

School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Assignment 5 solution

- Find the extremals of the functional

$$J\{y\} = \int_0^1 (y'^2 + x^2) dx,$$

subject to the conditions

$$y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y^2 dx = 2.$$

We add the integral constraint to the original functional to form the new functional

$$H\{y\} = \int_0^1 (y'^2 + x^2 + \lambda y^2) dx.$$

The Euler-Lagrange equation is

$$\frac{\partial h}{\partial y} - \frac{d}{dx} \frac{\partial h}{\partial y'} = 2\lambda y - \frac{d}{dx}(2y') = 0.$$

From which we derive the second order, linear, constant coefficient ODE

$$y'' - \lambda y = 0.$$

There are three cases to consider depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

Case 1: $\lambda > 0$

In this case let $\lambda = \mu^2$, where $\mu \in \mathbb{R}^+$. The general solution is given by

$$y = A \cosh \mu x + B \sinh \mu x.$$

Applying the boundary conditions $y(0) = 0$ implies $A = 0$, and $y(1) = 0$ implies $B = 0$ and so we get the trivial solution which cannot possibly satisfy the integral constraint.

Case 2: $\lambda = 0$

In this case the general solution is given by

$$y = A + Bx.$$

Applying the boundary conditions $y(0) = 0$ implies $A = 0$, and $y(1) = 0$ implies $B = 0$ and so again we get the trivial solution which cannot possibly satisfy the integral constraint.

Case 3: $\lambda < 0$

In this case let $\lambda = -\mu^2$, where $\mu \in \mathbb{R}^+$. The general solution is given by

$$y = A \cos \mu x + B \sin \mu x.$$

Applying the boundary conditions $y(0) = 0$ implies $A = 0$, and $y(1) = 0$ implies either $B = 0$ or $\sin \mu = 0$. The first option would again lead to a trivial solution which is no good and so we look for roots of

$$\sin \mu = 0, \quad \Rightarrow \quad \mu = n\pi, \quad \text{where } n \in \mathbb{N} = \{1, 2, 3, \dots\}.$$

So the extremals are given by $y = B \sin(n\pi x)$. Now we look at the integral constraint to determine the arbitrary constant B .

$$\int_0^1 y^2 dx = \int_0^1 [B \sin(n\pi x)]^2 dx = 2$$

$$B^2 \int_0^1 \sin^2(n\pi x) dx = 2.$$

Using the orthogonality of trigonometric functions the integral has a fixed value of $1/2$ for all integer values of n . Therefore

$$B^2 \frac{1}{2} = 2$$

$$B^2 = 4$$

$$B = \pm 2.$$

So the extremals of the functional are given by

$$y = \pm 2 \sin(n\pi x).$$

Now substituting this into the original functional we get that

$$J\{y\} = \int_0^1 (y'^2 + x^2) dx = \int_0^1 [4n^2\pi^2 \cos^2(n\pi x) + x^2] dx.$$

Again using the orthogonality of trigonometric functions we have

$$J\{y\} = 4n^2\pi^2 \frac{1}{2} + \frac{1}{3} = 2n^2\pi^2 + \frac{1}{3}.$$

Obviously J is monotonically increasing over \mathbb{N} and therefore will take a minimum value for $n = 1$ and so the extremals for this functional are

$$y = \pm 2 \sin(\pi x).$$

2. Find the curve joining the points $(0, 0)$ and $(1, 0)$ for which the integral

$$\int_0^1 y''^2 dx,$$

is a minimum,

The Euler-Poisson equation is $\frac{d^2}{dx^2}(2y'') = 0$, which gives $y^{(4)} = 0$. The general solution to this ODE is

$$y = c_1x^3 + c_2x^2 + c_3x + c_4.$$

From the end-points we have $y(0) = 0$ and $y(1) = 0$ and so this lets us determine two arbitrary constants

$$c_4 = 0, \quad c_3 = -(c_1 + c_2),$$

which gives us the general solution

$$y = x(x - 1)[c_1(x + 1) + c_2].$$

if:

(a) $y'(0) = a$ and $y'(1) = b$.

