UNDERWATER ACOUSTICS AND SONAR
SIGNAL PROCESSING

SS 2018

ASSIGNMENT 8

AMBIGUITY FUNCTION

Date: 25/06/2018

by

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Introduction

SONAR transmitters perform the underwater detection of objects and in most cases, these objects or the SONAR transmitters are in motion. This means that the actual received signal differs from the expected results due to a phenomenon known as Doppler Shift, which is caused by the relative motion of the transmitter and the target. The signal received at the SONAR receivers also changes due to the time delay introduced in the reception of the signals owing to the path travelled by the waves from the transmitter to the target and back to the receiver. These phenomena introduce distortions in the signal received and hence a term called the Ambiguity function is used to study these effects.

The spectra of Rectangular Pulse, Linear Frequency Modulated Pulse (LFM) with a rectangular envelope and Linear Frequency Modulated Pulse(LFM) with a Gaussian envelope are obtained. The spectra and ambiguity function of all these three functions are plotted and discussed in this assignment

Theory

Ambiguity Function

The ambiguity is a two-dimensional function of delay and Doppler frequency showing the distortion of an uncompensated match filter due to the Doppler shift of the return from a moving target. It is useful for describing the behaviour of a radar or a sonar signal. This function appears to be a good tool to select good signals for the Range and Doppler estimation. It gives an idea of the amount of distortion present in the received signal. It is defined as the time response of a filter matched to a given finite energy signal when the signal is received with a delay (τ) and a doppler shift (v) relative to the nominal values expected by the filter. The ambiguity function is defined by $-$

$$\chi(\tau, v) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi vt}dt \tag{1}$$

It can be interpreted as the output of a matched filter designed for a Doppler frequency shift f if a signal with Doppler frequency shift $f_0 + v$ is received. Thus $\chi(\tau, v)$ can be understood as the point target response in Range/Doppler domain.

$$
\begin{aligned}
\chi(\tau, v) &= \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi vt}dt \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} S(\omega)e^{j\omega t}\frac{d\omega}{2\pi}\right)\left(\int_{-\infty}^{\infty} S^*(\omega^{'})e^{-j\omega^{'}(t-\tau)}\frac{d\omega^{'}}{2\pi}\right)e^{j2\pi vt}dt \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\omega)S^*(\omega')e^{j\omega^{'}\tau}e^{j(\omega-\omega^{'}+2\pi v)t}d\omega d\omega' dt \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\omega)S^*(\omega')e^{j\omega^{'}\tau}\delta(\omega-\omega^{'}+2\pi v)d\omega d\omega' \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} S(\omega'-2\pi v)S^*(\omega^{'})e^{j\omega^{'}\tau}d\omega^{'}
\end{aligned}
\tag{2}
$$

The ambiguity function depends on the transmitted signal and one of the transmitted signals used here is a Linear Frequency Modulated pulse (LFM). In such a signal, the frequency is increased linearly resulting in a particular ambiguity function which is discussed in detail in the following sections.

Fast Fourier Transform (FFT) is an algorithm which is used to find the spectrum of any signal. FFT is a much faster algorithm as it reduces the number of computations involved in the fourier transform of the signal

Ambiguity Function of particular waveforms

The spectra of three waveforms are obtained using the FFT algorithm and they are studied. The three waveforms used are:

1. A rectangular Pulse
2. A Linear Frequency Modulated Pulse (LFM) with a rectangular envelope
3. A Linear Frequency Modulated Pulse(LFM) with a Gaussian envelope

Rectangular Pulse

The simplest waveform for a radar system is probably a rectangular waveform, sometimes also referred to as single frequency waveform. For the rectangular waveform, the pulse width is the reciprocal of the bandwidth. The equation for the waveform is given by −

$$s(t) = \frac{1}{\sqrt{T}} 1_{(-T/2,T/2)}(t) \tag{3}$$

with $||S||^2 = 1$. Hence, the ambiguity function is given by

$$\begin{aligned}\chi(\tau, v) &= \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi vt}dt \\ &= \begin{cases} e^{j\pi v\tau}\left(1 - \frac{|\tau|}{T}\right)\frac{sin(\pi v(T-|\tau|))}{\pi v(T-|\tau|)}, & \text{for } |\tau| \leq T \\ 0, & \text{elsewhere} \end{cases}\end{aligned} \tag{4}$$

