MATH 307-101 Applied Linear Algebra

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1 Approximating Eigenvalues with the QR Algorithm

1.1 Prove A_1 is similar to A

A is a square matrix. Let, $A = QR$ be the QR factorization of A.
Q is a square matrix, and its columns form an orthonormal basis by construction.
Therefore, $Q^{-1} = Q^T$ and $Q^T Q = QQ^T = I$, i.e, Q is invertible
$A_1 = RQ$.
A matrix A is similar to another matrix B if there exists an invertible matrix P such that:

$$A = PBP^{-1}$$

For $B = A_1$ and $P = Q$

$$A = QA_1Q^T = Q(RQ)Q^T = QR$$

Therefore, A_1 is similar to A.
Now, to prove A_1 and A have the same eigenvectors.
A matrix has an eigenvalue λ and corresponding eigenvector v if

$$Av = \lambda v$$

Since A_1 is similar to A

$$Q(RQ)Q^T v = \lambda v$$

Left multiply both sides by Q^T

$$(Q^T Q)(RQ)(Q^T v) = Q^T \lambda v = \lambda(Q^T v)$$

Set $u = Q^T v$, and since $A_1 = RQ$. We get,

$$A_1 u = \lambda u$$

Since λ and v were arbitrary, we have shown that A and A_1 have the same eigenvalues.

1.2 Find A_1 and A, verify they have the same eigenvalues

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

Finding the eigenvalues from factoring the characteristic equation, we get

$$det(A - \lambda I) = (1 - \lambda)(3 - \lambda) = 0 \implies \lambda = 3, 1$$

The QR factorization of A results in $A = QR$ such that

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{18}} \end{bmatrix}, R = \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{18}}{2} \end{bmatrix}$$

Therefore, $A_1 = RQ$ is

$$A_1 = RQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{\sqrt{18}}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

Finding the eigenvalues from factoring the characteristic equation of A_1, we get

$$det(A_1 - \lambda I) = (\frac{5}{2} - \lambda)(\frac{3}{2} - \lambda) - \frac{3}{2 \times 2} = 0 \implies \lambda^2 - 4\lambda + 3 = 0$$
$$\implies (1-\lambda)(3-\lambda) = 0 \implies \lambda = 3, 1$$

Therefore, A_1 and A have the same eigenvalues

1.3 A_k is similar to A

Proof by induction.
Inductive Hypothesis: A_k is similar to A for $k \geq 1$
Base case:A_1 is similar to A
For an invertible matrix Q,
$A = QA_1Q^T = Q(RQ)Q^T = QR$

Assume inductive hypothesis is true, that A_k is similar to A for $k \geq 1$

To show: A_{k+1} is similar to A_k, and therefore, A

$A_{k+1} = R_kQ_k$
$A_k = Q_kA_{k+1}Q_k^T = Q_kR_kQ_kQ_k^T = Q_kR_k$
Clearly, A_{k+1} is similar to A_k. Therefore, A_{k+1} is similar to A_k (since $A_k = Q_kR_k = R_{k-1}Q_{k-1}$ and so forth) by the inductive hypothesis.

1.4 Continuing Problem 1.2

Python code was written to calculate A_2, A_3, A_4, A_5.
See attached file `Q1-4.ipynb`

$$A_2 = \begin{bmatrix} 3.12 & -0.47 \\ 0.53 & 0.88 \end{bmatrix} A_3 = \begin{bmatrix} 3.06 & 0.84 \\ -0.16 & 0.94 \end{bmatrix} A_4 = \begin{bmatrix} 3.02 & -0.95 \\ 0.05 & 0.98 \end{bmatrix} A_5 = \begin{bmatrix} 3.01 & 0.98 \\ -0.02 & 0.99 \end{bmatrix}$$

As we can clearly see, the diagonals of A_k approach the eigenvalues of A as k increases.

