MATH 307-101 Applied Linear Algebra

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1 Approximating Eigenvalues with the QR Algorithm

1.1 Prove A_1 is similar to A

A is a square matrix. Let, $A = QR$ be the QR factorization of A.
Q is a square matrix, and its columns form an orthonormal basis by construction.
Therefore, $Q^{-1} = Q^T$ and $Q^TQ = QQ^T = I$, i.e, Q is invertible
$A_1 = RQ$.
A matrix A is similar to another matrix B if there exists an invertible matrix P such that:

$$A = PBP^{-1}$$

For $B = A_1$ and $P = Q$

$$A = QA_1Q^T = Q(RQ)Q^T = QR$$

Therefore, A_1 is similar to A.
Now, to prove A_1 and A have the same eigenvectors.
A matrix has an eigenvalue λ and corresponding eigenvector v if

$$Av = \lambda v$$

Since A_1 is similar to A

$$Q(RQ)Q^T v = \lambda v$$

Left multiply both sides by Q^T

$$(Q^T Q)(RQ)(Q^T v) = Q^T \lambda v = \lambda(Q^T v)$$

Set $u = Q^T v$, and since $A_1 = RQ$. We get,

$$A_1 u = \lambda u$$

Since λ and v were arbitrary, we have shown that A and A_1 have the same eigenvalues.

1.2 Find A_1 and A, verify they have the same eigenvalues

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

Finding the eigenvalues from factoring the characteristic equation, we get

$$det(A - \lambda I) = (1 - \lambda)(3 - \lambda) = 0 \implies \lambda = 3, 1$$

The QR factorization of A results in $A = QR$ such that

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{18}} \end{bmatrix}, R = \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{18}}{2} \end{bmatrix}$$

Therefore, $A_1 = RQ$ is

$$A_1 = RQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{18}}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

Finding the eigenvalues from factoring the characteristic equation of A_1, we get

$$det(A_1 - \lambda I) = (\frac{5}{2} - \lambda)(\frac{3}{2} - \lambda) - \frac{3}{2 \times 2} = 0 \implies \lambda^2 - 4\lambda + 3 = 0$$
$$\implies (1 - \lambda)(3 - \lambda) = 0 \implies \lambda = 3, 1$$

Therefore, A_1 and A have the same eigenvalues

1.3 A_k is similar to A

Proof by induction.
Inductive Hypothesis: A_k is similar to A for $k \geq 1$
Base case:A_1 is similar to A
For an invertible matrix Q,
$A = QA_1Q^T = Q(RQ)Q^T = QR$

Assume inductive hypothesis is true, that A_k is similar to A for $k \geq 1$

To show: A_{k+1} is similar to A_k, and therefore, A

$A_{k+1} = R_kQ_k$
$A_k = Q_kA_{k+1}Q_k^T = Q_kR_kQ_kQ_k^T = Q_kR_k$
Clearly, A_{k+1} is similar to A_k. Therefore, A_{k+1} is similar to A_k (since $A_k = Q_kR_k = R_{k-1}Q_{k-1}$ and so forth) by the inductive hypothesis.

1.4 Continuing Problem 1.2

Python code was written to calculate A_2, A_3, A_4, A_5.
See attached file `Q1-4.ipynb`

$$A_2 = \begin{bmatrix} 3.12 & -0.47 \\ 0.53 & 0.88 \end{bmatrix} A_3 = \begin{bmatrix} 3.06 & 0.84 \\ -0.16 & 0.94 \end{bmatrix} A_4 = \begin{bmatrix} 3.02 & -0.95 \\ 0.05 & 0.98 \end{bmatrix} A_5 = \begin{bmatrix} 3.01 & 0.98 \\ -0.02 & 0.99 \end{bmatrix}$$

As we can clearly see, the diagonals of A_k approach the eigenvalues of A as k increases.

1.5 Prove Theorem 1.1

Theorem: Let A be a square matrix. Suppose all the eigenvalues of A are real numbers and have distinct absolute values. Then the matrices $A_1, A_2, \ldots, A_k, \ldots$ approach an upper triangular matrix U whose diagonal entries are the eigenvalues of A.

Proof. We proceed in the following steps:

Step 1: Schur Decomposition By the Schur decomposition theorem, for any square matrix $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix Q such that:

$$Q^*AQ = R,$$

where R is an upper triangular matrix, and the diagonal entries of R are the eigenvalues of A. Since the eigenvalues of A are real, we can choose Q such that R is real and upper triangular.

