Hochschule Bremen
City University of Applied Sciences

STOCHASTIC SIGNALS AND SYSTEMS

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ASSIGNMENT 2

FUNCTIONS OF RANDOM VARIABLES

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by

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Functions of random variables

1.1 Normal distribution and uniform distribution

Task: Transform the $N(\mu, \sigma^2)$ distributed random variable X into a standard normal distributed random variable. Transform on *[a, b]* uniformly distributed random variable X such that it is uniformly distributed on [-1/2,1/2].

Solution:

Description

The standard normal distribution is a special case of the normal distribution. It is the distribution that occurs when a normal random variable has a mean, (μ) of zero and a standard deviation, $(\sigma)$of one. The normal random variable of a standard normal distribution is called a standard score. Every normal random variable X can be transformed into a standard normal distribution via the following equation:

$$Y = \frac{X - \mu}{\sigma}$$

Where X is a normal random variable, μ is the mean of X, and σ is the standard deviation of X.

To transform a normal distribution of $X \sim N(\mu, \sigma)$ into into a standard normal distribution, $Y \sim N(0, 1)$ must be considered. The probability density function of normal distribution can be given as,

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad x \in R$$

The working of the transformation is possible when, $y = g(x) = \frac{x-\mu}{\sigma}$. When, the mean (μ) is 0, $y = g(x = \frac{x}{\sigma})$. Using the inverse property, it can be written as,

$$x = g^{-1}(y) = \sigma x + \mu$$

The probability density function of Y can be given as,

$$f_y(y) = \frac{f_x(x)}{|\frac{dF_Y}{dy}|} \quad at \quad x = g^{-1}(y)$$

The probability density function of X after inserting $x = g^{-1}(y)$ can be given as,

$$f_x(x) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\sigma y + \mu - \mu)^2}{2\sigma^2}\right\}}{\frac{1}{\sigma}}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2}\right\}, \quad y \in R$$

Taking the derivative of the function *g(x)*,

$$\frac{\partial g(x)}{\partial x} = \frac{\partial}{\partial x}(\frac{x}{\sigma} - \frac{\mu}{\sigma}) = \frac{1}{\sigma}$$

Substituting this in the probability density function of Y, we get

$$f_y(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\}, \quad y \in R$$

This is the transformed form of normal distribution variable to standard normal distribution variable.

Transformation of uniformly distributed random variable

Consider the random variable $X \sim R[a, b]$ which is a rectangular distribution between interval a and b. The maximum value of this function is $1/(b-a)$. Our aim is to obtain the desired random variable, $Y \sim R[-1/2, 1/2]$. In order to achieve this, we create another random variable $\tilde{X} = X - a$.

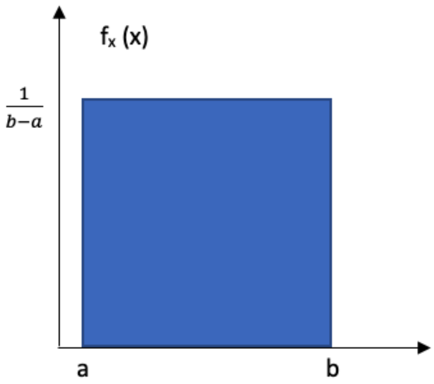
The corresponding density function has changed from rectangular function with limits $[a, b]$ to limits $[0, b-a]$. The height of both X and $\tilde{X}$ will be the same, i.e, $1/(b-a)$. Extending this further and taking another random variable, $\hat{X} = (X-a)/(b-a) = \tilde{X}/(b-a)$. Thus for this random variable, the graph envelops everything inside the rectangular region 0 and 1, with the maximum value of $\tilde{X}/(b-a)$.

Now, this random variable will help us in generating our desired random variable, $Y \sim R[-1/2, 1/2]$ For our desired variable, we consider

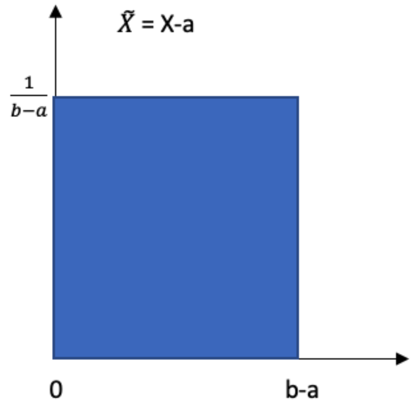
$$Y = \hat{X} - (1/2)$$

$$Y = ((X-a)/(b-a)) - 1/2$$

Hence we obtain the desired uniformly distributed random variable, i.e., if X is a random variable



(a)



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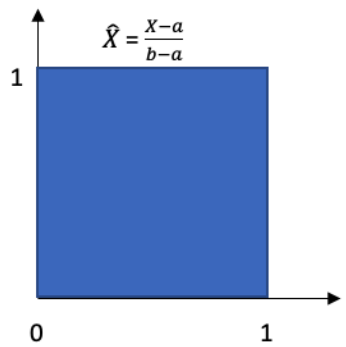

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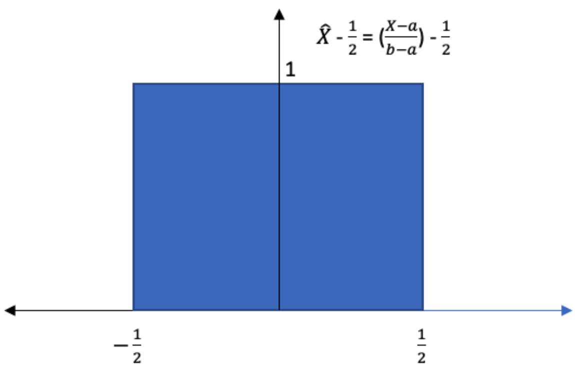

(d)

Figure 1.1: Transformation of uniformly distributed random variable

in [a, b], the Y is also a rectangular distribution in $[-1/2, 1/2]$. In practical this type of variable helps us in modelizing the quantization error, which jumps from one point to another over a region.

1.2 Exponential distribution

Task: Determine the expected value, the variance and the density function of $Y = -\frac{1}{a} lnX$ for $a > 0$, when X is uniformly distributed on (0, 1). Which other distribution has the same density function, as the exponential distribution for $\alpha = 1/2$?

Solution:

The distribution function $F_Y(y)$ defined by the probability of the random variable, $Y = -\frac{1}{\alpha}\ln(X)$ for $\alpha > 0$ and X is uniformly distributed on [0, 1] can be given as,

$$F_Y(y) = P(Y \leq y)$$

$$F_Y(y) = P(\frac{-1}{\alpha}\ln X \leq y) = P(e^{-\alpha y} \leq X)$$

$$F_Y(y) = 1 - e^{-\alpha y}$$

The relationship between the density and the distributed function in the used, i.e.,

$$f_Y = \frac{\partial F_Y}{\partial y}$$

Where, f_y is the probability density function and F_y is the distribution function of Y. Therefore,

$$F_Y = [1 - e^{-\alpha y}].1_{[0,1]}(y)$$

The derivative will be,

$$\frac{\partial F_Y}{\partial y} = \alpha e^{-\alpha y}.1_{[0,1]}(y)$$

Now, the probability density function is given as,

$$f_y(y) = \alpha e^{-\alpha y}$$

The expected value (mean):

