DEVELOPMENT OF FINITE ELEMENT MODEL
of 3D Bioheat Equation

Governing Equation

Developing finite element model of the governing heat transfer equation is one of the most reliable method to study heat transfer in human tissue as related to the local blood flow. Pennes' bio-heat equation is the foundation of bio-heat transfer which is expressed as,

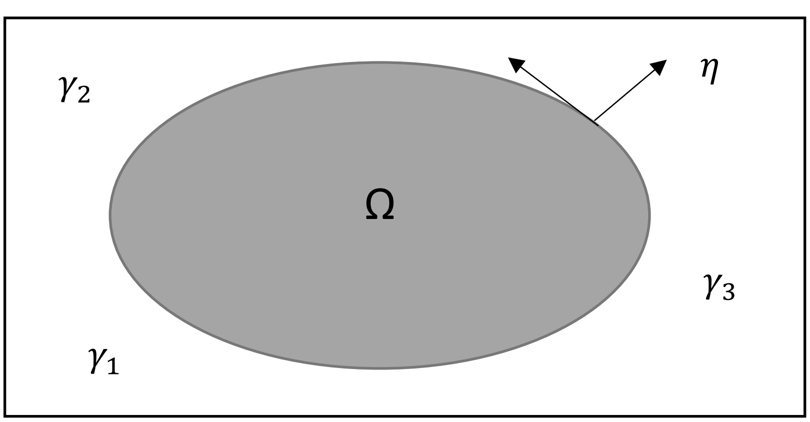
$$\mathrm{c}\rho\frac{\partial \mathrm{T}}{\partial \mathrm{t}} = \left\{\frac{\partial}{\partial \mathrm{x}}\left(\lambda\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right) + \frac{\partial}{\partial \mathrm{y}}\left(\lambda\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right) + \frac{\partial}{\partial \mathrm{y}}\left(\lambda\frac{\partial \mathrm{T}}{\partial \mathrm{z}}\right) + \omega_\mathrm{b}\rho_\mathrm{b}\mathrm{c}_\mathrm{b}(\mathrm{T_a} - \mathrm{T})\right\} \qquad (1)$$

Where, $\mathrm{T_a}$ is known arterial temperature and T is unknown tissue temperature. ρ, c and λ denote density, specific heat and thermal conductivity of the tissue respectively. $\omega_\mathrm{b}, \rho_\mathrm{b}, \mathrm{c}_\mathrm{b}$ are respectively blood perfusion, density and specific heat of blood.

Simple form of this equation is-

$$c\rho\frac{\partial T}{\partial t} = \left\{\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial z}\right) + f - BT\right\} \qquad (2)$$

Where, $\omega_b\rho_b c_b$ = B and BT_a= f.
Temperature regulation, $T = \bar{T}$ on γ_1
Heat flow regulation, $q = \bar{q}$ on γ_2
Heat transfer boundary (convention), $\mathrm{q} = \alpha_c(T - T_a)$ on γ_3



3.2 Development of Weak Form

The weak form of the differential equation can be expressed as,

$$0 = \int_{\Omega} W\left[\rho c\frac{\partial T}{\partial t} - \left\{\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)\right\} + BT - f\right] d\Omega$$

$$= \int_{\Omega} \rho c\frac{\partial T}{\partial t} W d\Omega - \int_{\Omega} \left\{\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)\right\} W d\Omega + B\int_{\Omega} TW d\Omega - \int_{\Omega} fW d\Omega \qquad (3)$$

Where, W is weighted function as it is weighted integral statement.

The Gauss divergence theorem is used to convert the volume integral to area integral, so that

$$\int_{\Omega} -w\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) d\Omega = \int_{\Omega} \lambda\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} d\Omega - \int_{\partial\Omega} \lambda W\frac{\partial T}{\partial x}\eta_x dS \qquad (4)$$

$$\int_{\Omega} -w\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) d\Omega = \int_{\Omega} \lambda\frac{\partial w}{\partial y}\frac{\partial T}{\partial y} d\Omega - \int_{\partial\Omega} \lambda W\frac{\partial T}{\partial y}\eta_y dS \qquad (5)$$

$$\int_{\Omega} -w\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right) d\Omega = \int_{\Omega} \lambda\frac{\partial w}{\partial z}\frac{\partial T}{\partial z} d\Omega - \int_{\partial\Omega} \lambda W\frac{\partial T}{\partial z}\eta_z dS \qquad (6)$$

So, the second term of right-hand side of the equation (3.3),

$$\int_{\Omega} \lambda\left(\frac{\partial W}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial T}{dy} + \frac{\partial W}{dz}\frac{\partial T}{dz} + \right) d\Omega - \int_{d\Omega} \lambda W\left(\eta_x\frac{\partial\theta}{dx} + \eta_y\frac{\partial\theta}{dy} + \eta_z\frac{\partial\theta}{dz}\right) dS \qquad (7)$$

For convective boundary, the natural boundary condition is a balance of energy transfer across the boundary due to conduction and/ or convection where Newton's law of cooling is applicable,

$$\eta_x\lambda\frac{\partial T}{dx} + \eta_y\lambda\frac{\partial T}{dy} + \eta_z\lambda\frac{\partial T}{dz} + \alpha_c(T - T_c) = \hat{q} \qquad (8)$$

Where, α_c is the convective heat transfer coefficient in W m^{-20}C^{-1}, $\hat{q}$ is the specific heat flow and, T_c is the ambient temperature. The first three terms with first order differentiation account for heat transfer by conduction, the fourth term for heat transfer by convection, while the term on the right hand side accounts for specific heat flux. The boundary integral should be modified to account for the convective heat transfer term.

So, the second term of the equation (7),

$$= \int_{\gamma_2} \hat{q} W dS - \int_{\gamma_3} \alpha_c (T - T_c) W dS \tag{9}$$

The final equation is expressed as,

$$\int_{\Omega} \rho\, c \frac{\partial T}{\partial t} W d\Omega + \int_{\Omega} \lambda \left(\frac{\partial T}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial W}{\partial y} + \frac{\partial T}{\partial z}\frac{\partial W}{\partial z} \right) d\Omega + \quad \int_{\gamma_3} \alpha_c W T dS + \int_{\Omega} B T W d\Omega = \int_{\Omega} f W\, d\Omega$$
$$+ \int_{\gamma_2} \hat{q} W ds + \int_{\gamma_3} \alpha_c W T_c dS \tag{10}$$

Time Discretization Scheme

Considered weak form discretization method for the equation (3.10),

$$\left\{ \phi K + \frac{1}{\Delta t} C \right\} T_{(t+\Delta t)} = \left\{ (\phi - 1) K + \frac{1}{\Delta t} C \right\} T_{(t)} + f \tag{11}$$

Where,

$T_{(t)}$[K] is temperature at time, t

$T_{(t+\Delta t)}$ [K] is temperature at time, $t + \Delta t$

Δt is time increment

Φ is a constant between 0 and 1.

