Indian Institute of Science Education and Research, Kolkata
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Introduction to Complex Dynamics and the Mandelbrot Set

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Chapter 1

Introduction

1.1 Equicontinuity and Normality

1.2 Completely Invariant Components

A domain D is called:

- *forward invariant* under the map f if $f(D) = D$.
- *backward invariant* under the map f if $f^{-1}(D) = D$.
- *completely invariant* under the map f if it is both forward and backward invariant under f i.e. $f(D) = D$ and $f^{-1}(D) = D$.

Definition 1.2.1 (Connectivity). *The connectivity of a domain $D \subset \hat{\mathbb{C}}$ is defined as the number of components of ∂D.*

Theorem 1.2.1. *The following are equivalent for a domain $D \subset \hat{\mathbb{C}}$:*

1. *D is simply connected.*
2. *D^c is connected.*
3. *∂D is connected or $c(D) = 1$.*

Theorem 1.2.2. *If R is a rational function, with $\deg(R) \geq 2$, and F_0 is a completely invariant Fatou component of R, then:*

1. *$\partial F_0 = J$.*
2. *F_0 is simply connected or infinitely connected.*
3. *All other components of F are simply connected.*

4. F_0 *is simply connected* $\iff$ J *is connected.*

Theorem 1.2.3. $\partial R(U) \subset R(\partial U)$

Lemma 1.2.4. *For a rational map* R*, if* F_1 *and* F_2 *are two Fatou components and* R *maps a point of* F_1 *to a point of* F_2*, then* $R(F_1) = F_2$*.*

Theorem 1.2.5. *The unbounded Fatou component of a polynomial* P*, i.e. the Fatou component containing* ∞ *is a completely invariant Fatou component. It is denoted by* $F_\infty(P)$ *or simply* F_∞ *when the context is clear.*

Theorem 1.2.6 (Vitali's Theorem).

Lemma 1.2.7. *If* α *is a (super)-attracting fixed point of a rational map* R *and* F_α *is the Fatou component containing* α *then* $R^{\circ n}(z) \to \alpha$ *locally uniformly in* F_α*.*

Theorem 1.2.8 (Riemann-Hurwitz Formula). *Let* F_0 *and* F_1 *be components of the Fatou set* F *of a rational map* R *and* R *maps* F_0 *into* F_1*. Then, for some integer* m*,* R *is an* m*-fold map of* F_0 *onto* F_1 *and*

$$\chi(F_0) + \delta_R(F_0) = m\chi(F_1).$$

1.3 Some properties of the Julia Sets

Let J denote the Julia set for a rational map R with $\deg(R) \geq 2$. Then we have the following properties:

Theorem 1.3.1. *J is infinite.*

Theorem 1.3.2 (Minimality of J).

Theorem 1.3.3. *J is a perfect set, and hence, uncountable.*

Chapter 2

Behaviour of analytic functions near fixed points

2.1 Behaviour near parabolic fixed points

A point p is called a parabolic fixed point of f if $f(p) = p$ and $f'(p) = e^{2\pi i t}$, where t is a rational number.

Lemma 2.1.1. *Suppose f is analytic and satisfies*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{p+2})$$

in some neighbourhood N of the origin. Let $\omega_1, \dots, \omega_p$ be the p-th roots of unity and let $\eta_1, \dots, \eta_p$ be the p-th roots of -1. Then for sufficiently small r_0 and θ_0,

1. *$|f(z)| < |z|$ on each sector*

$$S_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\omega_j)| < \theta_0\}.$$

2. *$|f(z)| > |z|$ on each sector*

$$\Sigma_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\eta_j)| < \theta_0\}.$$

Before stating the *Petal Theorem*, which discusses the behaviour of analytic functions near parabolic fixed points, we first define the notions of *petals*.

Definition 2.1.1 (Petals). *Let $p \in \mathbb{N}$. For each $k \in \{1, \dots, p-1\}$, define the sets as a function of a parameter $t > 0$ as follows,*

$$\Pi_k(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)); |\theta - 2k\pi/p| < \pi/p\}.$$

The sets $\Pi_k(t)$ are known as Petals.

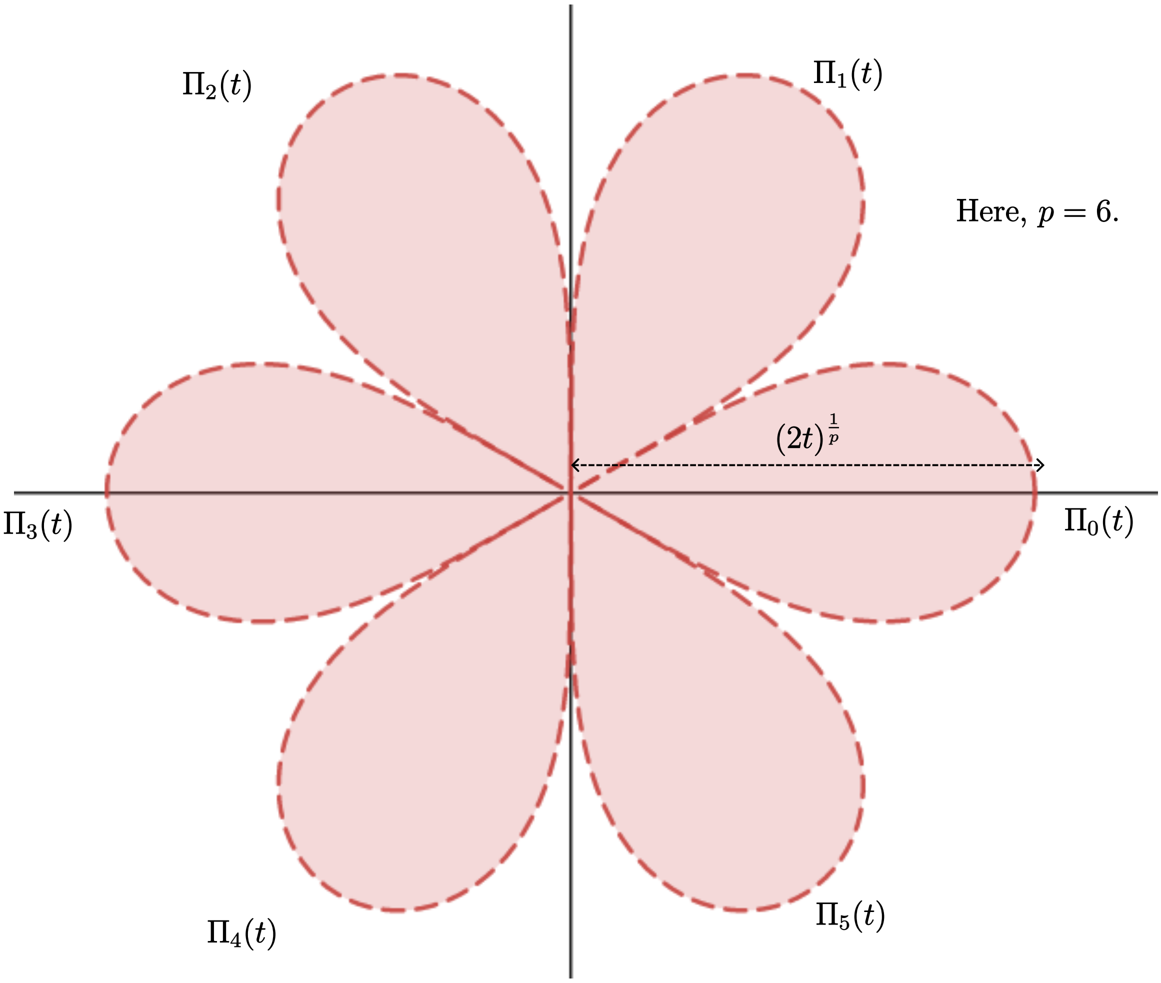

Figure 2.1: Six petals at the origin for $p = 6$.

