Tutorial 2 - Solution

Question 1

(a)

$$F\{y\} = \int_0^{\pi/2} (y^2 + y'^2 - 2y\sin x)\, dx, \quad y(0) = 0, \quad y(\pi/2) = 3/2.$$

The Euler-Lagrange equation gives

$$2y - 2\sin x - \frac{d}{dx}(2y') = 0.$$

$$\Rightarrow \quad y'' - y = -\sin x.$$

The solution to the homogeneous equation is

$$y_h = A\sinh x + B\cosh x.$$

Using the method of undetermined coefficients we have

$$\begin{aligned} y_p &= c_1 \sin x + c_2 \cos x \\ y_p'' &= -c_1 \sin x - c_2 \cos x. \end{aligned}$$

Substituting into the ODE we obtain

$$-c_1 \sin x - c_2 \cos x - c_1 \sin x - c_2 \cos x = -\sin x.$$

So $c_1 = 1/2$ and $c_2 = 0$. So the general solution is

$$y = A\sinh x + B\cosh x + \frac{1}{2}\sin x.$$

Now applying the end-point conditions

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad B = 0 \\ y(\pi/2) = 3/2 \quad &\Rightarrow \quad 3/2 = A\sinh(\pi/2) + 1/2 \\ &\Rightarrow \quad A = \frac{1}{\sinh(\pi/2)}. \end{aligned}$$

So the solution to this problem is

$$y = \frac{\sinh x}{\sinh(\pi/2)} + \frac{1}{2}\sin x.$$

(b)

$$F\{y\} = \int_1^2 \frac{y'^2}{x^3}\, dx, \quad y(1) = 0, \quad y(2) = 15.$$

The functional is y-absent so it satisfies the equation

$$\begin{aligned} \frac{\partial f}{\partial y'} &= c \\ \frac{2y'}{x^3} &= c \\ y' &= \frac{c}{2}x^3. \end{aligned}$$

Integrating we obtain

$$y = \frac{c}{8}x^4 + d.$$

Now applying the end-point conditions

$$\begin{aligned}
y(1) = 0 \quad &\Rightarrow \quad 0 = \frac{c}{8} + d \\
&\Rightarrow \quad y = d(1 - x^4). \\
y(2) = 15 \quad &\Rightarrow \quad 15 = d(1 - 16) \\
&\Rightarrow \quad d = -1.
\end{aligned}$$

So the solution to this problem is

$$y = x^4 - 1.$$

(c)

$$F\{y\} = \int_0^2 (xy' + y'^2)\, dx, \quad y(0) = 1, \quad y(2) = 0.$$

Again the functional is y-absent so it satisfies the equation

$$\begin{aligned}
\frac{\partial f}{\partial y'} &= c \\
x + 2y' &= c \\
y' &= \frac{c}{2} - \frac{x}{2}.
\end{aligned}$$

Integrating we obtain

$$y = d + \frac{c}{2}x - \frac{x^2}{4}.$$

Now applying the end-point conditions

$$\begin{aligned}
y(0) = 1 \quad &\Rightarrow \quad d = 1 \\
y(2) = 0 \quad &\Rightarrow \quad 0 = 1 + c - 1 \\
&\Rightarrow \quad c = 0.
\end{aligned}$$

So the solution to this problem is

$$y = 1 - \frac{x^2}{4}.$$

Question 2

Using Fermat's principle of least time, we want to design a material that gives circular arcs as optimal solutions to a variational problem. Following the notes we use the Calculus of Variations on the time functional

$$T\{y\} = \int \frac{\sqrt{1 + y'^2}}{c}\, dx,$$

but first let's convert to polar coordinates.

$$\begin{aligned}
x = r\cos\theta \quad &\Rightarrow \quad \frac{dx}{d\theta} = \dot{r}\cos\theta - r\sin\theta, \\
y = r\sin\theta \quad &\Rightarrow \quad \frac{dy}{d\theta} = \dot{r}\sin\theta + r\cos\theta, \\
\text{and so} \quad y' &= \frac{\dot{r}\sin\theta + r\cos\theta}{\dot{r}\cos\theta - r\sin\theta}.
\end{aligned}$$

Note well that $y' = dy/dx$, and $\dot{r} = dr/d\theta$. So a short calculation yields

$$1 + y'^2 = \frac{\dot{r}^2 + r^2}{(\dot{r}\cos\theta - r\sin\theta)^2}.$$

Now the functional becomes

$$
\begin{aligned}
T\{y\} &= \int \frac{\sqrt{1+y'^2}}{c}\,dx \\
\Rightarrow \quad T\{r\} &= \int \frac{\sqrt{\dot r^2 + r^2}}{c(\dot r\cos\theta - r\sin\theta)}(\dot r\cos\theta - r\sin\theta)\,d\theta \\
&= \int \frac{\sqrt{\dot r^2 + r^2}}{c}\,d\theta.
\end{aligned}
$$

