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Author(s): F. K. Hwang

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# ON STEINER MINIMAL TREES WITH RECTILINEAR DISTANCE\*

F. K. HWANG†

**Abstract.** We consider Steiner minimal trees in the plane with rectilinear distance. The rectilinear distance  $d(p_1, p_2)$  between two points  $p_1, p_2$  is  $|x_1 - x_2| + |y_1 - y_2|$ , where the  $(x_i, y_i)$  are the Cartesian coordinates of the  $p_i$ . For a given finite set  $P$  of points, let  $l_s$  denote the length of a Steiner minimal tree and  $l_m$  the length of a minimal spanning tree. The main result of the memorandum is that  $l_s/l_m \geq \frac{2}{3}$ .

**1. Introduction.** Let  $P$  be a set of points in the plane and let  $T$  be a tree in the plane with  $V$  as its set of vertices.  $T$  is called a *spanning tree* on  $P$  if  $V = P$  and a *Steiner tree* on  $P$  if  $V \supseteq P$ . The length of  $T$  is defined to be the total length of all edges in  $T$ . For given  $P$ , a *minimal spanning tree* is a spanning tree of minimal length and a *Steiner minimal tree* is a Steiner tree of minimal length. We use  $l_m$  to denote the length of the former and  $l_s$  that of the latter. Thus  $l_s \leq l_m$ .

Simple methods have been devised to construct minimal spanning trees [4], [6]. However, it is much harder to construct Steiner minimal trees [1], [5]; no computational method whose complexity is polynomially-bounded is known. Hence one may ask: "How much better than a minimal spanning tree is a Steiner minimal tree? Is it worthwhile to expend the extra effort?" For the Euclidean plane, Gilbert and Pollak [1] conjectured that  $l_s/l_m \geq \sqrt{3}/2$ . At present, however, this conjecture remains unsettled.

The rectilinear distance  $d(p_1, p_2)$  between two points  $p_1$  and  $p_2$  is defined by

$$d(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|,$$

where the  $(x_i, y_i)$  are the Cartesian coordinates of the  $p_i$ .

Applications of rectilinear distance occur in printed circuit technology where it is desired to electrically interconnect a set of points using the shortest possible total wire length with the wires restricted to run in just horizontal and vertical directions.

The methods known for constructing minimal spanning trees in the Euclidean plane ([4], [6]) are independent of the fact that the metric is Euclidean; hence, they apply to the rectilinear metric as well. On the other hand, the construction of Steiner minimal trees with rectilinear distance remains difficult ([2], [3], [7]), and no computational method whose complexity is polynomially-bounded is known.

In this paper, we prove  $l_s/l_m \geq \frac{2}{3}$  for rectilinear distance. Furthermore, this bound can be attained by an infinite set of values of  $|P|$ . In proving the bound, we also give many lemmas which characterize a minimal Steiner tree with rectilinear distance.

**2. Notation and general remarks.** A vertex of  $P$  is called a *node* and a vertex of  $V - P$  is called a *Steiner point*. An edge between two vertices may be a sequence of alternating vertical and horizontal lines. We call each turning point a *corner*. A *line* has only one direction but may contain a number of vertices on it.  $\overline{xy}$  denotes

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† Bell Telephone Laboratories, Murray Hill, New Jersey 07971.

the line section between two collinear points  $x$  and  $y$ ; and  $|xy|$  denotes its length.  $V_{xu}(V_{xd})$  denotes the vertical line at  $x$  which is above (below)  $x$  but excluding  $x$  itself. Similarly  $H_{xr}(H_{xl})$  denotes the horizontal line at  $x$  which extends to the right (left) of  $x$  but excluding  $x$  itself. If such a line ends at a node and contains no other vertex, we call it a *node line*; if it ends at a corner and contains no vertex, we call it a *corner line*.

We define two operations on  $T$ : *Shifting* a line means moving a line between two parallel lines until it is incident to a certain specified point. *Flipping* a corner  $\alpha$  between two specified points  $x$  and  $y$  adjacent to the corner means moving  $\overline{x\alpha}$  parallel to  $y$ , and  $\overline{y\alpha}$  parallel to  $x$  (see Fig. 1). Sometimes we omit the specification of  $x$  and  $y$ .

