1 Introduction

1.1 Completely Invariant Components

Definition 1.1.1 (Connectivity)**.** *The connectivity of a domain* $D \subset \hat{\mathbb{C}}$ *is defined as the number of components of* ∂D.

Theorem 1.1.1. *The following are equivalent for a domain* $D \subset \hat{\mathbb{C}}$*:*

1. D *is simply connected.*

2. D^c *is connected.*

3. ∂D *is connected or* $c(D) = 1$.

Theorem 1.1.2. *If* R *is a rational function, with* $\deg(R) \geq 2$, *and* F_0 *is a completely invariant Fatou component of* R, *then:*

1. $\partial F_0 = J$.

2. F_0 *is simply connected or infinitely connected.*

3. *All other components of* F *are simply connected.*

4. F_0 *is simply connected* $\iff$ J *is connected.*

Theorem 1.1.3. $\partial R(U) \subset R(\partial U)$

Lemma 1.1.4. *For a rational map* R, *if* F_1 *and* F_2 *are two Fatou components and* R *maps a point of* F_1 *to a point of* F_2, *then* $R(F_1) = F_2$.

Theorem 1.1.5. *The unbounded Fatou component of a polynomial* P, *i.e. the Fatou component containing* ∞ *is a completely invariant Fatou component. It is denoted by* $F_\infty(P)$ *or simply* F_∞ *when the context is clear.*

Theorem 1.1.6 (Vitali's Theorem).

Lemma 1.1.7. *If α is a (super)-attracting fixed point of a rational map R and F_α is the Fatou component containing α then $R^{\circ n}(z) \to \alpha$ locally uniformly in F_α.*

Theorem 1.1.8 (**Riemann-Hurwitz Formula**). *Let F_0 and F_1 be components of the Fatou set F of a rational map R and R maps F_0 into F_1. Then, for some integer m, R is an m-fold map of F_0 onto F_1 and*

$$\chi(F_0) + \delta_R(F_0) = m\chi(F_1).$$

1.2 Some properties of the Julia Sets

Let J denote the Julia set for a rational map R with $\deg(R) \geq 2$. Then we have the following properties:

Theorem 1.2.1. *J is infinite.*

Theorem 1.2.2 (**Minimality of J**).

Theorem 1.2.3. *J is a perfect set, and hence, uncountable.*

2 Behaviour of analytic functions near fixed points

2.1 Behaviour near parabolic fixed points

A point p is called a parabolic fixed point of f if $f(p) = p$ and $f'(p) = e^{2\pi i t}$, where t is a rational number.

Theorem 2.1.1 (The Petal Theorem). *Suppose that an analytic map f has the form:*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1})$$

near the origin. Then for sufficiently small t,

1. *f maps each $\Pi_k(t)$ into itself;*

2. *$f^{\circ n}(z) \to 0$ uniformly on each petal;*

3. *$\arg(f^{\circ n}(z)) \to 2k\pi/p$ locally uniformly on each petal;*

4. *$f : \Pi_k(t) \to \Pi_k(t)$ is conjugate to a translation.*

5. *$|f(z)| < |z|$ on a neighbourhood of the axis of each petal;*

Proof. For $0 < r_0 < 1$, define the sector S_0,

$$S_0 = \{re^{i\theta} : 0 < r < r_0, |\theta| < \pi/p\}$$

and the region W,

$$W = \{re^{i\theta} : r > \frac{1}{r_0^p}, |\theta| > \pi\}.$$

It is clear that the map $\sigma : z \mapsto \frac{1}{z^p}$ is a biholomorphism of S_0 onto W with $\sigma^{-1} : W \to S_0$ given by $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$. The branch of p-th root that we select determines which sector of width $2\pi/p$, the inverse map maps to. (The other sectors being $S_k = \{0 < r < r_0, |\theta - 2k\pi/p| < \pi/p\}$.)

