

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, KOLKATA  
DEPARTMENT OF MATHEMATICS AND STATISTICS  
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# Introduction to Complex Dynamics and the Mandelbrot Set

Aditya Dutta (18MS101)

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Supervisor: Dr. Sushil Gorai  
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# Chapter 1

## Introduction

### 1.1 Equicontinuity and Normality

### 1.2 Completely Invariant Components

A domain  $D$  is called:

- *forward invariant* under the map  $f$  if  $f(D) = D$ .
- *backward invariant* under the map  $f$  if  $f^{-1}(D) = D$ .
- *completely invariant* under the map  $f$  if it is both forward and backward invariant under  $f$  i.e.  $f(D) = D$  and  $f^{-1}(D) = D$ .

**Definition 1.2.1** (Connectivity). *The connectivity of a domain  $D \subset \hat{\mathbb{C}}$  is defined as the number of components of  $\partial D$ .*

**Theorem 1.2.1.** *The following are equivalent for a domain  $D \subset \hat{\mathbb{C}}$ :*

1.  $D$  is simply connected.
2.  $D^c$  is connected.
3.  $\partial D$  is connected or  $c(D) = 1$ .

**Theorem 1.2.2.** *If  $R$  is a rational function, with  $\deg(R) \geq 2$ , and  $F_0$  is a completely invariant Fatou component of  $R$ , then:*

1.  $\partial F_0 = J$ .
2.  $F_0$  is simply connected or infinitely connected.
3. All other components of  $F$  are simply connected.

4.  $F_0$  is simply connected  $\iff J$  is connected.

**Theorem 1.2.3.**  $\partial R(U) \subset R(\partial U)$

**Lemma 1.2.4.** For a rational map  $R$ , if  $F_1$  and  $F_2$  are two Fatou components and  $R$  maps a point of  $F_1$  to a point of  $F_2$ , then  $R(F_1) = F_2$ .

**Theorem 1.2.5.** The unbounded Fatou component of a polynomial  $P$ , i.e. the Fatou component containing  $\infty$  is a completely invariant Fatou component. It is denoted by  $F_\infty(P)$  or simply  $F_\infty$  when the context is clear.

**Theorem 1.2.6 (Vitali's Theorem).**

**Lemma 1.2.7.** If  $\alpha$  is a (super)-attracting fixed point of a rational map  $R$  and  $F_\alpha$  is the Fatou component containing  $\alpha$  then  $R^n(z) \rightarrow \alpha$  locally uniformly in  $F_\alpha$ .

**Theorem 1.2.8 (Riemann-Hurwitz Formula).** Let  $F_0$  and  $F_1$  be components of the Fatou set  $F$  of a rational map  $R$  and  $R$  maps  $F_0$  into  $F_1$ . Then, for some integer  $m$ ,  $R$  is an  $m$ -fold map of  $F_0$  onto  $F_1$  and

$$\chi(F_0) + \delta_R(F_0) = m\chi(F_1).$$

## 1.3 Some properties of the Julia Sets

Let  $J$  denote the Julia set for a rational map  $R$  with  $\deg(R) \geq 2$ . Then we have the following properties:

**Theorem 1.3.1.**  $J$  is infinite.

**Theorem 1.3.2 (Minimality of  $J$ ).**

**Theorem 1.3.3.**  $J$  is a perfect set, and hence, uncountable.

# Chapter 2

## Petal Theorem

A point  $p$  is called a parabolic fixed point of  $f$  if  $f(p) = p$  and  $f'(p) = e^{2\pi it}$ , where  $t$  is a rational number.

**Lemma 2.0.1.** *Suppose  $f$  is analytic and satisfies*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{p+2})$$

*in some neighbourhood  $N$  of the origin. Let  $\omega_1, \dots, \omega_p$  be the  $p$ -th roots of unity and let  $\eta_1, \dots, \eta_p$  be the  $p$ -th roots of  $-1$ . Then for sufficiently small  $r_0$  and  $\theta_0$ ,*

1.  $|f(z)| < |z|$  on each sector

$$S_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\omega_j)| < \theta_0\}.$$

2.  $|f(z)| > |z|$  on each sector

$$\Sigma_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\eta_j)| < \theta_0\}.$$

*Proof.* We have,

$$f(z)/z = 1 - z^p + \mathcal{O}(z^{p+1}) = 1 - z^p(1 + g(z)),$$

where  $g$  is analytic in  $N$  with  $g(0) = 0$ .

Now, consider the sector,

$$S = \{z \in \mathbb{C} : |z| < \frac{1}{2}; |\arg(z)| < \pi/4\}.$$

For small  $r_0$  and  $\theta_0$ ,  $z \in S_j \implies z^p(1 + g(z)) \in S$  and  $z \in \Sigma_j \implies -z^p(1 + g(z)) \in S$ .

This is because for small enough  $r_0$  and  $\theta_0$ ,  $z \mapsto z^p$  maps  $S_j$  onto the set

$$S_0 = \{z \in \mathbb{C} : |z| < r_0^p; |\arg(z)| < p\theta_0\} \subset S.$$

And,  $|z^p - z^p(1 + g(z))| = |z|^p |g(z)| \leq M|z|^{p+1} = M(|z|^p)^{1+\frac{1}{p}}$ . Hence, for any  $w \in S_0$ , the perturbation of any point is  $\leq M|w|^{1+1/p}$ .

□

Before stating the *Petal Theorem*, which discusses the behaviour of analytic functions near parabolic fixed points, we first define the notions of *petals*.

**Definition 2.0.1** (Petals). *Let  $p \in \mathbb{N}$ . For each  $k \in \{0, 1, \dots, p-1\}$ , define the sets as a function of a parameter  $t > 0$  as follows,*

$$\Pi_k(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)); |\theta - 2k\pi/p| < \pi/p\}.$$

*The sets  $\Pi_k(t)$  are known as Petals.*

We have shown a diagram of the petals  $\Pi_k(t)$  in [Figure 2.1](#) for  $p = 6$ . Note that all the petals are pairwise disjoint and each petal subtends an angle of  $2\pi/p$  at the origin.

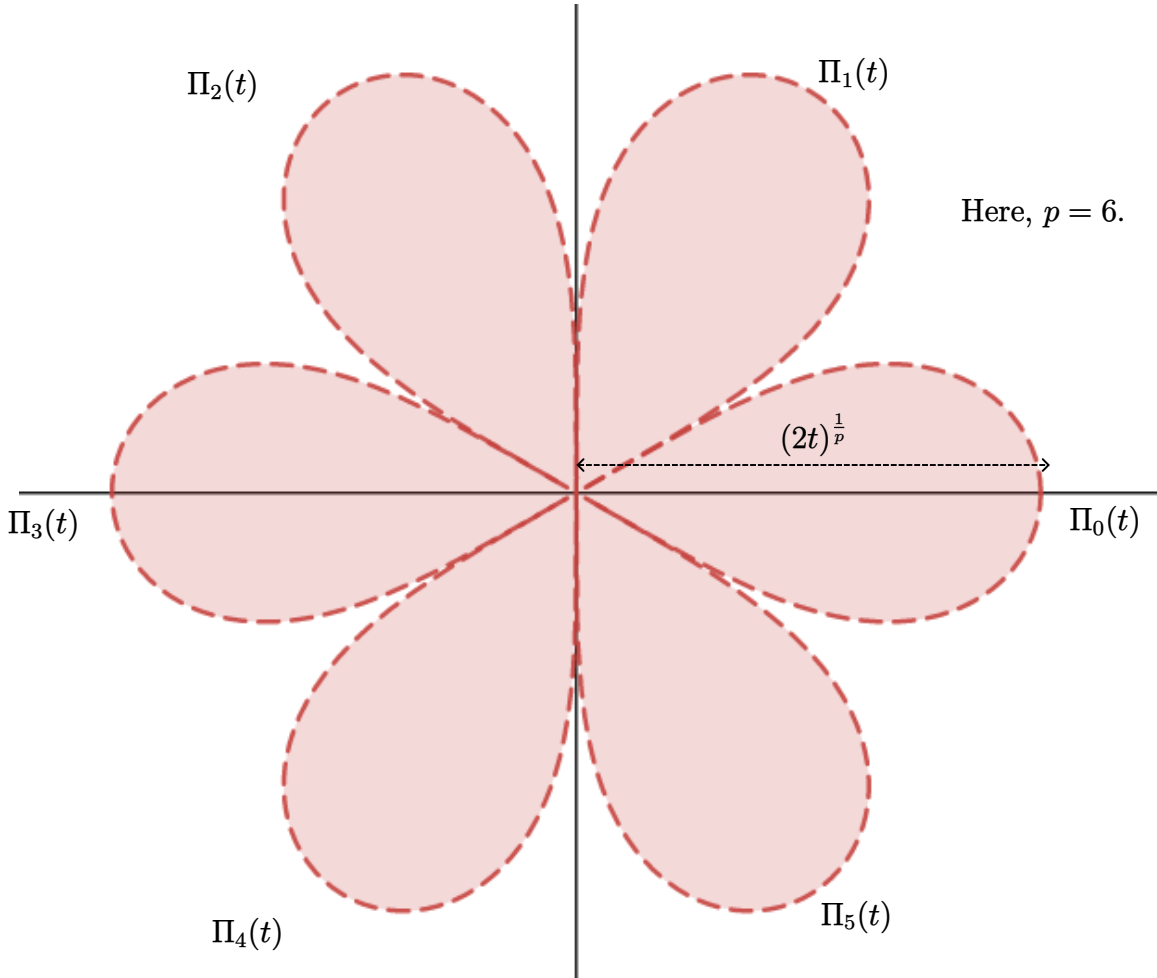


Figure 2.1: Six petals at the origin for  $p = 6$ .

