

# School of Mathematical Sciences

## APP MTH 3022 - Optimal Functions and Nanomechanics III

### Assignment 5 solution

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1. Find the extremals of the functional

$$J\{y\} = \int_0^1 (y'^2 + x^2) dx,$$

subject to the conditions

$$y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y^2 dx = 2.$$

We add the integral constraint to the original functional to form the new functional

$$H\{y\} = \int_0^1 (y'^2 + x^2 + \lambda y^2) dx.$$

The Euler-Lagrange equation is

$$\frac{\partial h}{\partial y} - \frac{d}{dx} \frac{\partial h}{\partial y'} = 2\lambda y - \frac{d}{dx}(2y') = 0.$$

From which we derive the second order, linear, constant coefficient ODE

$$y'' - \lambda y = 0.$$

There are three cases to consider depending on whether  $\lambda > 0$ ,  $\lambda = 0$  or  $\lambda < 0$ .

**Case 1:**  $\lambda > 0$

In this case let  $\lambda = \mu^2$ , where  $\mu \in \mathbb{R}^+$ . The general solution is given by

$$y = A \cosh \mu x + B \sinh \mu x.$$

Applying the boundary conditions  $y(0) = 0$  implies  $A = 0$ , and  $y(1) = 0$  implies  $B = 0$  and so we get the trivial solution which cannot possible satisfy the integral constraint.

**Case 2:**  $\lambda = 0$

In this case the general solution is given by

$$y = A + Bx.$$

Applying the boundary conditions  $y(0) = 0$  implies  $A = 0$ , and  $y(1) = 0$  implies  $B = 0$  and so again we get the trivial solution which cannot possible satisfy the integral constraint.

**Case 3:**  $\lambda < 0$

In this case let  $\lambda = -\mu^2$ , where  $\mu \in \mathbb{R}^+$ . The general solution is given by

$$y = A \cos \mu x + B \sin \mu x.$$

Applying the boundary conditions  $y(0) = 0$  implies  $A = 0$ , and  $y(1) = 0$  implies either  $B = 0$  or  $\sin \mu = 0$ . The first option would again lead to a trivial solution which is no good and so we look for roots of

$$\sin \mu = 0, \quad \Rightarrow \quad \mu = n\pi, \quad \text{where } n \in \mathbb{N} = \{1, 2, 3, \dots\}.$$

So the extremals are given by  $y = B \sin(n\pi x)$ . Now we look at the integral constraint to determine the arbitrary constant  $B$ .

$$\begin{aligned}\int_0^1 y^2 dx &= \int_0^1 [B \sin(n\pi x)]^2 dx = 2 \\ B^2 \int_0^1 \sin^2(n\pi x) dx &= 2.\end{aligned}$$

Using the orthogonality of trigonometric functions the integral has a fixed value of  $1/2$  for all integer values of  $n$ . Therefore

$$\begin{aligned}B^2 \frac{1}{2} &= 2 \\ B^2 &= 4 \\ B &= \pm 2.\end{aligned}$$

So the extremals of the functional are given by

$$y = \pm 2 \sin(n\pi x).$$

Now substituting this into the original functional we get that

$$J\{y\} = \int_0^1 (y'^2 + x^2) dx = \int_0^1 [4n^2\pi^2 \cos^2(n\pi x) + x^2] dx.$$

Again using the orthogonality of trigonometric functions we have

$$J\{y\} = 4n^2\pi^2 \frac{1}{2} + \frac{1}{3} = 2n^2\pi^2 + \frac{1}{3}.$$

Obviously  $J$  is monotonically increasing over  $\mathbb{N}$  and therefore will take a minimum value for  $n = 1$  and so the extremals for this functional are

$$y = \pm 2 \sin(\pi x).$$

2. Find the curve joining the points  $(0, 0)$  and  $(1, 0)$  for which the integral

$$\int_0^1 y''^2 dx,$$

is a minimum,

The Euler-Poisson equation is  $\frac{d^2}{dx^2}(2y'') = 0$ , which gives  $y^{(4)} = 0$ . The general solution to this ODE is

$$y = c_1 x^3 + c_2 x^2 + c_3 x + c_4.$$

From the end-points we have  $y(0) = 0$  and  $y(1) = 0$  and so this lets us determine two arbitrary constants

$$c_4 = 0, \quad c_3 = -(c_1 + c_2),$$

which gives us the general solution

$$y = x(x-1)[c_1(x+1) + c_2].$$

if:

- (a)  $y'(0) = a$  and  $y'(1) = b$ .  
So differentiating

$$y' = c_1(3x^2 - 1) + c_2(2x - 1).$$

The boundary conditions give

$$a = -c_1 - c_2, \quad b = 2c_1 + c_2.$$

from which we gave  $c_1 = a + b$ , and  $c_2 = -2a - b$ . So

$$y = x(x-1)[(a+b)(x+1) - 2a - b] = x(x-1)[(a+b)x - a].$$

- (b) no other conditions are prescribed.

If no other condition is prescribed then we have to use the natural boundary conditions for the problem, in this case

$$\left. \frac{\partial f}{\partial y''} \right|_{x=0} = 0, \quad \left. \frac{\partial f}{\partial y''} \right|_{x=1} = 0.$$

This equates to  $y''(0) = y''(1) = 0$ .

So differentiating again

$$y'' = 6c_1x + 2c_2.$$

So  $y''(0) = 0$  implies  $c_2 = 0$ , and  $y''(1) = 0$  implies  $c_1 = 0$ , and so we are left with the trivial solution

$$y = 0.$$

This is the extremal since it passes through the end points and minimises the functional (i.e. it gives a value of zero).

3. Among all the curves in  $C^2$  joining a given point  $(0, b)$  on the (positive)  $y$ -axis to a point on the (positive)  $x$ -axis, and enclosing a given area  $S$  together with the  $x$ - and  $y$ -axes, find the curve which generates the least area when rotated about the  $x$ -axis.

In this problem we consider  $x$  as the dependent variable and  $y$  the independent variable. We wish to find  $x(y)$  given that  $x(b) = 0$  and  $x(0) = a$  which is unspecified. We also have an integral constraint which is

$$G\{x\} = \int_0^b x \, dy = S.$$

Now the quantity we want to find the extremal for is the area of the surface of revolution around the  $x$ -axis. This is given by

$$F\{x\} = \int_0^b 2\pi y \, ds = \int_0^b 2\pi y \sqrt{1 + x'^2} \, dy$$

So forgetting about the constant  $2\pi$  multiplier and adding the integral constraint via a Lagrange multiplier we have

$$H\{x\} = \int_0^b \left( y \sqrt{1 + x'^2} + \lambda x \right) dy.$$

So now the Euler-Lagrange equation gives

$$\begin{aligned} \frac{d}{dy} \left( \frac{yx'}{\sqrt{1 + x'^2}} \right) &= \lambda \\ \frac{yx'}{\sqrt{1 + x'^2}} &= \lambda y + c_1. \end{aligned}$$

Now we consider the natural boundary condition at  $y = 0$ . Here

$$\left. \frac{\partial h}{\partial x'} \right|_{y=0} = 0 \quad \Rightarrow \quad \left. \frac{yx'}{\sqrt{1+x'^2}} \right|_{y=0} = 0.$$

Applying this in the previous equation for  $y = 0$  tells us that  $c_1 = 0$ . Therefore

$$\frac{yx'}{\sqrt{1+x'^2}} = \lambda y \quad \Rightarrow \quad \frac{x'}{\sqrt{1+x'^2}} = \lambda.$$

Simplifying we derive

$$x' = \frac{\lambda}{\sqrt{1-\lambda^2}} = \beta.$$

and so

$$x = \beta y + \gamma.$$

Applying the conditions  $x(0) = a$  and  $x(b) = 0$  we find  $\gamma = a$  and  $\beta = -a/b$  and so

$$x = a - \frac{a}{b}y, \quad \text{or in other words} \quad \frac{x}{a} + \frac{y}{b} = 1.$$

Finally we determine  $a$  from the integral constraint

$$\begin{aligned} \int_0^b x \, dy &= S \\ \int_0^b \left( a - \frac{a}{b}y \right) dy &= S \\ \left[ ay - \frac{a}{2b}y^2 \right]_0^b &= S \\ \frac{ab}{2} &= S \\ a &= \frac{2S}{b}, \end{aligned}$$

and so the solution is a straight line from  $(0, b)$  to  $(2S/b, 0)$ .