Now, the conjugate map of f on W is given by,

$$g(w) = \sigma f \sigma^{-1}(w) = f(w^{-\frac{1}{p}})^{-p}.$$

This just replaces the action of f on S by g on W, and we have the following commutative diagram:
Hence, we have the following estimates for g which will be crucial in everything that will follows:

$$g(w) = w + p + A/w + \theta(w), \text{ where } A \text{ is a constant and} \tag{2.1}$$
$$|\theta(w)| \leq B/|w|^{1+\frac{1}{p}}, B > 0. \tag{2.2}$$

Choose any K satisfying

$$K > \max\{1/r_0^p, 3(|A| + B)\} > 1 \text{ (as } r_0 < 1)$$

and let,

$$\Pi = \{x + iy : y^2 > 4K(K - x)\}.$$

Clearly, Π is bounded by a parabola and $\Pi \subset W$.

We have chosen this subset $\Pi \subset W$ because we will show that Π is nothing but the conformal image of $\Pi_0(t)$ under σ (for a suitable t) and g satisfies all the corresponding conditions that f should satisfy on $\Pi_0(t)$ according to the theorem.

Claim. Π is the conformal image of $\Pi_0(t)$ under σ for a suitable t.
The easiest way to see this is using polar coordinates. We write, $z = re^{i\theta}$ for $z \in S$ and $w = \rho e^{i\phi}$ for $w \in W$. Then, $\rho = \frac{1}{r^p}$ and $\phi = -p\theta$.

Now, we need to express Π in polar co-ordinates. To do so, we notice that points on the parabola are given by

$$\rho \text{ (distance from focus i.e. 0)} = 2K - \rho\cos\phi \text{ (distance from directrix i.e. } y = 2K).$$

Therefore, points on Π are given by

$$\rho > 2K - \rho\cos\phi.$$

Hence,

$$\Pi = \{\rho e^{i\phi} : 2K < \rho(1 + \cos\phi)\}.$$

Now, let $\Omega = \sigma^{-1}(\Pi)$. Then, Ω is given by

$$\Omega = \{re^{i\theta} : 2Kr^p < 1 + \cos(p\theta)\}.$$

Hence, $\Omega = \Pi_0\left(\frac{1}{2K}\right)$

Lemma 2.1.2. *g satisfies the following properties on Π:*

1. *Π is forward invariant under g.*

2. *$g^{\circ n}(w) \to \infty$ uniformly on Π.*

3. *$\arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π.*

4. *$g : \Pi \to \Pi$ is conjugate to a translation.*

Proof.
1. We write,

$$w = x + iy,\ g(w) = X + iY,\ A/w + \theta(w) = a + ib.$$

From Equation (2.1), we obtain,

$$\begin{aligned} X + iY &= (x + iy) + p + (a + ib) \\ \implies X &= x + p + a \text{ and } Y = y + b. \end{aligned}$$

Now, if $w \in \Pi$,

$$\begin{aligned} Y^2 - 4K(K - X) &= (y + b)^2 - 4K(K - x - p - a) \\ &= [y^2 - 4K(K - x)] + b^2 + 2yb + 4K(a + p) \\ &> 4Kp + (2yb + 4Ka) \\ &\geq |4Kp - |2yb + 4Ka||. \end{aligned}$$

Now, for $w \in \Pi$, $|w| > K > 1$. Hence we get,

$$|w||A/w + \theta(w)| \leq |w|(|A|/|w| + B/|w|^{1+\frac{1}{p}}) = |A| + B/|w|^{\frac{1}{p}} < |A| + B \tag{2.3}$$

(since for $|w| > 1$, $|w|^{\frac{1}{p}} > 1$). Therefore,

$$\begin{aligned}|2yb + 4Ka| &\le 2|yb| + 4K|a| \\ &\le 2|y||b| + 4K|a| \\ &\le 2|w||a+ib| + 4|w||a+ib| \\ &= 6|w||a+ib| \\ &< 2K < 2Kp.\end{aligned}$$

Therefore, we see that $Y^2 - 4K(K - X) > 0$ and hence, $g(w) \in \Pi$ for $w \in \Pi$. Hence, Π is forward invariant under g.

