Theory And Examples of “Fair” Game

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Fair game,Roulette,Python Simulation of Roulette,Martingale strategy, Martingale Stopping theorem,St. Petersburg Paradox,Solution of the St. Petersburg Paradox.

Abstract :

This paper discusses about fair game.Here the mathematics of roulette is discussed and some python simulation has been done to visualise the whole scenario. St. Petersburg Paradox can be resolved in various ways here some of them are discussed.

1 Introduction :

William Feller discussed about the Classical Definition of a “fair” game. So, Talking about a practical definition [2] and exhibiting some example [2.1],[3],[4] would be a good idea to get a overview of the whole concept .
Roulette is a game of probability so, win or loss is unpredictable but at the end of the day the house always wins. Why this happens is discussed here. But there are also some strategies to win over the house some of the strategies are discussed here, one of them is with the help

of Martingale Stopping theorem another intuitive solution is also given there with proper simulation to support the idea.
St. Petersburg paradox [4] remained a paradox for a long time . Daniel Bernoulli gave a intuitive solution to this paradox for the first time . But William Feller gave a mathematical explanation to this paradox . So, to get a good idea of the whole topic it necessary to discuss about the mathematical proof [4.1] .

Martingale strategy [6] is one of the strategies used in betting where the gambler must double the bet faced with a loss.It can mathematically proved that this strategy can make a roulette game almost fair game.

2 Fair Game :

A "fair game" or "fair bet" is a random game with a specified set of prizes and associated probabilities that has an expected value of zero.

2.1 Example.1 :

Flipping a fair coin with a friend for $ 1 (head you win $ 1, tail you lose $ 1). Let X = the random payoff of the game. The PDF of X is therefore,

States(p)	Probability(P)	VOP(x)
1(Head=Win)	0.5	+$1
2(Head=Win)	0.5	−$1

Thus, $E(X) = 0.5(+\$1) + 0.5(-\$1) = 0.5 - 0.5 = 0$.

2.2 Example.2 :

For \$ 8 a player can roll a pair of dice once. If the player gets a double, he is allowed to choose an envelop from a box containing four identical envelops which, however, contain different amounts of money. The amounts are \$ 5, \$ 25, \$ 50 and \$ 100.

- Is this a fair game ?
- If he played this game 25 times, how much would he expect to win/lose ?

Explanation:
Number of total possibilities of pairs : 36
Possible doubles : (1, 1); (2, 2); (3, 3); (4, 4); (5, 5); (6, 6)

$$p = \frac{6}{36} = \frac{1}{6}$$

The expected gain, E(x) = $\frac{1}{6} * \frac{1}{4}(5+25+50+100)\$ = \frac{1}{24}(180)\$ = 7.50\$$
Hence , the game is not fair .
The expected loss : (8.00-7.50) \$ = 0.50 \$
If he played this game 25 times, then the expected loss : $(25 \times 0.50)\$ = 12.50\$$

3 Example.3 : Roulette

Suppose the gambler goes to Las Vegas and makes \$1 bets on red or black in roulette. In this case, she will win \$1 with probability

$$\frac{18}{38} \approx 0.473$$

And she will lose \$1 with probability

$$\frac{20}{38} \approx 0.527$$

That's because the casinos and those bothersome green 0 and 00 to give the house a slight advantage.
At first glance (or after a few drinks), $\frac{18}{38}$ seems awfully close to $\frac{1}{2}$ and so our intuition tells us that the game is "almost fair". So we might expect the analysis we just did for the fair game to be "almost right" for the real game. For example, if the gambler starts with \$100 and quits when she gets ahead by \$100 in the fair game, then she goes home a winner with probability.

$$\frac{100}{200} \approx 0.5$$

And, if she wants to improve her chances of going home a winner, she could bring more money. If she brings \$1000 and quits when she gets ahead by \$100 in the fair game, then she goes home a winner with probability.

$$\frac{1000}{1100} \approx 0.91$$

So, given that the real game is "almost fair" we might expect the probabilities of going home a winner in these two scenarios to be "almost 50% and 91%", respectively.
Unfortunately for the gambler, all this "almost" reasoning will almost surely lead to disaster. Here are the grim facts for the real game where the gambler wins \$1 with probability $\frac{18}{38}$.

n=starting wealth	Probability that she reaches n+\$100 before 0
\$100	$\frac{1}{37649.619496\cdots}$
\$1000	$\frac{1}{37649.619496\cdots}$
\$1000	$\frac{1}{37649.619496\cdots}$

- She wins her first bet with probability p. She then has n+1 dollars.