Here, Φ is a weighting factor. For Crank-Nicolson method the value of Φ is 0.5 where for Galerkin method the value of Φ is 0.66.

And K, C and f are respectively heat conduction matrix, heat capacity matrix and heat flux vector.

$$K = \int_{\Omega} \lambda \left(\frac{\partial T_h}{\partial x}\frac{\partial W_h}{\partial x} + \frac{\partial T_h}{\partial y}\frac{\partial W_h}{\partial y} + \frac{\partial T_h}{\partial z}\frac{\partial W_h}{\partial z} \right) d\Omega + \int_{\Gamma_3} \alpha_c W_h T_h dS + B \int_{\Omega} W_h T_h\, d\Omega;$$

$$C = \int_{\Omega} \rho\, c T_h W_h d\Omega;$$

$$f = \int_{\Omega} fW_h \, d\Omega + \int_{\Gamma_2} qW_h ds + \int_{\Gamma_3} \alpha_c W_h T_c dS$$

Here, for equation (11) at initial time, the value of temperature is $T_{(t)}$. As time reach at $t + \Delta t$, the temperature of the node reach at $T_{(t+\Delta t)}$.

$$T(0) \rightarrow T(\Delta t) \rightarrow T(2\Delta t)$$

Here, K, C, and f value are needed to solve for three dimensional tetrahedron objects which is shown as-

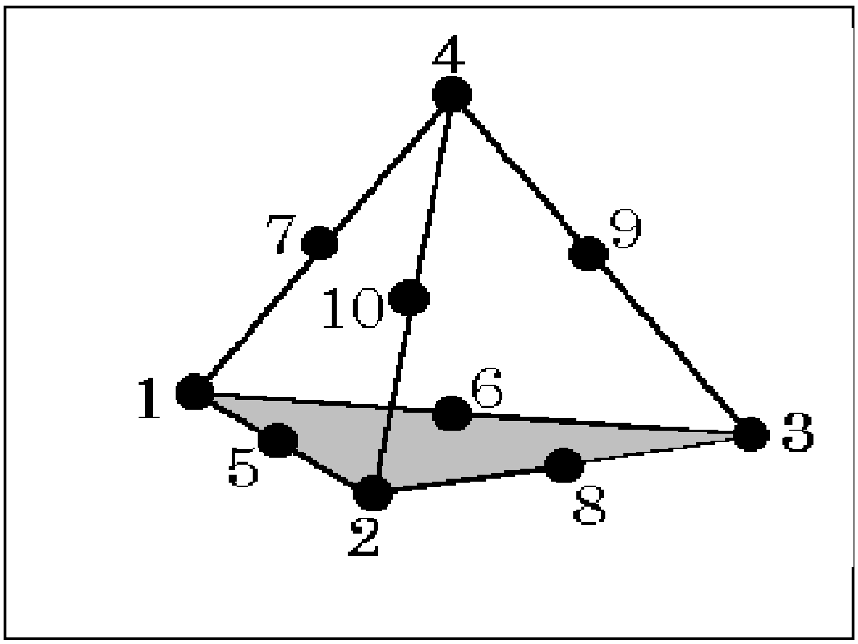


Figure 3.2 Three dimensional tetrahedral objects

Where, (L1, L2, L3, L4) are volume coordinates corresponding to that vertex and L_i is coordinate space.

Tetrahedral Approximation

For (x, y, z) coordinate

$$L_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z) \tag{12}$$

Where,

$$a_i = \begin{bmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{bmatrix}$$

$$b_i = -\begin{bmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{bmatrix}$$

$$c_i = -\begin{bmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_l & 1 & z_l \end{bmatrix}$$

$$d_i = -\begin{bmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_l & y_l & 1 \end{bmatrix}$$

The value of (x_i, y_j, z_k) for (i, j, k, l) given below,

Table 1

i	**j**	**k**	**l**
1	2	3	4
2	4	3	1
3	4	1	2
4	2	1	3

Those are natural number i, j, k, l are vertices in the direction in which the right-handed screw advances with respect to the triangular surface ijk.

Table 2

i	m	n
5	1	2
6	1	3
7	1	4
8	2	3
9	3	4
10	2	4

Finite Element method

This element information is in the element based on the finite element approximation of above method. First, element information is examined for the heat diffusion term and then, for the heat capacity term. Furthermore, by the heat transfer boundary.

Here the result matrix will show K, C and f. For the local node numbers are taken in the element as shown in Figure 3.1, the approximate value of T_h can be express as,

$$T_h = \sum_{i=1}^{10} N_i \, T_{h(i)}$$

$$= \sum_{i=1}^{4} L_i(2L_i - 1)T_{h(i)} + \sum_{i=5}^{10} 4L_{m(i)}L_{n(j)}\, T_{h(i)} \tag{13}$$

$$\begin{cases} \dfrac{\partial}{\partial x}[L_i(2L_i - 1)] = \dfrac{b_i}{6V}(4L_i - 1) \ , \\ \dfrac{\partial}{\partial y}[L_i(2L_i - 1)] = \dfrac{c_i}{6V}(4L_i - 1) \ , \\ \dfrac{\partial}{\partial z}[L_i(2L_i - 1)] = \dfrac{d_i}{6V}(4L_i - 1) \ . \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial x}\left[L_i L_j\right] = \frac{1}{6V}\left(b_i L_j + b_j L_i\right) \ , \\ \frac{\partial}{\partial y}\left[L_i L_j\right] = \frac{1}{6V}\left(c_i L_j + c_j L_i\right) \ , \\ \frac{\partial}{\partial z}\left[L_i L_j\right] = \frac{1}{6V}\left(d_i L_j + d_j L_i\right) \ . \end{cases}$$

Integral formula of volumetric coordinates,

$$\int_\Omega L_1^i L_2^j L_3^k L_4^l \, d\Omega = 6V \frac{i!j!k!l!}{(i+j+k+l+3)!} \tag{14}$$

And,

$$\int_\Omega L_i \, d\Omega = \frac{V}{4} \quad , \quad \int_\Omega L_i L_j \, d\Omega = \frac{V}{20} \quad , \int_\Omega {L_i}^2 \, d\Omega = \frac{V}{10}$$

