UM 204

With ForTheL

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Contents

ForTheL Preamble

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[set/-s] [element/-s] [belong/-s]

Signature Set. A set is a notion.
Signature Element.
    Let S be a set.
    An element of S is a notion.

Let x << S denote (x is an element of S).
Let x belongs to S denote (x is an element of S).

Definition DefSubset.
    Let S be a set.
    A subset of S is a set T such that every (x << T) belongs to S.

Let S [= T denote (S is a subset of T).

Definition DefPowerSet.
    Let S be a set.
    The power set of S is the set P such that
    for every (x << P) x is a subset of S
    and for every (T [= S) T is an element of P.

Let P = 2^S denote (P is the power set of S).

[pair/-s]

Signature Pair. A pair is a notion.

Signature FirstElementPair.
    Let Q be a pair.
    The first element of Q is a notion.

Signature SecondElementPair.
    Let Q be a pair.
    The second element of Q is a notion.

Let Q = (x,y) denote (Q is a pair and x is the first element of Q and y is the
   second element of Q).
Let (x,y) denote (for some pair Q x is the first element of Q and y is the second
   element of Q).

Definition DefCartesianProduct.
    Let S,T be sets.
    The cartesian product of S and T is the set U such that
    for every (x << S), (y << T) ((x,y) << U)
    and for every ((Q << U) such that (for some x, y such that Q = (x,y)))
    (x << S) and (y << T).

Let SxT denote (the cartesian product of S and T).

Definition DefRelationOn.
    Let S be a set.
    A relation on S is a subset R of SxS.

Let xRy denote ((x,y) << R).
```

1 Number Systems

1.1 Natural Numbers

1.1.1 Definition

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Definition DefNaturalNumbers.
    ...

Let N denote (the set of natural numbers
   ).
```

1.2 Relations

1.2.1 Definition

A relation R on A is a **partial order** if it satisfies

- Reflexivity, i.e., $xRy \; \forall x \in A$
- Anti-symmetry, i.e., $xRy \wedge yRx \implies x = y \; \forall x, y \in A$
- Transitivity, i.e., $xRy \wedge yRz \implies xRz \; \forall x, y, z \in A$

Additionally, if R satisfies

- xRy or $yRx \; \forall x, y \in A$

then R is an **order** (or total order) on A. A set with a partial order is called a partially ordered set or **poset**, and a set with a total order is called an ordered set or **totally ordered set**.

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Definition DefReflexiveRelationOn.
    Let S be a set.
    A reflexive relation on S is a
       relation R on S such that
    xRx for every (x << S).

Definition DefAntiSymmetricRelationOn.
    Let S be a set.
    An antisymmetric relation on S is a
       relation R on S such that
    (xRy and yRx => x = y) for every (x
       << S, y << S).

Definition DefTransitiveRelationOn.
    Let S be a set.
    A transitive relation on S is a
       relation R on S such that
    (xRy and yRz => xRz) for every (x <<
        S, y << S, z << S).

Definition DefPartialOrderOn.
    Let A be a set.
    A partial order on A is a relation R
        on A such that
    R is reflexive and R is
       antisymmetric and R is
       transitive.

Definition DefOrderOn.
    Let A be a set.
    An order on A is a relation R on A
       such that
    (R is a partial order on A) and ((
       xRy or yRx) for every (x << A, y
        << A)).

Let R is a total order on A denote (R is
     an order on A).

Definition DefPartiallyOrderedSet
    A partially ordered set is a pair (A
       ,R) such that A is a set and R
       is a partial order on A.

Let (A,R) is a poset denote ((A,R) is a
   partially ordered set).

Definition DefTotallyOrderedSet.
    A totally ordered set is a pair (A,R
       ) such that A is a set and R is
       a total order on A.

Let (A,R) is an ordered set denote ((A,R
   ) is a totally ordered set).
```

1.2.2 Definition

An **equivalence relation** on a set A is a relation R satisfying

- Reflexivity
- Symmetry, i.e., $xRy \implies yRx \; \forall x, y \in A$
- Transitivity

```
Definition DefSymmetricRelationOn.
    Let S be a set.
    A symmetric relation on S is a
        relation R on S such that
    (xRy => yRx) for every (x << S, y <<
         S).

Definition DefEquivalenceRelationOn.
    Let S be a set.
    An equivalence relation on S is a
        relation R on S such that
    R is reflexive and R is symmetric
        and R is transitive.
```

1.2.3 Definition

An **equivalence class** of an element x of a set A with respect to an equivalence relation R on A is defined as the set

$$[x]_R = \{y \in A \mid xRy\}$$

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Definition
   DefEquivalenceClassWithRespectTo.
    Let S be a set.
    Let R be an equivalence relation on
        S.
    Let x be an element of S.
    The equivalence class of x with
        respect to R is the set E
    such that for every (y << S) (y is
        an element of E <=> xRy).

Let [x](R) denote (the equivalence class
     of x with respect to R).
```

1.3 Integers

1.3.1 Definition

The set of integers is defined as

$$\mathbb{Z} = \{[x]_R \mid x \in \mathbb{N} \times \mathbb{N}\}$$

where R is the equivalence relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$(a, b)R(c, d) \iff a + d = b + c$$

```
Definition DefIntegers.
    Let R be the relation on NxN such
        that
    for every (a << NxN, b << NxN, c <<
        NxN, d << NxN)
    ((a,b)R(c,d) <=> a + d = b + c).
    The set of integers is the set Z
        such that
    for every x (x is an element of Z
        <=> (for some (a << NxN) x = [a
        ](R))).

Let Z denote (the set of integers).
```

1.4 Rational Numbers