**Theorem 2.0.2** (The Petal Theorem). *Suppose that an analytic map  $f$  has the form:*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1})$$

*near the origin. Then for sufficiently small  $t$ ,*

1.  $f$  maps each  $\Pi_k(t)$  into itself;
2.  $f^{\circ n}(z) \rightarrow 0$  uniformly on each petal;
3.  $\arg(f^{\circ n}(z)) \rightarrow 2k\pi/p$  locally uniformly on each petal;
4.  $f : \Pi_k(t) \rightarrow \Pi_k(t)$  is conjugate to a translation.
5.  $|f(z)| < |z|$  on a neighbourhood of the axis of each petal;

*Proof.* For  $0 < r_0 < 1$ , define the sector  $S_0$ ,

$$S_0 = \{re^{i\theta} : 0 < r < r_0, |\theta| < \pi/p\}$$

and the region  $W$ ,

$$W = \{re^{i\theta} : r > \frac{1}{r_0^p}, |\theta| > \pi\}.$$

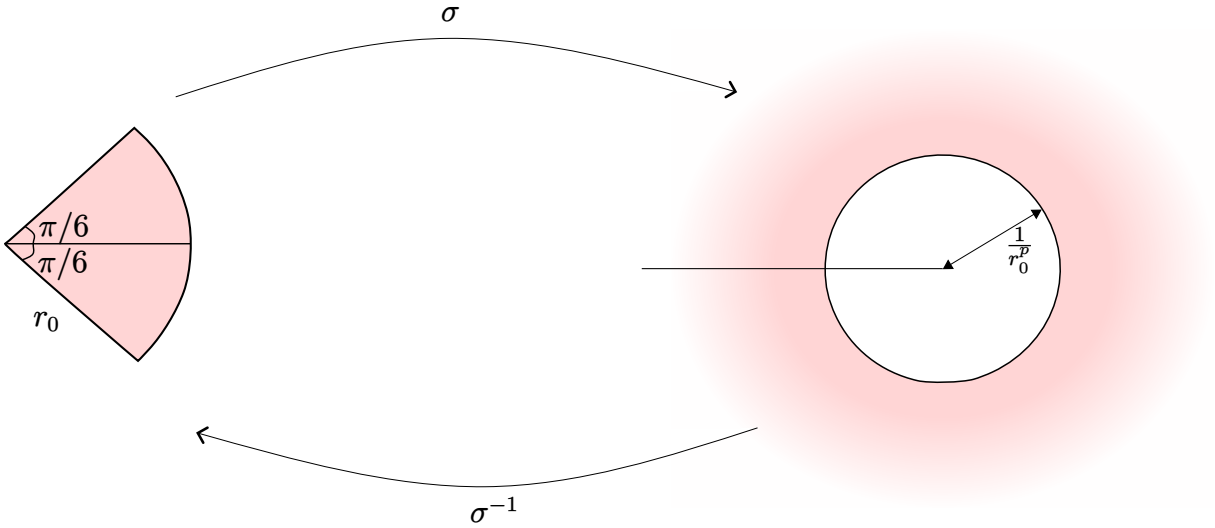


Figure 2.2:  $\sigma$  is a biholomorphism from  $S_0$  onto  $W$ .

It is clear that the map  $\sigma : z \mapsto \frac{1}{z^p}$  is a biholomorphism of  $S_0$  onto  $W$  with  $\sigma^{-1} : W \rightarrow S_0$  given by  $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$ . Actually,  $\sigma$  is a biholomorphism of each sector  $S_k = \{0 < r < r_0, |\theta - 2k\pi/p| < \pi/p\}$  onto  $W$ . The branch of  $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$  that we choose determines which sector the inverse map maps to.

Now, the conjugate map of  $f$  on  $W$  is given by,

$$g(w) = \sigma f \sigma^{-1}(w) = f(w^{-\frac{1}{p}})^{-p}.$$

This just replaces the action of  $f$  on  $S_0$  by  $g$  on  $W$ , and we have the following commutative diagram:

Now, we will use the power series expansion of  $f$  near the origin to get information about  $g$ .

First let us try to get a estimate of the power series expansion of  $f(z)^{-p}$ . We have,

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}) = z(1 - z^p + \mathcal{O}(z^{2p})) = z(1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + \dots).$$

So,

$$\frac{1}{f(z)^p} = z^{-p} \left( \frac{1}{1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + \dots} \right)^p.$$

Now, let  $\alpha(z) = z^p + a_0 z^{2p} + a_1 z^{2p+1} + \dots$ , then for  $r_0$  small enough such that  $|\alpha(z)| < 1$  on  $\{|z| < r_0\}$ , we can write,

$$\frac{1}{1 - \alpha(z)} = 1 + \alpha(z) + \alpha(z)^2 + \dots$$

Therefore,

$$\begin{aligned} \frac{1}{f(z)^p} &= z^{-p} (1 + \alpha(z) + \alpha(z)^2 + \dots)^p \\ &= z^{-p} (1 + pz^p + Az^{2p} + A_1 z^{2p+1} + \dots) \\ &= \frac{1}{z^p} + p + Az^p + v(z), \end{aligned}$$

where  $A$  is some constant and  $v(z)$  is holomorphic on  $\{|z| < r_0\}$ , and for some small  $r_0 > 0$ , it satisfies  $|v(z)| \leq B|z|^{p+1}$ ,  $B > 0$ .

Now, if  $w \in W$ , then  $\sigma^{-1}(w) \in S$ . Hence, by substituting  $z = \sigma^{-1}(w) = w^{-1/p}$ , we have,

$$\begin{aligned} g(w) &= \sigma f \sigma^{-1}(w) \\ &= \frac{1}{f(w^{-1/p})^p} \\ &= w + p + A/w + \theta(w), \end{aligned}$$

where  $|\theta(w)| = |v(w^{-1/p})| \leq B|w^{-1/p}|^{p+1} = B/|w|^{1+\frac{1}{p}}$ .

Hence, we have the following estimates for  $g$  which will be crucial in everything that will follow:

$$g(w) = w + p + A/w + \theta(w), \text{ where } A \text{ is a constant and} \quad (2.1)$$

$$|\theta(w)| \leq B/|w|^{1+\frac{1}{p}}, B > 0. \quad (2.2)$$

Choose any  $K$  satisfying

$$K > \max\{1/r_0^p, 3(|A| + B)\} > 1 \text{ (as } r_0 < 1)$$

and let,

$$\Pi = \{x + iy : y^2 > 4K(K - x)\}.$$

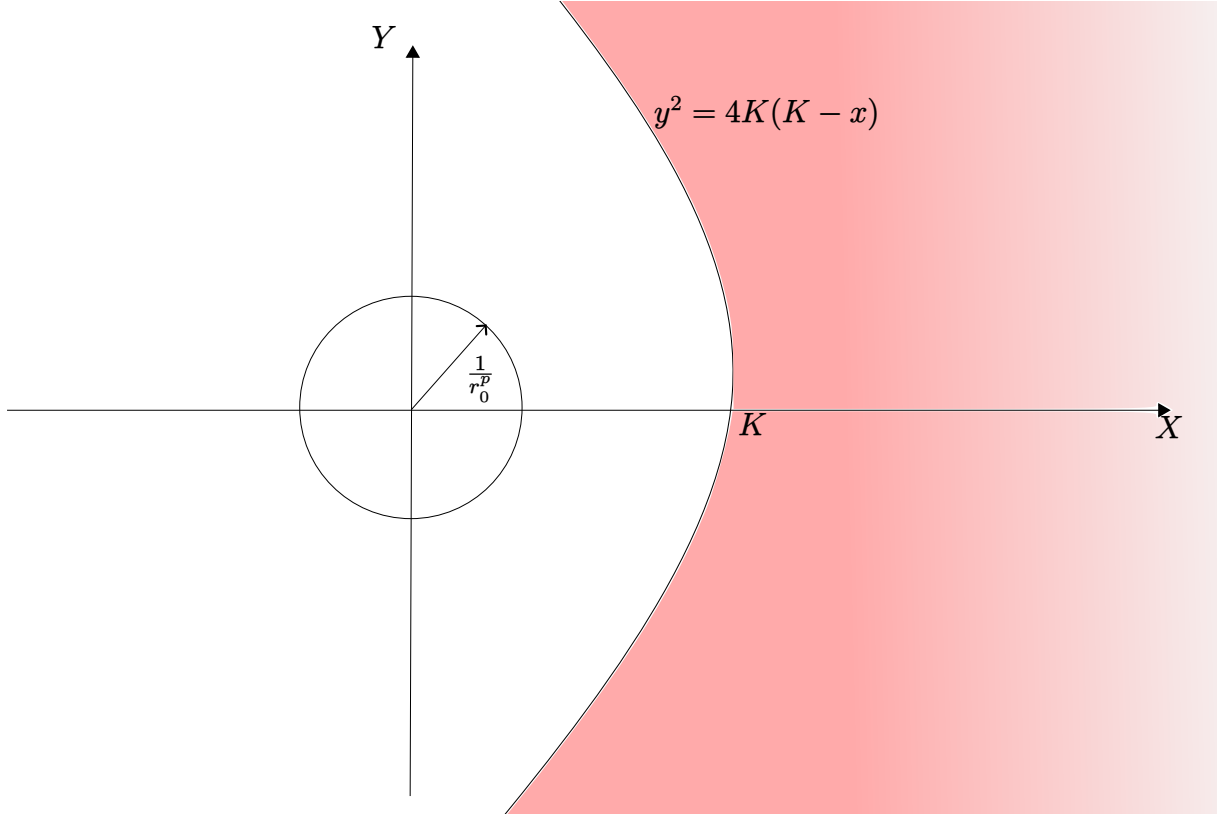


Figure 2.3:  $\Pi = \{(x, y) : y^2 > 4K(K - x)\}$ .

Clearly,  $\Pi$  is bounded by a parabola and  $\Pi \subset W$  (See [Figure 2.3](#)).

We have chosen this subset  $\Pi \subset W$  because we will show that  $\Pi$  is nothing but the conformal image of  $\Pi_0(t)$  under  $\sigma$  (for a suitable  $t$ ) and  $g$  satisfies all the corresponding conditions that  $f$  should satisfy on  $\Pi_0(t)$  according to the theorem.

**Claim.**  $\Pi$  is the conformal image of  $\Pi_0(t)$  under  $\sigma$  for a suitable  $t$ .

The easiest way to see this is using polar coordinates. We write,  $z = re^{i\theta}$  for  $z \in S_0$  and  $w = \rho e^{i\phi}$  for  $w \in W$ . Then,  $\rho = \frac{1}{r^p}$  and  $\phi = -p\theta$ .