1.5 Prove Theorem 1.1

// incomplete

1.6 QR Algorithm in Python

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	Eigenvalues
$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$	4.0, -1.0
$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$	2.41, -0.41
$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -4 & 0 & 1 \end{bmatrix}$	3.01, 1.99, -1.0
$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -2 & 4 & 2 \end{bmatrix}$	3.0, 2.0, 0.0

1.7 Another Example

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	2.0, -2.0

This is clearly incorrect as the eigenvalues should be 1 and -1. The basic QR algorithm has failed because the absolute values of the eigenvalues are non-distinct. The QR algorithm relies on seperating eigenvalues based on their magnitudes and since the eigenvalues are the same magnitude, the algorithm fails.

1.8 Using a Shift

Results from python code (QRAlgorithm(1.6-1.8).py):

Matrix	**Eigenvalues**
$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	1.0, -1.0

$B = A + \alpha I \implies \lambda$ is an eigenvalue of A $\iff$ $\lambda + \alpha$ is an eigenvalue of B.

Proof.
($\implies$) Let λ be an eigenvalue of A. Then there exists a non-zero vector x such that $Ax = \lambda x$.
Then $Bx = (A + \alpha I)x = Ax + \alpha x = \lambda x + \alpha x = (\lambda + \alpha)x$.
Thus, $\lambda + \alpha$ is an eigenvalue of B.

($\impliedby$) Let $\lambda + \alpha$ be an eigenvalue of B. Then there exists a non-zero vector x such that $Bx = (\lambda + \alpha)x$.
Then $Ax = (B - \alpha I)x = Bx - \alpha Ix = (\lambda + \alpha)x - \alpha x = \lambda x$.

Thus, λ is an eigenvalue of A.

$\square$

1.9 The QR factorization of A^{k+1}

Proof.
Let $Q_0 = Q$ and $R_0 = R$.
1. $Q_0Q_1 \cdots Q_{k-1}A_k = AQ_0Q_1 \cdots Q_{k-1}$ for all $k \geq 1$

Base Case: $k = 1$ so $A = Q_0R_0, A_1 = R_0Q_0$ then,
$Q_0A_1 = (Q_0R_0)Q_0 = AQ_0$
Inductive Step: Assume $Q_0Q_1 \cdots Q_{k-1}A_k = Q_0Q_1 \cdots Q_{k-1}R_{k-1}Q_{k-1} = AQ_0Q_1 \cdots Q_{k-1}$, then for $k+1$
$Q_0Q_1 \cdots Q_{k-1}Q_kA_{k+1} = Q_0Q_1 \cdots Q_{k-1}Q_kR_kQ_k$
$= Q_0Q_1 \cdots Q_{k-1}A_kQ_k$ (Since $A_k = Q_kR_k$)
$= (Q_0Q_1 \cdots Q_{k-1}R_{k-1}Q_{k-1})Q_k$ (Since $A_k = R_{k-1}Q_{k-1}$)
$= AQ_0Q_1 \cdots Q_{k-1}Q_k$ (By Inductive Hypothesis)

Thus $Q_0Q_1 \cdots Q_{k-1}A_k = AQ_0Q_1 \cdots Q_{k-1}$.

2. $(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0) = A(Q_0Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1R_0)$