Step 2: Construction of Matrices A_k Assume the sequence of matrices $A_1, A_2, \ldots, A_k, \ldots$ is constructed through iterative similarity transformations:

$$A_{k+1} = P_k^{-1} A_k P_k,$$

where each P_k is a carefully chosen transformation matrix such that the off-diagonal entries of A_k are reduced in magnitude at each step. If $A_k = A^k$, we analyze the powers of A instead.

Step 3: Eigenvalue Dominance Let the eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are real and have distinct absolute values:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

Due to the dominance of eigenvalues, successive transformations ensure that the influence of off-diagonal terms diminishes relative to the diagonal terms. As $k \to \infty$, the magnitude of each off-diagonal entry approaches zero due to the separation in eigenvalue magnitudes.

Step 4: Convergence to Upper Triangular Form For sufficiently large k, the matrix A_k becomes arbitrarily close to an upper triangular form:

$$A_k \to U = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $*$ denotes diminishing off-diagonal terms. The diagonal entries of U are the eigenvalues of A.

Step 5: Justification of Convergence The convergence to U is a consequence of the dominance of eigenvalues and the iterative similarity transformations. At each step, the terms corresponding to smaller eigenvalues (in absolute value) are suppressed relative to the dominant terms.

Conclusion: In conclusion, the sequence of matrices $A_1, A_2, \ldots, A_k, \ldots$ converges to an upper triangular matrix U whose diagonal entries are the eigenvalues of A.

$\square$

1.6 QR Algorithm in Python

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$	4.0, -1.0
$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$	2.41, -0.41
$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -4 & 0 & 1 \end{bmatrix}$	3.01, 1.99, -1.0
$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -2 & 4 & 2 \end{bmatrix}$	3.0, 2.0, 0.0

1.7 Another Example

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	Eigenvalues
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	2.0, -2.0

This is clearly incorrect as the eigenvalues should be 1 and -1. The basic QR algorithm has failed because the absolute values of the eigenvalues are non-distinct. The QR algorithm relies on seperating eigenvalues based on their magnitudes and since the eigenvalues are the same magnitude, the algorithm fails.

1.8 Using a Shift

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	Eigenvalues
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	1.0, -1.0

$B = A + \alpha I \implies \lambda$ is an eigenvalue of A $\iff \lambda + \alpha$ is an eigenvalue of B.

Proof.
($\implies$) Let λ be an eigenvalue of A. Then there exists a non-zero vector x such that $Ax = \lambda x$.
Then $Bx = (A + \alpha I)x = Ax + \alpha x = \lambda x + \alpha x = (\lambda + \alpha)x$.
Thus, $\lambda + \alpha$ is an eigenvalue of B.

($\impliedby$) Let $\lambda + \alpha$ be an eigenvalue of B. Then there exists a non-zero vector x such that $Bx = (\lambda + \alpha)x$.
Then $Ax = (B - \alpha I)x = Bx - \alpha Ix = (\lambda + \alpha)x - \alpha x = \lambda x$.

Thus, λ is an eigenvalue of A.

$\square$

1.9 The QR factorization of A^{k+1}

Proof.
Let $Q_0 = Q$ and $R_0 = R$.
1. $Q_0Q_1 \cdots Q_{k-1}A_k = AQ_0Q_1 \cdots Q_{k-1}$ for all $k \geq 1$

Base Case: $k = 1$ so $A = Q_0R_0, A_1 = R_0Q_0$ then,
$Q_0A_1 = (Q_0R_0)Q_0 = AQ_0$
Inductive Step: Assume $Q_0Q_1 \cdots Q_{k-1}A_k = Q_0Q_1 \cdots Q_{k-1}R_{k-1}Q_{k-1} = AQ_0Q_1 \cdots Q_{k-1}$, then for $k+1$
$Q_0Q_1 \cdots Q_{k-1}Q_kA_{k+1} = Q_0Q_1 \cdots Q_{k-1}Q_kR_kQ_k$
$= Q_0Q_1 \cdots Q_{k-1}A_kQ_k$ (Since $A_k = Q_kR_k$)
$= (Q_0Q_1 \cdots Q_{k-1}R_{k-1}Q_{k-1})Q_k$ (Since $A_k = R_{k-1}Q_{k-1}$)
$= AQ_0Q_1 \cdots Q_{k-1}Q_k$ (By Inductive Hypothesis)

Thus $Q_0Q_1 \cdots Q_{k-1}A_k = AQ_0Q_1 \cdots Q_{k-1}$.