Now, we need to express  $\Pi$  in polar co-ordinates. To do so, we notice that points on the parabola are given by

$$\rho \text{ (distance from focus i.e. } 0) = 2K - \rho \cos \phi \text{ (distance from directrix i.e. } y = 2K).$$

(See [Figure 2.4](#)). Therefore, points on  $\Pi$  are given by

$$\rho > 2K - \rho \cos \phi.$$

Hence,

$$\Pi = \{\rho e^{i\phi} : 2K < \rho(1 + \cos \phi)\}.$$



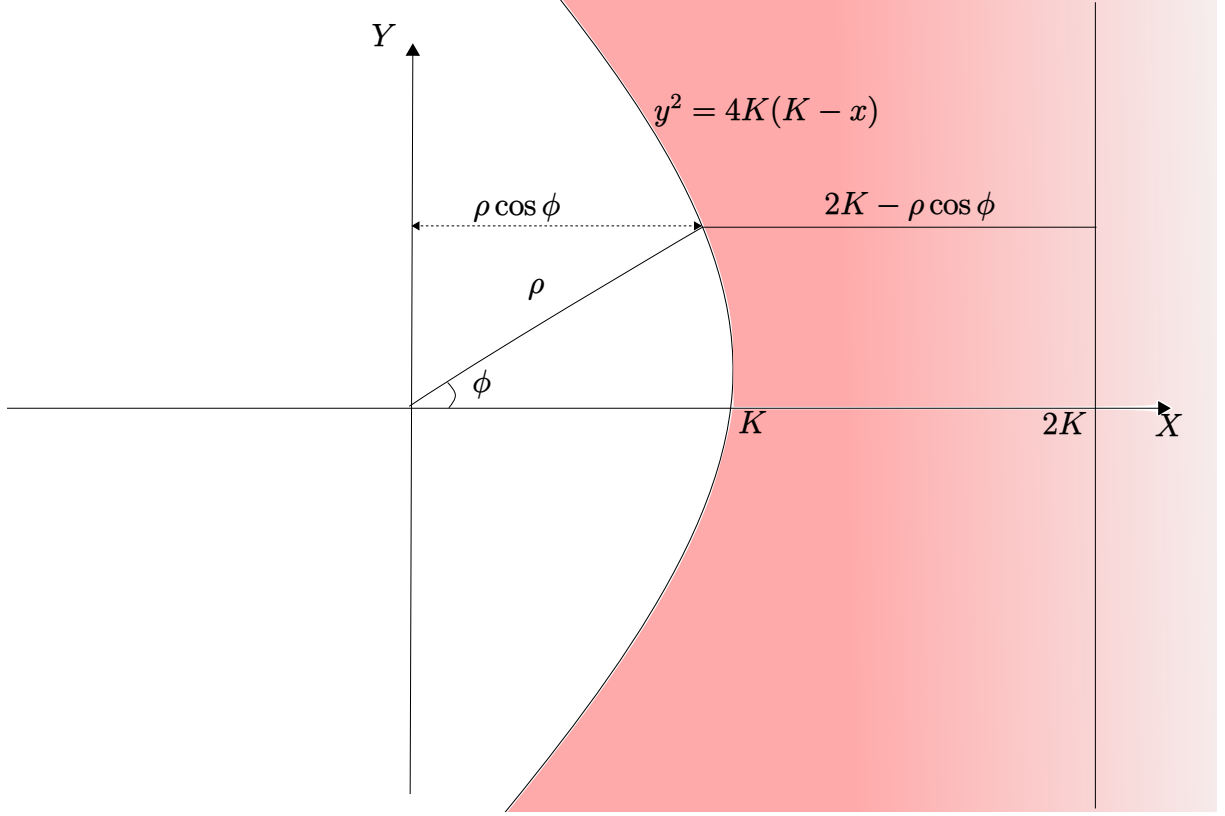


Figure 2.4:  $\Pi = \{\rho e^{i\phi} : \rho > 2K - \rho \cos \phi\}$ .

Now, let  $\Omega = \sigma^{-1}(\Pi)$ . Then,  $\Omega$  is given by

$$\Omega = \{re^{i\theta} : 2Kr^p < 1 + \cos(p\theta)\}.$$

Hence,  $\Omega = \Pi_0\left(\frac{1}{2K}\right)$ .

**Lemma 2.0.3.**  *$g$  satisfies the following properties on  $\Pi$ :*

1.  $\Pi$  is forward invariant under  $g$ .
2.  $g^{\circ n}(w) \rightarrow \infty$  uniformly on  $\Pi$ .
3.  $\arg(g^{\circ n}(w)) \rightarrow 0$  locally uniformly on  $\Pi$ .
4.  $g : \Pi \rightarrow \Pi$  is conjugate to a translation.

*Proof.*

1. We write,

$$w = x + iy, \quad g(w) = X + iY, \quad A/w + \theta(w) = a + ib.$$

From Equation (2.1), we obtain,

$$\begin{aligned} X + iY &= (x + iy) + p + (a + ib) \\ \implies X &= x + p + a \text{ and } Y = y + b. \end{aligned}$$

Now, if  $w \in \Pi$ ,

$$\begin{aligned} Y^2 - 4K(K - X) &= (y + b)^2 - 4K(K - x - p - a) \\ &= [y^2 - 4K(K - x)] + b^2 + 2yb + 4K(a + p) \\ &> 4Kp + (2yb + 4Ka). \end{aligned}$$

Now, for  $w \in \Pi$ ,  $|w| > K > 1$ . (It is clear for  $\operatorname{Re}(w) > K$ . For  $\operatorname{Re}(w) \leq K$ , we use the polar description  $\rho > 2K - \rho \cos \phi$  to get  $|w| > 2K - \operatorname{Re}(w) \geq K$ ). Hence we get,

$$|w||A/w + \theta(w)| \leq |w|(|A|/|w| + B/|w|^{1+\frac{1}{p}}) = |A| + B/|w|^{\frac{1}{p}} < |A| + B \quad (2.3)$$

(since for  $|w| > 1$ ,  $|w|^{\frac{1}{p}} > 1$ ). Therefore,

$$\begin{aligned} |2yb + 4Ka| &\leq 2|yb| + 4K|a| \\ &\leq 2|y||b| + 4K|a| \\ &\leq 2|w||a + ib| + 4|w||a + ib| \\ &= 6|w||a + ib| \\ &< 2K \leq 2Kp. \end{aligned}$$

Therefore, we see that  $Y^2 - 4K(K - X) > 0$  and hence,  $g(w) \in \Pi$  for  $w \in \Pi$ . Hence,  $\Pi$  is forward invariant under  $g$ .

**2.** Now, we will prove a stronger statement that for any  $t > 0$   $g$  maps  $\Pi + t$  into  $\Pi + t + p/2$ . This is simply because, for  $w \in \Pi + t$ , we have,  $y^2 - 4K(K + t - x) > 0$ . Hence,

$$\begin{aligned} Y^2 - 4K(K + t + p/2 - X) &= [y^2 - 4K(K + t - x)] + b^2 + 2yb + 4K(a - p/2) \\ &> 2Kp + (2yb + 4Ka) \\ &> 0. \end{aligned}$$

Therefore, if  $w \in \Pi$ ,  $g^{on}(w) \in \Pi + np/2$ . Hence,  $|g^{on}(w)| > \sqrt{n}$ . This is simply because, if  $x + iy \in \Pi + np/2$ , we have

$$x^2 + y^2 - n > x^2 + 4K(K + np/2 - x) - n = x^2 - 4Kx + (4K^2 + 2npK - n).$$

The discriminant of this quadratic equation in  $x$  is

$$16K^2 - 4(4K^2 + 2npK - n) = 4n(1 - 2pK) < 0$$

. Thus,  $x^2 + y^2 - n > 0$  for all  $x + iy \in \Pi + np/2$ .

Hence,  $g^{on}(w) \rightarrow \infty$  uniformly on  $\Pi$ .

3. Firstly, we see that,

$$g^{\circ k}(w) = g(g^{\circ k}(w)) = g^{\circ k}(w) + p + A/g^{\circ k}(w) + \theta(g^{\circ k}(w)).$$

By expanding the first term again and again, we obtain,

$$g^{\circ n}(w) = w + np + \sum_{k=0}^{n-1} \left( \frac{A}{g^{\circ k}(w)} + \theta(g^{\circ k}(w)) \right). \quad (2.4)$$

Also note that from [Equation \(2.3\)](#), we have,

$$|A/w + \theta(w)| < (|A| + B)/K < \frac{1}{3}.$$

Let  $Q$  be a compact subset of  $\Pi$ . From now, we will assume that  $w \in Q$  and we will use  $C_1, C_2, C_3, \dots$  to denote positive constants which will be dependent on  $Q$ .

Hence,

$$\begin{aligned} |g(w)| &= |w + p + A/w + \theta(w)| \geq ||w + p| - |A/w + \theta(w)|| \\ &= |w + p| - |A/w + \theta(w)| \\ &\geq |w| + p - \frac{1}{3}. \end{aligned}$$

Therefore, we obtain,

$$|g^{\circ n}(w)| \geq |w| + n(p - 1/3) \geq C_1 + C_2 n.$$

(Here,  $C_1 = \min\{|w| : w \in Q\} > 0$  and  $C_2 = p - \frac{1}{3} > 0$ .)

Hence,

$$|g^{\circ n}(w)| \geq C_2 n. \quad (2.5)$$

Next, with [Equation \(2.2\)](#), and the above inequality, we get,

$$|\theta(g^{\circ n}(w))| \leq B/|g^{\circ n}(w)|^{1+\frac{1}{p}} \leq C_3/n^{1+\frac{1}{p}}. \quad (2.6)$$

Finally, combining the above two inequalities and [Equation \(2.4\)](#), we obtain,

$$\begin{aligned} |g^{\circ n}(w) - np| &\leq |w| + |A/w + \theta(w)| + \frac{|A|}{C_2} \sum_{k=1}^{n-1} \frac{1}{k} + C_3 \sum_{k=1}^{n-1} \frac{1}{n^{1+\frac{1}{p}}} \\ &< C_4 + C_5 \sum_{k=1}^n \frac{1}{k}. \end{aligned}$$

(Here,  $C_4 = \max\{|w| : w \in Q\} + \frac{1}{3} + C_3 \sum_{n=1}^{\infty} 1/n^{1+\frac{1}{p}}$  and  $C_5 = |A|/C_2$ .)

We can select  $C_6$  large enough such that

$$|g^{\circ n}(w) - np| < C_6 \log n. \quad (2.7)$$

*Remark.* The above inequality follows from the fact that, if  $H_n = \sum_{k=1}^n \frac{1}{k}$ , then  $H_n - \log n \rightarrow \gamma$ . ( $\gamma$  is known as the Euler's constant). So, we have that

$$\begin{aligned}
P + QH_n &= P + Q(\log n + \gamma + \epsilon_n), \text{ where } \epsilon_n \rightarrow 0 \\
&\leq Q \log n + (P + Q \max\{\epsilon_n\} + Q\gamma) \\
&= Q \log n + R \\
&< S \log n
\end{aligned}$$

for  $S$  large enough.

