UM 204
With ForTheL

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Contents

ForTheL Preamble

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[set/-s] [element/-s] [belong/-s]

Signature Set. A set is a notion.
Signature Element.
    Let S be a set.
    An element of S is a notion.

Let x << S denote (x is an element of S).
Let x belongs to S denote (x is an element of S).

Definition DefSubset.
    Let S be a set.
    A subset of S is a set T such that every (x << T) belongs to S.

Let S [= T denote (S is a subset of T).

Definition DefPowerSet.
    Let S be a set.
    The power set of S is the set P such that
    for every (x << P) x is a subset of S.

Let P = 2^S denote (P is the power set of S).

[pair/-s]

Signature Pair. A pair is a notion.

Signature FirstElementPair.
    Let Q be a pair.
    The first element of Q is a notion.

Signature SecondElementPair.
    Let Q be a pair.
    The second element of Q is a notion.

Let Q = (x,y) denote (Q is a pair and x is the first element of Q and y is the
   second element of Q).
Let (x,y) denote (for some pair Q x is the first element of Q and y is the second
   element of Q).

Definition DefCartesianProduct.
    Let S,T be sets.
    The cartesian product of S and T is the set U such that
    ((x,y) << U) for every (x << S), (y << T).

Let SxT denote (the cartesian product of S and T).

Definition DefRelationOn.
    Let S be a set.
    A relation on S is a subset R of SxS.

Let xRy denote ((x,y) << R).
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1 Number Systems

1.1 Natural Numbers

1.1.1 Definition

A relation R on A is a **partial order** if it satisfies

- Reflexivity, i.e., $xRy\ \forall x \in A$
- Anti-symmetry, i.e., $xRy \wedge yRx \implies x = y\ \forall x, y \in A$
- Transitivity, i.e., $xRy \wedge yRz \implies xRz\ \forall x, y, z \in A$

Additionally, if R satisfies

- xRy or $yRx\ \forall x, y \in A$

then R is an **order** (or total order) on A. A set with a partial order is called a partially ordered set or **poset**, and a set with a total order is called an ordered set or **totally ordered set**.

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Definition DefReflexiveRelationOn.
    Let S be a set.
    A reflexive relation on S is a
       relation R on S such that
    xRx for every (x << S).

Definition DefAntiSymmetricRelationOn.
    Let S be a set.
    An antisymmetric relation on S is a
       relation R on S such that
    (xRy and yRx => x = y) for every (x
       << S, y << S).

Definition DefTransitiveRelationOn.
    Let S be a set.
    A transitive relation on S is a
       relation R on S such that
    (xRy and yRz => xRz) for every (x <<
        S, y << S, z << S).

Definition DefPartialOrderOn.
    Let A be a set.
    A partial order on A is a relation R
        on A such that
    R is reflexive and R is
       antisymmetric and R is
       transitive.

Definition DefOrderOn.
    Let A be a set.
    An order on A is a relation R on A
       such that
    (R is a partial order on A) and ((
       xRy or yRx) for every (x << A, y
        << A)).

Let R is a total order on A denote (R is
    an order on A).

Definition DefPartiallyOrderedSet
    A partially ordered set is a pair (A
       ,R) such that A is a set and R
       is a partial order on A.

Let (A,R) is a poset denote ((A,R) is a
   partially ordered set).

Definition DefTotallyOrderedSet.
    A totally ordered set is a pair (A,R
       ) such that A is a set and R is
       a total order on A.

Let (A,R) is an ordered set denote ((A,R
   ) is a totally ordered set).
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