1 Bottcher's Coordinates and Extension

1.1 Bottcher's Coordinates

A fixed point z_0 is called a super-attracting fixed point of f if $f'(z_0) = 0$.
If z_0 is a super-attracting fixed point for f, we can conjugate the map such that $z = 0$ becomes our super-attracting fixed point.

Thus, our map takes the form $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$ in a neighbourhood of 0, with $n \geq 2$ and $a_n \neq 0$, where the integer n is called the *local degree.*

Theorem 1.1.1 (**Bottcher's Theorem**). *With f as above, $\exists$ a local holomorphic change of coordinates $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to $w \mapsto w^n$ throughout some neighbourhood of* 0.
Furthermore, ϕ is unique upto multiplication by an $(n-1)$th root of unity.

Proof. **Existence.** Let $c \in \mathbb{C}$ be such that $c^{n-1} = a_n$. Then, the linearly conjugate map $cf(z/c)$ will have leading coefficient $+1$. Thus, without loss of generality, we will assume that our map f has the form $f(z) = z^n(1 + b_1 z + b_2 z^2 + \ldots) = z^n(1 + \eta(z))$, where $\eta(z) = (1 + b_1 z + b_2 z^2 + \ldots)$.

Choose $r \in (0, \frac{1}{2})$ such that $|\eta(z)| < \frac{1}{2}$ $\forall z \in \mathbb{D}_r$. This can be done since $\eta(0) = 0$ and η is continuous.
On this disc, we have two properties of f:

1. f maps this disc into itself:
   We have, $|f(z)| = |z^n||1 + \eta(z)| \leq |z|^n(1 + |\eta(z)|) < \frac{3}{2}|z|^n \leq \frac{3}{2^n}|z| \leq \frac{3}{4}|z|$ $\forall z \in \mathbb{D}_r$.
   Here we are using the fact that $n \geq 2$, $|z| < \frac{1}{2}$ and $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$.

2. $f(z) \neq 0$ $\forall z \in \mathbb{D}_r \backslash \{0\}$:
   This is simply because $|f(z)| = |z|^n|1 + \eta(z)|$ and since $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$, we can't have $\eta(z) = -1$.

The k-th iterate of f i.e. $f^{\circ k}$ also maps the $\mathbb{D}_r$ into itself and $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$.
Inductively, it can be shown that it has the form $f^{\circ k}(z) = z^{n^k}(1 + n^{k-1} b_1 z + \ldots)$.

The idea of the proof is to set,

$$\phi_k(z) = (f^{\circ k}(z))^{\frac{1}{n^k}} = z(1 + n^{k-1} b_1 z + \ldots)^{\frac{1}{n^k}}.$$

We choose z as our branch of holomorphic n^kth root of z^{n^k}.
Now, we can choose a holomorphic branch of $(1 + n^{k-1} b_1 z + \ldots)^{\frac{1}{n^k}}$ on $\mathbb{D}_r$ since $\mathbb{D}_r$ is simply connected and $(1 + n^{k-1} b_1 z + \ldots) \neq 0$ on $\mathbb{D}_r$ since $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$.

Therefore we set,

$$\phi_k(z) = z(1 + n^{k-1}b_1 z + \ldots)^{\frac{1}{n^k}} = z\left(1 + \frac{b_1}{n}z + \ldots\right),$$

where the expression on the right provides us an explicit choice of n^kth root.

We will show that the functions ϕ_k converge uniformly to a limit function ϕ on $\mathbb{D}_r$. To prove the convergence, we make the substitution $z = e^u$ where u ranges over the left half plane $\mathbb{H}_r := \{u : \mathrm{Re}(u) < \log r\}$. The exponential map maps $\mathbb{H}_r$ onto $\mathbb{D}_r \backslash \{0\}$.

The map f from $\mathbb{D}_r$ into itself corresponds to a map from $\mathbb{H}_r$ into itself given by $F(u) = \log f(e^u)$. We can select a holomorphic branch of the logarithm of $f(e^u)$ because $\mathbb{H}_r$ is simply connected and $f(e^u) \neq 0$ on $\mathbb{H}_r$.
Set $\eta = \eta(e^u) = b_1 e^u + b_2 e^{2u} + \ldots$, then since $|\eta| < \frac{1}{2}$, we see that F can be written as

$$F(u) = \log(e^{nu}(1+\eta)) = nu + \log(1+\eta) = nu + \left(\eta - \frac{\eta^2}{2} + \frac{\eta^3}{3} - + \ldots\right),$$

where the final expression provides us an explicit choice of which branch of logarithm we are using. Clearly, $F : \mathbb{H}_r \to \mathbb{H}_r$ is a well-defined holomorphic function.

