4

Step 5

The figure shows that, in step 2, the finger travels for 1 rod from G to H. In step 3, it travels for 1 rod from H to I. In step 4, it travels for 4 rods from I back to F and forward to G. And in step 5, it travels for 1 rod from G to H.

Furthermore, the right-to-left operation of this method may be said to run counter to the efficient left-to-right operation of the standard method.

3. Multiplication Beginning with the Highest Digits of the Multiplier and Multiplicand

It is also possible to do multiplication beginning with the highest digits of the multiplier and the multiplicand. For this method it is necessary to separate the multiplier and the multiplicand by as many rods plus two as there are digits in the multiplier. Otherwise the product will extend into the multiplicand, and operation will become impossible.



FIG. 57



FIG. 58



FIG. 59



FIG. 60

EXAMPLE 1: $43 \times 72 = 3096$

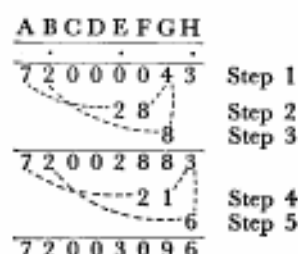
STEP 1: Set the multiplicand 43 on GH and the multiplier 72 on AB, with four vacant rods between them (Fig. 57).

STEP 2: Multiplying the 7 on A by the 4 on G, set the product 28 on EF (Fig. 58).

STEP 3: Multiplying the 2 on B by the same 4 on set the product 8 on G after clearing G of the 4, that is, the highest digit of the multiplier. This makes a total product of 288 on EFG (Fig. 59).



Fig. 61



STEP 4: Multiplying the 7 on A by the 3 on H, set the product 21 on FG. This makes a total product of 309 on EFG (Fig. 60).

STEP 5: Multiplying the 2 on B by the same 3 on set the product 6 on H after clearing H of the 3, that is, the second digit of the multiplier. This gives you, on EFGH, a total product of 3 096, which is the

answer (Fig. 61).

Advantages and Disadvantages

ADVANTAGES:

(1) When the multiplier is a whole number, the unit digit of the product forms on the unit rod of the multiplicand, so the necessity of searching for the unit rod of the product is eliminated. However, this rule does not hold when the multiplier ends in one or more zeros. The product moves to the right of the multiplicand by as many rods as there are zeros at the end of the multiplier.

(2) As the operation starts by multiplying the first digits of the multiplier and multiplicand, it is convenient for approximations.

DISADVANTAGES:

This method is not used very much for the following reasons.

(1) It necessitates counting the digits of the multiplier. The multiplier must be separated from the multiplicand by as many rods plus two as there are digits in the multiplier. When the multiplier is long, this method becomes rather awkward.

(2) It is inconvenient for the calculation of compound numbers not based on the decimal system. Try using this method on a problem involving hours, minutes, and seconds, for example.

(3) It has no corresponding method of division. In other words, there is no method of division where the quotient has the position of the multiplicand in this method of multiplication.

4. The Chinese Method of Multiplication

The following example introduces the method of multiplication widely practiced in China.

EXAMPLE 1: $345 \times 67 = 23\,115$

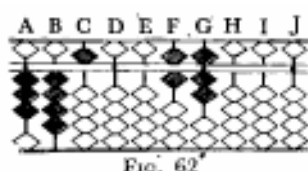


FIG. 62

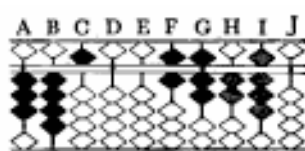


FIG. 63



FIG. 64

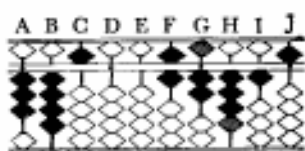


FIG. 65



FIG. 66



FIG. 67



FIG. 68

| A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 0 | 0 | 6 | 7 | 0 | 0 | 0 |
| | | | | | | | 2 | 8 | |
| | | | | | | | | 3 | 5 |
| | | | | | | | 2 | 1 | |
| | | | | | | | 3 | 4 | 5 |
| | | | | | | | 2 | 4 | |
| | | | | | | | | 3 | 0 |
| | | | | | | | 1 | 8 | |
| | | | | | | | 3 | 4 | 5 |
| | | | | | | | 0 | 2 | 3 |
| | | | | | | | | 1 | 1 |
| | | | | | | | | | 5 |

Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7

STEP 1: Set the multiplicand 67 on FG and the multiplier 345 on ABC (Fig. 62).

STEP 2: Multiplying the 4 on B by the 7 on G, set the product 28 on HI (Fig. 63).

STEP 3: Multiplying the 5 on C by the same 7 on set the product 35 on IJ. This gives you a total of 315 on HIJ (Fig. 64).

STEP 4: Multiplying the 3 on A by the same 7 on G, set the product 21 on GH after clearing G of its 7. This gives you a total product of 2 415 on GHIJ (Fig. 65).

STEP 5: Now multiplying the 4 on B by the 6 on set the product 24 on GH. This gives you a total of 4 815 on GHIJ (Fig. 66).

STEP 6: Multiplying the 5 on C by the same 6 on F, set the product 30 on HI. This gives you a total of 5 115 on GHIJ (Fig. 67).

STEP 7: Finally, multiplying the 3 on A by the same 6 on F, set the product 18 on FG, after clearing F of its 6. This gives you, on FGH IJ, a total product of 23 115, which is the answer (Fig. 68).

The reader will see that this method of multiplication forms its product in the same position as does the variant of the standard method of multiplication introduced at the beginning of this chapter.

NOTES: (a) When this method of multiplication is followed, it often facilitates calculation for the rods to have two five-unit counters when both the multiplier and multiplicand have large digits. This is one of the reasons why the Chinese abacus has two five-unit counters on each rod. (b) Also notice that in China the multiplicand is set on the left and the multiplier is set on the right.

Advantages and Disadvantages

ADVANTAGES:

In this example, suppose you had used the variant of the standard method of multiplication. Then, in step 2, rod G would have been cleared of the 7 before the product was set on GH. However, in this method, the 7 on G remains on the board until step 4, when all of the operations involving that 7 have been completed.

DISADVANTAGES:

In this example, the additions of the products proceed from left to right till step 3. But in step 4, the hand must shift back to the left to add the product. Since the operation proceeds in this way, it is somewhat slower than the standard method and its variant.

5. The Elimination of the Final Digit of a Multiplier Ending in One

Multiplication can be facilitated by leaving out of calculation the final digit one of the multiplier. The Japanese technical term for this kind of multiplication may be translated as the final-digit-one elimination multiplication.

This multiplication starts its calculation with the highest digits of the multiplier and the multiplicand. The multiplier and the multiplicand must be separated by as many rods plus one or two as there are digits in the multiplier; otherwise the product will extend into the multiplier.

EXAMPLE: $74 \times 31 = 2\,294$

STEP 1: Set the multiplicand 74 on GH and the 3 of the multiplier 31 on B (Fig. 69).



FIG. 69

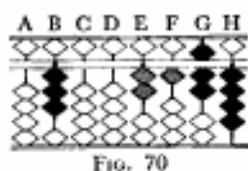


FIG. 70

NOTES: (a) The trouble of setting the final digit one of the multiplier is spared, because it is not used in this operation. (b) A beginner using this method is liable to make errors.

STEP 2: Multiplying the 3 on B by the 7 on G, set the product 21 on EF, and you get 2 174 on EFGH (Fig. 70).

NOTES: (a) In this method of multiplication, the multiplicand 74 is supposed to have been multiplied by the 1 of the multiplier 31, and is left intact on the board, and the products made by multiplying the 3 of 31, i.e. 30, by the two digits of the multiplicand each are added to the multiplicand 74. (b) When the multiplier is a whole number, the unit digit of the product forms on the unit rod of the multiplicand. (c) In step 2, the 3 on B is really 30 and the 7 on G is really 70. Accordingly, their product 21 is set on EF, because it is really 2 100.



FIG. 71

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| | 3 | | | | | 7 | 4 |
| | | | | 2 | 1 | | |
| | | | | 2 | 1 | 7 | 4 |
| | | | | | 1 | 2 | |
| | | | | 2 | 2 | 9 | 4 |

Step 1
Step 2
Step 3

STEP 3: Multiplying the same 3 on B by the 4 on H, add the product 12 to the 17 on FG. This gives you, on EFGH, a total of 2 294, which is the answer (Fig. 71).

NOTE: In this step, the 3 on B is really 30, but the 4 on H is a real 4. Accordingly, their product 12 is set on FG, because it is really 120.

This multiplication can be performed by subtracting 1 from a multiplier whose last digit is 2 or a larger digit, although this procedure does not improve calculations. An example will be given below.

EXAMPLE 2: $83 \times 42 = 3\,486$

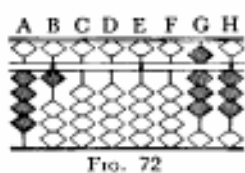


Fig. 72

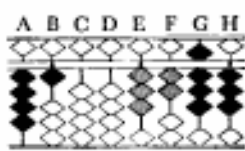


Fig. 73

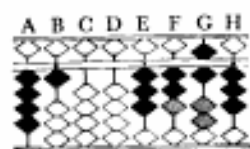


Fig. 74

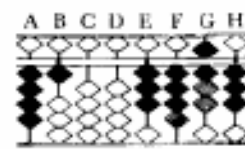


Fig. 75

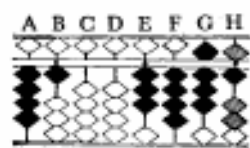


Fig. 76

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| 4 | 1 | 0 | 0 | 0 | 0 | 8 | 3 |
| | | | | 3 | 2 | | |
| | | | | | | 8 | |
| | | | | 4 | 1 | 0 | 0 |
| | | | | | | 3 | 6 |
| | | | | | | | 3 |
| | | | | | | 1 | 2 |
| | | | | | | | 3 |
| | | | | | | 4 | 1 |
| | | | | | | | 0 |
| | | | | | | | 0 |
| | | | | | | | 3 |
| | | | | | | | 4 |
| | | | | | | | 8 |
| | | | | | | | 6 |

Step 1
Step 2
Step 3
Step 4
Step 5

STEP 1: Set the multiplicand 83 on GH and the multiplier 41 on AB (Fig. 72).

NOTE: In this multiplication, the multiplicand is left intact on the board, and the operation performed can be expressed as $83 + [83 \times (42-1)]$.

STEP 2: Multiplying the 4 on A by the 8 on G, set the product 32 on EF. This gives you 3 283 on EFGH (Fig. 73).

STEP 3: Next, multiplying the 1 on B by the same 8 on G, set the product 8 on G. This gives you 3 363 on EFGH (Fig. 74).

STEP 4: Now, multiplying the 4 on B by the 3 on set the product 12 on FG. This gives you a total of 3 483 on EFGH (Fig. 75).

STEP 5: Finally, multiplying the 1 on B by the same 3 on H, set the product 3 on H. And you get, on EFGH, a total of 3 486, which is the answer (Fig. 76).

Advantages and Disadvantages

ADVANTAGE:

This method simplifies calculation by reducing the number of the digits in the multiplier by one.

DISADVANTAGES:

(1) This method necessitates counting the digits in the multiplier so as to separate the multiplier from the multiplicand by as many rods plus one or two as there are digits in the multiplier.

(2) The value of this method is limited to the case in which the final digit of the multiplier is one.

6. The Elimination of the Initial Digit of a Multiplier Beginning with One

Multiplication can also be simplified by leaving out the initial digit one of the multiplier. The Japanese technical term for this multiplication may be translated as the initial-digit-one elimination multiplication.

EXAMPLE 1: $23 \times 12 = 276$

STEP 1: Set the multiplicand 23 on EF and the 2 of the multiplier 12 on B, the third rod to the left of rod E (Fig. 77).

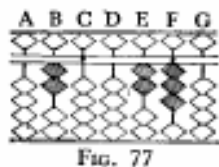


Fig. 77

NOTES: (a) There is no objection to setting the whole multiplier 12 on AB. However, experts who use this method do not customarily set the first digit 1 because it is not used in calculation, and they can remember that the 2 here is the 12 of the multiplier. However, this method of multiplication is apt to give the beginner trouble. (b) In this multiplication, 23 is supposed to have been multiplied by the 10 of the multiplier 12 and is regarded as 230, with G as its unit rod. The product is made by multiplying the 2 of 12 by each digit of 23 and adding the result to the 230 on EFG.

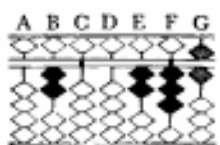


Fig. 78



Fig. 79

| A | B | C | D | E | F | G |
|---|---|---|---|---|---|----|
| | 2 | | | 2 | 3 | 0 |
| | | | | | | +6 |
| | | | | | | 6 |
| | | | | | | +4 |
| | | | | | | 4 |
| | | | | | | 6 |
| | | | | | | 6 |

STEP 2: Multiplying the 2 on B by the 3 on F, set the product 6 on G. This makes a total of 236 on EFG (Fig. 78).

NOTE: Since both the 2 on B and the 3 on F are in the unit place, the product 6 must be set on G, the unit rod of the product.

STEP 3: Next multiplying the same 2 on B by the 2 on E, add the product 4 to the 3 on F. This gives you, on EFG, a total of 276, which is the answer (Fig. 79).

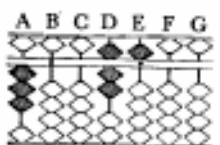


Fig. 80

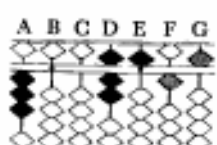


Fig. 81



Fig. 82

NOTES: (a) In this step, since the 2 on B is in the unit place and the 2 on E is in the tens place, the product 4 must be set on F, the tens rod of the product. (b) Operation must start with the last

digit of the multiplicand. Otherwise, the product will extend into the multiplicand and render calculation impossible. (c) This method simplifies calculation especially when the second digit of the multiplier is small, i.e., when the multiplier is a number such as 11, 107, 1 008, etc.

EXAMPLE 2: $75 \times 103 = 7\,725$

STEP 1: Set the multiplicand 75 on DE and the 3 of the multiplier 103 on A (Fig. 80).

NOTE: In this problem, the multiplicand 75 is supposed to have been multiplied by the 100 of the multiplier 103. Accordingly, the 75 on DE is regarded as the product 7 500 on DEFG, with G as the unit rod, F as the tens rod, and E as the hundreds rod.

| A | B | C | D | E | F | G |
|---|---|---|---|---|---|-----|
| | | | 7 | 5 | 0 | 0 |
| | | | | | | +15 |
| | | | | | | 15 |
| | | | | | | +21 |
| | | | | | | 21 |
| | | | | | | 25 |

STEP 2: Multiplying the 3 on A by the 5 on E, set the product 15 on FG. This makes a total of 7 515 on DEFG (Fig. 81).

STEP 3: Next multiplying the 3 on A by the 7 on add the product 21 to the 51 on EF. This gives you, on EFGH, a total of 7 725, which is the answer (Fig. 82).

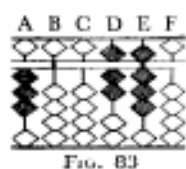


Fig. 83

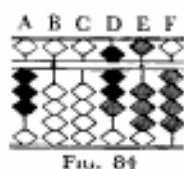


Fig. 84



Fig. 85

NOTE: Since the 7 on D is in the tens place, and the 3 on A in the unit place, the product 21 is really 210 and must be set on EF.

EXAMPLE 3: $78 \times 13 = 1\,014$

This example is to show that when the second digit in the multiplier is not a small one, this method becomes complicated.

| A | B | C | D | E | F |
|---|---|---|-----|---|---|
| 3 | 0 | 0 | 7 | 8 | 0 |
| 3 | 0 | 0 | 7 | 0 | 4 |
| | | | (1) | 2 | 1 |
| 3 | 0 | 1 | 0 | 1 | 4 |

Step 1
Step 2

Step 3

STEP 1: Set the multiplicand 78 on DE and the 3 of the multiplier 13 on A (Fig. 83).

NOTE: As the multiplier is a two-digit number, F becomes the unit rod of the product.

STEP 2: Multiplying the 3 on A by the 8 on E, set the product 24 on EF. This makes a total of 804 on DEF (Fig. 84).

NOTE: However, you must remember till the next step that the digit which was originally on D was not 8 but 7.

STEP 3: Multiplying the 3 on A by the 7 which you remember was on D, set the product 21 on DE. This gives you, on CDEF, a total of 1 014, which is the answer (Fig. 85).

EXAMPLE 4: An article was bought for \$250 and sold at a gain of 6.8%. Find the selling price.

Since the selling price is determined by adding the cost and the profit, it can be found by the following multiplication: $\$250 \times (1 + 0.068) \ \$250 \times 1.068 = \$267$

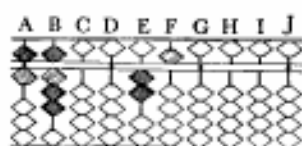


Fig. 86

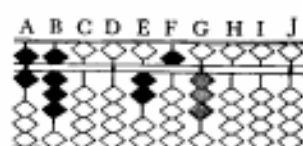


Fig. 87

There are two methods for working this multiplication. One is the standard method of multiplication, and the other the method of eliminating the initial digit one of the multiplier. The latter operation is shown below.

STEP 1: Set the dividend 250 on EFG, with G as the unit rod, and set the 68 of the divisor 1.068 on AB (Fig. 86).

NOTE: When the multiplier is a mixed decimal, it is generally advisable to set its unit figure on the unit rod. But experts usually do not bother to, since the digit 1 is not used in calculation. As long as you remember that 68 stands for 1.068, it matters little to set the unit digit of the multiplier on the unit rod.

STEP 2: Since the multiplier is a four-digit number, suppose that 250 has been multiplied by 1 000, producing 250 000 on EFGHIJ, with G as the unit rod (Fig. 87). Now multiplying the 6 on A by the 5 on F, set the product 30 on GH (Fig. 87).

NOTE: As both the 6 on A and the 5 on F are in the tens place, the product 30 must be set on GH.



Fig. 88



Fig. 89

STEP 3: Next multiplying the 8 on B by the same 5 on F, set the product 40 on HL. This gives you 34 on GH (Fig. 88).

STEP 4: Next multiplying the 6 on A by the 2 on E, add the product 12 to the 53 on FG. This gives you 654 on FGH (Fig. 89).



Fig. 90

| A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 8 | 0 | 0 | 2 | 5 | 0 | 0 | 0 | 0 |
| | | | | | 3 | 0 | | | |
| | | | | | | 4 | 0 | | |
| 6 | 8 | 0 | 0 | 2 | 5 | 3 | 4 | 0 | 0 |
| | | | | | 1 | 2 | | | |
| | | | | | | 1 | 6 | | |
| 6 | 8 | 0 | 0 | 2 | 6 | 7 | 0 | 0 | 0 |

Step 1
Step 2
Step 3
Step 4
Step 5

STEP 5: Finally multiplying the 8 on B by the same 2 on E, add the product 16 to the 54 on GH. This gives you, on EFG, a total of 267, which is the answer (Fig. 90).

NOTE: It should be noted that when the multiplier is a mixed decimal whose one whole figure is in the unit place, the unit rod of the multiplicand becomes that of the product. The reason for this is that the product may be regarded as having been multiplied by one. Accordingly, in this multiplication, it is quite easy to locate the unit rod of the product.

Rules for Finding the Unit Rod of the Product

RULE 1: When the multiplier is a mixed decimal whose whole figure one is in the unit place—i.e., 1—the unit rod of the quotient forms on that of the multiplicand.

RULE 2: Each time the value of this multiplier is raised by one place, the unit rod of the product shifts by one rod to the right of that of the multiplicand, and each time the value of this multiplier is reduced by one place, the unit rod of the product shifts by one rod to the left of that of the multiplicand.

Rule (1) means that when the multiplier is 1.05 or 1.023, the unit rod of the product forms on that of the multiplicand.

Rule (2) means that when the multiplier is 10.5 or 10.23, the unit rod of the product forms on the first rod to the right of that of the multiplicand, and that when the multiplier is 0.105 or 0.1023, the unit rod of the product forms on the first rod to the left of that of the multiplicand.

Advantages and Disadvantages

ADVANTAGES:

(1) This method of multiplication simplifies calculation especially when the second digit of the multiplier is zero, as in 103, 109, 1.082, etc.

When the multiplier is a number whose second digit is small, such as 11, 114, 125, 1.078, etc., calculation is often simplified. However, this situation sometimes requires a figure or figures which have to be carried over to the rod next on the left.

(2) As shown in Example 4, when the multiplier is a mixed decimal whose whole figure one is in the unit place, the multiplicand itself is used as the product. In other words, the unit digit of the product forms on the unit rod of the multiplicand. This enables the operator to locate easily the unit rod of the product and to simplify calculations.

Because of these two advantages, this method of multiplication is extensively used in the calculation of percentages in business.

