Overview

"Abacus I" introduced the basic operations of arithmetic. Now that you've successfully mastered adding, subtracting, multiplying, and dividing the abacus way, "Abacus II" explains how these four operations enable you to perform more advanced calculations.

The goal of "Abacus II" is to teach you how to solve more complex mathematical problems. You can even apply your skills to everyday problems like adding and subtracting quantities and making money transactions.

After completing this course, you will be able to use a Cranmer abacus to

- add and subtract fractions;
- multiply and divide fractions;
- find percents and work with units of measures;
- extract square roots; and
- subtract a larger number from a smaller one.

To achieve these objectives, the course is composed of five lessons. Lesson 1 explains how to reduce fractions, as well as how to add and subtract them. Lesson 2 explains how to multiply and divide fractions. Lesson 3 explains how to use percents and units of measures. Lesson 4 explains how to extract square roots. The last lesson, Lesson 5, explains how to subtract a larger number from a smaller one.

Lesson 1: Adding and Subtracting Fractions

Using the abacus, you can quickly and efficiently add, subtract, multiply, and divide fractions. This lesson explains how to reduce fractions as well as how to add and subtract them. Applying the knowledge you gained in "Abacus I" helps you to perform more advanced calculations, such as operations with fractions, the abacus way.

When you work with fractions, you can treat them as such or you can convert them to their decimal equivalents by simply dividing the numerator by the denominator. Thus, you could treat $\frac{7}{8}$ as 0.875, and $3\frac{7}{8}$ as 3.875.

You will be working with two kinds of fractions — common fractions and mixed numbers. Common fractions are written with a numerator and a denominator. Examples are $\frac{5}{7}$ and $\frac{32}{65}$. The numerator—the number above the fraction bar—shows how many fractional parts you have. The denominator—the number below the fraction bar—shows how many equal parts the whole is divided into. For example, in $\frac{5}{7}$ the whole is divided into 7 equal parts and you have 5 parts. Mixed numbers, for example $2\frac{3}{4}$ or $12\frac{5}{8}$, have a whole number and a common fraction.

Objectives

After completing this lesson, you will be able to use the Cranmer abacus to

- reduce fractions;
- add fractions; and
- subtract fractions.

Reducing Fractions

To reduce a fraction to its simplest form, look for a number that divides evenly into both the numerator and the denominator. This number is called a common factor. If there is more than one factor common to both the numerator and the denominator, look for the largest common factor. Then divide both the numerator and the denominator by the largest common factor, and replace each number by the quotient.

For instance, reduce $\frac{6}{8}$ to its simplest form using the abacus. Set the numerator, 6, immediately to the left of the first unit mark, on column 4, and set the denominator, 8, to the far right, on column 1. Now, think which numbers divide 6 and which divide 8. What factors are common to 6 and 8? The common factor is 2, so divide the numerator and the denominator by 2. Since $6 \div 2 = 3$, clear the numerator, 6, and set 3 in its place. Next, divide the denominator, 8, by 2. Since $8 \div 2 = 4$, clear 8 and set 4 in its place. The fraction is now reduced. You see that the simplest form of $\frac{6}{8}$ is $\frac{3}{4}$.

If the numerator and denominator still have a common factor, the fraction has not been reduced to its simplest form. Repeat the process. For example, to find the simplest form of $\frac{36}{96}$, set 36 to the left of the first 96 unit mark, on columns 5 and 4. Then set 96 to the far right, on columns 2 and 1. Since 3 is common to both 36 and 96, divide both numbers by 3. Since $36 \div 3 = 12$, clear 36 and set 12 in its place. Next, $99 \div 3 = 32$, so change 96 to 32. The reduced fraction is $\frac{12}{32}$. Since 4 is common to both 12 and 32, divide them both by 4. Since $12 \div 4 = 3$, change the numerator, 12, to 3 , and set 3 on column 4. Then divide 32 by 4, which is 8, so change the denominator, 32, to 8, and set 8 on column 1. Therefore, the simplest form of $\frac{36}{96}$ is $\frac{3}{8}$

Practice Exercises

Using the abacus, practice reducing these fractions to their simplest form. Record your answers so that you can compare them with those that follow the exercise.

- $\frac{4}{12}$
- $\frac{4}{6}$
- $\frac{8}{18}$
- $\frac{7}{15}$
- $\frac{7}{28}$
- $\frac{9}{15}$
- $\frac{21}{32}$
- $\frac{24}{54}$
- $\frac{77}{98}$
- $\frac{64}{96}$
- $\frac{12}{78}$
- $\frac{21}{48}$

Answers

Compare your answers with those that follow:

- $\frac{1}{3}$
- $\frac{2}{3}$
- $\frac{4}{9}$

- $\frac{7}{15}$
- $\frac{1}{4}$
- $\frac{3}{5}$
- $\frac{21}{32}$
- $\frac{4}{9}$
- $\frac{11}{14}$
- $\frac{2}{3}$
- $\frac{2}{13}$
- $\frac{7}{16}$

Adding Fractions

When adding two or more fractions, begin by reducing each fraction to its simplest form. Then, find the least common denominator, and express each fraction using the common denominator. Add any whole numbers. Then add the numerators.

When adding and subtracting fractions, the abacus is divided into three sections—whole numbers, numerators, and denominators. The first two unit marks at the right end of the abacus serve as section dividers. The left section is for whole numbers. It consists of the three columns to the left of the second unit mark, that is, columns 9, 8, and 7. The middle section is for numerators. It consists of the three columns to the left of the first unit mark, columns 6, 5, and 4. Finally, columns 3, 2, and 1 are for denominators. Since each section has three columns, take special care to place digits on the proper column in each section. The far right column within a section is for one-digit numbers. The two far right columns within a section are for two-digit numbers, and all three columns are used for three digit numbers. Because the unit marks separate the sections, you can easily distinguish whole numbers from numerators, and numerators from denominators. If a number has more than three digits, move a section to the left of the next unit mark so that numbers do not run into one another.

For example, add $2\frac{5}{8} + 16\frac{7}{12}$. Both fractions are already in simplest form, so your first step is to find a common denominator. To find the lowest common denominator, try multiplying the largest denominator by 2. Then check to see if the product can be divided by the other denominator or denominators. If it can, use that product as the lowest common denominator. If it cannot, multiply the largest denominator by 3, then 4, then 5, and so on, until you find a product that is divisible by all denominators.

In the example, $2\frac{5}{8} + 16\frac{7}{12}$, the denominators are 8 and 12. To find the lowest common denominator, try multiplying 12 by 2. Since you can divide the product, 24, by the other denominator, 8, use 24 as the common denominator. Set 24 in the denominator section on columns 2 and 1. Then set the whole number, 2, of the first mixed number, $2\frac{5}{8}$, in the whole number section. Since 2 is a one-digit number, place 2 on the column immediately to the left of the second unit mark, that is, on column 7.

Now that you have found the common denominator, change the numerator, 5, of $2\frac{5}{8}$. Set the numerator, 5, on column 13. Skip one column, then set the denominator, 8, on column 11. Next, divide the common denominator, 24, by the original denominator, 8, producing 3. Clear 8 and multiply the quotient, 3, by the original numerator, 5, producing 15. Then clear 5. Set the product, 15, in the numerator section, placing 15 on the two columns immediately left of the first unit mark, columns 5 and 4. Your abacus contains 2 in the whole-number section on column 7, 15 in the numerator section on columns 5 and 4, and 24 in the denominator section on columns 2 and 1.

You are now ready to add the second mixed number, $16\frac{7}{12}$. Add the whole number 16 to the 2 in 12 the whole-number section, making sure to set the 6 of 16 on the column already containing 2. The whole number section now shows 18 on columns 8 and 7. Now change the numerator, the 7 of $16\frac{7}{12}$. Set the numerator, 7, to the far left on column 13. Skip one column, then set the denominator, 12, on columns 11 and 10. Divide the common denominator, 24, by the original denominator, 12, which produces 2. Then clear 12. Now multiply the quotient, 2, by the original numerator, 7, which produces 14. Then clear 7. Next add 14 to the numerator section, adding it to the 15 already there. You now have 29 in the numerator section on columns 5 and 4.

Now that you have finished adding both fractions, examine the numerator and the denominator sections. The numerator, 29, is greater than the denominator, 24, which means you have an improper fraction. Therefore, you must change the improper fraction $\frac{29}{24}$ to a mixed number. To do this, divide the numerator, 29, by the denominator, 24. The quotient is 1. Add this 1 to the 18 already in the whole-number section. Then multiply this 1 by the denominator, 24. In the numerator section, subtract the product, 24, from the numerator, 29. You now have 19 in the whole-number section, 5 in the numerator section, and 24 in the denominator section. Your answer is $19\frac{5}{24}$. After working a problem, verify that the answer is in its simplest form. Otherwise, it should be reduced. In this problem, $19\frac{5}{24}$ is already in its simplest form because 5 and 24 have no common factors.

For additional practice in adding fractions, try solving this challenging problem. Add the fractions: $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}$

Here is a suggestion to find the lowest common denominator. Begin by finding the lowest common denominator for the first two fractions, $\frac{1}{2}$ and $\frac{1}{3}$, which is 6. Then find the lowest common denominator for 6 and the next denominator, 4, which is 12. Then find the lowest common denominator for 12 and the next denominator, 5. Continue until you find the lowest common denominator for all fractions. The correct answer is $1\frac{2089}{2520}$.

Practice Exercises

Practice adding fractions on the abacus by working the following problems. Record your answers so that you can compare them with those that follow the exercise.

- $3\frac{2}{9}+4\frac{5}{9}$
- $\frac{15}{16}+\frac{13}{16}$
- $\frac{1}{6}+\frac{5}{14}$
- $\frac{6}{7}+\frac{3}{4}$
- $6\frac{4}{57}+8$
- $4\frac{3}{8}+4\frac{3}{4}$
- $6\frac{7}{12}+3\frac{5}{9}$
- $\frac{231}{248}+\frac{3}{31}$
- $5\frac{7}{9}+\frac{7}{8}$
- $8\frac{2}{3}+7\frac{3}{4}$
- $5\frac{1}{2}+3\frac{4}{5}$
- $\frac{2}{3}+\frac{4}{7}$
- $\frac{4}{9}+6\frac{4}{27}$
- $8\frac{6}{7}+5\frac{5}{7}$
- $7\frac{2}{9}+9\frac{5}{6}$
- $23\frac{13}{20}+\frac{4}{8}$
- $8\frac{4}{7}+2\frac{5}{9}$
- $1\frac{2}{3}+4\frac{5}{6}$

Answers

Compare your answers with those that follow:

- $7\frac{7}{9}$
- $1\frac{3}{4}$
- $\frac{11}{21}$
- $1\frac{17}{28}$
- $14\frac{4}{57}$
- $9\frac{1}{8}$
- $10\frac{5}{36}$
- $1\frac{7}{248}$
- $6\frac{47}{72}$
- $16\frac{5}{12}$
- $9\frac{3}{10}$
- $1\frac{5}{21}$
- $6\frac{16}{27}$
- $14\frac{4}{7}$
- $17\frac{1}{18}$
- $24\frac{3}{20}$
- $11\frac{8}{63}$
- $6\frac{1}{2}$

Subtracting Fractions

You subtract fractions in much the same way as you add them, except that, instead of adding whole numbers to whole numbers and numerators to numerators, you subtract them. Whenever the numerator of the second fraction is greater than the numerator of the first fraction, you cannot subtract. Instead, borrow 1 from the whole-number section. Then convert the 1 to an equivalent fraction in which the numerator and denominator are both the same as the common denominator. Add the numerator of this fraction to the numerator section. Now you can subtract the numerators. If the subtraction still cannot be made, borrow another 1 from the whole-number section and repeat the process. For instance, if the common denominator is 24 and you borrow 1, change this borrowed 1 to its equivalent fraction, $\frac{24}{24}$. If the common denominator is 36, change 24 the borrowed 1 to $\frac{36}{36}$ You have not changed the value of the first mixed number—only its form. Although you borrowed 1 from the whole-number section, you returned it in fraction form to the numerator section.

