UNDERWATER ACOUSTICS AND SONAR SIGNAL PROCESSING

SS 2018

ASSIGNMENT 8

AMBIGUITY FUNCTION

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by

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Contents

Introduction

SONAR transmitters perform the underwater detection of objects and in most cases, these objects or the SONAR transmitters are in motion. This means that the actual received signal differs from the expected results due to a phenomenon known as Doppler Shift, which is caused by the relative motion of the transmitter and the target. The signal received at the SONAR receivers also changes due to the time delay introduced in the reception of the signals owing to the path travelled by the waves from the transmitter to the target and back to the receiver. These phenomena introduce distortions in the signal received and hence a term called the Ambiguity function is used to study these effects.

The spectra of Rectangular Pulse, Linear Frequency Modulated Pulse (LFM) with a rectangular envelope and Linear Frequency Modulated Pulse(LFM) with a Gaussian envelope are obtained. The spectra and ambiguity function of all these three functions are plotted and discussed in this assignment

Theory

Ambiguity Function

The ambiguity is a two-dimensional function of delay and Doppler frequency showing the distortion of an uncompensated match filter due to the Doppler shift of the return from a moving target. It is useful for describing the behaviour of a radar or a sonar signal. This function appears to be a good tool to select good signals for the Range and Doppler estimation. It gives an idea of the amount of distortion present in the received signal. It is defined as the time response of a filter matched to a given finite energy signal when the signal is received with a delay (τ) and a doppler shift (v) relative to the nominal values expected by the filter. The ambiguity function is defined by $-$

$$\chi(\tau, v) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi vt}dt \tag{1}$$

It can be interpreted as the output of a matched filter designed for a Doppler frequency shift f if a signal with Doppler frequency shift $f_0 + v$ is received. Thus $\chi(\tau, v)$ can be understood as the point target response in Range/Doppler domain.

$$
\begin{aligned}
\chi(\tau, v) &= \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi vt}dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S(\omega)e^{j\omega t}\frac{d\omega}{2\pi} \right) \left(\int_{-\infty}^{\infty} S^*(\omega^{'})e^{-j\omega^{'}(t-\tau)}\frac{d\omega^{'}}{2\pi} \right) e^{j2\pi vt}dt \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\omega)S^*(\omega')e^{j\omega^{'}\tau}e^{j(\omega-\omega^{'}+2\pi v)t}d\omega d\omega' dt \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\omega)S^*(\omega')e^{j\omega^{'}\tau}\delta(\omega-\omega^{'}+2\pi v)d\omega d\omega' \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} S(\omega'-2\pi v)S^*(\omega^{'})e^{j\omega^{'}\tau}d\omega^{'}
\end{aligned}
\tag{2}
$$

The ambiguity function depends on the transmitted signal and one of the transmitted signals used here is a Linear Frequency Modulated pulse (LFM). In such a signal, the frequency is increased linearly resulting in a particular ambiguity function which is discussed in detail in the following sections.

Fast Fourier Transform (FFT) is an algorithm which is used to find the spectrum of any signal. FFT is a much faster algorithm as it reduces the number of computations involved in the fourier transform of the signal

Ambiguity Function of particular waveforms

The spectra of three waveforms are obtained using the FFT algorithm and they are studied. The three waveforms used are:

1. A rectangular Pulse
2. A Linear Frequency Modulated Pulse (LFM) with a rectangular envelope
3. A Linear Frequency Modulated Pulse(LFM) with a Gaussian envelope

Rectangular Pulse

The simplest waveform for a radar system is probably a rectangular waveform, sometimes also referred to as single frequency waveform. For the rectangular waveform, the pulse width is the reciprocal of the bandwidth. The equation for the waveform is given by −

$$s(t) = \frac{1}{\sqrt{T}} 1_{(-T/2,T/2)}(t) \tag{3}$$

with $||S||^2 = 1$. Hence, the ambiguity function is given by

$$\begin{aligned} \chi(\tau, v) &= \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi vt}dt \\ &= \begin{cases} e^{j\pi v\tau}\left(1-\frac{|\tau|}{T}\right)\frac{sin(\pi v(T-|\tau|))}{\pi v(T-|\tau|)}, & \text{for } |\tau| \leq T \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \tag{4}$$