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy$$

Now substituting for $f_Y y$,

$$E(Y) = \int_{0}^{\infty} y\alpha e^{-\alpha y} dy$$

$$E(Y) = \alpha \left[-\frac{1}{\alpha} y e^{-\alpha y} \Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{\alpha} e^{-\alpha y} dy \right]$$

$$E(Y) = -\frac{1}{\alpha} e^{-\alpha y} \Big|_0^{\infty} = \frac{1}{\alpha}$$

The variance :

$${\sigma_Y}^2 = E(Y - E(Y))^2 = E(Y^2) - E(Y)^2$$

$E(Y^2)$ can be calculated as,

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_y(Y) dy = \int_0^{\infty} y^2 \alpha e^{-\alpha y} dy$$

$$E(Y^2) = \alpha \left[-\frac{1}{\alpha} y e^{-\alpha y} \Big|_0^{\infty} - \int_0^{\infty} -\frac{2}{\alpha} y e^{-\alpha y} dy \right]$$

$$E(Y^2) = 2 \int_0^{\infty} y e^{-\alpha y} dy = \frac{2}{\alpha^2}$$

$${\sigma_Y}^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

The other distribution function:

If we Suppose $X \sim X_n^2$ then the density function is given as

$$f_X(x) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}.x^{\frac{n}{2}-1}e^{-\frac{x}{2}}$$

When $n = 2$,

$$\Gamma(1) = 1$$

$$f_X(x) = \frac{1}{2}e^{\frac{-x}{2}}$$

The value of density function $f_Y(y)$ when $\alpha = 1/2$ is given as,

$$f_y(y) = \frac{1}{2}e^{\frac{-y}{2}}$$

Comparing $f_X(x)$ and $f_Y(y)$, we conclude that the other distribution function having the same density function as the exponential function is X_2^2 distribution.

1.3 Sum of Random Variables

Task: Determine the expected value, the variance and the density function of $Z = X + Y$, when X and Y are two stochastically independent, on [0,1] uniformly distributed random variables.

Solution:

Description:

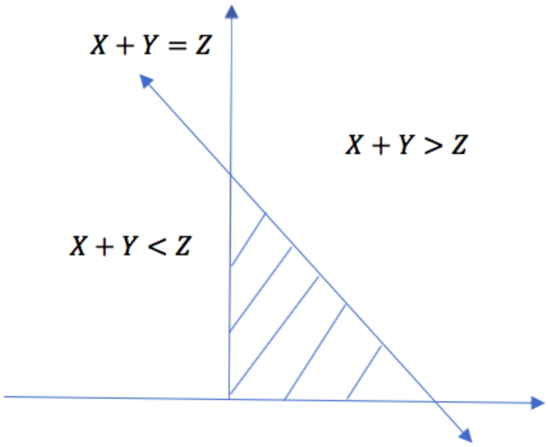

Figure 1.2: function $X + Y = Z$

Let X and Y be two independent random variables distributed on the interval [0, 1] that is, $X, Y \sim R(0, 1)$ and $Z = g(X, Y) = X + Y$. The distribution function for the random variable Z can be expressed as,

$$F_Z(z) = P(Z \leq z) = P(X + Y \leq z) = \int_{-\infty}^{\infty} \int_{\infty}^{z-x} f_{XY}(\text{xy}) \text{ dx dy}$$

The density function of the random variable Z is the derivative pf the distribution function,

$$f_Z(z) = \frac{\partial F_Z(z)}{\partial z}$$

Substituting $F_Z(z)$ in $f_Z(z)$,

$$f_Z(z) = \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(\text{x,y}) \text{ dy dx}$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z} \left(f_{XY}(\text{x, z-x}) - f_{XY}(\text{x}, -\infty) \right) \text{dx}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(\text{x, z-x}) \text{ dx}$$

If we assume X and Y as two independent random variables, then we get the density function for sum of two random variables as,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(\text{x}) f_Y(\text{z - x}) \text{dx}$$

This is a convolution integral and it represents the sum of two random variables.

$$f_Z(z) = \int_{-\infty}^{\infty} 1_{[0,1]}(\text{z-x}) \text{ dx} = \int_{0}^{1} 1_{[0,1]}(\text{z-x}) \text{ dx}$$

Consider $u = z - x$, $\Rightarrow dx = -du$

$$f_Z(z) = -\int_{z}^{z-1} 1_{[0,1]}(\text{u}) \text{ du} = \int_{z-1}^{z} 1_{[0,1]}(\text{u}) \text{ du}$$

The function $1_{[0,1]}(\text{u})$ is always zero expect when $0 \leq \text{u} \leq 1$ And, we can discuss the following cases:

Case 1: z<0 and z>2:

$$f_Z(z) = \int_{z-1}^{z} 1_{[0,1]}(\text{u}) \text{ du}$$

$$f_Z(z) = 0$$

Case 2: $0 \leq z \leq 1$:

$$f_Z(z) = \int_{z-1}^{z} 1_{[0,1]}(\mathrm{u})\ \mathrm{du} = \int_{z-1}^{0} 1_{[0,1]}(\mathrm{u})\ \mathrm{du} + \int_{0}^{z} 1_{[0,1]}(\mathrm{u})\ \mathrm{du} = u|_0^z$$

$$f_Z(z) = z$$

Case 3: $1 < z \leq 2$:

$$f_Z(z) = \int_{z-1}^{1} 1_{[0,1]}(\mathrm{u})\ \mathrm{du} = u|_{z-1}^{1} = 1 - z + 1$$

$$f_Z(z) = 2 - z$$

The resulting density function can be expressed as:

$$f_z(z) = \begin{cases} 0, & \text{otherwise} \\ \text{z}, & 0 \leq z \leq 1 \\ \text{2 - z}, & 1 \leq z \leq 2 \end{cases}$$

The expected value (mean):

$$E(Z) = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_0^1 z^2 dz + \int_1^2 z(2-z) dz$$

$$E(Z) = \left.\frac{z^3}{3}\right|_0^1 + \int_1^2 2z dz - \int_1^2 z^2 dz$$

$$E(Z) = \frac{1}{3} + z^2|_1^2 - \left.\frac{z^3}{3}\right|_1^2 = \frac{1}{3} + 3 - \frac{7}{3}$$

$$E(Z) = 1$$

For two independent random variables, E(Z) = E(X+Y) = E(X) + E(Y) and for uniformly distributed random variable on the interval [0,1], E(X) = E(Y) = $\frac{1}{2}$, hence E(Z) = $\frac{1}{2} + \frac{1}{2} = 1$.

The variance :

$${\sigma_Z}^2 = E(Z-1)^2 = \int_{-\infty}^{\infty} (Z-1) f_z(z) dz$$

We have Z=X+Y,

$${\sigma_Z}^2 = E(Z-1)^2 = E(X+Y-1)^2 = E\left(\left(X-\frac{1}{2}\right)+\left(Y-\frac{1}{2}\right)\right)^2$$

$${\sigma_Z}^2 = E(X+Y-1)^2 = E\left(\left(X-\frac{1}{2}\right)^2 + 2\left(X-\frac{1}{2}\right)\left(Y-\frac{1}{2}\right)+\left(Y-\frac{1}{2}\right)^2\right)$$

We know that, Cov(X,Y) = 0. Because X and Y are independent.

$${\sigma_Z}^2 = \frac{1}{12} + \frac{1}{12}$$

$${\sigma_Z}^2 = \frac{1}{6}$$

1.4 Product of random variables

Task: Calculate the expected value, the variance and the density function of $Z = XY$, when X and Y are two stochastically independent, on [0, 1] uniformly distributed random variables.