Consider this example: $5\frac{1}{4} - 2\frac{3}{4}$. You already have a common denominator because both fractions are expressed in fourths. So begin by setting the common denominator, 4, in the denominator section on column 1. Set the first fraction by setting 5 in the whole-number section, that is, on the column immediately to the left of the second unit mark, column 7. Then set the numerator, 1, in the numerator section on the column immediately left of the first unit mark, column 4. You are now ready to subtract $2\frac{3}{4}$ from $5\frac{1}{4}$. Subtract 2 from 5 in the whole-number section, leaving 3 in that section on column 7. Next, subtract the 3 of $2\frac{3}{4}$ from the numerator section, on column 4. Since you cannot subtract 3 from the 1 in that section, borrow 1 from the whole-number section by subtracting 1, which changes the 3 in that section to 2. Remember the common denominator is 4. This borrowed 1 is equal to $\frac{4}{4}$, so add 4 to the 1 in the numerator section. Now you have 5 in the numerator section on column 4. You can now make the subtraction in the numerator section. Subtract 3 from 5, which leaves 2 in the numerator section. The subtraction is now finished, giving you the difference, $2\frac{2}{4}$. The fraction $\frac{2}{4}$ can be reduced to $\frac{1}{2}$. The final answer is $2\frac{1}{2}$.

Now consider this example: $8\frac{7}{9} - 2\frac{5}{6}$. Your first step is to determine the lowest common denominator, which is 18. Set 18 to the far right on columns 2 and 1. Then, set the whole number 8 of the first mixed number, $8\frac{7}{9}$, immediately left of the second unit mark on column 7. To change the denominator of $\frac{7}{9}$ to 18, set the original numerator, 7, on the column to the far left, column 13. Skip one column, then set the original denominator, 9, on column 11. Next, divide the common denominator, 18, by the original denominator, 9. The quotient is 2. Then, clear 9. Now, multiply the quotient, 2, by the original numerator, 7. The product is 14. Clear 7. Set this product, 14, in the numerator section on the two columns immediately left of the first unit mark, columns 5 and 4. The abacus now contains $8\frac{14}{18}$. You are now ready to subtract the second mixed number, $2\frac{5}{6}$. Subtract 2 from 8 in the whole-number section, which leaves 6 on column 7. To subtract $\frac{5}{6}$ from $\frac{14}{18}$, you must change the 6 in to $\frac{5}{6}$ to 18. Set the original numerator, 5, on the column to the far left, column 13. Skip one column, then set the original denominator, 6, on column 11. Divide the common denominator, 18, by 6. The quotient is 3. Then clear 6. Multiply the resulting quotient, 3, by the original numerator, 5. The quotient is 15. Then clear 5.

You are now ready to subtract 15 from the numerator section. You cannot subtract 15 from the 14 already in that section, however, so borrow 1 from the whole-number section. This changes the 6 in the whole number section to 5. Remember the common denominator is 18. The borrowed 1 is equal to the fraction $\frac{18}{18}$, so add 18 to the 14 already set in the 18 numerator section. You now have 32 in the numerator section on columns 5 and 4. Now you can make the subtraction in the numerator section. Subtract 15 from 32 to get 17 in the numerator section. You have now completed the problem. The difference between $8\frac{7}{9}$ and $2\frac{5}{6}$ is $5\frac{17}{18}$. This fraction is already in simplest form, so you do not reduce it.

Practice Exercises

Practice subtracting fractions on the abacus by working the following problems. Record your answers so that you can compare them with those that follow the exercise.

- $\frac{11}{17} - \frac{8}{17}$
- $7\frac{5}{8} - 4\frac{3}{8}$
- $3\frac{1}{4} - \frac{3}{4}$
- $8\frac{4}{7} - 2\frac{4}{21}$
- $6\frac{7}{8} - 2\frac{3}{4}$
- $4\frac{5}{6} - 3\frac{4}{5}$
- $4\frac{3}{8} - 1\frac{7}{9}$
- $1\frac{2}{9} - \frac{7}{9}$
- $11\frac{1}{5} - \frac{3}{4}$
- $\frac{47}{336} - \frac{23}{336}$
- $10\frac{2}{3} - 1\frac{3}{4}$
- $8\frac{7}{8} - 2\frac{5}{6}$
- $9 - \frac{2}{5}$
- $23\frac{4}{9} - 7$
- $2\frac{1}{6} - \frac{3}{4}$

Answers

Compare your answers with those that follow:

- $\frac{3}{17}$
- $3\frac{1}{4}$
- $2\frac{1}{2}$
- $6\frac{8}{21}$
- $4\frac{1}{8}$
- $1\frac{1}{30}$
- $2\frac{43}{72}$
- $\frac{4}{9}$
- $10\frac{9}{20}$
- $\frac{1}{14}$
- $8\frac{11}{12}$
- $6\frac{1}{24}$
- $8\frac{3}{5}$
- $16\frac{4}{9}$
- $1\frac{5}{12}$

Assignment 1

Complete the following questions with all the correct answers. Don't assume that only one answer is correct. Sometimes two or three are needed to answer the question correctly. Questions 1-18 are worth 5 points each. Question 19 is worth 10 points.

- After simplifying $\frac{42}{56}$, you set
    - 42 on columns 5 and 4, and 56 on columns 2 and 1
    - 3 on column 7 and 4 on column 1
    - 3 on column 4 and 4 on column 1
- When adding mixed numbers on the abacus, you
    - add all the whole numbers in the whole-number section, then add all the numerators in the numerator section
    - find a common denominator (if necessary), set it in the denominator section, then set the whole number and numerator of the first mixed number before going to the next mixed number
    - set one entire mixed number in a right to left direction. Then add the second mixed number in a right to left direction

Questions 3-5 refer to the problem $\frac{1}{6} + \frac{7}{8}$

- The common denominator is 24. The first step is to set 24 on
    - columns 2 and 1
    - columns 3 and 2
    - columns 5 and 4
- When adding $\frac{1}{6} + \frac{7}{8}$, your next step is to
    - set the 1 of $\frac{1}{6}$ in the numerator section
    - change the numerator of $\frac{1}{6}$ and set it in the numerator section
    - set the second fraction, $\frac{7}{8}$
- Find $\frac{1}{6} + \frac{7}{8}$
    - $1\frac{1}{8}$

- $\frac{8}{14}$
- $1\frac{1}{24}$

Questions 6-8 refer to the problem $6\frac{2}{3}+8\frac{3}{4}$

- Your first step is to
    - set the first mixed number, $6\frac{2}{3}$ on your abacus
    - add the whole numbers, 6 and 8, in the far left section
    - set the common denominator in the far right section
- When adding $6\frac{2}{3}+8\frac{3}{4}$, your next step is to
    - set the 6 of $6\frac{2}{3}$ in the whole-number section
    - add the two numerators, 2 and 3
    - change $\frac{2}{3}$ to the common denominator and set the resulting fraction
- When adding $6\frac{2}{3}+8\frac{3}{4}$, you
    - simplify the answer to get $14\frac{17}{12}$
    - use the indirect method of addition three times
    - simplify the answer to get $15\frac{5}{12}$

Questions 9-12 refer to the problem $4\frac{3}{14}-\frac{6}{7}$

- Change $\frac{6}{7}$ to
    - $\frac{12}{14}$
    - $\frac{14}{14}$
    - $\frac{6}{14}$
- To subtract $\frac{6}{7}$ from $4\frac{3}{14}$, you must borrow 1. To do this,
    - change 4 to 3 in the whole-number section
    - add 14 to the 3 in the numerator section
    - change the borrowed 1 to $\frac{14}{14}$
- To subtract $\frac{6}{7}$ from $4\frac{3}{14}$, you actually subtract
    - 6 from 14.
    - 12 from 17.
    - 6 from 17.
- Find $4\frac{3}{14}-\frac{6}{7}$.
    - $3\frac{5}{14}$
    - $3\frac{11}{14}$
    - $4\frac{5}{14}$

Questions 13-15 refer to the problem $5\frac{7}{9}-1\frac{8}{9}$

- To work this problem, you must borrow
    - 1 from the whole number section.
    - 1 from the numerator section.
    - 2 from the whole number section.
- When subtracting $1\frac{8}{9}$ from $5\frac{7}{9}$, you
    - change the borrowed 1 to $\frac{8}{8}$
    - change the borrowed 1 to $\frac{9}{9}$.
    - add 9 to the numerator section.
- Find $5\frac{7}{9}-1\frac{8}{9}$
    - $4\frac{1}{9}$
    - $3\frac{8}{9}$ $$

    - $4\frac{8}{9}$
- To subtract $\frac{4}{9}$ from $\frac{11}{12}$
    - the common denominator is 36.
    - the common denominator is 18.
    - you must borrow 1.

Questions 17 and 18 refer to the problem $9\frac{3}{7} - \frac{6}{7}$

- To work this problem, you
    - must borrow 1 and add 7 to the numerator section.
    - subtract 6 from 9.
    - subtract 6 from 10.
- Find $9\frac{3}{7} - \frac{6}{7}$
    - $6\frac{3}{7}$
    - $8\frac{4}{7}$
    - $9\frac{3}{7}$
- Work the problem $1\frac{4}{5} + 3\frac{1}{2}$. Explain how you determined the common denominator, where the problem is set, and how you changed each fraction so it has the common denominator. Also, explain how you obtained the whole number part and fraction part of your final answer.

Lesson 2: Multiplying and Dividing Fractions

Lesson 1 described the steps to reduce, add, and subtract fractions. Lesson 2 describes how to multiply fractions, perform cancellation, and divide fractions. Gaining practice in manipulating fractions helps you to perform advanced calculations the abacus way.

Objectives

After completing this lesson, you will be able to use the Cranmer abacus to

- multiply fractions;
- cancel factors; and
- divide fractions.

Multiplying Fractions

When multiplying fractions, you multiply numerator by numerator, and denominator by denominator. For example, in the problem $\frac{2}{3} \times \frac{4}{5}$, multiply the numerators, 2 and 4, and the denominators, 3 and 5. The products give you the answer, $\frac{8}{15}$.

To set up a multiplication problem, divide the abacus into two sections. The left side is for numerators, and the right side is for denominators. In the previous example, $\frac{2}{3} \times \frac{4}{5}$, begin by setting the first fraction, $\frac{2}{3}$. Set the numerator, 2, on column 13. Set the denominator, 3, on column 1.