Linear Frequency Modulated Pulse (LFM) with a rectangular envelope

The equation for this waveform is given by −

$$s(t) = \frac{1}{\sqrt{T}} 1_{(-T/2,T/2)}(t) exp(j\pi k t^2) \tag{5}$$

with $k = b/T$. Hence, the ambiguity function is given by

$$\begin{aligned} \chi(\tau, v) &= \int_{-\infty}^{\infty} s(t) s^*(t-\tau) e^{j2\pi v t} dt \\ &= \begin{cases} e^{j\pi v \tau} \left(1 - \frac{|\tau|}{T}\right) \frac{sin(\pi(k\tau+v)(T-|\tau|))}{\pi(k\tau+v)(T-|\tau|)}, & \text{for } |\tau| \leq T \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \tag{6}$$

Determining the ambiguity function for a Linear Frequency Modulated Pulse (LFM) with a rectangular envelope

$$s(t) = e^{j(\pi k t^2)} \frac{1}{\sqrt{T}} rect(t)$$

$$\chi(\tau, v) = \frac{1}{\sqrt{T}} \frac{1}{\sqrt{T}} \int_{-\infty}^{\infty} rect(t) rect(t-\tau) e^{j(\pi k t^2)} e^{-j(\pi k (t-\tau)^2)} e^{j(2\pi k v t)} dt$$

Replacing: $t = t^{'} + \frac{\tau}{2}$

$$\rightarrow \chi(\tau, v) = \frac{1}{T} \int_{-\infty}^{\infty} rect(t^{'} + \frac{\tau}{2}) rect(t^{'} - \frac{\tau}{2}) e^{j(\pi k (t^{'} + \frac{\tau}{2})^2)} e^{-j(\pi k (t^{'} - \frac{\tau}{2})^2)} e^{j(2\pi k v (t^{'} + \frac{\tau}{2}))} dt^{'}$$

Since $(A+B)^2-(A-B)^2=4AB$

$$\begin{aligned} e^{j(\pi k(t^{'}+\frac{\tau}{2})^2)}e^{-j(\pi k(t^{'}-\frac{\tau}{2})^2)} &= e^{j\left(\pi k(t^{'}+\frac{\tau}{2})^2-(t^{'}-\frac{\tau}{2})^2\right)} \\ &= e^{j(4\pi kt^{'}\frac{\tau}{2})} \\ &= e^{j(2\pi kt^{'}\tau)} \end{aligned} \tag{7}$$

$$\rightarrow \chi(\tau,v)=\frac{1}{T}\int_{-\infty}^{\infty} rect(t^{'}+\frac{\tau}{2})rect(t^{'}-\frac{\tau}{2})e^{j(2\pi kt^{'}\tau)}e^{j(2\pi kt^{'})}dt^{'}$$

$$\rightarrow \chi(\tau,v)=\frac{1}{T}\int_{-\infty}^{\infty} rect(t^{'}+\frac{\tau}{2})rect(t^{'}-\frac{\tau}{2})e^{j(2\pi(k\tau+v)t^{'})}dt^{'}$$

Let $d(t)=\frac{T-|\tau|}{2}$

$$\rightarrow \chi(\tau,v)=\frac{1}{T}\int_{-d(t)}^{d(t)} e^{j(2\pi(k\tau+v)t^{'})}dt^{'}$$

Let $d(t)=\frac{T-|\tau|}{2}$

$$\rightarrow \chi(\tau,v)=\frac{1}{T}e^{j(\pi v\tau)}\frac{e^{j(2\pi(k\pi+v)(d(t)-(-d(t))))}}{j(2\pi(k\tau+v)(d(t)-(-d(t))))}$$

$$\rightarrow \chi(\tau,v)=\frac{1}{T}e^{j(\pi v\tau)}d(t^{'})\frac{j2sin(2\pi(k\pi+v)(\frac{T-|\tau|}{2}))}{j(2\pi(k\tau+v)(\frac{T-|\tau|}{2}))}$$

$$\chi(\tau,v)=\begin{cases} e^{j\pi v\tau}\left(1-\frac{|\tau|}{T}\right)\frac{sin(\pi(k\tau+v)(T-|\tau|))}{\pi(k\tau+v)(T-|\tau|)}, & \text{for } |\tau|\leq T \\ 0, & \text{elsewhere} \end{cases}$$