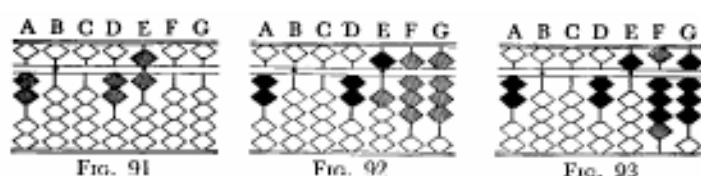
DISADVANTAGE:

When the multiplier is a number whose second digit is large, like 17, 189, etc., this method becomes awkward. It may involve remembering so many digits that calculation becomes extremely difficult. This is especially true when the multiplicand also is a number with many large digits.

7. Multiplication by Complementary Numbers

Multiplication can often be simplified by using the complement of a multiplier. For instance, take the problem 26×98 . The 26 is multiplied by 100, becoming 2 600. From this 26×2 , 52 is subtracted. In this way the answer 2 548 is obtained. The 2 is, of course, the complement of 98 with respect to 100. This method of computation is better than the ordinary method of multiplication when the multiplier is a number a little smaller than 100 or 1 000, etc., such as 97, 996, etc.

EXAMPLE 1: $26 \times 98 = 2\,548$



STEP 1: Set the multiplicand 26 on DE, and on A, set 2, the complement of 98 with respect to 100 (Fig. 91). When the multiplier is a two-figure number, the multiplicand is regarded as having

been multiplied by 100. So in this problem the unit rod of the product shifts to G.

| A | B | C | D | E | F | G |
|--------|---|---|---|----|---|---|
| 2 | 0 | 0 | 2 | 6 | 0 | 0 |
| Step 1 | | | | | | |
| | | | | -1 | 2 | |
| Step 2 | | | | | | |
| 2 | 0 | 0 | 2 | 5 | 8 | 8 |
| | | | | -4 | | |
| Step 3 | | | | | | |
| 2 | 0 | 0 | 2 | 5 | 4 | 8 |

STEP 2: Multiplying the 2 on A by the 6 on E, subtract the product 12, which is to be set on FG, from the 6 on E. This leaves 2 588 on DEFG (Fig. 92).

STEP 3: Multiplying the 2 on A by the 2 on D, subtract the product 4 from the 8 on F. This leaves 2 548 on DEFG (Fig. 93), the answer.

EXAMPLE 2: A manufacturer wishes to purchase a piece of machinery priced at \$7 250 on which there is a discount of 4% for cash. Find the price he pays in cash. The answer can be found by the following simplified multiplication.

$$\begin{aligned}
 &7\,250 \times 0.96 \\
 &= 7\,250 \times (1 - 0.04) \\
 &= 7\,250 - (7\,250 \times 0.04) \\
 &= 6\,960
 \end{aligned}$$



STEP 1: Set the multiplicand 7 250 on DEFG, and on A, set the 4 of 0.04, which is the complement of 0.96 with respect to 1 (Fig. 94).

NOTE: On the board you have this problem: $7\,250 - (7\,250 \times 0.04)$. In this problem, the actual multiplier is 0.96, i.e., a decimal with its first significant digit in the tenths place. So you may consider that the answer is obtained by keeping the multiplicand as it is and subtracting from it $7\,250 \times 0.04$. You should note that when the multiplier is a decimal with its first significant digit in the tenths place, the unit rod of the product remains that of the multiplicand.



Fig. 95



Fig. 96

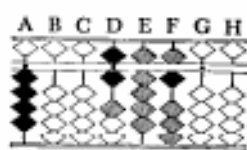


Fig. 97

| A | B | C | D | E | F | G | H | |
|---|---|---|---|---|----|---|---|--------|
| 4 | 0 | 0 | 7 | 2 | 5 | 0 | 0 | Step 1 |
| | | | | | -2 | 0 | | Step 2 |
| 4 | 0 | 0 | 7 | 2 | 4 | 8 | 0 | |
| | | | | | -8 | | | Step 3 |
| 4 | 0 | 0 | 7 | 2 | 4 | 0 | 0 | |
| | | | | | -2 | 8 | | Step 4 |
| 4 | 0 | 0 | 6 | 9 | 6 | 0 | 0 | |

STEP 2: Multiplying the 4 on A by the 5 on F, subtract the product 20, which is to be set on GH, from the 5 on F. This leaves 7 248 on DEFG (Fig. 95).

NOTE: As the 5 on F is in the tens place of the dividend, the product 20 must be set on GH.

STEP 3: Multiplying the 4 on A by the 2 on E, subtract the product 8 from the 8 on G. This leaves 724 on DEF (Fig. 96).

STEP 4: Multiplying the 4 on A by the 7 on D, subtract the product 28 from the 724 on DEF. This leaves 6 960 on DEFG (Fig. 97). The answer is \$6 960.

Rules for Finding the Unit Rod of the Product

RULE 1: When the multiplier is a decimal whose first significant digit is in the tenths place, the unit rod of the multiplicand remains that of the product.

RULE 2: Each time the value of this multiplier is raised by one digit, the unit rod of the product shifts by one rod to the right of that of the multiplicand, and each time the value of this multiplier is reduced by one digit, the unit rod of the product shifts by one rod to the left.

Rule (1) means that when the multiplier is 0.95 or 0.934, the unit rod of the product forms on that of the multiplicand.

Rule (2) means that when the multiplier is 9.5 or 9.34, the unit rod of the product forms on the first rod to the right of that of the multiplicand, and that when the multiplier is 0.095 or 0.0934, the unit rod of the product forms on the first rod to the left of that of the multiplicand.

Advantages and Disadvantages

ADVANTAGES:

(1) The reader can easily see in Example 2 that this method is easier than actually multiplying $7\,250 \times 0.96$.

(2) As stated in Rule 1 for finding the unit rod of the product, when the multiplier is a mixed decimal whose first significant digit is in the tenths place, the unit rod of the multiplicand remains that of the product. This saves the operator the trouble of finding the unit rod of the product. This is a real advantage.

For these reasons this method of multiplication is extensively employed in the calculation of percentages in business.

DISADVANTAGES:

This method has no particular disadvantages. However, it should be noted that its utility is limited to problems in which the multipliers are numbers a little smaller than 100, 1 000, etc. or numbers whose first digit is 9, such as 98, 987, 0.96, etc.

Exercises

- | | |
|-------------------------------|-----------------------------------|
| (1) $24 \times 88 = 2\,112$ | (4) $169 \times 987 = 166\,803$ |
| (2) $37 \times 98 = 3\,626$ | (5) $408 \times 996 = 406\,368$ |
| (3) $851 \times 99 = 84\,249$ | (6) $3\,600 \times 0.92 = 3\,312$ |

IV - OTHER METHODS OF DIVISION

There is an older method of division which is as fast as the standard method of division that I explained in my first book. But this method is no faster and it requires the memorization of a complicated division table. Although this method is still used by the older generation, it is generally becoming obsolete. Since this method is so complicated and since it is hardly used today, I will not spend time explaining it. Instead I shall present some methods of division which are popular for business calculations.

Historically speaking, however, the "standard method of division" is older than what I have called here the older method of division. In ancient China the "standard method of division" was used by means of reckoning blocks. However, the "older method of division," because of its convenience in the calculation of weights not based on the decimal system, became more popular than the "standard method of division." These weights were extremely important in Chinese life because they were used in the conversion of currency. When the abacus was introduced to Japan, it was the "older method of division" which accompanied it. The currency in Japan also was not based on the decimal system, so that although the "standard method of division" was strongly advocated by some mathematicians, it never won real public support. It was not until the new decimal-based currency of the Meiji era replaced the old Japanese currency that the "standard method of division" actually became standard.

1. The Elimination of the Initial Digit of a Divisor Beginning with One

There is a method which simplifies division by leaving out of calculation the initial digit, one, of the divisor. The Japanese technical term for this division may be translated as the initial-digit-one elimination division.

This division is particularly useful when the divisor is a number a little larger than 100, 1000 etc., such as 103, 1 014, etc. It is a counterpart of the multiplication method in which the initial digit one of the multiplier is left out of calculation.

Let us take a very simple example. If you divide 306 by 102, you get the quotient 3. In this case, since the first digit of the divisor 102 is 1, the first digit 3 of the dividend 306 may be used as the quotient. This principle is applied to this method of division on the abacus. On the abacus board, in each step of the division, the first digit of the dividend is used as the trial quotient, and thus the necessity of setting the quotient and that of removing the first digit of the dividend is eliminated, and calculation is considerably simplified and accelerated. Example 1 shows how this division, "306 / 102," is actually done on the board.

EXAMPLE 1: $306 / 102 = 3$



Fig. 98



Fig. 99

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|----|---|
| | | | | | 3 | 0 | 6 |
| 1 | 0 | 2 | 0 | 0 | 3 | 0 | 6 |
| | | | | | | -6 | |
| 1 | 0 | 2 | 0 | 0 | 3 | 0 | 0 |

Step 1
Step 2

STEP 1: Set the dividend 306 on FGH with H as the unit rod, and set the divisor 102 on ABC (Fig. 98).

STEP 2: Suppose that you have divided the 306 on FGH by the 100 of the 102 on ABC and have got the 3 on F as the quotient figure. Multiply the 2 on C by the 3 on F, and subtract the product 6 from the 6 on H. This clears the board of the dividend, and leaves the quotient 3, the answer, on F (Fig. 99).

NOTE: On the board, 306 is divided by 100. But since the actual divisor is 102, the product obtained by the multiplication of 2 (i.e., the difference between 100 and 102) and 3 (i.e., the quotient) must be subtracted from 306. On the board, the procedures of setting the first quotient figure 3 and of subtracting the first digit 3 of the dividend are saved or eliminated.

EXAMPLE 2: $2\ 575 / 103 = 25$

STEP 1: Set the dividend 2 575 on DEFG and the 3 of 103 on A (Fig. 100).

NOTE: There is no objection to setting the whole divisor 103 on the board. However, experts do not, because this digit is not actually used in calculation, and so long as they remember that the 3 on A stands for 103 it is not important to set the whole divisor on the board.

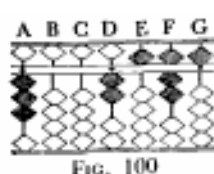


Fig. 100

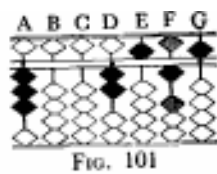


Fig. 101

STEP 2: As the divisor 103 is a three-digit number, suppose that you have divided the 257 of 2 575 by the 100 of 103 and have tried the 2 on D as the quotient figure. Multiplying the 3 on A by the 2 on D, subtract the product 6 from the 7 on F. This leaves, on EFG, 515 as the remainder of the dividend (Fig. 101).

NOTES: (a) As the second figure of the divisor 103 is zero, you must set the product 6 on F, skipping over E. (b) In this step, as the 2 on D is used as the first quotient figure, the procedures of setting the first quotient figure 2 and of subtracting the first digit 2 of the dividend 257 are eliminated.

STEP 3: Next, suppose that you have divided the 515 on EFG by 100 and have got the 5 on E as the second quotient figure. Multiplying the 3 on A by the 5 on E, subtract the product 15 from the 15 on FG. This clears the board of the remaining dividend, and leaves, on DEF, 25 as the quotient (Fig. 102).

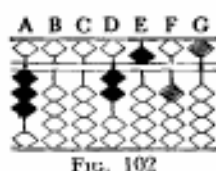


Fig. 102

| A | B | C | D | E | F | G |
|---|---|---|---|---|---|-----|
| 3 | 0 | 0 | 2 | 5 | 7 | 5 |
| | | | | | | -6 |
| 3 | 9 | 9 | 2 | 5 | 1 | 5 |
| | | | | | | -15 |
| 3 | 0 | 0 | 2 | 5 | 0 | 0 |

Step 1
Step 2
Step 3

As the dividend was practically divided by 100, the unit rod of the quotient moves to the second rod to the left of that of the dividend. Therefore, E becomes the unit rod of the quotient. The answer is 25.

NOTE: In this step, as the 5 on E is used as the second quotient figure, the procedures of setting the second quotient figure and of subtracting the first digit 5 of the dividend 515 are eliminated.

EXAMPLE 3: A merchant's sales increased 0.8% in the second month. The sales for the second month amounted to \$3 780. How much were the sales the first month?

This problem can be worked out by the following division: $\$3\,780 / 1.008 = \$3\,750$

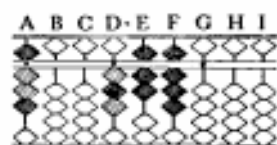


Fig. 103

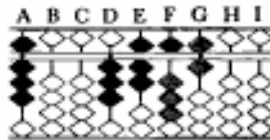


Fig. 104

STEP 1: Set the dividend 3 780 on DEFG and the 8 of the divisor 1.008 on A (Fig. 103).

NOTE: You must remember that 8 stands for 1.008.

STEP 2: As the divisor is a four-digit number, suppose that you have divided the 3 780 on DEFG by 1000 and have got the 3 on D as the first quotient figure. Multiplying the 8 on A by the 3 on D, subtract the product 24 from the 80 on FG. This leaves, on EFG, 756 as the remainder of the dividend (Fig. 104).

NOTE: As the second and third figures of 1.008 are zero, you must set the product 24 on FG and not on EF.



Fig. 105

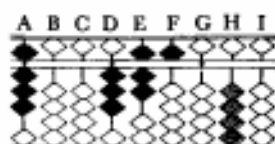


Fig. 106

STEP 3: Suppose that you have divided the 7 560 on EFGH by 1,000 and have got the 7 on E as the second quotient figure. Multiplying the 8 on A by the 7 on E, subtract the product 56 from the 560 on FGH. This leaves, on FGH, 504 as the remainder of the dividend (Fig. 105).

NOTE: Again, be sure to set the product 56 on GH.

STEP 4: Suppose that you have divided the 5 040 on FGH by 1 000, with the 5 on F as the third quotient figure. Multiplying the 8 on A by the 5 on F, subtract the product 40 from the 40 on HI. This clears the board of the dividend, and leaves 375 as the quotient on DEF (Fig. 106).

| A | B | C | D | E | F | G | H | I | |
|---|---|---|---|---|----|----|---|---|--------|
| 8 | 0 | 0 | 3 | 7 | 8 | 0 | 0 | 0 | Step 1 |
| | | | | | -2 | 4 | | | Step 2 |
| 8 | 0 | 0 | 3 | 7 | 5 | 6 | 0 | 0 | |
| | | | | | -5 | 6 | | | Step 3 |
| 8 | 0 | 0 | 3 | 7 | 5 | 0 | 4 | 0 | |
| | | | | | | -4 | 0 | | Step 4 |
| 8 | 0 | 0 | 3 | 7 | 5 | 0 | 0 | 0 | |

As the divisor 1.008 is a mixed decimal, whose whole figure one is in the unit place, the unit rod of the quotient is identical with that of the dividend. The answer is \$3 750.

When the divisor is a number whose second digit is not zero, such as 125, 137, etc., it often happens that a figure a little smaller than the first digit of the dividend must be tried as a quotient figure. In such cases this method is less efficient.

Here is an example showing how the division should be worked in such a case.

EXAMPLE 4: A man withdrew his savings from an account after they had earned 12% interest. If the amount he withdrew was \$5,376, what was his original investment?



Fig. 107

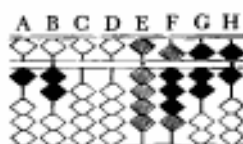


Fig. 108

This problem can be worked by the following division: $5\,376 / 1.12 = 4\,800$

STEP 1: Set the dividend 5 376 on EFGH and the last two digits 12 of the divisor 1.12 on AB (Fig. 107).

NOTE: When a divisor is a mixed decimal, it is advisable for the beginner to set its unit digit on a unit rod.

STEP 2: As the divisor is a three-digit number, sup-pose that you have divided the 537 on EFG by 100, with the 5 on E as the first quotient figure, and found that 5 is too large. It is because if you multiply the 1 on A by the 5 on E, the product 5 which is to be set on F is larger than the 3 on F. So try 4 as the first trial quotient figure in your mind. Then 1 is left on E as the first figure of the remaining dividend. Multiply the 1 on A by this 4 which is mentally on E, and you will see that you can subtract the product 4 from the 13 which remains mentally on EF. Now subtract the 4 from the 13 by setting 4 on E and adding 6 to the 3 on F. This gives you 49 on EF, with the 4 on E as the first quotient figure.

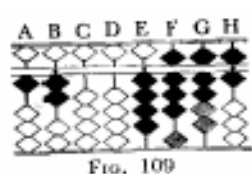


Fig. 109



Fig. 110

Of course, in this particular problem if the operator had looked at the problem carefully, he would have tried 4 the first time. An experienced operator can see at a glance that 4 is the proper first number of the quotient.

STEP 3: Now multiply the 2 on B by the 4 on E, and subtract the product 8 from the 97 on FG. This leaves, on FGH, 896 as the remaining dividend (Fig. 109).

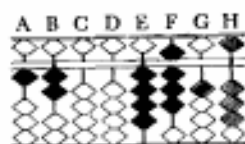


Fig. 111

STEP 4: Next suppose that, in dividing the 896 on FGH by 100, you have tried the 8 on F as the second quotient figure. Multiplying the 1 on A by the 8 on F, subtract the product 8 from the 9 on G. This leaves, on GH, 16 as the remaining subtrahend (Fig. 110).

| A | B | C | D | E | F | G | H |
|---|---|---|---|------|---|---|---|
| 1 | 2 | 0 | 0 | 5 | 3 | 7 | 6 |
| | | | | (-1) | | | |
| 1 | 2 | 0 | 0 | 4 | 3 | 7 | 6 |
| | | | | (1) | | | |
| | | | | -4 | | | |
| 1 | 2 | 0 | 0 | 4 | 9 | 7 | 6 |
| | | | | -8 | | | |
| 1 | 2 | 0 | 0 | 4 | 8 | 9 | 6 |
| | | | | -8 | | | |
| 1 | 2 | 0 | 0 | 4 | 8 | 1 | 6 |
| | | | | -1 | 6 | | |
| 1 | 2 | 0 | 0 | 4 | 8 | 0 | 0 |

Step 1

Step 2

Step 3

Step 4

Step 5

STEP 5: Finally multiplying the 2 on B by the 8 on F, subtract the product 16 from the 16 on GH. This clears the board of the subtrahend and leaves 48 as the quotient on EF (Fig. 111). The answer is \$4 800.

NOTE: As the divisor 1.12 is a mixed decimal whose whole figure one is in the unit place, the unit rod of the dividend remains that of the quotient.

Rules for Finding the Unit Rod of the Quotient

RULE 1: When the divisor is a mixed decimal whose whole digit one is in the unit place, the unit rod of the quotient forms on that of the dividend.

RULE 2: Each time the value of this divisor is raised by one digit, the unit rod of the quotient shifts one rod to the left of that of the dividend, and each time the value of this divisor is reduced by one digit, the unit rod of the quotient shifts one rod to the right of that of the dividend.

For instance, when the divisor is 1.05 or 1.023, the unit rod of the quotient forms on that of the dividend. When the divisor is 10.5 or 10.23, the unit rod of the quotient forms on the first rod to the left of that of the dividend. When the divisor is 0.105 or 0.1023, the unit rod of the quotient forms on the first rod to the right of that of the dividend.

This rule may be restated as follows:

RULE 2: The unit rod of the quotient forms on the rod to the left of that of the dividend by as many rods minus one as there are whole digits in the divisor.

The unit rod of the quotient forms on the rod immediately to the right of that of the dividend by as many rods plus one as there are zeros before the first significant decimal figure.

Advantages and Disadvantages

The advantages and disadvantages of this method of division are nearly identical with those of the method of multiplication which leaves the initial digit one of the multiplier out of calculation (see chapter 3, section 6).

ADVANTAGES:

(1) This method of division simplifies calculation when the second digit of the divisor is zero, as in 105, 1.08, etc. When the divisor is a number whose second digit is small, such as 13 or 1.12, etc., often this method simplifies calculation, although the sort of difficulties illustrated in Example 4 may arise.

(2) As shown in Examples 3 and 4, when the divisor is a mixed decimal whose whole digit one is in the units place, the unit rod of the quotient is identical with that of the dividend. This is a great advantage to the abacus operator, as he has no trouble in finding the unit rod of the quotient.

For these reasons this method of division is employed extensively in the calculation of percentages in business, that is, for finding the cost of goods from the selling price and the percent of profit.

DISADVANTAGES:

(1) When the divisor is a number whose second digit is large, such as 19, 183, etc., this method is awkward.

(2) When the divisor is a number whose second and successive digits are large, this method usually is much too complicated.

2. Division by Complementary Numbers

When the divisor is a number a little smaller than 100, 1 000, etc., such as 98, 997, etc., division can be accelerated by using the complement of the divisor with respect to 100, 1000, etc. Let us take two simple examples. 98 goes into 196 twice. So you can divide 196 by 100 with 2 as the quotient if you add, to 196, 4 which is obtained by multiplying 2, the difference between 100 and 98, and 2, the quotient. The 4 is the same number you have subtracted in excess. In the same way you can divide 294 by 100 with 3 as the quotient if you add, to 294, 6 which is obtained by multiplying the 2 and 3, the

quotient. The following two rules will help you find the quotient of this division easily, especially if the first digit of the divisor is 9.