2. Now, we will prove a stronger statement that for any $t > 0$ g maps $\Pi + t$ into $\Pi + t + p/2$. This is simply because, for $w \in \Pi + t$, we have, $y^2 - 4K(K + t - x) > 0$. Hence,

$$\begin{aligned}Y^2 - 4K(K + t + p/2 - X) &= [y^2 - 4K(K + t - x)] + b^2 + 2yb + 4K(a - p/2) \\ &> 2Kp + (2yb + 4Ka) \\ &\ge |2Kp - |2yb + 4Ka|| \\ &> 0.\end{aligned}$$

Therefore, if $w \in \Pi$, $g^{\circ n}(w) \in \Pi + np/2$. Hence, $|g^{\circ n}(w)| > \sqrt{n}$. This is simply because, $K + np/2 > 1 + n/2 > \sqrt{n}$ and hence $\Pi + np/2$ is disjoint from the disc $\{|z| \le \sqrt{n}\}$.

Hence, $g^{\circ n}(w) \to \infty$ uniformly on Π.

3. Firstly, we see that,

$$g^{\circ k}(w) = g(g^{\circ k}(w)) = g^{\circ k}(w) + p + A/g^{\circ k}(w) + \theta(g^{\circ k}(w)).$$

By expanding the first term again and again, we obtain,

$$g^{\circ n}(w) = w + np + \sum_{k=0}^{n-1} \left(\frac{A}{g^{\circ k}(w)} + \theta(g^{\circ k}(w)) \right). \tag{2.4}$$

Also note that form Equation (2.3), we have,

$$|A/w + \theta(w)| < (|A| + B)/K < \frac{1}{3}.$$

Let Q be a compact subset of Π. From now, we will assume that $w \in Q$ and we will use $C_1, C_2, C_3, \ldots$ to denote positive constants which will be dependent on Q.

Hence,

$$\begin{aligned} |g(w)| = |w + p + A/w + \theta(w)| &\geq ||w + p| - |A/w + \theta(w)|| \\ &= |w + p| - |A/w + \theta(w)| \\ &\geq |w| + p - \frac{1}{3}. \end{aligned}$$

Therefore, we obtain,

$$|g^{\circ n}(w)| \geq |w| + n(p - 1/3) \geq C_1 + C_2 n.$$

(Here, $C_1 = \min\{|w| : w \in Q\} > 0$ and $C_2 = p - \frac{1}{3} > 0$.)
We can select C_3 large enough such that

$$|g^{\circ n}(w)| \geq C_3 n \tag{2.5}$$

Next, with Equation (2.2), and the above inequality, we get,

$$|\theta(g^{\circ n}(w)| \leq B/|g^{\circ n}(w)|^{1+\frac{1}{p}} \leq C_4/n^{1+\frac{1}{p}}. \tag{2.6}$$

Finally, combining the above two inequalities and Equation (2.4), we obtain,

$$\begin{aligned} |g^{\circ n}(w) - np| &\leq |w| + |A/w + \theta(w)| + \frac{|A|}{C_3} \sum_{k=1}^{n-1} \frac{1}{k} + C_4 \sum_{k=1}^{n-1} \frac{1}{n^{1+\frac{1}{p}}} \\ &< C_5 + C_6 \sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

(Here, $C_5 = \max\{|w| : w \in Q\} + \frac{1}{3} + C_4 \sum_{n=1}^{\infty} 1/n^{1+\frac{1}{p}}$ and $C_6 = |A|/C_3$.)
We can select C_7 large enough such that

$$|g^{\circ n}(w) - np| < C_7 \log n. \tag{2.7}$$

Remark. The above inequality follows from the fact that, if $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then $H_n - \log n \to \gamma$. (γ is known as the Euler's constant). So, we have that

$$\begin{aligned} P + QH_n &= P + Q(\log n + \gamma + \epsilon_n), \text{where } \epsilon_n \to 0 \\ &\leq Q \log n + (P + Q \max\{\epsilon_n\} + Q\gamma) \\ &= Q \log n + R \\ &< S \log n \end{aligned}$$

for S large enough.