Now you are ready to set the second fraction, $\frac{4}{5}$. Skip one column to the right of the first numerator, 2, and set the second numerator, 4, on column 11. Then, skip one column to the left of the first denominator, 3, and set 5 on column 3. One unused column separates numerator from numerator, and denominator from denominator.

Now multiply the numerators. Since $2 \times 4 = 08$, set 08 on columns 12 and 11 in the first position. To do this, clear the numerator, 4, then set 8 in its place. Now clear the numerator, 2. This leaves 8 in the numerator section. Next, multiply the denominators. Since $5 \times 3 = 15$, set the product, 15, on columns 2 and 1 in the first position. To do this, clear the denominator, 3, then set 15 in its place. Next, clear the denominator, 5, and the multiplication is finished. Since 8 is in the numerator section and 15 is in the denominator section, the product is $\frac{8}{15}$.

When a multiplication problem involves a mixed number (i.e., a whole number and a fraction), first change the mixed number to an improper fraction (i.e., a fraction in which the numerator is larger than the denominator). Do this by multiplying the whole number of the fraction by its denominator; then, add this product to the numerator. For example, $3\frac{3}{4} = \frac{15}{4}$ and $9\frac{1}{2} = \frac{19}{2}$.

Practice Exercises

Practice changing mixed numbers to improper fractions on the abacus. Record your answers so that you can compare them with those that follow the exercise.

- $4\frac{2}{3}$
- $6\frac{1}{2}$
- $5\frac{3}{7}$
- $2\frac{3}{4}$
- $1\frac{6}{7}$

Answers

Compare your answers with those that follow:

- $\frac{14}{3}$
- $\frac{13}{2}$
- $\frac{38}{7}$
- $\frac{11}{4}$
- $\frac{13}{7}$

Canceling Factors

When you cancel factors, you divide out a common factor from a numerator and a denominator. For example, multiply $\frac{15}{9}$ by $3\frac{3}{4}$. Change the mixed numbers $1\frac{5}{9}$ and $3\frac{3}{4}$ to improper fractions. Do this by multiplying each whole number by its denominator, then adding the product to the numerator. In this example, $1\frac{5}{9}$. equals $\frac{14}{9}$. Set the numerator, 14, on columns 13 and 12 and 9 the denominator, 9, on column 1. Next, change $3\frac{3}{4}$ to $1\frac{14}{4}$. Skip one column immediately right of the first numerator, 14, and set the second numerator, 15, on columns 10 and 9. Skip one column immediately left of the denominator, 9, and set the second denominator, 4, on column 3. Now the problem is set.

You are now ready to cancel common factors. Place your left hand on the first numerator, 14, while your right hand checks the left denominator, 4. Since 14 and 4 have the common factor, 2, divide 4 by 2. Use your right hand to change the denominator, 4, to the quotient, 2. Then divide the numerator, 14, by 2, using your right hand to change 14 to the quotient, 7. Set the 7 on column 12. Next place your left hand on this 7, while your right hand checks the denominator, 9. Because 7 and 9 have no common factors, you cannot cancel. Move your left hand to the second numerator, 15, while your right hand checks the left denominator, 2. Because 15 and 2 have no common factors, move your right hand to the second denominator, 9. The common factor for 15 and 9 is 3. Since $9 \div 3 = 3$, use your right hand to change 9 to 3. Then move your right hand to the second numerator, 15. Since $15 \div 3 = 5$, change 15 to 5, and set the 5 on column 9. You now have 7 and 5 in the numerator section, and 2 and 3 in the denominator section.

You are now ready to multiply the numerators. As you say "7 times 5 equals 35," clear 5 and set 35 on the two columns immediately to the right of 7 (i.e., columns 11 and 10). Then clear 7. This leaves 35 in the numerator section. Next multiply the denominators, 2 and 3. As you say "2 times 3 equals 06," clear 3 and set 6 in its place on column 1. Then clear 2. Your answer is $\frac{35}{6}$.

Since $\frac{35}{6}$ is an improper fraction, change it to a mixed number. Do this by dividing the numerator, 35, by the denominator, 6. The quotient is 5. Set this 5 immediately left of 35 on column 12 in keeping with the rule for quotient figure placement that was first described in "Abacus I," Lesson 11 (e.g., **SAME, SKIP**, or **NOT THE SAME, DON'T SKIP**). Then multiply 6 by 5 and subtract 30 from 35. Your answer is $5\frac{5}{6}$.

Suppose you want to multiply fractions that contain numbers having two or more digits, such as $\frac{12}{37} \times \frac{12}{25}$. Begin by setting the first fraction, $\frac{12}{37}$. Set 12 to the far left on columns 13 and 12, and 37 to the far right on columns 2 and 1. To set the second fraction, $\frac{18}{25}$, skip one column to the right of 12 and set 18 on columns 10 and 9. Skip one column to the left of 37, and set 25 on columns 5 and 4.

There are no common factors to cancel, so you are ready to multiply the numerators, 12 and 18. Since 18 acts as the multiplier, clear 18 and retain it in memory. Now multiply 18 by the 2 of 12. Since $1 \times 2 = 02$, set 02 in the first position on the two columns immediately right of 12, columns 11 and 10. Then multiply 8 by 2. Set the product, 16, in the second position on the second and third columns to the right of 12, columns 10 and 9. Now clear the 2 of 12. The partial product is 36. Then multiply 18 by the 1 of 12. Since $1 \times 1 = 01$, set it in the first position on the two columns immediately right of 1, columns 12 and 11. Then multiply 8 by 1. Set 08 in the second position on the second and third columns to the right of 1, columns 11 and 10. To add 8 to the 3 already on column 10, clear 2 and set 1 left. Now clear the 1 of 12, and the multiplication is done. You now have 216 in the numerator section.

Now you are ready to multiply the denominators, 25 and 37. Since 37 acts as the multiplier, clear 37 and retain it in memory. Now multiply 37 by the 5 of 25. Set 15 in the first position and 35 in the second position. Then clear the 5 of 25. The partial product is 185. Next multiply 37 by the 2 of 25. Set 06 in the first position, adding 6 to 1. Set 14 in the second position, adding 1 to 7 and 4 to 8. To add 4 to 8, clear 6 and set 1 left. Next clear the 2 of 25, and the job is done. Your answer is $\frac{216}{925}$.

Practice Exercises

Practice multiplication of fractions and mixed numbers on the abacus by working the following problems. Make sure to cancel factors when necessary. Record your answers so that you can compare them with the ones that follow the exercise. If your answer is an improper fraction, make sure to change it to a mixed number.

- $\frac{3}{4} \times \frac{1}{4}$
- $\frac{3}{5} \times \frac{7}{9}$
- $\frac{6}{7} \times 1\frac{1}{2}$
- $3\frac{3}{4} \times \frac{7}{10}$
- $5\frac{2}{3} \times 1\frac{3}{8}$
- $2\frac{3}{4} \times 2\frac{2}{3}$
- $\frac{4}{21} \times \frac{14}{29}$
- $\frac{7}{8} \times \frac{4}{5}$
- $3\frac{4}{5} \times 6\frac{7}{8}$
- $\frac{6}{7} \times \frac{5}{9}$
- $3\frac{3}{8} \times \frac{7}{18}$

- $\frac{12}{27} \times \frac{6}{15}$
- $2\frac{1}{3} \times 3\frac{3}{4}$
- $2\frac{2}{3} \times 2\frac{1}{12}$
- $\frac{3}{14} \times \frac{7}{9}$

Answers

Compare your answers with those that follow:

- $\frac{3}{16}$
- $\frac{7}{15}$
- $1\frac{2}{7}$
- $2\frac{5}{8}$
- $7\frac{19}{24}$
- $7\frac{1}{3}$
- $\frac{8}{87}$
- $\frac{7}{10}$
- $26\frac{1}{8}$
- $\frac{10}{21}$
- $1\frac{5}{16}$
- $\frac{8}{45}$
- $8\frac{3}{4}$
- $5\frac{5}{9}$
- $\frac{1}{6}$

Dividing Fractions

Division is the inverse operation of multiplication. The steps for dividing fractions are almost the same as for multiplying them. First, set the first fraction. Then, invert the divisor. Finally, multiply numerator by numerator, and denominator by denominator. The inverted divisor is called the reciprocal of the divisor. Thus, when dividing fractions, you are actually multiplying the dividend by the reciprocal of the divisor. To divide $\frac{2}{3}$ by $\frac{3}{4}$ you invert the divisor $\frac{3}{4}$, by changing it to $\frac{4}{3}$. Then you multiply $\frac{2}{3}$ by $\frac{4}{3}$. The quotient is $\frac{8}{9}$.

For example, divide $2\frac{1}{4}$ by $1\frac{1}{5}$. Change the dividend, $2\frac{1}{4}$, to the improper fraction $\frac{9}{4}$. Set 9 to the far left on column 13 and 4 to the far right on column 1. Next, change the divisor, $1\frac{1}{5}$, to the improper fraction $\frac{6}{5}$.

Recall that when multiplying fractions you skip one column to the right of the first numerator and set the second numerator. Then, you skip one column to the left of the first denominator and set the second denominator. Since dividing fractions is the inverse of multiplying them, do just the opposite. In the problem, $2\frac{1}{4} \div 1\frac{1}{5}$, after your right hand has set the first denominator, 4, skip one column to the left. Set the numerator, 6, of $\frac{6}{5}$ on column 3 in the denominator section. Then move to the numerator section. Skip one column to the right of the first numerator, 9, and set the 5 of $\frac{6}{5}$ on column 11. By reversing the direction in which you enter the second fraction, you have inverted it, changing $\frac{6}{5}$ to its reciprocal, $\frac{5}{6}$. You have set 5 in the numerator section and 6 in the denominator section. Now, multiply as usual. Cancel common factors: divide 9 and 6 by 3, changing 9 to 3, and 6 to 2. No other factors can be canceled, so multiply the numerators, 3 and 5. Since $3 \times 5 = 15$, set 15 on columns 12 and 11 in the first position. To do this, clear the numerator, 5, and set 15 in its place. Now clear the numerator, 3.

Next multiply the denominators, 2 and 4. Since $2 \times 4 = 08$, set the product, 08, in the first position on columns 2 and 1. To do this, clear the denominator, 4, and set 08 in its place. Next clear the denominator, 2, and the multiplication is finished. The answer is $\frac{15}{8}$. Since $\frac{15}{8}$ is an improper fraction, divide 15 by 8. Set the quotient, 1, immediately left of 15. Then, multiply 8 by 1, and subtract 08 from $1\ 5$. The quotient is the mixed number, $1\frac{7}{8}$.

Practice Exercises

Practice dividing fractions and mixed numbers on the abacus by working the following problems. Make sure to cancel when necessary. Record your answers so that you can compare them with those that follow the exercise. If your answer is an improper fraction, change it to a mixed number.