Solution: Let $Z = XY$ be the product of two random variables X and Y, where X and Y are identical, independent and uniformly distributed random variables, i.e., $X, Y \sim R[0,1]$

In order to get the density function of the product of the two random variables, we introduce an auxiliary variable, $W = Y$. We also consider that the density of the bivariate function $f(x,y)$ is known. Therefore, Y is a function of $h(x,y)$,

$$x = \frac{z}{y} = \frac{z}{w} = g^{-1}(z,w)$$

$$y = w = h^{-1}(z,w)$$

To determine the distribution of the transformed random variables, the Jacobian matrix is emplyed,

$$J(x,y) = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ \\ 0 & 1 \end{bmatrix}$$

$$|\det(J(x,y))| = |y| = |w|$$

The bivariate density function is given by,

$$f_{Z,W}(z,w) = \frac{f_{XY}(g^{-1}(z,w), h^{-1}(z,w))}{|\det(J)|}$$

$$f_{Z,W}(z,w) = \frac{f_{XY}(z/w, w)}{|w|}$$

As the variable X and Y are independent, the probability function can be written as,

$$f_{XY} = f_X\left(\frac{z}{w}\right) f_Y(w)$$

The probability density function of Z is given by,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)dw$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{f_{X,Y}(\frac{z}{w},w)}{|w|}dw$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{f_X(\frac{z}{w})f_y(w)}{|w|}dw$$

$$f_Z(z) = \int_{0}^{1} \frac{f_X(\frac{z}{w})}{|w|}dw$$

Case 1: z<0 i.e., $\frac{\mathbf{z}}{\mathbf{w}}$<0:

$$f_Z(z) = 0$$

Case 2: $\mathbf{0 < z \leq 1}$, i.e, $\mathbf{z \leq w < 1}$: This means, $0 < \frac{z}{w} \leq 1$

$$f_Z(z) = \int_{z}^{1} \frac{1}{w}dw = \ln(w)|_z^1$$

$$f_Z(z) = -\ln(z)$$

Case 3: z>1 i.e., $\frac{\mathbf{z}}{\mathbf{w}}$>1:

$$f_Z(z) = 0$$

Then, the density function can be expressed by,

$$f_z(z) = \begin{cases} 0 & z < 0 \\ \text{-ln(z)} & 0 \leq z \leq 1 \\ 0 & z > 1 \end{cases}$$

The expected value (mean):

$$E[Z] = E[XY] = E[X].E[Y]$$

$$E(Z) = \int_{-\infty}^{\infty} z f_Z(z) dz = -\int_0^1 z\ln(z) dz = -\left(\frac{z^2}{2} ln(z) \Big|_0^1 - \int_0^1 \frac{z^2}{2} \frac{1}{z} dz \right) = \frac{z^2}{4} \Big|_0^1$$

$$E(Z) = \frac{1}{4}$$

The variance :

$$\sigma_Z{}^2 = E[Z^2] - (E[Z])^2$$

So,

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f_Z(z) dz = -\int_0^1 z^2 \ln(z) dz = \frac{z^3}{9} \Big|_0^1 = \frac{1}{9}$$

$$(E[Z])^2 = \frac{1}{16}$$

Hence, the variance is,

$$\sigma_Z{}^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

1.5 χ^2 distribution

Task: Determine the expected value, the variance and the density of

$$Z = \sum_{i=1}^{4} X_i^2$$

where X_i(i =1,2,3,4) are four stochastically independent, standard normally distributed random variables. How could one create a random variable from two stochastically independent exponentially distributed random variables that possess the same distribution as Z? Calculate the density of Z through convolution of the density of an exponential distribution.

Solution: χ^2 distribution defines a property such that its mean coincides with the number of degrees of freedom, n. It is mathematically defined as,

$$Z = \sum_{i=1}^{n} X_i^2$$

Where, X_i is a standard normally distributed random variable. Here, $n = 4$, therefore, the mean is 4 for χ_4^2 distribution. The variance of the distribution is nothing but, twice the degree of freedom. Hence, the variance is 8 for χ_4^2 distribution.

The expected value (mean): Conventionally, it can also be calculated by,

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_{0}^{1} z \left(\frac{1}{4}.z.e^{-\frac{z}{2}} \right) dz$$

$$E[Z] = 4$$

The variance: Conventionally, it can also be calculated by,

$$\sigma_Z{}^2 = E[Z^2] - (E[Z])^2$$

$$\sigma_Z{}^2 = \int_{-\infty}^{\infty} z^2 f_Z(z) dz - 16$$

$$\sigma_Z{}^2 = 8$$

Generating a random variable from two exponential distractions: Let us suppose we have two exponentially distributed random variables X_1 and X_2. Hence we can calculate the density of the sum of X_1 and X_2 by generating a third random variable Z. If X_i, $(i = 1, 2, 3, ..., n)$n are stochastically independent exponentially distributed random variables with rate parameter, α and mean, $1/\alpha$, then the probability density function is,

$$f_Z(x) = f_{X_1+X_2+...+X_n}(x) = \alpha e^{-\alpha x} \frac{(\alpha x)^{n-1}}{(n-1)!}$$

Thus for $n = 2$,

$$f_Z(x) = f_{X_1+X_2}(x) = \alpha e^{-\alpha x} \frac{(\alpha x)^{2-1}}{(2-1)!} = \alpha^2 x e^{-\alpha x}$$

If $\alpha = 1/2$,

$$f_Z(x) = \frac{1}{4} x e^{-\frac{1}{2}x} \tag{1}$$

Let us consider a random variable Z, which represents a function of a standard normally distributed random variables, such that

$$Z = \sum_{i=1}^{4} X_i^2$$

Its probability density function is given by,

$f_Z(x) = \frac{1}{2^n/2\Gamma(n/2)} Z^{\frac{n}{2}-1} e^{-z/2}, \quad \text{z} > 0$

$$f_Z(z) = \frac{1}{4} z e^{-\frac{1}{2}z} \tag{2}$$

From equation (1) and equation (2), we conclude that the density of random variable Z as a sum of two exponentially distributed random variables is the same as that ofχ^2 distributed random variable Z with four degrees of freedom, if $\alpha = 1/2$.

Probability Density as convolution of density of exponential random variables: We have four stochastically independent, standard normally distributed random variables, X_i where $(i = 1, 2, 3, 4)$. When these random variables are squared and grouped such that we have two exponentially distributed random variables X and Y where,

$X = X_1^2 + X_2^2$ and $Y = Y_3^2 + Y_4^2$

Therefore, the density function for the random variables can be expressed as,

$$f_X(x) = \frac{1}{2} e^{-\frac{x}{2}}$$

and

$$f_Y(y) = \frac{1}{2} e^{-\frac{y}{2}}$$

Now, if we assume that X and Y generate a new random variable Z, such that $Z = X + Y$. Then the bivariate density function for X and Y gives the probability distribution function of Z.

$$f_Z(z) = \int_{-\infty}^{\infty} f_{(X,Y)}(x, y) dx dy = \int_{-\infty}^{\infty} f_X(x) f_Y(y) dx dy$$

Substituting $Y = Z - X$,

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x) dx$$

$$f_Z(z) = \frac{1}{4}.z.e^{-\frac{z}{2}}$$

1.6 Normal distribution from uniform distribution

Task: Calculate the bivariate density $f_{Y_1Y_2}(y_1, y_2)$ of

$Y_1 = \sqrt{-2\ln X_1}\sin(2\pi X_2)$ and each on [0,1] uniformly distributed random variables.