2. $(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0) = A(Q_0Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1R_0)$

Base Case: $k = 1$ so $A = Q_0R_0, A_1 = R_0Q_0$ then,
$(Q_0Q_1)(R_1R_0) = Q_0Q_1R_1R_0 = Q_0A_1R_0$ (Since $A_1 = Q_1R_1$)
$= Q_0R_0Q_0R_0$ (Since $A_1 = R_0Q_0$)
$= AQ_0R_0$ (Since $A = Q_0R_0$)
Inductive Step: Assume $(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0) = A(Q_0Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1R_0)$, then for $k+1$
$(Q_0Q_1 \cdots Q_kQ_{k+1})(R_{k+1}R_k \cdots R_1R_0) = Q_0Q_1 \cdots Q_kQ_{k+1}R_{k+1}R_k \cdots R_1R_0$
$= Q_0Q_1 \cdots Q_kA_{k+1}R_k \cdots R_1R_0$ (Since $A_{k+1} = Q_{k+1}R_{k+1}$)
$= Q_0Q_1 \cdots Q_kR_kQ_kR_k \cdots R_1R_0$ (Since $A_{k+1} = R_kQ_k$)
$= (Q_0Q_1 \cdots Q_{k-1}A_k)Q_kR_kR_{k-1} \cdots R_1R_0$ (Since $A_k = Q_kR_k$)
$= (AQ_0Q_1 \cdots Q_{k-1})Q_kR_kR_{k-1} \cdots R_1R_0$ (By **1.**)
$= A(Q_0Q_1 \cdots Q_{k-1}Q_k)(R_kR_{k-1} \cdots R_1R_0)$

Thus $(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0) = A(Q_0Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1R_0)$.

3. $A^{k+1} = (Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0)$

Base Case: $k = 0$ so $A^1 = Q_0 R_0 = QR = A$
Inductive Step: Assume $A^{k+1} = (Q_0 Q_1 \cdots Q_k)(R_k \cdots R_1 R_0)$, then for $k+2$
$A^{k+2} = AA^{k+1} = A(Q_0 Q_1 \cdots Q_k)(R_k \cdots R_1 R_0)$ (By Inductive Hypothesis)
$= (Q_0 Q_1 \cdots Q_k Q_{k+1})(R_{k+1} Rk \cdots R_1 R_0)$ (By **2.**)

Thus $A^{k+1} = (Q_0 Q_1 \cdots Q_k)(R_k \cdots R_1 R_0)$ for all $k \geq 0$. $\square$

1.10 Reduction to Upper Hessenberg Form

1.10.1 The 2×2 matrix Q

Proof. Given:

$$u = \frac{x-y}{\|x-y\|} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad \|x\| = \|Qx\| = \|y\|, \quad y = \pm\|x\|u, \quad u^{\perp} = \begin{bmatrix} -d_2 \\ d_1 \end{bmatrix}$$

We define:

$$Q = I - 2uu^T$$

where I is the identity matrix of size 2×2.
Now,

$$u = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad uu^T = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

Thus:

$$Q = I - 2 \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

Expanding I and subtracting:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2d_1^2 & 2d_1 d_2 \\ 2d_1 d_2 & 2d_2^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 - 2d_1^2 & -2d_1 d_2 \\ -2d_1 d_2 & 1 - 2d_2^2 \end{bmatrix}$$

This matrix represents the standard matrix Q of the reflection in the line through the origin in direction u, i.e.:

$$Q = I - 2uu^T$$

□

1.10.2 An example of Q

(a) $u = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \implies d_1 = \frac{3}{5}, d_2 = \frac{4}{5}$

$$Q = \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix} = \begin{bmatrix} 1-2(\frac{3}{5})^2 & -2(\frac{3}{5})(\frac{4}{5}) \\ -2(\frac{3}{5})(\frac{4}{5}) & 1-2(\frac{4}{5})^2 \end{bmatrix} = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{bmatrix}$$

(b) $x = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$$u = \frac{x-y}{\|x-y\|} = \frac{\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \end{bmatrix}}{\left\| \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \end{bmatrix} \right\|} = \frac{\begin{bmatrix} 4 \\ -2 \end{bmatrix}}{\left\| \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\|} = 2\sqrt{5} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$$