RULE 1: The quotient can generally be found by adding 1 to the first digit of the dividend when the quotient is a one-digit number, as in the above examples, and when the final digit of the quotient is to be obtained.

RULE 2: In case the quotient is a more-than-one-digit number, the first digit of each dividend is generally used as the trial quotient figure, except in finding the final digit of the quotient.

EXAMPLE 1: $882 / 98 = 9$

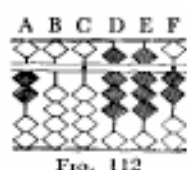


Fig. 112

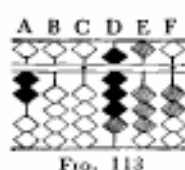


Fig. 113

| A | B | C | D | E | F |
|---|---|---|-----|---|---|
| 2 | 0 | 0 | 8 | 8 | 2 |
| | | | (9) | | |
| | | | +1 | 8 | |
| 2 | 0 | 0 | 9 | 0 | 0 |

Step 1

Step 2

STEP 1: Set the dividend 882 on DEF with F as the unit rod and set, on A, 2, which is the complementary number of 98 with respect to 100. It is important to note that although

100 is the divisor, only the complementary number of the real divisor with respect to 100 is set on the board.

STEP 2: Since the quotient is a one-digit number, you can apply the above-mentioned Rule 1 to this problem. Add 1 to the first digit 8 of the dividend 882 and you get 9. If 9 is the correct quotient, the following two equations should be found equal.

$$\text{Equation A: } 882 + (2 \times 9) = 900$$

$$\text{Equation B: } 100 \times 9 = 900$$

Since the above two equations are equal, 9 is the correct quotient. In equation A, 882 is the dividend, 2 is the complementary number, and 9 is the quotient. In equation B, 100 is the divisor.

Now, multiplying the 2 on A by the trial quotient figure 9, add the product 18 to the 82 on EF. This gives you 900 on DEF. Since the divisor 98 is a two-whole-digit number, the unit rod of the quotient is formed on D. The answer is 9 on D (Fig. 113).

EXAMPLE 2: $8160 / 96 = 85$



Fig. 114



Fig. 115



Fig. 116

STEP 1: Set the dividend 8160 on DEFG, and set on A, 4, which is the complement of 96 with respect to 100 (Fig. 114). Since the dividend is a two-whole-digit number, 100 is used as

the divisor.

| A | B | C | D | E | F | G |
|---|---|---|-----|---|---|---|
| 4 | 0 | 0 | 8 | 1 | 6 | 0 |
| | | | +3 | 2 | | |
| 4 | 0 | 0 | 8 | 4 | 8 | 0 |
| | | | (5) | | | |
| | | | +2 | 0 | | |
| 4 | 0 | 0 | 8 | 5 | 0 | 0 |

Step 1

Step 2

Step 3

STEP 2: Divide the 816 on DEF by 100, with the 8 on D as the first trial quotient figure. In this step, apply Rule 2. Now, multiplying the 4 on A by the 8 on D, add the product 32 to the 16 on EF. This gives you 480 on EFG (Fig. 115).

STEP 3: In this final step, apply Rule 1 and try 5, which is larger than the 4 on E by 1, as the next trial quotient figure. Multiplying the 4 on A by this 5, add the product 20 to the 80 on FG, and you get 500 on FGH. This number is equal to the 500 obtained by multiplying the divisor 100 by the trial quotient 5. As the divisor is a three-whole-digit number, the unit rod of the quotient is formed on the second rod to the left of that of the dividend. The answer is 85 on DE (Fig. 116).

EXAMPLE 3: $31\,428 \div 0.97 = 32\,400$



Fig. 117



Fig. 118

STEP 1: Set the dividend 31 428 on DEFGH, and set on A, 3, the complement of 97 with respect to 100 (Fig. 117).

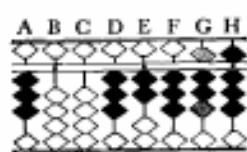


Fig. 119



Fig. 120

STEP 2: Divide the 314 on DEF by 100, with the 3 on D as the first quotient figure. Multiplying the 3 on A by the 3 on D, add the product 9 to the 4 on F. This leaves, on EFGH, 2 328 as the remainder of the dividend (Fig. 118).

STEP 3: Now divide the 232 on EFG by 100, with the 2 on E as the second quotient figure. Multiplying the 3 on A by the 2 on E, add the product 6 to the 2 on G. This leaves 388 on FGH as the remainder of the dividend (Fig. 119).

STEP 4: In this final step, apply Rule 1 and try 4, which is larger than the 3 on F by 1, as the final trial quotient figure. Add, to the 388 on FGH, the product 12 obtained by multiplying the 3 on A by this 4, and you get 400 on F. This product is the same as that obtained by multiplying the divisor 100 by this same 4. This shows that 4 is the correct third quotient figure. In this division, the divisor is a decimal fraction whose first significant digit is in the tenths' place. Therefore the unit rod of the quotient is identical with that of the dividend. The answer is 32 400 (Fig. 120).

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|-----|---|---|
| . | . | . | . | . | . | . | . |
| 3 | 0 | 0 | 3 | 1 | 4 | 2 | 8 |
| | | | | | +9 | | |
| 3 | 0 | 0 | 3 | 2 | 3 | 2 | 8 |
| | | | | | +6 | | |
| 3 | 0 | 0 | 3 | 2 | 3 | 8 | 8 |
| | | | | | (4) | | |
| | | | | | +12 | | |
| 3 | 0 | 0 | 3 | 2 | 4 | 0 | 0 |

Step 1
Step 2
Step 3
Step 4

Rules for Finding the Unit Rod of the Quotient

RULE 1: When the divisor is a decimal fraction whose first significant digit is in the tenths place, the unit rod of the dividend remains that of the quotient.

RULE 2: Each time the value of this divisor is raised by one place, the unit rod of the quotient shifts by one rod to the left of that of the dividend, and each time the value of this divisor is reduced by one place, the unit rod of the quotient shifts by one rod to the right of that of the dividend.

For instance, when the divisor is 0.95, the unit rod of the quotient forms on that of the dividend. When the divisor is 9.5, the unit rod of the quotient forms on the first rod to the left of that of the dividend. When the divisor is 0.095, the unit rod of the quotient forms on the first rod to the right of that of the dividend.

Rule 2 may be restated as follows:

RULE 2: The unit rod of the quotient forms on the rod immediately to the left of that of the dividend by as many rods as there are whole digits in the divisor.

The unit rod of the quotient forms on the rod immediately to the right of that of the dividend by as many rods as there are zeros before the first significant decimal figure.

Advantages and Disadvantages

The advantages and disadvantages of this method of division are nearly identical with those of the method of multiplication by means of complementary numbers.

ADVANTAGES:

(1) This method of division simplifies calculations when the divisor is a number a little smaller than 100, 1 000, 1, or 0.01, etc., or especially numbers whose first significant digit is 9, such as 98, 997, 0.96, or 0.0094, etc.

(2) As described in Rule 1 for finding the unit rod of the quotient, when the divisor is a decimal fraction whose first significant digit is in the tenths place, the unit rod of the quotient forms on that of the dividend. This is a great advantage to the abacus operator, as he has no trouble in finding the unit rod of the quotient.

Because of these advantages, this method is extensively used in business for the calculation of percentages. For example, it is useful in finding the regular price from the discount price and the discount rate.

DISADVANTAGE:

Its utility is limited to problems in which the divisor is a number just a little smaller than 100, 1 000, etc. When the divisor is a number whose first significant digit is smaller than 9—such as 75, 0.82, etc., Rules 1 and 2 given for finding trial quotient figures do not often apply. In such cases, this method of division is awkward, and the standard method of division is more efficient.

Exercises

- | | |
|-----------------------------|---------------------------------|
| (1) $2\,116 \div 92 = 23$ | (4) $382\,305 \div 993 = 385$ |
| (2) $4\,048 \div 88 = 46$ | (5) $705\,042 \div 897 = 786$ |
| (3) $13\,832 \div 988 = 14$ | (6) $1\,645 \div 0.94 = 1\,750$ |

V - MORE ABOUT DECIMALS

In my first book I included a chapter on decimals. I tried to show in that discussion the Japanese method of handling decimal fractions of various sorts. However, an astute reader may have seen two interesting things about my previous presentation. One thing was simply an oversight on my part, namely that I gave very few clues as to how such a multiplication problem as $0.2 \times 0.4 = 0.08$ would be done on the abacus, i.e. a problem having decimals both in the multiplicand and the multiplier. Also I neglected such division problems as $1.2 / 3 = 0.4$, i.e., a problem having decimals in the quotient.

The other thing that a Western reader in particular might have found strange is that I presented the following three problems as if they were different and each required a different kind of treatment: $31.36 / 0.32 = 98$, $3.136 / 0.032 = 98$, and $0.3136 / 0.0032 = 98$. The reason this might seem strange is that when an Occidental sets up any of these three problems on paper for long division, he immediately reduces them all to the same problem, namely $3136 / 32 = 98$. These are treated as distinct problems in Japan because of the abacus tradition of avoiding any mental effort and following the line of least resistance. The Western reader may not be satisfied with this solution and may want to work out his own compromise with the Japanese method. I think, however, that after a beginner has practiced this method enough so that it becomes mechanical, he will consider it better than any system which requires a non-mechanical, mental effort—better than the Western method of recognizing the essential simplicity of some problems.

Now let me turn to my original problem. In my first book, I explained how to do the following problems:

MULTIPLICATION

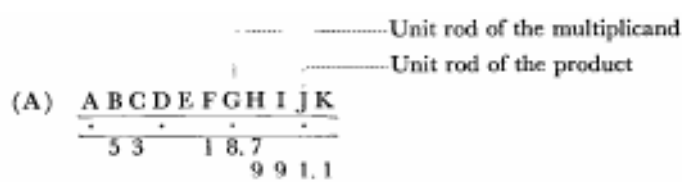
$4 \times 2 = 8$
 $25 \times 15 = 375$
 $405 \times 123 = 49\,815$
 $34 \times 1.2 = 40.8$
 $32 \times 0.4 = 12.8$
 $98 \times 0.32 = 31.36$
 $32 \times 0.04 = 1.28$
 $98 \times 0.032 = 3.136$
 $32 \times 0.004 = 0.128$
 $98 \times 0.0032 = 0.3136$

DIVISION

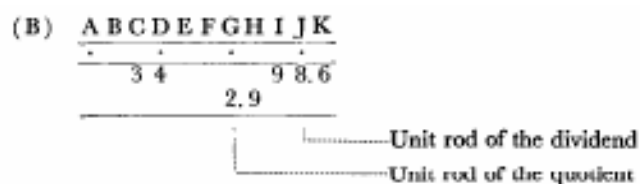
$8 / 2 = 4$
 $375 / 15 = 25$
 $49\,815 / 123 = 405$
 $40.8 / 1.2 = 34$
 $12.8 / 0.4 = 32$
 $31.36 / 0.32 = 98$
 $1.28 / 0.04 = 32$
 $3.136 / 0.032 = 98$
 $0.128 / 0.004 = 32$
 $0.3136 / 0.0032 = 98$

The following five examples should explain how to do any other normal multiplication and division problems.

EXAMPLE 1: (A) $18.7 \times 53 = 991.1$
 (B) $98.6 / 34 = 2.9$

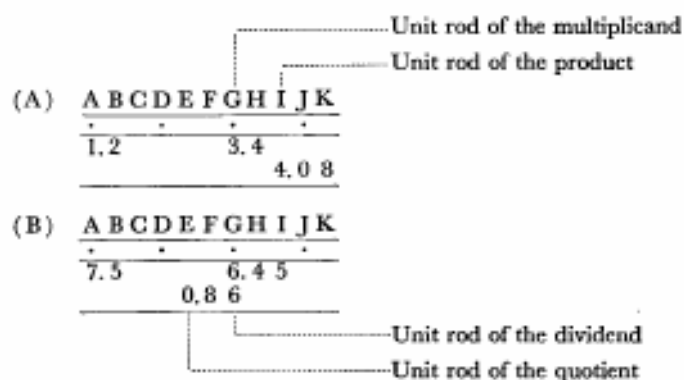


(A) shows that when the multiplier is a two-digit whole number, the unit digit of the product is formed on the third rod to the right of that of the multiplicand, i.e. on rod J.



(B) shows that when the divisor is a two-digit whole number, the unit digit of the quotient is formed on the third rod to the left of that of the dividend, i.e. on rod G.

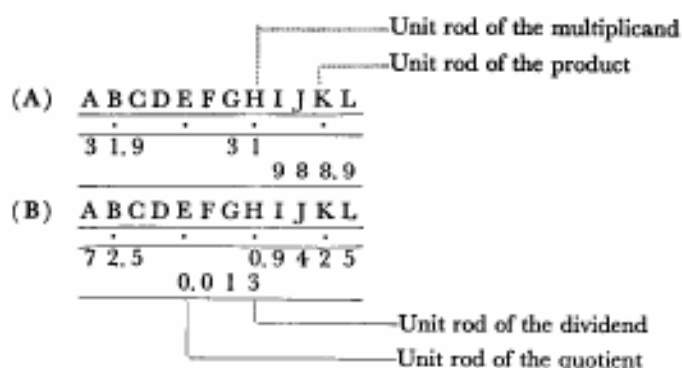
EXAMPLE 2: (A) $3.4 \times 1.2 = 4.08$
(B) $6.45 \div 7.5 = 0.86$



(A) shows that when the multiplier is a mixed number with only one whole digit in it, the unit rod of the product is formed on the second rod to the right of that of the multiplicand, i.e. on rod I.

(B) shows that when the divisor is a mixed number with one whole digit, the unit rod of the quotient forms on the second rod to the left of that of the dividend, E.

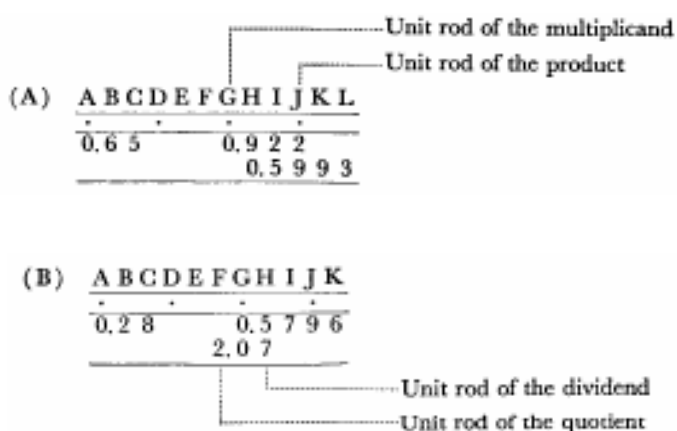
EXAMPLE 3: (A) $31 \times 31.9 = 988.9$
(B) $0.9429 \div 72.5 = 0.013$



(A) shows that when the multiplier is a mixed number with two whole digits, the unit rod of the product is formed on the third rod to the right of that of the multiplicand, i.e. rod K.

(B) shows that when the divisor is a mixed number with two whole digits, the unit rod of the quotient is formed on the third rod to the left of that of the dividend, i.e. rod E.

EXAMPLE 4: (A) $0.922 \times 0.65 = 0.5993$
(B) $0.5796 \div 0.28 = 2.07$



(A) shows that when the divisor is a decimal fraction with its first significant digit in the tenths place, the unit digit of the product is formed on the first rod to the right of that of the multiplicand, i.e. on rod H.

(B) shows that when the divisor is a decimal fraction with its first significant digit in the tenths place, the unit digit of the quotient is formed on the first rod to the left of that of the dividend, i.e. on rod F.

EXAMPLE 5: (A) $0.0289 \times 0.00741 = 0.000214149$
(B) $0.0374644 / 0.00409 = 9.16$

(A)
$$\begin{array}{cccccccccccccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} & \text{G} & \text{H} & \text{I} & \text{J} & \text{K} & \text{L} & \text{M} & \text{N} & \text{O} & \text{P} \\ \hline 0 & 0 & 0 & 7 & 4 & 1 & 0 & 0 & 2 & 8 & 9 & & & & & \\ \hline & & & & & 0 & 0 & 0 & 0 & 2 & 1 & 4 & 1 & 4 & 9 & \\ \hline \end{array}$$

(B)
$$\begin{array}{cccccccccccccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} & \text{G} & \text{H} & \text{I} & \text{J} & \text{K} & \text{L} & \text{M} & \text{N} & \text{O} & \text{P} & \text{Q} \\ \hline 0 & 0 & 0 & 4 & 0 & 9 & & & & 0 & 0 & 3 & 7 & 4 & 6 & 4 & 4 \\ \hline & & & & & & & & & 9 & 1 & 6 & & & & & \\ \hline \end{array}$$

(A) shows that when the multiplier is a decimal fraction with its first significant digit in the thousandths place, the unit rod of the product is formed on the first rod to the left of that of the multiplicand, i.e. on rod F.

(B) shows that when the divisor is a decimal fraction with its first significant digit in the thousandths place, the unit digit of the quotient is formed on the first rod to the right of that of the dividend, i.e., on rod K.

VI - CALCULATIONS INVOLVING MORE THAN ONE UNIT OF MEASUREMENT

1. Setting Compound Numbers on the Board

In setting a compound number, a measurement expressed in more than one unit, on the board, the part with the highest denomination is first set on a unit rod, and all the other parts are set so that they are separated by at least one vacant rod. Below are examples of how a compound number should be set on the board.



Fig. 121

EXAMPLE 1: £15 14 s. 9 d.

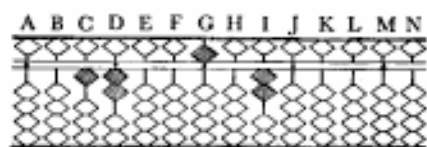


Fig. 122

EXAMPLE 2: 12 hr. 5 min. 20 sec.

In Example 1, 14 s. is separated from £15 by one vacant rod, while 9 d. is separated from 14 s. by two vacant rods (Fig. 121), because during calculations the number of pence may require the use of two digits and thus two rods. (There are 12 pence in a shilling and 20 shillings in a pound.)

In Example 2, 5 min. is separated from 12 hr. by two vacant rods, because during calculations we may have as many as 59 minutes and thereby need two rods for calculations and one for separation. The 20 seconds is separated from 5 min. by one vacant rod, because no additional rods will be necessary (Fig. 122).

It should be noted that the above arrangements of the compound numbers are the most convenient not only for addition and subtraction but also for multiplication and division. For further details, see Example 3.

EXAMPLE 3: 7 yd. 1 ft. 9 in.

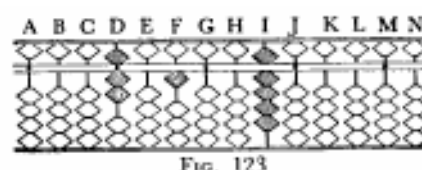


Fig. 123

In Example 3, 1 ft. is separated from 7 yd. by one vacant rod, while 9 in. is separated from 1 ft. by two vacant rods, because during calculations we may have as many as 11 inches and thereby two rods (Fig. 123).

When only addition and subtraction are to be performed, each unit of measurement is often given a separate unit rod, e.g., 1 ft. would be set on G and 9 in. on J. But when multiplication and division are also to be performed, it is more convenient to set the units as in the above examples.

2. Converting Compound Numbers

EXAMPLE 1: Find the value of £37 10 s. in dollars if one pound equals \$4.03.



Fig. 124

This problem can be simplified as follows:

$$10 \text{ s.} = £0.5 \text{ (NOTE: } 10 \text{ s.} / 20 \text{ s.} = 0.5)$$

$$£37 \text{ } 10 \text{ s.} = £37.5$$

$$£37.5 \times 4.03 = \$151.125$$

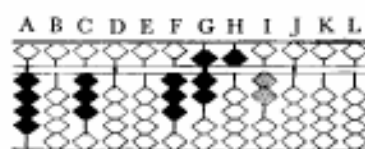


Fig. 125

STEP 1: Set the multiplicand 37.5 on FGH, with G as the unit rod. Set the multiplier 4.03 on ABC, leaving two rods vacant (Fig. 124).

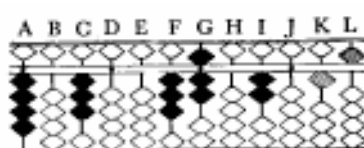


Fig. 126

NOTE: It is not really necessary to set the unit figure of the multiplier on a unit rod.

STEP 2: Multiplying 4 on A by 5 on H, set product 20 on IJ (Fig. 125).

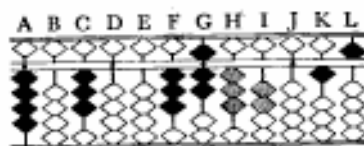


Fig. 127

STEP 3: Multiplying the 3 on C by the same 5 on H, set the product 15 on KL, and clear H of the 5. This makes a total of 2 015 on IJKL (Fig. 126).



Fig. 128

NOTE: As the second figure of the multiplier is zero, set the product 15 on KL, skipping JK.