Hence proved

Linear Frequency Modulated Pulse (LFM) with a Gaussian envelope

The equation for this waveform is given by −

$$s(t) = \frac{1}{\sqrt[4]{\pi\sigma^2}} exp\left(- \frac{t^2}{2\sigma^2} + j\pi k t^2 \right) \tag{8}$$

where k determines the slope of the LFM with $|k| = b/T$. After some manipulations, we obtain

$$\begin{aligned} \chi(\tau, v) &= \int_{-\infty}^{\infty} s(t) s^*(t-\tau) e^{j2\pi v t} dt \\ &= e^{j\pi v t} exp\Big(- \tau^2/(4\sigma^2) - \pi\sigma^2 (k\tau + v)^2 \Big) \end{aligned} \tag{9}$$

Experimental Research

Rectangular Pulse

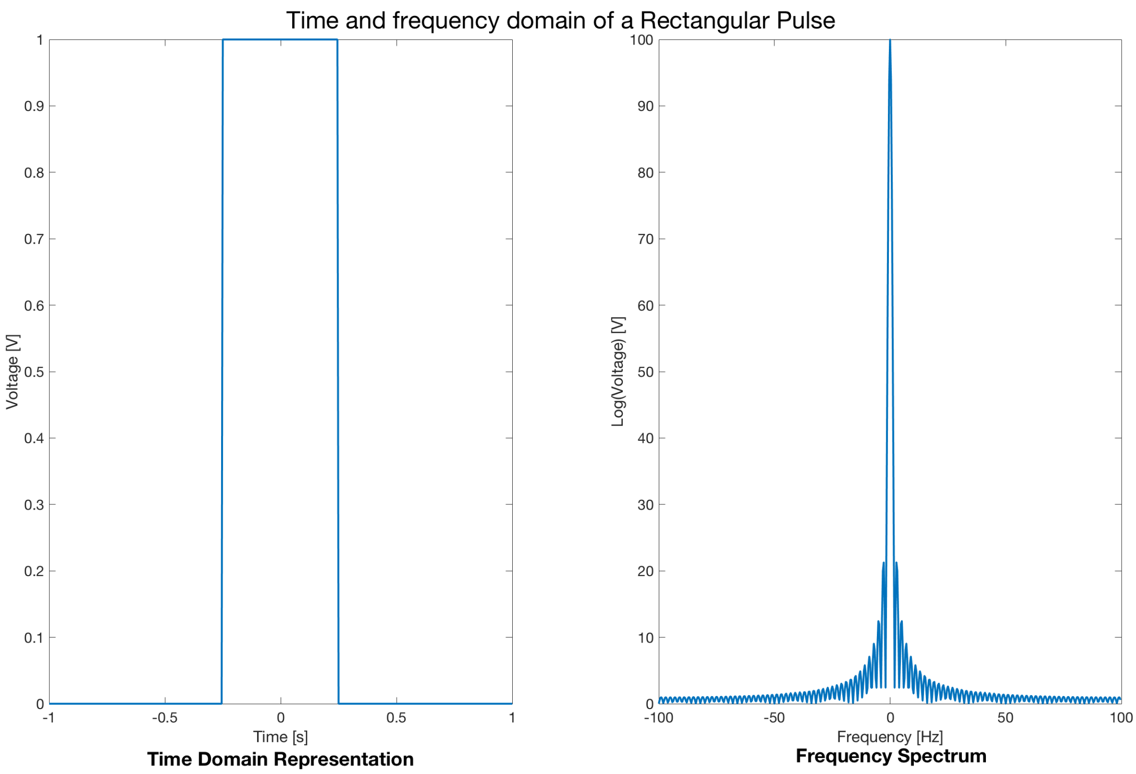

Figure 1: A rectangular pulse in time domain (left) along with its spectrum (right)

