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RANK TESTS FOR SOME LINEAR HYPOTHESES IN PAIRED COMPARISON DESIGNS[1]

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SUMMARY. The theory of rank order tests for the analysis of covariance problem in paired comparison designs, and an alternative class of tests when the model can be described by a smaller number of parameters are considered here. The asymptotic (Pitman-) efficiencies of the proposed tests with respect to the corresponding (normal-theory) variance-ratio tests are also studied.

1. Introduction

Consider $t\,(\geqslant 2)$ treatments in an experiment involving paired comparisons, and suppose that the pair (i, j) of treatments is in N_{ij} encounters which provide comparison differences

$$Z_{ij,s} = X_{is}-X_{js}, \quad s = 1, \ldots, N_{ij},\ 1 \leqslant i < j \leqslant t; \quad N = \sum_{1\leqslant i<j\leqslant t} N_{ij}. \qquad \ldots \quad (1.1)$$

In the usual paired comparison models, we assume that $Z_{ij,s}$, $s = 1, \ldots, N_{ij}$ are independent and identically distributed random variables (i.i.d.r.v.) having an absolutely continuous cumulative distribution function (c.d.f.) $F_{ij}(x)$, where

$$F_{ij}(x) = F(x-[\theta_i-\theta_j]), \quad 1 \leqslant i < j \leqslant t, \qquad \ldots \quad (1.2)$$

and we want to test the null hypothesis

$$H_0^{(1)} : \theta_1 = \ldots = \theta_t = \theta \text{ (unknown)}, \qquad \ldots \quad (1.3)$$

against the alternative that $\theta_1, \ldots, \theta_t$ are not all equal.

For the analysis of covariance problem, we assume that $Z_{ij,s}$ has the c.d.f.

$$F_{ij,s}(x) = F(x-[\theta_i-\theta_j]-\beta c^*_{ij,s}), \quad s = 1, \ldots, N_{ij};\ 1 \leqslant i < j \leqslant t, \qquad \ldots \quad (1.4)$$

where $c^*_{ij,s} = c_{ij,s}-\bar{c}_{ij}$ $(\bar{c}_{ij} = N_{ij}^{-1} \sum_{s=1}^{N_{ij}} c_{ij,s})$ are known constants (concomitant variables), β is an unknown regression parameter, and we want to test the null hypothesis

$$H_0^{(2)} : \quad \theta_1 = \ldots = \theta_t = \theta \quad \text{(unknown)} \qquad \ldots \quad (1.5)$$

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against the alternative that $\theta_1, \ldots, \theta_t$ are not equal, treating β as a nuisance parameter. The model (1.4) may arise, for example, in randomized block designs of two plots where there is a concomitant variate and one works with the difference of yields of the intra-block plots. We assume that in (1.4), $F \epsilon \mathscr{F}_0$, the class of all absolutely continuous symmetric c.d.f.'s with finite Fisher information

$$I(F) = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 dF(x) < \infty; \quad f(x) = F'(x). \qquad \ldots \quad (1.6)$$

Our second problem arises in situations where the t treatments represent the t quantitative levels $(l_1, \ldots, l_t)$ of a factor, so that the differences $\theta_i - \theta_j$, $1 \leqslant i < j \leqslant t$, can be described by some simple response surface models involving smaller number of parameters. For example, $\theta_i - \theta_j$ may be expressed as a q-th degree polynomial in $(l_i - l_j)$, where $q \leqslant t-1$. In such a case, we may write

$$\theta_i - \theta_j = \lambda_1 c_{ij}^{(1)} + \ldots + \lambda_q c_{ij}^{(q)}, \; 1 \leqslant i < j \leqslant t, \qquad \ldots \quad (1.7)$$

where $\lambda_1, \ldots, \lambda_q$ are unknown parameters and $c_{ij}^{(h)}$, $1 \leqslant h \leqslant q$, $1 \leqslant i < j \leqslant t$, are known constants (for example, orthogonal polynomials), depending on $l_1, \ldots, l_t$. In fact, (1.7) is a special case of the following more general model. Assume that $Z_{ij,s}$ has the c.d.f.