The first term of heat transfer matrix,

$$\int_{\Omega} \lambda\left(\frac{\partial \mathrm{T_h}}{\partial \mathrm{x}}\frac{\partial \mathrm{W_h}}{\partial \mathrm{x}}+\frac{\partial \mathrm{T_h}}{\partial \mathrm{y}}\frac{\partial \mathrm{W_h}}{\partial \mathrm{y}}+\frac{\partial \mathrm{T_h}}{\partial \mathrm{z}}\frac{\partial \mathrm{W_h}}{\partial \mathrm{z}}\right)\mathrm{d}\Omega$$

For equation 3.13,

$$\frac{\partial \mathrm{T_h}}{\partial \mathrm{x}}\frac{\partial \mathrm{W_h^*}}{\partial \mathrm{x}}=\frac{\partial}{\partial \mathrm{x}}\left[\sum_{\mathrm{j=1}}^{10}\mathrm{N_j T_{h(j)}}\right]\cdot\frac{\partial}{\partial \mathrm{x}}\left[\sum_{\mathrm{i=1}}^{10}\mathrm{N_i W_{h(i)}}\right]$$

For node i,

$$\frac{\partial}{\partial \mathrm{x}}\left[\sum_{\mathrm{j=1}}^{10}\mathrm{N_j T_{h(j)}}\right]\cdot\frac{\partial}{\partial \mathrm{x}}\mathrm{N_i}$$

For i=1-4;

$$\frac{\partial}{\partial \mathrm{x}}\left[\sum_{\mathrm{j=1}}^{10}\mathrm{N_j T_{h(j)}}\right]\cdot\frac{\partial}{\partial \mathrm{x}}\mathrm{N_i}=\frac{\mathrm{b_i}}{6\mathrm{V}}(4L_i-1)\left[\frac{1}{6V}\sum_{j=1}^{4}b_j\big(4L_j-1\big)T_{h(j)}+\frac{1}{6V}\sum_{j=5}^{10}4\big\{b_{m(j)}L_{n(j)}+b_{n(j)}L_{m(j)}\big\}T_{h(i)}\right]$$

$$=\frac{1}{36V^2}\left[\sum_{j=1}^{4}\underline{\underline{b_i}}b_j\left\{16\underline{\underline{\mathrm{L_i}}}L_j-4\left(\underline{\underline{\mathrm{L_i}}}+L_j\right)+1\right\}T_{h(j)}+\sum_{j=5}^{10}4\underline{\underline{b_i}}\left\{\left(4\underline{\underline{\mathrm{L_i}}}L_{n(j)}-L_{n(j)}\right)b_{m(j)}+\left(4\underline{\underline{\mathrm{L_i}}}L_{m(j)}-L_{m(j)}\right)b_{n(j)}\right\}\right]T_{(j)} \quad (15)$$

For value i =5-10,

$$\frac{\partial}{\partial \mathrm{x}}\left[\sum_{\mathrm{j=1}}^{10}\mathrm{N_j T_{h(j)}}\right]\cdot\frac{\partial}{\partial \mathrm{x}}\mathrm{N_i}=\frac{4}{6\mathrm{V}}\left\{b_{m(i)}L_{n(i)}+b_{n(i)}L_{m(i)}\right\}\left[\frac{1}{6V}\sum_{j=1}^{4}b_j\big(4L_j-1\big)T_{h(j)}+\frac{1}{6V}\sum_{j=5}^{10}4\big\{b_{m(j)}L_{n(j)}+b_{n(j)}L_{m(j)}\big\}T_{h(i)}\right]$$

$$=\frac{1}{36V^2}\left[\sum_{j=1}^{4}\underline{\underline{b_i}}b_j\left\{16\underline{\underline{\mathrm{L_i}}}L_j-4\left(\underline{\underline{\mathrm{L_i}}}+L_j\right)+1\right\}T_{h(j)}+\sum_{j=5}^{10}4\underline{\underline{b_i}}\left\{\left(4\underline{\underline{\mathrm{L_i}}}L_{n(j)}-L_{n(j)}\right)b_{m(j)}+\left(4\underline{\underline{\mathrm{L_i}}}L_{m(j)}-L_{m(j)}\right)b_{n(j)}\right\}T_{(j)}\right] \quad (3.16)$$

For each element,

$$\int_{\Omega} \lambda_0 \left(\frac{\partial \theta_h}{\partial x} \frac{\partial {\theta_h}^*}{\partial x} + \frac{\partial \theta_h}{\partial y} \frac{\partial {\theta_h}^*}{\partial y} + \frac{\partial \theta_h}{\partial z} \frac{\partial {\theta_h}^*}{\partial z} \right) d\Omega =$$

①	②^T
②	③

Where, T represents transport matrix.

① $\left(\times \frac{-\lambda}{180V} \right)$

$-3(b_1b_1+c_1c_1+d_1d_1)$			
$b_2b_1+c_2c_1+d_2d_1$	$-3(b_2b_2+c_2c_2+d_2d_2)$		
$b_3b_1+c_3c_1+d_3d_1$	$b_3b_2+c_3c_2+d_3d_2$	$-3(b_3b_3+c_3c_3+d_3d_3)$	
$b_4b_1+c_4c_1+d_4d_1$	$b_4b_2+c_4c_2+d_4d_2$	$b_4b_3+c_4c_3+d_4d_3$	$-3(b_4b_4+c_4c_4+d_4d_4)$

② $\left(\times \frac{-\lambda}{180V}\right)$

$b_1(b_1\text{-}3b_2)$ $+c_1(c_1\text{-}3c_2)$ $+d_1(d_1\text{-}3d_2)$	$b_2(\text{-}3b_1+b_2)$ $+c_2(\text{-}3c_1+c_2)$ $+d_2(\text{-}3d_1+d_2)$	$b_3(b_1+b_2)$ $+c_3(c_1+c_2)$ $+d_3(d_1+d_2)$	$b_4(b_1+b_2)$ $+c_4(c_1+c_2)$ $+d_4(d_1+d_2)$
$b_1(b_1\text{-}3b_3)$ $+c_1(c_1\text{-}3c_3)$ $+d_1(d_1\text{-}3d_3)$	$b_2(b_1+b_3)$ $+c_2(c_1+c_3)$ $+d_2(d_1+d_3)$	$b_3(\text{-}3b_1+b_3)$ $+c_3(\text{-}3c_1+c_3)$ $+d_3(\text{-}3d_1+d_3)$	$b_4(b_1+b_3)$ $+c_4(c_1+c_3)$ $+d_4(d_1+d_3)$
$b_1(b_1\text{-}3b_4)$ $+c_1(c_1\text{-}3c_4)$ $+d_1(d_1\text{-}3d_4)$	$b_2(b_1+b_4)$ $+c_2(c_1+c_4)$ $+d_2(d_1+d_4)$	$b_3(b_1+b_4)$ $+c_3(c_1+c_4)$ $+d_3(d_1+d_4)$	$b_4(\text{-}3b_1+b_4)$ $+c_4(\text{-}3c_1+c_4)$ $+d_4(\text{-}3d_1+d_4)$
$b_1(b_2+b_3)$ $+c_1(c_2+c_3)$ $+d_1(d_2+d_3)$	$b_2(b_2\text{-}3b_3)$ $+c_2(c_2\text{-}3c_3)$ $+d_2(d_2\text{-}3d_3)$	$b_3(\text{-}3b_2+b_3)$ $+c_3(\text{-}3c_2+c_3)$ $+d_3(\text{-}3d_2+d_3)$	$b_4(b_2+b_3)$ $+c_4(c_2+c_3)$ $+d_4(d_2+d_3)$