Solution: Given two random variables X_1 and X_2 which are independent, identical and uniformly distributed on the interval [0,1] that is $X_1 \sim R(0,1), Y_1 \sim R(0,1)$. The task is to generate bivariate density function of two random variables Y_1 and Y_2 where,

$$Y_1 = \sqrt{-2\ln(X_1)} sin(2\pi X_2) = g_1(X1, X2)$$

$$Y_2 = \sqrt{-2\ln(X_1)} cos(2\pi X_2) = g_2(X1, X2)$$

First step is relating Y_1 and Y_2 with X_1 and X_2 as follows:

$$Y_1^2 + Y_2^2 = -2\ln(X_1)$$

$$x_1 = g_1^{-1}(y_1, y_2) = \exp\left(-\frac{y_1^2 + y_2^2}{2}\right)$$

$$x_2 = g_2^{-1}(y_1, y_2) = \frac{1}{2\pi}\arctan\frac{y_1}{y_2}$$

To calculate the bivariate density function,

$$f_{Y_1Y_2}(y_1y_2) = \frac{f_{x_1x_2}(x_1,x_2)}{|\det(J)|}\Bigg|_{x_1=g_1^{-1}(y_1,y_2),\ x_2=g_2^{-1}(y_1,y_2)}$$

$$f_{Y_1Y_2}(y_1y_2) = f_{x_1x_2}(x_1,x_2)/\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} \tag{3}$$

where,

$$\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = |\det J|$$

Also, $f_{X_1X_2}(x_1x_2) = f_{X_1}(x_1)f_{Y_1}(y_1)$

To determine the determinant of the Jacobian Matrix, J,

$$J = \begin{bmatrix} \frac{\partial g_1(x_1,x_2)}{\partial x_1} & \frac{\partial g_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial g_2(x_1,x_2)}{\partial x_1} & \frac{\partial g_2(x_1,x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{sin(2\pi x_2)}{x_1\sqrt{-2\ln x_1}} & 2\pi\sqrt{-2\ln x_1}cos(2\pi x_2) \\ \frac{cos(2\pi x_2)}{x_1\sqrt{-2\ln x_1}} & -2\pi\sqrt{-2\ln x_1}sin(2\pi x_2) \end{bmatrix}$$

$$|\det(J)| = |\frac{2\pi}{x_1}|$$

$$|\det(J)| = \frac{2\pi}{\exp\left(-\frac{y_1^2+y_2^2}{2}\right)} \tag{4}$$

The joint probability density function of X_1 and X_2 will be:

$$f_{X_1X_2}(x_1x_2) = \begin{cases} 1 & 0 \leq x_1, x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Now, substituting equation (4) and equation (5) in equation (3), we get,

$$f_{x_1x_2}(x_1x_2) = \frac{1}{2\pi}exp(-\frac{y_1^2+y_2^2}{2})$$

Bivariate density function can be written as a product of two functions as shown below:

$$f_{y_1y_2}(y_1y_2) = \frac{1}{2\pi}\exp(-\frac{y_1^2}{2}).\frac{1}{2\pi}exp(-\frac{y_2^2}{2}) = f_{y_1}(y_1).f_{y_2}(y_2)$$

$$f_{y_1}(y1) = \frac{1}{2\pi}\exp(-\frac{y_1^2}{2})$$

$$f_{y_2}(y2) = \frac{1}{2\pi}\exp(-\frac{y_2^2}{2})$$

Each of these two functions represents a density function of a standard uniformly distributed random variable that is $Y_1 \sim N(0,1)$ and $Y_2 \sim N(0,1)$. Thus Y_1 and Y_2 can be termed now as independent and Standard Normally Distributed Random Variables.

Chapter 2

Exercises with Matlab

2.1 Standard normal distribution

Task: Load `dat1_3` containing the random sequences **x** and **y** of two bivariate normal distributed random variables. Transform the random samples such that they obey a standard normal distribution. Make sure that your transformation is correct by showing the theoretical and the via **density** estimated density of the transformed random sequence **x** in a diagram.

Solution: We have written separate functions in-order to have a clean code. First, the function **density** calculates the theoretical as well as the estimated density. Second, the function **plotGraphs**, plots outputs a single figure having a bar graph and two line graphs. And last, the function `display_results`, prints the mean and variances of the theoretical and estimated value.

MATLAB code:

```
clear all
close all
```

```
clc

load dat1_3;
std_normal_dist = (x - mean(x))/var(x);
[estimated_x, location, theoretical_x] = density (std_normal_dist);
plotGraphs(estimated_x,location,theoretical_x, ...
    'Standard normal density function');
display_results(std_normal_dist);



function [estimated,location,theoretical] = density(randomVariable)
% theoretical
n = -4:0.25:4;
theoretical = (1/sqrt(2*pi)) * exp(-(n.^2)/2);
% estimated
[height,location] = hist(randomVariable,length(n));
delta = location(2)-location(1);
estimated = height/(delta*length(randomVariable));
end

function plotGraphs(estimated,location,theoretical,str)
figure()
bar(location,estimated,'FaceColor',[0.9290, 0.6940, 0.1250], ...
    'EdgeColor','k','LineWidth',1)
set(gca,'Title',text('String',str,'FontAngle', 'italic', ...
    'FontWeight', 'bold'), ...
        'xlabel',text('String', 'range', 'FontAngle','italic'),...
        'ylabel',text('String', 'density','FontAngle','italic'), ...
        'FontSize',26)
hold on
grid on
plot(location,estimated,'r',location,theoretical,'--','LineWidth',3.0);
legend('Estimated value (bar graph)','Estimated value (line graph)', ...
    'Theoritical value');
hold off
end
```

```
function display_results(dist)
fprintf('Mean:')
fprintf('\n Theoretical value = %f',0)
fprintf('\n Expected value = %s',num2str(mean(dist)))
fprintf('\n\n Variance:')
fprintf('\n Theoretical value = %f',1)
fprintf('\n Estimated value = %f',var(dist))
end
```

Output: The output after execution of the code is shown below. The estimated value is shown using both, bar and line graphs. While, dotted line shows the theoretical value.