STEP 4: Next, multiplying the 4 on A by the 7 on G, set the product 28 on HI. This makes a total of 30 015 on HIJKL (Fig. 127).



Fig. 129

STEP 5: Multiplying the 3 on C by the same 7 on G, set the product 21 on JK, and clear G of the 7. This makes a total of 30 225 on HIJKL (Fig. 128).

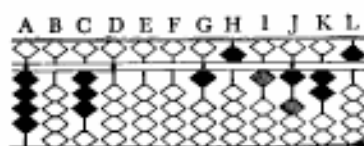


Fig. 130

STEP 6: Next, multiplying the 4 on A by the 3 on F, set the product 12 on GH. This makes a total of 150 225 on GHIJKL (Fig. 129).

STEP 7: Finally multiplying the 3 on C by the same 3 on F, set the product 9 on J, and clear F of the 3. This gives you a total of 151 125 on GHIJK (Fig. 130).

| A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|----|---|----|---|---|---|---|
| 4 | 0 | 3 | 0 | 0 | 3 | 7 | 5 | 0 | 0 | 0 | 0 |
| | | | | | +2 | 0 | | | | | |
| | | | | | | | +1 | 5 | | | |
| 4 | 0 | 3 | 0 | 0 | 3 | 7 | 0 | 2 | 0 | 1 | 5 |
| | | | | | +2 | 8 | | | | | |
| | | | | | | | +2 | 1 | | | |
| 4 | 0 | 3 | 0 | 0 | 3 | 0 | 3 | 0 | 2 | 2 | 5 |
| | | | | | +1 | 2 | | | | | |
| | | | | | | | +9 | | | | |
| 4 | 0 | 3 | 0 | 0 | 0 | 1 | 5 | 1 | 1 | 2 | 5 |

Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7

As the multiplier is a mixed decimal whose whole figure is in the unit place, the unit rod of the product forms on I, the second rod to the right of G, the unit rod of the multiplicand. The answer is \$151.125.

EXAMPLE 2: Find the value of the following compound number in terms of pence (12 pence = 1 shilling; 20 shillings = 1 pound). £12 16 s. 8 d.

Written calculations:

$$\begin{array}{r} \text{£} \\ 12 \\ \times 20 \\ \hline 240 \end{array}$$

$$\begin{array}{r} \text{s.} \\ 16 \\ + 240 \\ \hline 256 \\ \times 12 \\ \hline 512 \\ 256 \\ \hline 3072 \end{array}$$

$$\begin{array}{r} \text{d.} \\ 8 \\ + 3072 \\ \hline 3080 \end{array}$$

Ans.: 3,080 d.

On the abacus, the operation is performed much in the same way as in written calculation. First £12 is multiplied by 20 and the resulting 240 s. is added to 16 s. Next the sum 256 s. is multiplied by 12 and the product 3 072 d. is added to 8 d. The answer is 3 080 d.

However, on the abacus, both the multiplication and addition are performed simultaneously, and calculations are much faster.



Fig. 131



Fig. 132

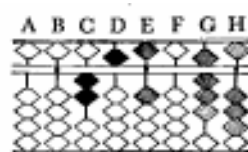


Fig. 133



Fig. 134

STEP 1: Set 12 on AB, with B as the unit rod, 16 on DE, with E as the unit rod, and 8 on H, with H as the unit rod (Fig. 131).

STEP 2: First multiply the 2 on B by 20, add the product 40 to the 16 on DE, and clear B of the 2. This makes a total of 56 on DE. Next multiplying the 1 on A by 20, set the product 20 on CD, and clear A of the 1. This gives you a total of 256 on CDE (Fig. 132).

NOTE: In this procedure multiply the 2 on B by 20 first and the 1 on A by 20 next. If you multiply the digit on A first, the product may extend to rod B. This will happen if the digit on A is large. If, for example, the number on A were 8, setting the product 16 on BC would be rather confusing.

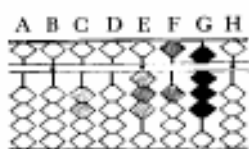


Fig. 135

STEP 3: Now the 256 on CDE must be multiplied by 12 and added to the 8 on H. First multiply the 6 on E by 10, and set the product 60 on GH. This makes a total of 68 on GH. Next multiplying the same 6 on E by 2, add the product 12 to the 68 on GH, and clear E of the 6. This gives you a total of 80 on GH (Fig. 133).

NOTE: Be sure to dispose of the 6 of 256 first, next the 5, and finally the 2.

$$\begin{array}{r} \text{A B C D E F G H} \\ 12016008 \\ +40 \\ +20 \\ \hline 00256008 \\ +60 \\ +12 \\ \hline 00250080 \\ +50 \\ +10 \\ \hline 00200680 \\ +20 \\ +4 \\ \hline 00003080 \end{array}$$

Step 1
Step 2

Step 3

Step 4

Step 5

STEP 4: Multiplying the 5 on D by 10, set the product 50 on FG. This makes a total of 580 on FGH. Next, multiplying the same 5 on D by 2, add the product 10 to the 58 on FG, and clear D of the 5. This gives you a total of 680 on FGH (Fig. 134).

STEP 5: Multiplying the 2 remaining on C by 10, set the product 20 on EF. This makes a total of 2 680 on EFGH.

Finally, multiplying the same 2 on C by 2, add the product 4 to the 6 on F, and clear C of the 2. This gives you a total of 3 080 on EFGH (Fig. 135). The answer is 3 080 d.

NOTE: Experts usually save the trouble of setting such simple multipliers as 20 and 12 on the board.

EXAMPLE 3: Find the value of 6 190 d. in terms of pounds and shillings.

Written calculation:

On the abacus, the operation is performed in the same way as in written calculation. First 6 190 is divided by 12, next the quotient 515 is divided by 20.

$$\begin{array}{r} 515 \\ 12 \overline{) 6190} \\ \underline{60} \\ 19 \\ \underline{12} \\ 70 \\ \underline{60} \\ 10 \end{array} \qquad \begin{array}{r} 25 \\ 20 \overline{) 515} \\ \underline{40} \\ 115 \\ \underline{100} \\ 15 \end{array} \qquad \text{Ans.: } \pounds 25 \ 15 \ s. \ 10 \ d.$$

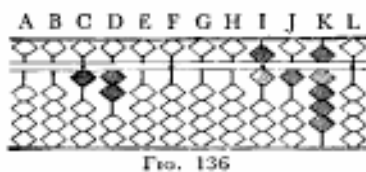


Fig. 136

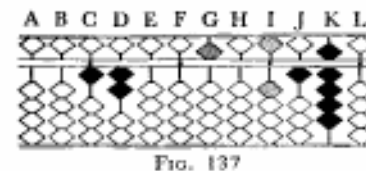


Fig. 137



Fig. 138

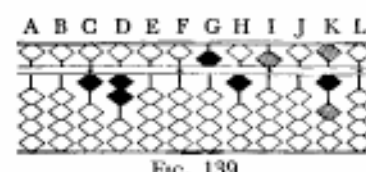


Fig. 139



Fig. 140



Fig. 141

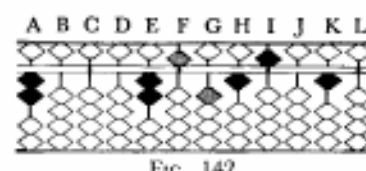


Fig. 142

STEP 1: Set 6 190 on IJKL, with L as the unit rod, and set 12 on CD. See that there are four vacant rods between the two numbers (Fig. 136).

STEP 2: Comparing the divisor 12 and the 61 in 6 190, set the quotient figure 5 on G. Now multiply the 1 on C by this 5, and subtract the product 5 from the 6 on I. This leaves 1 190 on IJKL. Next multiply the 2 on D by the same 5 on G and subtract the product 10 from the 11 on IJ. This leaves 190 on JKL (Fig. 137).

STEP 3: Comparing the divisor 12 and the 19 on JK, set the second quotient figure 1 on H. Now multiply the 1 on C by the 1 on H, and subtract the product 1 from the 1 on J. This leaves 90 on KL. Next multiply the 2 on D by the same 1 on H, and subtract the product 2 from the 9 on K. This leaves 70 on KL (Fig. 138).

STEP 4: Comparing the divisor 12 and the 70 on KL, set the third quotient figure 5 on I. Now multiply the 1 on C by this 5 on I, and subtract the product 5 from the 7 on K. This leaves 20 on KL. Next multiply the 2 on D by the same 5 on I, and subtract the product 10 from the 20 on KL. This leaves 10 on KL (Fig. 139).

STEP 5: Now leaving the 10 d. on KL as it is, you must divide the 515 s. on GHI by 20 to convert it into pounds. Clear CD of its 12, and set the divisor 20 on AB, leaving four vacant rods between it and the 515 on GHI (Fig. 140).

STEP 6: Now comparing the 2 on A and the 5 on G, set the quotient 2 on E. Multiplying the 2 on A by this 2 on E, subtract the product 4 from the 5 on G. This leaves 115 on GHI (Fig. 141).

| A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|----|----|---|---|---|---|
| 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 6 | 1 | 9 | 0 |
| | | | | | 5 | | | | | | |
| | | | | | | -5 | | | | | |
| | | | | | | -1 | 0 | | | | |
| 0 | 0 | 1 | 2 | 0 | 0 | 5 | 0 | 1 | 9 | 0 | |
| | | | | | | 1 | | | | | |
| | | | | | | | -1 | | | | |
| | | | | | | | -2 | | | | |
| 0 | 0 | 1 | 2 | 0 | 0 | 5 | 1 | 0 | 0 | 7 | 0 |
| | | | | | | 5 | | | | | |
| | | | | | | | -5 | | | | |
| | | | | | | | -1 | 0 | | | |
| 0 | 0 | 1 | 2 | 0 | 0 | 5 | 1 | 5 | 0 | 1 | 0 |
| | | | | | 2 | | | | | | |
| | | | | | | | -4 | | | | |
| 2 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 5 | 0 | 1 | 0 |
| | | | | | 5 | | | | | | |
| | | | | | | -1 | 0 | | | | |
| 2 | 0 | 0 | 0 | 2 | 5 | 0 | 1 | 5 | 0 | 1 | 0 |

STEP 7: Next compare the 2 on A and the 11 on GH and set the quotient figure 5 on F. Multiplying the 2 on A by this 5 on F, subtract the product 10 from the 11 on GH. This leaves 15 s. on HI. The answer is £25 15 s. 10 d. (Fig. 142).

NOTE: Why are the pounds, shillings, and pence figures set as they are in Examples 2 and 3?

The close arrangement of these figures is the most efficient arrangement for performing the four arithmetic operations. Any closer arrangement would cause confusion.

This arrangement affords another special advantage in that the standard methods of multiplication and division can be used most efficiently.

Examine step 2 in Example 2, where pounds are converted to shillings and you can see that the standard method of multiplication forms the unit digit of the product on the unit rod of the shillings figure on the board.

Examine steps 2, 3, and 4 in Example 3, where the pence are converted to shillings, and you can see that the standard method of division forms the unit digit of the quotient on the unit rod of the shillings figure on the board. Any other arrangement of the pounds, shillings, and pence figures would require the use of methods of multiplication and division other than the standard methods and would make calculations less efficient.

3. Adding and Subtracting Compound Numbers

EXAMPLE 1:

| | | | |
|---------|-------|--------|--------------------------|
| 21 yd. | 1 ft. | 8 in. | |
| 15 yd. | 2 ft. | 7 in. | |
| +29 yd. | 2 ft. | 11 in. | |
| 67 | 1 | 2 | Ans.: 67 yd. 1 ft. 2 in. |

There are two ways of adding measures. One is to add each compound number from higher to lower denominations, and the other is to add up column by column

from lower to higher denominations as in written calculation. The former method is generally used when both addition and subtraction are to be performed, and it must always be followed when successive compound numbers are dictated to the operator. The latter is considered to be a little faster if it can be used.

METHOD A: Adding from higher to lower denominations:

| yd. | ft. | in. | |
|-----|-----|-----|--------|
| 21 | 1 | 8 | Step 1 |
| +15 | 2 | 7 | |
| 36 | 3 | 15 | Step 2 |
| +29 | 2 | 11 | |
| 65 | 5 | 26 | Step 3 |
| +2 | 7 | 2 | |
| 67 | 7 | 2 | Step 4 |
| | 2 | 12 | |
| | 3 | 26 | Step 5 |
| | 7 | 24 | |
| | 6 | 2 | |
| | 1 | | |

STEP 1: Set 21 yd. on AB, with B as the unit rod. Set 1 ft. on D and 8 in. on G (Fig. 143). Refer to Fig. 123 at the beginning of this chapter.

STEP 2: Add 15 to the 21 on AB. This makes a total of 36 on AB. Now add 2 to the 1 on D. This makes a total of 3 on D.

Next add 7 to the 8 on G. This makes a total of 15 on FG (Fig. 144).



Fig. 143

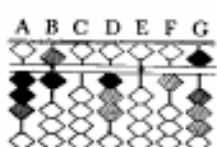


Fig. 144

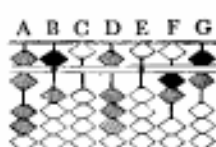


Fig. 145

STEP 3: Add 29 to the 36 on AB. This makes a total of 65 on AB. Next add 2 to the 3 on D. This makes a total of 5 on D. Finally add 11 to the 15 on FG. This makes a total of 26 on FG (Fig. 145).

STEP 4: Now divide the 26 on FG by 12, and add the quotient 2 to the 5 on D, leaving the remainder 2 on G. This makes a total of 7 on D (Fig. 146).

NOTE: In dividing the 26 on FG, experienced operators generally do not set the divisor 12 on the board.



Fig. 146



Fig. 147

STEP 5: Finally divide the 7 on D by 3, and add the quotient 2 to the 65 on AB, leaving the remainder 1 on D. This makes a total of 67 on AB. The answer is 67 yd. 1 ft. 2 in. (Fig. 147).

METHOD B: Adding from lower to higher denominations:

| yd. | ft. | in. |
|-----------|--------------|----------------|
| 2 | 2 | |
| 21 | 1 | 8 |
| 15 | 2 | 7 |
| +29 | 2 | 11 |
| 67 Step 5 | 7 Step 3 | 26 Step 1 |
| | 2 | 2 |
| | 3) 7 Step 4 | 12) 26 Step 2 |
| | 6 | 24 |
| | 1 | 2 |

STEP 1: Set 8 on unit rod G. Add 7 and 11 to the 8 on G, and you get a total of 26 on FG (Fig. 148).

STEP 2: Divide the 26 on FG by 12, and set the quotient 2 on D, which is the proper rod on which to set the quotient. This gives you 2 on D and leaves 2 on G (Fig. 149).



Fig. 148



Fig. 149



Fig. 150

STEP 3: Add 1, 2, and 2 successively to the 2 on D. This makes a total of 7 on D (Fig. 150).

STEP 4: Divide the 7 on D by 3, and set the quotient 2 on B, which is the proper rod on which to set the quotient. This gives you 2 on B and leaves 1 on D (Fig. 151).



Fig. 151



Fig. 152

STEP 5: Finally add 21, 15, and 29 successively to the 2 on B. This makes a total of 67 on AB. The answer is 67 yd. 1 ft. 2 in. on AB, D, and G respectively (Fig. 152).

NOTE: Why are the yards, feet, and inches figures set as they are in this example?

The great advantage of this arrangement of these figures is that the standard methods of multiplication and division can be used. Notice the use of the standard method of division in steps 4 and 5 of Method A, and in steps 2 and 4 of Method B in this example. For further details see Examples 2 and 3, section 2 of this chapter.

EXAMPLE 2: Subtract 12 days 17 hours 48 minutes from 25 days 12 hours 43 minutes.

| da. | hr. | min. |
|-----|-----|------|
| 25 | 12 | 43 |
| -12 | -17 | -48 |
| 12 | 18 | 55 |

Ans.: 12 da. 18 hr. 55 min.



FIG. 153



FIG. 154

STEP 1: Set 25 on AB, with B as the unit rod, set 12 on DE, with E as the unit rod, and set 43 on GH, with H as the unit rod (Fig. 153).

STEP 2: Subtract 12 from the 25 on AB. This leaves 13 on AB (Fig. 154).

STEP 3: Since you cannot subtract 17 from the 12 on DE, borrow 1 from B, and subtracting 17 from 24, add 7 to the 12 on DE. This makes 19 on DE (Fig. 155).



FIG. 155



FIG. 156

| A | B | C | D | E | F | G | H |
|----|----|---|---|----|---|---|---|
| 2 | 5 | 0 | 1 | 2 | 0 | 4 | 3 |
| -1 | 2 | | | | | | |
| 1 | 3 | 0 | 1 | 2 | 0 | 4 | 3 |
| -1 | | | | | | | |
| | 2 | 4 | | | | | |
| | -1 | 7 | | | | | |
| 1 | 2 | 0 | 1 | 9 | 0 | 4 | 3 |
| | | | | -1 | | | |
| | | | | 6 | 0 | | |
| | | | | -4 | 8 | | |
| 1 | 2 | 0 | 1 | 8 | 0 | 5 | 5 |

Step 4

NOTE: In borrowing from B, you may add 24 to the 12 on DE, and subtract 17 from the sum 36. This gives you the same result 19.

STEP 4: Since you cannot subtract 48 from the 43 on GH, borrow 1 from the 9 on E, and subtracting 48 from 60, add the result 12 to the 43 on GH. This makes 55 on GH. Now you have 12 on AB, 18 on DE, and 55 on GH. The answer is 12 days 18 hours 55 minutes (Fig. 156).

EXAMPLE 3:

| £ | s. | d. |
|-----|----|----|
| 95 | 7 | 3 |
| +12 | 8 | 5 |
| -7 | 19 | 10 |
| +26 | 2 | 7 |
| 125 | 18 | 5 |

Ans.: £125 18 s. 5 d.

In the above example, the second and fourth compound numbers are to be added, and the third to be subtracted.

When both additions and subtractions are to be performed, calculations generally move from higher to lower denominations on each row. But they can also move down each column—completing each column before moving to the higher denomination.

METHOD A: Calculating from higher to lower denominations:

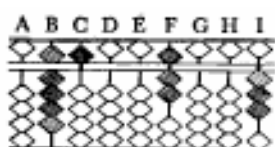


FIG. 157

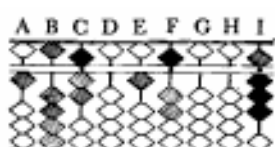


FIG. 158



FIG. 159



FIG. 160

STEP 1: Set £95 on BC, with C as the unit rod, set 7 s. on unit rod F, and set 3 d. on unit rod I (Fig. 157).

STEP 2: Add 12 to the 95 on BC, producing 107 on ABC. Add 8 to the 7 on F, yielding 15 on EF. Add 5 to the 3 on I producing 8 on I (Fig. 158).

STEP 3: Subtract 7 from the 107 on ABC, leaving 100 on ABC. As you cannot subtract

19 from the 15 on EF, borrow 1 from rod C. Subtracting 19 from 20, add 1 to the 15 on EF. This gives you 16 on EF and 99 on BC (Fig. 159).

STEP 4: As you cannot subtract 10 from the 8 on I, borrow 1 from the 16 on EF, and subtracting 10 from 12, add the result 2 to the 8 on I. This gives you 10 on HI and 15 on EF (Fig. 160).

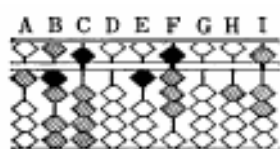


FIG. 161

STEP 5: Now add 26 to the 99 on BC, and you get 125 on ABC. Add 2 to the 15 on EF, leaving 17 on EF. Add 7 to the 10 on HI yielding 17 on HI. Finally subtract 12 from this 17 and carry 1 to the 17 on EF. This gives you 5 on I and 18 on EF. The answer is £125 18 s. 5 d. (Fig. 161).

METHOD B: Calculating from lower to higher denominations:

An alternative is to calculate column by column from lower to higher denominations.

First do the pence column on HI by means of complementary numbers, yielding 5 d. on I.

Next do the shillings column on EF by means of complementary numbers, and you get minus 2 s. on F. So borrow £1 from L95 to rectify this minus quantity, and you get 18 s. on EF and £94 left on BC.

In subtracting 19 s. from 15 s., you may borrow L1 instead of calculating by means of complementary numbers.

Finally setting L94 on BC, calculate the pound column, which leaves £ 125 on ABC.

This gives you the same result, £125, 18 s., and 5 d. on ABC, EF, and I respectively.

| | | | |
|------|-------|-----------------|--------|
| £ 4 | 2 s. | 6 d. | Step 1 |
| × 18 | × 18 | × 18 | Step 2 |
| 72 | 36 | 12) 108 (9 | Step 3 |
| + 2 | + 9 | 108 | |
| 74 | 45 (2 | 0 | |
| | 40 | | |
| | 5 | | |
| | | Ans. : £74 5 s. | |



FIG. 162

4. Multiplying Compound Numbers

EXAMPLE 1: Multiply the following compound number by 18: £4 2s. 6d.



FIG. 163

On the abacus the computation is made mostly in the same way as in the following written calculation.