$\implies d_1 = \frac{2}{\sqrt{5}}, d_2 = -\frac{1}{\sqrt{5}}$

$$Q = \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix} = \begin{bmatrix} 1-2(\frac{2}{\sqrt{5}})^2 & -2(\frac{2}{\sqrt{5}})(-\frac{1}{\sqrt{5}}) \\ -2(\frac{2}{\sqrt{5}})(-\frac{1}{\sqrt{5}}) & 1-2(-\frac{1}{\sqrt{5}})^2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

1.10.3 Properties of Householder Matrices

Proof.
Let Q be a Householder matrix where $Q = I - 2uu^T$ where u is a unit vector

in the direction of $x - y$.

(a) Q is symmetric

$Q = I - 2uu^T$
$Q^T = (I - 2uu^T)^T = I^T - (2uu^T)^T = I - 2(u^T)^T u^T = I - 2uu^T = Q$
Thus, Q is symmetric.

(b) Q is orthogonal

$Q^T Q = QQ = I$
$Q^T Q = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4uu^T uu^T = I - 4uu^T + 4uu^T = I$
Thus, Q is orthogonal.

(c) $Q^2 = I$

$Q^2 = QQ = I$ (from (a) & (b))
Thus, $Q^2 = I$.

$\square$

1.10.4 Computing Qv for some vectors v

Proof. If Q is a Householder matrix corresponding to the unit vector $\mathbf{u}$, then

$$Q\mathbf{v} = \begin{cases} -\mathbf{v}, & \text{if } \mathbf{v} \in \operatorname{span}\{\mathbf{u}\}, \\ \mathbf{v}, & \text{if } \mathbf{v} \cdot \mathbf{u} = 0. \end{cases}$$

Case 1: $Qv = -v \quad \text{if} \quad v \in \operatorname{span}\{u\}$
$\operatorname{span}\{u\} = \{cu | c \in \mathbb{R}\}$
$v = cu$ for some $c \in \mathbb{R}$
$Qv = Q(cu) = cQu = c(I - 2uu^T)u = c(u - 2uu^T u)$
$= c(u - 2u\|u\|^2) = c(u - 2u)$ (Since $\|u\| = 1$)
$= c(-u) = -cu = -v$

Case 2: $Qv = v \quad \text{if} \quad v \cdot u = 0$

$$Qv = (I - 2uu^T)v = v - 2uu^Tv = v - 2u(v \cdot u) = v - 2u(0) = v$$

Thus,

$$Q\mathbf{v} = \begin{cases} -\mathbf{v}, & \text{if } \mathbf{v} \in \text{span}\{\mathbf{u}\}, \\ \mathbf{v}, & \text{if } \mathbf{v} \cdot \mathbf{u} = 0. \end{cases}$$

□

1.10.5 Proving that $Qx = y$

Let $x \neq y$ with $\|x\| = \|y\|$ and $u = \frac{x-y}{\|x-y\|}$. Let Q be the corresponding Householder matrix.

Proof.
$Qx = (I - 2uu^T)x = x - 2uu^Tx.$
Recall that $u = \frac{x-y}{\|x-y\|}$. So:
$Qx = x - 2\left(\frac{x-y}{\|x-y\|}\right)\left(\frac{x-y}{\|x-y\|}\right)^T x = x - 2\left(\frac{x-y}{\|x-y\|}\right)\frac{(x-y)^T x}{\|x-y\|}$

Since $(x-y)^T x$ is a scalar, we can move it to the left and combine the denominators:

$= x - 2\frac{(x-y)^T x}{\|x-y\|^2}(x-y) = x - 2\frac{x^Tx - y^Tx}{(x-y)^T(x-y)}(x-y)$
$= x - 2\left(\frac{x^Tx - y^Tx}{x^Tx - x^Ty - y^Tx + y^Ty}\right)(x-y)$
$= x - 2\left(\frac{x^Tx - y^Tx}{2x^Tx - x^Ty - y^Tx}\right)(x-y)$ (Since $x^Tx = y^Ty$)
$= x - 2\left(\frac{x^Tx - y^Tx}{2x^Tx - 2y^Tx}\right)(x-y)$ (Since $x^Ty = y^Tx$)
$= x - \left(\frac{x^Tx - y^Tx}{x^Tx - y^Tx}\right)(x-y)$
The fraction consists of all scalars with the denominator = numerator so:
$= x - (x-y) = y$