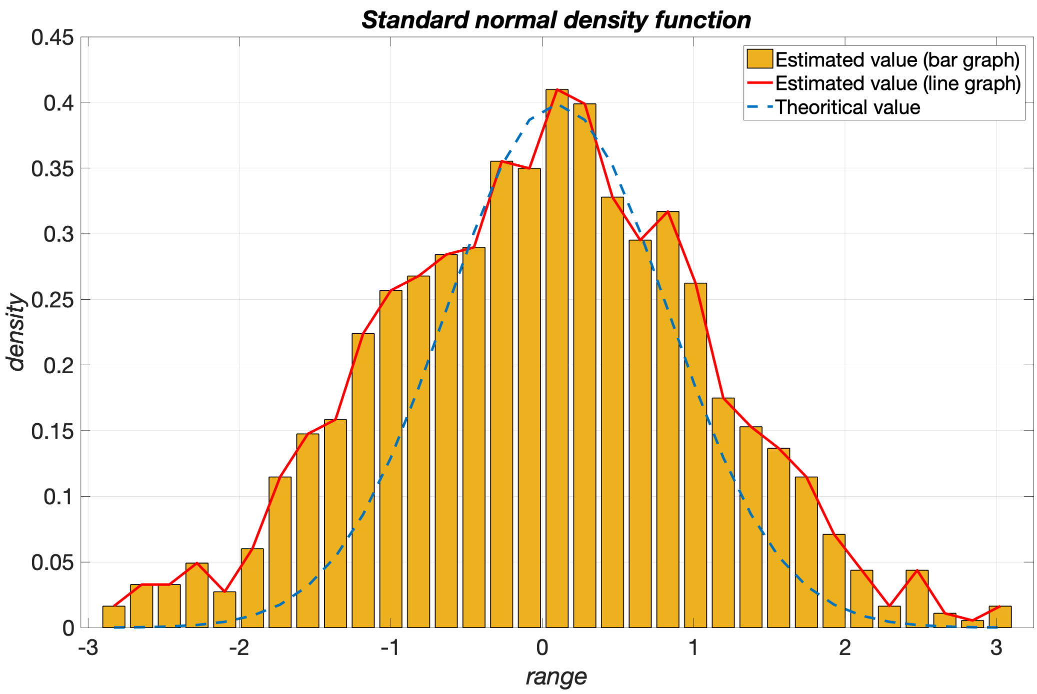

Figure 2.1: Standard normal density function

The output below shows the comparison of the theoretical value (from section 1.1.) and the estimated value (from the MATLAB code) of the mean and variances.

```
Mean:
 Theoretical value = 0.000000
 Expected value = -7.2459e-16

 Variance:
 Theoretical value = 1.000000
 Estimated value = 1.123375>>
```

Inference: We can infer that the theoretical value almost coincides with the estimated value. We can observe that the estimated function is very much close to the Gaussian bell obtained by the theoretical density function.

2.2 Exponential distribution

Task: Load the on [0, 1] uniformly distributed random sequence from `dat1_1` and transform it as explained in section 1.2 with $\alpha = 1/2$. Calculate the sample mean and sample variance of the transformed sequence and compare the results with the theoretical values from section 1.2. Show the in section 1.2 calculated and via `density` estimated density function of the transformed random sequence in a figure.

Solution: `density` A uniformly distributed random sequence from `dat1_1` is loaded and transformed as explained in section 1.2 with $\alpha = 1/2$. .The uniform density function as well as the normal density function is estimated by using the function `hist`. Line graph of both theoretical and estimated values are plotted in-order to compare the two.

MATLAB code:

```
clear all
close all
clc
```

```
alpha = 1/2;
load dat1_1;
exponential_dist = -2 * log(x);
n = 0:0.2:10;
[estimated_x, location, theoretical_x] = density (exponential_dist);
plotGraphs(estimated_x,location,theoretical_x, ...
    'Exponential density function');
display_results(exponential_dist, alpha);

function [estimated,location,theoretical] = density(randomVariable)
%theoretical
n = 0:0.2:10;
theoretical = 0.5 * exp((-0.5)*n);
%estimated
[height,location] = hist(randomVariable,n);
delta = location(2)-location(1);
estimated = height/(delta*length(randomVariable));
end

function plotGraphs(estimated,location,theoretical,str)
figure()
bar(location,estimated,'FaceColor',[0.9290, 0.6940, 0.1250], ...
    'EdgeColor','k','LineWidth',1)
set(gca,'Title',text('String',str,'FontAngle', 'italic', ...
    'FontWeight', 'bold'), ...
        'xlabel',text('String', 'range', 'FontAngle','italic'),...
        'ylabel',text('String', 'density','FontAngle','italic'), ...
        'FontSize',26)
hold on
grid on
plot(location,estimated,'r',location,theoretical,'--','LineWidth',3.0);
legend('Estimated value (bar graph)','Estimated value (line graph)', ...
    'Theoritical value');
hold off
end
```

```
function display_results(dist , alpha)
fprintf('Mean: ')
fprintf('\n Theoretical value = %f', 1/alpha)
fprintf('\n Expected value = %s',num2str(mean(dist)))
fprintf('\n\n Variance: ')
fprintf('\n Theoretical value = %f',1/alpha^2)
fprintf('\n Estimated value = %f',var(dist))

end
```

Output: The output after execution of the code is shown below. The estimated value is shown using both, bar and line graphs. While, dotted line shows the theoretical value.

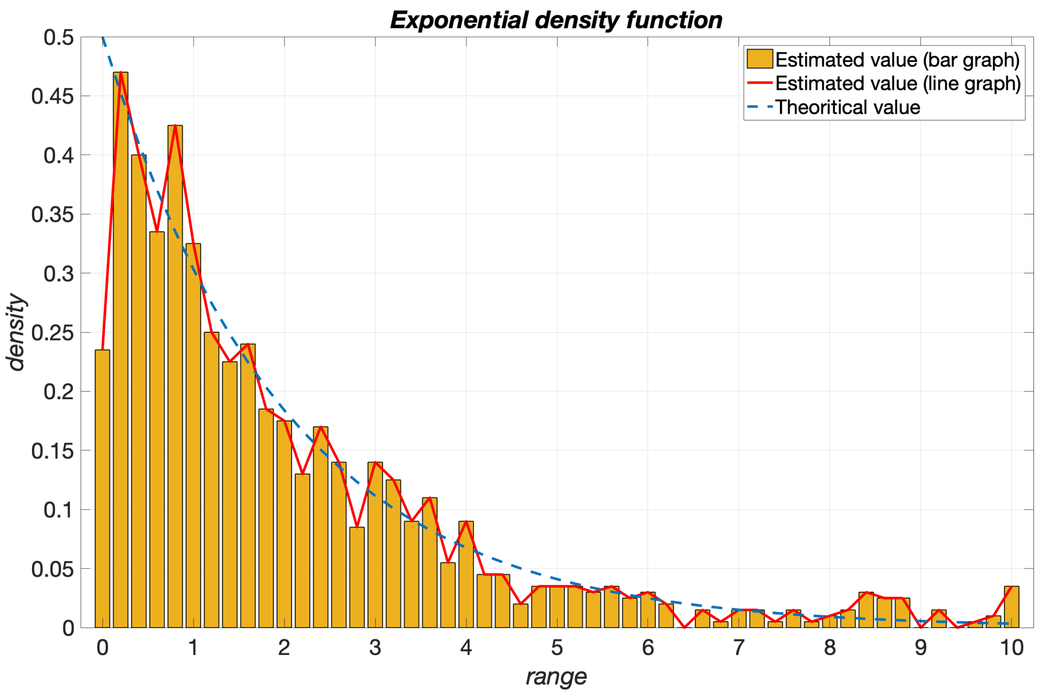

Figure 2.2: Exponential density function

The output below shows the comparison of the theoretical value (from section 1.2) and the estimated value (from the MATLAB code) of the mean and variances.

```
Mean:
 Theoretical value = 2.000000
 Expected value = 2.0275

 Variance:
 Theoretical value = 4.000000
 Estimated value = 4.451815>>
```

Inference: We can infer that the theoretical value almost coincides with the estimated value. From the output, we can observe small change in the values which is due to the length of the sample.