STEP 1: Set 4 on unit rod E, and 2 and 6 on H and K respectively. Set the multiplier 18 on AB (Fig. 162).

NOTE: Experts, however, often do not actually set the multiplier.

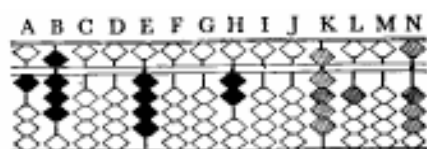


FIG. 164



Fig. 165



Fig. 166



Fig. 167

STEP 2: First multiply the 6 on K by 18, and you get the product 108 on LMN. Now clear K of its 6 (Fig. 163).

STEP 3: Now divide the 108 on LMN by 12 and you get the quotient 9 on K, having cleared LMN of the 108. Here you should remember the divisor 12 without setting it on the board (Fig. 164).

STEP 4: Next you must multiply the 2 on H by 18, and add the product 36 to the 9 on K. In this procedure, first multiply the 1 on A by the 2 on H, then set the product 2 on J. This gives you 29 on JK. Next multiply the 8 on B by the same 2 on H; add the product 16 to the 29 on JK, and clear H of its 2. This gives you a total of 45 on JK (Fig. 165).

STEP 5: Now divide the 45 on JK by 20, and you get the quotient 2 on H, with 5 left over on K (Fig. 166).

STEP 6: Finally you must multiply the 4 on E by 18, and add the product 72 to the 2 on H. In this process, first multiply the 1 on A by the 4 on E, and set the product 4 on G. This gives you 42 on GH. Next multiply the 8 on B by the same 4 on E, add the product 32 to the 42 on GH, and clear E of its 4. This gives you a total of 74 on GH. The answer is £74 5 s. (Fig. 167).

NOTE: An alternative method for working this problem is to convert the numerical figures of this problem to pence, and then to multiply the result by 18. At the end of the problem the product is reconverted to pounds, shillings, and pence. This method is often used when the multiplier is a decimal fraction.

5. Dividing Compound Numbers

EXAMPLE 1: Divide the following compound number by 14: £243 2s. 8d.

| | | |
|---------------------|---------------|--------------|
| £243 Step 1 | 2 s. Step 1 | 8 d. Step 1 |
| | + 5 × 20 | + 4 × 12 |
| | 102 | 56 |
| 17 | 7 | 4 |
| 14)243 Step 2 | 14)102 Step 4 | 14)56 Step 6 |
| 14 | 98 | 56 |
| 103 | 4 | 0 |
| 98 | | |
| 5 | | |
| Ans. : £17. 7s. 4d. | | |



Fig. 168

On the abacus the division is performed much the same way as in the following written calculation:

STEP 1: Set 243 on GHI, with I as the unit rod, and set 2 on unit rod L and 8 on unit rod O. The divisor 14 may be set either on AB or on BC, though preferably on the former with four vacant rods between B and G (Fig. 168).

STEP 2: Divide the 243 on GHI by 14, and you get the quotient 17 on EF and a remainder of 5 on I (Fig. 169).

STEP 3: Multiply the 5 on I by 20, and add the product 100 to the 2 on L, clearing the 5 on I, and you get 102 on JKL (Fig. 170).

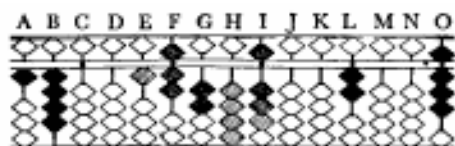


Fig. 169



Fig. 170



Fig. 171



Fig. 172



Fig. 173

STEP 4: Divide the 102 on JKL by 14, leaving a quotient of 7 on I and a remainder of 4 on L (Fig. 171).

STEP 5: Multiply the 4 on L by 12, and add the product 48 to the 8 on O, clearing the 4 on L, and you get 56 on NO (Fig. 172).

STEP 6: Divide the 56 on NO by 14, and you get the quotient 4 on L with no remainder. The answer is £17 7 s. 4 d. (Fig. 173).

NOTES: (a) An alternative method for this problem is to convert everything to pence and then to divide the result by 14. This result is converted to pounds, shillings, and pence. When the figure of the highest denomination is smaller than the divisor, division cannot be performed unless it is converted into that of the lower denomination and makes a figure larger than the divisor. (b) On the board, multiplication is performed simultaneously with addition, and division simultaneously with subtraction. This means that to multiply numbers is to add numbers and that to divide numbers is to subtract numbers. Accordingly, multiplication and division are all the faster on the board than on paper.

VII - EXTRACTING SQUARE ROOTS

There are several methods of finding both the square and cube roots of numbers on the abacus. They are all adaptations of algebraic methods.

Since the methods of finding cube roots are complicated, and are beyond the scope of the interest and need of ordinary abacus operators, this book only concerns itself with three typical methods of extracting square roots.

The first is the exact algebraic method. The second is considered the best by some authoritative experts. The third is the most representative traditional method and is the most widely accepted by modern abacus operators.

The method of finding the two-figure square root of a number is founded upon the method of finding the square of an ordinary binomial (or two-membered) expression, $(a + b)$. The number is analyzed into $(a^2 + 2ab + b^2)$ as in algebra, and the square root is obtained.

Let us take the number 625 as an example of finding a two-figure square root.

In all the three methods introduced here, the reader will see that the first step consists in subtracting the square of a or 20 ($= a^2$ or 400), the next of subtracting $2ab$ or $2 \times 20 \times 5 (=200)$, and the third of subtracting the square of b or 5 ($= b^2$ or 25)

As a refresher, the standard written method of extracting square roots is as follows:

$$\begin{array}{r} 25 \\ \sqrt{625} \\ \underline{4} \\ 22 \\ \underline{20} \\ 25 \\ \underline{25} \\ 0 \end{array}$$

625 is divided into pairs of digits beginning from the decimal point and moving right and left (thus 18 967.103 is divided 1'89'67.10'30). Then the largest perfect square which matches the first set of numbers is selected—4 is the largest perfect square smaller than or equal to 6.2—and the square root of 4, 2, is written both above the 6 and to the side. The product, 4, is written below the 6. The 4 is subtracted from 6 and the next pair of numbers is brought down—making 225. The number which is above the square root sign, 2, is doubled and brought down to the side—here the 4 of the 45 at the side. Then a number is selected, 5, such that when placed with 4 to make a two-digit number, 45, and that number is multiplied by the original number selected, 45×5 , the result is the largest number less than or equal to the “quotient,” which at this point is 225.

The following analysis will also help the reader to find the square root of 625 on the abacus. Note that in the series of algebraic equations given here, a is in substance 20 and b is 5.

$$\begin{aligned} & (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &= 20^2 + (2 \times 20 \times 5) + 5^2 \\ &= 400 + 200 + 25 \\ &= 625 \end{aligned}$$

Both the abacus method and this written method are based on algebraic equations. In a simple problem (one involving a two-digit answer) both methods of extracting a square

root can be described by the binomial equation $(a+b)^2 - a^2 - 2ab - b^2 = 0$. In a slightly more complicated problem (one involving a three-digit answer) both can be described by the equation $(a + b + c)^2 - a^2 - 2ab - b^2 - 2ac - 2bc - c^2 = 0$. To understand this compare the examples below, which show the standard written method (left) and the abacus method (right).

| | |
|--|---|
| $\begin{array}{r} 2 \quad 5. \\ \sqrt{6 \overline{) 2 \quad 5.}} \\ 2 \quad 4 \\ \hline 45 \quad 2 \quad 2 \quad 5 \end{array}$ | $\begin{array}{l} (a+b) \\ (a+b)^2 \\ (-a^2) \\ (-2ab-b^2) \end{array}$ |
| $\begin{array}{r} 2 \quad 5. \\ \sqrt{6 \overline{) 2 \quad 5.}} \\ 2 \quad 4 \\ \hline 4 \quad 2 \quad 0 \\ \hline 5 \quad 2 \quad 5 \end{array}$ | $\begin{array}{l} (a+b) \\ (a+b)^2 \\ (-a^2) \\ (-2ab) \\ (-b^2) \end{array}$ |

In a trinomial problem this is what happens. At left is the written method and at right the abacus method.

| | |
|---|---|
| $\begin{array}{r} 3 \quad 4 \quad 6. \\ \sqrt{1 \overline{) 1 \quad 9 \quad 7 \quad 1 \quad 6.}} \\ 3 \quad 9 \\ \hline 64 \quad 2 \quad 5 \quad 6 \\ \hline 686 \quad 4 \quad 1 \quad 1 \quad 6 \end{array}$ | $\begin{array}{l} (a+b+c) \\ (a+b+c)^2 \\ (-a^2) \\ (-2ab-b^2) \\ (-2ac-2bc-c^2) \end{array}$ |
| $\begin{array}{r} 3 \quad 4 \quad 6. \\ \sqrt{1 \overline{) 1 \quad 9 \quad 7 \quad 1 \quad 6.}} \\ 3 \quad 9 \\ \hline 6 \quad 2 \quad 4 \\ \hline 4 \quad 1 \quad 6 \\ \hline 6 \quad 3 \quad 6 \\ \hline 8 \quad 4 \quad 8 \\ \hline 6 \quad 3 \quad 6 \end{array}$ | $\begin{array}{l} (a+b+c) \\ (a+b+c)^2 \\ (-a^2) \\ (-2ab) \\ (-b^2) \\ (-2ac) \\ (-2bc) \\ (-c^2) \end{array}$ |

This means that the numbers handled in the Japanese abacus method as opposed to the Western written method are smaller and simpler, on the other hand, the number of manipulations required is larger. Nevertheless, once an operator becomes accustomed to using this Japanese method he can obviously do a problem faster than he would have been able to on paper.

EXAMPLE 1: Find the square root of 625.

METHOD A:



STEP 1: Set 625 on EFG, with G as the unit rod (Fig. 174). Mark off the digits in sets of twos. The largest perfect square less than 6 is 4.

STEP 2: Set, on C, 2, the root of the 4. Square this 2, and subtract the product 4 from the 6 on E. This leaves 225 on EFG (Fig. 175).

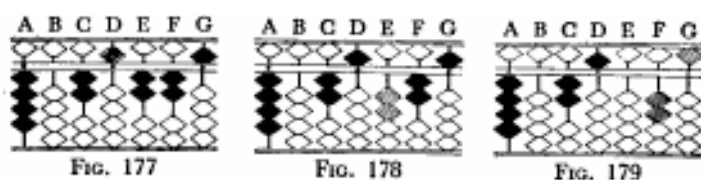
NOTE: The 2 on C corresponds to the a of the expression $(a^2 + 2ab + b^2)$.

STEP 3: Double the 2 on C, and set the product 4 on A (Fig. 176).

NOTE: Now the 4 on A corresponds with the $2a$ of the algebraic expression.

STEP 4: The 4 on A divides into the 22 on EF five times. Set the second trial quotient figure 5 on D (Fig. 177).

NOTE: The 5 on D corresponds with the b of the algebraic expression.



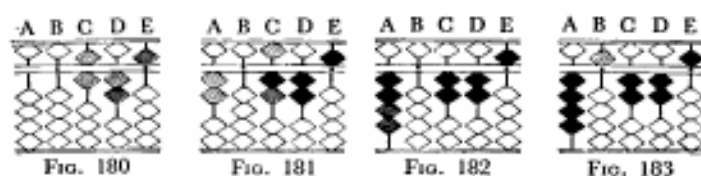
STEP 5: Multiplying the 4 on A by the 5 on D, subtract the product 20 from the 22 on EF. This leaves 25 on FG (Fig. 178).

NOTE: (4 x 5) corresponds with the $2ab$ of the expression.

STEP 6: Square the 5 on D, and subtract the product 25 from the 25 on FG. This clears the board of 625, and shows that the 5 on D is the correct second quotient figure. The answer is 25 on CD (Fig. 179).

NOTE: 52 or 25 corresponds with the b^2 of the expression. Also note that this method corresponds more strictly to the written algebraic method than the other methods introduced below.

METHOD B:



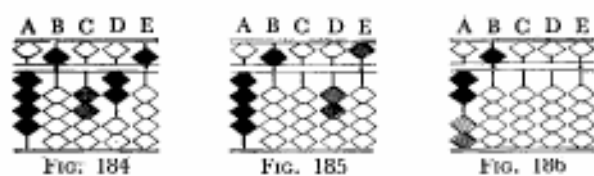
STEP 1: Set 625 on CDE, with E as the unit rod (Fig. 180). Mark off the digits in sets of twos. The largest square less than 6 is 4.

STEP 2: Set, on A, 2, the root of the 4. Square this 2, and subtract the product 4 from the 6 on C. This leaves 225 on CDE (Fig. 181).

STEP 3: Double the 2 on A, making 4 on A (Fig. 182).

STEP 4: The 4 on A goes into the 22 on CD five times. Set the second trial quotient figure 5 on B (Fig. 183).

NOTE: The 5 on B corresponds with the b of the expression.



STEP 5: Multiply the 4 on A by the 5 on B, and subtract the product 20 from the 22 on CD. This leaves 25 on DE (Fig. 184).

STEP 6: Square the 5 on B, and subtract the product 25 from the 25 on DE, and the board is cleared of the dividend (Fig. 185).

STEP 7: Finally halve the 4 on A into its original digit 2. Our square root is 25 on AB (Fig. 186).

METHOD C:

STEP 1: Set 625 on CDE, with E as the unit rod (Fig. 187). Mark off the digits in sets of twos. The largest square less than 6 is 4.

STEP 2: Set, on A, 2, the root of the 4. Square this 2, and subtract the product 4 from the 6 on C. This leaves 225 on CDE (Fig. 188).

NOTE: The 225 on CDE corresponds to the $(2ab + b^2)$ of the algebraic expression.



Fig. 187

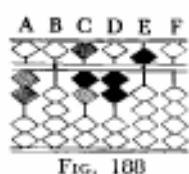


Fig. 188

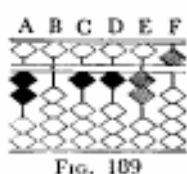


Fig. 189

STEP 3: Multiply the 225 on CDE by 0.5, and set the product 112.5 on CDEF (Fig. 189).

NOTES: (a) If you find it hard to do this division mentally, you may first set the product 112.5 on FGHI and later set it on CDEF. (b) The 112.5 on CDEF corresponds to $(ab + b^2 / 2)$ of the algebraic expression.



Fig. 190

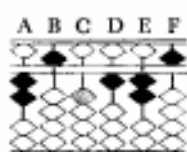


Fig. 191



Fig. 192

STEP 4: The 2 on A divides into the 11 on CD five times. Set the second trial quotient figure 5 on B (Fig. 190).

NOTE: The 5 on B corresponds with the b of the expression.

STEP 5: Multiply the 2 on A by the 5 on B, and subtract the product 10 from the 11 on CD. This leaves 12.5 on DEF (Fig. 191).

NOTE: The 12.5 on DEF corresponds to $b^2 / 2$ of the expression.

STEP 6: Square the 5 on B mentally, and you get 25. Next either divide this 25 by 2 or multiply it by 0.5, and you get 12.5. If you find it hard to do this calculation mentally you can do it on the board.

Now subtract this 12.5 from the 12.5 on DEF. This clears the board of the 625, and leaves the answer 25 on AB (Fig. 192).

EXAMPLE 2: Find the square root of 4 489 (Method B).



Fig. 193



Fig. 194



Fig. 195

STEP 1: Set 4 489 on DEFG, with G as the unit rod (Fig. 193). Mark off the digits in sets of twos. In the first pair 44, the highest square root is 6.

STEP 2: Set, on B, 6, which is the root of the 44 on DE. Square this 6, and subtract the product 36 from the 44 on DE. This leaves 889 on EFG (Fig. 194).

STEP 3: Double the 6 on B into 12 on AB (Fig. 195).

STEP 4: The 12 on AB divides into the 88 On EF seven times. Set the second trial divisor figure 7 on C (Fig. 196).

NOTE: The 7 on C corresponds with the b of the expression.

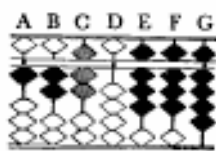


FIG. 196



FIG. 197



FIG. 198

STEP 5: Now multiplying the 12 on AB by the 7 on C, subtract the product 84 from the 88 on EF. This leaves 49 on FG (Fig. 197).

NOTE: The 12 on AB multiplied by 7 corresponds with $2ab$ or (120×7) .

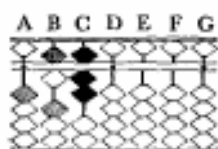


FIG. 199

STEP 6: Next square the 7 on C, and subtract the product 49 from the 49 on FG. This clears the board of the remainder of the number or the b^2 of the expression (Fig. 198).

STEP 7: Finally halve the 12 on AB into its original 6 on B. Our square root is 67 on BC (Fig. 199).

EXAMPLE 3: Find the square root of 119 716.

The method of finding the three-digit square root of a number is founded upon the algebraic method of finding the square of an ordinary trinomial (or three-membered) expression, $(a + b + c)$. A number is analyzed as in algebra, into $(a^2 + 2ab + b^2 + 2ac + 2bc + c^2)$ and the square root is found.

STEP 1: Set 119 716 on CDEFGH, and mark off the digits in sets of twos. In the first pair 11, the highest square root is 3 (Fig. 200).



FIG. 200



FIG. 201

STEP 2: Set, on A, 3, the first trial figure in the root. Square the 3, and subtract the product 9 from the 11 on CD. This leaves 29 716 on DEFGH (Fig. 201).

NOTE: 3 corresponds to the a of the algebraic expression $(a^2 + 2ab + b^2 + 2ac + 2bc + c^2)$.

STEP 3: Double the 3 on A making 6 on A (Fig. 202).



FIG. 202



FIG. 203

STEP 4: Figuring how many times the 6 on A goes into the 29 on DE, we find that 4 is correct. So we set 4 on B as the second trial figure in the root (Fig. 203).

NOTE: The 4 on B corresponds to the b of the algebraic expression.

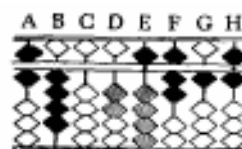


FIG. 204

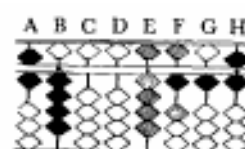


FIG. 205

STEP 5: Multiply the 6 on A by the 4 on 13, and subtract the product 24 from the 29 on DE. This leaves 5 716 on EFGH (Fig. 204).

NOTE: (4×6) or 24 corresponds to the $2ab$ of the algebraic expression.

STEP 6: Now square the 4 on 13, and subtract the product 16 from the 57 on EF. This leaves 4 116 on EFGH (Fig. 205).

NOTE: The square of 4 corresponds to the b^2 of the algebraic expression.

STEP 7: Double the 4 on B, making 8 (Fig. 206).



Fig. 206



Fig. 207

NOTE: When the second trial divisor figure is 5 or larger, move the first divisor figure to the first rod to the left, and set double the second trial quotient figure on AB.

STEP 8: Figuring how many times the 6 on A goes into the 41 on EF, we find that 6 is the proper figure. Set 6 on C as the third trial figure in the root (Fig. 207).

NOTE: The 6 on C corresponds to c of the expression.

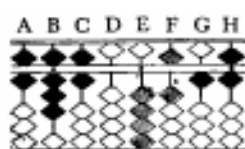


Fig. 208



Fig. 209

STEP 9: Multiply the 6 on A by the 6 on C, and subtract the product 36 from the 41 on EF. This leaves 516 on FGH (Fig. 208).

NOTE: 6×6 corresponds to the $2ac$ of the algebraic expression.

STEP 10: Multiply the 8 on B by the same 6 on C, and subtract the product 48 from the 51 on FG. This leaves you 36 on GH (Fig. 209).

NOTE: $(8 \text{ on B} \times 6 \text{ on C})$ corresponds to the $2bc$ of the algebraic expression.



Fig. 210



Fig. 211

STEP 11: Square the 6 on C, and subtract the product 36 from the 36 on GH. This clears the board of the remainder of the number, which corresponds to the c^2 of the algebraic expression (Fig. 210).

STEP 12: Now halve the 6 on A and the 8 on B into the original 3 and 4 respectively. Our square root is 346 on ABC (Fig. 211).

NOTE: The 68 which was on AB corresponded to the $2ab$ of the algebraic expression.

VIII - MORE EXERCISES

The exercises for the first, second, and third grades were actually used in recent national abacus license examinations and are here reprinted through the courtesy of the Japan Chamber of Commerce and Industry.