We have shown a diagram of the petals $\Pi_k(t)$ in Figure 2.1 for $p = 6$.
Note that all the petals are pairwise disjoint and each petal subtends an angle of $2\pi/p$ at the origin.

Theorem 2.1.2 (The Petal Theorem). *Suppose that an analytic map f has the form:*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1})$$

near the origin. Then for sufficiently small t,

1. *f maps each $\Pi_k(t)$ into itself;*
2. *$f^{\circ n}(z) \to 0$ uniformly on each petal;*
3. *$\arg(f^{\circ n}(z)) \to 2k\pi/p$ locally uniformly on each petal;*
4. *$f : \Pi_k(t) \to \Pi_k(t)$ is conjugate to a translation.*

5. *$|f(z)| < |z|$ on a neighbourhood of the axis of each petal;*

Proof. For $0 < r_0 < 1$, define the sector S_0,

$$S_0 = \{re^{i\theta} : 0 < r < r_0, |\theta| < \pi/p\}$$

and the region W,

$$W = \{re^{i\theta} : r > \frac{1}{r_0^p}, |\theta| > \pi\}.$$

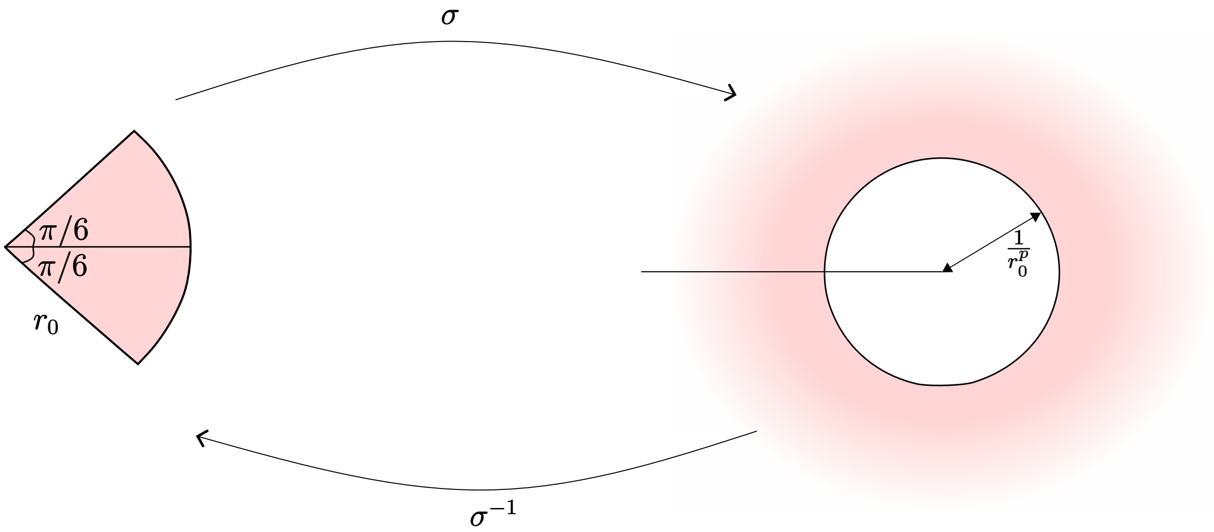

Figure 2.2: σ is a biholomorphism from S onto W.

It is clear that the map $\sigma : z \mapsto \frac{1}{z^p}$ is a biholomorphism of S_0 onto W with $\sigma^{-1} : W \to S_0$ given by $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$. The branch of p-th root that we select determines which sector of width $2\pi/p$, the inverse map maps to. (The other sectors being $S_k = \{0 < r < r_0, |\theta - 2k\pi/p| < \pi/p\}$.)

Now, the conjugate map of f on W is given by,

$$g(w) = \sigma f \sigma^{-1}(w) = f(w^{-\frac{1}{p}})^{-p}.$$

This just replaces the action of f on S by g on W, and we have the following commutative diagram:
Hence, we have the following estimates for g which will be crucial in everything that will follows:

$$g(w) = w + p + A/w + \theta(w), \text{ where } A \text{ is a constant and} \tag{2.1}$$

$$|\theta(w)| \leq B/|w|^{1+\frac{1}{p}}, B > 0. \tag{2.2}$$

Choose any K satisfying

$$K > \max\{1/r_0^p, 3(|A| + B)\} > 1 \text{ (as } r_0 < 1)$$

and let,

$$\Pi = \{x + iy : y^2 > 4K(K - x)\}.$$

Clearly, Π is bounded by a parabola and $\Pi \subset W$ (See Figure 2.3).

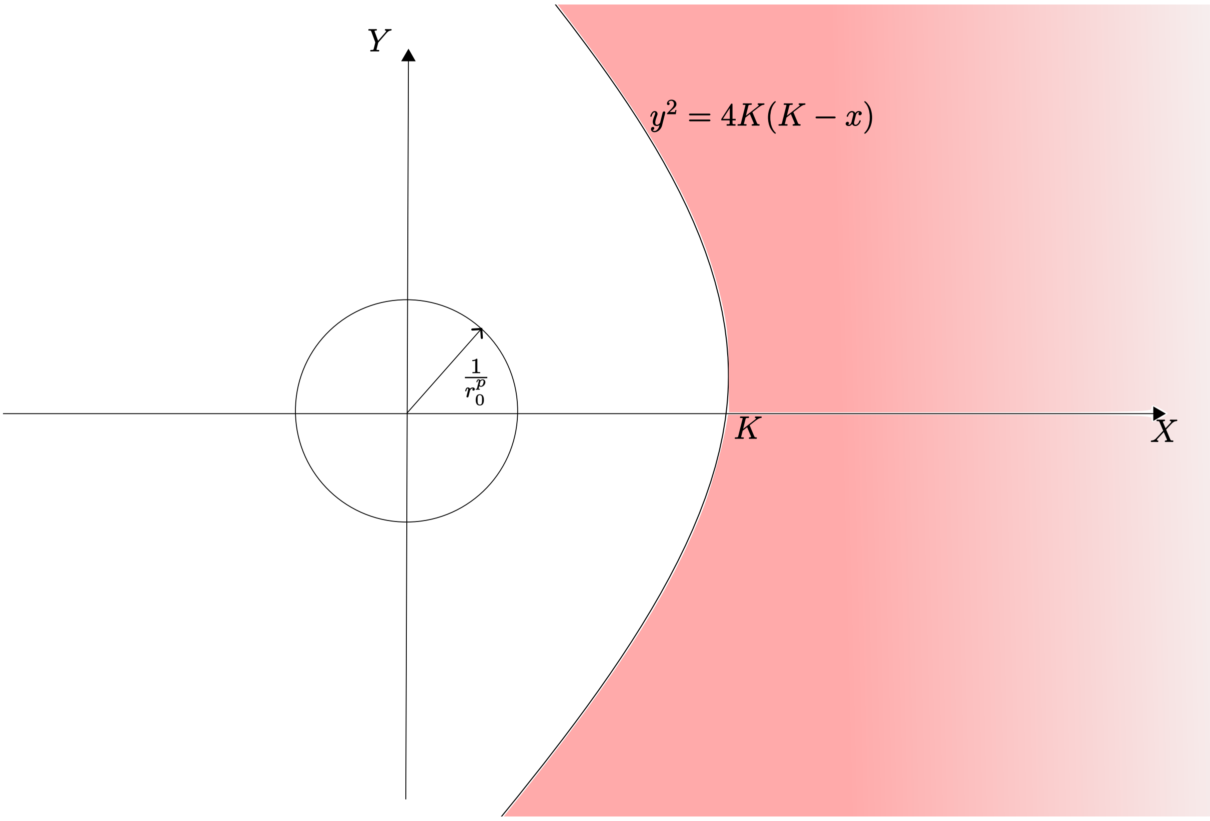

Figure 2.3: $\Pi = \{(x, y) : y^2 > 4K(K - x)\}$.