- $\frac{1}{5} \div \frac{1}{4}$
- $\frac{4}{5} \div \frac{3}{7}$
- $\frac{7}{8} \div \frac{9}{16}$
- $\frac{7}{12} \div \frac{3}{4}$
- $2\frac{1}{2} \div \frac{2}{3}$
- $4\frac{1}{2} \div \frac{2}{5}$
- $\frac{5}{6} \div \frac{4}{21}$
- $\frac{7}{8} \div \frac{3}{8}$
- $9\frac{1}{3} \div 4\frac{2}{3}$
- $\frac{9}{10} \div \frac{6}{25}$
- $2\frac{1}{5} \div \frac{9}{10}$
- $\frac{4}{9} \div \frac{3}{4}$
- $1\frac{7}{8} \div 2\frac{1}{4}$
- $\frac{2}{3} \div \frac{4}{9}$
- $2\frac{3}{4} \div \frac{6}{11}$

Answers

Compare your answers with those that follow:

- $\frac{4}{5}$
- $1\frac{13}{15}$
- $1\frac{5}{9}$
- $\frac{7}{9}$
- $3\frac{3}{4}$
- $11\frac{1}{4}$
- $4\frac{3}{8}$
- $2\frac{1}{3}$
- 2
- $3\frac{3}{4}$
- $2\frac{4}{9}$
- $\frac{16}{27}$
- $\frac{5}{6}$
- $1\frac{1}{2}$
- $5\frac{1}{24}$

Assignment 2

Complete the following questions with all the correct answers. Don't assume that only one answer is correct. Sometimes two or three are needed to answer the question correctly. Questions 1-18 are worth 5 points each. Question 19 is worth 10 points.

- When multiplying fractions on the abacus, the abacus is divided into two sections. Which of the following statements is true?
    - The far left section is for the first fraction, and the far right section is for the second fraction.
    - The far left section is for numerators, and the far right section is for denominators.
    - The far left section is for denominators, and the far right section is for numerators.

Questions 2-4 refer to the problem $\frac{3}{4} \times \frac{5}{7}$

- To set up this problem, you first set
    - both of the numerators on the appropriate columns, after which you set both denominators.
    - the first fraction $\frac{3}{4}$, setting 3 on column 13, and 4 on column 1.
    - both of the denominators on the appropriate columns, after which you set both numerators.
- After setting the problem $\frac{3}{4} \times \frac{5}{7}$ on your abacus, the 5of the second fraction, appears on
    - column 3.
    - column 12.
    - column 11.
- Find $\frac{3}{4} \times \frac{5}{7}$
    - $\frac{15}{28}$
    - $\frac{15}{24}$
    - $\frac{21}{20}$

Questions 5-8 refer to the problem $2\frac{1}{4} \times 3\frac{1}{2}$

- Your first step is to
    - find a common denominator.
    - multiply the whole numbers, 2 and 3.
    - change $2\frac{1}{4}$ to $\frac{9}{5}$ and set on the abacus.
- When multiplying $2\frac{1}{4}$ by $3\frac{1}{2}$, your next step is to
    - change $3\frac{1}{2}$ to $\frac{7}{2}$ and set $\frac{7}{2}$ on your abacus.
    - multiply the 1 of $2\frac{1}{4}$ by the 1 of $3\frac{1}{2}$
    - cancel the numerators.
- When multiplying $2\frac{1}{4}$ by $3\frac{1}{2}$, you multiply
    - the numerators, 9 and 7.
    - the numerators, 1 and 1.
    - the denominators, 2 and 4.
- When multiplying $2\frac{1}{4}$ by $3\frac{1}{2}$, the product of the numerators, 9 and 7, is set on
    - columns 13 and 12.
    - columns 12 and 11.
    - columns 2 and 1.

Questions 9-11 refer to the problem $\frac{6}{7} \times \frac{4}{9}$

- To begin this problem, set

- $\frac{6}{7}$ so that 6 is on column 13 and 7 is on column 1.
- $\frac{4}{9}$ so that 4 is on column 11 and 9 is on column 3.
- $\frac{6}{7}$ so that 6 is on column 13 and 7 is on column 11.

- After setting up the problem $\frac{6}{7} \times \frac{4}{9}$ your next step is to
    - multiply 6 by 4.
    - cancel 6 and 4 by 2.
    - cancel 6 and 9 by 3.
- Find $\frac{6}{7} \times \frac{4}{9}$
    - $\frac{8}{21}$
    - $\frac{2}{21}$
    - $\frac{8}{63}$

Questions 12-14 refer to the problem $\frac{7}{67} \times \frac{9}{34}$.

- To work this problem,
    - set 7 on column 13 and 67 on columns 2 and 1. Then set 9 on column 11, and 34 on columns 5 and 4.
    - cancel 7 and 67.
    - set the product of the numerators on columns 12 and 11.
- In the problem $\frac{7}{67} \times \frac{9}{34}$ when you multiply the denominators, 67 and 34,
    - treat 34 as the multiplier, clear it, and retain it in memory.
    - treat 67 as the multiplier, clear it, and retain it in memory.
    - set the products 6×4 and 6×3 in the correct first positions.
- Find the product of $\frac{7}{67}$ and $\frac{9}{34}$
    - $\frac{13}{2287}$
    - $\frac{63}{2278}$
    - $\frac{54}{2178}$

Questions 15-17 refer to the problem $\frac{3}{7} \div \frac{8}{9}$

- To work this problem, set the dividend so that 3 is on column 13 and 7 is on column 1. Then set the divisor so that
    - 8 is on column 11 and 9 is on column 3.
    - 8 is on column 3 and 9 is on column 11.
    - 8 is on column 10 and 9 is on column 4.
- After setting up the problem $\frac{3}{7} \div \frac{8}{9}$, your next step is to
    - cancel 3 and 9 by 3.
    - divide 9 by 3.
    - multiply 3 by 9.
- Find $\frac{3}{7} \div \frac{8}{9}$
    - $\frac{27}{56}$
    - $\frac{24}{63}$
    - $\frac{3}{7}$
- When dividing $1\frac{1}{5}$ by $1\frac{2}{7}$, change $1\frac{1}{5}$ to the improper fraction $\frac{6}{5}$ and $1\frac{2}{7}$ to $\frac{9}{7}$. Set 6 on column 13 and 5 on column 1. $$ To work this problem,
    - Set 9 on column 11 and 7 on column 3.
    - Set 9 on column 3 and 7 on column 11.
    - Your answer is $\frac{14}{15}$.

- Work the problem $1\frac{1}{2}+1\frac{1}{4}$. Explain where you set each numerator and denominator. If you canceled or inverted, explain how and state where each product was set. Also, explain how you obtained the final answer if you had a mixed number.

Lesson 3: Using Percents and Units of Measure

Lesson 1 explained how to add and subtract fractions, while Lesson 2 explained how to multiply and divide them. This lesson describes how to find a percent of a number and how to add and subtract measurements, for example, $5\ yards,\ 2\ feet + 4\ yards,\ 6\ inches$. Practicing calculations with percents and units of measure helps you reach your goal of performing advanced calculations on the abacus.

Objectives

After completing this lesson, you will be able to use the Cranmer abacus to

- find percents; and
- use units of measure in mathematical operations.

Finding Percents

You know that a cent is $\frac{1}{100}$ of a dollar and that "percent" means "per hundred." A percent is really a fraction whose denominator is 100. You can treat a percent as either a fraction or a decimal. You can express any percent as a fraction having the percent as its numerator and 100 as the denominator. You can also change a percent to its decimal form by moving the decimal point two places to the left. For example, you can express 17% as 0.17 or $\frac{17}{100}$; 9% as 0.09 or $\frac{9}{100}$; 99% as 0.99 or $\frac{99}{100}$; and 999% as 9.99 or $9\frac{99}{100}$, and so on.

Suppose that you want to find 12% of 64. You have two alternatives. The first alternative is to treat the problem as a fraction. Since 12% equals $\frac{12}{100}$, which is reduced to $\frac{3}{25}$, multiply $\frac{3}{25}$ by $\frac{64}{1}$. Then enter the numerators, 3 and 64, to the left and the denominators, 25 and 1, to the right. Multiply the numerators: $3\times64=192$. Next multiply the denominators: $1\times25=25$. The product is $\frac{192}{25}$, Finally, divide 192 by 25 and carry out your answer out to two decimal places. The final result is 7.68.

The other, less cumbersome, way is to change 12% to its decimal form, 0.12, then multiply 0.12 by 64. Set 12 on columns 13 and 12, then set 64 on columns 5 and 4. Next multiply 12 by 64, which is 768. Because the total number of decimal places in 0.12 and 64 is two, point off two decimal places in 768. Your answer is 7.68.

Some percents have fractional equivalents that you may already be familiar with, such as the following:

- 10% is $\frac{1}{10}$
- $12\frac{1}{2}\%$ is $\frac{1}{8}$
- 20% is $\frac{1}{5}$
- 25% is $\frac{1}{4}$
- $33\frac{1}{3}\%$ is $\frac{1}{3}$
- 50% is $\frac{1}{2}$
- $66\frac{2}{3}\%$ is $\frac{2}{3}$
- 75% is $\frac{3}{4}$

It is easier and quicker to treat these percents as fractions rather than decimals.

Practice Exercises

Practice using percents as fractions and decimals by working the following calculations. Record your answers so that you can compare them with those that follow the exercise.

- Find 25% of 176. Treat 25% as a fraction.
- Find 29% of 78. Treat 29% as a decimal and carry your answer out to two decimal places.
- Find 16% of 84. Treat 16% as a decimal and carry your answer out to two decimal places.
- Find $66\frac{2}{3}\%$ of 279. Treat $66\frac{2}{3}$ as a fraction.
- Find 3% of $1,092$. Treat 3% as a decimal and carry your answer out to two decimal places.
- Find 20% of 362. Treat 20% as a fraction and carry your answer out to one decimal place.

Answers

Compare your answers with those that follow:

- 44
- 22.62
- 13.44
- 186
- 32.76
- 72.4

Using Units of Measure

In the measurement $6~feet,~5~inches$ 16 and 5 are quantities. Feet and inches are units of measure or just "units." When adding and subtracting measurements with units like yards, feet, inches, or degrees, minutes, seconds, divide the abacus into three sections as you did when adding and subtracting fractions. Each section contains three columns. The unit marks divide the abacus into sections. Set the quantity of the largest unit (yards or degrees) in the left section, the quantity of the next unit (feet or minutes) in the middle section, and the quantity of the smallest unit (inches or seconds) in the right section.

For example, add $5~yards,~1~foot,~6~inches$ to $12~yards,~2~feet,~9~inches$. Begin by dividing the abacus into three sections for yards, feet, and inches. Then, set 12 in the yards section; that is, on the two columns immediately left of the second unit mark from the right, columns 8 and 7. Next, set 2 in the feet section on the column immediately left of the first unit mark, column 4. Finally, set 9 in the inches section on the column appearing at the far right of the abacus, column 1.

You are now ready to add 5 yards, 1 foot, 6 inches. Start with the largest unit of measure. Add 5 to the 12 in the yards section, 1 to the 2 in the feet section, and 6 to the 9 in the inches section. You now have 17 in the yards section, 3 in the feet section, and 15 in the inches section.