Linear Frequency Modulated Pulse (LFM) with a rectangular envelope

The equation for this waveform is given by −

$$s(t) = \frac{1}{\sqrt{T}} 1_{(-T/2,T/2)}(t) exp(j\pi k t^2) \tag{5}$$

with $k = b/T$. Hence, the ambiguity function is given by

$$\begin{aligned}\chi(\tau, v) &= \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi vt}dt \\ &= \begin{cases} e^{j\pi v\tau}\left(1-\frac{|\tau|}{T}\right)\frac{sin(\pi(k\tau+v)(T-|\tau|))}{\pi(k\tau+v)(T-|\tau|)}, & \text{for } |\tau| \leq T \\ 0, & \text{elsewhere} \end{cases}\end{aligned} \tag{6}$$

Determining the ambiguity function for a Linear Frequency Modulated Pulse (LFM) with a rectangular envelope

$$s(t) = e^{j(\pi k t^2)} \frac{1}{\sqrt{T}} rect(t)$$

$$\chi(\tau, v) = \frac{1}{\sqrt{T}}\frac{1}{\sqrt{T}} \int_{-\infty}^{\infty} rect(t)rect(t-\tau)e^{j(\pi k t^2)}e^{-j(\pi k(t-\tau)^2)}e^{j(2\pi kvt)}dt$$

Replacing: $t = t^{'} + \frac{\tau}{2}$

$$\rightarrow \chi(\tau, v) = \frac{1}{T}\int_{-\infty}^{\infty} rect(t^{'}+\frac{\tau}{2})rect(t^{'}-\frac{\tau}{2})e^{j(\pi k(t^{'}+\frac{\tau}{2})^2)}e^{-j(\pi k(t^{'}-\frac{\tau}{2})^2)}e^{j(2\pi kv(t^{'}+\frac{\tau}{2}))}dt^{'}$$

Since $(A+B)^2-(A-B)^2=4AB$

$$\begin{aligned} e^{j(\pi k(t^{'}+\frac{\tau}{2})^2)}e^{-j(\pi k(t^{'}-\frac{\tau}{2})^2)} &= e^{j\left(\pi k(t^{'}+\frac{\tau}{2})^2-(t^{'}-\frac{\tau}{2})^2\right)} \\ &= e^{j(4\pi k t^{'}\frac{\tau}{2})} \\ &= e^{j(2\pi k t^{'}\tau)} \end{aligned} \tag{7}$$

$$\rightarrow \chi(\tau, v) = \frac{1}{T}\int_{-\infty}^{\infty} rect(t^{'}+\frac{\tau}{2})rect(t^{'}-\frac{\tau}{2})e^{j(2\pi k t^{'}\tau)}e^{j(2\pi k t^{'})}dt^{'}$$

$$\rightarrow \chi(\tau, v) = \frac{1}{T}\int_{-\infty}^{\infty} rect(t^{'}+\frac{\tau}{2})rect(t^{'}-\frac{\tau}{2})e^{j(2\pi(k\tau+v)t^{'})}dt^{'}$$

Let $d(t)=\frac{T-|\tau|}{2}$

$$\rightarrow \chi(\tau, v) = \frac{1}{T}\int_{-d(t)}^{d(t)} e^{j(2\pi(k\tau+v)t^{'})}dt^{'}$$

Let $d(t)=\frac{T-|\tau|}{2}$

$$\rightarrow \chi(\tau, v) = \frac{1}{T}e^{j(\pi v\tau)}\frac{e^{j(2\pi(k\pi+v)(d(t)-(-d(t))))}}{j(2\pi(k\tau+v)(d(t)-(-d(t))))}$$

$$\rightarrow \chi(\tau, v) = \frac{1}{T}e^{j(\pi v\tau)}d(t^{'})\frac{j2sin(2\pi(k\pi+v)(\frac{T-|\tau|}{2}))}{j(2\pi(k\tau+v)(\frac{T-|\tau|}{2}))}$$

$$\chi(\tau, v) = \begin{cases} e^{j\pi v\tau}\left(1-\frac{|\tau|}{T}\right)\frac{sin(\pi(k\tau+v)(T-|\tau|))}{\pi(k\tau+v)(T-|\tau|)}, & \text{for } |\tau| \leq T \\ 0, & \text{elsewhere} \end{cases}$$

Hence proved

Linear Frequency Modulated Pulse (LFM) with a Gaussian envelope

The equation for this waveform is given by −

$$s(t) = \frac{1}{\sqrt[4]{\pi\sigma^2}} exp\left(-\frac{t^2}{2\sigma^2} + j\pi k t^2 \right) \tag{8}$$

where k determines the slope of the LFM with $|k| = b/T$. After some manipulations, we obtain

$$\begin{aligned} \chi(\tau, v) &= \int_{-\infty}^{\infty} s(t) s^*(t-\tau) e^{j2\pi vt} dt \\ &= e^{j\pi vt} exp\Big(-\tau^2/(4\sigma^2) - \pi\sigma^2 (k\tau + v)^2 \Big) \end{aligned} \tag{9}$$

Experimental Research

Rectangular Pulse

The spectrum of a rectangular pulse is shown in Figure 1. The output is obtained as a result of the Fast Fourier Transform algorithm using 512 points. Thus the frequency range corresponding to the time range on the left has been divided into 512 points in the spectrum on the right.The voltage of the spectrum is expressed in logarithm.

