School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Assignment 3 solution

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1. Find the form of extremals of the following functionals

   (a) $F\{y(x), z(x)\} = \int_{x_0}^{x_1} (8yz - 5y^2 + y'^2 - 4z'^2)\, dx.$

   $$f(x, y, y', z, z') = 8yz - 5y^2 + y'^2 - 4z'^2.$$

   Here we have two dependent variables and one independent variable and so we get two Euler-Lagrange equations.

   $$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \Rightarrow \quad 8z - 10y - 2y'' = 0, \quad \text{and}$$
   $$\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = 0 \quad \Rightarrow \quad 8y + 8z'' = 0.$$

   From the second equation we have the relation that $y = -z''$. Substituting this in the first Euler-Lagrange equation we arrive at the fourth order equation

   $$z'''' + 5z'' + 4z = 0.$$

   Since this is homogeneous, linear and constant coefficient we can find a solution by using the trial solution $z = e^{\lambda x}$ from which we derive the characteristic equation

   $$\lambda^4 + 5\lambda^2 + 4 = 0,$$

   and this conveniently factorises into

   $$(\lambda^2 + 4)(\lambda^2 + 1) = 0 \quad \Rightarrow \quad (\lambda + 2i)(\lambda - 2i)(\lambda + i)(\lambda - i) = 0.$$

   Since the eigenvalues are purely imaginary the general solution is

   $$z(x) = A\cos 2x + B\sin 2x + C\cos x + D\sin x.$$

   Now from the relation $y = -z''$ we can readily derive y which is given by

   $$y(x) = 4A\cos 2x + 4B\sin 2x + C\cos x + D\sin x,$$

   and the constants A, B, C and D will be determined by the value of the functions $y(x)$ and $z(x)$ at the end-points $x = x_0$ and $x = x_1$.

   (b) $F\{\boldsymbol{q}\} = \int_{t_0}^{t_1} (\dot{q}_1 q_2 + \dot{q}_2 q_3 + q_1\dot{q}_3 - \dot{q}_1^2)\, dt.$

   $$f(t, q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3) = \dot{q}_1 q_2 + \dot{q}_2 q_3 + q_1\dot{q}_3 - \dot{q}_1^2.$$

Here we have three dependent variables and so we get three Euler-Lagrange equations.

$$\frac{\partial f}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_1}\right) = 0 \quad \Rightarrow \quad \dot{q}_3 - \dot{q}_2 + 2\ddot{q}_1 = 0,$$
$$\frac{\partial f}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_2}\right) = 0 \quad \Rightarrow \quad \dot{q}_1 - \dot{q}_3 = 0,$$
$$\frac{\partial f}{\partial q_3} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_3}\right) = 0 \quad \Rightarrow \quad \dot{q}_2 - \dot{q}_1 = 0.$$

From the second and third equations we have that $\dot{q}_1 = \dot{q}_2 = \dot{q}_3$. Substitution into the first Euler-Lagrange equation we arrive at the second order equation

$$\ddot{q}_1 = 0.$$

This may be integrated once to derive $\dot{q}_1 = A$, and since $\dot{q}_1 = \dot{q}_2 = \dot{q}_3$ we may write

$$\dot{\boldsymbol{q}} = (A, A, A),$$

and integrating again we arrive at the final solution

$$\boldsymbol{q}(t) = (At + B, At + C, At + D),$$

where the constants A, B, C and D will be determined by end-points.

2. Find the extremal of the following functional

$$F\{y\} = \int_0^1 (y''^2 - 360x^2 y)\, dx,$$

subject to $y(0) = 0$, $y'(0) = 1$, $y(1) = 1$ and $y'(1) = 5/2$.

$$f(x, y, y', y'') = y''^2 - 360x^2 y.$$

Here we have a problem with a functional that depends on a second derivative and so we have to employ the Euler-Poisson equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0 \quad \Rightarrow \quad -360x^2 + 2y^{(4)} = 0.$$

From this we arrive at the fourth order equation

$$y^{(4)} = 180x^2,$$

which we may integrate four times to arrive at the general solution

$$y(x) = \frac{1}{2}x^6 + Ax^3 + Bx^2 + Cx + D.$$

Now examining boundary condition we have

$$y(0) = 0 \quad \Rightarrow \quad D = 0.$$
$$y'(0) = 1 \quad \Rightarrow \quad C = 1.$$
$$y(1) = 1 \quad \Rightarrow \quad A + B = -\frac{1}{2}.$$
$$y'(1) = \frac{5}{2} \quad \Rightarrow \quad 3A + 2B = -\frac{3}{2}.$$