$b_1(b_3+b_4)$ $+c_1(c_3+c_4)$ $+d_1(d_3+d_4)$	$b_2(b_3+b_4)$ $+c_2(c_3+c_4)$ $+d_2(d_3+d_4)$	$b_3(b_3-3b_4)$ $+c_3(c_3-3c_4)$ $+d_3(d_3-3d_4)$	$b_4(-3b_3+b_4)$ $+c_4(-3c_3+c_4)$ $+d_4(-3d_3+d_4)$
$b_1(b_2+b_4)$ $+c_1(c_2+c_4)$ $+d_1(d_2+d_4)$	$b_2(b_2-3b_4)$ $+c_2(c_2-3c_4)$ $+d_2(d_2-3d_4)$	$b_3(b_2+b_4)$ $+c_3(c_2+c_4)$ $+d_3(d_2+d_4)$	$b_4(-3b_2+b_4)$ $+c_4(-3c_2+c_4)$ $+d_4(-3d_2+d_4)$

③ $\left(\times \frac{4\lambda}{180V}\right)$

$2b_1b_1+b_1b_2+b_2b_1$ $+2b_2b_2+2c_1c_1+c_1c_2$ $+c_2c_1+2c_2c_2+2d_1d_1$ $+d_1d_2+d_2d_1+2d_2d_2$					
$b_1b_1+b_1b_2+b_3b_1+$ $2b_3b_2+c_1c_1+c_1c_2$ $+c_3c_1+2c_3c_2+d_1d_1$ $+d_1d_2+d_3d_1+2d_3d_2$	$2b_1b_1+b_1b_3+b_3b_1$ $+2b_3b_3+2c_1c_1+c_1c_3$ $+c_3c_1+2c_3c_3+2d_1d_1$ $+d_1d_3+d_3d_1+2d_3d_3$				

$b_1b_1+b_1b_2+b_4b_1+$ $2b_4b_2+c_1c_1+c_1c_2$ $+c_4c_1+2c_4c_2+d_1d_1$ $+d_1d_2+d_4d_1+2d_4d_2$	$b_1b_1+b_1b_3+b_4b_1+$ $2b_4b_3+c_1c_1+c_1c_3$ $+c_4c_1+2c_4c_3+d_1d_1$ $+d_1d_3+d_4d_1+2d_4d_3$	$2b_1b_1+b_1b_4+b_4b_1$ $+2b_4b_4+2c_1c_1+c_1c_4$ $+c_4c_1+2c_4c_4+2d_1d_1$ $+d_1d_4+d_4d_1+2d_4d_4$			
$b_2b_1+b_2b_2+2b_3b_1$ $+b_3b_2+c_2c_1+c_2c_2$ $+2c_3c_1+c_3c_2+d_2d_1$ $+d_2d_2+2d_3d_1+d_3d_2$	$2b_2b_1+b_2b_3+b_3b_1$ $+b_3b_3+2c_2c_1+c_2c_3$ $+c_3c_1+c_3c_3+2d_2d_1$ $+d_2d_3+d_3d_1+d_3d_3$	$b_2b_1+b_2b_4+b_3b_1$ $+b_3b_4+c_2c_1+c_2c_4$ $+c_3c_1+c_3c_4+d_2d_1$ $+d_2d_4+d_3d_1+d_3d_4$	$2b_2b_2+b_2b_3+b_3b_2$ $+2b_3b_3+2c_2c_2+c_2c_3$ $+c_3c_2+2c_3c_3+2d_2d_2$ $+d_2d_3+d_3d_2+2d_3d_3$		
$b_3b_1+b_3b_2+b_4b_1$ $+b_4b_2+c_3c_1+c_3c_2$ $+c_4c_1+c_4c_2+d_3d_1$ $+d_3d_2+d_4d_1+d_4d_2$	$b_3b_1+b_3b_3+2b_4b_1$ $+b_4b_3+c_3c_1+c_3c_3$ $+2c_4c_1+c_4c_3+d_3d_1$ $+d_3d_3+2d_4d_1+d_4d_3$	$2b_3b_1+b_3b_4+b_4b_1$ $+b_4b_4+2c_3c_1+c_3c_4$ $+c_4c_1+c_4c_4+2d_3d_1$ $+d_3d_4+d_4d_1+d_4d_4$	$b_3b_2+b_3b_3+2b_4b_2$ $+b_4b_3+c_3c_2+c_3c_3$ $+2c_4c_2+c_4c_3+d_3d_2$ $+d_3d_3+2d_4d_2+d_4d_3$	$2b_3b_3+b_3b_4+b_4b_3$ $+2b_4b_4+2c_3c_3+c_3c_4$ $+c_4c_3+2c_4c_4+2d_3d_3$ $+d_3d_4+d_4d_3+2d_4d_4$	
$b_2b_1+b_2b_2+2b_4b_1$ $+b_4b_2+c_2c_1+c_2c_2$ $+2c_4c_1+c_4c_2+d_2d_1$ $+d_2d_2+2d_4d_1+d_4d_2$	$b_2b_1+b_2b_3+b_4b_1$ $+b_4b_3+c_2c_1+c_2c_3$ $+c_4c_1+c_4c_3+d_2d_1$ $+d_2d_3+d_4d_1+d_4d_3$	$2b_2b_1+b_2b_4+b_4b_1$ $+b_4b_4+2c_2c_1+c_2c_4$ $+c_4c_1+c_4c_4+2d_2d_1$ $+d_2d_4+d_4d_1+d_4d_4$	$b_2b_2+b_2b_3+b_4b_2$ $+2b_4b_3+c_2c_2+c_2c_3$ $+c_4c_2+2c_4c_3+d_2d_2$ $+d_2d_3+d_4d_2+2d_4d_3$	$2b_3b_2+b_3b_4+b_4b_2$ $+b_4b_4+2c_3c_2+c_3c_4$ $+c_4c_2+c_4c_4+2d_3d_2$ $+d_3d_4+d_4d_2+d_4d_4$	$2b_2b_2+b_2b_4+b_4b_2$ $+2b_4b_4+2c_2c_2+c_2c_4$ $+c_4c_2+2c_4c_4+2d_2d_2$ $+d_2d_4+d_4d_2+2d_4d_4$

