1 Behaviour of analytic functions near fixed points

1.1 Behaviour near parabolic fixed points

A point p is called a parabolic fixed point of f if $f(p) = p$ and $f'(p) = e^{2\pi i t}$, where t is a rational number.H

Theorem 1.1.1 (The Petal Theorem). *Suppose that an analytic map f has the form:*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1})$$

near the origin. Then for sufficiently small t,

1. *f maps each $\Pi_k(t)$ into itself;*
2. *$f^{\circ n}(z) \to 0$ uniformly on each petal;*
3. *$\arg(f^{\circ n}(z)) \to 2k\pi/p$ locally uniformly on each petal;*
4. *$|f(z)| < |z|$ on a neighbourhood of the axis of each petal;*
5. *$f : \Pi_k(t) \to \Pi_k(t)$ is conjugate to a translation.*

Proof. For $0 < r_0 < 1$, define the sector S_0,

$$S_0 = \{re^{i\theta} : 0 < r < r_0, |\theta| < \pi/p\}$$

and the region W,

$$W = \{re^{i\theta} : r > \frac{1}{r_0^p}, |\theta| > \pi\}.$$

It is clear that the map $\sigma : z \mapsto \frac{1}{z^p}$ is a biholomorphism of S_0 onto W with $\sigma^{-1} : W \to S_0$ given by $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$. The branch of p-th root that we select determines which sector of width $2\pi/p$, the inverse map maps to (the other sectors being $S_k = \{0 < r < r_0, |\theta - 2k\pi/p| < \pi/p\}$.

Now, the conjugate map of f on W is given by,

$$g(w) = \sigma f \sigma^{-1}(w) = f(w^{-\frac{1}{p}})^{-p}.$$

This just replaces the action of f on S by g on W, and we have the following commutative diagram

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