$$F_{ij,s}(x) = F(x - \Delta_{ij,s}), \; \Delta_{ij,s} = \sum_{h=1}^{q} \lambda_h c_{ij,s}^{(h)}, \quad F \epsilon \mathscr{F}_0, \qquad \ldots \quad (1.8)$$

for $s = 1, \ldots, N_{ij}$, $1 \leqslant i < j \leqslant t$, where the $c_{ij,s}^{(h)}$ are known constants. We want to test

$$H_0^{(3)} : \lambda_1 = \ldots = \lambda_q = 0 \;\text{ vs } \lambda_j \neq 0 \text{ for at least one } j = 1, \ldots, q. \qquad \ldots \quad (1.9)$$

It may be noted that $H_0^{(3)}$ is analogous to $H_0^{(1)}$ defined in (1.3), but involves smaller number of parameters. Similarly, analogous to (1.4), we may consider the following model : let $Z_{ij,s}$ have the c.d.f.

$$F_{ij,s}\;(x) = F(x - \Delta^*_{i\;,s}), \; \Delta^*_{ij,s} = \sum_{h=1}^{q} \lambda_h c_{ij}^{(h)} + \beta c^*_{ij,s} \qquad \ldots \quad (1.10)$$

for $s = 1, \ldots, N_{ij}$, $1 \leqslant i < j \leqslant t$, $q \leqslant t-1$, where $\lambda_1, \ldots, \lambda_q$ and β are unknown parameters, and the $c_{ij}^{(h)}$ and $c_{ij,s}$ [defined in (1.4)] are known constants. The hypothesis to be tested is

$$H_0^{(4)} : \lambda_1 = \ldots = \lambda_q = 0 \;\text{ vs } \lambda_j \neq 0 \; \text{ for some } j = 1, \ldots, q, \qquad \ldots \quad (1.11)$$

where we treat β as a nuisance parameter.

The tests (parametric as well as nonparametric) for $H_0^{(1)}$ have been considered by various authors (see for example, Puri and Sen (1969a) and the references cited therein). The purpose of the present paper is to derive some procedures for testing $H_0^{(2)}$, $H_0^{(3)}$ and $H_0^{(4)}$, and study their asymptotic relative efficiency (ARE) properties. We remark that $H_0^{(2)}$ is a special case of $H_0^{(4)}$ when $q = t-1$, and hence, will not be treated separately.

2. Preliminary notions

For any specified real b, denote by

$$Z_{ij,s}(b) = Z_{ij,s} - bc^*_{ij,s}, \quad s = 1, \ldots, N_{ij}, \quad 1 \leqslant i < j \leqslant t, \qquad \ldots \quad (2.1)$$

and let $R^+_{ij,s}(b)$ be the rank of $|Z_{ij,s}(b)|$ among the N values $|Z_{i'j',s'}(b)|$ for $s' = 1, \ldots, N_{i'j'}$, $1 \leqslant i' < j' \leqslant t$. Also, let $R_{ij,s}(b)$ be the rank of $Z_{ij,s}(b)$ among $Z_{ij,1}(b), \ldots, Z_{ij,N_{ij}}(b)$, for $s = 1, \ldots, N_{ij}$ and $1 \leqslant i < j \leqslant t$. Define the statistics

$$T_{N,ij}(b) = C^{-1}_{N,ij} \sum_{s=1}^{N_{ij}} c^*_{ij,s} J^*(R_{ij,s}(b)/(N_{ij}+1)), \quad 1 \leqslant i < j \leqslant t, \qquad \ldots \quad (2.2)$$

where the $c^*_{ij,s}$ are defined in (1.4), $C^2_{N,ij} = \sum_{s=1}^{N_{ij}} (c^*_{ij,s})^2$, and $J^*(u)$ is a specified non-decreasing score function defined over $(0, 1)$, satisfying

$$\int_0^1 J^*(u)du = 0 \quad \text{and} \quad 0 < A^2(J^*) = \int_0^1 [J^*(u)]^2 du < \infty. \qquad \ldots \quad (2.3)$$

Further, denote by

$$T^0_N(b) = \sum_{1 \leqslant i < j \leqslant t} C_{N,ij} T_{N,ij}(b)/C_N; \quad C^2_N = \sum_{1 \leqslant i < j \leqslant t} C^2_{N,ij}. \qquad \ldots \quad (2.4)$$