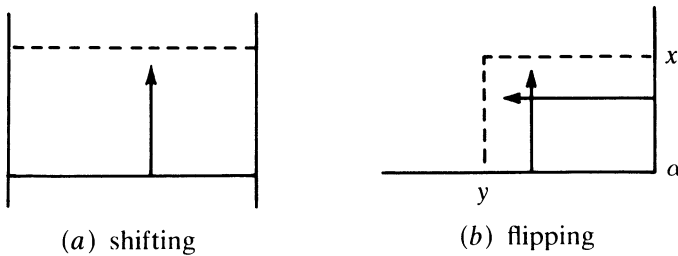


FIG. 1

It is clear that after shiftings and flippings, the resultant graph is still a tree, and of the same length. Moreover, if the lines moved do not contain a node, (hereafter we assume that this is always true), then the resultant tree is still a spanning tree (a Steiner tree) if  $T$  was. If  $T'$  is obtained from  $T$  through shiftings and flippings, we say  $T'$  is equivalent to  $T$ . Let  $E(T)$  denote the set of all trees equivalent to the tree  $T$ .

From now on, we will consider  $P$  to be fixed. Let  $S$  be the set of all Steiner minimal trees on  $P$  and  $s \in S$ . We partition  $S$  into  $S_1 + S_2$  where  $s \in S_1$  if and only if all nodes in  $s$  have degree one and  $s \in S_2$  otherwise.

Steiner points will be labeled by capital italic letters and represented by circles in a graph; nodes by small italic letters and squares; vertices also by small italic letters but black circles; and points which are not necessarily vertices by Greek letters.

We will prove the main theorem,  $l_s/l_m \geq \frac{2}{3}$  for rectilinear distance, by induction on number of nodes. It turns out that for  $s \in S_2$ , we can split  $s$  into two components at a node with degree two or more and apply induction on each component independently. For  $s \in E(S_2) = \bigcup_{s \in S_2} E(s)$ , then, there exists an  $s' \in E(s)$  such that  $s' \in S_2$ . The  $\frac{2}{3}$ -bound can be proved by working on  $s'$ . Hence  $s \notin E(S_2)$ , or equivalently,  $s \in (S_1 - E(S_2))$  is the only case which requires a proof. In § 3, we show that for any  $s \notin E(S_2)$ , the induced subgraph of (the subgraph of  $T$  generated by) its Steiner points is a chain. In § 4, we prove various properties of this chain. In § 5, we prove the  $\frac{2}{3}$ -bound using results obtained in § 3 and § 4.

**3. The induced subgraph of Steiner points.** It is easy to see that any edge consisting of more than two lines can be changed to one with at most two lines without increasing the length of the edge. From now on, we assume that an edge has at most two lines.

LEMMA 1 (Hanan [2]). *All Steiner points either have degree three or degree four.*

LEMMA 2. *Let  $A, B$  be two adjacent Steiner points in  $s$  where  $s \notin E(S_2)$ . Suppose that  $AB$  is a horizontal line and both  $V_{Au}, V_{Bu}$  exist. Then  $|V_{Bu}| \geq |V_{Au}|$  implies that  $V_{Au}$  is a corner line which turns away from  $V_{Bu}$ .*

*Proof.* (See Fig. 2). Suppose  $A$  is to the left of  $B$ .

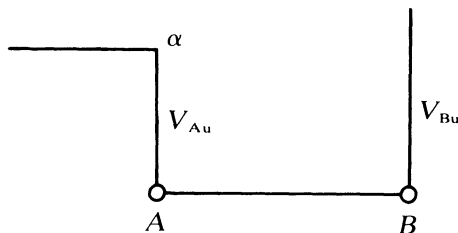


FIG. 2

(i)  $V_{Au}$  can contain no node, for otherwise we can shift  $\overline{AB}$  to that node and obtain an  $S_2$  tree, a contradiction to  $s \notin E(S_2)$ .

