1.(a)

Given,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$R = 0.1, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$$

$$t_f = \infty$$

∴ ARE:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{1}{0.1}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} P_1 + P_2 & P_2 + P_3 \\ P_1 - 2P_2 & P_2 - 2P_3 \end{bmatrix} + \begin{bmatrix} P_1 + P_2 & P_1 - 2P_2 \\ P_2 + P_3 & P_2 - 2P_3 \end{bmatrix} - 10\begin{bmatrix} P_1 + 2P_2 \\ P_2 + 2P_3 \end{bmatrix}\begin{bmatrix} P_1 + 2P_2 & P_2 + 2P_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2P_1 + 2P_2 & P_1 - P_2 + P_3 \\ P_1 - P_2 + P_3 & 2P_2 - 4P_3 \end{bmatrix} - 10\begin{bmatrix} P_1^2 + 4P_1P_2 + 4P_2^2 & P_1P_2 + 2P_2^2 + 2P_1P_3 + 4P_2P_3 \\ P_1P_2 + 2P_1P_3 + 2P_2^2 + 4P_2P_3 & P_2^2 + 4P_2P_3 + 4P_3^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = 0$$

$$\Rightarrow \begin{aligned} &2P_1 + 2P_2 - 10P_1^2 - 40P_1P_2 - 40P_2^2 + 1 = 0 \\ &P_1 - P_2 + P_3 - 10P_1P_2 - 20P_2^2 - 20P_1P_3 - 40P_2P_3 = 0 \\ &2P_2 - 4P_3 - 10P_2^2 - 40P_2P_3 - 40P_3^2 + 10 = 0 \end{aligned}$$

$$\Rightarrow \quad 2P_1 + 2P_2 - 10(P_1 + 2P_2)^2 + 1 = 0$$

$$P_1 - P_2 + P_3 - 10(P_1 + 2P_2)(P_2 + 2P_3) = 0 \quad \cdots\cdots \quad (1)$$

$$2P_2 - 4P_3 - 10(P_2 + 2P_3)^2 + 10 = 0$$

Let,

$$P_1 + 2P_2 = x \qquad P_1 = x - y - z$$

$$P_2 + 2P_3 = y \quad \Rightarrow \quad P_2 = \frac{1}{2}(y + z)$$

$$P_2 - 2P_3 = z \qquad P_3 = \frac{1}{4}(y - z)$$

eqn-1 $\Rightarrow$

$$2x - 2y - 2z + y + z - 10x^2 + 1 = 0$$

$$x - y - z - \frac{1}{2}(y + z) + \frac{1}{4}(y - z) - 10xy = 0$$

$$2z - 10y^2 + 10 = 0$$

$$2x - y - z - 10x^2 + 1 = 0 \quad \cdots\cdots \quad (2)$$

$$\Rightarrow \quad x - \frac{5}{4}y - \frac{7}{4}z - 10xy = 0 \quad \cdots\cdots \quad (3)$$

$$z = 5y^2 - 5 \quad \text{———} \quad (4)$$

substituting z to (2) and (3) we get,

$$2x - y - 5y^2 + 5 - 10x^2 + 1 = 0 \quad \text{———} \quad (5)$$

$$x - \frac{5}{4}y - \frac{35}{4}y^2 + \frac{35}{4} - 10xy = 0 \quad \text{———} \quad (6)$$

From (6)

$$x(1 - 10y) = \frac{5}{4}(7y^2 + y - 7)$$

$$\Rightarrow x = \frac{5}{4} \, \frac{7y^2 + y - 7}{1 - 10y}$$

substituting the value of x to equation (5) we get,

$$6 - \frac{5(7y^2+y-7)}{2(10y-1)} - \frac{125(7y^2+y-7)^2}{8(10y-1)^2} - 5y^2 - y = 0$$

using Matlab we get,

$$y = \begin{cases} -1.2 \\ -0.77 \\ 0.68 \\ 0.98 \end{cases} \quad \therefore x = \begin{cases} 0.18 \\ -0.52 \\ \cancel{-0.7121} \\ 0.66 \\ -0.0998 \end{cases} \quad \therefore Z = \begin{cases} 2.2 \\ -2.03 \\ -2.688 \\ -0.198 \end{cases}$$

$$P_1 = \begin{cases} -0.82 \\ 2.29 \\ 2.67 \\ -0.88 \end{cases}, \quad P_2 = \begin{cases} 0.5 \\ -1.4 \\ -1 \\ 0.39 \end{cases}, \quad P_3 = \begin{cases} -0.85 \\ 0.32 \\ 0.84 \\ 0.29 \end{cases}$$

For, $P \geq 0$

$$P_1 = 2.67, \quad P_2 = -1, \quad P_3 = 0.84$$

$$\therefore P = \begin{bmatrix} 2.67 & -1 \\ -1 & 0.84 \end{bmatrix}$$

$$\therefore K = R^{-1} B^T P$$

$$= 10 \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2.67 & -1 \\ -1 & 0.84 \end{bmatrix}$$

$$= \begin{bmatrix} 6.7 & 6.8 \end{bmatrix}$$

$\therefore$ The control law, $u^* = -Kx = -(6.7x_1 + 6.8x_2)$

(Ans)