2.3 Sum of random numbers

Task: Generate two on [0, 1] uniformly distributed random sequences of the length 1000 after setting the initial value to 0. Save both random sequences in `dat2_1`. Then add the two sequences element-by-element. Calculate the sample mean and sample variance of the sum and compare the results with the theoretical values from section 2.1.3. Display the in section 2.1.3 calculated and via `density` estimated density function in a diagram.

Solution: First, two standard normal distributed random sequences of length 1000 were generated using the function `randn` and these were saved in `dat2_1.` .The two random sequences from `dat2_1` were added element-by-element and this density function of the resulting product was calculated.

MATLAB code:

```
clear all
close all
clc

rand('state',0)
randomSequences = rand(1000,2);
```

```
randomSeq_1 = randomSequences(:,1);
randomSeq_2 = randomSequences(:,2);
save dat2_1 randomSeq_1 randomSeq_2
sum_Seq = randomSeq_1 + randomSeq_2;
[estimated_sum_seq, location, theoretical_sum_seq] = density (sum_Seq);
plotGraphs(estimated_sum_seq,location,theoretical_sum_seq, ...
    'Density function of sum of uniformly distributed random variables');
display_results(sum_Seq);


function [estimated,location,theoretical] = density(randomVariable)
%theoretical
k = 0:.1:2;
for i = length(k) : -1 : 1
    if k(i) <= 1
        theoretical(i) = k(i);
    else
        theoretical(i) = 2 - k(i);
    end
end
%estimated
[height,location] = hist(randomVariable,length(k));
delta = location(2)-location(1);
estimated = height/(delta*length(randomVariable));
end

function plotGraphs(estimated,location,theoretical,str)
figure()
bar(location,estimated,'FaceColor',[0.9290, 0.6940, 0.1250], ...
    'EdgeColor','k','LineWidth',1)
set(gca,'Title',text('String',str,'FontAngle', 'italic', ...
    'FontWeight', 'bold'), ...
        'xlabel',text('String', 'range', 'FontAngle','italic'),...
        'ylabel',text('String', 'density','FontAngle','italic'), ...
        'FontSize',26)
hold on
```

```
grid on
plot(location,estimated,'r',location,theoretical,'--','LineWidth',3.0);
legend('Estimated value (bar graph)','Estimated value (line graph)', ...
    'Theoritical value');
hold off
end

function display_results(dist)
fprintf('Mean:')
fprintf('\n Theoretical value = %f', 1)
fprintf('\n Expected value = %s',num2str(mean(dist)))
fprintf('\n\n Variance:')
fprintf('\n Theoretical value = %f',1/6)
fprintf('\n Estimated value = %s',num2str(var(dist)))
end
```

Output: The output after execution of the code is shown below. The estimated value is shown using both, bar and line graphs. While, dotted line shows the theoretical value.

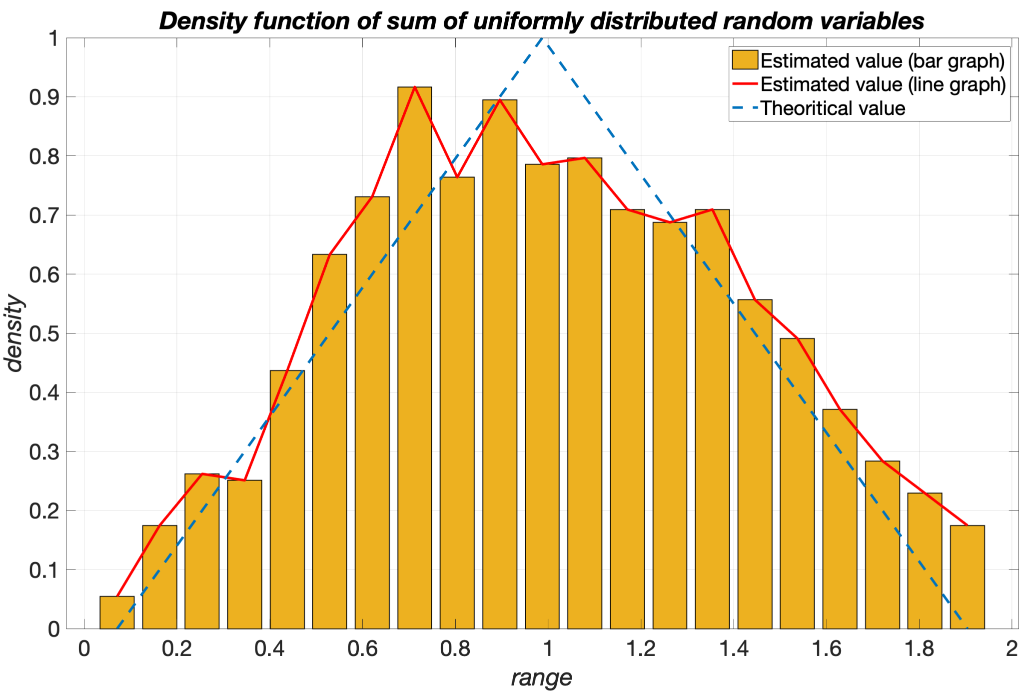

Figure 2.3: Density function of sum of uniformly distributed random variables

The output below shows the comparison of the theoretical value (from section 1.3) and the estimated value (from the MATLAB code) of the mean and variances of the sum of random sequences.

```
Mean:
 Theoretical value = 0.000000
 Expected value = -7.2459e-16

 Variance:
 Theoretical value = 1.000000
 Estimated value = 1.123375>>
```

Inference: We can infer that the theoretical value almost coincides with the estimated value. The small variation in the value is due to the sample length.

2.4 Product of random numbers

Task: Multiply both on [0, 1] uniformly distributed random sequences from `dat2_1` element by element. Calculate the sample mean and sample variance of the product and compare the results with the theoretical values derived in section 1.4. Depict the in section 1.4 calculated and via density estimated density function in a figure.

Solution: The two random sequences from `dat2_1` were multiplied element-by-element and this density function of the resulting product was calculated.

MATLAB code:

```
clear all
close all
clc

load dat2_1;
product_Seq = randomSeq_1 .* randomSeq_2;
```

```
[estimated_product_seq, location, theoretical_product_seq] = density (product_Seq);
plotGraphs(estimated_product_seq,location,theoretical_product_seq, ...
    'Density function of product of uniformly distributed random variables');
display_results(product_Seq);



function [estimated,location,theoretical] = density(randomVariable)
%theoretical
k = 0 : 0.025 : 1;
theoretical = -log(k);

%estimated
[height,location] = hist(randomVariable,length(k));
delta = location(2)-location(1);
estimated = height/(delta*length(randomVariable));

theoretical(1,1) = estimated(1,1);
end

function plotGraphs(estimated,location,theoretical,str)
figure()
bar(location,estimated,'FaceColor',[0.9290, 0.6940, 0.1250], ...
    'EdgeColor','k','LineWidth',1)
set(gca,'Title',text('String',str,'FontAngle', 'italic', ...
    'FontWeight', 'bold'), ...
        'xlabel',text('String', 'range', 'FontAngle','italic'),...
        'ylabel',text('String', 'density','FontAngle','italic'), ...
       'FontSize',26)
hold on
grid on
plot(location,(estimated),'r',location,(theoretical),'--','LineWidth',3.0);
ylim([0 inf])
legend('Estimated value (bar graph)','Estimated value (line graph)', ...
    'Theoritical value');
hold off
end
```

```
function display_results(dist)
fprintf('Mean:')
fprintf('\n Theoretical value = %f', 1/4)
fprintf('\n Expected value = %s',num2str(mean(dist)))
fprintf('\n\n Variance:')
fprintf('\n Theoretical value = %f',7/144)
fprintf('\n Estimated value = %s',num2str(var(dist)))
end
```

Output: The output after execution of the code is shown below. The estimated value is shown using both, bar and line graphs. While, dotted line shows the theoretical value.

