Mini-project #2

ELEN-865

Prepared by,

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(a)

$$r = 3\,, Q = \begin{bmatrix} 90 & 0 \\ 0 & 30 \end{bmatrix}$$

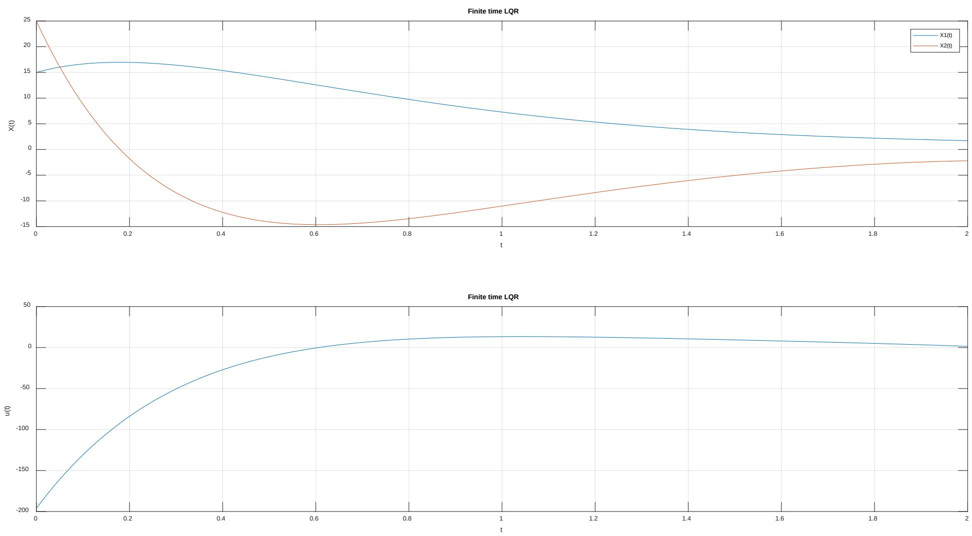

Figure 1: Plot of state trajectory and optimal controller for finite time LQR with $t_f = 2s$

(b)

$$r = 500\,, Q = \begin{bmatrix} 80 & 0 \\ 0 & 1 \end{bmatrix}$$

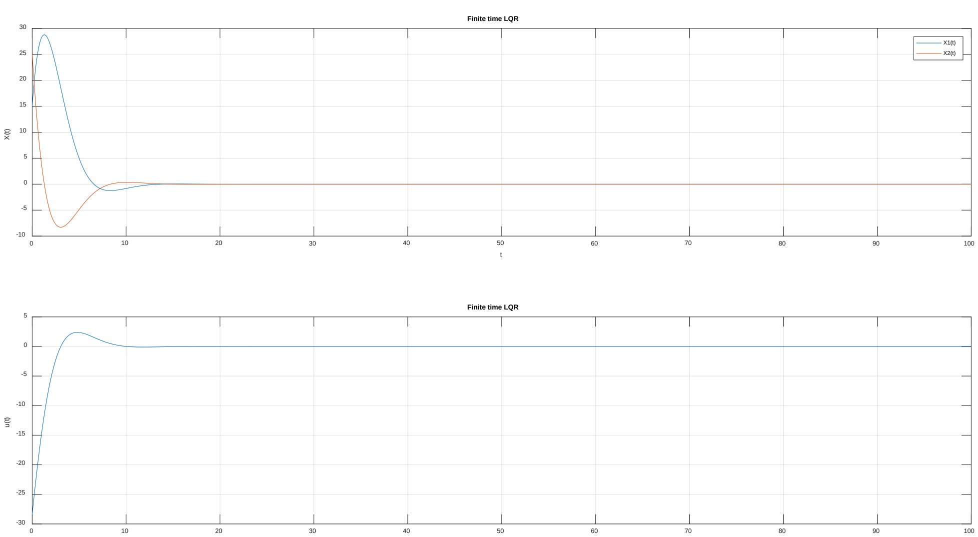

Figure 2: Plot of state trajectory and optimal controller for finite time LQR with $t_f = 100s$