Similarly, the map ϕ_k corresponds to a map, $\Phi_k(u) = \log \phi_k(e^u)$.

$$\Phi_k(u) = \log \phi_k(e^u) = \log f^{\circ k}(e^u)^{\frac{1}{n^k}} = \frac{1}{n^k}\log f^{\circ k}(e^u).$$

Since we have already fixed the branch of logarithm that we are using, we see that,

$$\log f^{\circ k}(e^u) = \log f(f^{\circ k-1}(e^u)) = \log f(e^{\log f^{\circ k-1}(e^u)}) = F(\log f^{\circ k-1}(e^u)).$$

Hence, inductively we can see that $\log f^{\circ k}(e^u) = F^{\circ k}(u)$.
Therefore, $\Phi_k(u) = F^{\circ k}(u)/n^k$. It is clear from this expression that $\Phi_k : \mathbb{H}_r \to \mathbb{H}$.

Now since $|\eta| < \frac{1}{2}$, we have

$$|F(u) - nu| = |\log(1+\eta)| < \log 2 < 1.$$

Hence,

$$|\Phi_{k+1}(u) - \Phi_k(u)| = \frac{1}{n^{k+1}}|F^{\circ k+1}(u) - nF^{\circ k}(u)| < \frac{1}{n^{k+1}},$$

by the above inequality.

We have, $\phi_k(e^u) = e^{\Phi_k(u)}$. Since, the exponential map, $e^{\square} : \mathbb{H} \to \mathbb{D}$ from the left half plane to the unit disc decreases distance, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r \backslash \{0\}.$$

Since $\phi_k(0) = 0$ for all k, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r.$$

Hence, the maps ϕ_k converge uniformly to some limit function ϕ on $\mathbb{D}_r$ by the Cauchy criterion for uniform convergence.
Clearly, $\phi(0) = 0$ and ϕ is holomorphic on $\mathbb{D}_r$ by Weierstrass convergence theorem.

It is clear that each $\phi_k : \mathbb{D}_r \to \mathbb{D}$. This is because $\phi_k(e^u) = e^{\Phi_k(u)}$ and $\Phi_k : \mathbb{H}_r \to \mathbb{H}$ and $e^{\square} : \mathbb{H} \to \mathbb{D}\backslash\{0\}$. Hence, $\phi : \mathbb{D}_r \to \mathbb{D}$. (Clearly $\mathrm{Im}(\phi)$ cannot contain points from $\partial\mathbb{D}$ because ϕ is holomorphic, hence it is an open map).

Now, it can be easily seen that, $\phi_k(f(z)) = \phi_{k+1}(z)^n$.
Hence, $\lim_{k\to\infty} \phi_k(f(z)) = \lim_{k\to\infty} \phi_{k+1}(z)^n \implies \phi(f(z)) = \phi(z)^n$ by continuity of nth power map.

Also, since $\phi_k'(0) = 1 \; \forall k \in \mathbb{N}$ (from the power series of ϕ_k), we have $\phi'(0) = 1$. Hence, ϕ is invertible in some neighbourhood of 0.

Therefore, we have a holomorphic change of coordinates in some neighbourhood of 0 which conjugates f to the nth power map. In this neighbourhood, ϕ is one-to-one, $f(z) \neq 0$ for $z \neq 0$ (i.e. no other point maps to the super-attracting fixed point via f) and f maps this neighbourhood into itself.

Uniqueness. It suffices to study the special case $f(z) = z^n$. If we can prove that any map which conjugates $z \mapsto z^n$ to itself is just multiplication by $(n-1)$th root of unity, then for any general map $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, if we have two maps ϕ and ψ which conjugate it to $z \mapsto z^n$, then $\phi \circ \psi^{-1}$ is a map which conjugates $z \mapsto z^n$ to itself. Hence, $\phi \circ \psi^{-1} = cz$, where $c^{n-1} = 1$. Therefore, $\phi = c\psi$, where c is a $(n-1)$th root of unity.

