Remark

We assumed that the process noise ($w(t)$) and measurment noise both are white noise.

white noise with infinite power ~~spe~~ does not exist in real life

Instead we can use a colored noise which is a noise with finite energy. and finite frequency components.

Colored noise can be modeled as the output of a linear system with an injection of white noise into its inputs

i.e. colored noise can be generated by a white noise.

white noise $\rightarrow$ $\boxed{\text{linear system}}$ $\rightarrow$ colored noise

ex. a low pass filter

Ex.

$$\begin{cases} \dot{x} = Ax + Bu + Gw \\ y = Cx + v \end{cases}$$

assume that $v(t)$ is a white noise but $w(t)$ is a color noise

$$\begin{cases} \dot{x}_w = A_w x_w + B_w n \\ w = C_w x_w + D_w n \end{cases}$$

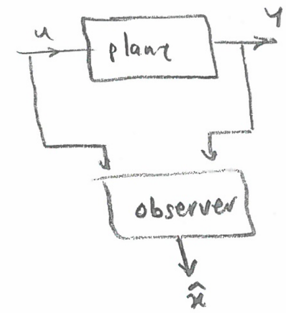
where $n(t)$ is a white noise

$\Rightarrow$ Augmented system:

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_w \end{bmatrix} = \begin{bmatrix} A & GC_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x \\ x_w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} GD_w \\ B_w \end{bmatrix} n \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + v \end{cases}$$

Filter interpretation

Observer:



$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

to find the transfer function from y to $\hat{x}$

let $u=0$ $\Rightarrow$ $\dot{\hat{x}} = (A - LC)\hat{x} + LY$

consider the scalar system:

$\Rightarrow$ $\hat{X}(s) = (sI - (A-LC))^{-1} LY(s)$

estimation

in case of siso $\Rightarrow$ $\dfrac{\hat{X}(s)}{Y(s)} = \dfrac{L}{(sI - (A-LC))}$ which is a low pass filter

measurment

$L = P_\infty C^T R^{-1}$

- if $R\downarrow \Rightarrow L\uparrow \Rightarrow$ pushes the poles out $\Rightarrow$ higher bandwidth

in this case the DC gain $\dfrac{\hat{X}(0)}{Y(0)} \rightarrow \dfrac{1}{C}$

Remark

- Estimation is an important concept of its own
  - Estimator is not always part of the control systems
  - Estimation is a critical issue for guidance and navigation systems.

Duality of LQR and LQE

LQR	LQE
$\dot{x}=(A-BK)x$	$\dot{\tilde{x}}=(A-LC)x+Gw-Lv$ $R_{ww}(\tau)=Q\,\delta(t-\tau)$, $R_{vv}(\tau)=R\,\delta(t-\tau)$
$J=\frac{1}{2}\int_0^\infty (x^TQx+u^TRu)\,dt$ $Q\geq 0\ ,\ R>0$	$J=E\{\tilde{x}^T(t)\,\tilde{x}(t)\}$ $Q\geq 0\ ,\ R>0$
$K=R^{-1}B^TP_\infty$	$L=P_\infty C^TR^{-1}$
$A^TP+PA-PBR^{-1}B^TP+Q=0$	$PA^T+AP-PC^TR^{-1}CP+GQG^T=0$

$$A \longleftrightarrow A^T$$
$$B \longleftrightarrow C^T$$
$$\sqrt{Q} \longleftrightarrow G\sqrt{Q}$$
$$R \longleftrightarrow R$$
$$K \longleftrightarrow L^T$$
$$P \longleftrightarrow P$$

LQR	LQE
If the plant is stabilizable $\Rightarrow$ it is possible to find K s.t. the C.L. controlled system matrix $\bar{A}=A-BK$ becomes stable.	If the plant is detectable $\Rightarrow$ it is possible to find L s.t. the C.L. observer matrix $\bar{A}=A-LC$ becomes stable
LQR loop acheives 60° phase margine and infinite gain margine	LQE loop acheives 60° phase margine and infinite gain margine

Remark

$P(t) = E\{\tilde{x}(t)\, \tilde{x}(t)^T\}$ = Error covariance : it is a function of time