Just as you must simplify your answers in fractions, you must also simplify your answer if they are measurements. In the answer 17 yards, 3 feet, 15 inches, some measurements are not in simplest form. Because there are 12 inches in a foot, the quantity, 15 inches, is too large. Also, because there are 3 feet in a yard, the quantity, 3 feet, is too large. To simplify your answer, start with the smallest unit of measure—inches—the section on the far right. Since 12 inches equal 1 foot, divide 15 in the inches section by 12. Set the quotient, 1, in the feet section, adding it to 3. Next, multiply this 1 by 12. Subtract the product, 12, from the 15 in the inches section. Now you have 3 in the inches section and 4 in the feet section. Since 3 feet equal 1 yard, divide 4 feet by 3. Set the quotient, 1, in the yards section, adding it to 17. Then multiply this 1 by 3. Subtract the product, 3, from 4 in the feet section. The job is done. Your answer is 18 yards, 1 foot, 3 inches.

This time, subtract $6~yards,~2~feet,~8~inches$ from $28~yards,~1~foot,~7~inches$. Begin by setting 28 in the yards section on the two columns immediately left of the second unit mark, columns 8 and 7. Then set 1 in the feet section on the column immediately left of the first unit mark, column 4. Finally, set 7 in the inches section on the far right column of the abacus, column 1. Subtract 6 from 28, leaving 22 in the yards section.

You are now ready to subtract 2 from the feet section. You cannot subtract 2 from the 1 in that section. Instead, borrow 1 from the yards section, changing 22 to 21. This borrowed 1 yard equals 3 feet, so add 3 to the feet section. You now have 4 in the feet section, so you can subtract 2 feet, which leaves 2 in the feet section.

You are now ready to subtract 8 from the inches section. You cannot subtract 8 from the 7 already in that section, so borrow or subtract 1 from the 2 in the feet section. Change this borrowed 1 foot to 12 inches, and add 12 to the 7 in the inches section. You now have 19 in the inches section, so you can subtract 8 from 19. The job is now finished. The answer is 21 yards, 1 foot, and 11 inches.

Practice Exercises

Practice using units of measure on the abacus by working the following problems. Record your answers so that you can compare them with those that follow the exercise.

- $2~feet,~6~inches+5~yards,~1~foot,~3~inches$
- $34~degrees,~25~minutes,~9~seconds-6~ degrees,~7~minutes,~8~seconds$ (Note: There are 60 seconds in a minute; 60 minutes in a degree.)
- $14~yards,~2~feet,~7~inches-4~feet,~9~inches$
- $12~degrees,~16~minutes,+27~minutes,~42~ seconds$
- $8~yards,~1~foot,~11~inches+2~feet,~3~inches$

Answers

Compare your answers with those that follow:

- $46~yards,~9 inches$
- $28~degrees,~18~minutes,~1~second$
- $13~yards,~10~inches$
- $12~degrees,~43~minutes,~42~seconds$
- $9~yards,~1~foot,~2~inches$

Assignment 3

Complete the following questions with all the correct answers. Don't assume that only one answer is correct. Sometimes two or three choices are needed to answer the question correctly. Questions 1-10 are worth 10 points each.

- Find $12\frac{1}{2}\%$ of 99. Treat $12\frac{1}{2}\%$ as the fraction, $\frac{1}{8}$ and carry your answer out to three decimal places.
    - 11.888
    - 12.384
    - 12.375

- Find 11% of 524. Treat 11% as a decimal and carry your answer out to two decimal places.
    - 57.64
    - 576.40
    - 5.76
- Find 75% of 876. Treat 75% as a fraction or a decimal.
    - 627
    - 332
    - 657

Questions 4 and 5 refer to the problem $4~yards,~2~feet,~9~inches+3~yards,~1~foot,~7~inches$.

- You first add
    - 3 to 4.
    - 1 to 2.
    - 7 to 9.
- Add $4~ yards, 2~ feet, 9~ inches+3~ yards, 1~ foot, 7~ inches$. The answer, in simplest form, is
    - $7~yards, ~3 ~feet, ~16 ~inches$
    - $8 ~yards, ~1 ~foot, ~4 ~inches$
    - $7 ~yards, ~4 ~feet, ~4 ~inches$

Questions 6 and 7 refer to the problem $9~yards,~1~foot-1~yard,~2~feet,~3~inches$.

- When you subtract 2 feet from the feet section, you must borrow 1 from the yards section. After borrowing this 1 and returning the required number to the feet section, the abacus shows
    - $8~yards,~4~feet$.
    - $7~yards,~4~feet$.
    - $7~yards,~5~feet$.
- Continue to work the problem $9~yards,~1~foot-1~yard,~2~feet,~3~inches$. When you subtract 3 inches from the inches section, you must borrow 1 from the feet section. After borrowing this 1 and adding the required number to the inches section, the abacus shows
    - $8~yards,~2~feet,~12~inches$.
    - $7~yards,~3~feet,~12~inches$.
    - $7~yards,~1~foot,~12~inches$.

Questions 8 and 9 refer to the problem $8~degrees,~3~seconds+11~minutes,~27~seconds$.

- You must
    - set 8 on column 4.
    - set 8 on column 7.
    - set 3 on column 1.
- Add: $8~degrees,~3~seconds+11~minutes,~27~seconds$.
    - $8~degrees,~11~minutes,~30~seconds$
    - $19~degrees,~27~minutes,~3~seconds$
    - $19~degrees,~30~seconds$
- Work the problem $17~yards,~7~inches-4~yards,~2~feet,~8~inches$. Explain where the problem is set and how each subtraction is made, including borrowing. Explain how you simplified the answer.

Lesson 4: Extracting Square Roots

Lesson 1 explained adding and subtracting fractions, while Lesson 2 explained multiplying and dividing them. Lesson 3 described how to find percents and work with units of measure. This lesson explains how to use the abacus to extract square roots (i.e., a factor of a number that when squared gives the number; e.g., the square root of 9 is 3.) Extracting square roots will help you achieve your goal of performing advanced calculations the abacus way.

Objectives

After completing this lesson, you will be able to use the Cranmer abacus to

- extract square roots of whole and decimal numbers that are perfect squares; and
- extract square roots of whole numbers that are not perfect squares.

Extracting Square Roots of Perfect Squares

You can find the square root of a number much like you do long division, except that you either halve the dividend or double the quotient. This lesson describes the method that doubles the quotient because it is easier to use.

When you extract the square root of a number on the abacus, the divisor changes with each round. Begin by setting the number at the far right end of the abacus. Separate its digits into imaginary groups of two digits each, from right to left, starting at the decimal point. If the number contains an odd number of digits, the group at the left end will have only one digit. For example, you separate the digits in 525 into the two groups, 5 and 25. You separate the digits in $16,489$ into three groups, 1, 64, and 89. The square root has one digit for each group. So the square root of 525 will have 2 digits, and the square root of $16,489$ will have 3.

In long division, you find the digits of the quotient one by one. When you extract a root, you also find the digits one by one. First find the largest perfect square that is smaller than the first group at the left of the number. Set its square root to the left of the number as if the root were a quotient in division. Use the rule for quotient figure placement. This is the first digit of the square root you are seeking.

Next, square this first quotient figure and subtract its product from the dividend in keeping with the rules of positioning. Then double the first quotient figure. This will be your second divisor.

Proceed as in long division—dividing, multiplying, and subtracting. Divide the dividend by the doubled quotient, and set the resulting quotient figure on the appropriate column to the left of the dividend. (Note: This will be the second digit of the square root you are seeking.) Then multiply the doubled first quotient figure and the second quotient figure by the second quotient figure. (That is, first multiply the doubled first quotient figure by the second quotient figure. Then multiply the second quotient figure by itself.) Subtract the resulting products from the dividend. Then double the second quotient figure.

For the third and any subsequent rounds of division, continue in this manner. That is, divide the dividend by the entire doubled quotient. Multiply the doubled quotient and the newest quotient figure by the newest quotient figure. (That is, first multiply all doubled quotient figures by the newest quotient figure. Then multiply the newest quotient figure by itself.) Subtract the resulting products from the dividend. Then double the newest quotient figure until the dividend is cleared or until you have carried the quotient to as many decimal places as you wish. If the dividend is cleared, then the number whose root you are extracting is a perfect square.

As a final step, halve each quotient figure—except the final one because it was not doubled. Now you have the square root that you are seeking. To verify your calculations, square the square root that you have extracted. You should find the number you started with and whose root you wanted to extract.

For example, extract the square root of $2,209$. Set $2,209$ on columns 4, 3, 2, and 1, separating the digits in your mind into two groups, 22 and 09. Find the largest perfect square in the first group, 22. It is 16, and the square root of 16 is 4. Set 4 immediately to the left of 22, keeping with the rules for quotient figure placement (**NOT THE SAME, DO NOT SKIP**). Now square 4 and subtract 16 from 22. This leaves 609 in the dividend.

For your second round, double the first quotient figure, 4, changing it to 8 on the abacus, and divide the 60 of 609 by 8. Set the quotient figure, 7, immediately left of 609, in keeping with the rules for quotient figure placement (**NOT THE SAME, DO NOT SKIP**). Multiply 87 by 7. To do this, subtract 56 from 60 and 49 from 49. This clears the dividend, so you know that $2,209$ is a perfect square. As a final step, halve the 8 of 87 (since it was the only quotient figure that was doubled) by changing 8 back to 4. The square root of $2,209$ is 47. To verify the result, square 47. You get $2,209$.

For another example, extract the square root of 676. Set 676 on columns 3, 2, and 1, and mentally separate it into two groups, 6 and 76. The first group has only one digit, 6. The largest perfect square in 6 is 4; the square root of 4 is 2. Set 2 on the second column to the left of 6, in keeping with the rules for quotient figure placement (**SAME, SKIP**). Square 2, and subtract 04 from 06, leaving 276 in the dividend. Now double 2, changing it to 4 on the abacus, and divide the 27 of 276 by 4. Set the resulting quotient figure, 6, immediately left of 27, in keeping with the rules for quotient figure placement (**NOT THE SAME, DO NOT SKIP**). Then multiply 46 by 6. Subtract 24 from 27 and 36 from 36, which clears the dividend. You see that 676 is a perfect square. Finally halve the 4 of 46 (since it was the only quotient figure that was doubled). The square root of 676 is 26.

As you have already seen, extracting square roots is done in much the same way as long division. When extracting roots, however, you do not have a divisor in the first round of division. Instead, you use the square root of the largest perfect square that is in the first group. Because of this, an exception occurs whereby the first quotient figure may not always be placed according to the rule for quotient figure placement. The exception is that whenever you find that the first quotient figure is 5 or more, enter that first quotient figure one column farther to the left than it normally would be placed. Still make the first subtraction from the two digits in the first group. Thereafter, apply the original rule for quotient figure placement and the rules for positioning.

For example, extract the square root of $8,649$. As you set $8,649$ on columns 4, 3,2, and 1, separate it mentally into two groups, 86 and 49. The largest perfect square in 86 is 81, and 9 is the square root of 81. If you use the rule for quotient figure placement, you would set this 9 on the column immediately left of 86 (**NOT THE SAME, DO NOT SKIP**). However, because the first quotient figure is 9, apply the exception and move one column farther to the left. That is, place 9 on the second column to the left of 86, column 6.