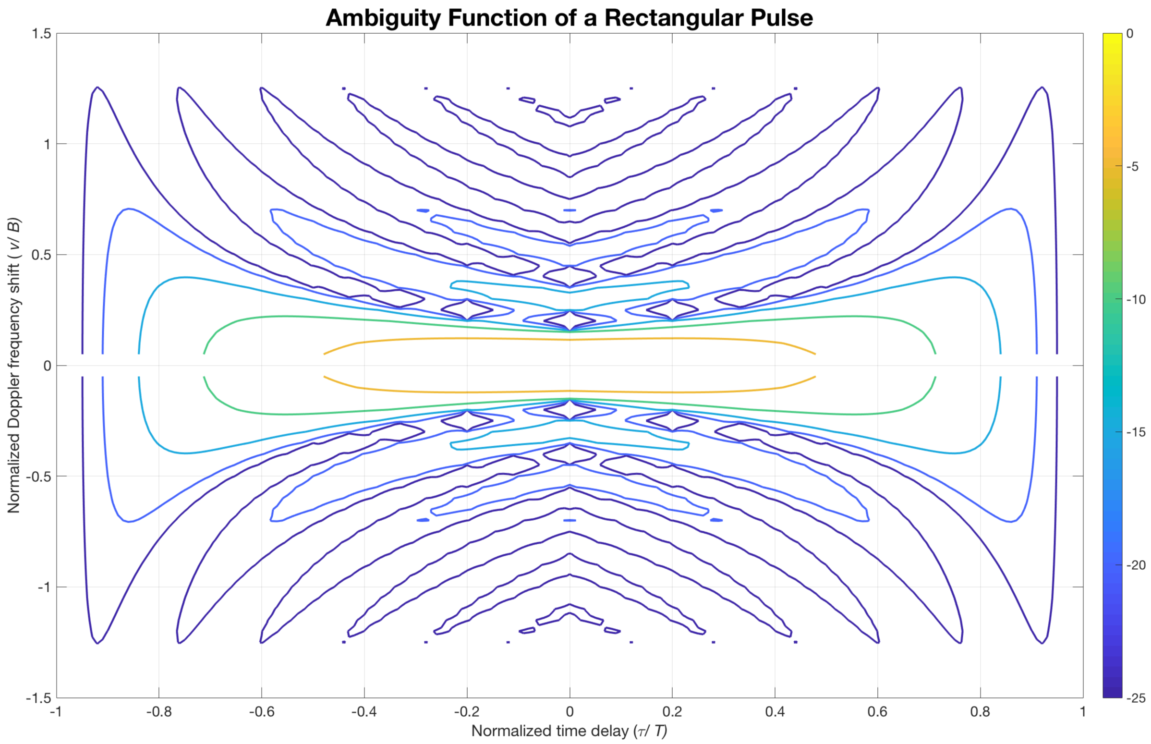

Figure 2: Ambiguity function of a rectangular pulse

Linear Frequency Modulated pulse with a Rectangular Envelope

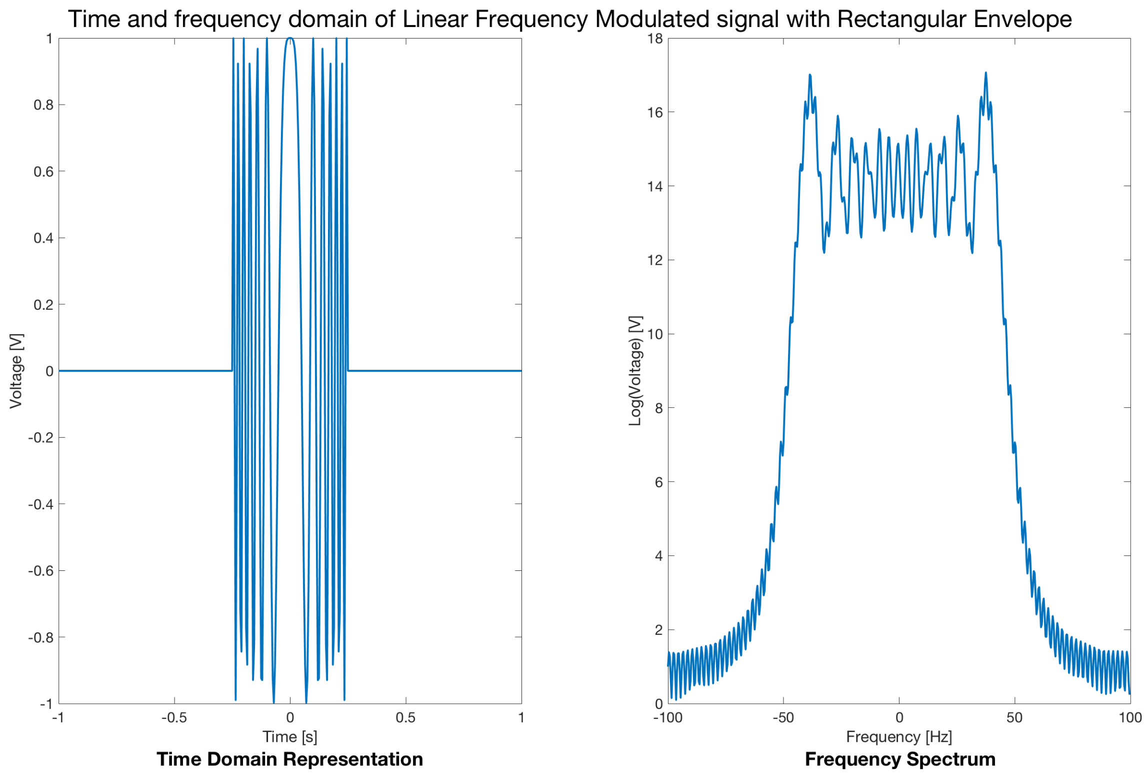

Figure 3: A Linear Frequency Modulated (LFM) pulse with rectangular envelope in time domain (left) along with its spectrum (right)

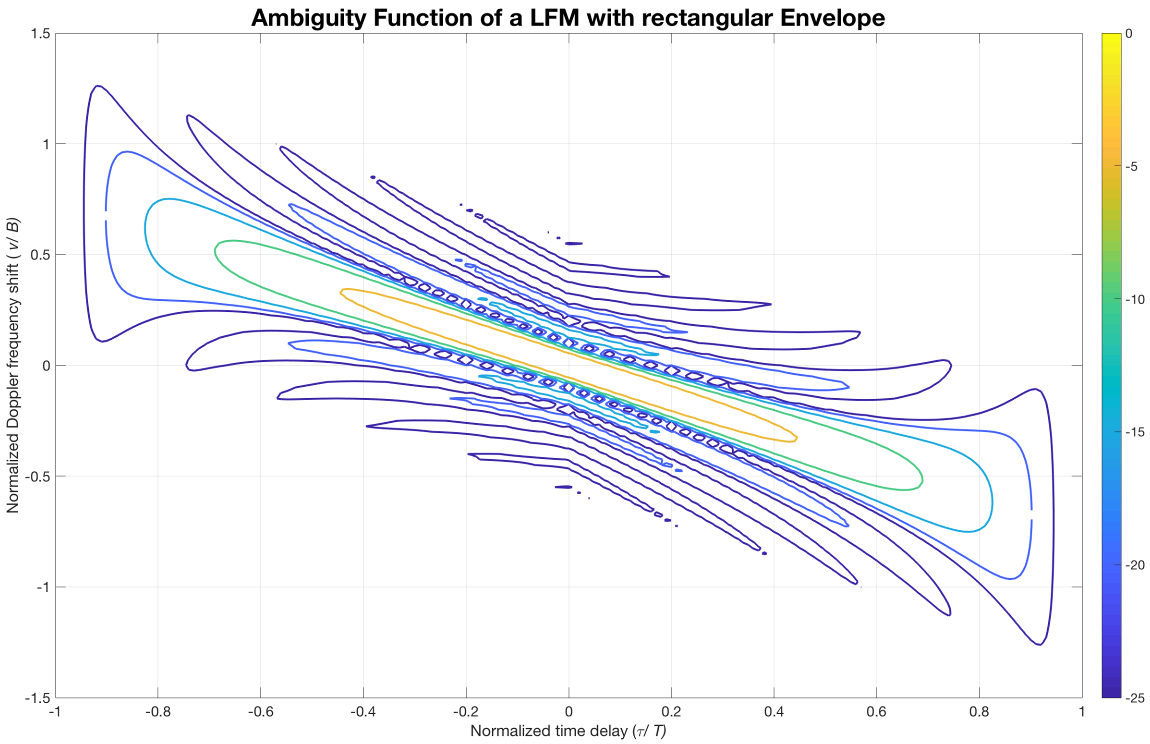

Figure 4: The ambiguity function of a Linear Frequency Modulated (LFM) pulse with rectangular envelope