4. In Newton's aero-dynamical problem we minimized resistance

$$F\{y\} = \int_0^R \frac{x}{1+y'^2} dx,$$

subject to  $y(0) = L$  and  $y(R) = 0$  (and  $y' \leq 0$  and  $y'' \geq 0$ ).

In nose-cone design this is sometimes approximated by assuming that the nose-cone will be long and thin, so  $y'$  will be large (and negative in our formulation). In that case, we may approximate  $1 + y'^2$  by  $y'^2$  and simplify the problem.

- (a) Using this approximation, derive an alternative formulation of the problem where we don't specify the length of the nose-cone, and instead we specify the total surface area (often called the "wetted area") of the nose-cone.

Let's say  $y(0) = L$  is unspecified but that the wetted area ( $W$  say) is specified and given by

$$W = 2\pi \int_0^R x \sqrt{1+y'^2} dx.$$

So adding this as an isoperimetric constraint we obtain the functional

$$H\{y\} = \int_0^R \left( \frac{x}{1+y'^2} + \lambda x \sqrt{1+y'^2} \right) dx.$$

Now we seek an approximate answer by approximating  $1 + y'^2 \approx y'^2$  then our approximate functional  $\tilde{H}\{y\}$  is given more simply by

$$\tilde{H}\{y\} = \int_0^R \left( \frac{x}{y'^2} + \lambda x y' \right) dx.$$

Note that we also have the natural boundary condition at  $x = 0$  which is that

$$\left. \frac{\partial \tilde{h}}{\partial y'} \right|_0 = 0.$$

- (b) Using the Calculus of Variations, find the optimal profile of the nose-cone with this formulation. The problem is  $y$ -absent and so the Euler-Lagrange equation simplifies to

$$\frac{d}{dx} \left( \frac{\partial \tilde{h}}{\partial y'} \right) = 0.$$

Integrating we obtain that

$$\frac{\partial \tilde{h}}{\partial y'} = \text{const.}$$

However from our natural boundary condition we know that this must equal zero at  $x = 0$  and therefore the constant must be zero. Therefore our Euler-Lagrange equation is given by

$$\begin{aligned} \frac{\partial \tilde{h}}{\partial y'} &= 0 \\ -2 \frac{x}{y'^3} + \lambda x &= 0 \\ y'^3 &= \frac{2}{\lambda} \\ y' &= \left( \frac{2}{\lambda} \right)^{1/3} = \alpha, \quad \text{a constant.} \end{aligned}$$

Integrating we obtain

$$y = \alpha(x - \beta).$$

Since  $y(R) = 0$  we find that  $\beta = R$  and to determine alpha we need to use our wetted area, isoperimetric constraint (and we are not going to make the approximation  $1 + y'^2 \approx y'^2$  here)

$$\begin{aligned} 2\pi \int_0^R x \sqrt{1+y'^2} dx &= W \\ 2\pi \int_0^R x \sqrt{1+\alpha^2} dx &= W \\ 2\pi \sqrt{1+\alpha^2} \int_0^R x dx &= W, \end{aligned}$$

and the integral is simply  $R^2/2$  and so

$$\begin{aligned} 2\pi\sqrt{1+\alpha^2}\frac{R^2}{2} &= W \\ \sqrt{1+\alpha^2} &= \frac{W}{\pi R^2} \\ \alpha^2 &= \left(\frac{W}{\pi R^2}\right)^2 - 1. \end{aligned}$$

Now since  $y' \leq 0$  we take alpha to be the negative root of this and so the final solution to the approximate problem is

$$y = (R-x)\sqrt{\left(\frac{W}{\pi R^2}\right)^2 - 1}.$$

Note that in this case the nose-cone is a right circular cone with length  $L = y(0)$  given by

$$L = R\sqrt{\left(\frac{W}{\pi R^2}\right)^2 - 1}.$$

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