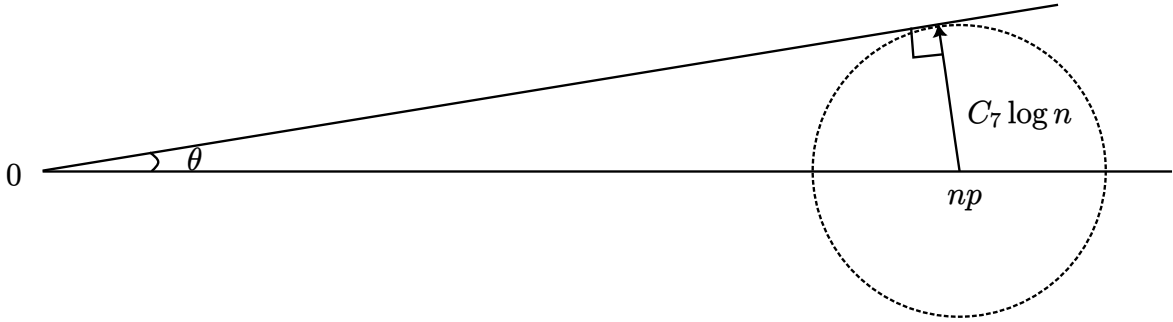


Figure 2.5:  $|\arg(g^{\circ n}(w))| \leq \sin^{-1}(\frac{C_6 \log n}{np})$ .

From,  $|g^{\circ n}(w) - np| < C_6 \log n$ , it follows that  $|\arg(g^{\circ n}(w))| < \sin^{-1}(\frac{C_6 \log n}{np})$  for  $n$  large enough. Hence,  $\arg(g^{\circ n}(w)) \rightarrow 0$  uniformly on  $Q$ , and consequently, locally uniformly on  $\Pi$ .

4. Define,

$$u_n(w) = g^{\circ n}(w) - np - (A/p) \log n.$$

**Claim.**  $u_n(w)$  converges locally uniformly on  $\Pi$  to a holomorphic function  $u$ , that is one-to-one on  $\Pi$ .

$$u_{n+1}(w) - u_n(w) = [g^{\circ n+1}(w) - g^{\circ n}(w)] - p - (A/p) \log \left( \frac{n+1}{n} \right).$$

From [Equation \(2.2\)](#), we obtain,

$$\begin{aligned}
u_{n+1}(w) - u_n(w) &= [g^{\circ n}(w) + p + A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - g^{\circ n}(w)] \\
&\quad - p - (A/p) \log(1 + 1/n) \\
&= A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - (A/p) \log(1 + 1/n) \\
&= A(1/g^{\circ n}(w) - 1/np) + \theta(g^{\circ n}(w)) + (A/p)(1/n - \log(1 + 1/n)).
\end{aligned}$$

Now, let  $Q$  is a compact subset of  $\Pi$  and  $w \in Q$ . We need to prove that  $u_n$  converges uniformly in  $Q$ . From the above equation, to prove that  $u_n$  converges uniformly in  $Q$ , we need to show that each of the following series converges uniformly in  $Q$ :

$$\sum_n |1/g^{\circ n}(w) - 1/np|, \sum_n |\theta(g^{\circ n}(w))|, \sum_n |1/n - \log(1 + 1/n)|.$$

Let us look at the first series. We have, (using [Equations \(2.5\)](#) and [\(2.7\)](#))

$$|1/g^{\circ n}(w) - 1/np| = \frac{|g^{\circ n}(w) - np|}{|g^{\circ n}(w)|np} \leq \frac{C_6 \log n}{C_2 n^2 p} = C_7 \log n/n^2.$$

(Here  $C_7 = C_6/(pC_2)$ ).

From [Equation \(2.6\)](#), it is clear that  $\sum_n |\theta(g^{\circ n}(w))|$  converges.

Now,  $0 < x - \log(1 + x) \leq x^2$  for  $x > 0$ .

This is because, it is zero at  $x = 0$  and  $\frac{d}{dx}(x - \log(1 + x)) = 1 - \frac{1}{1+x} > 0$  for  $x > 0$ .

Also,  $x^2 - x + \log(1 + x)$  is zero at  $x = 0$  and  $\frac{d}{dx}(x^2 - x + \log(1 + x)) = 2x - 1 + \frac{1}{1+x} > 0$  for  $x > 0$ .

Putting  $x = \frac{1}{n}$ , we get,

$$|1/n - \log(1 + 1/n)| < 1/n^2.$$

Therefore,  $u_n$  converges locally uniformly to some holomorphic function  $u$  on  $\Pi$ .

Now, from  $u_n(w) = g^{\circ n}(w) - np - (A/p) \log n$ , we get that,

$$\begin{aligned} (n+1)p + (A/p) \log(n+1) + u_{n+1}(w) &= g^{\circ n+1}(w) \\ &= g^{\circ n}(g(w)) \\ &= np + (A/p) \log n + u_n(g(w)) \\ \implies p + (A/p) \log(1 + 1/n) + u_{n+1}(w) &= u_n(g(w)). \end{aligned}$$

Taking limit  $n \rightarrow \infty$ , we get,

$$p + u(w) = u(g(w)).$$

Since  $f$  is injective near the origin,  $g$  is injective on  $\Pi$ , (if  $K$  is chosen large enough). Therefore,  $g^{\circ n}$  is injective on  $\Pi$  and hence, so is  $u_n$ . By Hurwitz Theorem,  $u$  is either injective or constant, but it is clearly not a constant since it satisfies the above equation.

This shows that  $g : \Pi \rightarrow \Pi$  is conjugate to the map  $z \mapsto z + p$  of  $u(\Pi)$  into itself.  $\square$

Coming back to our original theorem, we see that our original theorem is also proved as we had just replaced the action of  $f$  on  $\Pi_0$  by the action of its conjugate  $g$  on  $\Pi$  and we just proved all the parts of the theorem that the conjugate of  $f$ , i.e.  $g$  must satisfy.

From,  $g = \sigma f \sigma^{-1}$ , we get,  $g^{\circ n} = \sigma f^{\circ n} \sigma^{-1} \implies g^{\circ n} \sigma = \sigma f^{\circ n}$ . Writing,  $\sigma(z) = w$ , we have,

$$g^{\circ n}(w) = \frac{1}{f^{\circ n}(z)^p} \implies g^{\circ n}(w)(f^{\circ n}(z))^p = 1. \quad (2.8)$$

1. Since,  $g$  maps  $\Pi$  into itself,  $f$  maps  $\Pi_0$  into itself.
2. Now, since  $|g^{on}(w)| > \sqrt{n}$ ,  $|f^{on}(z)| < \frac{1}{n^{1/2p}}$  from Equation (2.8). Hence,  $f^{on}(z) \rightarrow 0$  uniformly on  $\Pi_0$ .
3. Also,  $\arg(f^{on}(z)) = \left(-\frac{1}{p}\right) \arg(g^{on}(w))$  from Equation (2.8). Since,  $\arg(g^{on}(w)) \rightarrow 0$  locally uniformly on  $\Pi$ ,  $f^{on}(z) = \left(-\frac{1}{p}\right) \arg(g^{on}(w)) \rightarrow 0$  locally uniformly on  $\Pi_0$ .
4. Since,  $g : \Pi \rightarrow \Pi$  is conjugate to a translation, and  $g$  is conjugate to  $f$ ,  $f : \Pi_0 \rightarrow \Pi_0$  is also conjugate to a translation.
5. It is immediate from Lemma 2.0.1 that  $|f(z)| < |z|$  on the axis of  $\Pi_0$ .

□

**Theorem 2.0.4.** Suppose that  $f$  has the power series expansion near 0 as,

$$f(z) = z + az^{p+1} + \mathcal{O}(z^{p+2}).$$

Then,  $f$  is conjugate near 0 to a function

$$F(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}).$$

*Proof.* First, we conjugate  $f$  by the map  $z \mapsto \lambda z$ , where  $\lambda^p = a$ . Then, we get that  $f$  is conjugate to the map,

$$\tilde{f} = \lambda f(z/\lambda) = \lambda z/\lambda + \lambda a z^{p+1}/\lambda^{p+1} + \mathcal{O}(z^{p+2}) = z + z^{p+1} + \mathcal{O}(z^{p+2}).$$

We will now proceed via induction over a finite number of steps. Let,

$$f_k(z) = z + z^{p+1} + bz^{p+k+1} + \dots, b \neq 0.$$

Here  $k \geq 1$ . Also if  $k \geq p$ , our theorem is proved. Hence, we assume,  $1 \leq k < p$ .

Now, define the polynomial,

$$\sigma(z) = z + \alpha z^{k+1},$$

where  $\alpha = \frac{b}{p-k}$  and let  $\sigma^{-1}$  be its inverse near 0 (We can do this because  $\sigma'(0) = 1$ ).

Now, we will show that we obtain  $f_r$  (for some  $r \geq k+1$ ) by conjugating  $f_k$  with  $\sigma$ . Hence, let

$$g = \sigma f_k \sigma^{-1}$$

and we need to show that  $g = f_r$  (for some  $r \geq k+1$ ). Since,  $g'(0) = f'_k(0) = 1$ , we let,

$$g(z) = z + \sum_{m=2}^{\infty} a_m z^m.$$

Now, we will use the identity,  $g\sigma = \sigma f_k$ .

$$\begin{aligned}\sigma f_k(z) &= (z + z^{p+1} + bz^{p+k+1} + \dots) + \alpha(z + z^{p+1} + bz^{p+k+1})^{k+1} \\ &= z + \alpha z^{k+1} + z^{p+1} + (b + \alpha(k+1))z^{p+k+1} + \mathcal{O}(z^{p+k+2}) \\ &= z + \alpha z^{k+1} + z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

The last equality follows because,

$$\alpha(p-k) = b \implies \alpha(p+1) - \alpha(k+1) = b \implies \alpha(p+1) = b + \alpha(k+1).$$

Now,

$$\begin{aligned}g\sigma(z) &= (z + \alpha z^{k+1}) + \sum_{m=2}^{\infty} a_m (z + \alpha z^{k+1})^m \\ &= z + \alpha z^{k+1} + \sum_{m=2}^{p+k+1} a_m (z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

Now, equating  $\sigma f_k(z) = g\sigma(z)$ , we get,

$$z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) = \sum_{m=2}^{p+k+1} a_m (z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}).$$

Firstly, we see that on the right hand side, the coefficient of  $z^2$  will be  $a_2$ , the coefficient of  $z^3$  will be some linear combination of  $a_2$  and  $a_3$ , the coefficient of  $z^4$  will be some linear combination of  $a_2, a_3$  and  $a_4$  and so on upto the coefficient of  $z^p$  will be some linear combination of  $a_2, a_3, \dots, a_p$ . Since, the coefficient of  $z^2, \dots, z^p$  is zero on the left hand side, it follows that  $a_2 = a_3 = \dots, a_p = 0$ . (This argument follows assuming  $p \geq 2$ , but if  $p = 1$  the coefficient of  $z^p = z$  i.e.  $a_1$  is automatically 0).