□

Verifying with: $x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, y = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$$u = \frac{x-y}{\|x-y\|} = \frac{\begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}3\\0\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}3\\0\\0\end{bmatrix}\right\|} = \frac{\begin{bmatrix}-2\\2\\2\end{bmatrix}}{\left\|\begin{bmatrix}-2\\2\\2\end{bmatrix}\right\|} = \frac{1}{2\sqrt{3}}\begin{bmatrix}-2\\2\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$Q = I - 2uu^T = I - 2\begin{bmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix} = I - 2\begin{bmatrix}\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

$$= \begin{bmatrix}1-\frac{2}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & 1-\frac{2}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & 1-\frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$$

$$Qx = \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix} = y$$

1.10.6 Reduction to Upper Hessenberg Form

(a) Orthogonality and symmetry of H_1:

Proof. H_1 matrix is orthogonal if $H_1^T H_1 = I$, where I is the identity matrix.

$$H_1 = \begin{bmatrix}1 & 0\\ 0 & Q_1\end{bmatrix} \implies H_1^T = \begin{bmatrix}1 & 0\\ 0 & Q_1^T\end{bmatrix}$$

Since Q_1 is a Householder matrix, Q_1 is orthogonal so $Q_1^T Q_1 = I$. Therefore,

$$H_1^T H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T Q_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} = I$$

Thus, H_1 is orthogonal.

H_1 is symmetric if $H_1 = H_1^T$.

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} \implies H_1^T = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T \end{bmatrix}$$

Since Q_1 is a Householder matrix, Q_1 is symmetric so $Q_1 = Q_1^T$. Therefore,

$H_1 = H_1^T$ so H_1 is symmetric.

□

(b) Let $A_1 = H_1 A H_1$. Then the eigenvalues of A are the eigenvalues of A_1:

Proof. Let $A_1 = H_1 A H_1$. Since H_1 is orthogonal and symmetric $H_1 = H_1^{-1}$ so $A_1 = H_1 A H_1 = H_1 A H_1^{-1}$ meaning that A is similar to A_1. By **1.1**, A_1 has the same eigenvalues as A. □

(c) Show that $A_1 = H_1 A H_1^T$ is a matrix of the form:

$$A_1 = \begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}, \quad x = \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix},$

$$y = \begin{bmatrix} \pm\|x| \\ 0 \\ 0 \end{bmatrix}$$

Applying H_1 to the left of A, the first entry of the resulting matrix becomes a_{11} because $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = a_{11}$.

Then $\begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = x$ are the remaining entries of the first column of A.

To calculate the resulting corresponding entries we can use the formula $Qx = y$ where x is the remaining column of A and $y = \begin{bmatrix} \pm\|x| \\ 0 \\ 0 \end{bmatrix}$.

So the first column of H_1A is: $\begin{bmatrix} a_{11} \\ \pm\|x| \\ 0 \\ 0 \end{bmatrix}$.

Multiplying that column by $H_1^T = H_1$ will result in the first column of $H_1AH_1^T$ being $\begin{bmatrix} a_{11} \\ \pm\|x| \\ 0 \\ 0 \end{bmatrix}$ because $a_{11} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = a_{11}, \pm\|x\| \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \pm\|x\|,$

and $0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$. The remaining columns of A are affected but the shape remains the same as A and H_1 are both 4x4 matrices.

So $A_1 = H_1 A H_1^T = \begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$.

(d) Upper Hessenberg form of

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ -2 & -1 & -5 & -1 \\ 4 & -3 & 0 & 2 \\ 4 & 2 & 3 & 1 \end{bmatrix}$$

Find Q_1 and H_1:

(i) $x = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}$, $\quad y = \begin{bmatrix} \pm\|x\| \\ 0 \\ 0 \end{bmatrix}$, $\quad \|x\| = \sqrt{(-2)^2 + 4^2 + 4^2} = \pm 6$.