Next, square 9 and subtract 81 from the first group, that is, from 86. This leaves 549 in the dividend. Now double 9 by clearing 1 on column 6 and setting 1 left on column 7. Then divide the 54 of 549 by 18, resulting in 3. Set this 3 on the second column to the left of 54 (**SAME, SKIP**). Next, multiply 183 by 3. Subtract 03 from 05, 24 from 24, and 09 from 09. This clears the dividend, so $8,649$ is a perfect square. Finally, halve the 18 of 183 since it was the only quotient figure that was doubled, by changing 18 to 9. The square root of $8,649$ is 93.

Note that you divided the 54 of 549 by the two-digit divisor, 18. Normally, you would divide by a one-digit divisor. However, since the original divisor, 9, was doubled to 18, for division purposes only, this 18 is thought of as a single unit. But when comparing the digits in the divisor with those in the dividend to determine where to set the quotient figure, the 1 8 is still considered a two-digit divisor.

Now do another example. Extract the square root of the six-digit number $455,625$. Set the dividend on columns 6, 5, 4, 3, 2, and 1, separating it mentally into three groups, 45, 56, and 25. Since there are three groups, there are three digits in the square root. The largest perfect square in 45 is 36, and the square root of 36 is 6. Normally, this 6 would be set immediately left of 45 (**NOT THE SAME, DO NOT SKIP**). Notice the first quotient figure is 6, which is 5 or more, so the exception applies. Move one column farther to the left. That is, set 6 on the second column to the left of 45, column 8.

Then square 6 and subtract 36 from 45, leaving $95,625$ in the dividend. After doubling 6 by setting 1, clearing 5 on column 8, and setting 1 left on column 9, divide 95 by 12. Set the resulting quotient figure, 7, on the second column to the left of 95 (**SAME, SKIP**). Remember, this is the second quotient figure and the exception applies only to the first. Then, multiply 127 by 7. Subtract 07 from 09, 14 from 25, and 49 from 16. Now the dividend contains $6,725$. Now double the 7 by setting 2 and clearing 5 on column 7, and setting 1 left on column 8. You now have 134 on columns 9, 8, and 7. Next, divide 672 by 134, and set the quotient figure, 5, on the second column to the left, column 6 (**SAME, SKIP**). Multiply 1345 by 5 and subtract 05 from 06, 15 from 17, 20 from 22, and 25 from 25, thereby clearing the dividend. Now halve the 134 of the 1345 because 134 came from doubling. The square root of $455,625$ is 675.

Note that you had a two-digit divisor, 12, and a three-digit divisor, 134, because you doubled the single digit divisors. When you determine where to set the quotient figures, apply the rules (**SAME, SKIP**, or **NOT THE SAME, DON'T SKIP**) to the numbers you were actually dividing by, 12 and 134, not to the quotient figures that had not yet been doubled. Sometimes, when finding square roots on the abacus, you must use downward correction. See the next example.

Extract the square root of $1,444$. Set $1,444$ on columns 4, 3, 2, and 1, and mentally separate it into two groups, 14 and 44. The largest perfect square in 14 is 9, and 3 is the square root of 9. Set 3 immediately left of 14 (**NOT THE SAME, DO NOT SKIP**). Square 3. Subtract 09 from 14, leaving 544 in the dividend. Next, double 3 to get 6 and set on the abacus. Divide 54 by 6. The quotient is 9. Set 9 immediately left (**NOT THE SAME, DO NOT SKIP**). Multiply 69 by 9 and subtract 54 from 54. You cannot subtract 81 from 04, so the quotient, 9, is too high. Reduce it by 1, changing 9 to 8, and return 06 in the first position, that is, in columns 3 and 2. Your dividend now shows 64. Now multiply the reduced quotient, 8, by 8 and subtract 64 from 64. This clears the dividend. Finally, halve the 6 of 68 since it was the only quotient figure that was doubled. The square root of $1,444$ is 38.

When finding square roots on the abacus, upward correction cannot be used. Here is an example that shows the alternative to upward correction. Extract the square root of $2,401$. Set $2,401$ on columns 4, 3, 2, and 1 and mentally separate it into two groups, 24 and 01. The largest perfect square in 24 is 16, and the square root of 16 is 4. Set 4 immediately left of 24 (**NOT THE SAME, DO NOT SKIP**). Square 4. Subtract 16 from 24, leaving 801 in the dividend. Now double the 4 in column 5 to get 8. Divide the 8 of 801 by 8 to get 1.

Normally, this 1 would be set on the second column to the left of 801, column 5 (**SAME, SKIP**). But column 5 already contains 8. Two quotient figures would be set on the same column (i.e., upward correction). Upward correction cannot be used when working square root problems on the abacus. Thus, an exception occurs. When a quotient figure is to be set on a column where one has already been placed, immediately change the newest quotient figure to 9. Then, set this 9 immediately to the left of the dividend.

Therefore, in this example, do not set 1 on column 5 but set 9 immediately left of 801, on column 4. The quotient figure is now 89. Now multiply 89 by 9. Subtract 72 from 80, and 81 from 81. This clears the dividend, so once again you had a perfect square. Finally, halve the 8 of 89, since it was the only quotient figure that was doubled. The square root of $2,401$ is 49.

Doubling can be done at the far left end of your abacus or on a separate abacus. This way, the quotient or actual square root will not be disturbed as it is being developed. You will not need to halve the quotient. For example, extract the square root of 6.25. Set 6 and 25. There will be as many whole digits to the left of the decimal point in the square root as there are groups to the left of the decimal point in 6.25. There will be as many decimal digits to the right of the decimal point in the square root as there are groups to the right of the decimal point in 6.25. Since 4 is the largest perfect square in 6, and 2 is the square root of 4, set 2 on the second column to the left of 6 (**SAME, SKIP**) and also on column 13. Then square 2 and subtract 04 from 06. This leaves 225 in the dividend.

Next, double the 2 on column 13 of your abacus, but do not disturb the 2 on column 5. Then, divide the 22 of 225 by 4. Set the quotient figure, 5, immediately left of 22 (**NOT THE SAME, DO NOT SKIP**) and also on column 12. You now have 45 on columns 13 and 12.

Next, multiply 45 by 5. Subtract 20 from 22, and 25 from 25, thereby clearing the dividend. Now clear 45 from the far left side. You do not need to halve the quotient, 25, because it was never doubled. Because there was one group in 6.25 to the left of the decimal point, the square root has one digit to the left of the decimal point. Because there was one group in 6.25 to the right of the decimal point, the square root has one digit to the right of the decimal point. Therefore, the square root of 6.25 is 2.5.

Practice Exercises

Extract the square roots of these numbers. Record your answers so that you can compare them with those that follow the exercise.

- $9,216$
- $12,321$
- $18,671,041$
- $585,225$
- 324
- $4,937,284$
- $514,089$
- $2,226,064$
- $2,809$
- $54,756$
- $95,481$
- $1,616,522,436$
- $3,611,168,649$
- $79,524$
- 151.29

Answers

Compare your answers with those that follow:

- 96
- 111
- $4,321$
- 765

- 18
- $2,222$
- 717
- $1,492$
- 53
- 234
- $3094
- $40,206$
- $60,093$
- 282
- 12.3

Extracting the Square Root of a Number That Is Not a Perfect Square

Most numbers are not perfect squares. This section describes how to find their square roots. Extract the square root of 55 and carry the root out to two decimal places. To do so, leave one group of two zeros each for each decimal place you intend to carry the root—in this case, two groups. You will be extracting the square root of 55.0000.

Set 550000 on columns 6, 5, 4, 3, 2, and 1, and mentally separate it into three groups, 55, 00, and 00. There will be as many whole digits to the left of the decimal point in the square root as there are groups to the left of the decimal point in 55.0000. In this case, there is one group. There will be as many decimal digits to the right of the decimal point in the square root as there are groups to the right of the decimal point. In this case, there are two groups.

The largest perfect square in 55 is 49, and 7 is its square root. Using the rule for quotient figure placement, you would set the 7 immediately left (**NOT THE SAME, DO NOT SKIP**). But because the quotient figure is 7, remember the exception to the rule and skip one column to the left. Set 7 on column 8 and also on column 12. Leave column 13 for doubling. Then, square 7 and subtract 49 from 55. This leaves 60000 in the dividend. Now double the 7 on column 12, by setting 2 and clearing 5 on column 12, and setting 1 left on column 13. Divide 60 by 14. The quotient is 4. Set this 4 on the second column to the left of 60, column 7 (**SAME, SKIP**), and also on column 11. You now have 144 on columns 13, 12, and 11.

Next, multiply 144 by 4. Subtract 04 from 06, 16 from 20, and 16 from 40. This leaves 2400 in the dividend. Next, double the 4 on column 11. You now have 148 on columns 13, 12, and 11. Divide 240 by 148. Set the quotient figure, 1, on the second column to the left of 2400 (i.e., column 6 **SAME, SKIP**) and also on column 10. You now have 1481 on columns 13, 12, 11, and 10.

Next, multiply 1481 by 1. Subtract 01 from 02, 04 from 14, 08 from 00, and 01 from 20. The dividend, 919, is still not cleared, but you can go no further because you cannot divide 919 by 1481. Clear 1481 from the far left. The quotient now shows 741 with a remainder of 919. Since there is one group in 55.0000 to the left of its decimal point there is one whole number in its square root. So, the square root of 55 is 7.41 with a remainder of 919.

Practice Exercises

Practice extracting the square root of the following numbers, which are not perfect squares. Record your answers so that you can compare them with those that follow the exercise.

- $7,598$
- 846
- $416,403$

- $9,114,899$
- 50 carried out to 2 decimal places

Answers

Compare your answers with those that follow:

- $87~remainder~29$
- $29~remainder~5$
- $645~remainder~378$
- $3,019~remainder~538$
- $7.07~remainder~151$

Assignment 4

Complete the following questions with all the correct answers. Don't assume that only one answer is correct. Sometimes two or three are needed to answer the question correctly. Questions 1-18 are worth 5 points each. Question 19 is worth 10 points.

Questions 1-4 refer to the problem of extracting the square root of $86,436$.

- How many digits are in the square root of $86,436$?
  - One
  - Two
  - Three
- When extracting the square root of $86,436$, the largest perfect square in the first group is
  - 9
  - $4.$
  - $2.$
- When extracting the square root of $86,436$, the first quotient figure is
  - 2, set on the column immediately left of $86,436$.
  - 2, set on the second column to the left of $86,436$.
  - 4, set on the second column to the left of $86,436$.
- Find the square root of $86,436$.
  - 294
  - 584
  - 494

Questions 5-7 refer to the problem of extracting the square root of $1,156$.

- After the first complete round ending in subtraction, what is the dividend?
  - 556
  - 856
  - 256
- As you extract the square root of $1,156$, your next step after the first round is to divide
  - 6 into $25.$
  - 3 into $25.$
  - 9 into $25.$
- Find the square root of $1,156$.
  - 34$
  - 64
  - 68

Questions 8-10 refer to the problem of extracting the square root of $7,421$.

- Your first quotient figure is
    - 8, set on the second column left of the dividend.
    - 7, set on the second column left of the dividend.
    - 8, set immediately left of the dividend.
- When extracting the square root of $7,421$, the second quotient figure is
    - 6, set on the column immediately left of the dividend.
    - 6, set on the second column left of the dividend.
    - 1, set immediately left of the dividend.
- Find the square root of $7,421$.
    - $166~ remainder~ 25$
    - $86~ remainder~ 25$
    - $83~ remainder~ 25$

Questions 11 and 12 refer to the problem of extracting the square root of $16,652.2436$.