So, let $\phi(z) = c_1 z + c_k z^k + \ldots$, $(c_1 \neq 0)$ be a map which conjugates $z \mapsto z^n$ to itself. Then, we should have $\phi(z^n) = \phi(z)^n$. Now,

$$\phi(z^n) = c_1 z^n + c_k z^{nk} + \ldots,$$

and

$$\phi(z)^n = c_1^n z^n + n c_1^{n-1} c_k z^{n+k-1} + \ldots.$$

Comparing coefficients, we get $c_1^n = c_1$ and $n c_1^{n-1} c_k = 0$ since $nk > n + k - 1$ for $k \geq 2$. Therefore, we get $c_1^{n-1} = 1$ and $c_k = 0$. The form $\phi(z) = c_1 z + c_k z^k + \ldots$ can be modified to any $k \geq 2$ to get $c_k = 0$ by the same process.

Therefore, $\phi(z) = cz$, where c is a $(n-1)$th root of unity. □

1.2 Extension of Bottcher's coordinates

We might hope to extend the Bottcher's coordinates to the whole of the immediate basin of attraction of the super-attracting fixed point. But this is not always possible. This is because, it requires computing expressions of the form $z \mapsto (\phi(f^{\circ k}(z))^{\frac{1}{n^k}}$, which is not always possible. For example, the basin may not be simply connected, or some point may map directly onto the super-attracting fixed point, in which case we can't take n^k-th roots, because $\phi(f^{\circ k}(z))$ must be zero at those points.

Theorem 1.2.1 (Extension of $|\phi|$). *If f has a super-attracting fixed point p, with immediate basin of attraction $\mathcal{A}$, then the function $z \mapsto |\phi(z)|$ of the above theorem extends* *uniquely to a continuous map* $|\phi| : \mathcal{A} \to [0,1)$ *which satisfies* $|\phi|(f(z)) = |\phi|(z)^n$. *Furthermore,* $|\phi|$ *is real analytic except at the iterated preimages of* p*, where it takes the value* 0.

Proof. Set $|\phi|(z) = |\phi(f^{\circ k}(z))|^{\frac{1}{n^k}}$ for large enough k for each $z \in \mathcal{A}$. ϕ is only defined in a some small neighbourhood of p. But since, $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, after k many iterates for some large k, $f^{\circ k}(z)$ belongs to the domain of definition of ϕ, which we shall call $\hat{U}$.

It is independent of the value of k (if k is large enough). Note that, if $f^{\circ k}(z) \in \hat{U}$, then so does $f^{\circ k+1}(z)$, since f maps $\hat{U}$ into itself.
Suppose we choose k minimal such that $f^{\circ k}(z) \in \hat{U}$. Then,

$$|\phi(f^{\circ k+1}(z)|^{\frac{1}{n^{k+1}}} = |\phi(f(f^{\circ k}(z)))|^{\frac{1}{n^{k+1}}} = |\phi(f^{\circ k}(z))^n|^{\frac{1}{n^{k+1}}} = |\phi(f^{\circ k}(z))|^{\frac{1}{n^k}} = |\phi|(z).$$

In the proof of the Bottcher's theorem, we saw that $\phi(z) \in \mathbb{D}\ \forall z \in \hat{U}$ Ḣence, $|\phi|(z) = |\phi(f^{\circ k}(z))| < 1\ \forall z \in \mathcal{A}$. Therefore, $|\phi| : \mathcal{A} \to [0,1)$.

Also,

$$\begin{aligned}
|\phi|(f(z)) &= |\phi(f^{\circ k}(f(z))|^{\frac{1}{n^k}} \\
&= |\phi(f(f^{\circ k}(z)))|^{\frac{1}{n^k}} \\
&= |\phi(f^{\circ k}(z))^n|^{\frac{1}{n^k}} \\
&= |\phi(f^{\circ k}(z))|^{\frac{n}{n^k}} \\
&= |\phi|(z)^n.
\end{aligned}$$

It is also clear that $|\phi| = 0$ only at p and its iterated preimages.
If q is an iterated preimage of p, say $f^{\circ k}(q) = p$, then we have $|\phi|(q) = |\phi(f^{\circ k}(q)|^{\frac{1}{n^k}} = |\phi(p)|^{\frac{1}{n^k}} = 0$.
Now, Suppose $|\phi|(z) = 0$ for some z. Then, $|\phi|(z)^{n^k} = 0\ \forall k \implies |\phi|(f^{\circ k}(z)) = 0\ \forall k$. But for some large k, $f^{\circ k}(z)$ belongs to the domain of definition of ϕ. But that means, $f^{\circ k}(z) = p$, since no other point in that domain is mapped to zero by ϕ. Hence, z is an an iterated preimage of p.