1. Eighth-Grade Operator

Group A

(1 set per minute, or entire group with 70% accuracy in 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 62 | 35 | 71 | 20 | 86 | 14 | 90 | 57 | 82 | 49 |
| 15 | 70 | 45 | 38 | 92 | 20 | 26 | 69 | 54 | 80 |
| 40 | 29 | 36 | 92 | 50 | 97 | -83 | 32 | -73 | 21 |
| 71 | 86 | 50 | 13 | 29 | 65 | 59 | 70 | -38 | 68 |
| 84 | 43 | -92 | 56 | -74 | 28 | 42 | 31 | 10 | 32 |
| 26 | 58 | -27 | 90 | 51 | 73 | -15 | 46 | 29 | 14 |
| 97 | 10 | 83 | 84 | 63 | 41 | -60 | 18 | 76 | 95 |
| 80 | 64 | 19 | 47 | -30 | 56 | 47 | 94 | -61 | 70 |
| 39 | 17 | -60 | 75 | -48 | 30 | 31 | 25 | 40 | 53 |
| <u>53</u> | <u>92</u> | <u>84</u> | <u>61</u> | <u>17</u> | <u>89</u> | <u>78</u> | <u>80</u> | <u>95</u> | <u>76</u> |
| 567 | 504 | 209 | 576 | 236 | 513 | 215 | 522 | 214 | 558 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) $45 \times 8 = \mathbf{360}$ | (1) $72 \div 2 = \mathbf{36}$ |
| (2) $73 \times 4 = \mathbf{292}$ | (2) $135 \div 5 = \mathbf{27}$ |
| (3) $52 \times 2 = \mathbf{104}$ | (3) $246 \div 3 = \mathbf{82}$ |
| (4) $38 \times 7 = \mathbf{266}$ | (4) $126 \div 9 = \mathbf{14}$ |
| (5) $94 \times 3 = \mathbf{282}$ | (5) $413 \div 7 = \mathbf{59}$ |
| (6) $15 \times 6 = \mathbf{90}$ | (6) $300 \div 4 = \mathbf{75}$ |
| (7) $86 \times 9 = \mathbf{774}$ | (7) $744 \div 8 = \mathbf{93}$ |
| (8) $67 \times 5 = \mathbf{335}$ | (8) $288 \div 6 = \mathbf{48}$ |
| (9) $21 \times 8 = \mathbf{168}$ | (9) $183 \div 3 = \mathbf{61}$ |
| (10) $79 \times 6 = \mathbf{474}$ | (10) $435 \div 5 = \mathbf{87}$ |
| (11) $134 \times 3 = \mathbf{402}$ | (11) $8\,829 \div 9 = \mathbf{981}$ |
| (12) $509 \times 2 = \mathbf{1\,018}$ | (12) $420 \div 4 = \mathbf{105}$ |
| (13) $876 \times 8 = \mathbf{7\,008}$ | (13) $2\,634 \div 6 = \mathbf{439}$ |
| (14) $623 \times 4 = \mathbf{2\,492}$ | (14) $752 \div 2 = \mathbf{376}$ |
| (15) $258 \times 9 = \mathbf{2\,322}$ | (15) $4\,336 \div 8 = \mathbf{542}$ |
| (16) $480 \times 6 = \mathbf{2\,880}$ | (16) $4\,050 \div 5 = \mathbf{810}$ |
| (17) $942 \times 5 = \mathbf{4\,710}$ | (17) $1\,906 \div 2 = \mathbf{953}$ |
| (18) $715 \times 7 = \mathbf{5\,005}$ | (18) $2\,124 \div 3 = \mathbf{708}$ |
| (19) $301 \times 4 = \mathbf{1\,204}$ | (19) $1\,068 \div 4 = \mathbf{267}$ |
| (20) $697 \times 3 = \mathbf{2\,091}$ | (20) $4\,368 \div 7 = \mathbf{624}$ |

2. Seventh-Grade Operator

Assortment 1**Group A**

(1 set per minute, or entire group with 70% accuracy in 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 45 | 63 | 87 | 31 | 56 | 94 | 26 | 79 | 70 | 12 |
| 78 | 95 | 60 | 19 | 27 | 70 | 98 | 16 | 45 | 60 |
| 30 | 59 | 31 | 40 | 68 | 19 | -20 | 84 | 61 | 85 |
| 59 | -80 | 48 | 95 | 90 | 51 | -57 | 40 | 13 | -32 |
| 61 | -43 | 24 | -87 | 31 | 35 | 41 | 62 | 90 | -78 |
| 96 | 78 | 12 | -28 | 47 | 60 | 93 | 97 | 24 | 45 |
| 20 | 51 | 39 | 30 | 82 | 89 | -65 | 23 | 57 | 90 |
| 83 | -27 | 50 | 72 | 43 | 46 | 10 | 95 | 38 | 76 |
| 52 | 82 | 67 | 16 | 10 | 18 | 74 | 60 | 47 | -53 |
| 17 | 90 | 29 | -57 | 24 | 42 | 58 | 35 | 96 | 10 |
| 93 | -64 | 15 | 20 | 93 | 37 | -89 | 21 | 80 | 34 |
| 40 | -37 | 98 | 65 | 51 | 68 | -13 | 74 | 12 | -89 |
| 62 | 10 | 54 | -43 | 80 | 27 | 30 | 83 | 35 | -71 |
| 81 | 46 | 70 | -64 | 69 | 50 | 76 | 18 | 26 | 94 |
| 74 | 12 | 36 | 98 | 75 | 23 | 42 | 50 | 89 | 67 |
| 891 | 335 | 720 | 207 | 846 | 729 | 304 | 837 | 783 | 250 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 7 832 x 6 = 46 992 | (1) 402 / 2 = 201 |
| (2) 4 096 x 7 = 28 672 | (2) 6 776 / 7 = 968 |
| (3) 6 348 x 8 = 50 784 | (3) 3 282 / 6 = 547 |
| (4) 1 573 x 4 = 6 292 | (4) 2 430 / 3 = 810 |
| (5) 8 105 x 9 = 72 945 | (5) 5 536 / 8 = 692 |
| (6) 2 619 x 3 = 7 857 | (6) 1 380 / 4 = 345 |
| (7) 5 980 x 6 = 35 880 | (7) 3 180 / 6 = 530 |
| (8) 3 467 x 5 = 17 335 | (8) 7 101 / 9 = 789 |
| (9) 9 024 x 9 = 81 216 | (9) 3 408 / 8 = 426 |
| (10) 2 751 x 2 = 5 502 | (10) 865 / 5 = 173 |
| (11) 3 647 x 6 = 21 882 | (11) 3 896 / 4 = 974 |
| (12) 9 502 x 4 = 38 008 | (12) 966 / 7 = 138 |
| (13) 6 138 x 3 = 18 414 | (13) 1 184 / 2 = 592 |
| (14) 8 420 x 9 = 75 780 | (14) 4 830 / 6 = 805 |
| (15) 4 975 x 2 = 9 950 | (15) 807 / 3 = 269 |
| (16) 1 769 x 8 = 14 152 | (16) 2 250 / 5 = 450 |
| (17) 7 081 x 4 = 28 324 | (17) 2 468 / 4 = 617 |
| (18) 2 153 x 2 = 4 306 | (18) 682 / 2 = 341 |
| (19) 5 304 x 5 = 26 520 | (19) 5 648 / 8 = 706 |
| (20) 8 296 x 7 = 58 072 | (20) 7 407 / 9 = 823 |

Assortment 2
Group A
 (70% accuracy, 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 61 | 47 | 21 | 86 | 34 | 53 | 95 | 78 | 16 | 41 |
| 72 | 16 | 90 | 47 | 68 | 61 | 59 | 40 | 83 | 68 |
| 30 | 28 | 64 | 91 | -50 | 42 | 10 | 61 | -24 | 90 |
| 84 | 60 | -39 | 20 | -37 | 90 | -46 | 35 | -70 | 42 |
| 12 | 59 | -70 | 35 | 64 | 57 | -89 | 84 | 25 | 37 |
| 89 | 32 | 94 | 58 | 15 | 32 | 71 | 27 | 93 | 56 |
| 93 | 75 | 18 | 24 | 40 | 18 | 20 | 95 | 41 | 27 |
| 10 | 39 | 56 | 10 | -83 | 25 | 64 | 50 | -67 | 10 |
| 56 | 40 | -82 | 73 | 29 | 70 | -23 | 32 | 80 | 96 |
| 34 | 18 | 25 | 69 | 71 | 64 | 87 | 19 | 35 | 53 |
| 59 | 83 | 43 | 26 | 97 | 86 | 30 | 94 | 78 | 39 |
| 67 | 71 | 80 | 49 | -21 | 39 | 15 | 87 | -46 | 28 |
| 75 | 50 | -67 | 18 | -60 | 80 | -78 | 23 | -59 | 70 |
| 20 | 94 | -71 | 30 | 58 | 74 | -62 | 16 | 90 | 14 |
| 48 | 26 | 53 | 75 | 92 | 91 | 34 | 60 | 12 | 85 |
| 810 | 738 | 215 | 711 | 317 | 882 | 187 | 801 | 287 | 756 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 2 604 x 3 = 7 812 | (1) 4 077 / 9 = 453 |
| (2) 5 219 x 7 = 36 533 | (2) 2 667 / 7 = 381 |
| (3) 8 045 x 5 = 4 0 225 | (3) 1 612 / 2 = 806 |
| (4) 1 573 x 6 = 9438 | (4) 2 810 / 5 = 562 |
| (5) 3 987 x 2 = 7 974 | (5) 882 / 3 = 294 |
| (6) 6 820 x 4 = 27 280 | (6) 5 096 / 7 = 728 |
| (7) 9 751 x 9 = 87 759 | (7) 680 / 4 = 170 |
| (8) 7 406 x 8 = 59 248 | (8) 957 / 3 = 319 |
| (9) 4 132 x 3 = 12 396 | (9) 7 240 / 8 = 905 |
| (10) 9 368 x 7 = 65 576 | (10) 3 882 / 6 = 647 |
| (11) 6 972 x 6 = 41 832 | (11) 5 481 / 7 = 783 |
| (12) 3 285 x 3 = 9 855 | (12) 2 100 / 6 = 350 |
| (13) 5 049 x 2 = 10 098 | (13) 6 246 / 9 = 694 |
| (14) 1 856 x 4 = 7 424 | (14) 1 195 / 5 = 239 |
| (15) 8 391 x 8 = 67 128 | (15) 3 488 / 4 = 872 |
| (16) 4 703 x 9 = 42 327 | (16) 435 / 3 = 145 |
| (17) 2 168 x 4 = 8 672 | (17) 1 054 / 2 = 527 |
| (18) 9 027 x 3 = 27 081 | (18) 900 / 5 = 180 |
| (19) 7 430 x 7 = 52 010 | (19) 7 744 / 8 = 968 |
| (20) 6 514 x 5 = 32 570 | (20) 832 / 2 = 416 |

3. Sixth-Grade Operator

Assortment 1**Group A**

(1 set per minute, or entire group with 70% accuracy in 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---------------|---------------|--------------|---------------|--------------|---------------|--------------|---------------|--------------|---------------|
| 35 | 64 | 9 017 | 781 | 29 | 83 | 459 | 6 082 | 54 | 207 |
| 409 | 5 013 | 928 | 350 | 67 | 962 | 104 | 94 | 592 | 18 |
| 852 | 937 | 241 | 19 | 8 024 | 38 | -76 | 350 | 9 364 | 173 |
| 14 | 790 | -58 | 28 | 71 | 5 093 | -395 | 71 | -108 | 14 |
| 3 267 | 28 | -703 | 974 | -438 | 126 | 7 201 | 259 | -46 | 3 059 |
| 198 | 576 | 342 | 6 831 | -92 | 40 | 68 | 37 | 15 | 86 |
| 41 | 102 | 96 | 402 | 684 | 57 | 890 | 25 | 687 | 542 |
| 620 | 58 | 430 | 56 | 305 | 814 | 24 | 68 | 30 | 69 |
| 93 | 4 681 | 73 | 98 | 586 | 71 | 16 | 1 704 | 879 | 250 |
| 8 065 | 25 | 16 | 34 | 90 | 2 708 | -73 | 219 | 13 | 67 |
| 47 | 872 | 65 | 2 097 | -1 632 | 345 | -248 | 73 | 205 | 938 |
| 971 | 34 | -809 | 516 | -741 | 96 | -3 091 | 854 | -36 | 8 401 |
| 53 | 609 | -7 521 | 45 | -53 | 607 | 67 | 380 | -4 071 | 97 |
| 702 | 91 | -86 | 27 | 910 | 54 | 582 | 69 | -728 | 45 |
| 86 | 43 | 54 | 603 | 57 | 219 | 35 | 146 | 92 | 623 |
| 15 453 | 13 923 | 2 085 | 12 861 | 7 867 | 11 313 | 5 563 | 10 431 | 6 942 | 14 589 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 46 x 754 = 34 684 | (1) 1 288 / 56 = 23 |
| (2) 703 x 13 = 9 139 | (2) 1 568 / 98 = 16 |
| (3) 28 x 972 = 27 216 | (3) 1 554 / 21 = 74 |
| (4) 159 x 69 = 10 971 | (4) 1 505 / 35 = 43 |
| (5) 82 x 407 = 33 374 | (5) 4 030 / 62 = 65 |
| (6) 514 x 18 = 9 252 | (6) 4 324 / 47 = 92 |
| (7) 37 x 230 = 8 510 | (7) 2 573 / 83 = 31 |
| (8) 950 x 51 = 48 450 | (8) 1 102 / 19 = 58 |
| (9) 64 x 805 = 51 520 | (9) 1 406 / 74 = 19 |
| (10) 821 x 36 = 29 556 | (10) 8 004 / 92 = 87 |
| (11) 798 x 84 = 67 032 | (11) 516 / 43 = 12 |
| (12) 45 x 289 = 13 005 | (12) 918 / 27 = 34 |
| (13) 172 x 52 = 8 944 | (13) 5 472 / 72 = 76 |
| (14) 630 x 65 = 40 950 | (14) 1 302 / 14 = 93 |
| (15) 84 x 906 = 76 104 | (15) 3 936 / 89 = 41 |
| (16) 209 x 47 = 9 823 | (16) 4 930 / 58 = 85 |
| (17) 51 x 743 = 37 893 | (17) 5 963 / 89 = 67 |
| (18) 306 x 19 = 5 814 | (18) 899 / 31 = 29 |
| (19) 97 x 308 = 29 876 | (19) 3 640 / 65 = 56 |
| (20) 63 x 261 = 16 443 | (20) 2 592 / 54 = 48 |

Assortment 2
Group A
 (70% accuracy, 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 586 | 41 | 873 | 268 | 184 | 725 | 92 | 308 | 657 | 468 |
| 93 | 705 | 345 | 981 | 510 | 86 | 743 | 96 | 14 | 274 |
| 610 | 539 | 62 | 705 | 61 | 97 | 10 | 473 | 280 | 59 |
| 139 | 285 | 204 | -53 | 647 | 830 | 451 | 89 | 906 | 412 |
| 304 | 129 | 583 | -178 | 79 | 15 | -809 | 21 | 45 | 160 |
| 78 | -450 | 691 | 94 | 803 | 957 | -65 | 140 | -12 | 358 |
| 42 | -683 | 78 | 360 | 52 | 401 | 586 | 897 | -138 | 907 |
| 175 | -42 | 306 | 412 | 923 | 648 | 31 | 205 | -879 | 13 |
| 762 | 801 | 452 | -29 | 490 | 312 | 708 | 34 | 923 | 76 |
| 407 | 93 | 18 | -506 | 735 | 60 | -342 | 516 | 498 | 802 |
| 863 | 217 | 790 | -635 | 96 | 502 | -74 | 925 | 60 | 35 |
| 51 | -964 | 29 | 840 | 318 | 243 | -623 | 650 | 327 | 791 |
| 920 | -78 | 145 | 27 | 574 | 89 | 290 | 874 | -705 | 623 |
| 98 | 360 | 67 | 491 | 26 | 173 | 968 | 731 | -46 | 580 |
| 245 | 76 | 910 | 73 | 208 | 694 | 157 | 62 | 531 | 49 |
| 5 373 | 1 029 | 5 553 | 2 850 | 5 706 | 5 832 | 2 123 | 6 021 | 2 461 | 5 607 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 403 x 73 = 29 419 | (1) 888 / 74 = 12 |
| (2) 26 x 208 = 5 408 | (2) 782 / 23 = 34 |
| (3) 589 x 74 = 43 586 | (3) 3 953 / 59 = 67 |
| (4) 81 x 861 = 69 741 | (4) 476 / 17 = 28 |
| (5) 170 x 49 = 8 330 | (5) 7 990 / 85 = 94 |
| (6) 68 x 617 = 41 956 | (6) 2 576 / 46 = 56 |
| (7) 945 x 14 = 13 230 | (7) 6 643 / 91 = 73 |
| (8) 307 x 32 = 9 824 | (8) 1 558 / 38 = 41 |
| (9) 69 x 506 = 34 914 | (9) 5 518 / 62 = 89 |
| (10) 72 x 985 = 70 920 | (10) 945 / 27 = 35 |
| (11) 34 x 194 = 6 596 | (11) 4 335 / 85 = 51 |
| (12) 512 x 83 = 42 496 | (12) 3 570 / 42 = 85 |
| (13) 49 x 508 = 24 892 | (13) 882 / 14 = 63 |
| (14) 301 x 27 = 8 127 | (14) 2 652 / 68 = 39 |
| (15) 758 x 91 = 68 978 | (15) 7 068 / 93 = 76 |
| (16) 93 x 609 = 56 637 | (16) 5 586 / 57 = 98 |
| (17) 206 x 35 = 7 210 | (17) 783 / 29 = 27 |
| (18) 87 x 402 = 34 974 | (18) 994 / 71 = 14 |
| (19) 65 x 253 = 16 445 | (19) 1 512 / 36 = 42 |
| (20) 124 x 76 = 9 424 | (20) 684 / 19 = 36 |

4. Fifth-Grade Operator

Assortment 1**Group A**

(1 set per minute, or entire group with 70% accuracy in 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 576 | 8 294 | 603 | 3 918 | 702 | 423 | 186 | 307 | 9 684 | 237 |
| 2 089 | 435 | 481 | 706 | 9 318 | 357 | 534 | 1 273 | 509 | 7 815 |
| 961 | 3 108 | 894 | 327 | 529 | 4 108 | 129 | 918 | 763 | 592 |
| 846 | 912 | 5 021 | 986 | 437 | 689 | 9 705 | 134 | -2 157 | 346 |
| 417 | 593 | 372 | 8 031 | -654 | 1 594 | 671 | 8 059 | -406 | 954 |
| 6 752 | 237 | 9 140 | 795 | -231 | 380 | 830 | 462 | -813 | 5 081 |
| 408 | 1 689 | 975 | 641 | -3 048 | 516 | -3 284 | 391 | 6 045 | 478 |
| 7 813 | 860 | 256 | 4 823 | 197 | 7 634 | -157 | 4 907 | 526 | 643 |
| 359 | 7 046 | -417 | 590 | 6 851 | 290 | 5 021 | 746 | 270 | 9 075 |
| 9 210 | 572 | -8 306 | 254 | 960 | 825 | 789 | 598 | 3 481 | 380 |
| 783 | 904 | 421 | 1 085 | -893 | 2 067 | -493 | 7 610 | 934 | 162 |
| 5 034 | 745 | 6 359 | 143 | -2 786 | 942 | -7 260 | 852 | -317 | 6 239 |
| 192 | 8 061 | -738 | 562 | 402 | 8 759 | -948 | 625 | -8 059 | 170 |
| 624 | 753 | -2 069 | 476 | 1 975 | 301 | 6 402 | 583 | 298 | 601 |
| 305 | 621 | -847 | 7 209 | 064 | 176 | 365 | 2 064 | 172 | 4 928 |
| 36 369 | 34 830 | 12 045 | 30 546 | 18 823 | 29 061 | 12 490 | 29 529 | 10 930 | 37 701 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 601 x 138 = 82 938 | (1) 9 180 / 108 = 85 |
| (2) 247 x 375 = 92 625 | (2) 8 232 / 42 = 196 |
| (3) 5 630 x 87 = 489 810 | (3) 50 934 / 653 = 78 |
| (4) 415 x 746 = 309 590 | (4) 49 282 / 82 = 601 |
| (5) 879 x 213 = 187 227 | (5) 15 141 / 309 = 49 |
| (6) 358 x 601 = 215 158 | (6) 28 917 / 51 = 567 |
| (7) 926 x 950 = 879 700 | (7) 66 822 / 74 = 903 |
| (8) 104 x 594 = 61 776 | (8) 26 236 / 937 = 28 |
| (9) 72 x 8 062 = 580 464 | (9) 7 550 / 25 = 302 |
| (10) 983 x 429 = 421 707 | (10) 30 784 / 416 = 74 |
| (11) 459 x 946 = 434 214 | (11) 14 196 / 39 = 364 |
| (12) 703 x 217 = 152 551 | (12) 8 333 / 641 = 13 |
| (13) 198 x 683 = 135 234 | (13) 5 508 / 204 = 27 |
| (14) 942 x 825 = 777 150 | (14) 70 590 / 78 = 905 |
| (15) 61 x 3 901 = 237 961 | (15) 45 568 / 512 = 89 |
| (16) 205 x 472 = 96 760 | (16) 51 085 / 85 = 601 |
| (17) 862 x 730 = 629 260 | (17) 9 280 / 16 = 580 |
| (18) 3 807 x 69 = 262 683 | (18) 65 016 / 903 = 72 |
| (19) 174 x 504 = 87 696 | (19) 22 372 / 47 = 476 |
| (20) 536 x 158 = 84 688 | (20) 31 850 / 325 = 98 |