(ii) No Steiner point on  $V_{Au}$  can have a right line, for otherwise we can replace  $\overline{AB}$  by extending that line (to meet  $V_{Bu}$ ) and save, an absurdity.

(iii) Therefore the upper endpoint of  $V_{Au}$  cannot be a Steiner point since it has no right line and upward line, hence it must be a corner turning left.

(iv)  $V_{Au}$  can contain no Steiner point, for let  $C$  be such a point; then  $H_C$  does not exist and hence  $H_{Cl}$  must exist. But then  $V_{Au}$  has at least two left lines ( $H_{Cl}$  and the corner turning left) and no right lines, which implies we can shift  $V_{Cu}$  to left and save, an absurdity.

COROLLARY. *Suppose  $V_{Bu}$  contains a vertex. Then  $V_{Au}$  must be a corner line turning away from  $V_{Bu}$  and  $|V_{Au}| < |V_{Bu}|$ .*

LEMMA 3. *Suppose  $V_{xu}$  ( $x$  a vertex) is a corner line turning left (right). Then  $H_{xl}(H_{xr})$  does not exist.*

*Proof.* (See Fig. 3). Let  $y$  be the first node on the horizontal line that  $V_{xu}$  turns

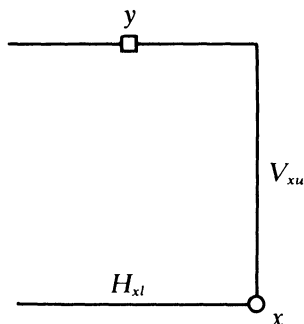


FIG. 3

to. If  $H_{xt}$  exists, then we can flip the corner of  $V_{xu}$  between  $x$  and  $y$  to cause overlapping with  $H_{xt}$ , an absurdity.

LEMMA 4. *No Steiner point can have more than one corner line.*

*Proof.* Figure 4 shows the four possible combinations of a Steiner point with two corner lines. (a) cannot exist by Lemma 3. (b) cannot exist because the third line at the Steiner point cannot exist by Lemma 3. (c) and (d) can be transformed to (a) and (b) respectively by flipping a corner.

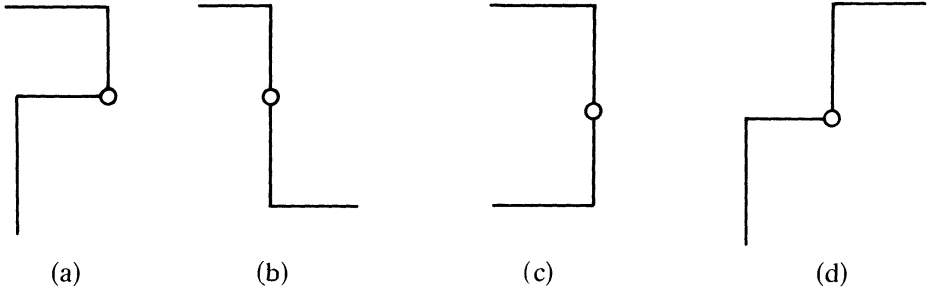


FIG. 4

THEOREM 1. *If  $s \notin E(S_2)$ , then the induced subgraph of Steiner points in  $s$  is a chain.*

*Proof.* For  $s \in S_1$ , the induced subgraph of Steiner points in  $s$  is connected, for otherwise some Steiner points have to be connected by nodes of degree two or more. Therefore we need only to prove that no Steiner point in  $s$  is adjacent to more than two other Steiner points. Suppose the contrary, and let  $A$  be such a Steiner point. Then from Lemma 3 and Lemma 4, the connection between  $A$  and its adjacent Steiner points must have one of the three forms found in Fig. 5.