Provided c does not depend explicitly on θ then this functional is θ-absent and therefore the extremal will satisfy

$$
H = \dot r\frac{\partial f}{\partial \dot r} - f = \alpha, \quad \text{a constant.}
$$

So let's assume that c is a function of r alone, that is $c = c(r)$, then

$$
H = \dot r\frac{\dot r}{c(r)\sqrt{\dot r^2 + r^2}} - \frac{\sqrt{\dot r^2 + r^2}}{c(r)} = -\frac{r^2}{c(r)\sqrt{\dot r^2 + r^2}} = \alpha.
$$

So solving for $c(r)$ we obtain

$$
c(r) = -\frac{r^2}{\alpha\sqrt{\dot r^2 + r^2}}.
$$

Now for a circular path we would require $\dot r = 0$ and therefore

$$
c(r) = -\frac{r}{\alpha},
$$

but how to find α?

Question 3

The cone can be thought of as a surface in polar spherical coordinates

$$
x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,
$$

where $\phi = \alpha$, a constant. So the surface is parameterised by

$$
\boldsymbol{r}(r,\theta) = (r\cos\theta\sin\alpha, r\sin\theta\sin\alpha, r\cos\alpha).
$$

Following the formulæ from lectures

$$
\begin{aligned}
P &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \\
&= \cos^2\theta\sin^2\alpha + \sin^2\theta\sin^2\alpha + \cos^2\alpha = 1. \\
Q &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
&= -r\sin\theta\cos\theta\sin^2\alpha + r\sin\theta\cos\theta\sin^2\alpha + 0 = 0. \\
R &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\
&= r^2\sin^2\theta\sin^2\alpha + r^2\cos^2\theta\sin^2\alpha + 0 = r^2\sin^2\alpha.
\end{aligned}
$$

So we can choose between either of the following formulations

$$
L\{\theta(r)\} = \int_{r_0}^{r_1} \sqrt{1 + \theta'^2 r^2\sin^2\alpha}\,dr, \quad \text{or} \quad L\{r(\theta)\} = \int_{\theta_0}^{\theta_1} \sqrt{r'^2 + r^2\sin^2\alpha}\,d\theta.
$$

Considering the second formulation we note that it is θ-absent (or autonomous) and therefore

$$H = r'\frac{\partial f}{\partial r'} - f = -c, \quad \text{(a constant)}.$$

Therefore

$$\begin{aligned}
\frac{r'^2}{\sqrt{r'^2 + r^2\sin^2\alpha}} - \sqrt{r'^2 + r^2\sin^2\alpha} &= -c \\
r^2\sin^2\alpha &= c\sqrt{r'^2 + r^2\sin^2\alpha} \\
r^4\sin^4\alpha &= c^2(r'^2 + r^2\sin^2\alpha) \\
r^2\sin^2\alpha\left(\frac{r^2\sin^2\alpha}{c^2} - 1\right) &= r'^2 \\
\frac{dr}{d\theta} &= r\sin\alpha\left(\frac{r^2\sin^2\alpha}{c^2} - 1\right)^{1/2} \\
\frac{d\theta}{dr} &= \frac{1}{r\sin\alpha\left(r^2\sin^2\alpha/c^2 - 1\right)^{1/2}}.
\end{aligned}$$

So integrating we have

$$\begin{aligned}
\theta - d &= -\frac{1}{\sin\alpha}\tan^{-1}\left(\frac{1}{\sqrt{r^2\sin^2\alpha/c^2 - 1}}\right) \\
\sin\alpha(d-\theta) &= \tan^{-1}\left(\frac{1}{\sqrt{r^2\sin^2\alpha/c^2 - 1}}\right) \\
\tan\left(\sin\alpha(d-\theta)\right) &= \frac{1}{\sqrt{r^2\sin^2\alpha/c^2 - 1}} \\
\sqrt{r^2\sin^2\alpha/c^2 - 1} &= \cot\left(\sin\alpha(d-\theta)\right) \\
r^2\sin^2\alpha/c^2 &= 1 + \cot^2\left(\sin\alpha(d-\theta)\right) \\
r^2\sin^2\alpha/c^2 &= \csc^2\left(\sin\alpha(d-\theta)\right) \\
r\sin\alpha/c &= \csc\left(\sin\alpha(d-\theta)\right).
\end{aligned}$$