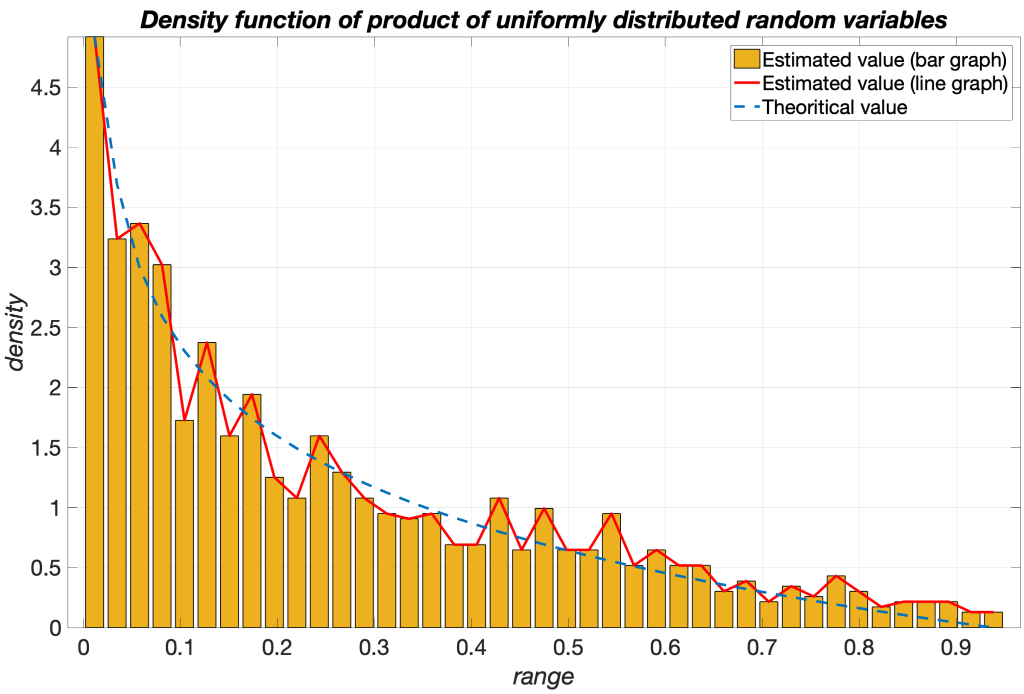

Figure 2.4: Density function of product of uniformly distributed random variables

he output below shows the comparison of the theoretical value (from section 1.4) and the estimated value (from the MATLAB code) of the mean and variances of the product of random sequences.

```
Mean:
 Theoretical value = 0.250000
 Expected value = 0.25444

 Variance:
 Theoretical value = 0.048611
 Estimated value = 0.053547>>
```

Inference: We can infer that the theoretical value almost coincides with the estimated value. The small variation in the value is due to the sample length. The theoretical curve depicts a smooth decaying curve whereas, the nature of the estimated curve is a decaying curve with many variations.

2.5 χ^2 -distribution

Task: Generate four standard normal distributed random sequences of the length 1000, where before the initial value has to be set to 0. Save the data in `dat2_2`. Determine the sum of squares of the four random sequences element-by-element. Calculate the sample mean and sample variance of the sum and compare the results with the theoretical values obtained in section 1.5. Present the in section 1.5 calculated and via density estimated `density` function in a diagram.

Solution: First, four standard normal distributed random sequences of length 1000 were generated using the function `randn` and these were saved in `dat2_2`. The sum of squares of the four random sequences element-by-element was determined and the density function of it was calculated.

MATLAB code:

```
clear all
close all
clc
```

```
w = randn(1000,1);
x = randn(1000,1);
y = randn(1000,1);
z = randn(1000,1);
save data2_2 w x y z
sum_of_squares_Seq = w.^2+x.^2+y.^2+z.^2;
[estimated_sum_of_squares_Seq, location, ...
    theoretical_sum_of_squares_Seq] = ...
    density (sum_of_squares_Seq);
plotGraphs(estimated_sum_of_squares_Seq, ...
    location,theoretical_sum_of_squares_Seq, ...
    '\chi^2 uniformly distributed density function');
display_results(sum_of_squares_Seq);


function [estimated,location,theoretical] = density(randomVariable)
%theoretical
k = 0 : .5 : 20;
theoretical = (k/4) .* exp(-k/2);

%estimated
[height,location] = hist(randomVariable,length(k));
delta = location(2)-location(1);
estimated = height/(delta*length(randomVariable));
end

function plotGraphs(estimated,location,theoretical,str)
figure()
bar(location,estimated,'FaceColor',[0.9290, 0.6940, 0.1250], ...
    'EdgeColor','k','LineWidth',1)
set(gca,'Title',text('String',str,'FontAngle', 'italic', ...
    'FontWeight', 'bold'), ...
        'xlabel',text('String', 'range', 'FontAngle','italic'),...
        'ylabel',text('String', 'density','FontAngle','italic'), ...
        'FontSize',26)
hold on
```

```
grid on
plot(location,estimated,'r',location,theoretical,'--','LineWidth',3.0);
legend('Estimated value (bar graph)','Estimated value (line graph)', ...
    'Theoritical value');
hold off
end

function display_results(dist)
fprintf('Mean:')
fprintf('\n Theoretical value = %f', 4)
fprintf('\n Expected value = %s',num2str(mean(dist)))
fprintf('\n\n Variance:')
fprintf('\n Theoretical value = %f',8)
fprintf('\n Estimated value = %s',num2str(var(dist)))
end
```

Output: The output of the code is shown below. The estimated value is shown using both, bar and line graphs. While, dotted line shows the theoretical value.