Base Case: $k = 1$ so $A = Q_0R_0, A_1 = R_0Q_0$ then,
$(Q_0Q_1)(R_1R_0) = Q_0Q_1R_1R_0 = Q_0A_1R_0$ (Since $A_1 = Q_1R_1$)
$= Q_0R_0Q_0R_0$ (Since $A_1 = R_0Q_0$)
$= AQ_0R_0$ (Since $A = Q_0R_0$)
Inductive Step: Assume $(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0) = A(Q_0Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1R_0)$, then for $k+1$
$(Q_0Q_1 \cdots Q_kQ_{k+1})(R_{k+1}R_k \cdots R_1R_0) = Q_0Q_1 \cdots Q_kQ_{k+1}R_{k+1}R_k \cdots R_1R_0$
$= Q_0Q_1 \cdots Q_kA_{k+1}R_k \cdots R_1R_0$ (Since $A_{k+1} = Q_{k+1}R_{k+1}$)
$= Q_0Q_1 \cdots Q_kR_kQ_kR_k \cdots R_1R_0$ (Since $A_{k+1} = R_kQ_k$)
$= (Q_0Q_1 \cdots Q_{k-1}A_k)Q_kR_kR_{k-1} \cdots R_1R_0$ (Since $A_k = Q_kR_k$)
$= (AQ_0Q_1 \cdots Q_{k-1})Q_kR_kR_{k-1} \cdots R_1R_0$ (By **1.**)
$= A(Q_0Q_1 \cdots Q_{k-1}Q_k)(R_kR_{k-1} \cdots R_1R_0)$

Thus $(Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0) = A(Q_0Q_1 \cdots Q_{k-1})(R_{k-1} \cdots R_1R_0)$.

3. $A^{k+1} = (Q_0Q_1 \cdots Q_k)(R_k \cdots R_1R_0)$

Base Case: $k = 0$ so $A^1 = Q_0 R_0 = QR = A$
Inductive Step: Assume $A^{k+1} = (Q_0 Q_1 \cdots Q_k)(R_k \cdots R_1 R_0)$, then for $k+2$
$A^{k+2} = AA^{k+1} = A(Q_0 Q_1 \cdots Q_k)(R_k \cdots R_1 R_0)$ (By Inductive Hypothesis)
$= (Q_0 Q_1 \cdots Q_k Q_{k+1})(R_{k+1} Rk \cdots R_1 R_0)$ (By **2.**)

Thus $A^{k+1} = (Q_0 Q_1 \cdots Q_k)(R_k \cdots R_1 R_0)$ for all $k \geq 0$. □

1.10 Reduction to Upper Hessenberg Form

1.10.1 The 2×2 matrix Q

Proof. Given:

$$u = \frac{x-y}{\|x-y\|} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad \|x\| = \|Qx\| = \|y\|, \quad y = \pm\|x\|u, \quad u^{\perp} = \begin{bmatrix} -d_2 \\ d_1 \end{bmatrix}$$

We define:

$$Q = I - 2uu^T$$

where I is the identity matrix of size 2×2.
Now,

$$u = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad uu^T = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

Thus:

$$Q = I - 2 \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

Expanding I and subtracting:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2d_1^2 & 2d_1 d_2 \\ 2d_1 d_2 & 2d_2^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 - 2d_1^2 & -2d_1 d_2 \\ -2d_1 d_2 & 1 - 2d_2^2 \end{bmatrix}$$

This matrix represents the standard matrix Q of the reflection in the line through the origin in direction u, i.e.:

$$Q = I - 2uu^T$$

$\square$

1.10.2 An example of Q

(a) $u = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \implies d_1 = \frac{3}{5}, d_2 = \frac{4}{5}$

$$Q = \begin{bmatrix} 1 - 2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1 - 2d_2^2 \end{bmatrix} = \begin{bmatrix} 1 - 2(\frac{3}{5})^2 & -2(\frac{3}{5})(\frac{4}{5}) \\ -2(\frac{3}{5})(\frac{4}{5}) & 1 - 2(\frac{4}{5})^2 \end{bmatrix} = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{bmatrix}$$

(b) $x = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$$u = \frac{x-y}{\|x-y\|} = \frac{\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \end{bmatrix}}{\left\| \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \end{bmatrix} \right\|} = \frac{\begin{bmatrix} 4 \\ -2 \end{bmatrix}}{\left\| \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\|} = 2\sqrt{5} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$$

$\implies d_1 = \frac{2}{\sqrt{5}}, d_2 = -\frac{1}{\sqrt{5}}$

$$Q = \begin{bmatrix} 1 - 2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1 - 2d_2^2 \end{bmatrix} = \begin{bmatrix} 1 - 2(\frac{2}{\sqrt{5}})^2 & -2(\frac{2}{\sqrt{5}})(-\frac{1}{\sqrt{5}}) \\ -2(\frac{2}{\sqrt{5}})(-\frac{1}{\sqrt{5}}) & 1 - 2(-\frac{1}{\sqrt{5}})^2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

1.10.3 Properties of Householder Matrices

Proof.
Let Q be a Householder matrix where $Q = I - 2uu^T$ where u is a unit vector

in the direction of $x - y$.