Let's relabel our arbitrary constants $\mu = c/\sin\alpha$ and $\nu = d\sin\alpha$ then we have

$$r = \mu\csc\left(\nu - \theta\sin\alpha\right).$$

So we have a solution that has two arbitrary constants which we can resolve with the end-point conditions

$$r(\theta_0) = r_0, \quad \text{and} \quad r(\theta_1) = r_1.$$

Question 4

$$F\{y\} = \int_0^R \frac{x}{y'^2}\,dx.$$

This functional is y-absent and therefore the Euler-Lagrange equation is

$$\begin{aligned}
\frac{\partial f}{\partial y'} &= \text{const} \\
-2\frac{x}{y'^3} &= \text{const} \\
y'^3 &= -\frac{2x}{\text{const}} \\
y' &= \alpha x^{1/3}.
\end{aligned}$$

So integrating

$$y = \frac{3\alpha}{4} x^{4/3} + \beta.$$

Now applying the end-point conditions

$$\begin{aligned}
y(0) = L \quad &\Rightarrow \quad \beta = L, \\
y(R) = 0 \quad &\Rightarrow \quad 0 = \frac{3\alpha}{4} R^{4/3} + L, \\
&\Rightarrow \quad \alpha = -\frac{4L}{3R^{4/3}}.
\end{aligned}$$

$$\text{So} \quad y = L\left[1 - \left(\frac{x}{R}\right)^{4/3}\right].$$

Lets assume $L = 1$, and $R = 1$ then

$$y = 1 + x^{4/3}, \quad y'^2 = \frac{16}{9} x^{2/3},$$

$$\text{So} \quad F = \int_0^1 \frac{x}{1 + \frac{16}{9} x^{2/3}} \, dx \approx 0.2200 \ldots,$$

but remember that this is not in the $L \gg R$ regime.

Question 5

(a)

If $f(y, y')$ does not explicitly depend on x then by the chain rule

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \\
&= y' f_y + y'' f_{y'} \\
\Rightarrow \quad y' f_y &= \frac{df}{dx} - y'' f_{y'}.
\end{aligned} \tag{1}$$

From the Euler-Lagrange equation

$$\begin{aligned}
f_y - \frac{d}{dx} f_{y'} &= 0 \\
y' f_y - y' \frac{d}{dx} f_{y'} &= 0.
\end{aligned}$$

and substitution from (1) yields

$$\frac{df}{dx} - y'' f_{y'} - y' \frac{d}{dx} f_{y'} = 0.$$

But the last two terms are a derivative of a product and therefore

$$\frac{df}{dx} - \frac{d}{dx}\left(y' f_{y'}\right) = 0,$$

and so integrating

$$f - y' f_{y'} = \text{const},$$

as required.

(b)

If $f(y, y', y'')$ does not explicitly depend on x then by the chain rule

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial y''}\frac{dy''}{dx} \\
&= y' f_y + y'' f_{y'} + y''' f_{y''} \\
\Rightarrow \quad y' f_y &= \frac{df}{dx} - y'' f_{y'} - y''' f_{y''}. \qquad (2)
\end{aligned}$$

From the Euler-Poisson equation

$$\begin{aligned}
f_y - \frac{d}{dx} f_{y'} + \frac{d^2}{dx^2} f_{y''} = 0 \\
y' f_y - y' \frac{d}{dx} f_{y'} + y' \frac{d^2}{dx^2} f_{y''} = 0.
\end{aligned}$$

and substitution from (2) yields

$$\frac{df}{dx} - y'' f_{y'} - y''' f_{y''} - y' \frac{d}{dx} f_{y'} + y' \frac{d^2}{dx^2} f_{y''} = 0.$$

But the second and fourth terms are a derivative of a product and therefore

$$\frac{df}{dx} - \frac{d}{dx}(y' f_{y'}) - y''' f_{y''} + y' \frac{d^2}{dx^2} f_{y''} = 0.$$

Now we add and subtract $y''\, d/dx(f_{y''})$

$$\frac{df}{dx} - \frac{d}{dx}(y' f_{y'}) - y''' f_{y''} - y'' \frac{d}{dx} f_{y''} + y' \frac{d^2}{dx^2} f_{y''} + -y'' \frac{d}{dx} f_{y''} = 0,$$

and now all the terms are also derivatives of products and so we obtain

$$\frac{df}{dx} - \frac{d}{dx}(y' f_{y'}) - \frac{d}{dx}(y'' f_{y''}) + \frac{d}{dx}\left(y' \frac{d}{dx} f_{y''} \right) = 0,$$

and so integrating

$$f - y' f_{y'} - y'' f_{y''} + y' \frac{d}{dx} f_{y''} = \text{const},$$

as required.