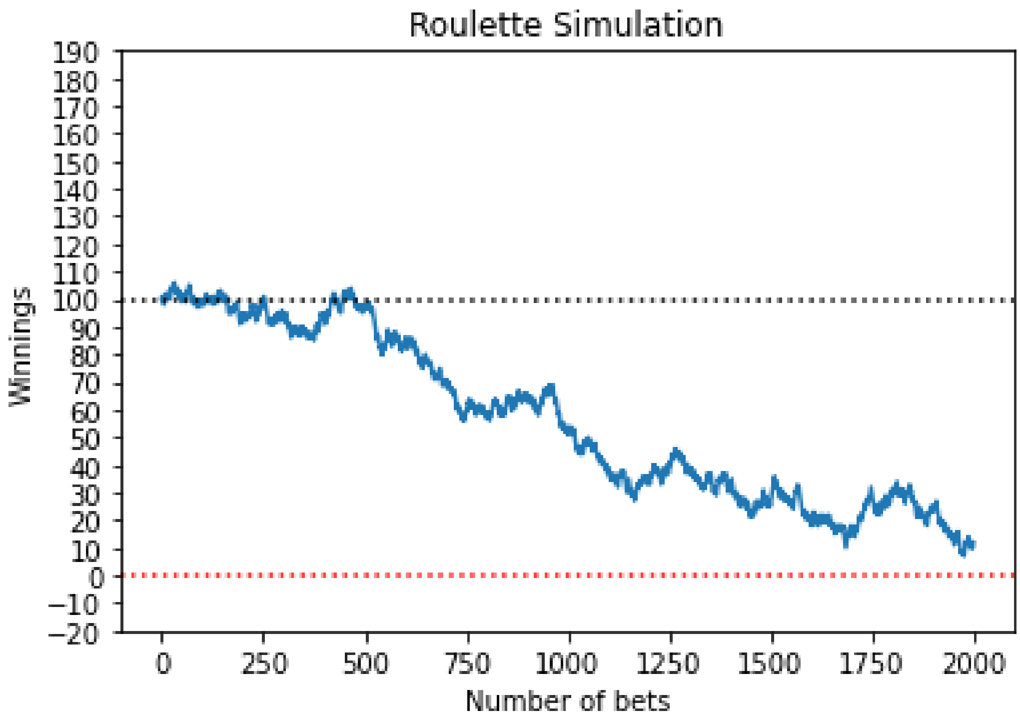

Figure 1: Roulette Simulation

- She loses her first bet with probability $(1-p)$.This leaves her with $n-1$ dollars and probability R_{n-1} of reaching her goal.

Plugging these facts into the Total Probability Theorem gives the equation:

$$R_n = pR_{n+1} + (1-p)R_{n-1}.$$

Solving this we get :

$$R_n = \frac{(\frac{1-p}{p})^n - 1}{(\frac{1-p}{p})^w - 1} \approx (\frac{1-p}{p})^{n-w} = (\frac{1-p}{p})^{-m}.$$

If m=100 then $p = \frac{18}{38}$ and her probability of success is : $(\frac{10}{9})^{-100} \approx \frac{1}{37648.619496}$.

So, it is not even close to a fair game. And the probability also does not depends on the initial money (n).It only depends on the goal

3.1 If the Roulette was a fair game :

Now take $p = \frac{1}{2}$
So,the equation :

$$R_n = pR_{n+1} + (1-p)R_{n-1}.$$

Has a double root . This leads to the general solution

$$R_n = a + bn$$

now let's consider the boundary conditions

- $$R_0 = 0 = a + b * 0 \Rightarrow a = 0$$

- $$R_w = 1 = 0 + b * w \Rightarrow b = \frac{1}{w}$$

This gives the solution

$$R_n = \frac{n}{w}$$

Here the probability of reaching her goal depends on her initial money and the goal both.

No particular prediction can be made here. It totally depends on the fortune of the gambler. But it can easily be shown that if the gambling goal is too high with respect to the initial money then it might not be reached. But if the gambling goal is small enough with respect to the initial money then the the probability of reaching the goal is too high.
Let the gambler starts with 100\$(n) and his/her goal is to reach 101\$(w). So the probability of reaching the goal is : $\frac{100}{101} \approx 0.990099$.