From, $|g^{\circ n}(w) - np| < C_7 \log n$, it follows that $|\arg(g^{\circ n}(w)| < \sin^{-1}\left(\frac{C_7 \log n}{np}\right)$ for n large enough. Hence, $\arg(g^{\circ n}(w)) \to 0$ uniformly on Q, and consequently, locally uniformly on Π.

4. Define,

$$u_n(w) = g^{\circ n}(w) - np - (A/p)\log n.$$

Claim. $u_n(w)$ converges locally uniformly on Π to a holomorphic function u, that is one-to-one on Π.

$$u_{n+1}(w) - u_n(w) = [g^{\circ n+1}(w) - g^{\circ n}(w)] - p - (A/p)\log\left(\frac{n+1}{n}\right).$$

From Equation (2.2), we obtain,

$$\begin{aligned} u_{n+1}(w) - u_n(w) &= [g^{\circ n}(w) + p + A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - g^{\circ n}(w)] \\ &\qquad - p - (A/p)\log(1 + 1/n) \\ &= A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - (A/p)\log(1 + 1/n) \\ &= A(1/g^{\circ n}(w) - 1/np) + \theta(g^{\circ n}(w)) + (A/p)(1/n - \log(1 + 1/n)). \end{aligned}$$

Now, let Q is a compact subset of Π and $w \in Q$. We need to prove that u_n converges uniformly in Q. From the above equation, to prove that u_n converges uniformly in Q, we need to show that each of the following series converges uniformly in Q:

$$\sum_n |1/g^{\circ n}(w) - 1/np|, \ \sum_n |\theta(g^{\circ n}(w)), \ \sum_n |1/n - \log(1 + 1/n)|.$$

Let us look at the first series. We have, (using Equations (2.5) and (2.7))

$$|1/g^{\circ n}(w) - 1/np| = \frac{|g^{\circ n}(w) - np|}{|g^{\circ n}(w)|np} \leq \frac{C_7 \log n}{C_3 n^2 p} = C_8 \log n/n^2.$$

(Here $C_8 = C_7/(pC_3)$).

From Equation (2.6), it is clear that $\sum_n |\theta(g^{\circ n}(w))|$ converges.

Now, $0 < x - \log(1 + x) \leq x^2$ for $x > 0$.

This is because, it is zero at $x = 0$ and $\frac{d}{dx}(x - \log(1 + x)) = 1 - \frac{1}{1+x} > 0$ for $x > 0$.

Also, $x^2 - x + \log(1 + x)$ is zero at $x = 0$ and $\frac{d}{dx}(x^2 - x + \log(1 + x)) = 2x - 1 + \frac{1}{1+x} > 0$ for $x > 0$.

Putting $x = \frac{1}{n}$, we get,

$$|1/n - \log(1 + 1/n)| < 1/n^2.$$

Therefore, u_n converges locally uniformly to some holomorphic function u on Π.
Now, from $u_n(w) = g^{\circ n}(w) - np - (A/p)\log n$, we get that,

$$\begin{aligned}(n+1)p + (A/p)\log(n+1) + u_{n+1}(w) &= g^{\circ n+1}(w)\\ &= g^{\circ n}(g(w))\\ &= np + (A/p)\log n + u_n(g(w))\\ \implies p + (A/p)\log(1+1/n) + u_{n+1}(w) &= u_n(g(w)).\end{aligned}$$

Taking limit $n \to \infty$, we get,

$$p + u(w) = u(g(w)).$$

Since f is injective near the origin, g is injective on Π, (if K is chosen large enough). Therefore, $g^{\circ n}$ is injective on Π and hence, so is u_n. By Hurwitz Theorem, u is either injective or constant, but it is clearly not a constant since it satisfies the above equation.