Hence, now we have,

$$\begin{aligned}z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) &= \sum_{m=p+1}^{p+k+1} a_m (z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}) \\ &= a_{p+1}z^{p+1} + \dots + a_{p+k+1}z^{p+k+1} + \\ &\quad a_{p+1}\alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

Therefore, we obtain

$$a_{p+1} = 1, a_{p+2} = \dots = a_{p+k} = 0 \text{ and } a_{p+k+1} + a_{p+1}\alpha(p+1) = \alpha(p+1).$$

Hence,  $a_{p+k+1} = 0$ . This gives that  $f_k$  is conjugate to the map

$$g(z) = z + z^{p+1} + \mathcal{O}(z^{p+k+2}).$$

Thus,  $g = f_r$  for some  $r \geq k+1$ . Continuing the induction process, we get that  $f$  is conjugate near 0 to a map

$$z \mapsto z + z^{p+1} + \mathcal{O}(z^{2p+1}).$$

Now, we can again conjugate this map with the map,  $z \mapsto \lambda z$ , where  $\lambda^p = -1$  to get that  $f$  is conjugate to a map,

$$F(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}).$$

□

At the end we consider the most general situation, i.e. when  $R$  is a rational function and

$$R(z) = az + bz^{p+1} + \dots, \text{ where } a \neq 1 \text{ but } a = e^{2\pi ip/q}.$$

In this case,  $R^{\circ q}(z)$  is of the form,

$$R^{\circ q}(z) = z + cz^{r+1} + \dots, \text{ for some } r \in \mathbb{N}.$$

Then  $R^{\circ q}$  has  $r$  petals at the origin. Now,  $F(R^{\circ q}) = F(R)$ .



# Chapter 3

## Bottcher's Theorem and its extension

### 3.1 Bottcher's Coordinates

A fixed point  $p$  is called a super-attracting fixed point of  $f$  if  $f'(p) = 0$ .

If  $p$  is a super-attracting fixed point for  $f$ , we can conjugate the map such that  $z = 0$  becomes our super-attracting fixed point.

Thus, our map takes the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$$

in a neighbourhood of 0, with  $n \geq 2$  and  $a_n \neq 0$ . Here the integer  $n$  is called the *local degree* of  $f$  at 0.

**Theorem 3.1.1 (Bottcher's Theorem).** *With  $f$  as above,  $\exists$  a local holomorphic change of coordinates  $w = \phi(z)$ , with  $\phi(0) = 0$ , which conjugates  $f$  to  $w \mapsto w^n$  throughout some neighbourhood of 0.*

*Furthermore,  $\phi$  is unique upto multiplication by an  $(n-1)$ th root of unity.*

**Proof. Existence.** Let  $c \in \mathbb{C}$  be such that  $c^{n-1} = a_n$ . Then, the linearly conjugate map  $cf(z/c)$  will have leading coefficient  $+1$ . Thus, without loss of generality, we will assume that our map  $f$  has the form

$$f(z) = z^n(1 + b_1 z + b_2 z^2 + \dots) = z^n(1 + \eta(z)), \text{ where } \eta(z) = (1 + b_1 z + b_2 z^2 + \dots).$$

Choose  $r \in (0, \frac{1}{2})$  such that  $|\eta(z)| < \frac{1}{2} \forall z \in \mathbb{D}_r$ . This can be done since  $\eta(0) = 0$  and  $\eta$  is continuous.

On this disc, we have two properties of  $f$  :

1.  $f$  maps this disc into itself: We have,  $|f(z)| = |z|^n |1 + \eta(z)| \leq |z|^n (1 + |\eta(z)|) < \frac{3}{2} |z|^n \leq \frac{3}{2^n} |z| \leq \frac{3}{4} |z| \forall z \in \mathbb{D}_r$ . Here we are using the fact that  $n \geq 2, |z| < \frac{1}{2}$  and  $|\eta(z)| < \frac{1}{2}$  on  $\mathbb{D}_r$ .
2.  $f(z) \neq 0 \forall z \in \mathbb{D}_r \setminus \{0\}$ . This is simply because  $|f(z)| = |z|^n |1 + \eta(z)|$  and since  $|\eta(z)| < \frac{1}{2}$  on  $\mathbb{D}_r$ , we can't have  $\eta(z) = -1$ .

The  $k$ -th iterate of  $f$  i.e.  $f^{\circ k}$  also maps the  $\mathbb{D}_r$  into itself and  $f^{\circ k}(z) \neq 0$  on  $\mathbb{D}_r \setminus \{0\}$ . Inductively, it can be shown that it has the form  $f^{\circ k}(z) = z^{n^k} (1 + n^{k-1} b_1 z + \dots)$ .

The idea of the proof is to set,

$$\phi_k(z) = \left(f^{\circ k}(z)\right)^{\frac{1}{n^k}} = z \left(1 + n^{k-1} b_1 z + \dots\right)^{\frac{1}{n^k}}$$

We choose  $z$  as our branch of holomorphic  $n^k$  th root of  $z^{n^k}$ .

Now, we can choose a holomorphic branch of  $\left(1 + n^{k-1} b_1 z + \dots\right)^{\frac{1}{n^k}}$  on  $\mathbb{D}_r$  since  $\mathbb{D}_r$  is simply connected and  $\left(1 + n^{k-1} b_1 z + \dots\right) \neq 0$  on  $\mathbb{D}_r$  since  $f^{\circ k}(z) \neq 0$  on  $\mathbb{D}_r \setminus \{0\}$ . Therefore we set,

$$\phi_k(z) = z \left(1 + n^{k-1} b_1 z + \dots\right)^{\frac{1}{n^k}} = z \left(1 + \frac{b_1}{n} z + \dots\right)$$

where the expression on the right provides us an explicit choice of  $n^k$  th root.

We will show that the functions  $\phi_k$  converge uniformly to a limit function  $\phi$  on  $\mathbb{D}_r$ . To prove the convergence, we make the substitution  $z = e^u$  where  $u$  ranges over the left half plane  $\mathbb{H}_r := \{u : \operatorname{Re}(u) < \log r\}$ . The exponential map maps  $\mathbb{H}_r$  onto  $\mathbb{D}_r \setminus \{0\}$ .

The map  $f$  from  $\mathbb{D}_r$  into itself corresponds to a map from  $\mathbb{H}_r$  into itself given by  $F(u) = \log f(e^u)$ . We can select a holomorphic branch of the logarithm of  $f(e^u)$  because  $\mathbb{H}_r$  is simply connected and  $f(e^u) \neq 0$  on  $\mathbb{H}_r$ .

Set  $\eta = \eta(e^u) = b_1 e^u + b_2 e^{2u} + \dots$ , then since  $|\eta| < \frac{1}{2}$ , we see that  $F$  can be written as

$$F(u) = \log(e^{nu}(1 + \eta)) = nu + \log(1 + \eta) = nu + \left(\eta - \frac{\eta^2}{2} + \frac{\eta^3}{3} - \dots\right)$$

where the final expression provides us an explicit choice of which branch of logarithm we are using. Clearly,  $F : \mathbb{H}_r \rightarrow \mathbb{H}_r$  is a well-defined holomorphic function.

Similarly, the map  $\phi_k$  corresponds to a map,  $\Phi_k(u) = \log \phi_k(e^u)$ .

$$\Phi_k(u) = \log \phi_k(e^u) = \log f^{\circ k}(e^u)^{\frac{1}{n^k}} = \frac{1}{n^k} \log f^{\circ k}(e^u).$$

Since we have already fixed the branch of logarithm that we are using, we see that,

$$\log f^{\circ k}(e^u) = \log f(f^{\circ k-1}(e^u)) = \log f(e^{\log f^{\circ k-1}(e^u)}) = F(\log f^{\circ k-1}(e^u))$$

Hence, inductively we can see that  $\log f^{\circ k}(e^u) = F^{\circ k}(u)$ .

Therefore,  $\Phi_k(u) = F^{\circ k}(u)/n^k$ . It is clear from this expression that  $\Phi_k : \mathbb{H}_r \rightarrow \mathbb{H}$ .

Now since  $|\eta| < \frac{1}{2}$ , we have

$$|F(u) - nu| = |\log(1 + \eta)| < \log 2 < 1$$

Hence,

$$|\Phi_{k+1}(u) - \Phi_k(u)| = \frac{1}{n^{k+1}} |F^{\circ k+1}(u) - nF^{\circ k}(u)| < \frac{1}{n^{k+1}}$$

by the above inequality.

We have,  $\phi_k(e^u) = e^{\Phi_k(u)}$ . Since, the exponential map,  $e^\square : \mathbb{H} \rightarrow \mathbb{D}$  from the left half plane to the unit disc decreases distance, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r \setminus \{0\}.$$

Since  $\phi_k(0) = 0$  for all  $k$ , we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r$$

Hence, the maps  $\phi_k$  converge uniformly to some limit function  $\phi$  on  $\mathbb{D}_r$  by the Cauchy criterion for uniform convergence.

Clearly,  $\phi(0) = 0$  and  $\phi$  is holomorphic on  $\mathbb{D}_r$  by Weierstrass convergence theorem.

It is clear that each  $\phi_k : \mathbb{D}_r \rightarrow \mathbb{D}$ . This is because  $\phi_k(e^u) = e^{\Phi_k(u)}$  and  $\Phi_k : \mathbb{H}_r \rightarrow \mathbb{H}$  and  $e^\square : \mathbb{H} \rightarrow \mathbb{D} \setminus \{0\}$ . Hence,  $\phi : \mathbb{D}_r \rightarrow \mathbb{D}$ . (Clearly  $\text{Im}(\phi)$  cannot contain points from  $\partial\mathbb{D}$  because  $\phi$  is holomorphic, hence it is an open map).