$\int_{\Gamma_3} \alpha_c W_h T_h dS$ is the second term of heat transfer matrix, k

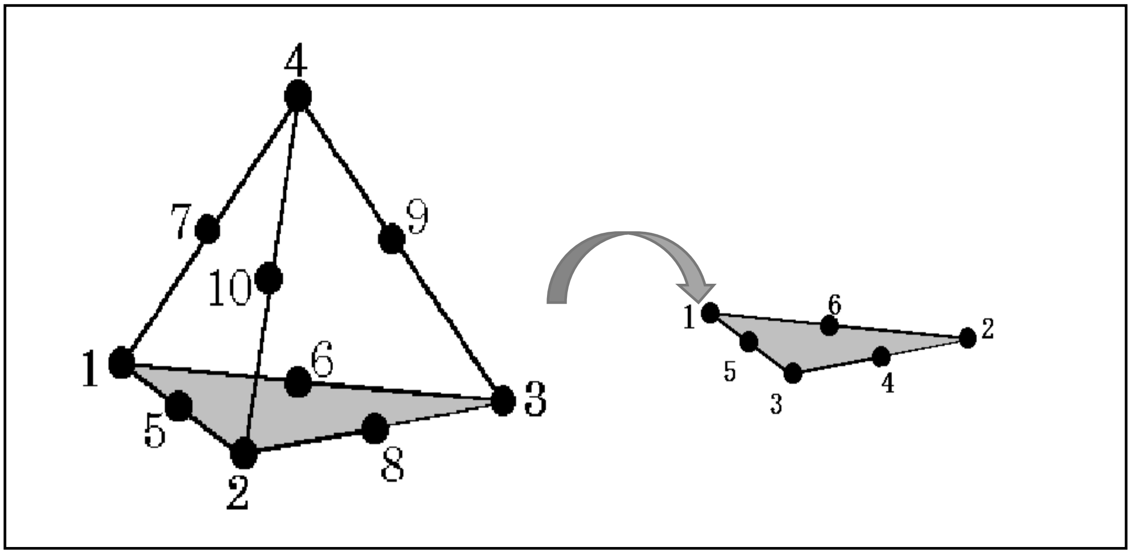


Figure 3.3 1-2-3 surface for heat conduction matrix

For, quadratic shape function, $\Phi \equiv (\varphi_1 \ldots \varphi_6)^T$

$\varphi_1 = L_1(2L_1 - 1), \quad \varphi_2 = L_2(2L_2 - 1), \quad \varphi_3 = L_3(2L_3 - 1)$

$\varphi_4 = 4L_2L_3, \quad \varphi_5 = 4L_1L_3, \quad \varphi_6 = 4L_1L_2,$

Properties of area coordinate is $L_1 + L_2 + L_3 = 1 \varphi_1 = L_1^2 - L_1L_2 - L_1L_3, \quad \varphi_2 = L_2^2 - L_2L_1 - L_2L_3, \quad \varphi_3 = L_3^2 - L_3L_1 - L_3L_2$ (3.17)

Where, $\Phi = Og$

So, it can be written as,

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad g = \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_2L_3 \\ L_3L_1 \\ L_1L_2 \end{bmatrix}$$

An approximation by the tetrahedron quadratic element of the function,

$$T_{\hbar} = \Phi^T T = g^T O^T T$$

So,

$$\begin{aligned} \int_{\Gamma_3} \alpha_c W_{\hbar} T_{\hbar} dS &= \int_{\Gamma_3} \alpha_c W^T \Phi^T \Phi T dS \\ &= \int_{\Gamma_3} \alpha_c W^T O g g^T O^T T dS \end{aligned} \tag{3.18}$$

Here,

$$g \cdot g^T = \begin{bmatrix} L_1^4 & L_1^2L_2^2 & L_1^2L_3^2 & L_1^2L_2L_3 & L_1^3L_3 & L_1^3L_2 \\ L_1^2L_2^2 & L_2^4 & L_2^2L_3^2 & L_2^3L_3 & L_2^2L_1L_3 & L_2^3L_1 \\ L_1^2L_3^2 & L_2^2L_3^2 & L_3^4 & L_3^3L_2 & L_3^3L_1 & L_3^2L_1L_2 \\ L_1^2L_2L_3 & L_2^3L_3 & L_3^3L_2 & L_2^2L_3^2 & L_3^2L_1L_2 & L_2^2L_1L_3 \\ L_1^3L_3 & L_2^2L_1L_3 & L_3^3L_1 & L_3^2L_1L_2 & L_1^2L_3^2 & L_1^2L_2L_3 \\ L_1^3L_2 & L_2^3L_1 & L_3^2L_1L_2 & L_2^2L_1L_3 & L_1^2L_2L_3 & L_1^2L_2^2 \end{bmatrix}$$

$$\int_{\Gamma_2} g g^T dS = \frac{S}{180} \begin{bmatrix} 12 & 2 & 2 & 1 & 3 & 3 \\ 2 & 12 & 2 & 3 & 1 & 3 \\ 2 & 2 & 12 & 3 & 3 & 1 \\ 1 & 3 & 3 & 2 & 1 & 1 \\ 3 & 1 & 3 & 1 & 2 & 1 \\ 3 & 3 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$
Ogg^TO^T = \frac{S}{180}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 12 & 2 & 2 & 1 & 3 & 3 \\ 2 & 12 & 2 & 3 & 1 & 3 \\ 2 & 2 & 12 & 3 & 3 & 1 \\ 1 & 3 & 3 & 2 & 1 & 1 \\ 3 & 1 & 3 & 1 & 2 & 1 \\ 3 & 3 & 1 & 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & 0 \\ -1 & 0 & -1 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & 0 & 4 \end{bmatrix}
$$

$$
= \frac{S}{180}\begin{bmatrix} 6 & -1 & -1 & -4 & 0 & 0 \\ -1 & 6 & -1 & 0 & -4 & 0 \\ -1 & -1 & 6 & 0 & 0 & -4 \\ -4 & 0 & 0 & 32 & 16 & 16 \\ 0 & -4 & 0 & 16 & 32 & 16 \\ 0 & 0 & -4 & 16 & 16 & 32 \end{bmatrix}
$$

$$
\int_{\Gamma_3} \mathrm{\alpha_c T_h W_h dS} = \frac{\alpha_c S}{180}\begin{bmatrix} 6 & -1 & -1 & -4 & 0 & 0 \\ -1 & 6 & -1 & 0 & -4 & 0 \\ -1 & -1 & 6 & 0 & 0 & -4 \\ -4 & 0 & 0 & 32 & 16 & 16 \\ 0 & -4 & 0 & 16 & 32 & 16 \\ 0 & 0 & -4 & 16 & 16 & 32 \end{bmatrix}
$$