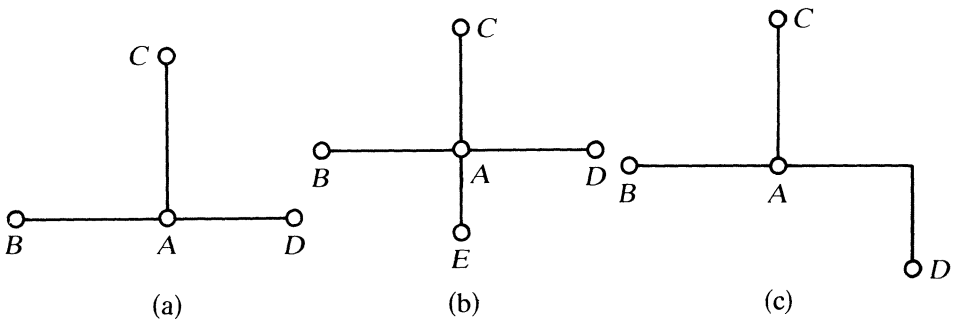


FIG. 5

First consider Fig. 5(a) and (b). Suppose  $H_{Ct}$  exists. Then from the corollary of Lemma 2,  $H_{Ct}$  is a corner line turning up and  $|H_{Ct}| < |H_{At}|$  (the latter may be longer than  $|AB|$ ). Similarly, if  $H_{Cn}$  exists, then it is also a corner line turning up. Since  $C$  is a Steiner point, hence at least two of the three lines  $H_{Ct}$ ,  $H_{Cn}$ , and  $V_{Cu}$  exist. But regardless which two (or all three) exist, we obtain a contradiction to either Lemma 3 or Lemma 4.

Next consider Fig. 5(c). The argument on  $H_{Cl}$  is the same as the other two cases. If  $H_{Cr}$  exists and  $|H_{Cr}| \leq |H_{Ar}|$ , then the argument on  $H_{Cr}$  is again the same. So we need only discuss the case  $|H_{Cr}| > |H_{Ar}|$  (see Fig. 6).

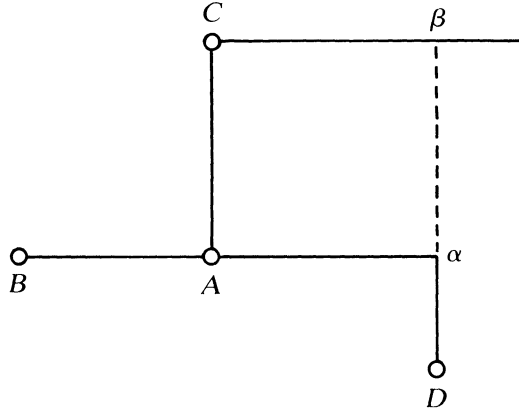


FIG. 6

Let  $\alpha$  be the corner on the edge connecting  $A$  and  $D$ . Shift  $\overline{AC}$  to  $\alpha$  and let the new line meet  $H_{Cr}$  at  $\beta$ . Then  $\alpha$  and  $\beta$  replace  $A$  and  $C$  as Steiner points in the new graph. But the new graph contains a Steiner point  $\alpha$  which is adjacent and connected to three other Steiner points  $\beta$ ,  $B$  and  $D$  in the form of Fig. 5(a), a graph contained only in  $S_2$  trees. We obtain a contradiction to  $s \notin E(S_2)$ .

We call the chain of Steiner points the Steiner chain.

#### 4. Steiner minimal trees not belonging to $E(S_2)$ .

LEMMA 5. Suppose  $s \notin E(S_2)$ . Then its Steiner chain cannot contain the subgraph found in Fig. 7, where  $B$  is adjacent to  $A$  and  $C$ .

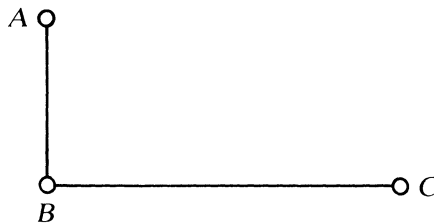


FIG. 7

*Proof.* Suppose  $H_{Ar}$  exists. Then from the corollary of Lemma 2,  $H_{Ar}$  is a corner line. Since we can flip the corner, we assume without loss of generality that  $H_{Ar}$  does not exist. Therefore both  $V_{Au}$  and  $H_{Al}$  exists. From Lemma 3,  $H_{Al}$  cannot be a corner line. But  $H_{Al}$  cannot contain any Steiner point or more than one node, hence  $H_{Al}$  is a node line. By symmetry,  $V_{Cd}$  exists and is a node line. If