(a) Q is symmetric

$Q = I - 2uu^T$
$Q^T = (I - 2uu^T)^T = I^T - (2uu^T)^T = I - 2(u^T)^T u^T = I - 2uu^T = Q$
Thus, Q is symmetric.

(b) Q is orthogonal

$Q^T Q = QQ = I$
$Q^T Q = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4uu^T uu^T = I - 4uu^T + 4uu^T = I$
Thus, Q is orthogonal.

(c) $Q^2 = I$

$Q^2 = QQ = I$ (from (a) & (b))
Thus, $Q^2 = I$.

□

1.10.4 Computing Qv for some vectors v

Proof. If Q is a Householder matrix corresponding to the unit vector $\mathbf{u}$, then

$$Q\mathbf{v} = \begin{cases} -\mathbf{v}, & \text{if } \mathbf{v} \in \text{span}\{\mathbf{u}\}, \\ \mathbf{v}, & \text{if } \mathbf{v} \cdot \mathbf{u} = 0. \end{cases}$$

Case 1: $Qv = -v \quad \text{if} \quad v \in \text{span}\{u\}$
$\text{span}\{u\} = \{cu | c \in \mathbb{R}\}$
$v = cu$ for some $c \in \mathbb{R}$
$Qv = Q(cu) = cQu = c(I - 2uu^T)u = c(u - 2uu^T u)$
$= c(u - 2u\|u\|^2) = c(u - 2u)$ (Since $\|u\| = 1$)
$= c(-u) = -cu = -v$

Case 2: $Qv = v \quad \text{if} \quad v \cdot u = 0$

$$Qv = (I - 2uu^T)v = v - 2uu^Tv = v - 2u(v \cdot u) = v - 2u(0) = v$$

Thus,

$$Q\mathbf{v} = \begin{cases} -\mathbf{v}, & \text{if } \mathbf{v} \in \text{span}\{\mathbf{u}\}, \\ \mathbf{v}, & \text{if } \mathbf{v} \cdot \mathbf{u} = 0. \end{cases}$$

□

1.10.5 Proving that $Qx = y$

Let $x \neq y$ with $\|x\| = \|y\|$ and $u = \frac{x-y}{\|x-y\|}$. Let Q be the corresponding Householder matrix.

Proof.
$Qx = (I - 2uu^T)x = x - 2uu^Tx$.
Recall that $u = \frac{x-y}{\|x-y\|}$. So:
$Qx = x - 2\left(\frac{x-y}{\|x-y\|}\right)\left(\frac{x-y}{\|x-y\|}\right)^T x = x - 2\left(\frac{x-y}{\|x-y\|}\right)\frac{(x-y)^Tx}{\|x-y\|}$

Since $(x-y)^Tx$ is a scalar, we can move it to the left and combine the denominators:

$= x - 2\frac{(x-y)^Tx}{\|x-y\|^2}(x-y) = x - 2\frac{x^Tx-y^Tx}{(x-y)^T(x-y)}(x-y)$
$= x - 2\left(\frac{x^Tx-y^Tx}{x^Tx-x^Ty-y^Tx+y^Ty}\right)(x-y)$
$= x - 2\left(\frac{x^Tx-y^Tx}{2x^Tx-x^Ty-y^Tx}\right)(x-y)$ (Since $x^Tx = y^Ty$)
$= x - 2\left(\frac{x^Tx-y^Tx}{2x^Tx-2y^Tx}\right)(x-y)$ (Since $x^Ty = y^Tx$)
$= x - \left(\frac{x^Tx-y^Tx}{x^Tx-y^Tx}\right)(x-y)$
The fraction consists of all scalars with the denominator = numerator so:
$= x - (x-y) = y$