Now, since $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, for each $a \in \mathcal{A}$, we have a neighbourhood W_a and a constant $k \in \mathbb{N}$ such that $f^{\circ k}(z) \in \hat{U}\ \forall z \in W_a$.
Hence, for $z \in W_a$, we can define $|\phi|(z) = |\phi(f^{\circ k}(z))| = |g(z)|$, where $g = \phi \circ f^{\circ k}|_{W_a}$. Therefore, $|\phi|_{W_a} = |g|$, where g is some holomorphic function defined on W_a.

It is clear from this that $|\phi|$ is continuous in $\mathcal{A}$.

Now, if h is any holomorphic function, then $|h(z)|$ is real-analytic everywhere in its domain except at those z, where $h(z) = 0$.
Since, $|g| = |\phi|_{W_a}$ is zero only at the iterated preimages of f in W_a, $|\phi|_{W_a}$ is real analytic everywhere in W_a except at the iterated preimages of p.
Therefore, $|\phi|$ is real analytic everywhere in $\mathcal{A}$ except at the iterated preimages of p.

□

Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with a super-attracting fixed point p. Then the associated Bottcher map ϕ carries a neighbourhood of p biholomorphically onto a neighbourhood of zero, conjugating f to the nth power map, where n is the local degree of f near p. ϕ has a local inverse ψ_ϵ which maps the ϵ-disc around zero to a neighbourhood of p.

Theorem 1.2.2 (Extending ψ_ϵ)**.** *There exists a unique open disc of maximal radius $0 < r \leq 1$ such that ψ_ϵ extends holomorphically to a map $\psi : \mathbb{D}_r \to \mathcal{A}$, where $\mathcal{A}$ is the immediate basin of attraction of p.*

1. *If $r = 1$, then ψ maps the open unit disc $\mathbb{D}$ onto $\mathcal{A}$ biholomorphically.*
2. *If $0 < r < 1$, then ψ maps $\mathbb{D}_r$ onto its image biholomorphically and there exists atleast one other critical point in $\mathcal{A}$ on the boundary of $\psi(\mathbb{D}_r)$.*

If ψ_ϵ is extended biholomorphically in this way to the map ψ defined on $\mathbb{D}_r$, then the inverse map $\psi^{-1} : \psi(\mathbb{D}_r) \to \mathbb{D}_r$ must be the extension of ϕ from some neighbourhood of p to $\psi(\mathbb{D}_r)$ (since ψ^{-1} agrees with ϕ on some neighbourhood of p).

Proof. Let us try to extend ψ_ϵ along radial lines by analytic continuation. Then, we can't extend it indefinitely as it would yeild a holomorphic map ψ from the entire complex plane onto an open set $\psi(\mathbb{C}) \subset \mathcal{A} \subsetneq \mathbb{C}_\infty$. ($\mathcal{A}$ cannot be the whole of $\mathbb{C}_\infty$ since the Julia set of f cannot be empty as $\deg(f) \geq 2$). We can conjugate f such that $\infty \notin \mathcal{A}$. Then the corresponding map ψ will map the whole of the complex plane into $\mathcal{A} \subset \mathbb{C}$. By Louiville's theorem, since the map ψ cannot be a constant, $\psi(\mathbb{C}) = \mathbb{C} = \mathcal{A}$. Therefore, $\mathbb{C}_\infty \backslash \mathcal{A} = \{\infty\}$. This too is not possible since the Julia set of f must be an infinite set since $\deg(f) \geq 2$.
Thus, there must be some largest radius r so that ψ_ϵ extends analytically throughout the open disc $\mathbb{D}_r$.

Also, $|\phi|(\psi(w)) = |\phi(\psi(w))| = |w|$ near 0, hence for all $w \in \mathbb{D}_r$ by analytic continuation.???
Since, $|\phi| : \mathcal{A} \to [0, 1)$, this proves that for any $w \in \mathbb{D}_r$, $|\phi|(\psi(w)) = |w| < 1$. Therefore, ψ can be defined only on $\mathbb{D}_r$ for $r \leq 1$.