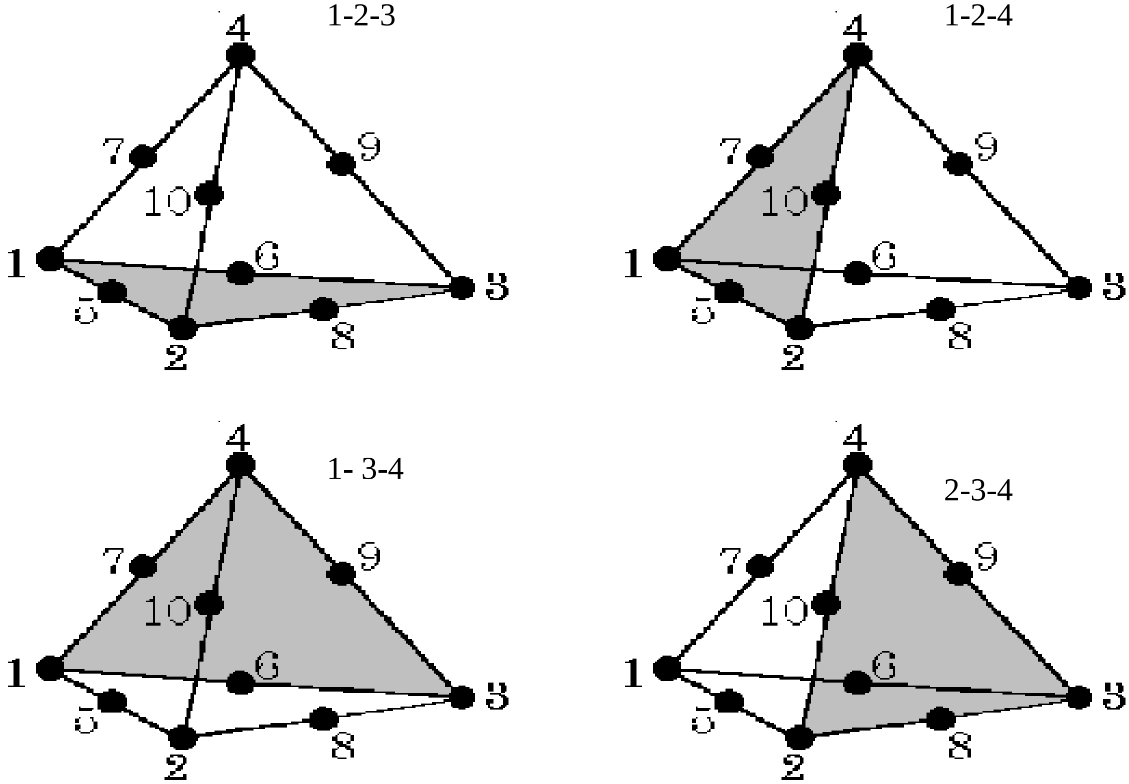


Figure 3.4 Different surface for heat conduction matrix

For 1-2-3 surface,

$$\int_{\Gamma_3} \alpha_c T_h W_h dS = \frac{\alpha_c S}{180}\begin{bmatrix} 6 & -1 & -1 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ -1 & -1 & 6 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 32 & 16 & 0 & 16 & 0 & 0 \\ 0 & -4 & 0 & 0 & 16 & 32 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 16 & 16 & 0 & 32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For 1-2-4 surface,

$$\int_{\Gamma_3} \alpha_c T_h W_h dS = \frac{\alpha_c S}{180}\begin{bmatrix} 6 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -4 \\ -1 & 6 & 0 & -1 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 6 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 32 & 0 & 16 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 16 & 0 & 32 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 16 & 0 & 16 & 0 & 0 & 32 \end{bmatrix}$$

For 1-3-4 surface,

$$\int_{\Gamma_3} \alpha_c T_h W_h dS = \frac{\alpha_c S}{180}\begin{bmatrix} 6 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 6 & -1 & 0 & -4 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 6 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 & 0 & 32 & 16 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & 32 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 16 & 16 & 0 & 32 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For 2-3-4 surface,

$$\int_{\Gamma_3} \alpha_c T_h W_h dS = \frac{\alpha_c S}{180}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & -1 & -1 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & -1 & 6 & -1 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & -1 & -1 & 6 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 32 & 16 & 16 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 & 16 & 32 & 16 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 16 & 16 & 32 \end{bmatrix}$$

The next term of heat transfer matrix, $B\int_{\Omega} W_h T_h \, d\Omega$where the shape function is

$\Phi \equiv (\varphi_1 \ldots \varphi_{10})^T$.

So,

$$\varphi_1 = L_1(2L_1 - 1),\ \varphi_2 = L_2(2L_2 - 1),\ \varphi_3 = L_3(2L_3 - 1),\ \varphi_4 = L_4(2L_4 - 1),$$

And,

$$\varphi_5 = 4L_1L_2, \quad \varphi_6 = 4L_1L_3, \quad \varphi_7 = 4L_1L_4,$$
$$\varphi_8 = 4L_2L_3, \quad \varphi_9 = 4L_3L_4, \quad \varphi_{10} = 4L_2L_4,$$

It is established from the figure that, $L_1 + L_2 + L_3 + L_4 = 1$

Where, $\Phi = Og$

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad g = \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_4^2 \\ L_1L_2 \\ L_1L_3 \\ L_1L_4 \\ L_2L_3 \\ L_3L_4 \\ L_2L_4 \end{bmatrix}$$