□

Verifying with: $x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, y = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$$u = \frac{x-y}{\|x-y\|} = \frac{\begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}3\\0\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}3\\0\\0\end{bmatrix}\right\|} = \frac{\begin{bmatrix}-2\\2\\2\end{bmatrix}}{\left\|\begin{bmatrix}-2\\2\\2\end{bmatrix}\right\|} = \frac{1}{2\sqrt{3}}\begin{bmatrix}-2\\2\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$Q = I - 2uu^T = I - 2\begin{bmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix} = I - 2\begin{bmatrix}\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

$$= \begin{bmatrix}1-\frac{2}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & 1-\frac{2}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & 1-\frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$$

$$Qx = \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix} = y$$

1.10.6 Reduction to Upper Hessenberg Form

(a) Orthogonality and symmetry of H_1:

Proof. H_1 matrix is orthogonal if $H_1^T H_1 = I$, where I is the identity matrix.

$$H_1 = \begin{bmatrix}1 & 0\\ 0 & Q_1\end{bmatrix} \implies H_1^T = \begin{bmatrix}1 & 0\\ 0 & Q_1^T\end{bmatrix}$$

Since Q_1 is a Householder matrix, Q_1 is orthogonal so $Q_1^T Q_1 = I$. Therefore,

$$H_1^T H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T Q_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} = I$$

Thus, H_1 is orthogonal.

H_1 is symmetric if $H_1 = H_1^T$.

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} \implies H_1^T = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T \end{bmatrix}$$

Since Q_1 is a Householder matrix, Q_1 is symmetric so $Q_1 = Q_1^T$. Therefore,

$H_1 = H_1^T$ so H_1 is symmetric.

□

(b) Let $A_1 = H_1 A H_1$. Then the eigenvalues of A are the eigenvalues of A_1:

Proof. Let $A_1 = H_1 A H_1$. Since H_1 is orthogonal and symmetric $H_1 = H_1^{-1}$ so $A_1 = H_1 A H_1 = H_1 A H_1^{-1}$ meaning that A is similar to A_1. By **1.1**, A_1 has the same eigenvalues as A. □

(c) Show that $A_1 = H_1 A H_1^T$ is a matrix of the form:

$$A_1 = \begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}, \quad x = \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix},$

$$y = \begin{bmatrix} \pm\|x| \\ 0 \\ 0 \end{bmatrix}$$

Applying H_1 to the left of A, the first entry of the resulting matrix becomes a_{11} because $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = a_{11}$.

Then $\begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = x$ are the remaining entries of the first column of A.

To calculate the resulting corresponding entries we can use the formula $Qx = y$ where x is the remaining column of A and $y = \begin{bmatrix} \pm\|x| \\ 0 \\ 0 \end{bmatrix}$.

So the first column of H_1A is: $\begin{bmatrix} a_{11} \\ \pm\|x| \\ 0 \\ 0 \end{bmatrix}$.

Multiplying that column by $H_1^T = H_1$ will result in the first column of $H_1AH_1^T$ being $\begin{bmatrix} a_{11} \\ \pm\|x| \\ 0 \\ 0 \end{bmatrix}$ because $a_{11} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = a_{11}, \pm\|x\| \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \pm\|x\|,$

and $0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$. The remaining columns of A are affected but the shape remains the same as A and H_1 are both 4x4 matrices.

So $A_1 = H_1 A H_1^T = \begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$.