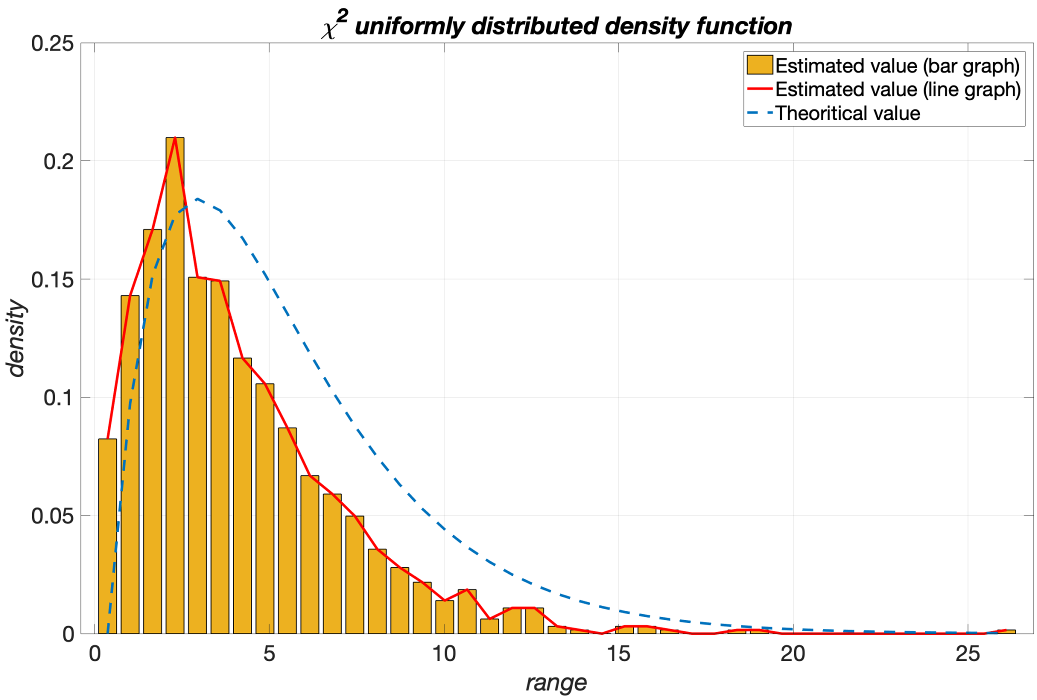

Figure 2.5: χ^2 uniformly distributed density function

The output below shows the comparison of the theoretical value (from section 1.5) and the estimated value (from the MATLAB code) of the mean and variances of the sum of squares of the four random sequences.

```
Mean:
 Theoretical value = 4.000000
 Expected value = 4.076

 Variance:
 Theoretical value = 8.000000
 Estimated value = 7.8996>>
```

Inference: We can infer that the theoretical value almost coincides with the estimated value. The small variation in the value is due to the sample length.

2.6 Normal distribution from uniform distribution

Task: Now use again `dat2_1` containing two on [0, 1] uniformly distributed random sequences. Build **`y1`** as explained in section 1.6. Calculate the sample mean and sample variance of **`y1`** and compare the results with the theoretical values determined in section 2.1.6. Show the in section 1.6 calculated and via `density` estimated density function in a figure.

Solution: `dat2_1` was loaded as mentioned in the task. `y1` was considered to be the first column of `dat2_1`, i.e, `random_Seq_1`. The estimated density of `y1` was calculated by the function called `density`.

MATLAB code:

```
clear all
close all
clc
```

```
load dat2_1
y1 = sqrt(-2*log(randomSeq_1)) .* sin(2*pi*randomSeq_2);
[estimated_y1, location, theoretical_y1] = density (y1);
plotGraphs(estimated_y1,location,theoretical_y1, ...
    'Normal distribution from uniform distribution');
display_results(y1);



function [estimated,location,theoretical_y1] = density(randomVariable)
%theoretical
k = -5 : 0.5 : 5;
theoretical_y1 = (1/sqrt(2*pi)) * exp(-k.^2/2);

%estimated
[height,location] = hist(randomVariable,length(k));
delta = location(2)-location(1);
estimated = height/(delta*length(randomVariable));
end

function plotGraphs(estimated,location,theoretical,str)
figure()
bar(location,estimated,'FaceColor',[0.9290, 0.6940, 0.1250], ...
    'EdgeColor','k','LineWidth',1)
set(gca,'Title',text('String',str,'FontAngle', 'italic', ...
    'FontWeight', 'bold'), ...
        'xlabel',text('String', 'range', 'FontAngle','italic'),...
        'ylabel',text('String', 'density','FontAngle','italic'), ...
        'FontSize',26)
hold on
grid on
plot(location,estimated,'r',location,theoretical,'--','LineWidth',3.0);
legend('Estimated value (bar graph)','Estimated value (line graph)', ...
    'Theoritical value');
hold off
end
```

```
function display_results(dist)
fprintf('Mean:')
fprintf('\n Theoretical value = %f', 0)
fprintf('\n Expected value = %s',num2str(mean(dist)))
fprintf('\n\n Variance:')
fprintf('\n Theoretical value = %f',1)
fprintf('\n Estimated value = %s',num2str(var(dist)))
end
```

Output: The output of the code is shown below. The estimated value is shown using both, bar and line graphs. While, dotted line shows the theoretical value.

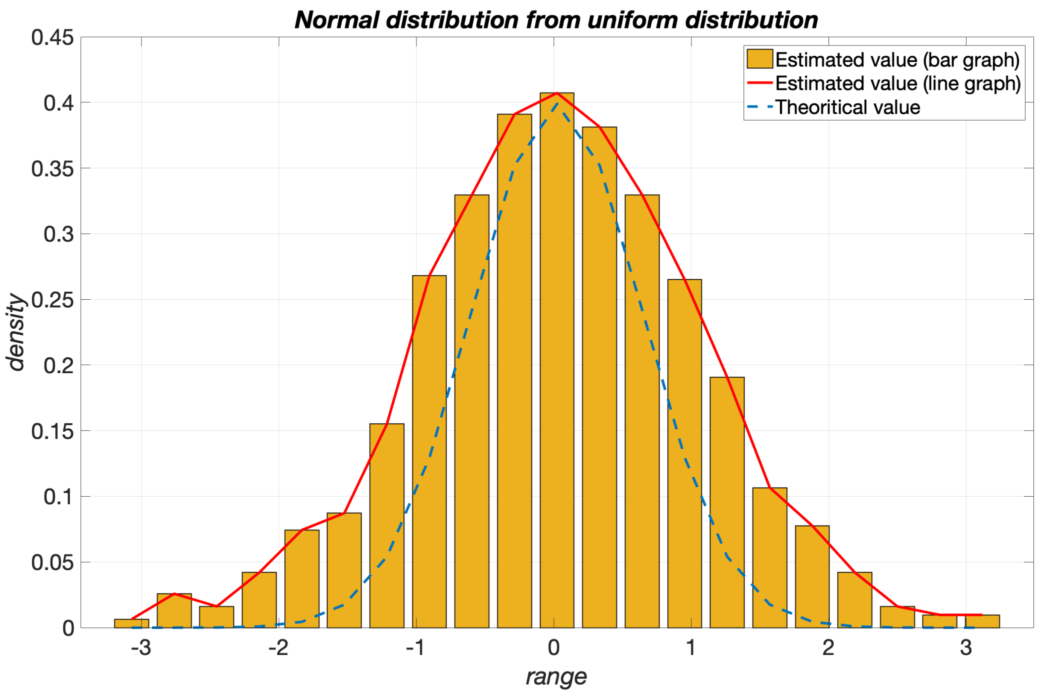

Figure 2.6: Normal distribution from uniform distribution

The output below shows the comparison of the theoretical value (from section 1.6) and the estimated value (from the MATLAB code) of the mean and variances of the transformed y1.

```
Mean:
 Theoretical value = 0.000000
 Expected value = 0.037115

 Variance:
 Theoretical value = 1.000000
 Estimated value = 1.0174>>
```

Inference: We can infer that the theoretical value almost coincides with the estimated value. The estimated graph of `y1` shows a normal uniformly distributed curve, whereas the obtained theoretical graph is a bivariate density function curve with mean as 0 and variance as 1.

Bibliography

[1] Prof. Dr.-Ing. Dieter Kraus, *"Stochastic Signals and Systems- Probability Theory* lecture notes.

[2] https://de.mathworks.com/help/matlab/

[3] Intuitive Probability and Random Processes using MATLAB, Steven M. Kay, Kluwer Academic Publishers, 2006