Now from these last two equations we find $A = -1/2$ and $B = 0$ and so the final answer is given by

$$y(x) = \frac{1}{2}x^6 - \frac{1}{2}x^3 + x.$$

3. Consider the integral

$$\int_0^1 t^{z-1}(1-t)^{z-1}\,dz.$$

From the integral definition of the beta function and its relationship to the gamma function, use the substitution $2t = 1+s$ in the above integral to derive the duplication formula, given by

$$\Gamma(1/2)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2).$$

From the definition of the beta function we have

$$B(z,z) = \frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = \int_0^1 t^{z-1}(1-t)^{z-1}\,dt.$$

Making the substitution $2t = 1+s$, we find that $dt = \frac{1}{2}ds$ and therefore

$$\frac{\Gamma^2(z)}{\Gamma(2z)} = \frac{1}{2}\int_{-1}^{1}\left(\frac{1+s}{2}\right)^{z-1}\left(\frac{1-s}{2}\right)^{z-1} ds = \frac{1}{2^{2z-1}}\int_{-1}^{1}(1-s^2)^{z-1}\,ds.$$

Now noting that this integrand is even in s, we may write

$$\frac{\Gamma^2(z)}{\Gamma(2z)} = \frac{2}{2^{2z-1}}\int_0^1 (1-s^2)^{z-1}\,ds,$$

and making the final substitution of variable $r = s^2$, so that $ds = \frac{1}{2}r^{-1/2}dr$ and therefore

$$\frac{\Gamma^2(z)}{\Gamma(2z)} = \frac{1}{2^{2z-1}}\int_0^1 r^{-1/2}(1-r)^{z-1}\,dr.$$

We note that this last integral is again in the form of the fundamental integral definition of a beta function, specifically $B(1/2,z) = \Gamma(1/2)\Gamma(z)/\Gamma(z+1/2)$ and therefore

$$\frac{\Gamma^2(z)}{\Gamma(2z)} = \frac{1}{2^{2z-1}}\frac{\Gamma(1/2)\Gamma(z)}{\Gamma(z+1/2)}.$$

Finally, after some minor rearrangement, we arrive at the duplication formula

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \Gamma(1/2)\Gamma(2z).$$

Although the question doesn't require it we recall that $\Gamma(1/2) = \sqrt{\pi}$.

4. (a) Using the integral definitions given in class show that

$$K(k) = \frac{\pi}{2}F(1/2,1/2;1;k^2).$$

From class we have that the complete elliptic integral of the first kind is given by

$$K(k) = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

So making the substitution $t = \sin^2\varphi$ then $dt = 2\sin\varphi\cos\varphi\,d\varphi = 2t^{1/2}(1-t)^{1/2}d\varphi$ and therefore

$$K(k) = \frac{1}{2}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-k^2t)^{-1/2}\,dt.$$

Now this is in fundamental integral (Euler) form for the hypergeometric function which is generally given by

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt = B(b,c-b)F(a,b;c;z).$$

Solving for the parameters we find that $a = 1/2$, $b = 1/2$, and $c = 1$. Furthermore the argument $z = k^2$ and writing the beta function in terms of gamma functions we have

$$K(k) = \frac{1}{2}\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} F\left(\frac{1}{2},\frac{1}{2};1;k^2\right),$$

and recalling that $\Gamma(1/2) = \sqrt{\pi}$ we have

$$K(k) = \frac{\pi}{2} F\left(\frac{1}{2},\frac{1}{2};1;k^2\right).$$

(b) The incomplete beta function can be defined by the integral

$$B_z(x,y) = \int_0^z t^{x-1}(1-t)^{y-1}\,dt,$$

where $\Re(x) > 0$, $\Re(y) > 0$, and $z \in [0,1]$. Derive an expression for $B_z(x,y)$ in terms of one hypergeometric function and no other special functions.

In this case we need to deal with the upper limit of integration so making the streatching substitution $t = zu$ then $dt = z\,du$ and the limits are now from 0 to 1, namely

$$B_z(x,y) = z\int_0^1 (zu)^{x-1}(1-zu)^{y-1}\,du = z^x \int_0^1 u^{x-1}(1-zu)^{y-1}\,du.$$

This final integral is in the fundamental form and although the $(1-t)$ term is absent, but this just means that $c - b - 1 = 0$. Therefore solving for the parameters we find $a = 1-y$, $b = x$, $c = x+1$ and the argument is simply z. Therefore we have

$$B_z(x,y) = z^x \frac{\Gamma(x)\Gamma(1)}{\Gamma(x+1)} F(1-y,x;x+1;z),$$

and using the recurrence relation for the gamma function we have $\Gamma(x+1) = x\Gamma(x)$ and therefore

$$B_z(x,y) = \frac{z^x}{x} F(1-y,x;x+1;z),$$

and due to the symmetry of the hypergeometric function, we could also write this as

$$B_z(x,y) = \frac{z^x}{x} F(x,1-y;x+1;z),$$

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