School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Assignment 2 solution

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1. Find the extremals of the following functionals:

   (a) $F\{y\} = \displaystyle\int_0^1 (y^2 + y'^2 + 2ye^x)\,dx, \quad y(0) = 0, \quad y(1) = 1.$

$$f(x, y, y') = y^2 + y'^2 + 2ye^x.$$

Substitution in the standard Euler-Lagrange equation gives

$$\begin{aligned}\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 0\\ 2y + 2e^x - \frac{d}{dx}(2y') &= 0\\ 2y + 2e^x - 2y'' &= 0.\end{aligned}$$

Putting this equation is standard form we have

$$y'' - y = e^x, \tag{1}$$

A second order, linear, constant coefficient but nonhomogeneous ordinary differential equation. The homogeneous solution is

$$y_h = A\sinh x + B\cosh x.$$

Now using the method of undetermined coefficients we look for a particular integral of the form

$$y_p = c_1 x e^x.$$

Differentiating we obtain

$$y_p' = c_1(1 + x)e^x, \quad y_p'' = c_1(2 + x)e^x,$$

and substituting in (1) we have

$$c_1(2 + x)e^x - c_1 x e^x = e^x, \quad \Rightarrow \quad c_1 = \frac{1}{2}.$$

So the general solution is

$$y = y_h + y_p = A\sinh x + B\cosh x + \frac{1}{2}xe^x.$$

Now employing the boundary conditions $y(0) = 0$, $y(1) = 1$.

$$\begin{aligned}y(0) = 0, &\quad \Rightarrow \quad B = 0.\\ y(1) = 1, &\quad \Rightarrow \quad A = \frac{1 - e/2}{\sinh 1}.\end{aligned}$$

So the final answer is the extremal is given by

$$y = \left(\frac{1 - e/2}{\sinh 1}\right)\sinh x + \frac{1}{2}xe^x.$$

(b) $F\{y\} = \int_0^1 (y^2 - y'^2 - 2y\sin x)\, dx, \quad y(0) = 0, \quad y(1) = 1.$

$$f(x, y, y') = y^2 - y'^2 - 2y\sin x.$$

Substitution in the standard Euler-Lagrange equation gives

$$\begin{aligned}\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 0\\ 2y - 2\sin x - \frac{d}{dx}(-2y') &= 0\\ 2y - 2\sin x + 2y'' &= 0.\end{aligned}$$

Putting this equation is standard form we have

$$y'' + y = \sin x, \tag{2}$$

A second order, linear, constant coefficient but nonhomogeneous ordinary differential equation. The homogeneous solution is

$$y_h = A\sin x + B\cos x.$$

Now using the method of undetermined coefficients we look for a particular integral of the form

$$y_p = c_1 x\sin x + c_2 x\cos x.$$

Differentiating we obtain

$$y_p' = c_1(\sin x + x\cos x) + c_2(\cos x - x\sin x), \quad y_p'' = c_1(2\cos x - x\sin x) - c_2(2\sin x + x\cos x),$$

and substituting in (2) we have

$$c_1(2\cos x - x\sin x) - c_2(2\sin x + x\cos x) + c_1 x\sin x + c_2 x\cos x = \sin x, \quad \Rightarrow \quad c_1 = 0, \quad c_2 = -\frac{1}{2}.$$

So the general solution is

$$y = y_h + y_p = A\sin x + B\cos x - \frac{1}{2}x\cos x.$$

Now employing the boundary conditions $y(0) = 0$, $y(1) = 1$.

$$\begin{aligned}y(0) = 0, &\quad \Rightarrow \quad B = 0.\\ y(1) = 1, &\quad \Rightarrow \quad A = \frac{1 + \frac{1}{2}\cos 1}{\sin 1}.\end{aligned}$$

So the final answer is the extremal is given by

$$y = \left(\frac{1 + \frac{1}{2}\cos 1}{\sin 1}\right)\sin x - \frac{1}{2}x\cos x.$$

2. Consider a fixed end-point functional of the general form

$$F\{y\} = \int_a^b g(x)\sqrt{1+y'^2}\,dx,$$

which arises in a number of problems.