From the Figure 1, we can conclude that the Fourier transform of rectangular pulse is a sinc function in frequency domain. We know also from the Fourier transformation that the bandwidth of a rectangular pulse is given as $b = 1/T$ from this relation we deduce that $b * T = 1$ for all pulse durations. This CW signal exhibits some interference patterns in its ambiguity diagrams which could affect its range resolution. This pulse is having a good Doppler resolution which can be seen in the Figure 1.

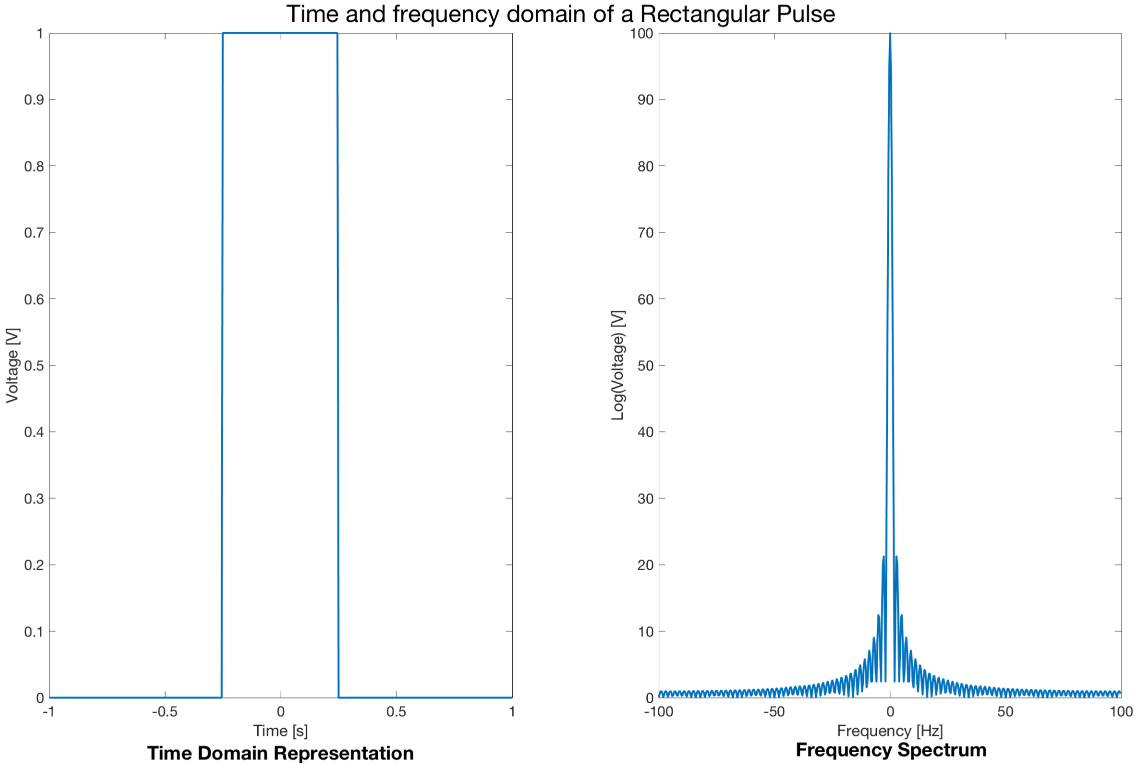

Figure 1: A rectangular pulse in time domain (left) along with its spectrum (right)

The ambiguity function plot above provides a three dimensional view of the amount of distortion that is involved with a rectangular pulse with a Doppler shift and a time delay. The peak value can be observed at the center while the values gradually decrease around the sides. This resembles a sinc function in three dimensions exactly like the Fourier transform of the rectangular pulse shown in Figure 1. In Figure 2 we can observe that the output of a matched filter depends on the Doppler Effect. We can also say that the range resolution of the rectangular pulse is relatively poor due to the zero Doppler shifts ($v = 0$). T he matched filter produces a high output values for a big τ range. A lot of interferences could be observed for values of v> 0.