We have chosen this subset $\Pi \subset W$ because we will show that Π is nothing but the conformal image of $\Pi_0(t)$ under σ (for a suitable t) and g satisfies all the corresponding conditions that f should satisfy on $\Pi_0(t)$ according to the theorem.

Claim. Π is the conformal image of $\Pi_0(t)$ under σ for a suitable t.
The easiest way to see this is using polar coordinates. We write, $z = re^{i\theta}$ for $z \in S$ and $w = \rho e^{i\phi}$ for $w \in W$. Then, $\rho = \frac{1}{r^p}$ and $\phi = -p\theta$.

Now, we need to express Π in polar co-ordinates. To do so, we notice that points on the parabola are given by

$$\rho \text{ (distance from focus i.e. 0)} = 2K - \rho\cos\phi \text{ (distance from directrix i.e. } y = 2K).$$

(See Figure 2.4). Therefore, points on Π are given by

$$\rho > 2K - \rho\cos\phi.$$

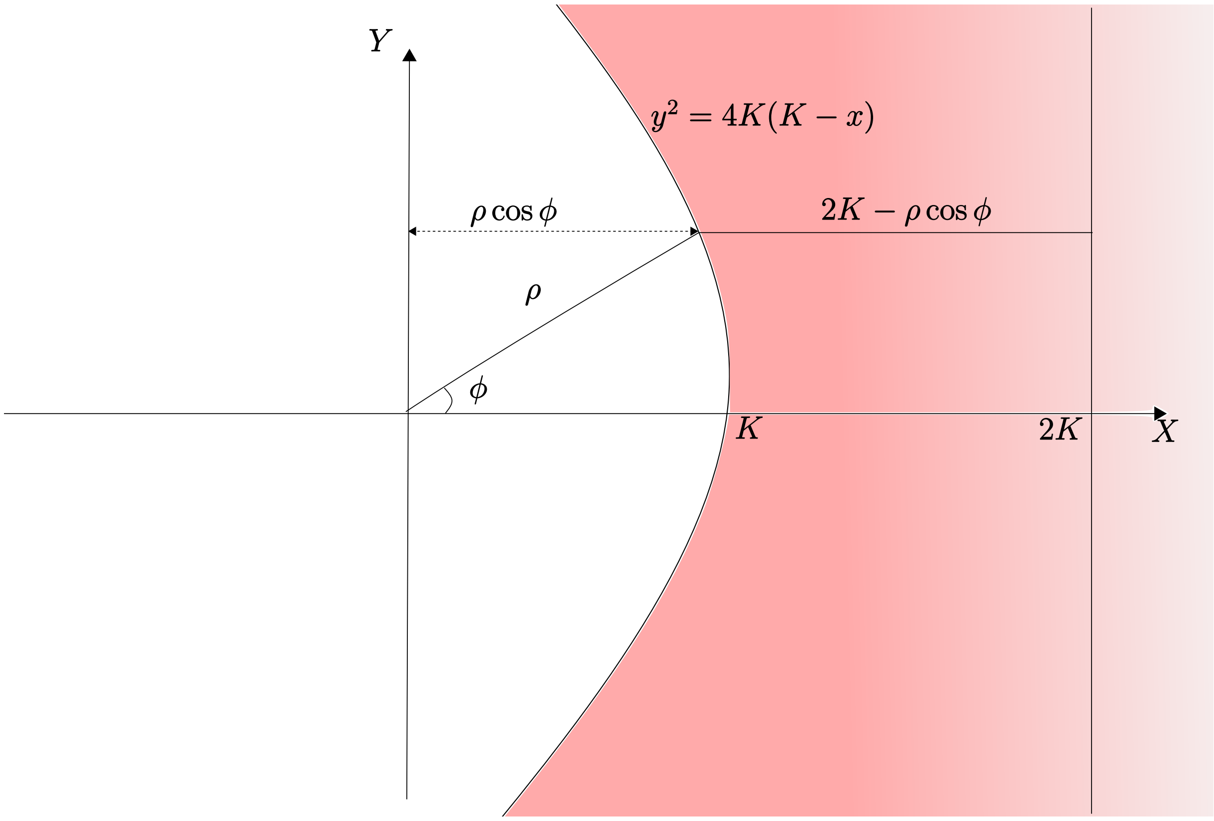

Figure 2.4: $\Pi = \{\rho e^{i\phi} : \rho > 2K - \rho \cos \phi\}$.

Hence,

$$\Pi = \{\rho e^{i\phi} : 2K < \rho(1 + \cos \phi)\}.$$

Now, let $\Omega = \sigma^{-1}(\Pi)$. Then, Ω is given by

$$\Omega = \{re^{i\theta} : 2Kr^p < 1 + \cos(p\theta)\}.$$

Hence, $\Omega = \Pi_0\left(\frac{1}{2K}\right)$.

Lemma 2.1.3. *g satisfies the following properties on Π:*

1. *Π is forward invariant under g.*
2. *$g^{\circ n}(w) \to \infty$ uniformly on Π.*
3. *$\arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π.*
4. *$g : \Pi \to \Pi$ is conjugate to a translation.*

Proof.

1. We write,

$$w = x + iy, \;\; g(w) = X + iY, \;\; A/w + \theta(w) = a + ib.$$

From Equation (2.1), we obtain,

$$\begin{aligned} X + iY &= (x+iy) + p + (a+ib) \\ \implies X &= x + p + a \text{ and } Y = y + b. \end{aligned}$$

Now, if $w \in \Pi$,

$$\begin{aligned} Y^2 - 4K(K-X) &= (y+b)^2 - 4K(K-x-p-a) \\ &= [y^2 - 4K(K-x)] + b^2 + 2yb + 4K(a+p) \\ &> 4Kp + (2yb + 4Ka) \\ &\geq |4Kp - |2yb + 4Ka||. \end{aligned}$$

Now, for $w \in \Pi$, $|w| > K > 1$. Hence we get,

$$|w||A/w + \theta(w)| \leq |w|(|A|/|w| + B/|w|^{1+\frac{1}{p}}) = |A| + B/|w|^{\frac{1}{p}} < |A| + B \tag{2.3}$$

(since for $|w| > 1$, $|w|^{\frac{1}{p}} > 1$). Therefore,

$$\begin{aligned} |2yb + 4Ka| &\leq 2|yb| + 4K|a| \\ &\leq 2|y||b| + 4K|a| \\ &\leq 2|w||a+ib| + 4|w||a+ib| \\ &= 6|w||a+ib| \\ &< 2K < 2Kp. \end{aligned}$$

Therefore, we see that $Y^2 - 4K(K-X) > 0$ and hence, $g(w) \in \Pi$ for $w \in \Pi$. Hence, Π is forward invariant under g.

2. Now, we will prove a stronger statement that for any $t > 0$ g maps $\Pi + t$ into $\Pi + t + p/2$. This is simply because, for $w \in \Pi + t$, we have, $y^2 - 4K(K + t - x) > 0$. Hence,

$$\begin{aligned} Y^2 - 4K(K + t + p/2 - X) &= [y^2 - 4K(K + t - x)] + b^2 + 2yb + 4K(a - p/2) \\ &> 2Kp + (2yb + 4Ka) \\ &\geq |2Kp - |2yb + 4Ka|| \\ &> 0. \end{aligned}$$

Therefore, if $w \in \Pi$, $g^{\circ n}(w) \in \Pi + np/2$. Hence, $|g^{\circ n}(w)| > \sqrt{n}$. This is simply because, $K + np/2 > 1 + n/2 > \sqrt{n}$ and hence $\Pi + np/2$ is disjoint from the disc $\{|z| \leq \sqrt{n}\}$.