(a) Find a general expression for the function $y(x)$ that is an extremal of F.
The integrand if

$$f(x, y') = g(x)\sqrt{1+y'^2},$$

which is y-absent and so the Euler-Lagrange equation simplifies to

$$\begin{aligned}\frac{\partial f}{\partial y'} &= c_1,\\ \frac{g(x)y'}{\sqrt{1+y'^2}} &= c_1.\end{aligned}$$

So rearranging we can isolate y' as

$$\begin{aligned}g(x)y' &= c_1\sqrt{1+y'^2},\\ g^2(x)y'^2 &= c_1^2(1+y'^2),\\ \frac{g^2(x)}{c_1^2}y'^2 &= 1+y'^2,\\ y'^2 &= \frac{1}{\frac{g^2(x)}{c_1^2}-1},\\ y' &= \frac{1}{\sqrt{\frac{g^2(x)}{c_1^2}-1}},\\ y(x) &= c_2 + \int \frac{dx}{\sqrt{\frac{g^2(x)}{c_1^2}-1}},\end{aligned}$$

where c_1 and c_2 are arbitrary constant to be determined from the end-point conditions.

(b) Find the explicit solution $y(x)$ for the cases where
There is more than one way to write these as explicit functions of x. The solutions here are perhaps the simplest, but equivalent answers scored full marks.

i. $g(x) = \sqrt{x}$.

$$y(x) = c_2 + \int \frac{dx}{\sqrt{\frac{x}{c_1^2}-1}}, \quad \Rightarrow \quad y(x) = c_2 + 2c_1\sqrt{x-c_1^2}.$$

ii. $g(x) = x$.

$$y(x) = c_2 + \int \frac{dx}{\sqrt{\frac{x^2}{c_1^2}-1}}, \quad \Rightarrow \quad y(x) = c_2 + c_1\cosh^{-1}\left(\frac{x}{c_1}\right).$$

iii. $g(x) = e^x$.

$$y(x) = c_2 + \int \frac{dx}{\sqrt{\frac{e^{2x}}{c_1^2}-1}}, \quad \Rightarrow \quad y(x) = c_2 + \cos^{-1}\left(c_1 e^{-x}\right).$$

3. Consider the functional

$$F\{y\} = \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)\, dx, \quad y(0) = 0, \quad y(1) = \frac{3}{2}.$$

(a) Determine a differential expression H that takes a constant value for extremals of F.
In this case the integrand is x-absent, so our functional is autonomous and therefore

$$H = f - y'\frac{\partial f}{\partial y'} = \text{const}.$$

Substituting we have

$$\begin{aligned} \frac{1}{2}y'^2 + yy' + y' + y - y'\left(y' + y + 1\right) &= \text{const}, \\ -\frac{1}{2}y'^2 + y &= \text{const}, \\ y'^2 - 2y &= c_1, \end{aligned}$$

where c_1 is an arbitrary constant.

(b) Derive the function $y(x)$ which is an extremal of F.
From part (a) we can rearrange to give

$$y' = \sqrt{2y + c_1},$$

which is separable. So

$$\frac{dy}{\sqrt{2y + c_1}} = dx, \quad \Rightarrow \quad \sqrt{2y + c_1} = x + c_2,$$

and rearranging toform an explicit solution for $y(x)$ we obtain

$$y(x) = \frac{1}{2}\left[(x + c_2)^2 - c_1\right].$$

Expanding and reassigning arbitrary constants to d_1 and d_2 we may write

$$y(x) = \frac{1}{2}x^2 + d_1 x + d_2.$$

Now turning to the boundary conditions we have

$$\begin{aligned} y(0) = 0, &\quad \Rightarrow \quad d_2 = 0. \\ y(1) = \frac{3}{2}, &\quad \Rightarrow \quad d_1 = 1. \end{aligned}$$