So differentiating

$$y' = c_1(3x^2 - 1) + c_2(2x - 1).$$

The boundary conditions give

$$a = -c_1 - c_2, \quad b = 2c_1 + c_2.$$

from which we gave $c_1 = a + b$, and $c_2 = -2a - b$. So

$$y = x(x - 1)[(a + b)(x + 1) - 2a - b] = x(x - 1)[(a + b)x - a].$$

(b) no other conditions are prescribed.

If no other condition is prescribed then we have to use the natural boundary conditions for the problem, in this case

$$\frac{\partial f}{\partial y''} \Big|_{x=0} = 0, \quad \frac{\partial f}{\partial y''} \Big|_{x=1} = 0.$$

This equates to $y''(0) = y''(1) = 0$.

So differentiating again

$$y'' = 6c_1x + 2c_2.$$

So $y''(0) = 0$ implies $c_2 = 0$, and $y''(1) = 0$ implies $c_1 = 0$, and so we are left with the trivial solution

$$y = 0.$$

This is the extremal since it passes through the end points and minimises the functional (i.e. it gives a value of zero).

3. Among all the curves in C^2 joining a given point $(0, b)$ on the (positive) y -axis to a point on the (positive) x -axis, and enclosing a given area S together with the x - and y -axes, find the curve which generates the least area when rotated about the x -axis.

In this problem we consider x as the dependent variable and y the independent variable. We wish to find $x(y)$ given that $x(b) = 0$ and $x(0) = a$ which is unspecified. We also have an integral constraint which is

$$G\{x\} = \int_0^b x \, dy = S.$$

Now the quantity we want to find the extremal for is the area of the surface of revolution around the x -axis. This is given by

$$F\{x\} = \int_0^b 2\pi y \, ds = \int_0^b 2\pi y \sqrt{1 + x'^2} \, dy$$

So forgetting about the constant 2π multiplier and adding the integral constraint via a Lagrange multiplier we have

$$H\{x\} = \int_0^b \left(y \sqrt{1 + x'^2} + \lambda x \right) \, dy.$$

So now the Euler-Lagrange equation gives

$$\begin{aligned} \frac{d}{dy} \left(\frac{yx'}{\sqrt{1 + x'^2}} \right) &= \lambda \\ \frac{yx'}{\sqrt{1 + x'^2}} &= \lambda y + c_1. \end{aligned}$$

Now we consider the natural boundary condition at $y = 0$. Here

$$\frac{\partial h}{\partial x'} \Big|_{y=0} = 0 \quad \Rightarrow \quad \frac{yx'}{\sqrt{1+x'^2}} \Big|_{y=0} = 0.$$

Applying this in the previous equation for $y = 0$ tells us that $c_1 = 0$. Therefore

$$\frac{yx'}{\sqrt{1+x'^2}} = \lambda y \quad \Rightarrow \quad \frac{x'}{\sqrt{1+x'^2}} = \lambda.$$

Simplifying we derive

$$x' = \frac{\lambda}{\sqrt{1-\lambda^2}} = \beta.$$

and so

$$x = \beta y + \gamma.$$

Applying the conditions $x(0) = a$ and $x(b) = 0$ we find $\gamma = a$ and $\beta = -a/b$ and so

$$x = a - \frac{a}{b}y, \quad \text{or in other words} \quad \frac{x}{a} + \frac{y}{b} = 1.$$

Finally we determine a from the integral constraint

$$\begin{aligned} \int_0^b x \, dy &= S \\ \int_0^b \left(a - \frac{a}{b}y \right) \, dy &= S \\ \left[ay - \frac{a}{2b}y^2 \right]_0^b &= S \\ \frac{ab}{2} &= S \\ a &= \frac{2S}{b}, \end{aligned}$$

and so the solution is a straight line from $(0, b)$ to $(2S/b, 0)$.