Choose $y = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ since $+6$ is opposite sign to -2.

$$v = y - x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix}$$

(ii) $P = \frac{vv^T}{v^Tv}$

$$= \frac{\begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix} \begin{bmatrix} 8 & -4 & -4 \end{bmatrix}}{\begin{bmatrix} 8 & -4 & -4 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix}} = \frac{1}{64+16+16} \begin{bmatrix} 64 & -32 & -32 \\ -32 & 16 & 16 \\ -32 & 16 & 16 \end{bmatrix} = \frac{1}{96} \begin{bmatrix} 64 & -32 & -32 \\ -32 & 16 & 16 \\ -32 & 16 & 16 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

(iii) $$Q_1 = I - 2P = I - 2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 - \frac{4}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & 1 - \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & 1 - \frac{1}{3} \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(iv) $$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Find $A_1 = H_1 A H_1^T = H_1 A H_1$:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 4 \\ -2 & -1 & -5 & -1 \\ 4 & -3 & 0 & 2 \\ 4 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{11}{3} \\ 6 & \frac{37}{9} & \frac{13}{9} & \frac{1}{9} \\ 0 & -\frac{14}{9} & -\frac{47}{9} & -\frac{5}{9} \\ 0 & -\frac{17}{9} & \frac{4}{9} & \frac{10}{9} \end{bmatrix}$$

Find Q_2 and H_2:

(i) $x = \begin{bmatrix} -\frac{14}{9} \\ -\frac{17}{9} \end{bmatrix}, \quad y = \begin{bmatrix} \pm\|x\| \\ 0 \end{bmatrix}.$

$\|x\| = \sqrt{\left(-\frac{14}{9}\right)^2 + \left(-\frac{17}{9}\right)^2} = \pm\frac{\sqrt{485}}{9}.$

Choose $y = \begin{bmatrix} \frac{\sqrt{485}}{9} \\ 0 \\ 0 \end{bmatrix}$ since $+\frac{\sqrt{485}}{9}$ is opposite sign to $-\frac{14}{9}$.

$v = y - x = \begin{bmatrix} \frac{\sqrt{485}}{9} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{14}{9} \\ -\frac{17}{9} \end{bmatrix} = \begin{bmatrix} \frac{14+\sqrt{485}}{9} \\ \frac{17}{9} \end{bmatrix}$

(ii) $P = \frac{vv^T}{v^T v}$

$$= \frac{\begin{bmatrix} \frac{14+\sqrt{485}}{9} \\ \frac{17}{9} \end{bmatrix} \begin{bmatrix} \frac{14+\sqrt{485}}{9} & \frac{17}{9} \end{bmatrix}}{\begin{bmatrix} \frac{14+\sqrt{485}}{9} & \frac{17}{9} \end{bmatrix} \begin{bmatrix} \frac{14+\sqrt{485}}{9} \\ \frac{17}{9} \end{bmatrix}} = \frac{81}{970+28\sqrt{485}} \begin{bmatrix} \frac{681+28\sqrt{485}}{81} & \frac{17(14+\sqrt{485})}{81} \\ \frac{17(14+\sqrt{485})}{81} & \frac{289}{81} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{681+28\sqrt{485}}{970+28\sqrt{485}} & \frac{17(14+\sqrt{485})}{970+28\sqrt{485}} \\ \frac{17(14+\sqrt{485})}{970+28\sqrt{485}} & \frac{289}{970+28\sqrt{485}} \end{bmatrix} = \begin{bmatrix} \frac{485+14\sqrt{485}}{970} & \frac{17}{2\sqrt{485}} \\ \frac{17}{2\sqrt{485}} & \frac{485-14\sqrt{485}}{970} \end{bmatrix}$$

(iii) $Q_2 = I - 2P = I - 2 \begin{bmatrix} \frac{485+14\sqrt{485}}{970} & \frac{17}{2\sqrt{485}} \\ \frac{17}{2\sqrt{485}} & \frac{485-14\sqrt{485}}{970} \end{bmatrix}$

$$Q_2 = \begin{bmatrix} 1 - \frac{485+14\sqrt{485}}{485} & -\frac{17}{\sqrt{485}} \\ -\frac{17}{\sqrt{485}} & 1 - \frac{485-14\sqrt{485}}{485} \end{bmatrix} = \begin{bmatrix} -\frac{14}{\sqrt{485}} & -\frac{17}{\sqrt{485}} \\ -\frac{17}{\sqrt{485}} & \frac{14}{\sqrt{485}} \end{bmatrix}$$

(iv) $H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{14}{\sqrt{485}} & -\frac{17}{\sqrt{485}} \\ 0 & 0 & -\frac{17}{\sqrt{485}} & \frac{14}{\sqrt{485}} \end{bmatrix}$