Assortment 2
Group A
 (70% accuracy, 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|--------------|---------------|
| 1 063 | 274 | 319 | 4 657 | 530 | 6 841 | 805 | 956 | 728 | 4 059 |
| 542 | 8 029 | 753 | 208 | 3 714 | 265 | 154 | 7 209 | 156 | 236 |
| 951 | 367 | 8 406 | 431 | 862 | 1 079 | 7 061 | 567 | 4 809 | 941 |
| 674 | 3 510 | 837 | -1 629 | 495 | 682 | 142 | 498 | 237 | 1 308 |
| 8 502 | 781 | 264 | -370 | 7 023 | 236 | 869 | 3 015 | 416 | 179 |
| 349 | -608 | 1 472 | 593 | 971 | 8 903 | 237 | 947 | 6 038 | 897 |
| 175 | -2 153 | 950 | 8 042 | 839 | 579 | 9 740 | 572 | 345 | 2 610 |
| 928 | -467 | 3 029 | 416 | 9 506 | 713 | -392 | 1 086 | -293 | 798 |
| 6 730 | 892 | 145 | 359 | 648 | 2 054 | -128 | 621 | -9 507 | 562 |
| 481 | 9 046 | 581 | 6 283 | 187 | 928 | -5 093 | 835 | 184 | 8 205 |
| 7 209 | 431 | 936 | 107 | 2 368 | 435 | 817 | 2 690 | 451 | 386 |
| 856 | 584 | 7 620 | -925 | 240 | 197 | 6 284 | 723 | 7 069 | 463 |
| 293 | -5 670 | 814 | -768 | 571 | 4 780 | 356 | 134 | -273 | 5 017 |
| 4 018 | -239 | 6 095 | -5 890 | 124 | 305 | -3 479 | 408 | -690 | 423 |
| 367 | 915 | 728 | 174 | 6 059 | 146 | -605 | 8 341 | -5 182 | 574 |
| 33 138 | 15 692 | 32 949 | 11 688 | 34 137 | 28 143 | 16 768 | 28 602 | 4 488 | 26 658 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 723 x 815 = 589 245 | (1) 51 874 / 74 = 701 |
| (2) 38 x 4 079 = 155 002 | (2) 45 301 / 509 = 89 |
| (3) 901 x 687 = 618 987 | (3) 14 118 / 26 = 543 |
| (4) 467 x 942 = 439 914 | (4) 42 151 / 691 = 61 |
| (5) 235 x 351 = 82 485 | (5) 7 585 / 37 = 205 |
| (6) 6 702 x 14 = 93 828 | (6) 8 676 / 482 = 18 |
| (7) 149 x 530 = 78 970 | (7) 29 056 / 908 = 32 |
| (8) 584 x 603 = 352 152 | (8) 3 302 / 13 = 254 |
| (9) 816 x 296 = 241 536 | (9) 37 060 / 85 = 436 |
| (10) 950 x 728 = 691 600 | (10) 68 482 / 706 = 97 |
| (11) 257 x 498 = 127 986 | (11) 11 856 / 624 = 19 |
| (12) 183 x 532 = 97 356 | (12) 40 356 / 57 = 708 |
| (13) 49 x 6 705 = 328 545 | (13) 52 136 / 931 = 56 |
| (14) 802 x 103 = 82 606 | (14) 33 615 / 405 = 83 |
| (15) 716 x 789 = 564 924 | (15) 59 512 / 86 = 692 |
| (16) 364 x 217 = 78 988 | (16) 2 856 / 14 = 204 |
| (17) 9 408 x 31 = 261 648 | (17) 26 274 / 302 = 87 |
| (18) 521 x 856 = 445 976 | (18) 8 428 / 28 = 301 |
| (19) 390 x 264 = 102 960 | (19) 77 104 / 79 = 976 |
| (20) 675 x 940 = 634 500 | (20) 6 885 / 153 = 45 |

Assortment 3
Group A
 (70% accuracy, 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---------------|--------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 741 | 2 908 | 456 | 1 063 | 601 | 397 | 8 509 | 516 | 907 | 371 |
| 6 102 | 847 | 321 | 759 | 2 738 | 206 | 971 | 4 089 | 132 | 653 |
| 819 | 758 | 9 604 | 3 941 | 573 | 421 | 128 | 934 | 8 021 | 492 |
| 4 057 | 3 416 | 259 | 528 | 697 | 1 035 | 613 | 5 129 | 346 | 6 104 |
| 938 | 673 | 512 | 709 | 528 | 647 | 2 760 | 761 | 238 | 589 |
| 692 | -105 | 7 364 | 247 | 7 089 | 531 | 597 | 830 | -176 | 4 068 |
| 8 720 | -369 | 947 | 5 482 | 316 | 785 | 381 | 6 208 | -2 759 | 815 |
| 143 | -5 280 | 396 | 620 | -8 172 | 4 190 | 435 | 792 | -681 | 7 920 |
| 9 567 | 721 | 4 018 | 135 | -450 | 928 | -7 043 | 831 | 3 460 | 9 173 |
| 285 | 954 | 130 | 8 062 | 849 | 6 273 | -816 | 627 | 594 | 258 |
| 306 | -1 470 | 685 | 594 | 9 403 | 806 | 564 | 7 283 | -815 | 304 |
| 928 | -536 | 8 207 | 317 | -961 | 5 469 | 9 308 | 345 | -5 092 | 167 |
| 5 034 | 182 | 793 | 486 | -245 | 714 | -726 | 590 | 4 978 | 5 049 |
| 476 | 6 093 | 278 | 901 | -3 016 | 852 | -4 952 | 1 054 | 703 | 726 |
| 315 | 942 | 1 850 | 6 873 | 254 | 3 098 | -240 | 476 | 564 | 832 |
| 39 123 | 9 734 | 35 820 | 30 717 | 10 204 | 26 352 | 10 489 | 30 465 | 10 420 | 37 521 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 415 x 948 = 393 420 | (1) 55 647 / 81 = 687 |
| (2) 952 x 852 = 811 104 | (2) 56 870 / 605 = 94 |
| (3) 381 x 176 = 67 056 | (3) 7 599 / 149 = 51 |
| (4) 64 x 6 307 = 403 648 | (4) 36 868 / 52 = 709 |
| (5) 890 x 219 = 194 910 | (5) 2 808 / 216 = 13 |
| (6) 167 x 563 = 94 021 | (6) 67 200 / 75 = 896 |
| (7) 523 x 490 = 256 270 | (7) 8 289 / 307 = 27 |
| (8) 7 048 x 75 = 528 600 | (8) 41 495 / 43 = 965 |
| (9) 279 x 184 = 79 236 | (9) 28 952 / 94 = 308 |
| (10) 306 x 301 = 92 106 | (10) 5 376 / 128 = 42 |
| (11) 675 x 369 = 249 075 | (11) 18 666 / 306 = 61 |
| (12) 1 948 x 54 = 105 192 | (12) 7 605 / 15 = 507 |
| (13) 403 x 890 = 358 670 | (13) 30 848 / 64 = 482 |
| (14) 21 x 4 701 = 98 721 | (14) 7 884 / 219 = 36 |
| (15) 906 x 548 = 496 488 | (15) 72 071 / 743 = 97 |
| (16) 587 x 132 = 77 484 | (16) 82 156 / 92 = 893 |
| (17) 862 x 607 = 523 234 | (17) 34 983 / 507 = 69 |
| (18) 359 x 286 = 102 674 | (18) 57 024 / 81 = 704 |
| (19) 730 x 913 = 666 490 | (19) 8 136 / 452 = 18 |
| (20) 124 x 725 = 89 900 | (20) 7 790 / 38 = 205 |

5. Fourth-Grade Operator

Assortment 1**Group A**

(1 set per minute, or entire group with 70% accuracy in 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 629 | 7 084 | 13 472 | 8 563 | 698 | 40 615 | 293 | 5 471 | 90 386 | 3 257 |
| 3 167 | 653 | 4 081 | 795 | 5 907 | 123 | 6 748 | 186 | 419 | 60 382 |
| 82 036 | 3 471 | 958 | 6 041 | 654 | 7 850 | 913 | 42 305 | 3 504 | 935 |
| 578 | 49 680 | 65 412 | 359 | 3 761 | 584 | 1 095 | -627 | 927 | 78 106 |
| 9 247 | -8 163 | 7 504 | 7 261 | 80 342 | 1 609 | 78 932 | -6 741 | 4 036 | -679 |
| 15 402 | -794 | 963 | 50 178 | 531 | 90 476 | 689 | 20 567 | 71 285 | -1 538 |
| 139 | 162 | 80 721 | 98 432 | 4 270 | 6 398 | 17 504 | 492 | 598 | -82 041 |
| 8 576 | 27 905 | 3 852 | -7 510 | 65 492 | 213 | 9 827 | 91 058 | 32 960 | 467 |
| 30 951 | 8 741 | 139 | -42 087 | 789 | 27 064 | 451 | -9 683 | 4 187 | 50 219 |
| 745 | 91 036 | 26 598 | -623 | 10 928 | 947 | 82 076 | -539 | 651 | -4 983 |
| 2 810 | 56 327 | 9 605 | 39 401 | 5 163 | 54 801 | 1 463 | -31 870 | 14 706 | -759 |
| 71 684 | 209 | 437 | 896 | 73 095 | 1 326 | 60 852 | 5 324 | 3 542 | 27 130 |
| 493 | -852 | 70 216 | -8 304 | 817 | 37 892 | 3 174 | 908 | 80 269 | 6 814 |
| 4 508 | -30 915 | 8 647 | -712 | 1204 | 935 | 52 630 | 3 216 | 723 | 4 096 |
| 60 392 | -2 548 | 390 | 26 945 | 42 386 | 8 752 | 405 | 80 947 | 5 871 | 425 |
| 291357 | 201996 | 292995 | 179653 | 296037 | 279585 | 317052 | 201014 | 314064 | 141831 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 4 321 x 607 = 2 622 847 | (1) 130 977 / 567 = 231 |
| (2) 9 108 x 938 = 8 543 304 | (2) 440 700 / 975 = 452 |
| (3) 7 685 x 452 = 3 473 620 | (3) 336 226 / 341 = 986 |
| (4) 2 934 x 120 = 352 080 | (4) 81 624 / 456 = 179 |
| (5) 5 017 x 543 = 2 724 231 | (5) 622 800 / 720 = 865 |
| (6) 8 652 x 789 = 6 826 428 | (6) 67 402 / 134 = 503 |
| (7) 3 476 x 204 = 709 104 | (7) 502 854 / 802 = 627 |
| (8) 1 809 x 671 = 1 213 839 | (8) 90 746 / 289 = 314 |
| (9) 6 523 x 316 = 2 061 268 | (9) 294 240 / 613 = 480 |
| (10) 7 940 x 895 = 7 106 300 | (10) 724 584 / 908 = 798 |
| (11) 2 463 x 271 = 667 473 | (11) 821 215 / 851 = 965 |
| (12) 6 289 x 613 = 3 855 157 | (12) 99 144 / 408 = 243 |
| (13) 3 045 x 492 = 1 498 140 | (13) 493 425 / 675 = 731 |
| (14) 8 251 x 507 = 4 183 257 | (14) 77 089 / 127 = 607 |
| (15) 7 836 x 324 = 2 538 864 | (15) 31 968 / 296 = 108 |
| (16) 5 970 x 749 = 4 471 530 | (16) 248 992 / 502 = 496 |
| (17) 9 012 x 830 = 7 479 960 | (17) 84 940 / 310 = 274 |
| (18) 4 758 x 186 = 884 988 | (18) 567 207 / 963 = 589 |
| (19) 8 617 x 905 = 7 798 385 | (19) 136 090 / 439 = 310 |
| (20) 1 394 x 658 = 917 252 | (20) 667 968 / 784 = 852 |

Assortment 2
Group A
 (70% accuracy, 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---------------|--------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 4 068 | 371 | 93 416 | 8 506 | 74 289 | 1 903 | 6 081 | 25 937 | 813 | 56 209 |
| 975 | 30 142 | 6 904 | 75 243 | 315 | 246 | 943 | 3 216 | 51 402 | 392 |
| 43 106 | 5 489 | 21 586 | 759 | 1 082 | 36 152 | 3 257 | 64 108 | 2 769 | 60 528 |
| 1 279 | 901 | 7 843 | 60 138 | 471 | 580 | 87 109 | 967 | 80 624 | 137 |
| 50 813 | 6 492 | 397 | -7 865 | 16 208 | 29 365 | 548 | -7 091 | 6 157 | 7 630 |
| 497 | 72 054 | 6 052 | -971 | 4 539 | 5 078 | 20 816 | -459 | 47 098 | -293 |
| 7160 | -523 | 87 931 | 1 084 | 293 | 8 914 | 9 457 | 620 | 834 | -8 145 |
| 9 582 | -64 370 | 728 | 32 416 | 90 614 | 57 681 | 872 | 30 415 | 3 571 | -94 028 |
| 251 | -7 865 | 240 | -9 302 | 5 367 | 409 | 5 139 | 732 | 18 290 | 3 684 |
| 86 304 | 218 | 5 072 | -615 | 86 970 | 1 853 | 726 | -54 803 | 419 | 10 957 |
| 629 | 45 806 | 53 419 | -48 127 | 9 156 | 60 729 | 43 065 | -1 278 | 2 365 | 479 |
| 20 374 | -9 762 | 805 | 793 | 304 | 498 | 2 493 | -564 | 942 | -4 851 |
| 8 735 | -689 | 70 693 | 6 540 | 762 | 7 031 | 76 340 | 8 925 | 6 087 | -713 |
| 281 | 10 935 | 2 581 | 924 | 8 045 | 247 | 629 | 90 183 | 356 | 5 046 |
| 95 346 | 8 317 | 164 | 20 398 | 57 823 | 46 372 | 15 084 | 7 846 | 79 503 | 81 762 |
| 329400 | 97516 | 357831 | 139921 | 356238 | 257058 | 272259 | 168754 | 301230 | 118794 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 9 634 x 426 = 4 104 084 | (1) 346 800 / 850 = 408 |
| (2) 2 761 x 593 = 1 637 273 | (2) 64 390 / 274 = 235 |
| (3) 4 508 x 180 = 811 440 | (3) 419 152 / 536 = 782 |
| (4) 7 985 x 867 = 6 922 995 | (4) 89 027 / 701 = 127 |
| (5) 3 120 x 305 = 951 600 | (5) 80 850 / 165 = 490 |
| (6) 6 842 x 751 = 5 138 342 | (6) 250 158 / 482 = 519 |
| (7) 1 076 x 918 = 987 768 | (7) 130 472 / 347 = 376 |
| (8) 8 453 x 472 = 3 989 816 | (8) 586 112 / 608 = 964 |
| (9) 5 917 x 609 = 3 603 453 | (9) 787 319 / 923 = 853 |
| (10) 3 209 x 234 = 750 906 | (10) 95 559 / 159 = 601 |
| (11) 6 013 x 801 = 4 816 413 | (11) 368 193 / 623 = 591 |
| (12) 3 497 x 487 = 1 703 039 | (12) 26 520 / 195 = 136 |
| (13) 9 084 x 712 = 6 467 808 | (13) 216 832 / 308 = 704 |
| (14) 4 526 x 206 = 932 356 | (14) 603 880 / 974 = 620 |
| (15) 8 761 x 658 = 5 764 738 | (15) 140 322 / 546 = 257 |
| (16) 2 905 x 923 = 2 681 315 | (16) 666 462 / 831 = 802 |
| (17) 5 430 x 165 = 895 950 | (17) 88 665 / 257 = 345 |
| (18) 1 258 x 594 = 747 252 | (18) 402 047 / 409 = 983 |
| (19) 3 872 x 370 = 1 432 640 | (19) 326 876 / 782 = 418 |
| (20) 7 619 x 439 = 3 344 741 | (20) 469 090 / 610 = 769 |

Assortment 3
Group A
 (70% accuracy, 10 minutes)

| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 2 819 | 314 | 80 593 | 4 295 | 63 017 | 7 694 | 579 | 18 024 | 9 072 | 798 |
| 15 038 | 2 639 | 921 | 50 973 | 4 361 | 853 | 4 106 | 9 132 | 265 | 34 019 |
| 946 | 708 | 1 650 | 391 | 25 703 | 4 081 | 95 248 | 389 | 21 403 | 5 628 |
| 71 405 | 4 512 | 789 | -8 406 | 892 | 725 | 134 | 65 708 | 561 | 60 937 |
| 571 | 98 056 | 6 432 | -829 | 7 210 | 28 139 | 3 079 | -8 421 | 70 134 | 146 |
| 2 795 | 67 321 | 958 | 37 146 | 982 | 4 506 | 18 620 | -968 | 8 372 | -4 503 |
| 346 | -485 | 29 076 | 1 065 | 80 124 | 13 472 | 493 | 90 614 | 45 031 | -654 |
| 49 630 | -7 249 | 513 | 94 752 | 6 873 | 956 | 7 851 | 2 837 | 483 | -52 890 |
| 7 512 | -30 967 | 71 408 | 218 | 58 907 | 2 780 | 89 315 | -756 | 2 917 | 7 652 |
| 86 057 | 8 293 | 4 817 | -15 630 | 764 | 91 352 | 407 | -36 571 | 596 | 21 345 |
| 983 | 51 420 | 43 160 | -728 | 9 658 | 6 041 | 1 986 | -4 720 | 13 609 | 761 |
| 6 108 | -137 | 5 249 | -6 203 | 435 | 467 | 50 732 | 945 | 4 928 | -1 803 |
| 729 | -6 905 | 763 | 871 | 42 059 | 36 908 | 276 | 51 092 | 86 745 | -472 |
| 20 463 | 584 | 67 024 | 60 534 | 913 | 219 | 6 023 | 7 653 | 680 | 97 081 |
| 8 234 | 10 876 | 2 385 | 9 487 | 1 546 | 70 583 | 42 568 | 304 | 7 859 | 8 293 |
| 273636 | 198980 | 315738 | 227936 | 303444 | 268776 | 321417 | 195262 | 272655 | 176338 |

| Group B (70 % accuracy, 10 minutes.) | Group C (70% accuracy, 10 minutes.) |
|---|--|
| (1) 1 975 x 419 = 827 525 | (1) 93 063 / 463 = 201 |
| (2) 4 702 x 165 = 775 830 | (2) 653 650 / 850 = 769 |
| (3) 6 819 x 382 = 2 604 858 | (3) 45 310 / 197 = 230 |
| (4) 8 431 x 674 = 5 682 494 | (4) 141 360 / 304 = 465 |
| (5) 3 587 x 503 = 1 804 261 | (5) 546 336 / 672 = 813 |
| (6) 1 064 x 926 = 985 264 | (6) 894 132 / 918 = 974 |
| (7) 5 293 x 708 = 3 747 444 | (7) 102 910 / 205 = 502 |
| (8) 7 648 x 853 = 6 523 744 | (8) 72 954 / 386 = 189 |
| (9) 9 320 x 497 = 4 632 040 | (9) 337 608 / 521 = 648 |
| (10) 2 056 x 201 = 413 256 | (10) 267 393 / 749 = 357 |
| (11) 3 674 x 159 = 584 166 | (11) 213 852 / 502 = 426 |
| (12) 7 831 x 840 = 6 578 040 | (12) 74 889 / 471 = 159 |
| (13) 9 502 x 765 = 7 269 030 | (13) 756 286 / 943 = 802 |
| (14) 1 425 x 238 = 339 150 | (14) 199 080 / 316 = 630 |
| (15) 2 368 x 406 = 961 408 | (15) 694 848 / 704 = 987 |
| (16) 5 093 x 687 = 3 498 891 | (16) 97 552 / 268 = 364 |
| (17) 8 956 x 924 = 8 275 344 | (17) 498 261 / 921 = 541 |
| (18) 2 709 x 173 = 468 657 | (18) 84 525 / 805 = 105 |
| (19) 6 147 x 302 = 1 856 394 | (19) 524 286 / 657 = 798 |
| (20) 4 081 x 591 = 2 411 871 | (20) 37 674 / 138 = 273 |