$H_{Bl}$  exists, then from the corollary of Lemma 2, it must be a corner line turning downward. Therefore without loss of generality we assume  $H_{Bl}$  does not exist and  $H_{Bd}$  exists. Since  $V_{Cd}$  contains a vertex, by the corollary of Lemma 2,  $H_{Bd}$  must be a corner (denoted by  $\beta$ ) turning left. But this is impossible, for otherwise we can shift  $BC$  to  $\beta$  and  $A\beta$  to  $b$  (see Fig. 8) to obtain an  $S_2$  tree.

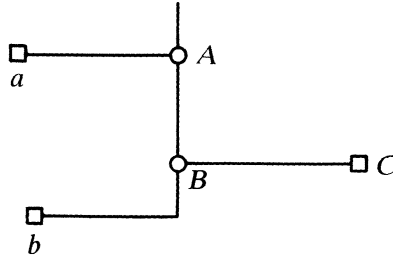


FIG. 8

Define a staircase to be a continuous path of alternating vertical lines and horizontal lines such that their projections on the vertical and horizontal axes have no overlapping intervals.

LEMMA 6. Suppose  $s \notin E(S_2)$ . Then the Steiner chain is a staircase.

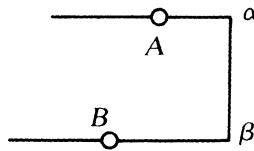


FIG. 9

*Proof.* Suppose the Steiner chain bends back as in Fig. 9, where  $A, B$  are the Steiner points on the upper and lower horizontal lines closest to the turning points  $\alpha$  and  $\beta$ . There must be at least two Steiner points on  $\alpha\beta$ , for otherwise we can shift  $\alpha\beta$  to the left and save, an absurdity. From Lemma 5, neither  $a$  nor  $\beta$  can be a Steiner point. From Lemma 3, the horizontal line of any Steiner point on  $\alpha\beta$  must be a node line. Therefore, from the corollary of Lemma 2, the adjacent Steiner points on  $\alpha\beta$  cannot have horizontal lines on the same side. Now it is clear that  $\alpha\beta$  must have more left lines than right lines ( $A\alpha$  and  $B\beta$  are left lines), which implies we can shift  $\alpha\beta$  to the left and save, an absurdity.

LEMMA 7. Suppose  $s \notin E(S_2)$ . Then the Steiner chain cannot contain a corner with more than one Steiner point on each surrounding line.

*Proof.* Suppose to the contrary that  $s$  contains the subgraph found in Fig. 10. Here  $B$  is adjacent to  $A$  and  $C$  and  $C$  is adjacent to  $D$ .

From Lemma 3,  $V_{Ca}$  does not exist. Hence  $V_{Cd}$  exists and is a node line (again from Lemma 3). Suppose  $V_{Da}$  exists. Then from the corollary of Lemma 2,  $V_{Da}$  is a corner line. As a result,  $H_{Dr}$  does not exist (Lemma 3). Therefore  $V_{Da}$  and  $H_{Dr}$

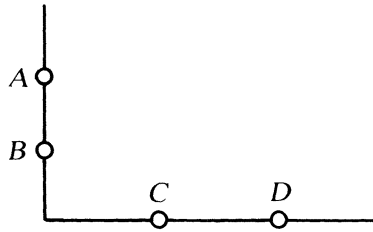


FIG. 10

cannot both exist, hence  $V_{Du}$  must exist and is a node line either from Lemma 3 or Lemma 4. Flip the corner  $\alpha$  between  $B$  and  $C$  to form the new corner  $\beta$ . If  $|V_{Du}| \leq |C\beta|$ , then we can shift  $\overline{CD}$  to the node on  $V_{Du}$  and obtain an  $S_2$  graph, a contradiction to  $s \notin E(S_2)$ . If  $|V_{Du}| > |C\beta|$ , then we can shift  $\overline{CD}$  to  $\beta$ , and  $\beta$  becomes a Steiner point. But the induced subgraph between  $AB\beta$  is contained only in  $E(S_2)$  trees by Lemma 5, a contradiction to  $s \notin E(S_2)$ .