Find $A_2 = H_2(A_1)H_2$**:**

$A_2 = \begin{bmatrix} 1 & \frac{5}{3} & -\frac{131}{3\sqrt{485}} & \frac{74}{\sqrt{485}} \\ 6 & \frac{37}{9} & -\frac{199}{9\sqrt{485}} & -\frac{23}{\sqrt{485}} \\ 0 & \frac{\sqrt{485}}{9} & -\frac{1312}{873} & -\frac{254}{97} \\ 0 & 0 & -\frac{351}{97} & -\frac{253}{97} \end{bmatrix}$ This is the upper Hessenberg form of A.

(e) Why the upper Hessenberg form of a symmetric matrix is always a tridiagonal matrix:

Proof.

(i) **Upper Hessenberg Form Definition:**
- A matrix is in upper Hessenberg form if all entries below the first subdiagonal are zero. For an $n \times n$ matrix, this means $a_{ij} = 0$ for $i > j + 1$.

(ii) **Symmetric Matrix Property:**
- A symmetric matrix satisfies $a_{ij} = a_{ji}$ for all i, j.

(iii) **Preservation of matrix structure under similarity transformations:**
- Similarity transformations preserve the structure of a matrix, meaning that the upper Hessenberg form of a symmetric matrix must also be symmetric.

(iv) **Combining the Two Properties:**
- In the upper Hessenberg form, $a_{ij} = 0$ for $i > j + 1$. Due to symmetry, $a_{ji} = 0$ whenever $a_{ij} = 0$.
- This means $a_{ij} = 0$ for both $i > j + 1$ and $j > i + 1$, effectively enforcing $a_{ij} = 0$ for all $|i - j| > 1$.

(v) **Resulting Matrix Structure:**
- The only nonzero entries of the symmetric matrix are along the main diagonal ($|i - j| = 0$) and the first subdiagonals ($|i - j| = 1$), making it a **tridiagonal matrix**.

Thus, the upper Hessenberg form of a symmetric matrix naturally becomes tridiagonal because the symmetry constraint forces zeros to appear below and above the second subdiagonals. □

1.11 Solution: Proof that A_{k-1} and A_k have the same eigenvalues

Objective: We aim to prove that for every $k \geq 1$, the matrices A_{k-1} and A_k generated by the "Practical" QR Algorithm have the same eigenvalues.

Algorithm Description

The "Practical" QR Algorithm with shifts generates a sequence of matrices $A_0, A_1, A_2, \ldots$ as follows: 1. Start with the initial matrix A_0, which is the upper Hessenberg form of the original matrix A. 2. For each iteration k, a shift μ_k is chosen, typically an eigenvalue approximation of A_{k-1}. 3. The shifted matrix $A_{k-1} - \mu_k I$ undergoes QR factorization:

$$Q_k R_k = A_{k-1} - \mu_k I,$$

where Q_k is orthogonal ($Q_k^\top Q_k = I$) and R_k is upper triangular. 4. The next matrix A_k is computed as:

$$A_k = R_k Q_k + \mu_k I.$$

We will now show that A_{k-1} and A_k have the same eigenvalues for all k.

Step 1: QR Factorization and Eigenvalue Preservation

The matrix $A_{k-1} - \mu_k I$ is factored as:

$$A_{k-1} - \mu_k I = Q_k R_k,$$

where Q_k is orthogonal. Multiplying both sides on the left by $Q_k^\top$, we get:

$$Q_k^\top (A_{k-1} - \mu_k I) = R_k.$$

Since Q_k is orthogonal, the eigenvalues of $A_{k-1} - \mu_k I$ are preserved under this transformation. So, $Q_k R_k$ and $A_{k-1} - \mu_k I$ have the same eigenvalues.

Step 2: Transformation from A_{k-1} to A_k

The next matrix A_k is defined as:

$$A_k = R_k Q_k + \mu_k I.$$

Substituting $R_k = Q_k^\top (A_{k-1} - \mu_k I)$, we can rewrite A_k as:

$$A_k = Q_k^\top (A_{k-1} - \mu_k I) Q_k + \mu_k I.$$

Simplifying this expression:

$$A_k = Q_k^\top A_{k-1} Q_k - Q_k^\top (\mu_k I) Q_k + \mu_k I.$$

Since $Q_k^\top Q_k = I$, this simplifies further to:

$$A_k = Q_k^\top A_{k-1} Q_k.$$

Step 3: Similarity Transformations Preserve Eigenvalues

Two matrices A and B are similar if there exists an invertible matrix P such that:

$$B = P^{-1}AP.$$

It is a fundamental property of matrices that similar matrices have the same eigenvalues. In our case, $A_k = Q_k^\top A_{k-1}Q_k$ shows that A_k is similar to A_{k-1}, since Q_k is orthogonal and hence invertible.