Now, it can be easily seen that,  $\phi_k(f(z)) = \phi_{k+1}(z)^n$ .

Hence,  $\lim_{k \rightarrow \infty} \phi_k(f(z)) = \lim_{k \rightarrow \infty} \phi_{k+1}(z)^n \implies \phi(f(z)) = \phi(z)^n$  by continuity of  $n$ th power map.

Also, since  $\phi'_k(0) = 1 \forall k \in \mathbb{N}$  (from the power series of  $\phi_k$ ), we have  $\phi'(0) = 1$ . Hence,  $\phi$  is invertible in some neighbourhood of 0.

Therefore, we have a holomorphic change of coordinates in some neighbourhood of 0 which conjugates  $f$  to the  $n$ th power map. In this neighbourhood,  $\phi$  is one-to-one,  $f(z) \neq 0$  for  $z \neq 0$  (i.e. no other point maps to the super-attracting fixed point via  $f$ ) and  $f$  maps this neighbourhood into itself.

**Uniqueness.** It suffices to study the special case  $f(z) = z^n$ . If we can prove that any map which conjugates  $z \mapsto z^n$  to itself is just multiplication by  $(n-1)$ th root of unity, then for any general map  $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$ , if we have two maps  $\phi$  and  $\psi$  which conjugate it to  $z \mapsto z^n$ , then  $\phi \circ \psi^{-1}$  is a map which conjugates  $z \mapsto z^n$  to itself.

Hence,  $\phi \circ \psi^{-1} = cz$ , where  $c^{n-1} = 1$ . Therefore,  $\phi = c\psi$ , where  $c$  is a  $(n-1)$  th root of unity.

So, let  $\phi(z) = c_1z + c_kz^k + \dots, (c_1 \neq 0)$  be a map which conjugates  $z \mapsto z^n$  to itself. Then, we should have  $\phi(z^n) = \phi(z)^n$ . Now,

$$\phi(z^n) = c_1z^n + c_kz^{nk} + \dots$$

and

$$\phi(z)^n = c_1^n z^n + nc_1^{n-1}c_k z^{n+k-1} + \dots$$

Comparing coefficients, we get  $c_1^n = c_1$  and  $nc_1^{n-1}c_k = 0$  since  $nk > n+k-1$  for  $k \geq 2$ . Therefore, we get  $c_1^{n-1} = 1$  and  $c_k = 0$ . The form  $\phi(z) = c_1z + c_kz^k + \dots$  can be modified to any  $k \geq 2$  to get  $c_k = 0$  by the same process.

Therefore,  $\phi(z) = cz$ , where  $c$  is a  $(n-1)$  th root of unity. □

## 3.2 Extension of Bottcher's coordinates

We might hope to extend the Bottcher's coordinates to the whole of the immediate basin of attraction of the super-attracting fixed point. But this is not always possible. This is because, it requires computing expressions of the form  $z \mapsto \left(\phi(f^{\circ k}(z))\right)^{\frac{1}{n^k}}$ , which is not always possible. For example, the basin may not be simply connected, or some point may map directly onto the super-attracting fixed point, in which case we can't take  $n^k$ -th roots, because  $\phi(f^{\circ k}(z))$  must be zero at those points.

**Theorem 3.2.1 (Extension of  $|\phi|$ ).** *If  $f$  has a super-attracting fixed point  $p$ , with immediate basin of attraction  $\mathcal{A}$ , then the function  $z \mapsto |\phi(z)|$  of the above theorem extends uniquely to a continuous map  $|\phi| : \mathcal{A} \rightarrow [0, 1)$  which satisfies  $|\phi|(f(z)) = |\phi|(z)^n$ .*

*Furthermore,  $|\phi|$  is real analytic except at the iterated preimages of  $p$ , where it takes the value 0.*

*Proof.* Set  $|\phi|(z) = \left|\phi(f^{\circ k}(z))\right|^{\frac{1}{n^k}}$  for large enough  $k$  for each  $z \in \mathcal{A}$ .  $\phi$  is only defined in a some small neighbourhood of  $p$ . But since,  $f^{\circ k} \rightarrow p$  locally uniformly in  $\mathcal{A}$ , after  $k$  many iterates for some large  $k$ ,  $f^{\circ k}(z)$  belongs to the domain of definition of  $\phi$ , which we shall call  $\hat{U}$ .

It is independent of the value of  $k$  (if  $k$  is large enough). Note that, if  $f^{\circ k}(z) \in \hat{U}$ , then so does  $f^{\circ k+1}(z)$ , since  $f$  maps  $\hat{U}$  into itself.

Suppose we choose  $k$  minimal such that  $f^{\circ k}(z) \in \hat{U}$ . Then,

$$|\phi(f^{\circ k+1}(z))|^{\frac{1}{n^{k+1}}} = |\phi(f(f^{\circ k}(z)))|^{\frac{1}{n^{k+1}}} = |\phi(f^{\circ k}(z))^n|^{\frac{1}{n^{k+1}}} = |\phi(f^{\circ k}(z))|^{\frac{1}{n^k}} = |\phi|(z).$$

In the proof of the Bottcher's theorem, we saw that  $\phi(z) \in \mathbb{D} \forall z \in \hat{U}$ . Hence,  $|\phi|(z) = |\phi(f^{\circ k}(z))|^{\frac{1}{n^k}} < 1 \forall z \in \mathcal{A}$ . Therefore,  $|\phi| : \mathcal{A} \rightarrow [0, 1]$ .

Also,

$$\begin{aligned} |\phi|(f(z)) &= |\phi(f^{\circ k}(f(z)))|^{\frac{1}{n^k}} \\ &= |\phi(f(f^{\circ k}(z)))|^{\frac{1}{n^k}} \\ &= |\phi(f^{\circ k}(z))^n|^{\frac{1}{n^k}} \\ &= |\phi(f^{\circ k}(z))|^{\frac{n}{n^k}} \\ &= |\phi|(z)^n. \end{aligned}$$

It is also clear that  $|\phi| = 0$  only at  $p$  and its iterated preimages.

If  $q$  is an iterated preimage of  $p$ , say  $f^{\circ k}(q) = p$ , then we have  $|\phi|(q) = |\phi(f^{\circ k}(q))|^{\frac{1}{n^k}} = |\phi(p)|^{\frac{1}{n^k}} = 0$ .

Now, Suppose  $|\phi|(z) = 0$  for some  $z$ . Then,  $|\phi|(z)^{n^k} = 0 \forall k \implies |\phi|(f^{\circ k}(z)) = 0 \forall k$ . But for some large  $k$ ,  $f^{\circ k}(z)$  belongs to the domain of definition of  $\phi$ . But that means,  $f^{\circ k}(z) = p$ , since no other point in that domain is mapped to zero by  $\phi$ . Hence,  $z$  is an iterated preimage of  $p$ .

Now, since  $f^{\circ k} \rightarrow p$  locally uniformly in  $\mathcal{A}$ , for each  $a \in \mathcal{A}$ , we have a neighbourhood  $W_a$  and a constant  $k \in \mathbb{N}$  such that  $f^{\circ k}(z) \in \hat{U} \forall z \in W_a$ .

Hence, for  $z \in W_a$ , we can define  $|\phi|(z) = |\phi(f^{\circ k}(z))|^{\frac{1}{n^k}} = |g(z)|$ , where  $g = \phi \circ f^{\circ k}|_{W_a}$ . Therefore,  $|\phi|_{W_a} = |g|$ , where  $g$  is some holomorphic function defined on  $W_a$ .

It is clear from this that  $|\phi|$  is continuous in  $\mathcal{A}$ .

Now, if  $h$  is any holomorphic function, then  $|h(z)|$  is real-analytic everywhere in its domain except at those  $z$ , where  $h(z) = 0$ .

Since,  $|g| = |\phi|_{W_a}$  is zero only at the iterated preimages of  $f$  in  $W_a$ ,  $|\phi|_{W_a}$  is real analytic everywhere in  $W_a$  except at the iterated preimages of  $p$ .

Therefore,  $|\phi|$  is real analytic everywhere in  $\mathcal{A}$  except at the iterated preimages of  $p$ . Let  $f : \mathbb{C}_{\infty} \rightarrow \mathbb{C}_{\infty}$  be a rational map with a super-attracting fixed point  $p$ . Then the associated Bottcher map  $\phi$  carries a neighbourhood of  $p$  biholomorphically onto a neighbourhood of zero, conjugating  $f$  to the  $n$ th power map, where  $n$  is the local degree of  $f$  near  $p$ .  $\phi$  has a local inverse  $\psi_{\epsilon}$  which maps the  $\epsilon$ -disc around zero to a neighbourhood of  $p$ .  $\square$

**Theorem 3.2.2 (Extending  $\psi_\epsilon$ ).** *There exists a unique open disc of maximal radius  $0 < r \leq 1$  such that  $\psi_\epsilon$  extends holomorphically to a map  $\psi : \mathbb{D}_r \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  is the immediate basin of attraction of  $p$ .*

1. *If  $r = 1$ , then  $\psi$  maps the open unit disc  $\mathbb{D}$  onto  $\mathcal{A}$  biholomorphically.*
2. *If  $0 < r < 1$ , then  $\psi$  maps  $\mathbb{D}_r$  onto its image biholomorphically and there exists atleast one other critical point in  $\mathcal{A}$  on the boundary of  $\psi(\mathbb{D}_r)$ .*

If  $\psi_\epsilon$  is extended biholomorphically in this way to the map  $\psi$  defined on  $\mathbb{D}_r$ , then the inverse map  $\psi^{-1} : \psi(\mathbb{D}_r) \rightarrow \mathbb{D}_r$  must be the extension of  $\phi$  from some neighbourhood of  $p$  to  $\psi(\mathbb{D}_r)$  (since  $\psi^{-1}$  agrees with  $\phi$  on some neighbourhood of  $p$ ).

*Proof.* Let us try to extend  $\psi_\epsilon$  along radial lines by analytic continuation. Then, we can't extend it indefinitely as it would yeild a holomorphic map  $\psi$  from the entire complex plane onto an open set  $\psi(\mathbb{C}) \subset \mathcal{A} \subsetneq \mathbb{C}_\infty$ . ( $\mathcal{A}$  cannot be the whole of  $\mathbb{C}_\infty$  since the Julia set of  $f$  cannot be empty as  $\deg(f) \geq 2$ ). We can conjugate  $f$  such that  $\infty \notin \mathcal{A}$ . Then the corresponding map  $\psi$  will map the whole of the complex plane into  $\mathcal{A} \subset \mathbb{C}$ . By Liouville's theorem, since the map  $\psi$  cannot be a constant,  $\psi(\mathbb{C}) = \mathbb{C} = \mathcal{A}$ . Therefore,  $\mathbb{C}_\infty \setminus \mathcal{A} = \{\infty\}$ . This too is not possible since the Julia set of  $f$  must be an infinite set since  $\deg(f) \geq 2$ .