Linear Frequency Modulated pulse with a Gaussian Envelope

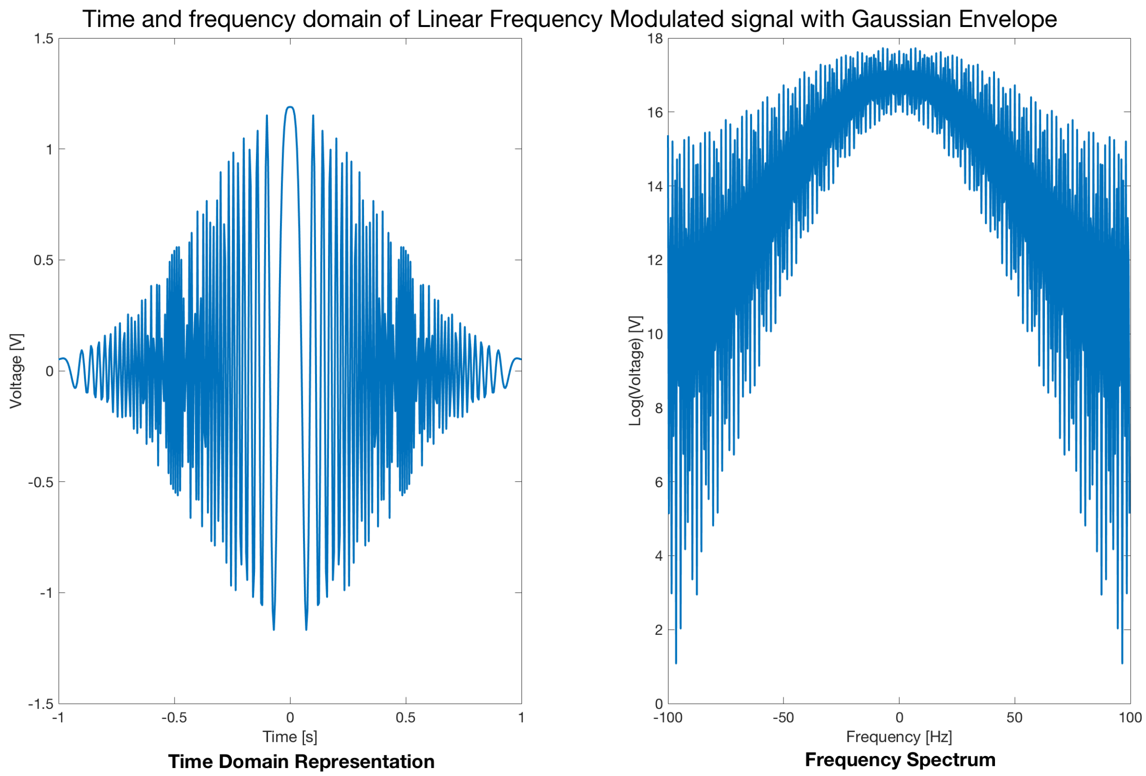

Figure 5: A Linear Frequency Modulated (LFM) pulse with Gaussian envelope in time domain (left) along with its spectrum (right)

