MATH 307-101 Applied Linear Algebra 2024 Winter Term 1 (Sep–Dec 2024)

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1 Approximating Eigenvalues with the QR Algorithm

1.1 Prove A_1 is similar to A

A is a square matrix. Let, $A = QR$ be the QR factorization of A.
Q is a square matrix, and its columns form an orthonormal basis by construction.
Therefore, $Q^{-1} = Q^T$ and $Q^TQ = QQ^T = I$, i.e, Q is invertible
$A_1 = RQ$.
A matrix A is similar to another matrix B if there exists an invertible matrix P such that:

$$A = PBP^{-1}$$

For $B = A_1$ and $P = Q$

$$A = QA_1Q^T = Q(RQ)Q^T = QR$$

Therefore, A_1 is similar to A.
Now, to prove A_1 and A have the same eigenvectors.
A matrix has an eigenvalue λ and corresponding eigenvector v if

$$Av = \lambda v$$

Since A_1 is similar to A

$$Q(RQ)Q^T v = \lambda v$$

Left multiply both sides by Q^T

$$(Q^T Q)(RQ)(Q^T v) = Q^T \lambda v = \lambda(Q^T v)$$

Set $u = Q^T v$, and since $A_1 = RQ$. We get,

$$A_1 u = \lambda u$$

Since λ and v were arbitrary, we have shown that A and A_1 have the same eigenvalues.

1.2 Find A_1 and A, verify they have the same eigenvalues

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

Finding the eigenvalues from factoring the characteristic equation, we get

$$det(A - \lambda I) = (1 - \lambda)(3 - \lambda) = 0 \implies \lambda = 3, 1$$

The QR factorization of A results in $A = QR$ such that

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{18}} \end{bmatrix}, R = \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{18}}{2} \end{bmatrix}$$

Therefore, $A_1 = RQ$ is

$$A_1 = RQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{18}}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

Finding the eigenvalues from factoring the characteristic equation of A_1, we get

$$det(A_1 - \lambda I) = (\frac{5}{2} - \lambda)(\frac{3}{2} - \lambda) - \frac{3}{2 \times 2} = 0 \implies \lambda^2 - 4\lambda + 3 = 0$$

$$\implies (1-\lambda)(3-\lambda) = 0 \implies \lambda = 3, 1$$

Therefore, A_1 and A have the same eigenvalues

1.3 A_k is similar to A

Proof by induction.
Inductive Hypothesis: A_k is similar to A for $k \geq 1$
Base case:A_1 is similar to A
For an invertible matrix Q,
$A = QA_1Q^T = Q(RQ)Q^T = QR$

Assume inductive hypothesis is true, that A_k is similar to A for $k \geq 1$

To show: A_{k+1} is similar to A_k, and therefore, A

$A_{k+1} = R_kQ_k$
$A_k = Q_kA_{k+1}Q_k^T = Q_kR_kQ_kQ_k^T = Q_kR_k$
Clearly, A_{k+1} is similar to A_k. Therefore, A_{k+1} is similar to A_k (since $A_k = Q_kR_k = R_{k-1}Q_{k-1}$ and so forth) by the inductive hypothesis.

1.4 Continuing Problem 1.2

Python code was written to calculate A_2, A_3, A_4, A_5.

$$A_2 = \begin{bmatrix} 3.12 & -0.47 \\ 0.53 & 0.88 \end{bmatrix} A_3 = \begin{bmatrix} 3.06 & 0.84 \\ -0.16 & 0.94 \end{bmatrix} A_4 = \begin{bmatrix} 3.02 & -0.95 \\ 0.05 & 0.98 \end{bmatrix} A_5 = \begin{bmatrix} 3.01 & 0.98 \\ -0.02 & 0.99 \end{bmatrix}$$

As we can clearly see, $A_k \to A$ as k increases

1.5

1.6 QR Algorithm in Python

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$	4.0, -1.0
$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$	2.41, -0.41
$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -4 & 0 & 1 \end{bmatrix}$	3.01, 1.99, -1.0
$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -2 & 4 & 2 \end{bmatrix}$	3.0, 2.0, 0.0

1.7 Another Example

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	2.0, -2.0

This is clearly incorrect as the eigenvalues should be 1 and -1. The basic QR algorithm has failed because the absolute values of the eigenvalues are non-distinct. The QR algorithm relies on seperating eigenvalues based on their magnitudes and since the eigenvalues are the same magnitude, the algorithm fails.

1.8 Using a Shift

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	Eigenvalues
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	1.0, -1.0

$B = A + \alpha I \implies \lambda$ is an eigenvalue of A $\iff \lambda + \alpha$ is an eigenvalue of B.

Proof.
($\implies$) Let λ be an eigenvalue of A. Then there exists a non-zero vector x such that $Ax = \lambda x$.
Then $Bx = (A + \alpha I)x = Ax + \alpha x = \lambda x + \alpha x = (\lambda + \alpha)x$.
Thus, $\lambda + \alpha$ is an eigenvalue of B.