Finally, let

$$U_{N,h}(b) = \sum_{1 \leqslant i < j \leqslant t} \sum_{s=1}^{N_{ij}} c^{(h)}_{ij,s} J(R^+_{ij,s}(b)/(N+1)) \operatorname{sgn} Z_{ij,s}, \qquad \ldots \quad (2.5)$$

for $h = 1, \ldots, q$, where the $c^{(h)}_{ij,s}$ are defined in (1.8), and let

$$\boldsymbol{U}_N(b) = (U_{N,1}(b), \ldots, U_{N,q}(b))'. \qquad \ldots \quad (2.6)$$

In (2.5), $J(u)$ is a non-decreasing score function defined over $(0, 1)$, satisfying

$$0 < A^2(J) = \int_0^1 J^2(u)du < \infty. \qquad \ldots \quad (2.7)$$

We assume that J^* is skew symmetric i.e., $J^*(u) + J^*(1-u) = 0$ for all $0 < u < 1$, and let

$$J(u) = J^*((1+u)/2) \text{ for all } u \epsilon [0, 1]. \qquad \ldots \quad (2.8)$$

Important special cases of $J^*(u)$ are the Wilcoxon and normal scores functions for which J^* is the inverse of the rectangular $[-1, 1]$ and the standard normal c.d.f., respectively.

3. Rank order tests for $H_0^{(3)}$

We consider the case only briefly for its relevance to our general case of $H_0^{(4)}$. By suitable reparametrization, we shall reduce the problem as a special case of a general linear model considered in Puri and Sen (1969b), and thereby, we avoid the details. We relabel the subscript (i, j), $1 \leqslant i < j \leqslant t$ as follows. Denote the pair (i, j) by $\alpha = (i-1)t+j-\binom{i+1}{2}$, $1 \leqslant i < j \leqslant t$. Then $Z_{\alpha 1}, \ldots, Z_{\alpha N_\alpha}$ are independent random variables with respective c.d.f.'s $F_{\alpha,s}(x)$, $s = 1, \ldots, N_\alpha$, defined by (1.8), with (i, j) being replaced by α, for $\alpha = 1, \ldots, c = \binom{t}{2}$. Next, denote the random variables $(Z_{11}, \ldots, Z_{1N_1}, \ldots, Z_{c1}, \ldots, Z_{cN_c})$ by $(Z_1, \ldots, Z_N)$ and the corresponding sequence of constants by $(c_1^{(h)}, \ldots, c_N^{(h)})$, for $h = 1, \ldots, q$. Then, we may rewrite the model (1.8) as

$$P\{Z_r \leqslant x\} = F_r(x) = F(x - \sum_{h=1}^{q} \lambda_h c_r^{(h)}), \quad r = 1, \ldots, N. \qquad \ldots \quad (3.1)$$

Furthermore, the statistics $U_{N,h}$, $h = 1, \ldots, q$, defined in (2.5), can be written as

$$U_{N,h} = U_{N,h}(0) = \sum_{r=1}^{N} c_r^{(h)} J(R_r^+(0)/(N+1)) \operatorname{sgn} Z_r, \quad h = 1, \ldots, q. \qquad \ldots \quad (3.2)$$

Define then

$$\boldsymbol{K}_N = ((k_{N,gh})) = \left(\left(\frac{1}{N} \sum_{r=1}^{N} c_r^{(g)} c_r^{(h)}\right)\right)_{g,h=1,\ldots,q} \qquad \ldots \quad (3.3)$$

and assume that the following conditions hold :

(i) $\lim_{N \to \infty} \boldsymbol{K}_N = \boldsymbol{K} = ((k_{gh}))$ exists and is positive definite, ... (3.4)

(ii) $\lim_{N \to \infty} N_\alpha / N = \rho_\alpha : 0 < \rho_\alpha < 1, \quad \alpha = 1, \ldots, c;$ ... (3.5)

(iii) $\max_{1 \leqslant i < j \leqslant t} \max_{1 \leqslant s \leqslant N_{ij}} |c_{ij,s}^*| / C_N = o(1),$ ... (3.6)