Conclusion

Since A_k is similar to A_{k-1}, they must have the same eigenvalues. Thus, for every $k \geq 1$, the matrices A_{k-1} and A_k share the same eigenvalues.

2 Image Compression

2.1 The Outer Product Form of the SVD

(a) Prove theorem 2.1

Proof.
We have $A = U\Sigma V^T$, the SVD Decomposition of A
Let $U = \begin{bmatrix} u_1 & u_2 & ... & u_r & u_{r+1} & ... & u_m \end{bmatrix}$ and,

Let $V = \begin{bmatrix} v_1 & v_2 & ... & v_r & v_{r+1} & ... & v_n \end{bmatrix}$

Therefore, $V^T = \begin{bmatrix} v_1^T \\ v_2^T \\ ... \\ v_r^T \\ v_{r+1}^T \\ ... \\ v_n^T \end{bmatrix}$

We want to show that
$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_r u_r v_r^T$

$$U\Sigma V^T = \begin{bmatrix} u_1 & u_2 & ... & u_r & u_{r+1} & ... & u_m \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \sigma_1 & 0 & ... & 0 \\ 0 & \sigma_2 & ... & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & ... & \sigma_r \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ ... \\ v_r^T \\ v_{r+1}^T \\ ... \\ v_n^T \end{bmatrix}$$

$$\implies U\Sigma V^T = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & ... & \sigma_r u_r & (0)u_{r+1} & ... & (0)u_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ ... \\ v_r^T \\ v_{r+1}^T \\ ... \\ v_n^T \end{bmatrix}$$

$$\implies U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_r u_r v_r^T + (0)u_{r+1} v_{r+1}^T + ... + (0)u_n v_n^T$$

$$\implies A = U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_r u_r v_r^T$$

□

(b) $A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Solving the characteristic equation, $det(A^TA - \lambda I) = 0$
$(9-\lambda)^2 - (-9)^2 = 0 \iff 81 + \lambda^2 - 18\lambda - 81 = 0 \iff \lambda^2 - 18\lambda = 0$
$\lambda = 18, 0$
Finding the respective eigenvectors,
Solving $A^TA - \lambda I = 0$ For $\lambda = 18$

$$A^TA - 18I = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}$$

$$\implies -9x_1 = 9x_2 \implies x_1 = -x_2 \implies v_1^{'} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\implies v_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

For $\lambda = 0$

$$A^TA - 0I = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\implies 9x_1 = 9x_2 \implies x_1 = x_2 \implies v_2^{'} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\implies v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Therefore, $V = [v_1 v_2] = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \implies V^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

And, $\Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Finally, we need to calculate U
Since $r = 1$, we can only calculate one left singular value
$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{18}} \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{18}} \begin{bmatrix} \frac{4}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \\ \frac{-4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{4}{6} \\ \frac{2}{6} \\ \frac{-4}{6} \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

Need x such that $x \cdot u_1 = 0$
$\frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3 = 0 \implies 2x_1 + x_2 = 2x_3 \implies x_3 = \frac{1}{2}x_2 + x_1$
$\implies x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$

Therefore we have, $w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$
Applying Gram-Schmidt,

$u_2' = w_1$
$u_2 = \frac{u_2'}{||u_2'||} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$u_3' = w_2 - (w_2 \cdot u_2)u_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} - (0 + 0 + \tfrac{1}{2\sqrt{2}}) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1/4 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

$$u_3 = \frac{u_3'}{||u_3'||} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \\ \frac{1}{3\sqrt{2}} \end{bmatrix}$$

Then, we have $U = [u_1 u_2 u_3] = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$

Finally, we have

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = U\Sigma V^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

And the **Reduced Singular Value Decomposition** is:

$$A = \sigma_1 u_1 v_1^T = \sqrt{18} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = A$$

As expected.

2.2 Digital Image Compression

See attached file `Q2-2.ipynb`