- How many whole number digits are in its square root?
    - Three
    - Four
    - Five
- When extracting the square root of $16,652.2436$, the first quotient figure is
    - 1, set on the second column to the left of the dividend.
    - 4, set on the column immediately to the left of the dividend.
    - 3, set on the column immediately to the left of the dividend.

Questions 13 and 14 refer to the problem of extracting the square root of $1,681$.

- The first quotient figure is
    - 1, set immediately to the left of $1,681$.
    - 4, set on the second column to the left of $1,681$.
    - 4, set on the column immediately left of $1,681$.
- Find the square root of $1,681$.
    - 81
    - 41
    - 401

Questions 15 and 16 refer to the problem of extracting the square root of $7,666$.

- Which statement(s) about the first two quotient figures is(are) true?
    - The first quotient figure is 8, set on the column immediately left of $7,666$.
    - The first quotient figure is 8, set on the second column to the left of $7,666$.
    - The second quotient figure is 7, set on the column immediately left of $7,666$.
- Find the square root of $7,666$.
    - $87~ remainder~ 97$
    - $167~ remainder~ 97$
    - $47~ remainder~ 7$

Questions 17 and 18 refer to the problem of finding the square root of 633.

- To carry the root out to three decimal places, you set
    - 633.000000 on columns 9, 8, 7, 6, 5, 4, 3, 2, and 1.
    - 633.000 on columns 6, 5, 4, 3, 2, and 1.

- 633.00000000 on columns 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, and 1.
- After completing the first round ending in subtraction, when finding the square root of 633, you divide by
  - 2.
  - 4.
  - 16.
- Extract the square root of $165,649$. Explain where the problem is set, how many digits will be in the answer and why, as well as where each quotient figure is set and why. Make sure to note the dividend after each round of division, multiplication, and subtraction. Point out when you double or halve. Explain how you get the final answer.

Lesson 5: Subtracting the Larger Number from a Smaller Number

Lesson 1 explained how to add and subtract fractions, while Lesson 2 demonstrated how to multiply and divide them. Lesson 3 described how to find percents and work with units of measure. Lesson 4 explained how to extract square roots. This lesson, which concludes "Abacus II," demonstrates how to subtract a larger number from a smaller one and how to subtract a series of numbers.

You might think it is difficult to subtract a larger number from a smaller one (e.g., 39 from 15). Using the steps in this lesson, however, you will find it is simple, fast, and accurate. Subtracting larger numbers from smaller numbers is used when banking, balancing checkbooks, accounting, and shopping. Learning to subtract a larger number from a smaller one will help you achieve your goal of performing advanced calculations the abacus way.

Objectives

After completing this lesson, you will be able to use the Cranmer abacus to

- identify complementary numbers, active numbers, and inactive numbers;
- subtract a larger number from a smaller one; and subtract a series of numbers.

Identifying Complementary, Active, and Inactive Numbers

To subtract a larger number from a smaller one, you use complementary numbers. In Lesson 2 of "Abacus I," you were introduced to pairs of complementary numbers, such as the complementary pairs of 5 and 10. Now, consider the complementary pairs of 9. They are 9 and 0, 8 and 1, 7 and 2, 6 and 3, and 5 and 4. It is said that 5 is the complement of 4 and 4 is the complement of 5 because when added they complete 9. When you use the abacus, each column contains two numbers—the number set against the separation bar and its complementary number away from the bar (i.e., against the frame). The beads set against the bar are called active; they represent positive numbers. The beads away from the bar (i.e., against the frame) are called inactive; they represent negative numbers. On any column, the number set against the bar has a complementary number against the frame. These two numbers always add up to 9. When using the Cranmer abacus, 9 is the largest value that you can set on a single column because in base 10, the base used by most number systems, the greatest single-digit value is 9.

Whenever you set a number on the abacus, the beads away from the bar (i.e., against the frame), are the complements of the number you set on that column (i.e., against the bar). Until now, you considered only the active beads, the ones set against the bar. In this section, you also consider the inactive beads, the ones against the frame. Practice reading the inactive beads by checking the beads against the frame at both the top and the bottom of each column. For example, set 9 on column 1 of your abacus. Notice that all the beads are as close to the bar as possible; none are against the frame. So, you have nine active beads set against the bar and the complement, zero inactive beads, against the frame. Clear one bead from this 9, which changes 9 to 8. You now have eight active beads set against the bar and the complement, one inactive bead, set against the frame. Clear one bead again, changing 8 to 7. You now have seven active beads set against the bar and the complement, two inactive beads, set against the frame. Clear one bead again. You have six active beads set against the bar and the complement, three inactive beads, set against the frame.

Next, clear the five bead. One active bead is set against the bar and the complement, eight inactive beads, is set against the frame. The five bead is against the top of the frame and three lower beads are against the bottom of the frame. Now, clear your abacus and set 99 on columns 2 and 1 . From 99, subtract 34. Your result, 65, is against the bar; and 34, which you subtracted, is away from the bar and against the frame. The two-digit number represented by the active beads is 65 and its complement, 34, is the two-digit number represented by the inactive beads. The complement of 65 is 34. If your number has two digits, its complement, found against the frame, is the two-digit number that you add to make the total, 99. Similarly, if your number has three digits, its complement, found against the frame, is the three-digit number that you add to make the total 999, and so on.

Suppose you have a three-digit number. What number do you add to total $1,000$? Simply set your number. Then, clear 1 on column 1. Clearing 1 from the active beads adds 1 to the beads that represent the complement.

For example, subtract $1,000$ from 412. When you set 412 on columns 3, 2, and 1, you automatically form its complement against the frame. The inactive beads against the frame show 587. Column 3, which contains four active beads, shows five inactive beads against the frame. Column 2, which contains one active bead, shows eight inactive beads. Column 1, which contains two active beads, shows seven inactive beads. You are 1 short of $1,000$ because $412+587=999$. Since you want the total to be $1,000$, not 999, clear 1 in column 1. This gives you 588, which you add to 412, for a total of $1,000$.

Remember that the active beads set against the bar represent positive numbers. Similarly, the inactive beads against the frame represent negative numbers—numbers that have a negative sign in front of them. Thus, $412-1,000=-588$.

Negative numbers are useful in everyday life. If you write checks for $\$1,000$ against a checkbook balance of $\$412$, you are overdrawn by $\$588$ or -588 dollars. Or, suppose you buy an item that costs $\$4.12$. You give the sales clerk $\$10$ to pay for your purchase. Therefore, the clerk owes you $\$5.88$ in change or -5 dollars and 88 cents. The easiest and quickest way to verify this is to set $\$4.12$ on your abacus, clear 1 on column 1, and read the inactive beads. Try another example. Subtract $1,000$ from 843—think of it as giving a $\$10$ bill to pay for an $\$8.43$ purchase. Set 843 on columns 3, 2, and 1 of your abacus. Then, read its complement, 156, by checking the inactive beads against the frame, both top and bottom. Finally, clear 1 in column 1. Your result is -157. Thus, $843-1,000=-157$. In the case of money, the salesclerk would owe you $\$1.57$ in change.

Practice Exercises

Practice identifying complementary numbers by working the following problems on your abacus. Record your answers so that you can compare them with those that follow the exercise. Find the complement of the following numbers:

- 23
- $9,085$
- $1,671$
- $54,089$
- How much change would you receive if you purchase a candy bar for $\$0.78$ and you give the cashier $\$1.00$?
- How much change would you receive if you purchase groceries for $\$54.08$ and you give the cashier $\$100.00$?

Answers

Compare your answers with those that follow:

- 76
- 914
- $8,328$
- $45,910$
- $\$0.22$
- $\$45.92$

Subtracting a Larger Number from a Smaller One

When you subtract a larger number from a smaller one, you will find a column where you cannot make a subtraction because not enough value is set on that column to subtract from. Borrow an imaginary 1 from the column immediately to the left. Then, continue subtracting using the methods of subtraction you already know. Finally, clear 1 on column 1 and read the inactive beads.

You have used 10, 100, 1,000, or another base 10 number as the larger number. What happens when you subtract $2,000$ from 634? Set 634 on columns 3, 2, and 1. From this number, subtract 2,000. The 2 of 2,000 must be subtracted in the thousands column (column 4). This column contains no value. To subtract 2 from 0, clear 1 left and set 8. You cannot actually clear 1 on column 5 because it contains no value. Just pretend it is there and clear it. Then set 8 on column 4. Clear 1 from column 1, then read the inactive beads to find that $634-2000=-1,366$. In the same way, if you had $\$6.34$ in your checkbook and you wrote a check for $\$20$, you would subtract $\$20$ from $\$6.34$. You are $\$13.66~ overdrawn$, that is, your balance is $-13~ dollars~ and ~66~ cents$.

Now, instead of subtracting $\$20$ from $\$6.34$, subtract $\$50$ from $\$6.34$. Apply the indirect method for subtracting 5 by clearing an imaginary 1 from column 5 and setting 5 on column 4. Clear 1 in column 1. Then, read the inactive beads to find that $\$6.34-\$50.00=-\$43 ~and ~66 ~cents$.

Similarly, subtract $8,000$ from 634. Clear an imaginary 1 from column 5 and set 2 on column 4. Clear 1 from column 1. Read the inactive beads to find $-7,366$.

For another example, imagine your bank balance shows $\$422$ and you write checks for $\$600$. To find out how much you are overdrawn, set 422 on columns 3, 2, and 1 and subtract 600. To subtract the 6 of 600 from the 4 of 422, clear an imaginary 1 from column 4. Then set 5 and clear 1 on column 3. Clear 1 from column 1. Now read the inactive beads. You are overdrawn by $178.

So far, you have subtracted larger numbers like 600, 1,000, 2,000, 5,000, and 8,000 from smaller numbers. This method also works when the larger number does not end in zeros. For example, subtract 39 from 15. Set 15 on columns 2 and 1. To subtract the 3 of 39 from the 1 of 15, borrow 100 and clear an imaginary 1 from column 3. Then, set 7 on column 2. To subtract 9 from 5 on column 1, clear 1 left from column 2 and set 1 on column 1. Now clear 1 on column 1. Read the inactive beads. Your answer is -24.

Next, work the problem $234 - 567$. Set 234 on columns 3, 2, and 1. To subtract the 5 of 567 from the 2 of 234, clear an imaginary 1 from column 4 and set 5 on column 3. The abacus now shows $+734$ set against the bar. To subtract the 6 of 567 from the 3 of 234, clear 1 left from the 7 on column 3. Then set 5 and clear 1 on column 2. Now, your abacus shows $+674$. To subtract the 7 of 567 from the 4 of 234, clear 1 left from column 2. Then set 5 and clear 2 on column 1. Clear 1 on column 1. Finally, read the inactive beads against the frame. The result is -333.