Hence, $g^{\circ n}(w) \to \infty$ uniformly on Π.

3. Firstly, we see that,

$$g^{\circ k}(w) = g(g^{\circ k}(w)) = g^{\circ k}(w) + p + A/g^{\circ k}(w) + \theta(g^{\circ k}(w)).$$

By expanding the first term again and again, we obtain,

$$g^{\circ n}(w) = w + np + \sum_{k=0}^{n-1}\left(\frac{A}{g^{\circ k}(w)} + \theta(g^{\circ k}(w))\right). \tag{2.4}$$

Also note that form Equation (2.3), we have,

$$|A/w + \theta(w)| < (|A| + B)/K < \frac{1}{3}.$$

Let Q be a compact subset of Π. From now, we will assume that $w \in Q$ and we will use $C_1, C_2, C_3, \ldots$ to denote positive constants which will be dependent on Q.

Hence,

$$\begin{aligned} |g(w)| = |w + p + A/w + \theta(w)| &\geq ||w + p| - |A/w + \theta(w)|| \\ &= |w + p| - |A/w + \theta(w)| \\ &\geq |w| + p - \frac{1}{3}. \end{aligned}$$

Therefore, we obtain,

$$|g^{\circ n}(w)| \geq |w| + n(p - 1/3) \geq C_1 + C_2 n.$$

(Here, $C_1 = \min\{|w| : w \in Q\} > 0$ and $C_2 = p - \frac{1}{3} > 0$.)
We can select C_3 large enough such that

$$|g^{\circ n}(w)| \geq C_3 n. \tag{2.5}$$

Next, with Equation (2.2), and the above inequality, we get,

$$|\theta(g^{\circ n}(w)| \leq B/|g^{\circ n}(w)|^{1+\frac{1}{p}} \leq C_4/n^{1+\frac{1}{p}}. \tag{2.6}$$

Finally, combining the above two inequalities and Equation (2.4), we obtain,

$$\begin{aligned} |g^{\circ n}(w) - np| &\leq |w| + |A/w + \theta(w)| + \frac{|A|}{C_3}\sum_{k=1}^{n-1}\frac{1}{k} + C_4\sum_{k=1}^{n-1}\frac{1}{n^{1+\frac{1}{p}}} \\ &< C_5 + C_6\sum_{k=1}^{n}\frac{1}{k}. \end{aligned}$$

(Here, $C_5 = \max\{|w| : w \in Q\} + \frac{1}{3} + C_4\sum_{n=1}^{\infty} 1/n^{1+\frac{1}{p}}$ and $C_6 = |A|/C_3$.)
We can select C_7 large enough such that

$$|g^{\circ n}(w) - np| < C_7 \log n. \tag{2.7}$$

Remark. The above inequality follows from the fact that, if $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then $H_n - \log n \to \gamma$. (γ is known as the Euler's constant). So, we have that

$$\begin{aligned} P + QH_n &= P + Q(\log n + \gamma + \epsilon_n), \text{where } \epsilon_n \to 0 \\ &\leq Q\log n + (P + Q\max\{\epsilon_n\} + Q\gamma) \\ &= Q\log n + R \\ &< S\log n \end{aligned}$$

for S large enough.

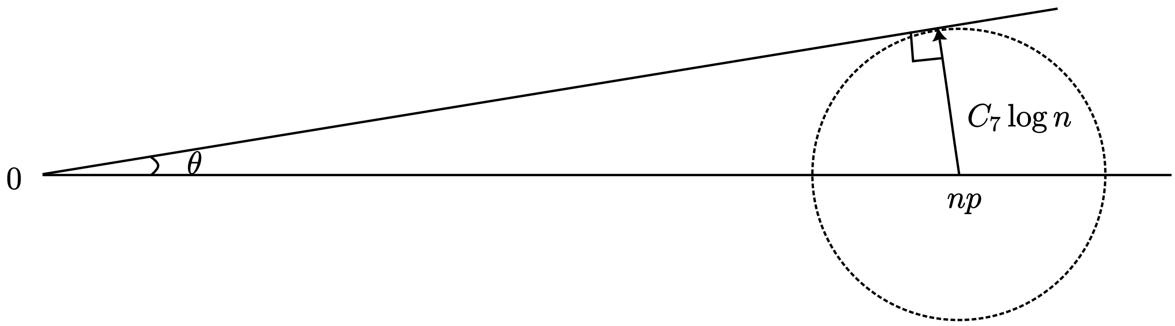

Figure 2.5: $|\arg(g^{\circ n}(w))| \leq \sin^{-1}(\frac{C_7\log n}{np})$.

From, $|g^{\circ n}(w) - np| < C_7\log n$, it follows that $|\arg(g^{\circ n}(w)| < \sin^{-1}\left(\frac{C_7\log n}{np}\right)$ for n large enough. Hence, $\arg(g^{\circ n}(w)) \to 0$ uniformly on Q, and consequently, locally uniformly on Π.

4. Define,

$$u_n(w) = g^{\circ n}(w) - np - (A/p)\log n.$$

Claim. $u_n(w)$ converges locally uniformly on Π to a holomorphic function u, that is one-to-one on Π.

$$u_{n+1}(w) - u_n(w) = [g^{\circ n+1}(w) - g^{\circ n}(w)] - p - (A/p)\log\left(\frac{n+1}{n}\right).$$

From Equation (2.2), we obtain,

$$\begin{aligned} u_{n+1}(w) - u_n(w) &= [g^{\circ n}(w) + p + A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - g^{\circ n}(w)] \\ &\qquad - p - (A/p)\log(1 + 1/n) \\ &= A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - (A/p)\log(1 + 1/n) \\ &= A(1/g^{\circ n}(w) - 1/np) + \theta(g^{\circ n}(w)) + (A/p)(1/n - \log(1 + 1/n)). \end{aligned}$$

Now, let Q is a compact subset of Π and $w \in Q$. We need to prove that u_n converges uniformly in Q. From the above equation, to prove that u_n converges uniformly in Q, we need to show that each of the following series converges uniformly in Q:

$$\sum_n |1/g^{\circ n}(w) - 1/np|, \sum_n |\theta(g^{\circ n}(w)), \sum_n |1/n - \log(1+1/n)|.$$

Let us look at the first series. We have, (using Equations (2.5) and (2.7))

$$|1/g^{\circ n}(w) - 1/np| = \frac{|g^{\circ n}(w) - np|}{|g^{\circ n}(w)|np} \leq \frac{C_7 \log n}{C_3 n^2 p} = C_8 \log n/n^2.$$

(Here $C_8 = C_7/(pC_3)$).
From Equation (2.6), it is clear that $\sum_n |\theta(g^{\circ n}(w))|$ converges.
Now, $0 < x - \log(1+x) \leq x^2$ for $x > 0$.
This is because, it is zero at $x = 0$ and $\frac{d}{dx}(x - \log(1+x)) = 1 - \frac{1}{1+x} > 0$ for $x > 0$.
Also, $x^2 - x + \log(1+x)$ is zero at $x = 0$ and $\frac{d}{dx}(x^2 - x + \log(1+x)) = 2x - 1 + \frac{1}{1+x} > 0$ for $x > 0$.