(iv) $\max_{1 \leqslant h \leqslant q} \max_{1 \leqslant r \leqslant N} |c_r^{(h)}| / k_{N,hh} = o(N^{1/2});$ ... (3.7)

(v) $\lim\inf_{N} N^{-1} C_N^2 > 0.$ ... (3.8)

Then, proceeding as in Puri and Sen (1969b), we consider the following test statistic

$$M_N = N[\boldsymbol{U}_N' \boldsymbol{K}_N^{-1} \boldsymbol{U}_N]/A^2(J), \text{ where } \boldsymbol{U}_N' = (U_{N,1}(0), \ldots, U_{N,q}(0))'. \qquad \ldots \quad (3.9)$$

Under $H_0^{(3)}$, M_N is strictly distribution-free, and its distribution is generated by the 2^N equally likely sign-inversions of $(Z_1, \ldots, Z_N)$. Asymptotically, under $H_0^{(3)}$, M_N has a chi-square c.d.f. with q degrees of freedom (d.f.) which provides a large sample approximation of the critical point. Asymptotic non-null distribution theory and ARE results also follow from those in Puri and Sen (1969b). Since, these will be studied in the more general case of $H_0^{(4)}$, in view of the essential similarity, we omit the details.

4. Rank order tests for $H_0^{(4)}$

If β were known, one could have worked with the variables $Z_{ij,s}(\beta)$, and constructed a test as in Section 3. Since β is not known, we consider aligned observations $Z_{ij,s}(\beta_N^*)$, based on the following estimator of β, and propose our test on these aligned observations.

First, by Theorem 6.1 of Sen (1969), each $T_{N,ij}(b)$ is non-increasing in $b : -\infty < b < \infty$. Further, since $E[T_N^0(\beta)] = 0$ independently of the $\Delta_{ij,s}$, defined in (1.10), as in Sen (1969), we have the following consistent, translation-invariant and robust estimator of β : let

$$\beta_{N,1}^* = \sup \{b : T_N^0(b) > 0\}, \ \beta_{N,2}^* = \inf\{b : T_N^0(b) < 0\}; \qquad \ldots \quad (4.1)$$

$$\beta_N^* = [\beta_{N,1}^* + \beta_{N,2}^*]/2. \qquad \ldots \quad (4.2)$$

Our aligned rank order statistic vector is then

$$\hat{\boldsymbol{U}}_N = (\hat{U}_{N,1}, \ldots, \hat{U}_{N,q})' = \boldsymbol{U}_N(\beta_N^*) = (U_{N,1}(\beta_N^*), \ldots, U_{N,q}(\beta_N^*))', \qquad \ldots \quad (4.3)$$

and as in (3.9), we propose the following test statistic

$$\hat{M}_N = N[\hat{\boldsymbol{U}}_N' \boldsymbol{K}_N^{-1} \hat{\boldsymbol{U}}_N]/A^2(J). \qquad \ldots \quad (4.4)$$

From Theorem 4.1 (to follow) we conclude that under $H_0^{(4)}$, $\hat{M}_N$ has asymptotically chi-square distribution with q d.f., which provide a large sample approximation to the critical point of $\hat{M}_N$. To study its large sample properties, we confine ourselves to the sequence of (Pitman-) alternatives $\{H_N\}$, specified by

$$H_N; \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_q)' = N^{-1/2}\boldsymbol{\tau}, \quad \boldsymbol{\tau} = (\tau_1, \ldots, \tau_q)', \qquad \ldots \quad (4.5)$$

where the τ_i are fixed numbers. Then, we have the following.

Theorem 4.1 : *Under* (3.4)–(3.8), *if* $F \epsilon \mathscr{F}_0$, *then under* $\{H_N\}$ *in* (4.5), $\hat{M}_N$ *has asymptotically* (*as* $N \to \infty$) *a non-central chi-square distribution with* q *d.f. and non-centrality parameter*

$$\Delta_M = (\boldsymbol{\mu}' \boldsymbol{K}^{-1} \boldsymbol{\mu})/A^2(J) = \left(\int_0^1 \psi(u) J(u) du\right)^2 (\boldsymbol{\tau}' \boldsymbol{K} \boldsymbol{\tau})/A^2(J), \qquad \ldots \quad (4.6)$$

where $\boldsymbol{K}$ *is defined by* (3.4), $\boldsymbol{\mu} = (\boldsymbol{\tau}' \boldsymbol{K})\left(\int_0^1 \psi(u) J(u) du\right)$ *and*

$$\psi(u) = -f'(F^{-1}((1+u)/2)/f(F^{-1}((1+u)/2), \quad 0 < u < 1. \qquad \ldots \quad (4.7)$$