(d) Upper Hessenberg form of

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ -2 & -1 & -5 & -1 \\ 4 & -3 & 0 & 2 \\ 4 & 2 & 3 & 1 \end{bmatrix}$$

Find Q_1 and H_1:

(i) $x = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}$, $\quad y = \begin{bmatrix} \pm\|x\| \\ 0 \\ 0 \end{bmatrix}$, $\quad \|x\| = \sqrt{(-2)^2 + 4^2 + 4^2} = \pm 6$.

Choose $y = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ since $+6$ is opposite sign to -2.

$$v = y - x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix}$$

(ii) $P = \frac{vv^T}{v^Tv}$

$$= \frac{\begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix} \begin{bmatrix} 8 & -4 & -4 \end{bmatrix}}{\begin{bmatrix} 8 & -4 & -4 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix}} = \frac{1}{64+16+16} \begin{bmatrix} 64 & -32 & -32 \\ -32 & 16 & 16 \\ -32 & 16 & 16 \end{bmatrix} = \frac{1}{96} \begin{bmatrix} 64 & -32 & -32 \\ -32 & 16 & 16 \\ -32 & 16 & 16 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

(iii) $Q_1 = I - 2P = I - 2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 - \frac{4}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & 1 - \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & 1 - \frac{1}{3} \end{bmatrix} =$

$$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(iv) $H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Find $A_1 = H_1 A H_1^T = H_1 A H_1$:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 4 \\ -2 & -1 & -5 & -1 \\ 4 & -3 & 0 & 2 \\ 4 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{11}{3} \\ 6 & \frac{37}{9} & \frac{13}{9} & \frac{1}{9} \\ 0 & -\frac{14}{9} & -\frac{47}{9} & -\frac{5}{9} \\ 0 & -\frac{17}{9} & \frac{4}{9} & \frac{10}{9} \end{bmatrix}$$

Find Q_2 and H_2:

(i) $x = \begin{bmatrix} -\frac{14}{9} \\ -\frac{17}{9} \end{bmatrix}, \quad y = \begin{bmatrix} \pm\|x\| \\ 0 \end{bmatrix}.$

$\|x\| = \sqrt{\left(-\frac{14}{9}\right)^2 + \left(-\frac{17}{9}\right)^2} = \pm\frac{\sqrt{485}}{9}.$

Choose $y = \begin{bmatrix} \frac{\sqrt{485}}{9} \\ 0 \\ 0 \end{bmatrix}$ since $+\frac{\sqrt{485}}{9}$ is opposite sign to $-\frac{14}{9}$.

$v = y - x = \begin{bmatrix} \frac{\sqrt{485}}{9} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{14}{9} \\ -\frac{17}{9} \end{bmatrix} = \begin{bmatrix} \frac{14+\sqrt{485}}{9} \\ \frac{17}{9} \end{bmatrix}$

(ii) $P = \frac{vv^T}{v^T v}$

$$= \frac{\begin{bmatrix} \frac{14+\sqrt{485}}{9} \\ \frac{17}{9} \end{bmatrix} \begin{bmatrix} \frac{14+\sqrt{485}}{9} & \frac{17}{9} \end{bmatrix}}{\begin{bmatrix} \frac{14+\sqrt{485}}{9} & \frac{17}{9} \end{bmatrix} \begin{bmatrix} \frac{14+\sqrt{485}}{9} \\ \frac{17}{9} \end{bmatrix}} = \frac{81}{970+28\sqrt{485}} \begin{bmatrix} \frac{681+28\sqrt{485}}{81} & \frac{17(14+\sqrt{485})}{81} \\ \frac{17(14+\sqrt{485})}{81} & \frac{289}{81} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{681+28\sqrt{485}}{970+28\sqrt{485}} & \frac{17(14+\sqrt{485})}{970+28\sqrt{485}} \\ \frac{17(14+\sqrt{485})}{970+28\sqrt{485}} & \frac{289}{970+28\sqrt{485}} \end{bmatrix} = \begin{bmatrix} \frac{485+14\sqrt{485}}{970} & \frac{17}{2\sqrt{485}} \\ \frac{17}{2\sqrt{485}} & \frac{485-14\sqrt{485}}{970} \end{bmatrix}$$