We will now show that ψ is actually one-to-one on $\mathbb{D}_r$. Suppose $\psi(w_1) = \psi(w_2)$. Applying $|\phi|$, we see that $|w_1| = |w_2|$. Choose such a pair such that $\psi(w_1) = \psi(w_2)$ ($w_1 \neq w_2$) with $|w_1| = |w_2|$ minimal. A minimal pair exists because for $|w| < \epsilon$, $\psi = \psi_\epsilon$ which is one-to-one as it is invertible.
Now, ψ is an open mapping. Choose a sufficiently small neighbourhood U_{w_2} of w_2. Then, $\psi(U_{w_2})$ is a small neighbourhood of $\psi(w_1) = \psi(w_2)$. Hence, for any w_1' sufficiently close to w_1, $\psi(w_1') \in \psi(U_{w_2})$. Hence, we can find w_2' sufficiently close to w_2 such that $\psi(w_1') = \psi(w_2')$. Choosing $|w_1'| < |w_1|$, we get a contradiction.
Hence, ψ maps $\mathbb{D}_r$ onto its image biholomorphically.

In case when $r = 1$, $U = \psi(\mathbb{D}) = \mathcal{A}$. If not then we would have some boundary point of U, say $z_0 \in \mathcal{A}$. We can approximate z_0 by points of $\psi(w_j)$, where $|w_j| \to 1$.
Now, $\lim_{j\to\infty} \psi(w_j) = z_0$. Hence,

$$\lim_{j\to\infty} |\phi|(\psi(w_j)) = |\phi|(z_0) \implies \lim_{j\to\infty} |w_j| = |\phi|(z_0) \implies |\phi|(z_0) = 1,$$

which is impossible.

Now, let $0 < r < 1$. We need to prove that ∂U, where $U = \psi(\mathbb{D}_r)$ must contain a critical point of f. Suppose, $w_0 \in \partial\mathbb{D}_r$ and let $(w_j)_{j=1}^{\infty} \subset \mathbb{D}_r$ such that $w_j \to w_0$. Let $\psi(w_j) \to z_0$. Then $z_0 \in \partial U$ because ψ maps $\mathbb{D}_r$ onto U biholomorphically.
If z_0 is not a critical point of f, then f maps a neighbourhood of z_0, say A onto a neighbourhood of $f(z_0)$, say B biholomorphically.

It should be noted that B can be chosen such that $B \subset U$. This is because $f(z_0) \in U$. We have, $\lim_{j\to\infty} \psi(w_j) = z_0 \implies \lim_{j\to\infty} f(\psi(w_j)) = f(z_0) \implies \lim_{j\to\infty} \psi(w_j^n) = f(z_0) \implies \psi(w_0^n) = f(z_0)$. Since, $|w_0| = r < 1$, $|w_0|^n < r^n < r$. Hence, $w_0 \in \mathbb{D}_r$. Therefore, $\psi(w_0^n) = f(z_0) \in U$.

Let g be the local inverse of f near $f(z_0)$. Then, ψ can be extended throughout a neighbourhood of w_0 by

$$w \mapsto g(\psi(w^n)).$$

We have, $\psi(w_0^n) = f(z_0) \implies w_0^n = \phi(f(z_0))$. Since, $\phi(B)$ is a neighbourhood of $\phi(f(z_0))$ lying inside $\mathbb{D}_r$, choose a small enough neighbourhood of w_0, say C such that $w^n \in \phi(B)$, for all $w \in C$. In this neighbourhood, C our newly defined map agrees with ψ on $C \cap \mathbb{D}_r$. This is because, for $w \in C \cap \mathbb{D}_r$, $f(\psi(w)) = \psi(w^n) \in B$. Therefore, $g(\psi(w^n))$ can be defined and $\psi(w) = g(\psi(w^n)) \in A$. Hence, our new map is an analytic continuation of ψ on the neighbourhood C.

Now, if none of the $z_0 \in \partial U$ are critical points, we can extend ψ to a neighbourhood of $w_0 \,\forall w_0 \in \partial\mathbb{D}_r$. Clearly, these continutations would patch together to define ψ in a strictly greater disc than $\mathbb{D}_r$, which is a contradiction. $\square$