This time, subtract $1,682$ from 425. Set 425 on columns 3, 2, and 1. First subtract the 1 of 1,682 from column 4. Because this column contains no value, you must borrow 1 from column 5. Clear an imaginary 1 left on column 5 and set 9 on column 4. Your abacus now shows $+9,425$. Next, subtract the 6 of $1,682$ from the 4 of 9,425 in the usual way; that is, clear 1 from column 4. Then, set 5 and clear 1 on column 3. Your abacus now shows $+8,825$. Subtract the 8 of $1,682$ from 2 on column 2 by clearing 1 left on column 3 and setting 2 on column 2. Your abacus now shows $+8,745$. To subtract the 2 of $1,682$ from 5 on column 1, set 3 and clear 5. The number $8,743$ is set against the bar. Its complementary number against the frame is $1,256$. Clear 1 on column 1. Thus, $425-1,682=-1,257$. If you want to prove your answer, add 425 to $1,257$. You get $1,682$.

Now, subtract $4,321$ from 56. Set 56 on columns 2 and 1. To subtract the 4 of $4,321$ from column 4, clear an imaginary 1 from column 5 and set 6 on column 4. The abacus now shows $+6,056$. To subtract the 3 of $4,321$ from the 0 of $6,056$, clear 1 from the 6 on column 4 and set 7 on column 3. The abacus shows $+5,756$. Subtract the 2 of $4,321$ from 5 on column 2 by setting 3 and clearing 5. You now have $+5,736$. Subtract the 1 of $4,321$ directly from the 6 on column 1. Clear 1 on column 1 by setting 4 and clearing 5. The job is done. Read the inactive beads. The answer is $-4,265$.

So far, you borrowed 1 in the very first step of the problem to subtract in the first column to the left. Occasionally, you can avoid borrowing 1 until you have partially worked the problem. For example, to subtract $4,321$ from $4,265$, set $4,265$ on your abacus. Subtract 4 from 4 directly on column 4, which leaves 265 in the active beads. To subtract 3 from 2 on column 3, clear an imaginary 1 from column 4 and set 7 on column 3. The abacus now shows $+965$. Now, subtract the 2 of $4,321$ from the 6 of 965 on column 2 by setting 3 and clearing 5. Your abacus now shows $+945$ against the bar. Subtract the 1 of $4,321$ from 5 on column 1 by setting 4 and clearing 5. Clear 1 on column 1. Read the inactive beads. The answer is -56.

Now, work the problem $781-789$. Set 781 on your abacus and subtract 7 from 7 and 8 from 8 directly. This leaves 1 on column 1. Since you cannot subtract 9 from 1, borrow 10. To do this, clear an imaginary 1 from column 2 and set 1 on column 1. Before you read the inactive beads, remember to clear 1 on column 1. Your final answer is -8.

Practice Exercises

Practice subtracting a larger number from a smaller one by working the following problems on your abacus. Record your answers so that you can compare them with those that follow the exercise.

- $23-98$
- $42-78$
- $54-242$
- $749-789$
- $28-1,971$
- $36-70,088$
- $432-8,642$
- $532-537$
- $789-798$
- $286-6,666$

Answers

Compare your answers with those that follow:

- -75
- -36
- -188
- -40
- $-1,943$
- $-70,052$
- $-8,210$
- -5
- -9
- $-6,380$

Subtracting a Series of Numbers

This section explains how to subtract a series of numbers, such as $11-75-4,348$. Set 11 on columns 2 and 1. You are now ready to subtract 75 from 11. To subtract 7 from 1, borrow 1 from column 3 by clearing an imaginary 1. Because the 1 was in column 3, you have borrowed 100. Then set 3 on column 2. Your abacus shows $+41$.

To subtract the 5 of 75 from 1, clear 1 left from the 4 on column 2 and set 5 on column 1. You now have 36 against the bar. Next, subtract $4,348$. To subtract 4 from column 4, you need to borrow. Your first thought might be to clear 1 from column 5, that is, to borrow $10,000$. Remember that you have already borrowed 100. The amount you borrow should total $10,000$. So, with this in mind, borrow $9,900$. Set 9 on both columns 4 and 3. The number $9,936$ is now against the bar. From this, subtract $4,348$. Clear 1 in column 1 before you read the inactive beads. Your answer is $-4,412$.

Remember that the amount you borrow must total $10, 100, $1,000, $10,000$, and so on. Therefore, if you borrow 1 and later discover that you must borrow again, borrow 9 from the appropriate column or columns so that the total borrowed is 100, $1,000$, and so on.

When you deal with a series of subtractions, you can sometimes think ahead and borrow enough the first time you borrow. Find the number with the greatest value of all the numbers to be subtracted. Then, borrow 1 from the column immediately to the left of that number.

For example, rework the problem, $11-75-4,348$. Set 11 on your abacus. To subtract 75 from it, last time you borrowed 100 and then you borrowed $9,900$ later on to subtract $4,348$. This time, avoid borrowing twice. Of the two numbers to subtract, $4,348$ is the largest. The 4 of $4,348$ will eventually be subtracted from column 4, so borrow 1 from the column immediately to the left (column 5). Now you can subtract 75. To subtract 7 from 1, clear this borrowed 1 on column 5. Then, change the 0s to 9s on columns 4 and 3, and set 3 on column 2. Your abacus now shows $+9,941$ against the bar. Next, subtract the 5 of 75 from column 1. Clear 1 from column 2 and set 5 on column 1. You now have $+9,936$ on your abacus. Subtract $4,348$ from $9,936$. Read the inactive beads, clearing one bead in column 1 as usual. Once again, your answer is $-4,412$.

For another example to compare borrowing twice and borrowing once, work the problem $50-98-76$ both ways. Set 50 on columns 2 and 1. Subtract 98 from 50 by borrowing 1 from column 3. Subtract the 9 of 98 by clearing this imaginary 1 from column 3 and setting 1 on column 2. Next, subtract the 8 of 98 by clearing 1 from column 2 and setting 2 on column 1.

Your abacus now shows $+52$. Since you cannot subtract 76 from 52, you have to borrow. You have already borrowed 100 when you borrowed 1 from column 3. So you must now borrow 900 to give you a total of $1,000$. Set 9 on column 3. To subtract the 7 of 76, clear 1 left from this 9 on column 3 and set 3 on column 2. Finally, subtract the 6 of 76 by clearing 1 from column 2, then on column 1, set 5 and clear 1. Clear 1 in column 1 and read the inactive beads. Your answer is -124.

You obtain the same result if you borrow $1,000$ the first time, instead of 100. If you had anticipated this, you would set 50, then subtract the 9 of 98 by borrowing 1 from column 4. Change 0 to 9 on column 3 and set 1 on column 2. You need not borrow again. Complete the subtractions in the usual way.

Finally, work a problem that has both addition and subtraction: $16-73+84$. Set 16 on columns 2 and 1. Subtract 73 from 16. To subtract the 7 of 73 from the 1 of 16, borrow 1 from column 3. Clear this borrowed 1 and set 3 on column 2. Next, subtract the 3 of 73 from the 6 of 16 by setting 2 and clearing 5 on column 1. You now have $+43$ set against the bar. Then, add 84 to 43. Add 8 to 4 by clearing 2 and setting 1 left. Finally, add the 4 of 84 to the 3 of 43 by setting 5 and clearing 1 on column 1. Your abacus shows $+127$ against the bar. Notice that this 127 is larger than the 100 you borrowed. Whenever the number against the bar is larger than the borrowed one, return or subtract the borrowed number. When you do so, your answer is a positive number—the number that appears against the bar. Remember, you borrowed 100 when you borrowed 1 from column 3. So, remove the 1 from 127, which leaves 27 against the bar. Because your answer is a positive number, ignore the inactive beads. The answer is $+27$.

Practice Exercises

Practice subtracting a series of numbers on the abacus by completing the following problems. Record your answers so that you can compare them with those that follow the exercise.

- $11-33-79$
- $98-76-54-725$
- $12-34+56-78$
- $15-43+89-67$
- $20-111-2,345+123$
- $33-55+77+99$
- $88-77-66+55$

Answers

Compare your answers with those that follow:

- -101
- -757
- -44
- -6
- $-2,313$
- $+154$
- 0

Assignment 5

Complete the following questions with all the correct answers. Don't assume that only one answer is correct. Sometimes two or three are needed to answer the question correctly. Questions 1-18 are worth 5 points each. Question 19 is worth 10 points.

- You purchase an item that costs $\$7.82$ and pay with a $\$10$ bill. The quickest way to determine how much change the salesclerk owes you is to set
    - $1,000 on your abacus and subtract 782.
    - 782 on your abacus, clear 1 from column 1 in the active beads, and read the inactive beads.
    - 782 on your abacus and read its complement.
      Questions 2 and 3 refer to the problem $27-400$.
- To work this problem
    - you must clear an imaginary 1 from column 4.
    - the first step is to set 27 on your abacus.
    - the last step is to clear 1 on column 1 and read the inactive beads.
- Find $27-400$.
    - $+373$
    - -372
    - -373
- Set 37 on your abacus and subtract 68 from it.
    - The answer is -31.
    - Read the active beads to determine the answer.
    - After setting 37, your next step is to borrow 100.
      Questions 5 and 6 refer to the problem $345-6,789$.
- Immediately after you subtract the 6 of 6,789, what number appears against the bar?
    - 4,345
    - 4,789
    - 6,345
- Find $345-6,789$.
    - $-3,556$
    - $-6,443$
    - $-6,444$

Questions 7-9 refer to the problem $88-10,234$.

- After setting 88, the next step is to
    - subtract 4 from 8.
    - borrow $100,000 in order to subtract 2 from 0.

    - borrow $100,000 in order to clear 1 left and set 9.
- When working the problem, $88-10,234$, what number appears against the bar immediately after you subtract the 2 of $10,234$?
    - $90,888$
    - $89,888$
    - $88,088$
- Find $88-10,234$.
    - $-10,144$
    - $-10,145$
    - $-10,146$

Questions 10 and 11 refer to the problem $27-1,589$.

- To work this problem, you must borrow
    - 100.
    - $1,000$.
    - $10,000$.
- In finding $27-1,589$, what number shows against the bar immediately after you subtract 8 from 2?
    - $8,447$
    - $1,552$
    - $8,427$

Questions 12-13 refer to the problem $25-88-399$.

- You can work this problem by borrowing either once or twice. To answer questions 12 and 13, however, borrow only once. When you finish subtracting 88 from 25, what number shows against the bar?
    - 37
    - 937
    - 925
- Find $25-88-399$.
    - -362
    - -461
    - -462

Questions 14 and 15 refer to the following problem: Set 24 on your abacus and subtract 53. Then add 97.

- Your last step is to
    - clear 1 on column 1.
    - remove 100.
    - subtract 7 from 1.
- Find $24-53+97$.
    - $+68$
    - $+168$
    - -832

Questions 16 through 18 refer to the problem $59-77+2,613$.

- To work this problem you must borrow
    - 100.
    - $1,000.

- $10,000.
- When working the problem, $59-77+2,613$, your final step is to
    - subtract the borrowed 100.
    - clear 1 on column 1 and read the inactive beads.
    - subtract the borrowed $10,000.
- Find $59-77+2,613$.
    - $-7,405$
    - $+1,695$
    - $+2,595$
- Subtract $3,028$ from 27. Explain where you set the problem, what number was borrowed, and how each subtraction was made. Also, explain how you determined the final answer.