(iii) $Q_2 = I - 2P = I - 2 \begin{bmatrix} \frac{485+14\sqrt{485}}{970} & \frac{17}{2\sqrt{485}} \\ \frac{17}{2\sqrt{485}} & \frac{485-14\sqrt{485}}{970} \end{bmatrix}$

$$Q_2 = \begin{bmatrix} 1 - \frac{485+14\sqrt{485}}{485} & -\frac{17}{\sqrt{485}} \\ -\frac{17}{\sqrt{485}} & 1 - \frac{485-14\sqrt{485}}{485} \end{bmatrix} = \begin{bmatrix} -\frac{14}{\sqrt{485}} & -\frac{17}{\sqrt{485}} \\ -\frac{17}{\sqrt{485}} & \frac{14}{\sqrt{485}} \end{bmatrix}$$

(iv) $H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{14}{\sqrt{485}} & -\frac{17}{\sqrt{485}} \\ 0 & 0 & -\frac{17}{\sqrt{485}} & \frac{14}{\sqrt{485}} \end{bmatrix}$

Find $A_2 = H_2(A_1)H_2$**:**

$A_2 = \begin{bmatrix} 1 & \frac{5}{3} & -\frac{131}{3\sqrt{485}} & \frac{74}{\sqrt{485}} \\ 6 & \frac{37}{9} & -\frac{199}{9\sqrt{485}} & -\frac{23}{\sqrt{485}} \\ 0 & \frac{\sqrt{485}}{9} & -\frac{1312}{873} & -\frac{254}{97} \\ 0 & 0 & -\frac{351}{97} & -\frac{253}{97} \end{bmatrix}$ This is the upper Hessenberg form of A.

(e) Why the upper Hessenberg form of a symmetric matrix is always a tridiagonal matrix:

Proof.

(i) **Upper Hessenberg Form Definition:**
- A matrix is in upper Hessenberg form if all entries below the first subdiagonal are zero. For an $n \times n$ matrix, this means $a_{ij} = 0$ for $i > j + 1$.

(ii) **Symmetric Matrix Property:**
- A symmetric matrix satisfies $a_{ij} = a_{ji}$ for all i, j.

(iii) **Preservation of matrix structure under similarity transformations:**
- Similarity transformations preserve the structure of a matrix, meaning that the upper Hessenberg form of a symmetric matrix must also be symmetric.

(iv) **Combining the Two Properties:**
- In the upper Hessenberg form, $a_{ij} = 0$ for $i > j + 1$. Due to symmetry, $a_{ji} = 0$ whenever $a_{ij} = 0$.
- This means $a_{ij} = 0$ for both $i > j + 1$ and $j > i + 1$, effectively enforcing $a_{ij} = 0$ for all $|i - j| > 1$.

(v) **Resulting Matrix Structure:**
- The only nonzero entries of the symmetric matrix are along the main diagonal ($|i-j| = 0$) and the first subdiagonals ($|i-j| = 1$), making it a **tridiagonal matrix**.

Thus, the upper Hessenberg form of a symmetric matrix naturally becomes tridiagonal because the symmetry constraint forces zeros to appear below and above the second subdiagonals. □

1.11 2.1 R

2 Image Compression

2.1 The Outer Product Form of the SVD

(a) Prove theorem 2.1

Proof.
We have $A = U\Sigma V^T$, the SVD Decomposition of A

Let $U = \begin{bmatrix} u_1 & u_2 & ... & u_r & u_{r+1} & ... & u_m \end{bmatrix}$ and,

Let $V = \begin{bmatrix} v_1 & v_2 & ... & v_r & v_{r+1} & ... & v_n \end{bmatrix}$