Putting $x = \frac{1}{n}$, we get,

$$|1/n - \log(1+1/n)| < 1/n^2.$$

Therefore, u_n converges locally uniformly to some holomorphic function u on Π.
Now, from $u_n(w) = g^{\circ n}(w) - np - (A/p)\log n$, we get that,

$$\begin{aligned}(n+1)p + (A/p)\log(n+1) + u_{n+1}(w) &= g^{\circ n+1}(w) \\ &= g^{\circ n}(g(w)) \\ &= np + (A/p)\log n + u_n(g(w)) \\ \implies p + (A/p)\log(1+1/n) + u_{n+1}(w) &= u_n(g(w)).\end{aligned}$$

Taking limit $n \to \infty$, we get,

$$p + u(w) = u(g(w)).$$

Since f is injective near the origin, g is injective on Π, (if K is chosen large enough). Therefore, $g^{\circ n}$ is injective on Π and hence, so is u_n. By Hurwitz Theorem, u is either injective or constant, but it is clearly not a constant since it satisfies the above equation.

This shows that $g : \Pi \to \Pi$ is conjugate to the map $z \mapsto z + p$ of $u(\Pi)$ into itself. $\square$

Coming back to our original theorem, we see that since g maps Π into itself, f also maps each $\Pi_k(t)$ into itself.

Now since, $|g^{\circ n}(w)| > \sqrt{n}$ for all $w \in \Pi$, $|\sigma f^{\circ n} \sigma^{-1}(w)| \to \infty$ uniformly on Π.

2.2 Behaviour near attracting fixed points

2.3 Behaviour near super-attracting fixed points

We will study the behaviour of analytic maps near super-attracting fixed points in the next chapter under Bottcher's theorem.

□

Chapter 3

Bottcher's Theorem and its extension

3.1 Bottcher's Coordinates

A fixed point z_0 is called a super-attracting fixed point of f if $f'(z_0) = 0$.

If z_0 is a super-attracting fixed point for f, we can conjugate the map such that $z = 0$ becomes our super-attracting fixed point.

Thus, our map takes the form $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$ in a neighbourhood of 0 , with $n \geq 2$ and $a_n \neq 0$, where the integer n is called the local degree.

Theorem 3.1.1 (**Bottcher's Theorem**). *With f as above, $\exists$ a local holomorphic change of coordinates $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to $w \mapsto w^n$ throughout some neighbourhood of 0 .*
Furthermore, ϕ is unique upto multiplication by an $(n-1)$ th root of unity.

Proof. **Existence.** Let $c \in \mathbb{C}$ be such that $c^{n-1} = a_n$. Then, the linearly conjugate map $cf(z/c)$ will have leading coefficient $+1$. Thus, without loss of generality, we will assume that our map f has the form $f(z) = z^n(1 + b_1 z + b_2 z^2 + \ldots) = z^n(1 + \eta(z))$, where $\eta(z) = (1 + b_1 z + b_2 z^2 + \ldots)$.

Choose $r \in \left(0, \frac{1}{2}\right)$ such that $|\eta(z)| < \frac{1}{2} \forall z \in \mathbb{D}_r$. This can be done since $\eta(0) = 0$ and η is continuous.

On this disc, we have two properties of f :

1. f maps this disc into itself:

We have, $|f(z)| = |z^n| |1 + \eta(z)| \leq |z|^n(1 + |\eta(z)|) < \frac{3}{2}|z|^n \leq \frac{3}{2^n}|z| \leq \frac{3}{4}|z| \forall z \in \mathbb{D}_r$. Here we are using the fact that $n \geq 2, |z| < \frac{1}{2}$ and $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$.

2. $f(z) \neq 0 \forall z \in \mathbb{D}_r \backslash \{0\}$.

This is simply because $|f(z)| = |z|^n|1+\eta(z)|$ and since $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$, we can't have $\eta(z) = -1$.

The k-th iterate of f i.e. $f^{\circ k}$ also maps the $\mathbb{D}_r$ into itself and $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$. Inductively, it can be shown that it has the form $f^{\circ k}(z) = z^{n^k}\left(1 + n^{k-1}b_1 z + \ldots\right)$.

The idea of the proof is to set,

$$\phi_k(z) = \left(f^{\circ k}(z)\right)^{\frac{1}{n^k}} = z\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}}$$

We choose z as our branch of holomorphic n^k th root of z^{n^k}.

Now, we can choose a holomorphic branch of $\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}}$ on $\mathbb{D}_r$ since $\mathbb{D}_r$ is simply connected and $\left(1 + n^{k-1}b_1 z + \ldots\right) \neq 0$ on $\mathbb{D}_r$ since $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$. Therefore we set,

$$\phi_k(z) = z\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}} = z\left(1 + \frac{b_1}{n}z + \ldots\right)$$

where the expression on the right provides us an explicit choice of n^k th root.

We will show that the functions ϕ_k converge uniformly to a limit function ϕ on $\mathbb{D}_r$. To prove the convergence, we make the substitution $z = e^u$ where u ranges over the left half plane $\mathbb{H}_r := \{u : \mathrm{Re}(u) < \log r\}$. The exponential map maps $\mathbb{H}_r$ onto $\mathbb{D}_r \backslash \{0\}$.

The map f from $\mathbb{D}_r$ into itself corresponds to a map from $\mathbb{H}_r$ into itself given by $F(u) = \log f\left(e^u\right)$. We can select a holomorphic branch of the logarithm of $f\left(e^u\right)$ because $\mathbb{H}_r$ is simply connected and $f\left(e^u\right) \neq 0$ on $\mathbb{H}_r$.

Set $\eta = \eta\left(e^u\right) = b_1 e^u + b_2 e^{2u} + \ldots$, then since $|\eta| < \frac{1}{2}$, we see that F can be written as

$$F(u) = \log\left(e^{nu}(1+\eta)\right) = nu + \log(1+\eta) = nu + \left(\eta - \frac{\eta^2}{2} + \frac{\eta^3}{3} - + \ldots\right)$$

where the final expression provides us an explicit choice of which branch of logarithm we are using. Clearly, $F : \mathbb{H}_r \to \mathbb{H}_r$ is a well-defined holomorphic function.

Similarly, the map ϕ_k corresponds to a map, $\Phi_k(u) = \log \phi_k\left(e^u\right)$.

$$\Phi_k(u) = \log \phi_k\left(e^u\right) = \log f^{\circ k}\left(e^u\right)^{\frac{1}{n^k}} = \frac{1}{n^k}\log f^{\circ k}\left(e^u\right).$$

Since we have already fixed the branch of logarithm that we are using, we see that,

$$\log f^{\circ k}\left(e^u\right) = \log f\left(f^{\circ k-1}\left(e^u\right)\right) = \log f\left(e^{\log f^{\circ k-1}\left(e^u\right)}\right) = F\left(\log f^{\circ k-1}\left(e^u\right)\right)$$

Hence, inductively we can see that $\log f^{\circ k}\left(e^u\right) = F^{\circ k}(u)$.

Therefore, $\Phi_k(u) = F^{\circ k}(u)/n^k$. It is clear from this expression that $\Phi_k : \mathbb{H}_r \to \mathbb{H}$.

Now since $|\eta| < \frac{1}{2}$, we have

$$|F(u) - nu| = |\log(1+\eta)| < \log 2 < 1$$

Hence,

$$|\Phi_{k+1}(u) - \Phi_k(u)| = \frac{1}{n^{k+1}} \left| F^{\circ k+1}(u) - nF^{\circ k}(u) \right| < \frac{1}{n^{k+1}}$$

by the above inequality.

We have, $\phi_k(e^u) = e^{\Phi_k(u)}$. Since, the exponential map, $e^{\square} : \mathbb{H} \to \mathbb{D}$ from the left half plane to the unit disc decreases distance, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r \backslash \{0\}.$$

Since $\phi_k(0) = 0$ for all k, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r$$

Hence, the maps ϕ_k converge uniformly to some limit function ϕ on $\mathbb{D}_r$ by the Cauchy criterion for uniform convergence.