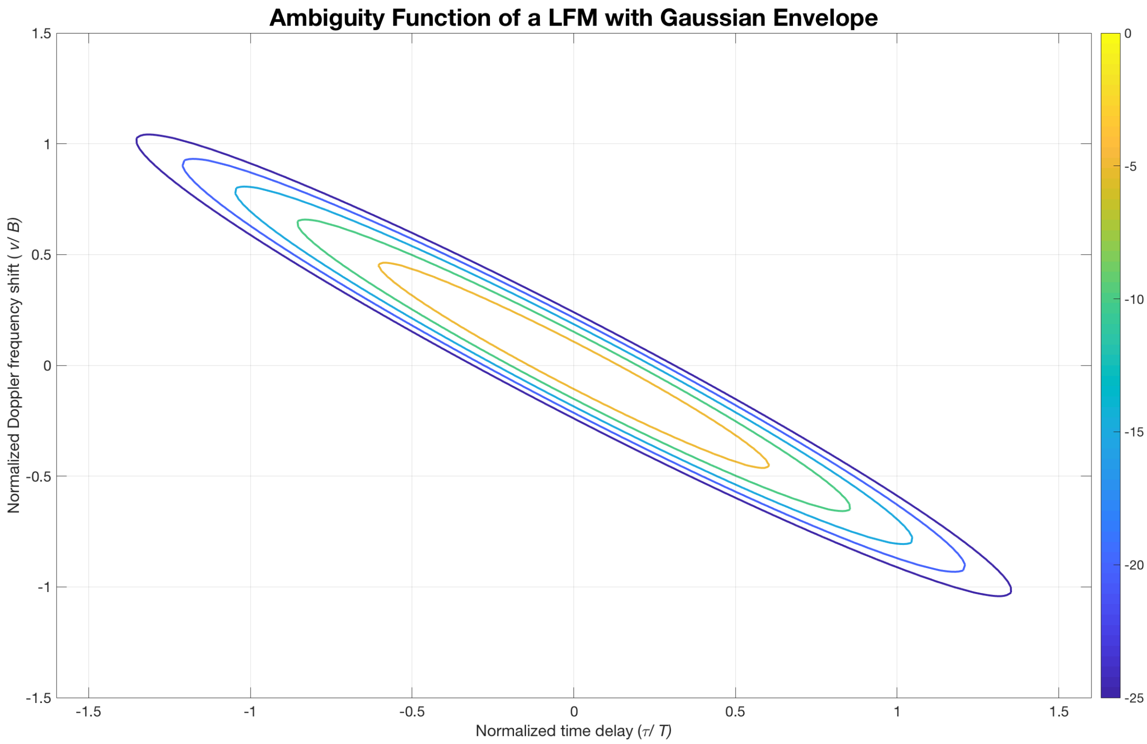

Figure 6: The ambiguity function of a Linear Frequency Modulated (LFM) pulse with Gaussian envelope

Conclusion

Appendix

MATLAB code to plot the spectrums for different waveforms

```
Fs = 200; %Sampling frequency
t = -1:1/Fs:1;
T = max(t); %pulse-width
B = 200; %bandwidth
k = B/T;
width = 0.5;
pulse = rectpuls(t,width);
figure('Color',[1 1 1]);
suptitle('\fontsize{25}Time and frequency domain of a Rectangular Pulse')
s1 = (1/sqrt(max(t))).*pulse;
subplot(1,2,1)
plot(t,s1,'LineWidth',2)
hold on
xlabel('Time [s]')
ylabel('Voltage [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Time Domain Representation','FontSize',20);
set(a,'Position',[0.00,-0.10],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
[f, fft_s1] = FFT(s1, Fs);
subplot(1,2,2)
```

```
plot(f, fft_s1 ,'LineWidth',2);
xlabel('Frequency [Hz]')
ylabel('Log(Voltage) [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Frequency Spectrum','FontSize',20);
set(a,'Position',[0.5,-9.5],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
figure('Color',[1 1 1]);
suptitle('\fontsize{25}Time and frequency domain of Linear Frequency Modulated signal with
    Rectangular Envelope')
s2 = (1/sqrt(max(t))).*pulse.*exp(1i*pi*k.*t.*t);
subplot(1,2,1)
plot(t,s2,'LineWidth',2)
xlabel('Time [s]')
ylabel('Voltage [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Time Domain Representation','FontSize',20);
set(a,'Position',[0.00,-1.20],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
[f, fft_s2] = FFT(s2, Fs);
subplot(1,2,2)
plot(f,fft_s2 ,'LineWidth',2);
xlabel('Frequency [Hz]')
ylabel('Log(Voltage) [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Frequency Spectrum','FontSize',20);
set(a,'Position',[0.5,-1.8],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
figure('Color',[1 1 1]);
```

```
suptitle('\fontsize{25}Time and frequency domain of Linear Frequency Modulated signal with Gaussian Envelope')
sig = T/sqrt(2*pi);
s3 = (1/sqrt(sqrt(pi*sig*sig)))*exp(-((t.*t)/(2*sig*sig))  ...
    + (1i*pi*k.*t.*t));
subplot(1,2,1)
plot(t,s3,'LineWidth',2)
xlabel('Time [s]')
ylabel('Voltage [V]')
ax = gca; % current axes
ax.FontSize = 15;
a =title('Time Domain Representation','FontSize',20);
set(a,'Position',[0.05,-1.81],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')
[f, fft_s3] = FFT(s3, Fs);
subplot(1,2,2)
plot(f, fft_s3,'LineWidth',2);
xlabel('Frequency [Hz]')
ylabel('Log(Voltage) [V]')
ax = gca; % current axes
ax.FontSize = 15;
a = title('Frequency Spectrum','FontSize',20);
set(a,'Position',[0.5,-1.81],...
    'VerticalAlignment','bottom', ...
    'HorizontalAlignment','center')

function [ f, fft_sig_val ] = FFT( sig, Fs )
% Discrete Fourier Transform
L = length(sig);
N = 512; fft_sig=fft(sig,N);
f = (-N/2:(N/2) - 1)*(Fs/N);
fft_sig_val = fftshift(fft_sig);
fft_sig_val = abs(fft_sig_val);
end
```