Thus, there must be some largest radius  $r$  so that  $\psi_\epsilon$  extends analytically throughout the open disc  $\mathbb{D}_r$ .

Also,  $|\phi|(\psi(w)) = |\phi(\psi(w))| = |w|$  near 0, hence for all  $w \in \mathbb{D}_r$  by analytic continuation.???

Since,  $|\phi| : \mathcal{A} \rightarrow [0, 1)$ , this proves that for any  $w \in \mathbb{D}_r$ ,  $|\phi|(\psi(w)) = |w| < 1$ . Therefore,  $\psi$  can be defined only on  $\mathbb{D}_r$  for  $r \leq 1$ .

We will now show that  $\psi$  is actually one-to-one on  $\mathbb{D}_r$ . Suppose  $\psi(w_1) = \psi(w_2)$ . Applying  $|\phi|$ , we see that  $|w_1| = |w_2|$ . Choose such a pair such that  $\psi(w_1) = \psi(w_2)$  ( $w_1 \neq w_2$ ) with  $|w_1| = |w_2|$  minimal. A minimal pair exists because for  $|w| < \epsilon$ ,  $\psi = \psi_\epsilon$  which is one-to-one as it is invertible.

Now,  $\psi$  is an open mapping. Choose a sufficiently small neighbourhood  $U_{w_2}$  of  $w_2$ . Then,  $\psi(U_{w_2})$  is a small neighbourhood of  $\psi(w_1) = \psi(w_2)$ . Hence, for any  $w'_1$  sufficiently close to  $w_1$ ,  $\psi(w'_1) \in \psi(U_{w_2})$ . Hence, we can find  $w'_2$  sufficiently close to  $w_2$  such that  $\psi(w'_1) = \psi(w'_2)$ . Choosing  $|w'_1| < |w_1|$ , we get a contradiction.

Hence,  $\psi$  maps  $\mathbb{D}_r$  onto its image biholomorphically.

In case when  $r = 1$ ,  $U = \psi(\mathbb{D}) = \mathcal{A}$ . If not then we would have some boundary point of  $U$ , say  $z_0 \in \mathcal{A}$ . We can approximate  $z_0$  by points of  $\psi(w_j)$ , where  $|w_j| \rightarrow 1$ .

Now,  $\lim_{j \rightarrow \infty} \psi(w_j) = z_0$ . Hence,

$$\lim_{j \rightarrow \infty} |\phi|(\psi(w_j)) = |\phi|(z_0) \implies \lim_{j \rightarrow \infty} |w_j| = |\phi|(z_0) \implies |\phi|(z_0) = 1$$

which is impossible.

Now, let  $0 < r < 1$ . We need to prove that  $\partial U$ , where  $U = \psi(\mathbb{D}_r)$  must contain a critical point of  $f$ . Suppose,  $w_0 \in \partial \mathbb{D}_r$  and let  $(w_j)_{j=1}^\infty \subset \mathbb{D}_r$  such that  $w_j \rightarrow w_0$ . Let  $\psi(w_j) \rightarrow z_0$ . Then  $z_0 \in \partial U$  because  $\psi$  maps  $\mathbb{D}_r$  onto  $U$  biholomorphically.

If  $z_0$  is not a critical point of  $f$ , then  $f$  maps a neighbourhood of  $z_0$ , say  $A$  onto a neighbourhood of  $f(z_0)$ , say  $B$  biholomorphically.

It should be noted that  $B$  can be chosen such that  $B \subset U$ . This is because  $f(z_0) \in U$ . We have,  $\lim_{j \rightarrow \infty} \psi(w_j) = z_0 \implies \lim_{j \rightarrow \infty} f(\psi(w_j)) = f(z_0) \implies \lim_{j \rightarrow \infty} \psi(w_j^n) = f(z_0) \implies \psi(w_0^n) = f(z_0)$ . Since,  $|w_0| = r < 1, |w_0|^n < r^n < r$ . Hence,  $w_0 \in \mathbb{D}_r$ . Therefore,  $\psi(w_0^n) = f(z_0) \in U$

Let  $g$  be the local inverse of  $f$  near  $f(z_0)$ . Then,  $\psi$  can be extended throughout a neighbourhood of  $w_0$  by

$$w \mapsto g(\psi(w^n))$$

We have,  $\psi(w_0^n) = f(z_0) \implies w_0^n = \phi(f(z_0))$ . Since,  $\phi(B)$  is a neighbourhood of  $\phi(f(z_0))$  lying inside  $\mathbb{D}_r$ , choose a small enough neighbourhood of  $w_0$ , say  $C$  such that  $w^n \in \phi(B)$ , for all  $w \in C$ . In this neighbourhood,  $C$  our newly defined map agrees with  $\psi$  on  $C \cap \mathbb{D}_r$ . This is because, for  $w \in C \cap \mathbb{D}_r, f(\psi(w)) = \psi(w^n) \in B$ . Therefore,  $g(\psi(w^n))$  can be defined and  $\psi(w) = g(\psi(w^n)) \in A$ . Hence, our new map is an analytic continuation of  $\psi$  on the neighbourhood  $C$ .

Now, if none of the  $z_0 \in \partial U$  are critical points, we can extend  $\psi$  to a neighbourhood of  $w_0 \forall w_0 \in \partial \mathbb{D}_r$ . Clearly, these continuations would patch together to define  $\psi$  in a strictly greater disc than  $\mathbb{D}_r$ , which is a contradiction.  $\square$

# Chapter 4

## Introduction to the Mandelbrot Set

We consider the set of quadratic polynomials  $\{f_c(z) = z^2 + c : c \in \mathbb{C}\}$ . It is enough to consider this set because every quadratic polynomial is linearly conjugate to a quadratic polynomial of the type  $f_c(z)$  for some unique  $c \in \mathbb{C}$ .

To prove this, let  $f(z) = pz^2 + qz + r$ ,  $p \neq 0$ . Consider the conjugation with the map,  $\sigma(z) = az + b$  with  $a = p$  and  $b = q/2$ . Then, we have

$$\begin{aligned}\sigma f \sigma^{-1}(z) &= a \left( p \left( \frac{z-b}{a} \right)^2 + q \left( \frac{z-b}{a} \right) + r \right) + b \\ &= (z-b)^2 + q(z-b) + ar + b \\ &= z^2 + b^2 - qb + ar + b \\ &= z^2 + q^2/4 - q^2/2 + pr + q/2 \\ &= z^2 + (pr + q/2 - q^2/4).\end{aligned}$$

This is of the form  $z^2 + c$  for  $c = pr + q/2 - q^2/4$ .

### 4.1 Definition of the Mandelbrot Set

First, we define as new type of set, known as the *Filled-in Julia Set* for a polynomial  $P$ .

**Definition 4.1.1** (Filled-in Julia Set). *The Filled-in Julia Set of a polynomial  $P$  is defined as  $K(P) = \hat{\mathbb{C}} \setminus F_\infty(P)$ . It is the union of the Julia set and the bounded Fatou components. It is denoted by  $K(P)$  or simply  $K$  when the context is clear.*

By [Lemma 1.2.7](#),  $K$  can also be defined as follows:

$$K = \{z \in \mathbb{C} : P^{on}(z) \text{ is bounded}\}.$$

**Notation.** We will use  $F_c$ ,  $J_c$  and  $K_c$  for the  $F_\infty(f_c)$ ,  $J(f_c)$  and  $K(f_c)$  respectively.



**Definition 4.1.2 (Mandelbrot Set).** *The Mandelbrot Set is defined as*

$$M = \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$

Now, we have the following two theorems for polynomials:

- For polynomials, since  $F_\infty$  is a completely invariant Fatou component (by [Theorem 1.2.5](#)),  $\partial F_\infty = J$  (by [Theorem 1.2.2](#)).
- And, from [Theorem 1.2.1](#), we have that  $F_\infty$  is simply connected  $\iff \hat{\mathbb{C}} \setminus F_\infty$  is connected  $\iff \partial F_\infty$  is connected.

Thus, for a polynomial,

$$F_\infty \text{ is simply connected} \iff K \text{ is connected} \iff J \text{ is connected}.$$

Hence, we have the following equivalent descriptions for the Mandelbrot Set:

$$\begin{aligned} M &= \{c \in \mathbb{C} : K_c \text{ is connected}\} \\ &= \{c \in \mathbb{C} : F_c \text{ is simply connected}\} \\ &= \{c \in \mathbb{C} : J_c \text{ is connected}\}. \end{aligned}$$

## 4.2 The Fundamental Dichotomy

**Theorem 4.2.1.** *For a polynomial  $P$ , the following are equivalent:*

1.  $F_\infty$  is simply connected  $\iff J$  is connected  $\iff K$  is connected.
2. There are no finite critical points of  $P$  in  $F_\infty$ .

*Proof.* First assume that  $F_\infty$  is simply connected  $\implies c(F_\infty) = 1$  and hence,  $\chi(F_\infty) = 2 - c(F_\infty) = 1$ . Now, since  $F_\infty$  is completely invariant and  $P$  is a polynomial of degree  $d$  (say),  $P$  is a  $d$ -fold map of  $F_\infty$  onto itself. Applying the Riemann-Hurwitz relation to the map  $P$  of  $F_\infty$  onto itself, we obtain,

$$\begin{aligned} \chi(F_\infty) + \delta_P(F_\infty) &= d\chi(F_\infty) \\ \implies 1 + \delta_P(F_\infty) &= d \\ \implies \delta_P(F_\infty) &= d - 1. \end{aligned}$$

Now,  $\delta_P(\infty) = d - 1$  and therefore,  $P$  does not have any finite critical points in  $F_\infty$ .