$\Rightarrow$ it is a nonstationary process

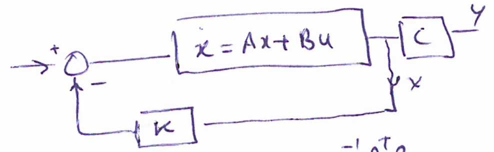
the smaller diagonal elements of $P(t)$ $\equiv$ the error is more closely distributed about its mean value (which is zero) $\equiv$ better estimation

$$L = P(t)\, C^T R^{-1}$$

★ Large uncertainty, $P(t) = E\{\tilde{x}(t)^T \tilde{x}(t)\}$, creates large ~~observer~~ estimator gain L to place emphasis on the corrective action of the filter

Linear Quadratic Gaussian (LQG) Design

LQR:



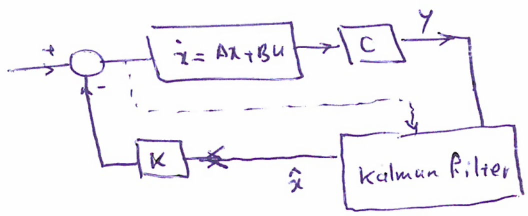
$u = -Kx$ $K = R^{-1}B^TP$

$\rightarrow$ assumption: All states are accessible!

if not $\Rightarrow$ LQG

$\downarrow$

LQG:



problem formulation:

plant: $\begin{cases} \dot{x} = Ax + Bu + Gw \\ y = Cx + v(t) \end{cases}$

$x(0) = (\bar{x}_0, P_0)$, $E\{w(t)\} = 0$ $E\{v(t)\} = 0$

$R_{ww}(\tau) = E\{w(t)\, w^T(t+\tau)\} = Q_e \delta(t-\tau)$ $Q_e = Q_e^T \geq 0$

$R_{vv}(\tau) = E\{v(t)\, v^T(t+\tau)\} = R_e \delta(t-\tau)$ $R_e = R_e^T > 0$

R nonsingular

observer: $\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$ $\hat{x}(0) = \bar{x}_0$

P.I.: $J = \lim_{T \to \infty} \frac{1}{T} E\left\{ \int_0^T (x^TQx + u^TRu)\, dt \right\}$

control law: $u = -K\hat{x} + Fr$

Solution: $K = R^{-1}B^TP$ where P is the solution of:

$PA + A^TP - PBR^{-1}B^TP + Q = 0$ $Q \geq 0$, $R > 0$

$L = P_eC^TR^{-1}$ where P_e is the solution of:

$P_eA^T + A^TP_e - P_eC^TR_e^{-1}CP_e + G^TQ_eG = 0$ $P_e > 0$

$F = \frac{1}{\text{DC gain}} = [C(A - BK)^{-1}B]^{-1}$

Separation principle:

In LQG, the controller and observer designed separately and then they were combined together. using this method:

1. The closed-loop poles are as the same as the LQR case with full state feedback with $u = -Kx + Fr$
   with $u = -K\hat{x} + Fr$

2. The transfer function for the LQG system with $u = -K\hat{x} + Fr$ from r to y is as the transfer function with $u = -Kx + Fr$

To prove these properties, lets find the C.L. system for the LQG system:

$$\begin{cases} \text{Plant:} \begin{cases} \dot{x} = Ax + Bu + Gw \\ y = Cx + v \end{cases} \\ \text{observer:} \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases} \Rightarrow \text{define } \tilde{x} = x - \hat{x} \\ \text{control law:} \quad u = -K\hat{x} + Fr \end{cases}$$

$$\dot{x} = (A - BK)x + BK\tilde{x} + BFr + Gw$$

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} + Gw - A\hat{x} - \cancel{Bu} - LC(x - \hat{x}) - Lv$$

$$= (A - LC)\tilde{x} + Gw - Lv$$

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & 0 \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} BF \\ 0 \end{bmatrix} r + \begin{bmatrix} G \\ G \end{bmatrix} w + \begin{bmatrix} 0 \\ -L \end{bmatrix} v \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + v \end{cases}$$

$$\Rightarrow H_{ry}(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI - (A - BK) & -BK \\ 0 & sI - (A - LC) \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$= C\,[sI - (A - BK)]^{-1} B$$