3-D Rotation Representations

Euler Angles (Z-Y-Z Euler Angles)

$$\mathbb{R} = \mathrm{Rot}(z,\alpha)\ \mathrm{Rot}(y,\beta)\ \mathrm{Rot}(z,\gamma).$$

[Fixed/Rotating?]

Roll-Pitch-Yaw Angles (Z-Y-X Euler Angles)

$$\mathbb{R} = \mathrm{Rot}(z,\alpha)\ \mathrm{Rot}(y,\beta)\ \mathrm{Rot}(z,\gamma).$$

[Fixed/Rotating?]

Axis-Angle Representation

$$\mathbb{R} = \mathrm{Rot}(k,\theta),$$

$$k = [k_x\ k_y\ k_z]^T, \quad \|k\| = 1.$$

Euler Parameters Representation

$$\epsilon = [\epsilon_1\ \ \epsilon_2\ \ \epsilon_3\ \ \epsilon_4]^T, \quad \|\epsilon\| = 1$$

$$\left(\sin\frac{\theta}{2}\right) k = [\epsilon_1\ \epsilon_2\ \epsilon_3]^T \cancel{\sin\frac{\theta}{2}}, \quad \cos\frac{\theta}{2} = \epsilon_4.$$

Direct Representation

$\rightleftharpoons$ Inverse Solution

Euler Angles (Z-Y-Z)

$$\mathbb{R} = Rot(z,\alpha)\, Rot(y,\beta)\, Rot(z,\gamma)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

Inverse problem: Given $\mathbb{R}$;

find α, β, γ.

[Two possible solutions]

$$\beta = \tan^{-1} \frac{\sqrt{r_{31}^2 + r_{32}^2}}{r_{33}}$$

$$\alpha = \tan^{-1} \frac{r_{23}/s\beta}{r_{13}/s\beta}$$

$$\gamma = \tan^{-1} \frac{r_{32}/s\beta}{\pm r_{31}/s\beta}$$

* For accuracy
* For $\beta \neq 0, \beta \neq \pi$
* Use Atan2 function

Degeneracy: $\beta = 0$ or π

Roll-Pitch-Yaw Angles (Z-Y-X)

$$\mathbb{R} = Rot(z,\alpha)\; Rot(y,\beta)\; Rot(x,\gamma)$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Inverse Solution:

$$\beta = \operatorname{atan2}\left(\pm r_{31},\ \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\alpha = \operatorname{atan2}\left(r_{21}/c\beta,\ r_{11}/c\beta\right)$$

$$\gamma = \operatorname{atan2}\left(r_{32}/c\beta,\ r_{33}/c\beta\right)$$

Degeneracy : $\beta = \pm \pi/2$

A special advantage:

Vector $\begin{bmatrix} \gamma \\ \beta \\ \alpha \end{bmatrix}$,

though meaningless for rotation representation,

upon differentiation will give

$$\begin{bmatrix} \dot\gamma \\ \dot\beta \\ \dot\alpha \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \omega \ \left(\begin{matrix}\text{Angular} \\ \text{Velocity}\end{matrix}\right)$$

Axis-Angle Representation

$$\mathbb{R} = \mathbb{R}_{align}\, Rot(z,\theta)\, \mathbb{R}_{restore}$$

$$= \begin{bmatrix} n_x & o_x & k_x \\ n_y & o_y & k_y \\ n_z & o_z & k_z \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & n_y & n_z \\ o_x & o_y & o_z \\ k_x & k_y & k_z \end{bmatrix}$$

$$= \begin{bmatrix} k_x^2 v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y \\ k_x k_y v\theta + k_z s\theta & k_y^2 v\theta + c\theta & k_y k_z v\theta - k_x s \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z^2 v\theta + c\theta \end{bmatrix}$$

Inverse Solution:

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$k = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Euler Parameters

$$\left(\sin\frac{\theta}{2}\right) k = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \frac{1}{4\cos\frac{\theta}{2}} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\cos\frac{\theta}{2} = \epsilon_4$$

$$\mathbb{R} = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_3\epsilon_3 + \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix}$$

Roll-Pitch-Yaw Angles (Z-Y-X)

$$\mathbb{R} = Rot(z,\alpha)\ Rot(y,\beta)\ Rot(x,\gamma)$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Inverse Solution:

$$\beta = atan2\left(\pm r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\alpha = atan2\left(r_{21}/c\beta,\ r_{11}/c\beta\right)$$

$$\gamma = atan2\left(r_{32}/c\beta,\ r_{33}/c\beta\right)$$

Degeneracy : $\beta = \pm \pi/2$

A special advantage:

Vector $\begin{bmatrix} \gamma \\ \beta \\ \alpha \end{bmatrix}$,

though meaningless for rotation representation,

upon differentiation will give

$$\begin{bmatrix} \dot\gamma \\ \dot\beta \\ \dot\alpha \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \omega \ \left(\begin{matrix}\text{Angular} \\ \text{Velocity}\end{matrix}\right)$$