**LEMMA 8.** *Suppose  $s \notin E(S_2)$ . If the number of Steiner points is greater than two, then either every vertical line (on the Steiner chain) contains more than one Steiner point (except perhaps the first and the last) and every horizontal line contains exactly one Steiner point, or vice versa.*

*Proof.* By Lemma 4, a Steiner point cannot have two corner lines. Hence at least two of the first three Steiner points (counting from either end) are collinear. Without loss of generality, suppose the first collinearity occurs on a vertical line. Let  $A$  be the first Steiner point (if any) not on this vertical line. Then  $A$  is connected to its preceding Steiner point through a corner, for otherwise Lemma 5 is contradicted. Let  $B$  be the Steiner point (if any) succeeding  $A$ . Then  $A$  and  $B$  must be on the same vertical line, for otherwise either Lemma 7 is contradicted (if  $A$  and  $B$  are on the same horizontal line), or  $A$  has two corner lines (if  $A$  and  $B$  are connected through a corner). If there are more Steiner points after  $B$ , then we repeat the above argument to prove Lemma 8.

Since our problem is not affected by a  $90^\circ$  rotation, we now assume that if  $s \notin E(S_2)$ , then the Steiner chain consists of a set of vertical lines where adjacent vertical lines are connected through a corner. We label the  $i$ th Steiner point on the chain counting from above by  $A_i$ .

**LEMMA 9.** *Suppose  $s \notin E(S_2)$ . Then every Steiner point must have a horizontal node line.*

*Proof.* Note that a horizontal line not on the Steiner chain cannot contain any Steiner points, nor can it contain more than one node since  $s \notin E(S_2)$ . Therefore it suffices to show that there exists a horizontal line which is not a corner line. The last statement can be easily verified by using Lemma 3 and Lemma 4 (details are omitted).

By the corollary of Lemma 2, two adjacent Steiner points cannot have node lines on the same side. Hence for  $s \notin E(S_2)$ , the node lines on the Steiner chain must alternate in the left-right direction.

**LEMMA 10.** *Suppose  $s \notin E(S_2)$ . Then a corner connecting  $A_i$  and  $A_{i+1}$  can be transferred to either one connecting  $A_{i-2}$  and  $A_{i-1}$ , or one connecting  $A_{i+2}$  and  $A_{i+3}$ ,*





LEMMA 11. Suppose  $s \notin E(S_2)$  and  $|P| = n$ . Then  $m$ , the number of Steiner points, is  $n - 2$  except when  $n = 4$ . Then  $m$  could be either 1 or 2.

**5. The main theorem.** For  $s$  belonging to type (ii) or type (iii) defined in the last section, let  $h_i$  be the length of the horizontal node line at  $A_i$  and let  $a_i$  be the node on it. If  $V_{A_1u}(V_{A_{n-2},d})$  is a corner line, then let  $h_0(h_{n-1})$  be the length of the horizontal line  $V_{A_1u}(V_{A_{n-2},d})$  turns to. Otherwise, define  $h_0(h_{n-1}) = 0$ . Define  $v_1 = |V_{A_1u}|$ ,  $v_i = |A_{i-1}A_i|$  for  $2 \leq i \leq n - 2$ , and  $v_{n-1} = |V_{A_{n-2}d}|$ .

LEMMA 12. Suppose  $2 \leq n \leq 4$ . Then

$$l_s/l_m \geq \frac{2}{3}.$$

*Proof.* For  $n = 2$  or all  $n$  points are collinear, the Steiner minimal tree is the minimal spanning tree.

Otherwise,  $s$  is either a tree in Fig. 5 with  $A$  as the only Steiner point or a tree shown in Fig. 12, where  $h_0$  and  $h_3$  can be zero in Fig. 12(a).