So, for a fair roulette if the gambler's goal is not high enough then it's possible for him/her to reach the goal (Here the sheer fall indicates that the gambler has reached her goal).

4 Martingale Strategy :

- The Martingale Strategy is a strategy of investing or betting introduced by French mathematician Paul Pierre Levy. It is considered a risky method of investing.

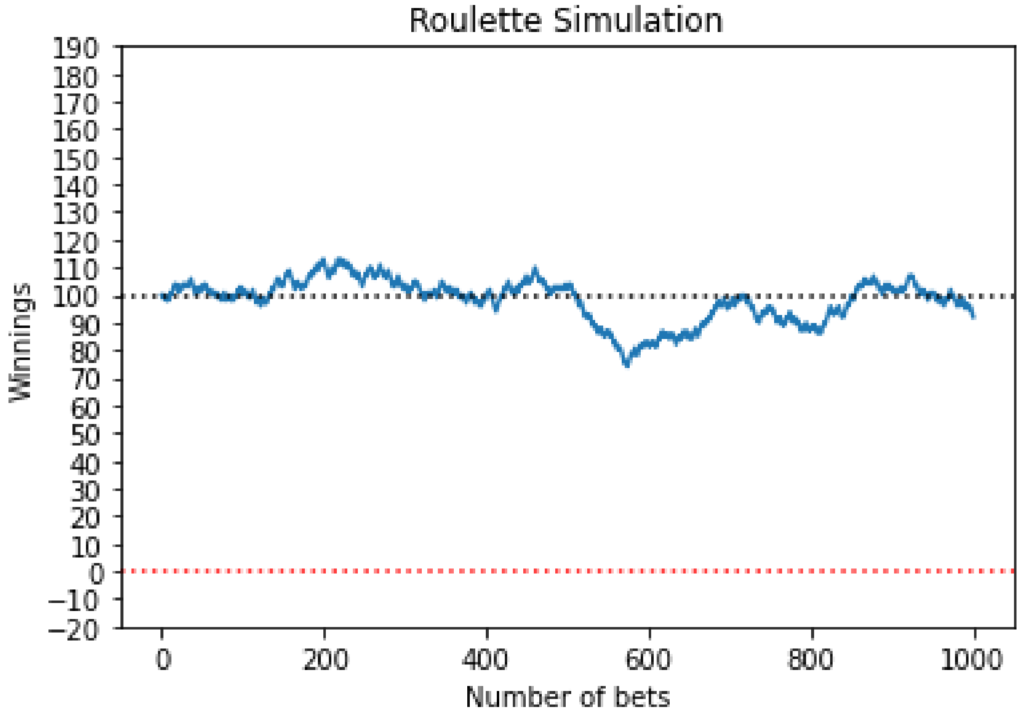

Figure 2:

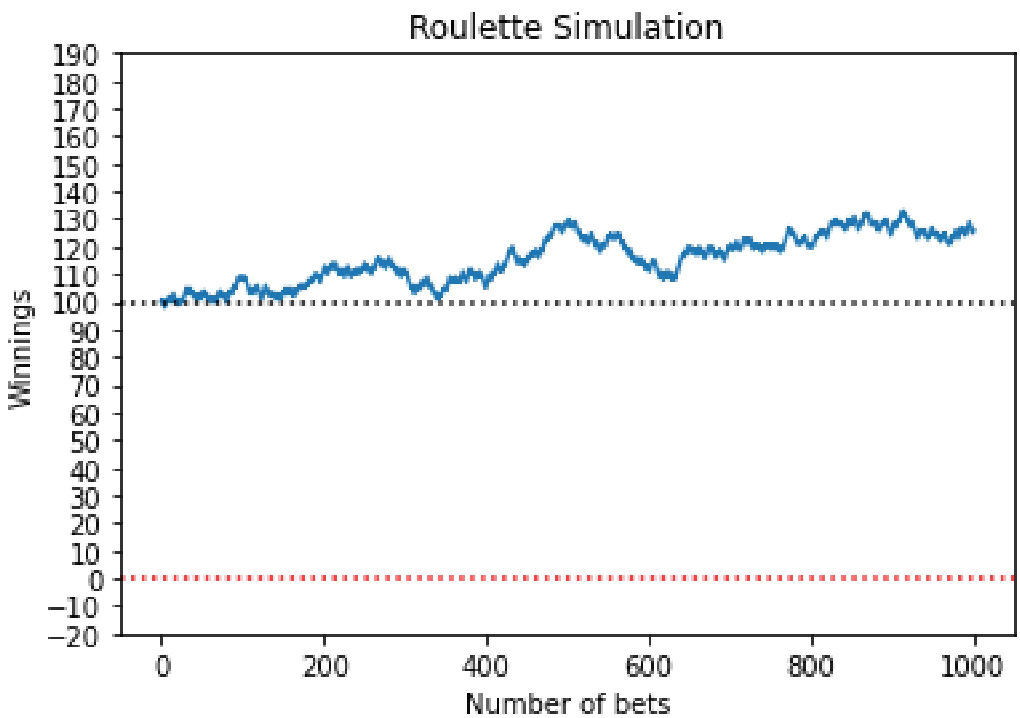

Figure 3:

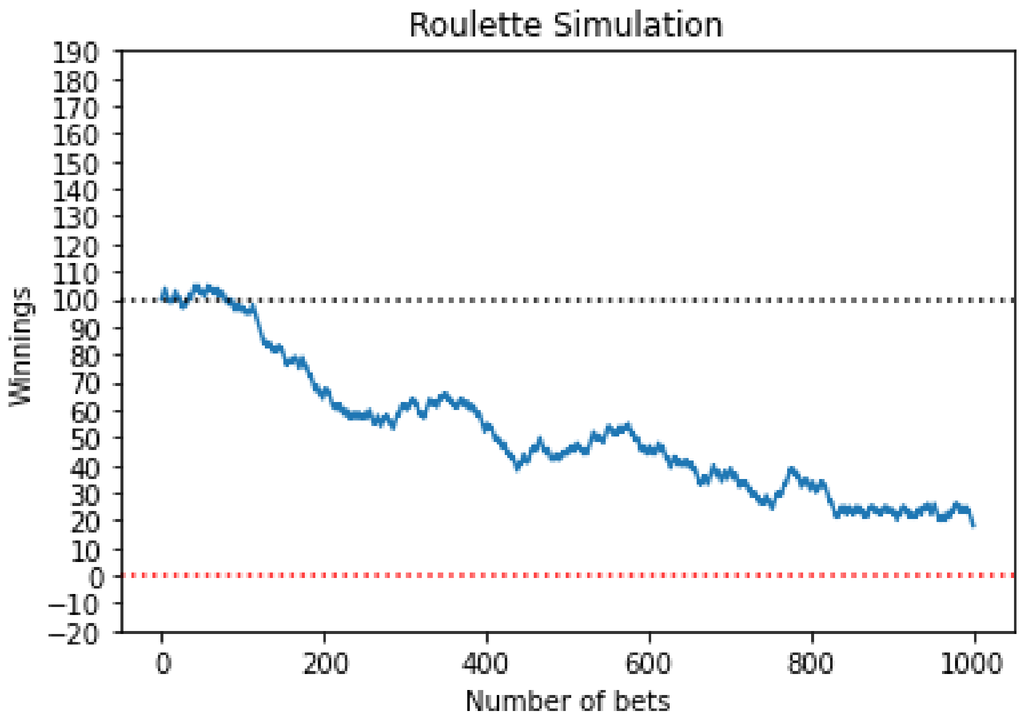
Roulette Simulation
Winnings
190
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
−10
−20
0
200
400
600
800
1000
Number of bets

Figure 4:

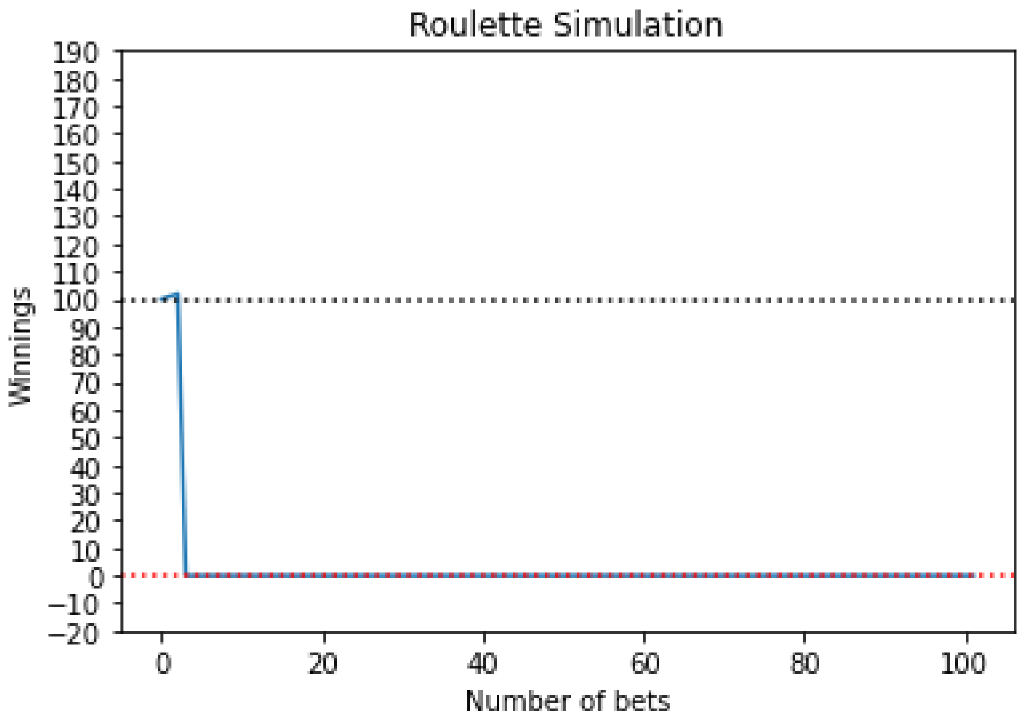

Figure 5: Gambler reaches the goal

- It is based on the theory of increasing the amount allocated for investments, even if its value is falling, in expectation of a future increase.
- When the Martingale Strategy is used in betting, the gambler must double the bet when faced with a loss.

A Martingale is (informally) a random process $X = \{X_n\}$ models a player's fortune in a fair game. That is to say, his expected fortune at time n given the history up to this point is equal to his current fortune :

$$E(X_n|X_1, X_2, \cdots, X_{n-1}) = X_{n-1}$$

This in turn implies that for all n,

$$E(X_n) = E(X_{n-1}) = \cdots = E(X_0)$$

So the player's expected fortune at any time is equal to his starting expected fortune

Martingale Stopping Theorem : A fair game that is stopped at a random time will remain fair to the end of the game if it is assumed that :

1. There is a finite amount of money in the world.
2. A player must stop if he wins all of this money or goes into debt by this amount.

If τ is a random stopping time and X_τ denotes the game stopped at this time, then

$$E(X_\tau|X_0) = X_0 \Rightarrow E(X_\tau) = E(X_0)$$

Example : Let us consider a game where a player will get 1 \$ if the outcome is head when a fair coin is tossed , either he will loose 1 \$. According to Martingale strategy , on the next bet , he should put 2 \$, if he looses , then again he should put 2^2 \$=4\$, on the very next bet and so on.
Let us assume that the player wins the game at (n+1) th bet, then he should put 2^n \$ and he earns 2^n \$ in this bet.
Till nth bet , the total loss = $(1 + 2 + 2^2 + \cdots + 2^{n-1}\$) = (2^n - 1)$.
Hence the expected gain : $(2^n - (2^n - 1))\$ = 1\$$
So let the player can play n times before getting bankrupt then $2^n = N \Rightarrow n = \log_2 N$ [Where N is the value of money with which the gambler started gambling.]

5 Example.5 : Petersburg Game

The St. Petersburg paradox is derived from the St.Petersburg game which is played as follows: A fair coin is flipped until it comes head for the first time.
Let the first head comes in the first flip so the gambler wins $\$2^1 = \$2$ with probability $\frac{1}{2}$. Now let in the first flip it lands tail and then in the second flip it comes head so the gambler gets $\$2^2 = \$4$ with probability $\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$ So, in this way if the first head comes in the nth trial then the gambler gets $\$2^k$ with probability $\frac{1}{2^k}$
Now, let's calculate the expected monetary value :
$\sum_{n=1}^{\infty} \frac{1}{2^n} * 2^n = 1 + 1 + \cdots + 1 = \infty$ So in order to make the game fair the gambler should pay an entry fees $e_n = \infty$ Dollars.