Clearly, $\phi(0) = 0$ and ϕ is holomorphic on $\mathbb{D}_r$ by Weierstrass convergence theorem.

It is clear that each $\phi_k : \mathbb{D}_r \to \mathbb{D}$. This is because $\phi_k(e^u) = e^{\Phi_k(u)}$ and $\Phi_k : \mathbb{H}_r \to \mathbb{H}$ and $e^{\square} : \mathbb{H} \to \mathbb{D}\backslash\{0\}$. Hence, $\phi : \mathbb{D}_r \to \mathbb{D}$. (Clearly $\operatorname{Im}(\phi)$ cannot contain points from $\partial\mathbb{D}$ because ϕ is holomorphic, hence it is an open map).

Now, it can be easily seen that, $\phi_k(f(z)) = \phi_{k+1}(z)^n$.

Hence, $\lim_{k\to\infty} \phi_k(f(z)) = \lim_{k\to\infty} \phi_{k+1}(z)^n \implies \phi(f(z)) = \phi(z)^n$ by continuity of nth power map.

Also, since $\phi_k'(0) = 1 \forall k \in \mathbb{N}$ (from the power series of ϕ_k), we have $\phi'(0) = 1$. Hence, ϕ is invertible in some neighbourhood of 0 .

Therefore, we have a holomorphic change of coordinates in some neighbourhood of 0 which conjugates f to the nth power map. In this neighbourhood, ϕ is one-to-one, $f(z) \neq 0$ for $z \neq 0$ (i.e. no other point maps to the super-attracting fixed point via f) and f maps this neighbourhood into itself.

Uniqueness. It suffices to study the special case $f(z) = z^n$. If we can prove that any map which conjugates $z \mapsto z^n$ to itself is just multiplication by $(n-1)$ th root of unity, then for any general map $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, if we have two maps ϕ and ψ which conjugate it to $z \mapsto z^n$, then $\phi \circ \psi^{-1}$ is a map which conjugates $z \mapsto z^n$ to itself. Hence, $\phi \circ \psi^{-1} = cz$, where $c^{n-1} = 1$. Therefore, $\phi = c\psi$, where c is a $(n-1)$ th root of unity.

So, let $\phi(z) = c_1 z + c_k z^k + \ldots, (c_1 \neq 0)$ be a map which conjugates $z \mapsto z^n$ to itself. Then, we should have $\phi(z^n) = \phi(z)^n$. Now,

$$\phi(z^n) = c_1 z^n + c_k z^{nk} + \ldots$$

and

$$\phi(z)^n = c_1^n z^n + n c_1^{n-1} c_k z^{n+k-1} + \ldots$$

Comparing coefficients, we get $c_1^n = c_1$ and $n c_1^{n-1} c_k = 0$ since $nk > n+k-1$ for $k \geq 2$. Therefore, we get $c_1^{n-1} = 1$ and $c_k = 0$. The form $\phi(z) = c_1 z + c_k z^k + \ldots$ can be modified to any $k \geq 2$ to get $c_k = 0$ by the same process.

Therefore, $\phi(z) = cz$, where c is a $(n-1)$ th root of unity.

□

3.2 Extension of Bottcher's coordinates

We might hope to extend the Bottcher's coordinates to the whole of the immediate basin of attraction of the super-attracting fixed point. But this is not always possible. This is because, it requires computing expressions of the form $z \mapsto \left(\phi\left(f^{\circ k}(z)\right)\right)^{\frac{1}{n^k}}$, which is not always possible. For example, the basin may not be simply connected, or some point may map directly onto the super-attracting fixed point, in which case we can't take n^k-th roots, because $\phi\left(f^{\circ k}(z)\right)$ must be zero at those points.

Theorem 3.2.1 (**Extension of** $|\phi|$). *If f has a super-attracting fixed point p, with immediate basin of attraction $\mathcal{A}$, then the function $z \mapsto |\phi(z)|$ of the above theorem extends uniquely to a continuous map $|\phi| : \mathcal{A} \to [0,1)$ which satisfies $|\phi|(f(z)) = |\phi|(z)^n$.*

Furthermore, $|\phi|$ is real analytic except at the iterated preimages of p, where it takes the value 0 .

Proof. Set $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{1}{n^k}}$ for large enough k for each $z \in \mathcal{A}$. ϕ is only defined in a some small neighbourhood of p. But since, $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, after k many iterates for some large k, $f^{\circ k}(z)$ belongs to the domain of definition of ϕ, which we shall call $\hat{U}$.

It is independent of the value of k (if k is large enough). Note that, if $f^{\circ k}(z) \in \hat{U}$, then so does $f^{\circ k+1}(z)$, since f maps $\hat{U}$ into itself.

Suppose we choose k minimal such that $f^{\circ k}(z) \in \hat{U}$. Then,

$$|\,\phi\left(f^{\circ k+1}(z)\right|^{\frac{1}{n^{k+1}}} = \left|\phi\left(f\left(f^{\circ k}(z)\right)\right)\right|^{\frac{1}{n^{k+1}}} = \left|\phi\left(f^{\circ k}(z)\right)^n\right|^{\frac{1}{n^{k+1}}} = \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{1}{n^k}} = |\phi|(z).$$

In the proof of the Bottcher's theorem, we saw that $\phi(z) \in \mathbb{D} \forall z \in \hat{U}$ Hence, $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right| < 1 \forall z \in \mathcal{A}$. Therefore, $|\phi| : \mathcal{A} \to [0,1)$.

Also,

$$\begin{aligned}
|\phi|(f(z)) &= \left| \phi\left(f^{\circ k}(f(z))\right|^{\frac{1}{n^k}} \right. \\
&= \left|\phi\left(f\left(f^{\circ k}(z)\right)\right)\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f^{\circ k}(z)\right)^n\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{n}{n^k}} \\
&= |\phi|(z)^n.
\end{aligned}$$

It is also clear that $|\phi| = 0$ only at p and its iterated preimages.

If q is an iterated preimage of p, say $f^{\circ k}(q) = p$, then we have $|\phi|(q) = \left| \phi\left(f^{\circ k}(q)\right|^{\frac{1}{n^k}} \right. = |\phi(p)|^{\frac{1}{n^k}} = 0$

Now, Suppose $|\phi|(z) = 0$ for some z. Then, $|\phi|(z)^{n^k} = 0 \forall k \implies |\phi|\left(f^{\circ k}(z)\right) = 0 \forall k$. But for some large k, $f^{\circ k}(z)$ belongs to the domain of definition of ϕ. But that means, $f^{\circ k}(z) = p$, since no other point in that domain is mapped to zero by ϕ. Hence, z is an an iterated preimage of p.

Now, since $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, for each $a \in \mathcal{A}$, we have a neighbourhood W_a and a constant $k \in \mathbb{N}$ such that $f^{\circ k}(z) \in \hat{U} \forall z \in W_a$.

Hence, for $z \in W_a$, we can define $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right| = |g(z)|$, where $g = \phi \circ f^{\circ k}\big|_{W_a}$. Therefore, $|\phi|_{W_a} = |g|$, where g is some holomorphic function defined on W_a.

It is clear from this that $|\phi|$ is continuous in $\mathcal{A}$.

Now, if h is any holomorphic function, then $|h(z)|$ is real-analytic everywhere in its domain except at those z, where $h(z) = 0$.

Since, $|g| = |\phi|_{W_a}$ is zero only at the iterated preimages of f in W_a, $|\phi|_{W_a}$ is real analytic everywhere in W_a except at the iterated preimages of p.