This shows that $g : \Pi \to \Pi$ is conjugate to the map $z \mapsto z + p$ of $u(\Pi)$ into itself. $\square$

Coming back to our original theorem, we see that since g maps Π into itself, f also maps each $\Pi_k(t)$ into itself.

Now since, $|g^{\circ n}(w)| > \sqrt{n}$ for all $w \in \Pi$, $|\sigma f^{\circ n} \sigma^{-1}(w)| \to \infty$ uniformly on Π.

$\square$

3 Introduction to the Mandelbrot Set

We consider the set of quadratic polynomials $\{f_c(z) = z^2 + c : c \in \mathbb{C}\}$. It is enough to consider this set because every quadratic polynomial is conjugate to a quadratic polynomial of the type $f_c(z)$ for some unique $c \in \mathbb{C}$.

To prove this, let $f(z) = az^2 + bz + c$, $a \neq 0$. And consider the conjugation, $\sigma(z) =$

3.1 Definition of the Mandelbrot Set

First, we define as new type of set, known as the *Filled-in Julia Set* for a polynomial P.

Definition 3.1.1 (Filled-in Julia Set). *The Filled-in Julia Set of a polynomial P is defined as $K(P) = \hat{\mathbb{C}} \backslash F_\infty(P)$. It is the union of the Julia set and the bounded Fatou components. It is denoted by $K(P)$ or simply K when the context is clear.*

By Lemma 1.1.7, K can also be defined as follows:

$$K = \{z \in \mathbb{C} : P^{\circ n}(z) \text{ is bounded}\}.$$

Notation. We will use F_c, J_c and K_c for the $F_\infty(f_c)$, $J(f_c)$ and $K(f_c)$ respectively.

Definition 3.1.2 (**Mandelbrot Set**). *The Mandelbrot Set is defined as*

$$M = \{c \in \mathbb{C} : K_c \textit{ is connected}\}.$$

Now, we have the following two theorems for polynomials:

- For polynomials, since F_∞ is a completely invariant Fatou component (by Theorem 1.1.5), $\partial F_\infty = J$ (by Theorem 1.1.2).
- And, from Theorem 1.1.1, we have that F_∞ is simply connected $\iff$ $\hat{\mathbb{C}} \backslash F_\infty$ is connected $\iff$ ∂F_∞ is connected.

Thus, for a polynomial,

$$F_\infty \text{ is simply connected } \iff K \text{ is connected } \iff J \text{ is connected.}$$

Hence, we have the following equivalent descriptions for the Mandelbrot Set:

$$\begin{aligned} M &= \{c \in \mathbb{C} : K_c \text{ is connected}\} \\ &= \{c \in \mathbb{C} : F_c \text{ is simply connected}\} \\ &= \{c \in \mathbb{C} : J_c \text{ is connected}\}. \end{aligned}$$

3.2 The Fundamental Dichotomy

Theorem 3.2.1. *For a polynomial P, the following are equivalent:*

1. *F_∞ is simply connected $\iff$ J is connected $\iff$ K is connected.*
2. *There are no finite critical points of P in F_∞.*

Proof. First assume that F_∞ is simply connected $\implies$ $c(F_\infty) = 1$ and hence, $\chi(F_\infty) = 2 - c(F_\infty) = 1$. Now, since F_∞ is completely invariant and P is a polynomial of degree d (say), P is a d-fold map of F_∞ onto itself. Applying the Riemann-Hurwitz relation to the map P of F_∞ onto itself, we obtain,

$$\begin{aligned} &\chi(F_\infty) + \delta_P(F_\infty) = d\,\chi(F_\infty) \\ \implies &1 + \delta_P(F_\infty) = d \\ \implies &\delta_P(F_\infty) = d - 1. \end{aligned}$$

Now, $\delta_P(\infty) = d - 1$??? and therefore, P does not have any finite critical points in F_∞.