($\impliedby$) Let $\lambda + \alpha$ be an eigenvalue of B. Then there exists a non-zero vector x such that $Bx = (\lambda + \alpha)x$.
Then $Ax = (B - \alpha I)x = Bx - \alpha Ix = (\lambda + \alpha)x - \alpha x = \lambda x$.
Thus, λ is an eigenvalue of A.

$\square$

1.9 The QR factorization of A^{k+1}

Proof.
Let $Q_0 = Q$ and $R_0 = R$.
1. $Q_0Q_1 \cdots Q_{k-1}A_k = AQ_0Q_1 \cdots Q_{k-1}$ for all $k \geq 1$

Base Case: $k = 1$ so $A = Q_0R_0, A_1 = R_0Q_0$ then,
$Q_0A_1 = (Q_0R_0)Q_0 = AQ_0$
Inductive Step: Assume $Q_0Q_1 \cdots Q_{k-1}A_k = Q_0Q_1 \cdots Q_{k-1}R_{k-1}Q_{k-1} = AQ_0Q_1 \cdots Q_{k-1}$, then for $k + 1$
$Q_0Q_1 \cdots Q_{k-1}Q_kA_{k+1} = Q_0Q_1 \cdots Q_{k-1}Q_kR_kQ_k$
$= Q_0Q_1 \cdots Q_{k-1}A_kQ_k$ (Since $A_k = Q_kR_k$)
$= (Q_0Q_1 \cdots Q_{k-1}R_{k-1}Q_{k-1})Q_k$ (Since $A_k = R_{k-1}Q_{k-1}$)
$= AQ_0Q_1 \cdots Q_{k-1}Q_k$ (By Inductive Hypothesis)

Thus $Q_0Q_1\cdots Q_{k-1}A_k = AQ_0Q_1\cdots Q_{k-1}$.

2. $(Q_0Q_1\cdots Q_k)(R_k\cdots R_1R_0) = A(Q_0Q_1\cdots Q_{k-1})(R_{k-1}\cdots R_1R_0)$

Base Case: $k=1$ so $A = Q_0R_0, A_1 = R_0Q_0$ then,
$(Q_0Q_1)(R_1R_0) = Q_0Q_1R_1R_0 = Q_0A_1R_0$ (Since $A_1 = Q_1R_1$)
$= Q_0R_0Q_0R_0$ (Since $A_1 = R_0Q_0$)
$= AQ_0R_0$ (Since $A = Q_0R_0$)
Inductive Step: Assume $(Q_0Q_1\cdots Q_k)(R_k\cdots R_1R_0) = A(Q_0Q_1\cdots Q_{k-1})(R_{k-1}\cdots R_1R_0)$, then for $k+1$
$(Q_0Q_1\cdots Q_kQ_{k+1})(R_{k+1}R_k\cdots R_1R_0) = Q_0Q_1\cdots Q_kQ_{k+1}R_{k+1}R_k\cdots R_1R_0$
$= Q_0Q_1\cdots Q_kA_{k+1}R_k\cdots R_1R_0$ (Since $A_{k+1} = Q_{k+1}R_{k+1}$)
$= Q_0Q_1\cdots Q_kR_kQ_kR_k\cdots R_1R_0$ (Since $A_{k+1} = R_kQ_k$)
$= (Q_0Q_1\cdots Q_{k-1}A_k)Q_kR_kR_{k-1}\cdots R_1R_0$ (Since $A_k = Q_kR_k$)
$= (AQ_0Q_1\cdots Q_{k-1})Q_kR_kR_{k-1}\cdots R_1R_0$ (By **1.**)
$= A(Q_0Q_1\cdots Q_{k-1}Q_k)(R_kR_{k-1}\cdots R_1R_0)$

Thus $(Q_0Q_1\cdots Q_k)(R_k\cdots R_1R_0) = A(Q_0Q_1\cdots Q_{k-1})(R_{k-1}\cdots R_1R_0)$.

3. $A^{k+1} = (Q_0Q_1\cdots Q_k)(R_k\cdots R_1R_0)$

Base Case: $k=0$ so $A^1 = Q_0R_0 = QR = A$
Inductive Step: Assume $A^{k+1} = (Q_0Q_1\cdots Q_k)(R_k\cdots R_1R_0)$, then for $k+2$
$A^{k+2} = AA^{k+1} = A(Q_0Q_1\cdots Q_k)(R_k\cdots R_1R_0)$ (By Inductive Hypothesis)
$= (Q_0Q_1\cdots Q_kQ_{k+1})(R_{k+1}Rk\cdots R_1R_0)$ (By **2.**)

Thus $A^{k+1} = (Q_0Q_1\cdots Q_k)(R_k\cdots R_1R_0)$ for all $k \geq 0$. $\square$