Indian Institute of Science Education and Research, Kolkata
Department of Mathematics and Statistics
Master's Thesis BS-MS Dual Degree Program

Introduction to Complex Dynamics and the Mandelbrot Set

Aditya Dutta (18MS101)

Supervisor:	Dr. Sushil Gorai
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1 Introduction

1.1 The Spherical metric

We begin by defining the extended complex plane, $\mathbb{C}_\infty$ simply as the union,

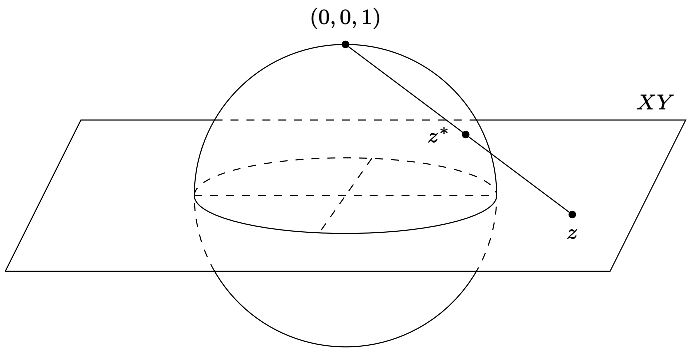
$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

To obtain, a metric on $\mathbb{C}_\infty$, we identify $\mathbb{C}$ with the XY plane in $\mathbb{R}^3$. And let S be the unit sphere centered at origin.
We then use the stereographic projection, $\pi : z \mapsto z^*$ by projecting each point z in $\mathbb{C}$ linearly towards (or away) $(0, 0, 1)$ until it meets S. We then define, $\pi(\infty) = \infty$. In this way, π is a bijective map from $\mathbb{C}_\infty$ onto S, and this is the reason why it $\mathbb{C}_\infty$ is also called the *Complex or Riemann Sphere.*

We now define a natural metric on $\mathbb{C}_\infty$ using this stereographic projection as,

$$\sigma(z, w) = |\pi(z) - \pi(w)| = |z^* - w^*|.$$

This is known as the *Spherical metric* on $\mathbb{C}_\infty$ and we will use this metric mainly to define equicontinuity ahead.



1.2 Rational maps

Definition 1.2.1 (Rational maps). *A rational map is a function of the form,*

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials but not simultaneously zero polynomials. If $Q(z) = 0$ and P is not the zero polynomial, then R is defined to be ∞. Also, we define $R(\infty)$ to be the limit of $R(z)$ as $z \to \infty$.

We shall always assume P and Q are co-prime. We define the **degree** of a rational map R as,

$$\deg(R) = \max\{\deg(P), \deg(Q)\}.$$

If R is a constant map (even ∞), we define, $\deg(R) = 0$.

It is a crucial fact that if R is a rational function of degree d, then R is a d-fold map of $\mathbb{C}_\infty$ onto itself.

1.3 Definition of Fatou and Julia sets in terms of equicontinuity

Definition 1.3.1 (Fatou and Julia Sets). *Let R be a non-constant rational function. The Fatou set of R denoted by $F(R)$ is the maximal open subset of $\mathbb{C}_\infty$ on which $\{R^{\circ n}\}$ is equicontinuous. The Julia set of R, denoted by $J(R)$ is the complement of $F(R)$ in $\mathbb{C}_\infty$.*

By definition, $F(R)$ is open and $J(R)$ is compact.
They are denoted by simply F or J when the context is clear.

1.4 Completely Invariant Components

If $f : X \to X$, then a subset $D \subset X$ is:

- *forward invariant* under the map f if $f(D) = D$.
- *backward invariant* under the map f if $f^{-1}(D) = D$.
- *completely invariant* under the map f if it is both forward and backward invariant under f i.e. $f(D) = D$ and $f^{-1}(D) = D$.

Note that if f is surjective, i.e. $f(X) = X$, then backward invariance implies complete invariance. This is because, $f(f^{-1}(D)) = D$ if f is surjective. Hence, if $f^{-1}(D) = D$, we have $f(D) = D$ i.e. forward invariance.

Theorem 1.4.1. *If $f : X \to X$ be a continuous, open and surjective map of a topological space X onto itself. If $D \subset X$ is completely invariant under f, then so are the complement $X \backslash D$, the interior D^0, the boundary ∂D and the closure $\overline{D}$.*

Proof. Firstly, note that it is enough to prove backward invariance since f is surjective. It is trivial to see that $X \backslash D$ is completely invariant.

Now, since f is a continuous map, $f^{-1}(D^0)$ is an open subset of $f^{-1}(D) = D$. Hence, $f^{-1}(D^0) \subset D^0$. Now, since f is an open map, $f(D^0)$ is an open subset of $f(D) = D$. Hence, $f(D^0) \subset D^0 \implies D^0 \subset f^{-1}(f(D^0)) \subset f^{-1}(D^0)$. Hence, $f^{-1}(D^0) = D^0$ and hence, D^0 is completely invariant.

From the general fact for continuous maps, $\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$. Hence, $\overline{D} \subset f^{-1}(\overline{D})$. Now, let $x \in f^{-1}(\overline{D})$ (or $f(x) \in \overline{D}$). If $x \notin \overline{D}$, then there exists and open set around x, say U such that $U \cap D = \phi$. Since f is an open map, $f(U)$ is an open set containing $f(x)$. Since, $f(x) \in \overline{D}$, $f(U) \cap D \neq \phi$. But since, $f^{-1}(D) = D$, $f^{-1}(f(U) \cap D) \subset D$. But, $f^{-1}(f(U) \cap D) \cap U \neq \phi \implies D \cap U \neq \phi$, which is a contradiction. Hence, $\overline{D} = f^{-1}(\overline{D})$. Hence, $\overline{D}$ is also completely invariant.

