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Given that,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x_1(0) = 15, \quad x_2(0) = 25$$

and $$J = \frac{1}{2} x^T(t_f) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x(t_f) + \frac{1}{2} \int_0^{t_f} \left(x^T \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} x + r u^2 \right) dt$$

here,

$$P(t_f) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q(t) = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad R(t) = r$$

For optimal solution, $P(t_f) \geq 0$, $Q(t) \geq 0$, $R(t) > 0$ and all are symmetric.

<u>Now the DRT:</u>

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q \quad ; \; t_0 \leq t \leq t_f$$

$$\Rightarrow -\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_2 & \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$+ \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ P_1 & P_2 \end{bmatrix} + \begin{bmatrix} 0 & P_1 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \frac{1}{r_c} \begin{bmatrix} P_2 & P_3 \end{bmatrix}$$

$$+ \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} q_1 & P_1 \\ P_1 & 2P_2 + q_2 \end{bmatrix} - \begin{bmatrix} \frac{P_2^2}{r_c} & \frac{P_2 P_3}{r_c} \\ \frac{P_2 P_3}{r_c} & \frac{P_3^2}{r_c} \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} q_1 - \frac{P_2^2}{r_c} & P_1 - \frac{P_2 P_3}{r_c} \\ P_1 - \frac{P_2 P_3}{r_c} & 2P_2 + q_2 - \frac{P_3^2}{r_c} \end{bmatrix}$$

$$\Rightarrow -\dot{P}_1 = q_1 - \frac{P_2^2}{r_c}$$

$$-\dot{P}_2 = P_1 - \frac{P_2 P_3}{r_c}$$

$$-\dot{P}_3 = 2P_2 + q_2 - \frac{P_3^2}{r_c}$$

NOW, $K = R^{-1} B^T P$

$$= \frac{1}{r_c} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$$

$$= \frac{1}{r_c} \begin{bmatrix} P_2 & P_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{P_2}{r_c} & \frac{P_3}{r_c} \end{bmatrix}$$

$\therefore$ The optimal controller, $u^* = -KX$

$$= -\begin{bmatrix} \frac{P_2}{r} & \frac{P_3}{r} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -\left(\frac{x_1 P_2}{r} + \frac{x_2 P_3}{r} \right)$$

For infinite time LQR:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ P_1 & P_2 \end{bmatrix} + \begin{bmatrix} 0 & P_1 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \frac{1}{r} \begin{bmatrix} P_2 & P_3 \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & P_1 \\ P_1 & 2P_2 \end{bmatrix} - \begin{bmatrix} \frac{P_2^2}{r} & \frac{P_2 P_3}{r} \\ \frac{P_2 P_3}{r} & \frac{P_3^2}{r} \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -\frac{P_2^2}{r} + q_1 & P_1 - \frac{P_2 P_3}{r} \\ P_1 - \frac{P_2 P_3}{r} & 2P_2 - \frac{P_3^2}{r} + q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad -\frac{P_2^2}{r} + q_1 = 0 \quad \text{——— (1)}$$

$$P_1 - \frac{P_2 P_3}{r} = 0 \quad \text{——— (2)}$$

$$2P_2 - \frac{P_3^2}{r} + q_2 = 0 \quad \text{——— (3)}$$

Let,

$$r = 500$$

$$Q = \begin{bmatrix} 80 & 0 \\ 0 & 1 \end{bmatrix}$$

From eqn. (1) we get,

$$-\frac{P_2^2}{500} + 80 = 0$$

$$\Rightarrow P_2 = \sqrt{80 \times 500}$$

$$\Rightarrow P_2 = 200$$

From eqn. (3) we get,

$$2 \times 200 - \frac{P_3^2}{500} + 1 = 0$$

$$\Rightarrow P_3 = \sqrt{401 \times 500}$$

$$\therefore P_3 = 447.77$$

From eqn. (2) we get,

$$P_1 - \frac{200 \times 447.77}{500} = 0$$

$$\Rightarrow P_1 = 179.11$$

Now,
$$K = R^{-1}B^{T}P$$
$$= \frac{1}{500}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}179.11 & 200 \\ 200 & 447.77\end{bmatrix}$$
$$= \frac{1}{500}\begin{bmatrix}200 & 447.77\end{bmatrix}$$
$$= \begin{bmatrix}0.4 & 0.8955\end{bmatrix}$$

$\therefore$ The suboptimal controller,
$$u = -KX$$
$$= -\begin{bmatrix}0.4 & 0.8955\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$
$$= -(0.4x_1 + 0.8955x_2)$$