5.1 Willam Fellar's solution to the paradox :

Let the player in the Petersburg game [4] at the k-th trial pay the amount $\log_2 k$. The accumulated entrance fees up to the n-th trial are $P_n \sim n \log_2 n$ the game is "fair" in the sense that the law of large

numbers holds. This requirement determines the entrance fees essentially uniquely (that is to say up to terms of smaller order of magnitude which, by definition, remain undetermined).
Proof : We use the method of truncation, defining the variables U_k and V_k $(k = 1, 2, \cdots, n)$

$$U_k = X_k, \qquad V_k = 0 \quad if X_k \leq n \log_2 n$$

$$U_k = 0, \qquad V_k = X_k \quad if X_k > n \log_2 n \qquad \cdots (1)$$

Then,

$$P\{|e_n^{-1} S_n - 1| > \epsilon\} \leq P\{|U_1 + U_2 + \cdots + U_n - e_n| > \epsilon e_n\} + P\{V_1 + V_2 + \cdots + V_n \neq 0\}. \qquad \cdots (2)$$

Because the event on the left can not occur unless at least one of the events on the right is realized. Now,

$$P\{V_1 + V_2 + \cdots + V_n \neq 0\} \leq nP\{X_1 > n \log_2 n\} \leq \frac{2}{\log_2 n} \to 0. \quad \cdots (3)$$

To verify (1) it suffices therefore to prove that

$$P\{|U_1 + U_2 + \cdots + U_n - n \log_2 n| > \epsilon n \log_2 n\} \to 0. \qquad \cdots (4)$$

Put $\mu_n = E(U_k)\ and\ \sigma_n^2 = Var(U_k)$; these quantities depend on n but are common to $U_1, U_2, \cdots, U_n$. If r is the largest integer such that $2^r \leq n \log_2 n$, then $\mu_n = r$ and hence for sufficiently n

$$\log_2 n < \mu_n \leq \log_2 log_2 n. \qquad \cdots (5)$$

similarly, $\sigma_n^2 < E(U_k^2) = 2 + 2^2 + \cdots + 2^r < 2^{r+1} \leq 2n \log_2 n. \quad \cdots (6)$
Since the sum $U_1 + U_2 + \cdots + U_n$ has mean nμ_n and variance $n\sigma_n^2$,

we have by **Chebyshev's** inequality

$$P\{|U_1+U_2+\cdots+U_n-n\mu_n| > \epsilon n\mu_n\} \leq \frac{n\sigma_n^2}{\epsilon^2 n^2 \mu_n^2} < \frac{2}{\epsilon^2 n^2 \log_2 n} \to 0. \cdots (7)$$

Now by (5) $\mu_n \sim \log_2 n$ and hence (7) is equivalent to (4)

5.2 Some other ways to solve the paradox :

5.2.1 Method 1 :

We can think about a simpler solution to The Petersburg Paradox i.e. we can fix a boundary condition to the process . Let's say the rule is : the game will be played 50 times then it's not a paradox any more the following calculation shows this : Now let's calculate the expected monetary value :

$$\sum_{n=0}^{50} \frac{1}{2^n} * 2^n = 1 + 1 + \cdots + 1 = 50$$

So in order to make the game fair the gambler should pay \$50 only instead of paying infinity

5.2.2 Method 2 (Bernoulli's Solution):

Does the St. Petersburg Paradox mean that the expected value must be infinite? - concept of diminishing marginal utility first discussed by Bernoulli can be used to deal with the problem. he suggests that a realistic measure of the utility of money might be given by the logarithm of the amount: utiles = log(\$)
Introduce diminishing marginal utility of wealth.

$$U(x) = \log(x)$$
$$U = E[(x)] \Rightarrow U = \sum_{k=1}^{n} U(X_k)p^k$$

$$U = \sum_{k=1}^{\infty} \log(2^k) * \frac{1}{2^k} = \log(2) \sum_{k=1}^{\infty} \frac{k}{2^k} < +\infty$$

6 Conclusion :

There are many applications of the modern probability theory.One of the most beneficial and practical based gambling games can be discussed by the probability theories which has been thoroughly discussed through this paper. Either the game is Fair or not , if not, the amount of expected gain or loss has been calculated . Some strategies has been discussed through which an biased or unfair game can be made fair or at least almost fair. And some proofs has been given to establish the intuitive assumptions.

7 References :

[1] Introduction to Probability SECOND EDITION ,Dimitri P. Bertsekas and John N. Tsitsiklis, Massachusetts Institute of Technology.

[2] The Martingale Stopping Theorem,Scott M. LaLonde ,February 27, 2013

[3] Introduction To Probability Theory (Vol. 1),William Feller.(Page :251-253).

[4] The Law of Large Numbers And “Fair” Games , William Feller, Cornell University.