Proof : From Theorem 4 of Jurečková (1971), it follows that for every finite $B(>0)$,

$$\sup_{b:\ |b|<B} |U_{N,h}(\beta - N^{-1/2}b) - U_{N,h}(\beta)| = o_p(N^{-1/2}), \quad h = 1, \ldots, q. \qquad \ldots \quad (4.8)$$

Let us now set

$$W_N = N^{1/2}(\beta_N^*-\beta) = [C_N(\beta_N^*-\beta)][N/C_N^2]^{1/2}. \qquad \dots \quad (4.9)$$

Then, it follows from Sen (1969) that $|C_N(\beta_N^*-\beta)| = O_p(1)$, so that by (3.8) and (4.9), we have

$$|W_N| = O_p(1) \text{ as } N \to \infty. \qquad \dots \quad (4.10)$$

As a result, upon noting that $\hat{U}_N = U_N(\beta_N^*)$, we obtain from (4.8) and (4.10) that

$$\hat{U}_{N,h} - U_{N,h}(\beta) = o_p(N^{-1/2}), \quad \text{for all } h = 1, \dots, q, \qquad \dots \quad (4.11)$$

both under the null hypothesis $H_0^{(4)}$ and $\{H_N\}$. Consequently, by (3.9), (4.4), (4.11) and the Sverdrup (1952) theorem, we conclude that M_N and $\hat{M}_N$ both have asymptorically the same limiting distribution, under $H_0^{(4)}$ as well as $\{H_N\}$. The theorem now follows from Theorem 3.1 and Corollary 3.1 of Puri and Sen (1969b).

We now consider the normal theory test based on the variance-ratio criterion with a view to obtaining the asymptotic power comparisons with the $\hat{M}_N$ test. Here, under the assumption that the c.d.f. F in (1.10) is normal with mean 0 and a finite and positive variance $\sigma^2(F)$, the analysis of variance test for $H_0^{(4)}$ in (1.11) is based on

$$\mathscr{F} = [(\boldsymbol{P}_N'\boldsymbol{K}_N^{-1}\boldsymbol{P}_N)/q]/[(S_0^2 - \boldsymbol{P}_N'\boldsymbol{K}_N^{-1}\boldsymbol{P}_N - P_0^2/C_N^2)/(N-q-1)], \qquad \dots \quad (4.12)$$

where C_N^2 and $\boldsymbol{K}_N$ are defined in (2.4) and (3.3), and

$$S_0^2 = \sum_{r=1}^N Z_r^2, \boldsymbol{P}_N = \left(N^{-1/2}\sum_{r=1}^N c_r^{(1)} Z_r, \dots, N^{-1/2}\sum_{r=1}^N c_r^{(q)} Z_r\right) \qquad \dots \quad (4.13)$$

$$P_0 = \sum_{\alpha=1}^c \sum_{s=1}^{N_\alpha} Z_{\alpha,s} c_{\alpha,s}^*. \qquad \dots \quad (4.14)$$

Let us now consider the case where F in (1.10) has a finite (positive) variance $\sigma^2(F)$, but it is not necessarily normal. Some simple computations as in Sen and Puri (1970) lead us to conclude that the denominator of $\mathscr{F}$ in (4.12) converges in probability to $\sigma^2(F)$, and hence, by the Slutsky Theorem, the statistic $\mathscr{F}$ in (4.12) is asymptotically equivalent (in probability) to

$$\mathscr{F}^* = (\boldsymbol{P}_N'\boldsymbol{K}_N^{-1}\boldsymbol{P}_N)/q\sigma^2(F). \qquad \dots \quad (4.15)$$