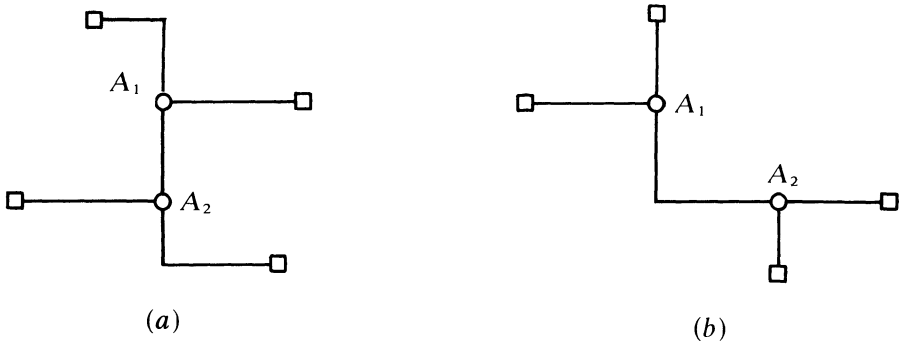


FIG. 12

In any case, form the enclosing rectangle of all nodes. Let  $L$  and  $W$  be the length and width of this rectangle. Then  $l_s \geq L + W$ . Let  $R$  be the boundary of the enclosing rectangle. Then  $R$  contains all the nodes in  $s$  and  $|R| = 2(L + W)$ . By deleting any link between two nodes adjacent on  $R$ , we obtain a spanning tree of the nodes. Suppose we delete the link of maximal length. Then

$$l_m \leq (1 - \frac{1}{4})|R| = \frac{3}{2}(L + W) \leq \frac{3}{2}l_s.$$

THEOREM 2.  $l_s/l_m \geq \frac{2}{3}$  for rectilinear distance.

*Proof.* The proof will proceed by induction on  $|P| = n$ . Theorem 2 is trivially true for  $n = 2$ . Suppose the theorem is true for all  $P'$  with  $|P'| < n$  and suppose  $|P| = n \geq 3$ . Let  $s \in E(S_2)$ . Then there exists a Steiner minimal tree  $s'$  which has a node  $x$  with degree two or more. We can split  $s'$  at  $x$  into two components and apply induction on each.

Suppose  $s \notin E(S_2)$ . From Lemma 11, we may assume  $n \geq 5$ . Our strategy here is to partition  $s$  at a Steiner point, say,  $A_q$ , into two subgraphs  $s_1$  and  $s_2$  such that  $s_1$  is the induced subgraph of  $\{a_0, a_1, \dots, a_{q-1}\}$  plus the edge  $\{A_{q-1}, A_q\}$  and  $s_2$  is the

induced subgraph of  $\{a_q, a_{q+1}, \dots, a_n\}$ . We will construct a path  $p_1$  on the set of points  $\{a_0, a_1, \dots, a_q\}$  such that

$$(5.1) \quad \frac{2}{3}(\text{the length of } p_1) \leq \text{the length of } s_1.$$

Let  $m_2$  be the minimal spanning tree on the set of points  $\{a_q, a_{q+1}, \dots, a_n\}$ . Then  $\frac{2}{3}l_{m_2} \leq l_{s_2}$  by the induction hypothesis. Hence  $p_1$  and  $m_2$  together are a spanning tree whose length is not greater than  $\frac{2}{3}(l_{s_1} + l_{s_2}) = \frac{2}{3}l_s$ . Theorem 2 is then proved.

The selection of  $p_1$  depends on whether there exists an  $s_1$  such that the total length of its horizontal lines are sufficiently small in proportion to the length of  $s_1$ . To be exact, suppose there exists a  $k$ ,  $1 \leq k \leq n-2$ , such that

$$(5.2) \quad \frac{2}{3} \left( \sum_{i=0}^{k-1} h_i + \sum_{i=1}^k h_i + \sum_{i=1}^k v_i \right) \leq \sum_{i=0}^{k-1} h_i + \sum_{i=1}^k v_i.$$

Set  $q = k$ . Then  $p_1$  can be selected as the path connecting  $a_0, a_1, \dots, a_k$  in that order. The length of  $p_1$  is the bracket of (5.2), the length of  $s_1$  is the right-hand side of (5.2). Clearly (5.1) is satisfied.