(c)

$$r = 500\,, Q = \begin{bmatrix} 80 & 0 \\ 0 & 1 \end{bmatrix}$$

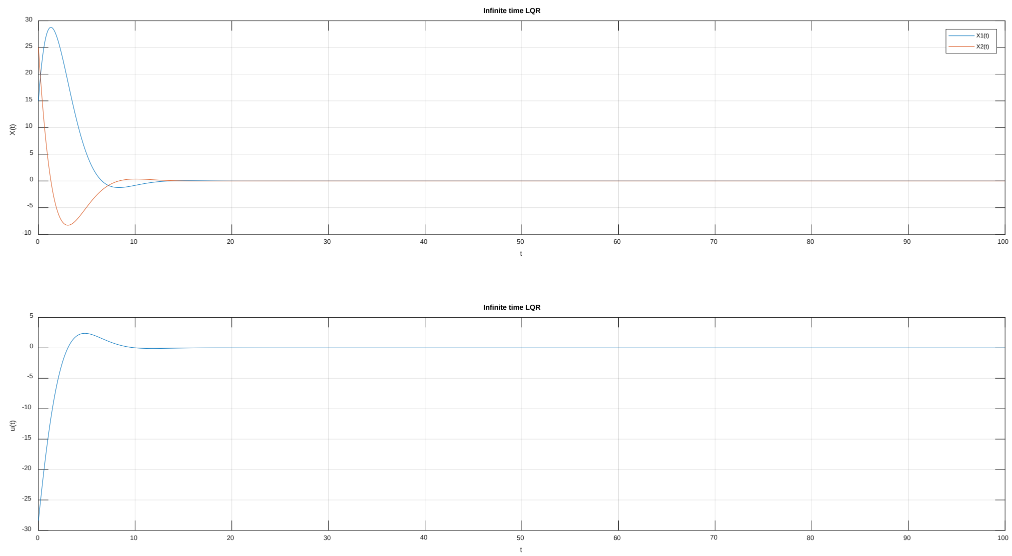

Figure 3: Plot of state trajectory and sub-optimal controller for infinite time LQR with t_f = 100s

(d)

In (a) the optimal controller needs to put a very high input signal to converge the trajectory to origin within 2 second but for (b) the optimal controller needs to put very low input signal compared to (a) to converge the trajectory to origin.

(e)

The trajectory and input signal are same for (c) and (b).

(f)

From (d) and (e), we can conclude that if time is not a concern for the control problem in hand we can use infinite time LQR than finite time LQR with less computational effort. But if the control problem needs very fast response and accuracy the finite time LQR is a must with burden cost of computational effort.

Appendix:

MATLAB Code:

```
%%
%clear all;
%close all;
%% finite time LQR
A = [0 1;...
     0 0];
B = [0;...
     1];
x0 = [15;...
      25];
tf = 100;
r = 500;

[X,u,pf,t] = simoptsys(A,B,r,x0,tf);

%%
clf
figure(1)
subplot(211)
plot(t,X(1,:))
hold on
plot(t,X(2,:))
xlabel('t')
ylabel('X(t)')
legend('X1(t)','X2(t)')
title('Finite time LQR')
grid on
subplot(212)
plot(t,u)
xlabel('t')
ylabel('u(t)')
grid on
title('Finite time LQR')

%% infinite time LQR
Q =[80 0;...
```

```
    0 1];
R = 500;
tf = 100;
k = lqr(A,B,Q,R);
t = 0:.01:tf;
X=[];
u=[];
X(:,1) = x0;
u(1) = -k*X(:,1);
for n=1:length(t)-1
    X(:,n+1)=expm((A-B*k)*(t(n+1)-t(n)))*X(:,n);
    u(n+1) = -k*X(:,n+1);
end

%%
figure(2)
subplot(211)
plot(t,X(1,:))
hold on
plot(t,X(2,:))
xlabel('t')
ylabel('X(t)')
legend('X1(t)','X2(t)')
title('Infinite time LQR')
grid on
subplot(212)
plot(t,u)
xlabel('t')
ylabel('u(t)')
title('Infinite time LQR')
grid on
```

```
function [X,u,pf,t]=simoptsys(A,B,r,x0,tf)
[tb,p]=ode45(@DRE,-tf:.001:0,[2;0;2]);
pf = flipud(p);
t = -flipud(tb);
k = (1/r)*pf(:,2:3);
X(:,1) = x0;
u(1) = -k(1,:)*X(:,1);
```

```
for n=1:length(t)-1
    X(:,n+1)=expm((A-B*k(n,:))*(t(n+1)-t(n)))*X(:,n);
    u(n+1) = -k(n+1,:)*X(:,n+1);
end
end
```

```
function pd=DRE(t,p)
r = 500;
q1 = 80;
q2 = 1;
pd=[q1-p(2)^2/r;...
    p(1)-p(2)*p(3)/r;...
    2*p(2)+q2-p(3)^2/r];
end
```

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Given that,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x_1(0) = 15, \quad x_2(0) = 25$$

and $$J = \frac{1}{2} X^T(t_f) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X(t_f) + \frac{1}{2} \int_0^{t_f} \left(X^T \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} X + r u^2 \right) dt$$

here,

$$P(t_f) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q(t) = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad R(t) = r$$

For optimal solution, $P(t_f) \geq 0$, $Q(t) \geq 0$, $R(t) > 0$ and all are symmetric.