Now, $\boldsymbol{P}_N$, defined by (4.13), is a vector of linear functions of the observations $(Z_1, \dots, Z_N)$ and hence, proceeding as in Theorem 2.2 of Sen and Puri (1970), it follows that $q\mathscr{F}^*$ (or equivalently, $q\mathscr{F}$) has asymptotically (as $N \to \infty$) (i) under $H_0^{(4)}$ in (1.11), chi-square distribution with q d.f., and (ii) under $\{H_N\}$ in (4.5), a non-central chi-square distribution with q d.f. and non-centrality parameter

$$\Delta^* = (\boldsymbol{\tau}'\boldsymbol{K}\boldsymbol{\tau})/\sigma^2(F) = \Delta_0/[I(F)\sigma^2(F)], \qquad \dots \quad (4.16)$$

where $\qquad \Delta_0 = (\boldsymbol{\tau}'\boldsymbol{K}\boldsymbol{\tau})I(F). \qquad \dots \quad (4.17)$

It may be remarked that Δ_0 is the non-centrality parameter of the likelihood-ratio test based on the true form of the c.d.f. F in (1.10); the corresponding test statistic [by the results of Wald (1943)] has asymptotically (central and non-central) chi-square distributions under $H_0^{(4)}$ and $\{H_N\}$. Thus, the ARE of $\hat{M}_N$ with resepct to the variance ratio test, as computed from (4.6) and (4.16), is given by

$$e_1 = \Delta_M/\Delta_0^* = \sigma^2(F)\left(\int_0^1 J(u)\psi(u)du\right)^2/A^2(J), \qquad \dots \quad (4.18)$$

and the ARE of $\hat{M}_N$ with respect to the likelihood-ratio test is given by

$$e_2 = \Delta_M/\Delta_0 = \left(\int_0^1 J(u)\psi(u)du\right)^2/[I(F)A^2(J)], \qquad \dots \quad (4.19)$$

and the latter is bounded from above by 1, by the Schwarz inequality. Also, $e_1 \geqslant e_2$, by the Cramér-Rao inequality.

Now, for specific $J(u)$, the expressions for e_1 and e_2 can be simplified and evaluated for spcific F. For example, for $J(u) = u$ (i.e., Wilcoxon scores), e_1 agrees with the ARE of the Wilcoxon test with respect to the t-test, and similarly, for $J^*(u) = \Phi^{-1}(u)$, where Φ is the standard normal c.d.f., (4.18) agrees with the ARE of the normal scores test with respect to the t-test. Since, these are studied in detail in Chapter 3 of Puri and Sen (1971), we omit the details. In passing, we may remark that for normal scores test, (4.18) is bounded below by 1, where the lower bound is attained only when F is itself a normal c.d.f.

Remark: As mentioned earlier, $H_0^{(2)}$ is a special case of $H_0^{(4)}$ with $q = t-1$. In this case, the proposed test statistic reduces to

$$\hat{S}_N = \frac{\left[\sum_{i=1}^{t}\left(\sum_{j=1(\neq i)}^{t} N_{ij}^{-1/2} \sum_{s=1}^{N_{ij}} J(R_{ij,s}^{+}(\beta_N^*)/(N+1))\,\mathrm{sgn}\, Z_{ij,s}\right)^2\right]}{tA^2(J)} \qquad \dots \quad (4.20)$$

where β_N^* is defined by (4.2) and $R_{ij,s}^{+}(b)$ after (2.1). We conclude this section by some comments on the relative performance of $\hat{S}_N$ and $\hat{M}_N$, when in fact, t is not very small and q may be small. In many cases, for t not very small, (1.7), (1.8) or (1.10) can be described by a small value of q (viz., a linear, quadratic or cubic equation). Thus, whereas $\hat{S}_N$ relates to a non-central chi-square c.d.f. with $t-1$ d.f., $\hat{M}_N$ relates to a non-central chi-square c.d.f. with q d.f. If the fit of such a lower degree equation is appropriate, Δ_M will not be substantially smaller than the corresponding non-centrality parameter of the limiting distribution of $\hat{S}_N$ [under $\{H_N\}$], and hence, using a result of Mann and Wald (1942), we may conclude that whenever $q/(t-1)$ is small, the tests for $H_0^{(3)}$ and $H_0^{(4)}$ based on M_N and $\hat{M}_N$ will be substantially better than the existing ones.

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