So the extremal is given by

$$y(x) = \frac{1}{2}x^2 + x.$$

4. Assuming fixed end-points, find the extremals of the functional

$$F\{y\} = \int \sqrt{x^2+y^2}\sqrt{1+y'^2}\,dx,$$

by making a change to polar coordinates. i.e. $x = r\cos\theta$, $y = r\sin\theta$.
Express your final answer in Cartesian coordinates in the form of y as an explicit function of x.
Our transform is given by

$$x(r,\theta) = r\cos\theta, \quad y(r,\theta) = r\sin\theta.$$

Let's choose r to be out new independent variable and θ as out new dependent variable. So from the chain rule

$$\frac{dx}{dr} = \cos\theta - \theta' r\sin\theta, \quad \frac{dy}{dr} = \sin\theta + \theta' r\cos\theta.$$

Now we note that $\sqrt{x^2+y^2} = r$, and examining the second square-root term we have

$$\begin{aligned}
1+y'^2 &= 1 + \left(\frac{dy/dr}{dx/dr}\right)^2, \\
&= 1 + \left(\frac{\sin\theta + \theta' r\cos\theta}{\cos\theta - \theta' r\sin\theta}\right)^2, \\
&= \frac{(\cos\theta - \theta' r\sin\theta)^2 + (\sin\theta + \theta' r\cos\theta)^2}{(\cos\theta - \theta' r\sin\theta)^2}, \\
&= \frac{1 + r^2\theta'^2}{(\cos\theta - \theta' r\sin\theta)^2}, \\
\sqrt{1+y'^2} &= \frac{\sqrt{1+r^2\theta'^2}}{\cos\theta - \theta' r\sin\theta}.
\end{aligned}$$

So our new functional $\tilde{F}\{\theta\}$ is specified by

$$\begin{aligned}
\tilde{F}\{\theta\} &= \int r\frac{\sqrt{1+r^2\theta'^2}}{\cos\theta - \theta' r\sin\theta}\frac{dx}{dr}\,dr, \\
&= \int r\sqrt{1+r^2\theta'^2}\,dr.
\end{aligned}$$

This is θ-absent and so we have that

$$\begin{aligned}
\frac{\partial\tilde{f}}{\partial\theta'} &= \text{const}, \\
\frac{r^3\theta'}{\sqrt{1+r^2\theta'^2}} &= \text{const}, \\
\frac{r^6\theta'^2}{1+r^2\theta'^2} &= \text{const}, \\
c_1^2 r^6\theta'^2 &= 1 + r^2\theta'^2, \\
r^2(c_1^2 r^4 - 1)\theta'^2 &= 1, \\
\theta' &= \frac{1}{r\sqrt{c_1^2r^4-1}} = \frac{1}{c_1 r^3\sqrt{1-1/c_1^2r^4}}, \\
\theta &= -\frac{1}{2}c_2 - \frac{1}{2}\arcsin\left(\frac{1}{c_1 r^2}\right).
\end{aligned}$$

Now rearranging we have

$$
\begin{aligned}
\sin(2\theta + c_2) &= \frac{1}{c_1 r^2}, \\
\sin(2\theta)\cos c_2 + \cos(2\theta)\sin c_2 &= \frac{1}{c_1 r^2}, \\
2\sin\theta\cos\theta\cos c_2 + (\cos^2\theta - \sin^2\theta)\sin c_2 &= \frac{1}{c_1 r^2}, \\
2r\sin\theta\, r\cos\theta\cos c_2 + (r^2\cos^2\theta - r^2\sin^2\theta)\sin c_2 &= \frac{1}{c_1}.
\end{aligned}
$$

Now we note that $x = r\cos\theta$ and $y = r\sin\theta$, and so if we relabel $\alpha = \pi/2 - c_2$ and $\beta = 1/c_1$ then we may write the extremal is given in Cartesian coordinates by

$$2xy\sin\alpha + (x^2 - y^2)\cos\alpha = \beta,$$

where α and β are arbitrary constants.

Finally, to find y as an explicit function of x we may use the quadratic formula to find that the extremals of F are

$$y(x) = x\tan\alpha \pm \sqrt{x^2\sec^2\alpha - \beta\sec\alpha},$$

and since β is arbitrary we can make one final reassignment of constants: $A = \tan\alpha$ and $B = \beta\sec\alpha$, to give

$$y(x) = Ax \pm \sqrt{(1 + A^2)x^2 - B},$$

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