MATH 307-101 Applied Linear Algebra 2024 Winter Term 1 (Sep–Dec 2024)

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1 Approximating Eigenvalues with the QR Algorithm

1.1

1.2

1.3

1.4

1.5

1.6 QR Algorithm in Python

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$	4.0, -1.0
$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$	2.41, -0.41
$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -4 & 0 & 1 \end{bmatrix}$	3.01, 1.99, -1.0
$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -2 & 4 & 2 \end{bmatrix}$	3.0, 2.0, 0.0

1.7 Another Example

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	2.0, -2.0

This is clearly incorrect as the eigenvalues should be 1 and -1. The basic QR algorithm has failed because the absolute values of the eigenvalues are non-distinct. The QR algorithm relies on seperating eigenvalues based on their magnitudes and since the eigenvalues are the same magnitude, the algorithm fails.

1.8 Using a Shift

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	1.0, -1.0

$B = A + \alpha I \implies \lambda$ is an eigenvalue of A $\iff \lambda + \alpha$ is an eigenvalue of B.

Proof.
($\implies$) Let λ be an eigenvalue of A. Then there exists a non-zero vector x such that $Ax = \lambda x$.
Then $Bx = (A + \alpha I)x = Ax + \alpha x = \lambda x + \alpha x = (\lambda + \alpha)x$.
Thus, $\lambda + \alpha$ is an eigenvalue of B.

($\impliedby$) Let $\lambda + \alpha$ be an eigenvalue of B. Then there exists a non-zero vector x such that $Bx = (\lambda + \alpha)x$.
Then $Ax = (B - \alpha I)x = Bx - \alpha Ix = (\lambda + \alpha)x - \alpha x = \lambda x$.
Thus, λ is an eigenvalue of A.

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