The National Examination for the Third-Grade Abacus License

Held under the auspices of the Japan Chamber of Commerce and Industry

Group A (Entire group of ten sets with 70% accuracy in ten minutes)

| (1) | (2) | (3) | (4) | (5) |
|------------------|------------------|----------------|------------------|------------------|
| 20 139 | 954 | 638 | 7 524 | 37 258 |
| 826 | 360 728 | 780 241 | 301 492 | 1 630 |
| 432 501 | 46 170 | 2 516 | 59 710 | 849 075 |
| 96 317 | 8 397 | 37 690 | 834 | 26 914 |
| 4 620 | 27 816 | -8 215 | 6 185 | -604 132 |
| 81 453 | 819 605 | -496 507 | 720 943 | -98 206 |
| 708 319 | 3 547 | -21 365 | 62 397 | 715 |
| 5 492 | 64 052 | 619 | 958 | 3 854 |
| 975 | 573 | 84 035 | 36 270 | 179 038 |
| 67 154 | 945 610 | 7 283 | 917 846 | 43 927 |
| 849 560 | 8 324 | 209 714 | 513 | -879 |
| 2 738 | 21 496 | -95 480 | 6 905 | -50 186 |
| 683 | 952 | -347 | 80 673 | -2 937 |
| 58 027 | 102 873 | 57 981 | 654 812 | 564 |
| 907 614 | 38 109 | 430 692 | 28 041 | 624 105 |
| 3 236 418 | 2 449 206 | 989 495 | 2 885 103 | 1 010 740 |

| (6) | (7) | (8) | (9) | (10) |
|------------------|------------------|------------------|------------------|------------------|
| 406 725 | 6 208 | 872 | 5 930 | 18 357 |
| 157 | 37 461 | 46 237 | 20 718 | 9 041 |
| 54 860 | 801 537 | 695 018 | 463 802 | 378 260 |
| 3 746 | 972 | 79 561 | 549 | 56 834 |
| 760 912 | -42 796 | 6 409 | -87 056 | 672 |
| 538 | -194 083 | 84 357 | -109 867 | 29 138 |
| 28 301 | -8 615 | 360 172 | 97 385 | 710 953 |
| 804 139 | 23 590 | 483 | 8 476 | 5 280 |
| 6 852 | 314 | 51 290 | 714 | 429 |
| 19 026 | 260 475 | 3 625 | 689 502 | 69 704 |
| 497 | 7 132 | 976 108 | -54 923 | 804 519 |
| 96 813 | -56 084 | 741 | -361 | 235 |
| 570 289 | -392 | 29 083 | -1 274 | 1 627 |
| 31 794 | 904 875 | 4 539 | 246 013 | 48 706 |
| 2 435 | 69 581 | 508 412 | 93 251 | 963 415 |
| 2 787 084 | 1 810 175 | 2 846 907 | 1 372 859 | 3 097 170 |

| Group B (70 % accuracy, 10 minutes. Calculate problems 1-10 to the nearest thousandth; 11-20 to the nearest yen.) | Group C (70 % accuracy, 10 minutes. Calculate problems 1-10 to the nearest thousandth; 11-20 to the nearest yen.) |
|--|--|
| (1) $2\,931 \times 819 = 2\,400.489$ | (1) $172\,894 / 631 = 274$ |
| (2) $6\,052 \times 235 = 1\,422.220$ | (2) $0.20592 / 3.96 = 0.052$ |
| (3) $84.67 \times 90.7 = 7\,679.569$ | (3) $361\,444 / 829 = 436$ |
| (4) $9\,234 \times 572 = 5\,281.848$ | (4) $76.296 / 408 = 0.187$ |
| (5) $70\,496 \times 86 = 6\,062.656$ | (5) $50.735 / 0.073 = 695$ |
| (6) $1\,507 \times 0.461 = 694.727$ | (6) $186\,296 / 584 = 319$ |
| (7) $218 \times 1\,059 = 230.862$ | (7) $212.795 / 26.5 = 8.03$ |
| (8) $0.3875 \times 6.48 = 2.511$ | (8) $0.47693 / 0.917 = 0.520$ |
| (9) $5\,689 \times 3.24 = 18\,432.36$ | (9) $639\,184 / 7\,024 = 91$ |
| (10) $0.4103 \times 0.073 = 0.030$ | (10) $11\,526 / 15 = 768.4$ |
| (11) $¥6\,379 \times 108 = ¥688.932$ | (11) $¥324\,563 / 463 = ¥701$ |
| (12) $¥2\,014 \times 0.429 = ¥864$ | (12) $¥185 / 0.625 = ¥296$ |
| (13) $¥1\,493 \times 963 = ¥1\,437.759$ | (13) $¥646\,919 / 7\,109 = ¥91$ |
| (14) $¥9\,584 \times 0.875 = ¥8.386$ | (14) $¥455 / 0.547 = ¥832$ |
| (15) $¥765 \times 504.8 = ¥386.172$ | (15) $¥341\,105 / 85 = ¥4\,013$ |
| (16) $¥2\,301 \times 637 = ¥1\,465.737$ | (16) $¥66 / 0.176 = ¥375$ |
| (17) $¥40\,678 \times 91 = ¥3\,701.698$ | (17) $¥78\,196 / 90.4 = ¥865$ |
| (18) $¥5\,896 \times 20.5 = ¥120.868$ | (18) $¥282\,048 / 312 = ¥904$ |
| (19) $¥8\,125 \times 0.736 = ¥5.980$ | (19) $¥1\,820 / 2.08 = ¥875$ |
| (20) $¥3\,207 \times 214 = ¥686.298$ | (20) $¥602\,196 / 938 = ¥642$ |

The National Examination for the Second-Grade Abacus License

Held under the auspices of the Japan Chamber of Commerce and Industry

Group A

(Entire group of ten sets with 80% accuracy in ten minutes)

| (1) | (2) | (3) | (4) | (5) |
|--------------------|-------------------|--------------------|-------------------|--------------------|
| 1 289 307 | 483 971 | 3 029 176 | 62 501 478 | 974 362 |
| 675 243 | 81 602 459 | 15 849 | 729 360 | 4 265 103 |
| 90 428 165 | 5 871 304 | 436 250 | 3 968 251 | 58 130 279 |
| 17 436 | -90 562 | 20 754 318 | 91 087 | 86 341 |
| 7 360 294 | -264 918 | 8 109 567 | -46 083 529 | 679 854 |
| 6 548 | -6 738 095 | 3 291 | -857 901 | 1 502 463 |
| 874 310 | 10 947 283 | 97 846 | 5 340 782 | 7 318 |
| 48 039 152 | 306 947 | 521 738 | 19 648 | 80 264 597 |
| 2 157 896 | 54 732 | 70 983 462 | 654 270 | 392 086 |
| 93 405 | 7 086 529 | 1 240 573 | -2 593 | 18 759 |
| 526 781 | -32 169 840 | 612 984 | -9 145 806 | 2 840 915 |
| 3 185 069 | -7 615 | 54 601 | -36 714 | 76 051 432 |
| 43 972 | 9 520 736 | 9 072 345 | 20 794 136 | 419 687 |
| 54 702 631 | 64 187 | 865 913 | 1 283 097 | 23 590 |
| 951 820 | 815 023 | 65 378 024 | 476 835 | 3 705 241 |
| 210 352 029 | 77 482 141 | 181 175 937 | 39 732 401 | 229 362 027 |

| (6) | (7) | (8) | (9) | (10) |
|--------------------|--------------------|--------------------|--------------------|--------------------|
| 537 641 | 312 650 | 8 029 137 | 86 374 | 80 625 749 |
| 7 241 930 | 1 287 094 | 79 630 821 | 10 342 569 | 79 853 |
| 35 082 469 | 64 238 | 948 675 | 493 720 | 5 986 072 |
| 697 508 | 40 175 982 | 52 149 | 6 205 917 | 395 846 |
| -4 803 621 | 3 674 | -2 407 536 | 8 253 | 10 793 |
| -28 159 | 5 214 803 | -163 904 | 274 068 | 9 753 261 |
| -90 154 682 | 793 165 | 60 245 817 | 31 679 | 148 507 |
| 763 240 | 83 620 941 | 74 396 | 7 052 841 | 73 062 918 |
| 17 563 | 19 576 | 5 720 681 | 54 189 302 | 2 814 730 |
| 2 058 374 | 6 048 239 | 819 570 | 924 516 | 2 684 |
| -386 019 | 852 497 | -6 438 | 8 760 193 | 490 172 |
| -9 785 | 41 056 | -31 057 294 | 17 245 | 36 405 |
| 10 964 827 | 9 536 718 | -98 763 | 536 980 | 41 205 396 |
| 8 435 719 | 27 418 305 | 4 502 698 | 3 875 421 | 6 897 013 |
| 70 296 | 907 523 | 381 052 | 90 641 835 | 541 628 |
| -29 512 699 | 176 296 461 | 126 671 061 | 183 440 913 | 222 051 027 |

| Group B (80 % accuracy, 10 minutes. Calculate problems 1-10 to the nearest thousandth; 11-20 to the nearest yen.) | Group C (80 % accuracy, 10 minutes. Calculate problems 1-10 to the nearest thousandth; 11-20 to the nearest yen.) |
|--|--|
| (1) $60\,937 \times 2\,154 = 131\,258\,298$ | (1) $8\,485\,074 / 2\,754 = 3\,081$ |
| (2) $42\,618 \times 7\,309 = 311\,494\,962$ | (2) $44\,080\,524 / 5\,436 = 8\,109$ |
| (3) $804\,752 \times 962 = 774\,171\,424$ | (3) $0.2059086 / 0.7068 = 0.2913$ |
| (4) $18\,509 \times 0.5817 = 10\,766.6853$ | (4) $52\,622\,371 / 61\,403 = 857$ |
| (5) $93.125 \times 6.048 = 563.22$ | (5) $0.6793975 / 9.371 = 0.0725$ |
| (6) $76\,084 \times 8\,539 = 649\,681\,276$ | (6) $2\,984\,785 / 482 = 6\,192.5$ |
| (7) $0.20436 \times 470.6 = 96.1718$ | (7) $1\,709.6483 / 3\,592 = 0.4760$ |
| (8) $8\,741 \times 39\,281 = 343\,355\,221$ | (8) $5.74902 / 0.0615 = 93.48$ |
| (9) $31\,597 \times 1.475 = 46\,605\,575$ | (9) $12\,426\,698 / 1\,987 = 6\,254$ |
| (10) $0.52963 \times 0.0623 = 0.0330$ | (10) $610.42124 / 82.09 = 7.436$ |
| (11) $¥47\,512 \times 8\,063 = ¥383\,089\,256$ | (11) $¥11\,393\,544 / 3\,918 = ¥2\,908$ |
| (12) $¥90\,386 \times 0.7421 = ¥67\,075$ | (12) $¥641 / 0.0784 = ¥8\,176$ |
| (13) $¥2\,975 \times 508.96 = ¥1\,514\,156$ | (13) $¥12\,945\,709 / 287 = ¥45\,107$ |
| (14) $¥76\,128 \times 0.4375 = ¥33\,306$ | (14) $¥2\,187 / 0.4306 = ¥5\,079$ |
| (15) $¥38\,604 \times 2\,869 = ¥110\,754\,876$ | (15) $¥52\,899 / 5.496 = ¥9\,625$ |
| (16) $¥65\,093 \times 0.0942 = ¥6\,132$ | (16) $¥6\,593\,836 / 1\,637 = ¥4\,028$ |
| (17) $¥17\,249 \times 3\,517 = ¥60\,664\,733$ | (17) $¥1\,131 / 0.8125 = ¥1\,392$ |
| (18) $¥504\,831 \times 658 = ¥332\,178\,798$ | (18) $¥59\,415\,129 / 92\,403 = ¥643$ |
| (19) $¥49\,625 \times 1.024 = ¥50.816$ | (19) $¥360\,804 / 70.25 = ¥5\,136$ |
| (20) $¥81\,307 \times 7\,391 = ¥600\,940\,037$ | (20) $¥51\,693\,213 / 6\,591 = ¥7\,843$ |

| Group D Mental Calculation (80 % accuracy, 2 minutes) | | | | | | | | | |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| 58 | 130 | 61 | 987 | 83 | 704 | 32 | 501 | 23 | 436 |
| 716 | 97 | 389 | 19 | 206 | 16 | 284 | 48 | 741 | 28 |
| 65 | 681 | 24 | 870 | 45 | 860 | 91 | 17 | 308 | 510 |
| 901 | 458 | 85 | 21 | 731 | 29 | 105 | 930 | 49 | 67 |
| 45 | 76 | 150 | 36 | 52 | 487 | 53 | 376 | 85 | 940 |
| 234 | 802 | 93 | 603 | 417 | 72 | 314 | 29 | 132 | 79 |
| 82 | 37 | 704 | 546 | 180 | 918 | 790 | 862 | 64 | 12 |
| 320 | 26 | 417 | 89 | 94 | 63 | 85 | 75 | 250 | 386 |
| 79 | 940 | 52 | 405 | 39 | 590 | 67 | 409 | 96 | 807 |
| 143 | 59 | 236 | 72 | 625 | 35 | 426 | 68 | 71 | 95 |
| 2 643 | 3 306 | 2 211 | 3 648 | 2 472 | 3 774 | 2 247 | 3 315 | 2 319 | 3 360 |

The National Examination for the First-Grade Abacus License

Held under the auspices of the Japan Chamber of Commerce and Industry

Group A

(Entire group of ten sets with 80% accuracy in ten minutes)

| (1) | (2) | (3) | (4) | (5) |
|-----------------------|----------------------|-----------------------|----------------------|-----------------------|
| 52 089 461 | 432 186 075 | 9 135 842 | 4 170 326 958 | 247 013 589 |
| 7 153 820 | 8 145 903 267 | 304 827 196 | 9 852 036 | 5 928 376 |
| 3 045 861 297 | 9 247 183 | 96 705 | 12 493 870 | 36 740 218 |
| 179 625 408 | -62 098 541 | 2 036 754 981 | -736 981 425 | 27 190 |
| 43 162 | -762 309 | 1 408 273 | -572 908 | 1 029 846 753 |
| 81 904 753 | -5 280 931 764 | 83 927 654 | 1 630 857 | 3 507 461 |
| 9 286 041 | 3 679 452 | 702 149 836 | 6 058 249 731 | 574 268 109 |
| 7 306 154 289 | 24 805 691 | 5 368 071 | 93 175 648 | 61 930 548 |
| 8 479 503 | 901 326 785 | 6 520 743 189 | 567 903 412 | 459 862 |
| 297 580 364 | 4 270 516 | 49 386 527 | -2 647 103 | 8 702 164 539 |
| 165 937 | 71 058 934 | 610 945 | -3 901 728 564 | 6 382 047 |
| 64 327 810 | -358 712 640 | 9 017 254 368 | -28 157 496 | 418 079 635 |
| 4 710 698 325 | -6 450 839 | 168 032 579 | 845 069 273 | 92 543 718 |
| 5 732 649 | 1 097 534 628 | 2 571 460 | 24 019 | 1 605 923 |
| 923 517 086 | 83 917 | 50 492 813 | 4 615 380 | 7 350 291 864 |
| 16 692 619 905 | 4 981 140 355 | 18 952 760 439 | 7 093 253 688 | 18 530 789 832 |

| (6) | (7) | (8) | (9) | (10) |
|---------------------|-----------------------|----------------------|-----------------------|-----------------------|
| 2 709 658 341 | 7 421 630 | 921 376 048 | 6 194 873 | 314 726 905 |
| 5 182 069 | 65 049 | 6 347 208 195 | 804 725 361 | 2 645 718 |
| 890 246 753 | 4 086 739 215 | -70 153 264 | 16 249 | 59 832 067 |
| -731 482 | 615 908 743 | -694 503 | 30 954 187 | 8 031 276 549 |
| -46 029 378 | 27 851 409 | 162 980 375 | 1 054 678 932 | 694 783 |
| -9 017 486 523 | 9 372 586 | 3 015 467 928 | 7 240 156 | 3 501 624 |
| 3 592 710 | 7 251 094 863 | 7 935 681 | 201 536 498 | 968 135 470 |
| 109 264 538 | 36 281 597 | 84 029 135 | 871 250 | 25 840 391 |
| 72 503 694 | 548 617 320 | 3 748 256 | 76 109 382 | 6 401 279 835 |
| 65 871 | 3 946 075 | -456 102 987 | 5 192 783 604 | 7 453 102 |
| -1 874 305 | 8 905 173 246 | -8 561 740 | 8 459 736 | 46 087 921 |
| -380 516 924 | 639 182 | -2 609 845 317 | 420 368 915 | 172 943 058 |
| 5 024 387 196 | 4 280 951 | 27 439 | 69 547 023 | 5 318 960 |
| 61 740 935 | 92 314 708 | 7 890 251 | 9 043 182 567 | 69 287 |
| 8 961 247 | 170 523 864 | 93 014 862 | 5 073 829 | 7 890 421 356 |
| -561 035 258 | 21 760 230 438 | 7 498 320 359 | 16 921 742 562 | 23 930 227 026 |

| Group B (80 % accuracy, 10 minutes. Calculate problems 1-10 to the nearest thousandth; 11-20 to the nearest yen.) | Group C (80 % accuracy, 10 minutes. Calculate problems 1-10 to the nearest thousandth; 11-20 to the nearest yen.) |
|--|--|
| (1) $432\,159 \times 68\,194 = 29\,470\,650\,846$ | (1) $9\,373\,655\,025 / 48\,237 = 194\,325$ |
| (2) $751\,083 \times 95\,036 = 71\,379\,923\,988$ | (2) $59\,530\,739\,332 / 70\,652 = 842\,591$ |
| (3) $628\,407 \times 23\,871 = 15\,000\,703\,497$ | (3) $0.1271344371 / 2.14863 = 0.05917$ |
| (4) $0.97815 \times 80.6253 = 78.86364$ | (4) $52\,274\,789\,792 / 56\,104 = 931\,748$ |
| (5) $360\,798 \times 0.17409 = 62\,811.32382$ | (5) $40.87907525 / 0.13807 = 296.075$ |
| (6) $584\,162 \times 75\,382 = 44\,035\,299\,884$ | (6) $330\,085.21078 / 69\,018 = 4.78260$ |
| (7) $23.6054 \times 40.617 = 958.78053$ | (7) $7\,534\,346\,391 / 35\,769 = 210\,639$ |
| (8) $8\,079\,231 \times 3\,925 = 31\,710\,981\,675$ | (8) $0.4164695782 / 0.07391 = 5.63482$ |
| (9) $127\,496 \times 5.4178 = 690\,747.8288$ | (9) $80\,233\,503\,322 / 9\,254 = 8\,670\,143$ |
| (10) $0.603945 \times 0.02649 = 0.01600$ | (10) $624.88295605 / 84.925 = 7.35806$ |
| (11) $¥576\,183 \times 42\,817 = ¥24\,670\,427\,511$ | (11) $¥3\,931\,366\,928 / 15\,896 = ¥247\,318$ |
| (12) $¥862\,054 \times 0.06759 = ¥58\,266$ | (12) $¥635\,615 / 0.98013 = ¥648\,501$ |
| (13) $¥3\,915\,478 \times 5\,603 = ¥21\,938\,423\,234$ | (13) $¥44\,731\,890 / 63.125 = ¥708\,624$ |
| (14) $¥740\,625 \times 0.84512 = ¥625\,917$ | (14) $¥7\,643\,415\,038 / 37\,142 = ¥205\,789$ |
| (15) $¥184\,397 \times 739.24 = ¥136\,313\,638$ | (15) $¥254\,071\,068 / 296.71 = ¥856\,294$ |
| (16) $¥902\,871 \times 0.28196 = ¥254\,574$ | (16) $¥46\,113 / 0.06459 = ¥713\,934$ |
| (17) $¥451\,063 \times 92\,031 = ¥41\,511\,778\,953$ | (17) $¥52\,348\,910\,242 / 58\,067 = ¥901\,526$ |
| (18) $¥273\,509 \times 54\,768 = ¥14\,979\,540\,912$ | (18) $¥37\,108\,174\,071 / 4\,057 = ¥9\,146\,703$ |
| (19) $¥68\,192 \times 360.875 = ¥24\,608\,788$ | (19) $¥174\,669 / 0.83424 = ¥209\,375$ |
| (20) $¥390\,246 \times 13\,904 = ¥5\,425\,980\,384$ | (20) $¥39\,176\,554\,843 / 729\,803 = ¥53\,681$ |

| Group D Mental Calculation (80 % accuracy, 2 minutes) | | | | | | | | | |
|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| 321 | 5 680 | 759 | 6 014 | 495 | 8 720 | 147 | 3 098 | 254 | 9 413 |
| 2 038 | 173 | 9 370 | 397 | 8 150 | 479 | 6 250 | 917 | 8 190 | 126 |
| 716 | 3 527 | 164 | 4 680 | -243 | 6 093 | 863 | 1 340 | -436 | 2 780 |
| 1 542 | 8 601 | -527 | 251 | -3 961 | 158 | -235 | 654 | -5 021 | 837 |
| 653 | 912 | 2 843 | 7 406 | 518 | 7 826 | 4 012 | 2 801 | 617 | 586 |
| 9 704 | 485 | 682 | 5 937 | 2 074 | 613 | 9 581 | 579 | 4 203 | 7 105 |
| 285 | 7 034 | 4 105 | 873 | 1 632 | 2 760 | 729 | 465 | 3 568 | 954 |
| 5 410 | 298 | -3 524 | 695 | -925 | 594 | -1 936 | 7 602 | -937 | 4 270 |
| 869 | 6 940 | -913 | 1 208 | 784 | 3 408 | -348 | 8 937 | 784 | 6 098 |
| 4 937 | 796 | 8 061 | 982 | 6 307 | 951 | 5 704 | 286 | 1 592 | 369 |
| 26 475 | 34 446 | 21 020 | 28 443 | 14 831 | 31 602 | 24 767 | 26 679 | 12 814 | 32 538 |