Therefore, $|\phi|$ is real analytic everywhere in $\mathcal{A}$ except at the iterated preimages of p. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with a super-attracting fixed point p. Then the associated Bottcher map ϕ carries a neighbourhood of p biholomorphically onto a neighbourhood of zero, conjugating f to the nth power map, where n is the local degree of f near p. ϕ has a local inverse ψ_ϵ which maps the ϵ-disc around zero to a neighbourhood of p. $\square$

Theorem 3.2.2 (Extending ψ_ϵ). *There exists a unique open disc of maximal radius $0 < r \leq 1$ such that ψ_ϵ extends holomorphically to a map $\psi : \mathbb{D}_r \to \mathcal{A}$, where $\mathcal{A}$ is the immediate basin of attraction of p.*

1. *If $r = 1$, then ψ maps the open unit disc $\mathbb{D}$ onto $\mathcal{A}$ biholomorphically.*

2. *If $0 < r < 1$, then ψ maps $\mathbb{D}_r$ onto its image biholomorphically and there exists atleast one other critical point in $\mathcal{A}$ on the boundary of $\psi(\mathbb{D}_r)$.*

If ψ_ϵ is extended biholomorphically in this way to the map ψ defined on $\mathbb{D}_r$, then the inverse map $\psi^{-1} : \psi(\mathbb{D}_r) \to \mathbb{D}_r$ must be the extension of ϕ from some neighbourhood of p to $\psi(\mathbb{D}_r)$ (since ψ^{-1} agrees with ϕ on some neighbourhood of p).

Proof. Let us try to extend ψ_ϵ along radial lines by analytic continuation. Then, we can't extend it indefinitely as it would yeild a holomorphic map ψ from the entire complex plane onto an open set $\psi(\mathbb{C}) \subset \mathcal{A} \subsetneq \mathbb{C}_\infty$. ($\mathcal{A}$ cannot be the whole of $\mathbb{C}_\infty$ since the Julia set of f cannot be empty as $\deg(f) \geq 2$). We can conjugate f such that $\infty \notin \mathcal{A}$. Then the corresponding map ψ will map the whole of the complex plane into $\mathcal{A} \subset \mathbb{C}$. By Louiville's theorem, since the map ψ cannot be a constant, $\psi(\mathbb{C}) = \mathbb{C} = \mathcal{A}$. Therefore, $\mathbb{C}_\infty \backslash \mathcal{A} = \{\infty\}$. This too is not possible since the Julia set of f must be an infinite set since $\deg(f) \geq 2$.

Thus, there must be some largest radius r so that ψ_ϵ extends analytically throughout the open disc $\mathbb{D}_r$.

Also, $|\phi|(\psi(w)) = |\phi(\psi(w))| = |w|$ near 0 , hence for all $w \in \mathbb{D}_r$ by analytic continuation.???

Since, $|\phi| : \mathcal{A} \to [0, 1)$, this proves that for any $w \in \mathbb{D}_r, |\phi|(\psi(w)) = |w| < 1$. Therefore, ψ can be defined only on $\mathbb{D}_r$ for $r \leq 1$.

We will now show that ψ is actually one-to-one on $\mathbb{D}_r$. Suppose $\psi(w_1) = \psi(w_2)$. Applying $|\phi|$, we see that $|w_1| = |w_2|$. Choose such a pair such that $\psi(w_1) = \psi(w_2)$ $(w_1 \neq w_2)$ with $|w_1| = |w_2|$ minimal. A minimal pair exists because for $|w| < \epsilon, \psi = \psi_\epsilon$ which is one-to-one as it is invertible.

Now, ψ is an open mapping. Choose a sufficiently small neighbourhood U_{w_2} of w_2. Then, $\psi(U_{w_2})$ is a small neighbourhood of $\psi(w_1) = \psi(w_2)$. Hence, for any w_1' sufficiently close to $w_1, \psi(w_1') \in \psi(U_{w_2})$. Hence, we can find w_2' sufficiently close to w_2 such that $\psi(w_1') = \psi(w_2')$. Choosing $|w_1'| < |w_1|$, we get a contradiction.

Hence, ψ maps $\mathbb{D}_r$ onto its image biholomorphically.

In case when $r = 1, U = \psi(\mathbb{D}) = \mathcal{A}$. If not then we would have some boundary point of U, say $z_0 \in \mathcal{A}$. We can approximate z_0 by points of $\psi(w_j)$, where $|w_j| \to 1$.

Now, $\lim_{j\to\infty} \psi(w_j) = z_0$. Hence,

$$\lim_{j\to\infty} |\phi|(\psi(w_j)) = |\phi|(z_0) \Longrightarrow \lim_{j\to\infty} |w_j| = |\phi|(z_0) \Longrightarrow |\phi|(z_0) = 1$$

which is impossible.

Now, let $0 < r < 1$. We need to prove that ∂U, where $U = \psi(\mathbb{D}_r)$ must contain a critical point of f. Suppose, $w_0 \in \partial\mathbb{D}_r$ and let $(w_j)_{j=1}^{\infty} \subset \mathbb{D}_r$ such that $w_j \to w_0$. Let

$\psi(w_j) \to z_0$. Then $z_0 \in \partial U$ because ψ maps $\mathbb{D}_r$ onto U biholomorphically.

If z_0 is not a critical point of f, then f maps a neighbourhood of z_0, say A onto a neighbourhood of $f(z_0)$, say B biholomorphically.

It should be noted that B can be chosen such that $B \subset U$. This is because $f(z_0) \in U$. We have, $\lim_{j\to\infty} \psi(w_j) = z_0 \implies \lim_{j\to\infty} f(\psi(w_j)) = f(z_0) \implies \lim_{j\to\infty} \psi\left(w_j^n\right) = f(z_0) \implies \psi(w_0^n) = f(z_0)$. Since, $|w_0| = r < 1, |w_0|^n < r^n < r$. Hence, $w_0 \in \mathbb{D}_r$. Therefore, $\psi(w_0^n) = f(z_0) \in U$

Let g be the local inverse of f near $f(z_0)$. Then, ψ can be extended throughout a neighbourhood of w_0 by

$$w \mapsto g(\psi(w^n))$$

We have, $\psi(w_0^n) = f(z_0) \implies w_0^n = \phi(f(z_0))$. Since, $\phi(B)$ is a neighbourhood of $\phi(f(z_0))$ lying inside $\mathbb{D}_r$, choose a small enough neighbourhood of w_0, say C such that $w^n \in \phi(B)$, for all $w \in C$. In this neighbourhood, C our newly defined map agrees with ψ on $C \cap \mathbb{D}_r$. This is because, for $w \in C \cap \mathbb{D}_r, f(\psi(w)) = \psi(w^n) \in B$. Therefore, $g(\psi(w^n))$ can be defined and $\psi(w) = g(\psi(w^n)) \in A$. Hence, our new map is an analytic continuation of ψ on the neighbourhood C.

Now, if none of the $z_0 \in \partial U$ are critical points, we can extend ψ to a neighbourhood of $w_0 \forall w_0 \in \partial\mathbb{D}_r$. Clearly, these continutations would patch together to define ψ in a strictly greater disc than $\mathbb{D}_r$, which is a contradiction. $\square$

Chapter 4

Introduction to the Mandelbrot Set

We consider the set of quadratic polynomials $\{f_c(z) = z^2 + c : c \in \mathbb{C}\}$. It is enough to consider this set because every quadratic polynomial is conjugate to a quadratic polynomial of the type $f_c(z)$ for some unique $c \in \mathbb{C}$.

To prove this, let $f(z) = az^2 + bz + c$, $a \neq 0$. And consider the conjugation, $\sigma(z) =$

4.1 Definition of the Mandelbrot Set

First, we define as new type of set, known as the *Filled-in Julia Set* for a polynomial P.