Therefore, $V^T = \begin{bmatrix} v_1^T \\ v_2^T \\ ... \\ v_r^T \\ v_{r+1}^T \\ ... \\ v_n^T \end{bmatrix}$

We want to show that
$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_r u_r v_r^T$

$$U\Sigma V^T = \begin{bmatrix} u_1 & u_2 & ... & u_r & u_{r+1} & ... & u_m \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \sigma_1 & 0 & ... & 0 \\ 0 & \sigma_2 & ... & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & ... & \sigma_r \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ ... \\ v_r^T \\ v_{r+1}^T \\ ... \\ v_n^T \end{bmatrix}$$

$$\implies U\Sigma V^T = \begin{bmatrix}\sigma_1 u_1 & \sigma_2 u_2 & ... & \sigma_r u_r & (0)u_{r+1} & ... & (0)u_n\end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \\ ... \\ v_r^T \\ v_{r+1}^T \\ ... \\ v_n^T \end{bmatrix}$$

$$\implies U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_r u_r v_r^T + (0)u_{r+1}v_{r+1}^T + ... + (0)u_n v_n^T$$

$$\implies A = U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_r u_r v_r^T$$

□

(b) $A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$

$$A^T A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Solving the characteristic equation, $det(A^T A - \lambda I) = 0$
$(9-\lambda)^2 - (-9)^2 = 0 \iff 81 + \lambda^2 - 18\lambda - 81 = 0 \iff \lambda^2 - 18\lambda = 0$
$\lambda = 18, 0$
Finding the respective eigenvectors,
Solving $A^T A - \lambda I = 0$ For $\lambda = 18$
$A^T A - 18I = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}$

$$\implies -9x_1 = 9x_2 \implies x_1 = -x_2 \implies v_1^{'} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\implies v_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

For $\lambda = 0$

$$A^T A - 0I = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\implies 9x_1 = 9x_2 \implies x_1 = x_2 \implies v_2^{'} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\implies v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Therefore, $V = [v_1 v_2] = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \implies V^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

And, $\Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Finally, we need to calculate U

Since $r = 1$, we can only calculate one left singular value

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{18}} \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{18}} \begin{bmatrix} \frac{4}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \\ \frac{-4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{4}{6} \\ \frac{2}{6} \\ \frac{-4}{6} \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

Need x such that $x \cdot u_1 = 0$

$\frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3 = 0 \implies 2x_1 + x_2 = 2x_3 \implies x_3 = \frac{1}{2}x_2 + x_1$

$$\implies x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

Therefore we have, $w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$

Applying Gram-Schmidt,

$u_2^{'} = w_1$

$u_2 = \frac{u_2^{'}}{||u_2^{'}||} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$u_3^{'} = w_2 - (w_2 \cdot u_2)u_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} - (0 + 0 + \frac{1}{2\sqrt{2}}) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1/4 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$

$u_3 = \frac{u_3^{'}}{||u_3^{'}||} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \\ \frac{1}{3\sqrt{2}} \end{bmatrix}$

Then, we have $U = [u_1 u_2 u_3] = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$

Finally, we have

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = U\Sigma V^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

And the **Reduced Singular Value Decomposition** is:

$$A = \sigma_1 u_1 v_1^T = \sqrt{18} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = A$$

As expected.

2.2 Digital Image Compression

See attached file `Q2-2.ipynb`

(a) Original Image

Figure 1: Ankkit

(b) Original image matrix dimensions: 400×400

(c) Rank 1 Approximation:

Figure 2: Rank 1 Approximation

(d) Retained Singular Values vs Compression Ratio

Singular Values Retained	**Compression Ratio**	**Image**
5	39.95006242197253	
10	19.975031210986266	
40	4.9937578027465666	
60	3.329171868497711	

Table 1: Singular Values Retained vs Compression Ratio

(e) Original Image vs Compressed Image

Original Image Compressed Image (k=60)

Figure 3: Original Image vs Compressed Image