Now the DRT:

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q \quad ; t_0 \leq t \leq t_f$$

$$\Rightarrow - \begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$$

$$+ \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ P_1 & P_2 \end{bmatrix} + \begin{bmatrix} 0 & P_1 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \frac{1}{r_c} \begin{bmatrix} P_2 & P_3 \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} q_1 & P_1 \\ P_1 & 2P_2 + q_2 \end{bmatrix} - \begin{bmatrix} \frac{P_2^2}{r_c} & \frac{P_2 P_3}{r_c} \\ \frac{P_2 P_3}{r_c} & \frac{P_3^2}{r_c} \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} q_1 - \frac{P_2^2}{r_c} & P_1 - \frac{P_2 P_3}{r_c} \\ P_1 - \frac{P_2 P_3}{r_c} & 2P_2 + q_2 - \frac{P_3^2}{r_c} \end{bmatrix}$$

$$\Rightarrow -\dot{P}_1 = q_1 - \frac{P_2^2}{r_c}$$

$$-\dot{P}_2 = P_1 - \frac{P_2 P_3}{r_c}$$

$$-\dot{P}_3 = 2P_2 + q_2 - \frac{P_3^2}{r_c}$$

NOW, $K = R^{-1} B^T P$

$$= \frac{1}{r_c} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$$

$$= \frac{1}{r_c} \begin{bmatrix} P_2 & P_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{P_2}{r_c} & \frac{P_3}{r_c} \end{bmatrix}$$

$\therefore$ The optimal controller, $u^* = -KX$

$$= -\begin{bmatrix} \frac{P_2}{r} & \frac{P_3}{r} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -\left(\frac{x_1 P_2}{r} + \frac{x_2 P_3}{r} \right)$$

For infinite time LQR:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ P_1 & P_2 \end{bmatrix} + \begin{bmatrix} 0 & P_1 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \frac{1}{r} \begin{bmatrix} P_2 & P_3 \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & P_1 \\ P_1 & 2P_2 \end{bmatrix} - \begin{bmatrix} \frac{P_2^2}{r} & \frac{P_2 P_3}{r} \\ \frac{P_2 P_3}{r} & \frac{P_3^2}{r} \end{bmatrix} + \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -\frac{P_2^2}{r} + q_1 & P_1 - \frac{P_2 P_3}{r} \\ P_1 - \frac{P_2 P_3}{r} & 2P_2 - \frac{P_3^2}{r} + q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad -\frac{P_2^2}{R} + q_1 = 0 \quad \text{———} \quad (1)$$

$$P_1 - \frac{P_2 P_3}{R} = 0 \quad \text{———} \quad (2)$$

$$2P_2 - \frac{P_3^2}{R} + q_2 = 0 \quad \text{———} \quad (3)$$

Let,

$$R = 500$$

$$Q = \begin{bmatrix} 80 & 0 \\ 0 & 1 \end{bmatrix}$$

From eqn. (1) we get,

$$-\frac{P_2^2}{500} + 80 = 0$$

$$\Rightarrow P_2 = \sqrt{80 \times 500}$$

$$\Rightarrow P_2 = 200$$

From eqn. (3) we get,

$$2 \times 200 - \frac{P_3^2}{500} + 1 = 0$$

$$\Rightarrow P_3 = \sqrt{401 \times 500}$$

$$\therefore P_3 = 447.77$$

From eqn. (2) we get,

$$P_1 - \frac{200 \times 447.77}{500} = 0$$

$$\Rightarrow P_1 = 179.11$$

Now,
$$K = R^{-1}B^{T}P$$

$$= \frac{1}{500}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}179.11 & 200 \\ 200 & 447.77\end{bmatrix}$$

$$= \frac{1}{500}\begin{bmatrix}200 & 447.77\end{bmatrix}$$

$$= \begin{bmatrix}0.4 & 0.8955\end{bmatrix}$$

$\therefore$ The suboptimal controller,

$$u = -KX$$

$$= -\begin{bmatrix}0.4 & 0.8955\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

$$= -(0.4x_1 + 0.8955x_2)$$