Definition 4.1.1 (Filled-in Julia Set). *The Filled-in Julia Set of a polynomial P is defined as $K(P) = \hat{\mathbb{C}} \backslash F_\infty(P)$. It is the union of the Julia set and the bounded Fatou components. It is denoted by $K(P)$ or simply K when the context is clear.*

By Lemma 1.2.7, K can also be defined as follows:

$$K = \{z \in \mathbb{C} : P^{\circ n}(z) \text{ is bounded}\}.$$

Notation. We will use F_c, J_c and K_c for the $F_\infty(f_c)$, $J(f_c)$ and $K(f_c)$ respectively.

Definition 4.1.2 (Mandelbrot Set). *The Mandelbrot Set is defined as*

$$M = \{c \in \mathbb{C} : K_c \textit{ is connected}\}.$$

Now, we have the following two theorems for polynomials:

- For polynomials, since F_∞ is a completely invariant Fatou component (by Theorem 1.2.5), $\partial F_\infty = J$ (by Theorem 1.2.2).
- And, from Theorem 1.2.1, we have that F_∞ is simply connected $\iff$ $\hat{\mathbb{C}} \backslash F_\infty$ is connected $\iff$ ∂F_∞ is connected.

Thus, for a polynomial,

$$F_\infty \text{ is simply connected } \iff K \text{ is connected } \iff J \text{ is connected.}$$

Hence, we have the following equivalent descriptions for the Mandelbrot Set:

$$\begin{aligned} M &= \{c \in \mathbb{C} : K_c \text{ is connected}\} \\ &= \{c \in \mathbb{C} : F_c \text{ is simply connected}\} \\ &= \{c \in \mathbb{C} : J_c \text{ is connected}\}. \end{aligned}$$

4.2 The Fundamental Dichotomy

Theorem 4.2.1. *For a polynomial P, the following are equivalent:*

1. *F_∞ is simply connected $\iff$ J is connected $\iff$ K is connected.*
2. *There are no finite critical points of P in F_∞.*

Proof. First assume that F_∞ is simply connected $\implies$ $c(F_\infty) = 1$ and hence, $\chi(F_\infty) = 2 - c(F_\infty) = 1$. Now, since F_∞ is completely invariant and P is a polynomial of degree d (say), P is a d-fold map of F_∞ onto itself. Applying the Riemann-Hurwitz relation to the map P of F_∞ onto itself, we obtain,

$$\begin{aligned} \chi(F_\infty) + \delta_P(F_\infty) &= d\,\chi(F_\infty) \\ \implies 1 + \delta_P(F_\infty) &= d \\ \implies \delta_P(F_\infty) &= d - 1. \end{aligned}$$

Now, $\delta_P(\infty) = d - 1$ and therefore, P does not have any finite critical points in F_∞.

For the converse part, assume there are no critical points of P in F_∞. Then, the Bottcher's map ϕ which conjugates P to the map, $z \mapsto z^d$ can be extended to the whole of F_∞ and $\phi : F_\infty \to \mathbb{D}$ is a biholomorphism. Hence, F_∞ is simply connected. $\square$

Now, quadratic maps have only one finite critical point and f_c have the critical point at 0 for all $c \in \mathbb{C}$. Hence, by the Fundamental Dichotomy, F_c is simply connected $\iff$ $0 \notin F_c$ or $0 \in K_c$. Using, c_n to denote $f_c^{\circ n}(0)$, we get,

$$\begin{aligned} M &= \{c \in \mathbb{C} : 0 \in K_c\} \\ &= \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}. \end{aligned}$$

Note that $c_0 = 0$ and $c_1 = f_c(0) = c$. So, (c_n) is also the forward orbit of c. Hence, in other words, the Mandelbrot Set consists of $c \in \mathbb{C}$ such that its forward orbit under the map f_c remains bounded.

4.3 The other end of the Dichotomy Theorem

By the Dichotomy Theorem, we can say that if even one finite critical point of P lies in F_∞, then K cannot be connected. But this theorem states that if all finite critical points of P lie in F_∞, then K is not only disconnected, but totally disconnected.

Definition 4.3.1 (Cantor set). *A subset $X \subset \hat{\mathbb{C}}$ is called a Cantor set if it is non-empty, closed, perfect and totally disconnected.*

Theorem 4.3.1. *Let R be a rational map with $\deg(R) \geq 2$. Let α be a super-attracting fixed point of R. If the Fatou component of R containing α, say F_α, contains all the critical points of R, then $J(R)$ is a Cantor set.*

Corollary 4.3.1.1. *If $c \notin M$, then J_c is a Cantor set.*

Proof. Since 0 is the only critical points of f_c (apart from ∞), if it belongs to $F_c = F_\infty(f_c)$, i.e. if $0 \notin K_c \iff c \notin M$, then J_c is a Cantor set. □

4.4 Some properties of the Mandelbrot Set

We know, $M = \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}$. This description for M can be strengthened significantly by the following theorem:

Theorem 4.4.1. $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Proof. Obviously, $\{c \in \mathbb{C} : |c_n| \leq 2\} \subseteq M$.

Now, Suppose that $c \in M$. We need to prove that $|f_c^{\circ n}(c)| = |c_n| \leq 2$ for all $n \in \mathbb{N}$. Consider the set $W_c = \{z \in \mathbb{C} : |z| \geq |c|, |z| > 2\}$. For $z \in W_c$,

$$|f_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| \geq |z|(|z| - 1) = |z|(1 + \epsilon)$$

for some $\epsilon > 0$ (as $|z| > 2$). Clearly, $|f_c^{\circ n}(z)| \geq |z|(1 + \epsilon)^n \implies z \notin K_c$.
This implies $|c| \leq 2$. Consequently, $|f_c^{\circ n}(c)| \leq 2$ for all $n \in \mathbb{N}$.
Hence, $M \subseteq \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Therefore, $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$. □

As $c_1 = c$, we have that $|c| \leq 2$ for all $c \in M$ i.e $M \subseteq \{c \in \mathbb{C} : |c| \leq 2\}$. This turns out to be the strongest bound possible for $|c|$ as $-2 \in M$. The orbit of 0 under the map $z \mapsto z^2 - 2$ is:

$$0 \mapsto -2 \mapsto 2 \mapsto 2$$

and hence is bounded.

Theorem 4.4.2. *The Mandelbrot set is compact and $\hat{\mathbb{C}} \backslash M$ is open and connected.*

Proof. Let, $c_n = f_c^{\circ n}(c) = Q_n(c)$ be a polynomial in c. Clearly, from Theorem 4.4.1

$$M = \cap_{n=1}^{\infty} Q_n^{-1}(\overline{\mathbb{D}_2}),$$

where $\overline{\mathbb{D}_2} = \{z \in \mathbb{C} : |z| \leq 2\}$. Thus, M is closed. It is already known that it is bounded. Hence, it is compact.

Now,

$$\hat{\mathbb{C}} \backslash M = \cup_{n=1}^{\infty} Q_n^{-1}(E)$$

where $E = \hat{\mathbb{C}} \backslash \overline{\mathbb{D}_2}$. Now, E is open and connected and since, Q_n are non-constant polynomials, $Q_n^{-1}(E)$ is open and connected for all $n \in \mathbb{N}$. Also, each one of them contains ∞ and hence, their union is also open and connected.
Therefore, $\hat{\mathbb{C}} \backslash M$ is open and connected.

$\square$

4.5 Plotting the Mandelbrot Set

Theorem 4.4.1 is also used to plot the Mandelbrot Set. A simple code in python would be

Chapter 5

Connectedness of the Mandelbrot Set