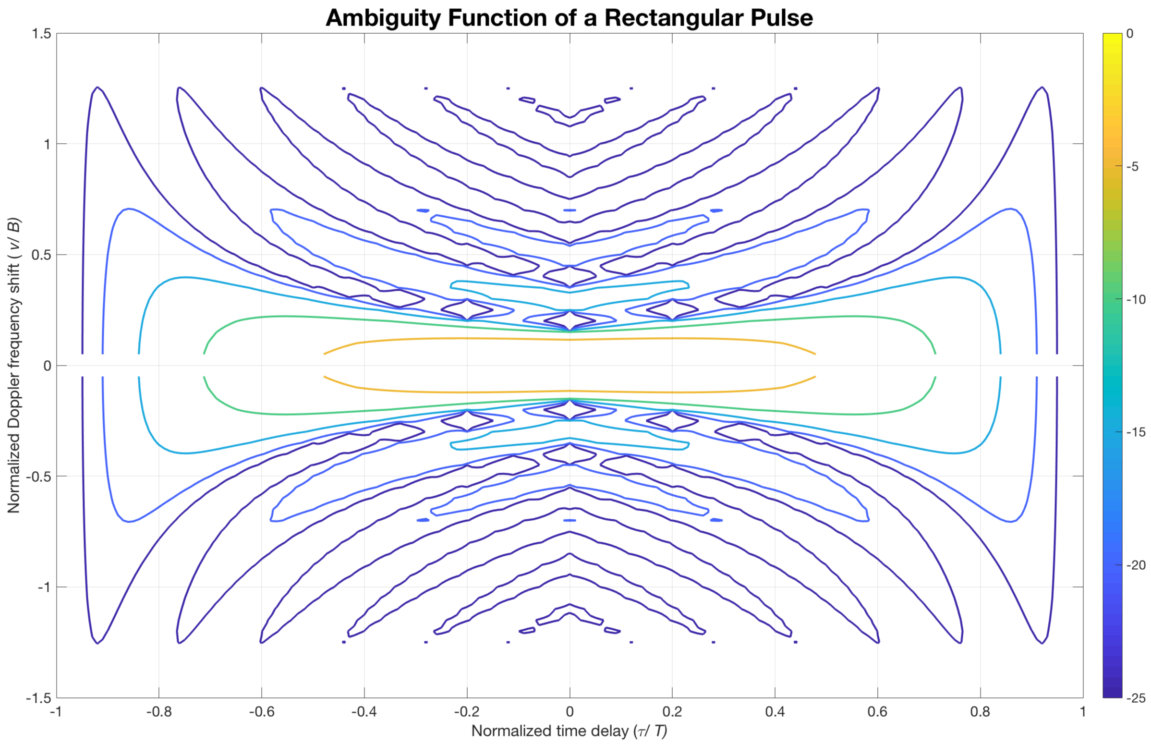

Figure 2: Ambiguity function of a rectangular pulse

Linear Frequency Modulated pulse with a Rectangular Envelope

Figure 3 displays the spectrum of a Linear Frequency Modulated pulse with a rectangular envelope. The voltage of the spectrum is expressed in logarithm.