Therefore let us now suppose that (5.2) does not hold for each  $k = 1, \dots, n-2$ , namely, we assume

$$(5.3) \quad \sum_{i=1}^k v_i \leq \sum_{i=1}^k h_i + (h_k - h_0), \quad 1 \leq k \leq n-2.$$

A tree of type (ii) or type (iii) remains so if we reverse the ordering of its Steiner points (for type (iii) tree, push the corner of the Steiner chain to the other end). Therefore we can assume the existence of a subscript  $j$ ,  $2 \leq j \leq [n/2] + 1$ , such that  $h_i > h_{i-2}$  for all  $i = 2, \dots, j-1$ , and  $h_j \leq h_{j-2}$ . Set  $q = j$ . Connect  $(a_j, a_{j-2}, a_{j-4}, \dots, a_0, \dots, a_{j-3}, a_{j-1}, a_j)$  in that order, i.e., connect all nodes on the same side in order before crossing to the other sides. We have a tour, call it  $t$ , on this set of  $j+1$  nodes. The length of  $t$  equals the periphery (denoted by  $|R|$ ) of the enclosing rectangle of the  $j+1$  nodes in  $t$ . We show that the desired  $p_2$  can be obtained by deleting a proper link in  $t$ . If  $j = 2$  or  $3$ , then the selection of  $p_2$  and the proof of (5.1) are essentially the same as done in the proof of Lemma 12. So let us consider  $j \geq 4$ . Let  $\sum_{i=0}^{j-3} h_i = \theta |R| \geq 0$ . Then the four links  $a_{j-4} \rightarrow a_{j-2} \rightarrow a_j \rightarrow a_{j-1} \rightarrow a_{j-3}$  in  $t$  have a total length of

$$\begin{aligned} & |R| - h_{j-3} - h_{j-4} - \sum_{i=1}^{j-4} v_i - \sum_{i=1}^{j-3} v_i \\ & \geq |R| - h_{j-3} - h_{j-4} - \left[ \sum_{i=1}^{j-4} h_i + (h_{j-4} - h_0) \right] \\ & \quad - \left[ \sum_{i=1}^{j-3} h_i + (h_{j-3} - h_0) \right] \quad (\text{by using (5.3)}) \\ & \geq |R| - 4 \sum_{i=0}^{j-3} h_i = |R|(1 - 4\theta). \end{aligned}$$

If we delete the link of maximal length among these four links from  $t$ , we obtain a path of the  $j+1$  nodes with length

$$|R| - \frac{1}{4}|R|(1-4\theta) = |R|(\frac{3}{4} + \theta).$$

Multiplying this length by  $\frac{2}{3}$ , we obtain

$$(5.4) \quad |R|(\frac{1}{2} + \frac{2}{3}\theta).$$

But the length of  $s_1$  is

$$(5.5) \quad \sum_{i=0}^{j-1} h_i + \sum_{i=1}^j v_i = \sum_{i=1}^{j-3} h_i + \frac{1}{2}|R| = |R|(\frac{1}{2} + \theta),$$

which is not less than (5.4). Theorem 2 is proved.

Note that the  $\frac{2}{3}$ -bound can be attained at an infinite number of values of  $n$ . For example, the  $\frac{2}{3}$ -bound is attained in the graph shown in Fig. 13, where all edges are of equal length, and graphs which consist of copies of Fig. 13. Now by adding  $n-4$  nodes within distance  $\varepsilon$  to any of the four nodes in Fig. 13, we obtain a graph of  $n$  nodes such that  $l_s/l_m$  is arbitrarily close to  $\frac{2}{3}$ .

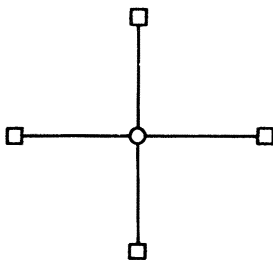


FIG. 13

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