MATLAB code to plot the ambiguity functions for different waveforms

```
tau = -5:0.05:5;
mu = -1.5:0.05:1.5;
T = max(tau);
B = max(mu);
sig = T/sqrt(2*pi);
k = B/T;
[tau,mu] = meshgrid(tau,mu);
amb_sig = abs(exp(1i*pi.*mu.*tau) .* (1 - abs(tau/T)) ...
    .* (sin(pi.*mu.*(T - abs(tau)))./(pi.*mu.*(T - abs(tau)))));
amb_sig = amb_sig.';
figure,contour(tau/5,mu,20*log10(amb_sig/max(max(amb_sig)))',-(0:5:25), ...
    'LineWidth',2)
colorbar
grid on
xlabel('Normalized time delay (\tau/\it T)','Interpreter','tex')
ylabel('Normalized Doppler frequency shift (\it v/\it B)','Interpreter','tex')
ax = gca; % current axes
ax.FontSize = 15;
title('\fontsize{25}Ambiguity Function of a Rectangular Pulse')
%LFM with rectangular envelope
tau = -5:0.05:5;
mu = -3:0.05:3;
[tau,mu] = meshgrid(tau,mu);
amb_sig = abs(exp(1i*pi.*mu.*tau) .* (1 - abs(tau/T)) ...
    .* (sin(pi.*(k.*tau + mu) .* (T - abs(tau)))./(pi.*(k.*tau + mu) ...
    .*(T - abs(tau)))));
amb_sig = amb_sig.';
figure,contour(tau/5,mu/2,20*log10(amb_sig/max(max(amb_sig)))',-(0:5:25), ...
    'LineWidth',2)
colorbar
grid on
xlabel('Normalized time delay (\tau/\it T)','Interpreter','tex')
```

```
ylabel('Normalized Doppler frequency shift (\it v/\it B)','Interpreter','tex')
ax = gca; % current axes
ax.FontSize = 15;
title('\fontsize{25}Ambiguity Function of a LFM with rectangular Envelope')
tau = -8:0.05:8;
mu = -3:0.05:3;
[tau,mu] = meshgrid(tau,mu);
amb_sig = abs(exp(1i*pi.*mu.*tau) .* exp((-(tau.^2)/(4*sig^2)) ...
    - (pi*sig^2.*(k.*tau + mu).^2)));
amb_sig = amb_sig.';
figure,contour(tau/5,mu/2,20*log10(amb_sig/max(max(amb_sig)))' ...
    ,-(0:5:25),'LineWidth',2)
colorbar
grid on
xlabel('Normalized time delay (\tau/\it T)','Interpreter','tex')
ylabel('Normalized Doppler frequency shift (\it v/\it B)','Interpreter','tex')
ax = gca; % current axes
ax.FontSize = 15;
title('\fontsize{25}Ambiguity Function of a LFM with Gaussian Envelope')
```