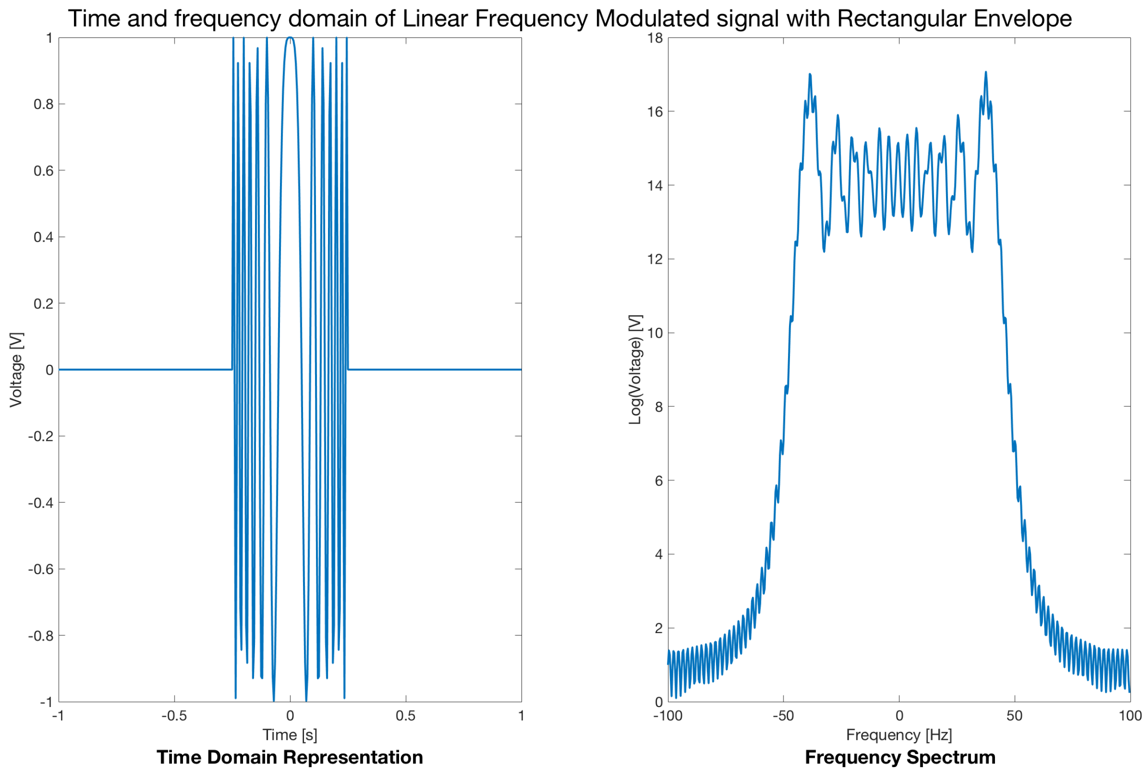

Figure 3: A Linear Frequency Modulated (LFM) pulse with rectangular envelope in time domain (left) along with its spectrum (right)

The signal shown on the left is a Linear Frequency Modulated pulse which is also known as a chirp. The Fourier Transform of a chirp with rectangular envelope is the figure shown on the right. The frequency variations can be seen in the peak of the spectrum around the center frequency.

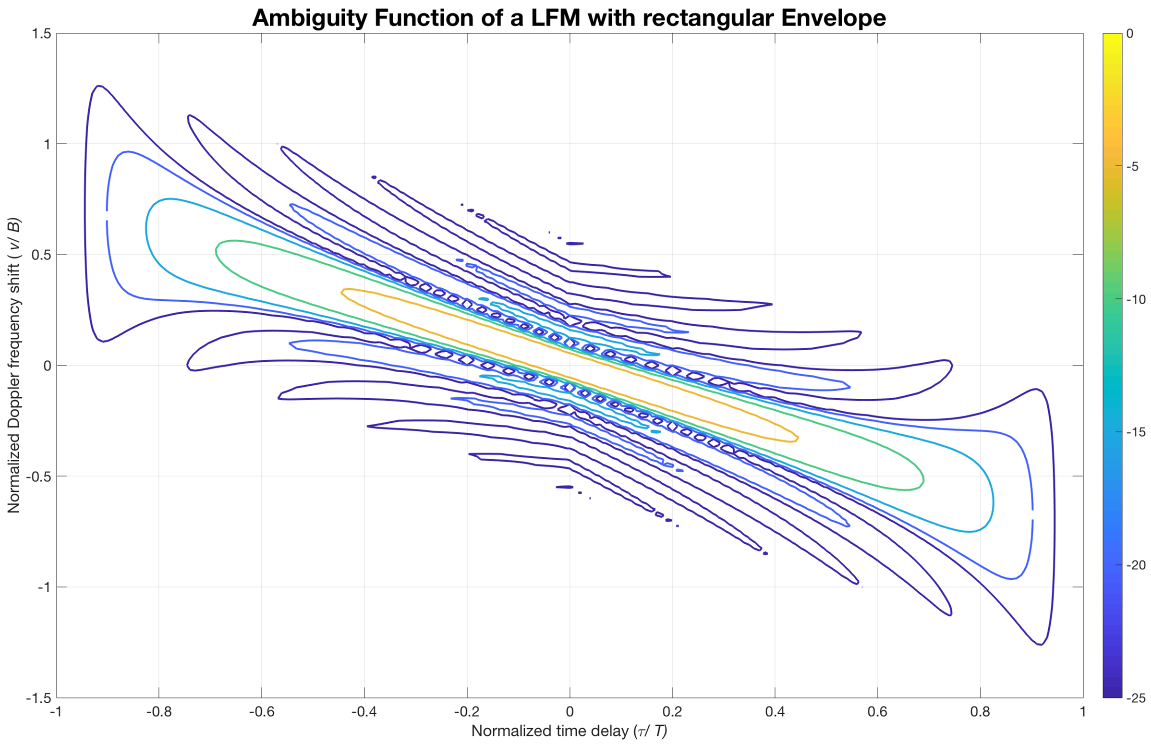

Figure 4: The ambiguity function of a Linear Frequency Modulated (LFM) pulse with rectangular envelope

The spectrum and ambiguity functions in Figure 3 and Figure 4 are similar to that of Figure 1 when the slope of the FM signal is 0. So, the characteristics are also same. The slope of the FM signal (Figure 3) causes better range and Doppler resolution than the CW rectangular pulse (Figure 1).