An approximation of tetrahedron quadratic element of the function,

$$T_h = \Phi^T \theta = g^T O^T T$$

Hence, B $\int_\Omega W_h T_h \, \mathrm{d}\Omega = \int_\Omega BW^T \phi^T \phi T d\Omega$

$$= \int_\Omega \mathrm{BW^T O g g^T O^T T d\Omega} \tag{3.19}$$

Again,

$$\int_\Omega L_i^4 d\Omega = \frac{V}{35}, \quad \int_\Omega L_i^2 L_j^2 d\Omega = \frac{V}{210}, \quad \int_\Omega L_i^3 L_j d\Omega = \frac{V}{140},$$
$$\int_\Omega L_i^2 L_j L_k d\Omega = \frac{V}{420}, \quad \int_\Omega L_i L_j L_k L_l d\Omega = \frac{V}{840}$$

Here,

$$g \cdot g^T = \begin{bmatrix} L_1^4 & L_1^2L_2^2 & L_1^2L_3^2 & L_1^2L_4^2 & L_1^3L_2 & L_1^3L_3 & L_1^3L_3 & L_1^2L_2L_3 & L_1^2L_3L_4 & L_1^2L_2L_4 \\ L_1^2L_2^2 & L_2^4 & L_2^2L_3^2 & L_2^2L_4^2 & L_2^3L_1 & L_2^2L_1L_3 & L_2^2L_1L_4 & L_2^3L_3 & L_2^2L_3L_4 & L_2^3L_4 \\ L_1^2L_3^2 & L_2^2L_3^2 & L_3^4 & L_3^2L_4^2 & L_3^2L_1L_2 & L_3^3L_1 & L_3^2L_1L_4 & L_3^3L_2 & L_3^3L_4 & L_3^2L_2L_4 \\ L_1^2L_4^2 & L_2^2L_4^2 & L_3^2L_4^2 & L_4^4 & L_4^2L_1L_2 & L_4^2L_1L_3 & L_4^3L_1 & L_4^2L_2L_3 & L_4^3L_3 & L_4^3L_2 \\ L_1^3L_2 & L_2^3L_1 & L_3^2L_1L_2 & L_4^2L_1L_2 & L_1^2L_2^2 & L_1^2L_2L_3 & L_1^2L_2L_4 & L_2^2L_1L_3 & L_1L_2L_3L_4 & L_2^2L_1L_4 \\ L_1^3L_3 & L_2^2L_1L_3 & L_3^3L_1 & L_4^2L_1L_3 & L_1^2L_2L_3 & L_1^2L_3^2 & L_1^2L_3L_4 & L_3^2L_1L_2 & L_3^2L_1L_4 & L_1L_2L_3L_4 \\ L_1^3L_3 & L_2^2L_1L_4 & L_3^2L_1L_4 & L_4^3L_1 & L_1^2L_2L_4 & L_1^2L_3L_4 & L_1^2L_4^2 & L_1L_2L_3L_4 & L_4^2L_1L_3 & L_4^2L_1L_2 \\ L_1^2L_2L_3 & L_2^3L_3 & L_3^3L_2 & L_4^2L_2L_3 & L_2^2L_1L_3 & L_3^2L_1L_2 & L_1L_2L_3L_4 & L_2^2L_3^2 & L_3^2L_2L_4 & L_2^2L_3L_4 \\ L_1^2L_3L_4 & L_2^2L_3L_4 & L_3^3L_4 & L_4^3L_3 & L_1L_2L_3L_4 & L_3^2L_1L_4 & L_4^2L_1L_3 & L_3^2L_2L_4 & L_3^2L_4^2 & L_4^2L_2L_3 \\ L_1^2L_2L_4 & L_2^3L_4 & L_3^2L_2L_4 & L_4^3L_2 & L_2^2L_1L_4 & L_1L_2L_3L_4 & L_4^2L_1L_2 & L_2^2L_3L_4 & L_4^2L_2L_3 & L_2^2L_4^2 \end{bmatrix}$$

From the value of above, $\int_\Omega g g^T d\Omega = \frac{V}{840} \begin{bmatrix} 24 & 4 & 4 & 4 & 6 & 6 & 6 & 2 & 2 & 2 \\ 4 & 24 & 4 & 4 & 6 & 2 & 2 & 6 & 2 & 6 \\ 4 & 4 & 24 & 4 & 2 & 6 & 2 & 6 & 6 & 2 \\ 4 & 4 & 4 & 24 & 2 & 2 & 6 & 2 & 6 & 6 \\ 6 & 6 & 2 & 2 & 4 & 2 & 2 & 2 & 1 & 2 \\ 6 & 2 & 6 & 2 & 2 & 4 & 2 & 2 & 2 & 1 \\ 6 & 2 & 2 & 6 & 2 & 2 & 4 & 1 & 2 & 2 \\ 2 & 6 & 6 & 2 & 2 & 2 & 1 & 4 & 2 & 2 \\ 2 & 2 & 6 & 6 & 1 & 2 & 2 & 2 & 4 & 2 \\ 2 & 6 & 2 & 6 & 2 & 1 & 2 & 2 & 2 & 4 \end{bmatrix}$

$$B\int_{\Omega} W_h T_h \, d\Omega =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 24 & 4 & 4 & 4 & 6 & 6 & 6 & 2 & 2 & 2 \\ 4 & 24 & 4 & 4 & 6 & 2 & 2 & 6 & 2 & 6 \\ 4 & 4 & 24 & 4 & 2 & 6 & 2 & 6 & 6 & 2 \\ 4 & 4 & 4 & 24 & 2 & 2 & 6 & 2 & 6 & 6 \\ 6 & 6 & 2 & 2 & 4 & 2 & 2 & 2 & 1 & 2 \\ 6 & 2 & 6 & 2 & 2 & 4 & 2 & 2 & 2 & 1 \\ 6 & 2 & 2 & 6 & 2 & 2 & 4 & 1 & 2 & 2 \\ 2 & 6 & 6 & 2 & 2 & 2 & 1 & 4 & 2 & 2 \\ 2 & 2 & 6 & 6 & 1 & 2 & 2 & 2 & 4 & 2 \\ 2 & 6 & 2 & 6 & 2 & 1 & 2 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

The solution of this matrix is,

$$B\int_{\Omega} W_h T_h \, d\Omega = \frac{BV}{420}\begin{bmatrix} 6 & 1 & 1 & 1 & -4 & -4 & -4 & -6 & -6 & -6 \\ 1 & 6 & 1 & 1 & -4 & -6 & -6 & -4 & -6 & -4 \\ 1 & 1 & 6 & 1 & -6 & -4 & -6 & -4 & -4 & -6 \\ 1 & 1 & 1 & 6 & -6 & -6 & -4 & -6 & -4 & -4 \\ -4 & -4 & -6 & -6 & 32 & 16 & 16 & 16 & 8 & 16 \\ -4 & -6 & -4 & -6 & 16 & 32 & 16 & 16 & 16 & 8 \\ -4 & -6 & -6 & -4 & 16 & 16 & 32 & 8 & 16 & 16 \\ -6 & -4 & -4 & -6 & 16 & 16 & 8 & 32 & 16 & 16 \\ -6 & -6 & -4 & -4 & 8 & 16 & 16 & 16 & 32 & 16 \\ -6 & -4 & -6 & -4 & 16 & 8 & 16 & 16 & 16 & 32 \end{bmatrix}$$