For the converse part, assume there are no critical points of P in F_∞. Then, the Bottcher's map ϕ which conjugates P to the map, $z \mapsto z^d$ can be extended to the whole of F_∞ and $\phi : F_\infty \to \mathbb{D}$ is a biholomorphism. Hence, F_∞ is simply connected. $\square$

Now, quadratic maps have only one finite critical point and f_c have the critical point at 0 for all $c \in \mathbb{C}$. Hence, by the Fundamental Dichotomy, F_c is simply connected $\iff$ $0 \notin F_c$ or $0 \in K_c$. Using, c_n to denote $f_c^{\circ n}(0)$, we get,

$$\begin{aligned} M &= \{c \in \mathbb{C} : 0 \in K_c\} \\ &= \{c \in \mathbb{C} : (c_n) \text{ is bounded}\} \end{aligned}$$

.

Note that $c_0 = 0$ and $c_1 = f_c(0) = c$. So, (c_n) is also the forward orbit of c. Hence, in other words, the Mandelbrot Set consists of $c \in \mathbb{C}$ such that its forward orbit under the map f_c remains bounded.

3.2.1 The other end of the Dichotomy Theorem

By the Dichotomy Theorem, we can say that if even one finite critical point of P lies in F_∞, then K cannot be connected. But this theorem states that if all finite critical points of P lie in F_∞, then K is not only disconnected, but totally disconnected.

3.3 Some properties of the Mandelbrot Set

We know, $M = \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}$. This description for M can be strengthened significantly by the following theorem:

Theorem 3.3.1. $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Proof. Obviously, $\{c \in \mathbb{C} : |c_n| \leq 2\} \subseteq M$.

Now, Suppose that $c \in M$. We need to prove that $|f_c^{\circ n}(c)| = |c_n| \leq 2$ for all $n \in \mathbb{N}$. Consider the set $W_c = \{z \in \mathbb{C} : |z| \geq |c|, |z| > 2\}$. For $z \in W_c$,

$$|f_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| \geq |z|(|z| - 1) = |z|(1 + \epsilon)$$

for some $\epsilon > 0$ (as $|z| > 2$). Clearly, $|f_c^{\circ n}(z)| \geq |z|(1 + \epsilon)^n \implies z \notin K_c$.
This implies $|c| \leq 2$. Consequently, $|f_c^{\circ n}(c)| \leq 2$ for all $n \in \mathbb{N}$.
Hence, $M \subseteq \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Therefore, $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$. □

As $c_1 = c$, we have that $|c| \leq 2$ for all $c \in M$ i.e $M \subseteq \{c \in \mathbb{C} : |c| \leq 2\}$. This turns out to be the strongest bound possible for $|c|$ as $-2 \in M$. The orbit of 0 under the map $z \mapsto z^2 - 2$ is:

$$0 \mapsto -2 \mapsto 2 \mapsto 2$$

and hence is bounded.

Theorem 3.3.2. *The Mandelbrot set is compact and $\hat{\mathbb{C}} \backslash M$ is open and connected.*

Proof. Let, $c_n = f_c^{\circ n}(c) = Q_n(c)$ be a polynomial in c. Clearly, from Theorem 3.3.1

$$M = \cap_{n=1}^{\infty} Q_n^{-1}(\overline{\mathbb{D}_2}),$$

where $\overline{\mathbb{D}_2} = \{z \in \mathbb{C} : |z| \leq 2\}$. Thus, M is closed. It is already known that it is bounded. Hence, it is compact.

Now,

$$\hat{\mathbb{C}} \backslash M = \cup_{n=1}^{\infty} Q_n^{-1}(E)$$

where $E = \hat{\mathbb{C}} \backslash \overline{\mathbb{D}_2}$. Now, E is open and connected and since, Q_n are non-constant polynomials, $Q_n^{-1}(E)$ is open and connected for all $n \in \mathbb{N}$. Also, each one of them contains ∞ and hence, their union is also open and connected.
Therefore, $\hat{\mathbb{C}} \backslash M$ is open and connected.

□

3.3.1 Plotting the Mandelbrot Set

Theorem 3.3.1 is also used to plot the Mandelbrot Set.