The ambiguity function plot above shown above is similar to the one obtained for the rectangular pulse. The obvious change is the gradient observed in this figure which is due to the factor (k) introduced in the expression for the ambiguity function. The bandwidth time product for this pulse is $b * T = k * T^2$. The Figure 4 shows that the output of a matched filter has only a small dependency on the Doppler Effect. On the contrary, there are no problems with interferences in this case.

Linear Frequency Modulated pulse with a Gaussian Envelope

Figure 5 displays the spectrum of a Linear Frequency Modulated pulse with a Gaussian envelope. The voltage of the spectrum is expressed in logarithm.

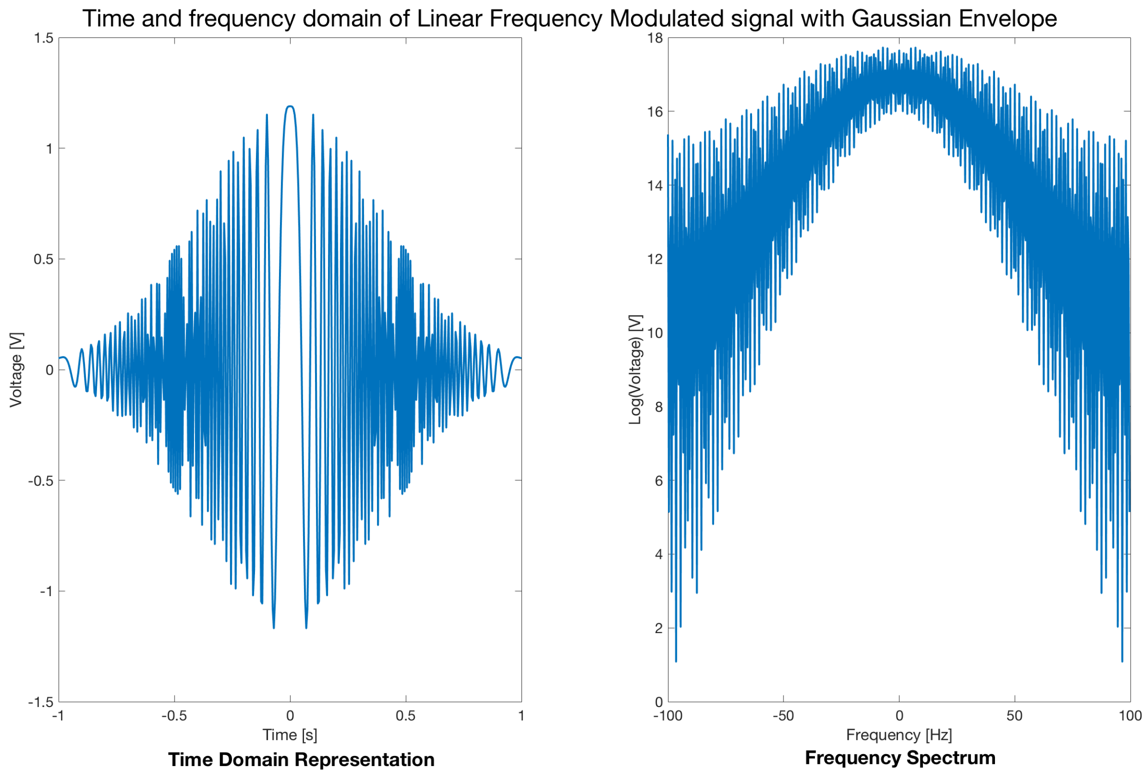

Figure 5: A Linear Frequency Modulated (LFM) pulse with Gaussian envelope in time domain (left) along with its spectrum (right)

The signal shown on the left is a Linear Frequency Modulated signal with a Gaussian envelope. The Fourier Transform of such a chirp is expected to yield a Gaussian envelope as is shown in the figure to the right. A Gaussian envelope with the frequency variations around it can be observed in the figure to the right.