4. In Newton's aero-dynamical problem we minimized resistance

$$F\{y\} = \int_0^R \frac{x}{1+y'^2} \, dx,$$

subject to $y(0) = L$ and $y(R) = 0$ (and $y' \leq 0$ and $y'' \geq 0$).

In nose-cone design this is sometimes approximated by assuming that the nose-cone will be long and thin, so y' will be large (and negative in our formulation). In that case, we may approximate $1+y'^2$ by y'^2 and simplify the problem.

- (a) Using this approximation, derive an alternative formulation of the problem where we don't specify the length of the nose-cone, and instead we specify the total surface area (often called the "wetted area") of the nose-cone.

Let's say $y(0) = L$ is unspecified but that the wetted area (W say) is specified and given by

$$W = 2\pi \int_0^R x \sqrt{1+y'^2} \, dx.$$

So adding this as an isoperimetric constraint we obtain the functional

$$H\{y\} = \int_0^R \left(\frac{x}{1+y'^2} + \lambda x \sqrt{1+y'^2} \right) dx.$$

Now we seek an approximate answer by approximating $1+y'^2 \approx y'^2$ then our approximate functional $\tilde{H}\{y\}$ is given more simply by

$$\tilde{H}\{y\} = \int_0^R \left(\frac{x}{y'^2} + \lambda xy' \right) dx.$$

Note that we also have the natural boundary condition at $x = 0$ which is that

$$\left. \frac{\partial \tilde{h}}{\partial y'} \right|_0 = 0.$$

- (b) Using the Calculus of Variations, find the optimal profile of the nose-cone with this formulation.
The problem is y -absent and so the Euler-Lagrange equation simplifies to

$$\frac{d}{dx} \left(\frac{\partial \tilde{h}}{\partial y'} \right) = 0.$$

Integrating we obtain that

$$\frac{\partial \tilde{h}}{\partial y'} = \text{const.}$$

However from our natural boundary condition we know that this must equal zero at $x = 0$ and therefore the constant must be zero. Therefore our Euler-Lagrange equation is given by

$$\begin{aligned} \frac{\partial \tilde{h}}{\partial y'} &= 0 \\ -2 \frac{x}{y'^3} + \lambda x &= 0 \\ y'^3 &= \frac{2}{\lambda} \\ y' &= \left(\frac{2}{\lambda} \right)^{1/3} = \alpha, \quad \text{a constant.} \end{aligned}$$

Integrating we obtain

$$y = \alpha(x - \beta).$$

Since $y(R) = 0$ we find that $\beta = R$ and to determine alpha we need to use our wetted area, isoperimetric constraint (and we are not going to make the approximation $1+y'^2 \approx y'^2$ here)

$$\begin{aligned} 2\pi \int_0^R x \sqrt{1+y'^2} dx &= W \\ 2\pi \int_0^R x \sqrt{1+\alpha^2} dx &= W \\ 2\pi \sqrt{1+\alpha^2} \int_0^R x dx &= W, \end{aligned}$$

and the integral is simply $R^2/2$ and so

$$\begin{aligned} 2\pi\sqrt{1+\alpha^2}\frac{R^2}{2} &= W \\ \sqrt{1+\alpha^2} &= \frac{W}{\pi R^2} \\ \alpha^2 &= \left(\frac{W}{\pi R^2}\right)^2 - 1. \end{aligned}$$

Now since $y' \leq 0$ we take alpha to be the negative root of this and so the final solution to the approximate problem is

$$y = (R-x)\sqrt{\left(\frac{W}{\pi R^2}\right)^2 - 1}.$$

Note that in this case the nose-cone is a right circular cone with length $L = y(0)$ given by

$$L = R\sqrt{\left(\frac{W}{\pi R^2}\right)^2 - 1}.$$

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