Which can finally be written as,

$$B\int_{\Omega} W_h T_h \, d\Omega = \frac{BV}{40}\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

For heat capacity matrix the value of $\int_\Omega W_h T_h \, d\Omega$ is same. So,

$$C = \rho c \int_\Omega W_h T_h \, d\Omega = \frac{c\rho V}{40}\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

Here, first term of heat flux matrix,

$$\int_\Omega f W_h \, d\Omega$$

Where, f = constant. For this condition,

$$\int_\Omega f W_h \, d\Omega = \sum_{i=1}^{10} \int_\Omega f N_i W_{h(i)} \, d\Omega \qquad (3.20)$$

Therefore, at a single node i, $\int_\Omega f \, N_i d\Omega$

Here for i =1～4,

$$N_i = L_i(2L_i - 1)$$

$$\int_\Omega f N_i d\,\Omega = \int_\Omega f(2L_i^2 - L_i) d\Omega = -\frac{1}{20} fV \qquad (3.21)$$

There for i =5～10,

$$N_i = 4L_{m(i)} L_{n(i)}$$

So, for each element,

$$\int_\Omega f W_h \, d\Omega = \frac{fV}{20}[-1 \quad -1 \quad -1 \quad -1 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4]^T$$

At heat flux matrix q, α_c, T_c are constant for $\int_{\Gamma_2} \mathrm{qW_h ds}$ and $\int_{\Gamma_3} \mathrm{\alpha_c W_h T_c dS}$. So, here both have same procedure $\int_{\Gamma_*} \mathrm{W_h dS}$ where different constant need to add where Γ_2 and Γ_3 got replaced with Γ_* for simplification. So,

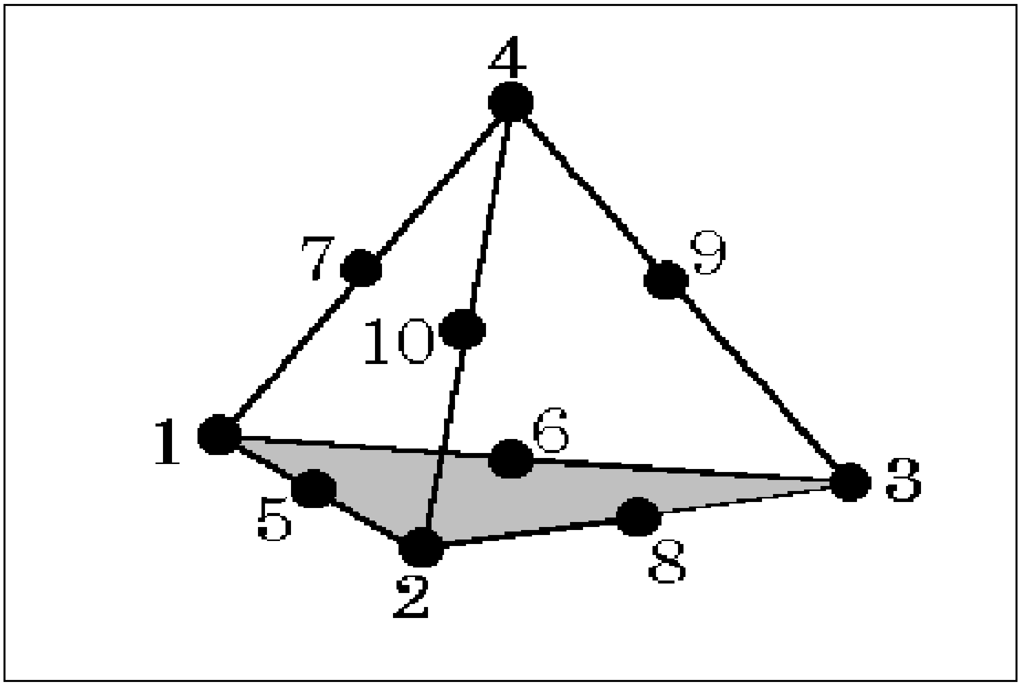


Figure 3.5 1-2-3surface for heat flux vector

$$\int_{\Gamma_*} \mathrm{WdS} = \sum_{i=1}^{10} \int_{\Gamma_*} \mathrm{N_i W_i dS} \tag{3.22}$$

Here, $\int_{\Gamma} L_1^i L_2^j L_3^k dS = 2S \frac{l!m!n!}{(l+m+n+2)!}$

And,

$$\int_{\Gamma} L_i \, dS = \frac{S}{3}, \quad \int_{\Gamma} L_i L_j dS = \frac{S}{12}, \quad \int_{\Gamma} {L_i}^2 dS = \frac{S}{6}, \quad \int_{\Gamma} L_i^4 dS = \frac{S}{15},$$

$$\int_{\Gamma} L_i^3 L_j dS = \frac{S}{60}, \quad \int_{\Gamma} L_i^2 L_j^2 dS = \frac{S}{90}, \quad \int_{\Gamma} L_i^2 L_j L_k dS = \frac{S}{180}$$

For i =1, 2, 3

$N_i = L_i(2L_i - 1)$

$$\Rightarrow \int_{\Gamma_*} N_i dS = \int_{\Gamma_*} (2L_i^2 - L_i) dS = 0 \qquad (3.23)$$

For, i= 5, 6, 8

$N_i = 4L_{m(i)}L_{n(i)}$

$$\Rightarrow \int_{\Gamma_*} N_i dS = \int_{\Gamma_*} 4L_{m(i)}L_{n(i)} dS = \frac{1}{3}S \qquad (3.24)$$

And, for i = 4, 7, 9, 10 the values are

$$\int_{\Gamma_*} N_i dS = 0$$

Therefore,

$$\int_{\Gamma_*} W dS = \frac{S}{3}[0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0]^T$$

And,

$$\int_{\Gamma_2} qW_\hbar dS = \frac{\bar{q}S}{3}[0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0]^T$$

$$\int_{\Gamma_3} \alpha_c T_c W_\hbar dS = \frac{\alpha_c T_a S}{3}[0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0]^T$$

All the necessary element information was collected above. The value of K, C and f are needed to set up in equation.