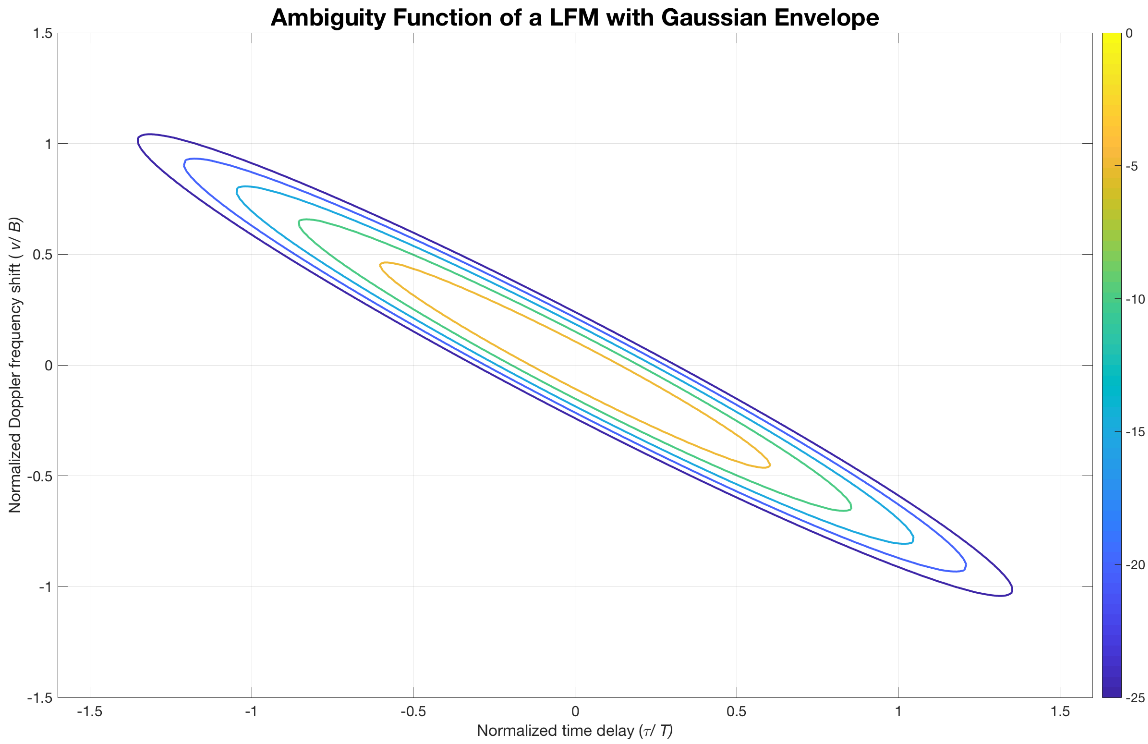

Figure 6: The ambiguity function of a Linear Frequency Modulated (LFM) pulse with Gaussian envelope

The ambiguity function plot shown above is similar to the spectrum of the LFM pulse with the Gaussian envelope. The three dimensional representation shown above also has a Gaussian shape with the peak around the center frequency.

Interference is not present as compared to the rectangular pulse. Furthermore, slope of the FM signal increases the range and Doppler resolutions considerably. The bandwidth time product for this pulse is $2b * T = k * T$. In Figure 6, we can observe that the output of a matched filter has only a small dependency on the Doppler Effect.

Conclusion

It can be concluded that although the ambiguity function gives an idea of the amount of distortion obtained at the matched filter, the above three ambiguity function plots prove that the ambiguity function of any transmitted signal is similar to the signal itself. The figures attain their maximum value at the origin of the plane, and the values of the origin reveal correlations in the signal structure with respect to both time and frequency shifts. And, the ambiguity function has symmetry with respect to the origin.