For the converse part, assume there are no critical points of  $P$  in  $F_\infty$ . Then, the Bottcher's map  $\phi$  which conjugates  $P$  to the map,  $z \mapsto z^d$  can be extended to the whole of  $F_\infty$  and  $\phi : F_\infty \rightarrow \mathbb{D}$  is a biholomorphism. Hence,  $F_\infty$  is simply connected.  $\square$

Now, quadratic maps have only one finite critical point and  $f_c$  have the critical point at 0 for all  $c \in \mathbb{C}$ . Hence, by the Fundamental Dichotomy,  $F_c$  is simply connected  $\iff 0 \notin F_c$  or  $0 \in K_c$ . Using,  $c_n$  to denote  $f_c^{\circ n}(0)$ , we get,

$$\begin{aligned} M &= \{c \in \mathbb{C} : 0 \in K_c\} \\ &= \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}. \end{aligned}$$

Note that  $c_0 = 0$  and  $c_1 = f_c(0) = c$ . So,  $(c_n)$  is also the forward orbit of  $c$ . Hence, in other words, the Mandelbrot Set consists of  $c \in \mathbb{C}$  such that its forward orbit under the map  $f_c$  remains bounded.

### 4.3 The other end of the Dichotomy Theorem

By the Dichotomy Theorem, we can say that if even one finite critical point of  $P$  lies in  $F_\infty$ , then  $K$  cannot be connected. But this theorem states that if all finite critical points of  $P$  lie in  $F_\infty$ , then  $K$  is not only disconnected, but totally disconnected.

**Definition 4.3.1** (Cantor set). *A subset  $X \subset \hat{\mathbb{C}}$  is called a Cantor set if it is non-empty, closed, perfect and totally disconnected.*

**Theorem 4.3.1.** *Let  $R$  be a rational map with  $\deg(R) \geq 2$ . Let  $\alpha$  be a super-attracting fixed point of  $R$ . If the Fatou component of  $R$  containing  $\alpha$ , say  $F_\alpha$ , contains all the critical points of  $R$ , then  $J(R)$  is a Cantor set.*

**Corollary 4.3.1.1.** *If  $c \notin M$ , then  $J_c$  is a Cantor set.*

*Proof.* Since 0 is the only critical points of  $f_c$  (apart from  $\infty$ ), if it belongs to  $F_c = F_\infty(f_c)$ , i.e. if  $0 \notin K_c \iff c \notin M$ , then  $J_c$  is a Cantor set.  $\square$

### 4.4 Some properties of the Mandelbrot Set

We know,  $M = \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}$ . This description for  $M$  can be strengthened significantly by the following theorem:

**Theorem 4.4.1.**  $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$ .

*Proof.* Obviously,  $\{c \in \mathbb{C} : |c_n| \leq 2\} \subseteq M$ .

Now, Suppose that  $c \in M$ . We need to prove that  $|f_c^{\circ n}(c)| = |c_n| \leq 2$  for all  $n \in \mathbb{N}$ . Consider the set  $W_c = \{z \in \mathbb{C} : |z| \geq |c|, |z| > 2\}$ . For  $z \in W_c$ ,

$$|f_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| \geq |z|(|z| - 1) = |z|(1 + \epsilon)$$

for some  $\epsilon > 0$  (as  $|z| > 2$ ). Clearly,  $|f_c^{\circ n}(z)| \geq |z|(1 + \epsilon)^n \implies z \notin K_c$ .

This implies  $|c| \leq 2$ . Consequently,  $|f_c^{\circ n}(c)| \leq 2$  for all  $n \in \mathbb{N}$ .

Hence,  $M \subseteq \{c \in \mathbb{C} : |c_n| \leq 2\}$ .

Therefore,  $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$ . □

As  $c_1 = c$ , we have that  $|c| \leq 2$  for all  $c \in M$  i.e  $M \subseteq \{c \in \mathbb{C} : |c| \leq 2\}$ . This turns out to be the strongest bound possible for  $|c|$  as  $-2 \in M$ . The orbit of 0 under the map  $z \mapsto z^2 - 2$  is:

$$0 \mapsto -2 \mapsto 2 \mapsto 2$$

and hence is bounded.

**Theorem 4.4.2.** *The Mandelbrot set  $M$  is compact and  $\hat{\mathbb{C}} \setminus M$  is open and connected.*

*Proof.* Let,  $c_n = f_c^{\circ n}(c) = Q_n(c)$  be a polynomial in  $c$ . Clearly, from [Theorem 4.4.1](#)

$$M = \cap_{n=1}^{\infty} Q_n^{-1}(\overline{\mathbb{D}_2}),$$

where  $\overline{\mathbb{D}_2} = \{z \in \mathbb{C} : |z| \leq 2\}$ . Thus,  $M$  is closed. It is already known that it is bounded. Hence, it is compact.

Now,

$$\hat{\mathbb{C}} \setminus M = \cup_{n=1}^{\infty} Q_n^{-1}(E)$$

where  $E = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}_2}$ . Now,  $E$  is open and connected and since,  $Q_n$  are non-constant polynomials,  $Q_n^{-1}(E)$  is open and connected for all  $n \in \mathbb{N}$ . Also, each one of them contains  $\infty$  and hence, their union is also open and connected.

Therefore,  $\hat{\mathbb{C}} \setminus M$  is open and connected. □

## 4.5 Plotting the Mandelbrot Set

[Theorem 4.4.1](#) is also used to plot the Mandelbrot Set. A simple code in python would

# Chapter 5

## Connectedness of the Mandelbrot Set

In the previous chapter, we proved that the Mandelbrot set is compact and  $\mathbb{C} \setminus M$  is open and connected. In this chapter, we will prove that  $\hat{\mathbb{C}} \setminus M$  is biholomorphic to  $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ , proving that  $\hat{\mathbb{C}} \setminus M$  is simply connected, thus implying that  $M$  is connected by [Theorem 1.2.1](#).

### 5.1 The Green's Function

From Bottcher's theorem, we know that near a super-attracting fixed point  $\alpha$ , an analytic map,  $f$  of local degree  $d$  is locally conjugate to the map  $z \mapsto z^d$  via the Bottcher's map  $\phi$ . Furthermore, the Bottcher's map  $\phi$  can be extended throughout the whole of the immediate basin of attraction of the super-attracting fixed point  $\alpha$ , say  $F_\alpha$  if there are no critical points in the basin and  $\phi(F_\alpha) = \mathbb{D}$ . If there are critical points in  $F_\alpha$ , then the Bottcher's map can be extended to a set,  $U$  such that  $\phi(U) = \mathbb{D}_r$  for some  $r < 1$  and  $\partial U$  contains some other critical point of  $f$ .

For polynomials,  $f_c$ , we take the super-attracting fixed point  $\infty$  and the Bottcher's map  $\phi_c = \frac{1}{\phi}$ . In this way, we get that,  $f_c$  is conjugate to the map  $z \mapsto z^2$  in a neighbourhood of  $\infty$ . Furthermore, we see that if  $c \in M$ , the only other critical point  $0 \notin F_c = F_\infty(f_c)$ . Thus,  $\phi_c$  can be extended to the whole of  $F_c$  and  $\phi_c(F_c) = \mathbb{C} \setminus \overline{\mathbb{D}}$ .

**Theorem 5.1.1.** *For a given  $c \in \mathbb{C}$ , the set on which  $\phi_c$  is defined is given by  $S_c = \{z \in \hat{\mathbb{C}} : G_c(z) > G_c(0)\}$*

**Theorem 5.1.2.** *The map,  $(z, c) \mapsto \Phi(z, c) := \phi_c(z)$  is analytic in two variables on the set  $\{(z, c) \in \hat{\mathbb{C}} \times \mathbb{C} : G_c(z) > G_c(0)\}$  and  $\Phi(c, c)/c \rightarrow 1$  as  $c \rightarrow \infty$ .*

**Theorem 5.1.3.**  *$(z, c) \mapsto G_c(z)$  is continuous in two variables.*

## 5.2 The Isomorphism by Douady and Hubbard

Douady and Hubbard proved that the Mandelbrot set is connected by defining an isomorphism  $\Psi : \mathbb{C} \setminus M \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$  given by,

$$\Psi(c) = \Phi(c, c) = \phi_c(c).$$

To prove that it is indeed an isomorphism, we will proceed in the following steps:

1.  $\Psi$  is a well defined map: For  $c \in \mathbb{C} \setminus M$ ,  $G_c(c) = 2G_c(0) > G_c(0)$ . Thus,  $c \in S_c$  and  $\phi_c(c)$  can be defined. Also,  $|\phi_c(c)| > 1$ .
2.  $\Psi$  is analytic: We already proved that  $(z, c) \mapsto \Phi(z, c)$  is analytic in two variables. Hence the map  $c \mapsto (c, c) \mapsto \Phi(c, c)$  is analytic.
3.  $|\Psi(c_n)| \rightarrow 1$  as  $c_n \rightarrow M$ : This is due to the continuity of  $(z, c) \mapsto G_c(z)$  in two variables. Hence,  $c \mapsto (c, c) \mapsto G_c(c)$  is continuous. Hence, as  $c_n \rightarrow c_0 \in M$ ,  $G_{c_n}(c_n) \rightarrow G_{c_0}(c_0) = 0$ . Hence,  $\log |\phi_{c_n}(c_n)| = \log |G_{c_n}(c_n)| \rightarrow 0 \implies |\phi_{c_n}(c_n)| \rightarrow 1$ .
4.  $\Psi$  can be extended to an analytic map  $\Psi : \hat{\mathbb{C}} \setminus M \rightarrow \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  by definiing  $\Psi(\infty) = \infty$ : This is due to the fact that  $\Psi(c)/c = \Phi(c, c)/c \rightarrow 1$  as  $c \rightarrow \infty$ .
5. This extension is a proper map: Let  $K$  be a compact subset of  $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ . Clearly,  $\Psi^{-1}(K)$  is a closed subset of  $\hat{\mathbb{C}} \setminus M$ . If  $\Psi^{-1}(K)$  is not compact, there is a sequence,  $(c_n)_{n=1}^\infty \subset \Psi^{-1}(K)$  such that  $c_n \rightarrow M$ . This implies  $|\Psi(c_n)| \rightarrow 1$  by point 3. This is not possible as  $K$  being a compact subset of  $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  is at a positive distance from  $\overline{\mathbb{D}}$ .
6. Now,  $\Psi$  being a proper holomorphic map, it is a branched covering of some degree  $d$ . As  $\Psi^{-1}(\infty) = \{\infty\}$  with multiplicity 1 (because  $\Psi(c)/c \rightarrow 1$  as  $c \rightarrow \infty$ ),  $d = 1$ .

Therefore,  $\Psi : \hat{\mathbb{C}} \setminus M \rightarrow \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  is an isomorphism and  $M$  is connected. Consecutively,  $\Psi : \mathbb{C} \setminus M \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$  is also an isomorphism.