Appendix

MATLAB code to plot the spectrums for different waveforms

```
Fs = 200; %Sampling frequency
t = -1:1/Fs:1;
T = max(t); %pulse-width
B = 200; %bandwidth
k = B/T;
width = 0.5;
pulse = rectpuls(t,width);
figure('Color',[1 1 1]);
suptitle('\fontsize{25}Time and frequency domain of a Rectangular Pulse')
s1 = (1/sqrt(max(t))).*pulse;
subplot(1,2,1)
plot(t,s1,'LineWidth',2)
hold on
xlabel('Time [s]')
ylabel('Voltage [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Time Domain Representation','FontSize',20);
set(a,'Position',[0.00,-0.10],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
[f, fft_s1] = FFT(s1, Fs);
subplot(1,2,2)
```

```
plot(f, fft_s1 ,'LineWidth',2);
xlabel('Frequency [Hz]')
ylabel('Log(Voltage) [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Frequency Spectrum','FontSize',20);
set(a,'Position',[0.5,-9.5],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
figure('Color',[1 1 1]);
suptitle('\fontsize{25}Time and frequency domain of Linear Frequency Modulated signal with Rectangular Envelope')
s2 = (1/sqrt(max(t))).*pulse.*exp(1i*pi*k.*t.*t);
subplot(1,2,1)
plot(t,s2,'LineWidth',2)
xlabel('Time [s]')
ylabel('Voltage [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Time Domain Representation','FontSize',20);
set(a,'Position',[0.00,-1.20],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
[f, fft_s2] = FFT(s2, Fs);
subplot(1,2,2)
plot(f,fft_s2 ,'LineWidth',2);
xlabel('Frequency [Hz]')
ylabel('Log(Voltage) [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Frequency Spectrum','FontSize',20);
set(a,'Position',[0.5,-1.8],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
figure('Color',[1 1 1]);
```

```
suptitle('\fontsize{25}Time and frequency domain of Linear Frequency Modulated signal with Gaussian
    Envelope')
sig = T/sqrt(2*pi);
s3 = (1/sqrt(sqrt(pi*sig*sig)))*exp(-((t.*t)/(2*sig*sig))  ...
    + (1i*pi*k.*t.*t));
subplot(1,2,1)
plot(t,s3,'LineWidth',2)
xlabel('Time [s]')
ylabel('Voltage [V]')
ax = gca; % current axes
ax.FontSize = 15;
a =title('Time Domain Representation','FontSize',20);
set(a,'Position',[0.05,-1.81],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
[f, fft_s3] = FFT(s3, Fs);
subplot(1,2,2)
plot(f, fft_s3,'LineWidth',2);
xlabel('Frequency [Hz]')
ylabel('Log(Voltage) [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Frequency Spectrum','FontSize',20);
set(a,'Position',[0.5,-1.81],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')

function [ f, fft_sig_val ] = FFT( sig, Fs )
% Discrete Fourier Transform
L = length(sig);
N = 512; fft_sig=fft(sig,N);
f = (-N/2:(N/2) - 1)*(Fs/N);
fft_sig_val = fftshift(fft_sig);
fft_sig_val = abs(fft_sig_val);
end
```

MATLAB code to plot the ambiguity functions for different waveforms

```
tau = -5:0.05:5;
mu = -1.5:0.05:1.5;
T = max(tau);
B = max(mu);
sig = T/sqrt(2*pi);
k = B/T;
[tau,mu] = meshgrid(tau,mu);
amb_sig = abs(exp(1i*pi.*mu.*tau) .* (1 - abs(tau/T)) ...
    .* (sin(pi.*mu.*(T - abs(tau)))./(pi.*mu.*(T - abs(tau)))));
amb_sig = amb_sig.';
figure,contour(tau/5,mu,20*log10(amb_sig/max(max(amb_sig)))',-(0:5:25), ...
    'LineWidth',2)
colorbar
grid on
xlabel('Normalized time delay (\tau/\it T)','Interpreter','tex')
ylabel('Normalized Doppler frequency shift (\it v/\it B)','Interpreter','tex')
ax = gca; % current axes
ax.FontSize = 15;
title('\fontsize{25}Ambiguity Function of a Rectangular Pulse')
%LFM with rectangular envelope
tau = -5:0.05:5;
mu = -3:0.05:3;
[tau,mu] = meshgrid(tau,mu);
amb_sig = abs(exp(1i*pi.*mu.*tau) .* (1 - abs(tau/T)) ...
    .* (sin(pi.*(k.*tau + mu) .* (T - abs(tau)))./(pi.*(k.*tau + mu) ...
    .*(T - abs(tau)))));
amb_sig = amb_sig.';
figure,contour(tau/5,mu/2,20*log10(amb_sig/max(max(amb_sig)))',-(0:5:25), ...
    'LineWidth',2)
colorbar
grid on
xlabel('Normalized time delay (\tau/\it T)','Interpreter','tex')
```

```
ylabel('Normalized Doppler frequency shift (\it v/\it B)','Interpreter','tex')
ax = gca; % current axes
ax.FontSize = 15;
title('\fontsize{25}Ambiguity Function of a LFM with rectangular Envelope')
tau = -8:0.05:8;
mu = -3:0.05:3;
[tau,mu] = meshgrid(tau,mu);
amb_sig = abs(exp(1i*pi.*mu.*tau) .* exp((-(tau.^2)/(4*sig^2)) ...
    - (pi*sig^2.*(k.*tau + mu).^2)));
amb_sig = amb_sig.';
figure,contour(tau/5,mu/2,20*log10(amb_sig/max(max(amb_sig)))' ...
    ,-(0:5:25),'LineWidth',2)
colorbar
grid on
xlabel('Normalized time delay (\tau/\it T)','Interpreter','tex')
ylabel('Normalized Doppler frequency shift (\it v/\it B)','Interpreter','tex')
ax = gca; % current axes
ax.FontSize = 15;
title('\fontsize{25}Ambiguity Function of a LFM with Gaussian Envelope')
```