Consequently, $\partial D = \overline{D} \backslash D^0$ is also completely invariant. □

Theorem 1.4.2. *For any rational function R, the Fatou and Julia sets of R i.e. $F(R)$ and $J(R)$ are completely invariant.*

Proof. First note that it is enough to prove only backward invariance because R is surjective. Also, we will only prove the complete invariance of $F(R)$, the complete invariance of $J(R)$ then follows from above theorem. We will use F to denote $F(R)$.

Let $z_0 \in R^{-1}(F)$ and let $w_0 = R(z_0) \in F$. By equicontinuity, for any $\epsilon > 0$, $\exists \delta > 0$ such that if $\sigma(z, z_0) < \delta$, then for all $n \in \mathbb{N}$, $\sigma(R^{\circ n}(w), R^{\circ n}(w_0)) < \epsilon$. By continuity of R, there exists $\delta' > w_0$ such that if $\sigma(z, w_0) < \delta'$, then $\sigma(R(z), w_0) < \delta$ and hence, $\sigma(R^{\circ n+1}(z), R^{\circ n+1}(z_0)) < \epsilon$ for all $n \in \mathbb{N}$. Hence, $\{R^{\circ n+1} : n \in \mathbb{N}\}$ is equicontinuous at z_0 and hence, so is $\{R^{\circ n} : n \in \mathbb{N}\}$. Therefore, $z_0 \in F$ and $R^{-1}(F) \subset F$.

Now, let $z_0 \in F$. To prove that $z_0 \in R^{-1}(F)$, we need to prove that $R(z_0) \in F$. Let $w_0 = R(z_0)$. We have by equicontinuity, that for any $\epsilon > 0$, $\exists \delta > 0$ such that for all $n \in \mathbb{N}$, if $\sigma(z, z_0) < \delta$, then $\sigma(R^{\circ n+1}(z), R^{\circ n+1}(z_0)) < \epsilon$. Now, $N = \{z : \sigma(z, z_0) < \delta\}$ is an open set containing z_0 and hence, $R(N)$ is an open set containing w_0. Now, if $w \in R(N)$ then $w = R(z)$ for some $z \in N$. Hence,

$$\sigma(R^{\circ n}(w), R^{\circ n}(w_0)) = \sigma(R^{\circ n+1}(z), R^{\circ n+1}(z_0)) < \epsilon.$$

Hence, $z_0 \in R^{-1}(F)$ and $F \subset R^{-1}(F)$.

Therefore, $R^{-1}(F) = F$ and $F(R)$ is completely invariant. □

Lemma 1.4.3. *For any rational map R and a domain $U \subset \mathbb{C}_\infty$, $\partial R(U) \subset R(\partial U)$.*

Proof. Let $w_0 \in \partial R(U)$ such that it is approximated by $R(z_n)$ for $(z_n)_{n=1}^{\infty} \subset U$. Now, assume $z_n \to z_0$ (after taking a subsequence). Now, z_0 cannot lie in U, otherwise $R(z_0) =$

$w_0 \in R(U)$. Since, R is an open map, $R(U)$ is an open set and is disjoint from $\partial R(U)$. Hence, $z_0 \in \partial U$ and $R(z_0) = w_0 \in R(\partial U)$. Therefore, $\partial R(U) \subset R(\partial U)$. □

Lemma 1.4.4. *For a rational map R, if F_1 and F_2 are two Fatou components and R maps a point of F_1 to a point of F_2, then $R(F_1) = F_2$.*

Proof. Clearly, $R(F_1) \subset F_2$ because of forward invariance of F under R and since F_1 and F_2 are connected components of F. If $R(F_1) \neq F_2$, then $\exists z \in \partial F_1$ such that $R(z) \in F_2$ and this is not possible as $z \in \partial F_1 \implies z \in J$ and J is completely invariant. Hence, $R(F_1) = F_2$. □

Theorem 1.4.5. *The unbounded Fatou component of a polynomial P, i.e. the Fatou component containing ∞ is a completely invariant Fatou component. It is denoted by $F_\infty(P)$ or simply F_∞ when the context is clear.*

Proof. First note that since $P(\infty) = \infty$, we have $P(F_\infty) = F_\infty$ by the above lemma. Hence, $F_\infty \subset P^{-1}(F_\infty)$. Now assume, some point $z_0 \in P^{-1}(F_\infty)$ but $z_0 \notin F_\infty$. By backward invariance of F, $z_0 \in F'$, where F' is some other Fatou component. Again, $P(F') = F_\infty$ by above lemma. But for polynomials, we have $P^{-1}(\infty) = \{\infty\}$. Hence, $\infty \in F'$ and F' must be F_∞ itself. Therefore, $P^{-1}(F_\infty) = F_\infty$ and F_∞ is completely invariant under P. □

Theorem 1.4.6. *Let R be rational map and let E be a finite set which is completely invariant under R. Then E has atmost two elements.*

Proof. Suppose E has k elements. Now, R acts as a permutation of E and hence for some integer s, $R^{\circ s}$ acts as an identity map on E. Now, suppose $R^{\circ s}$ has degree d. It follows that for any $z_0 \in E$, $R^{\circ s}(z) = z_0$ has solution z_0 with multiplicity d. Applying the Riemann-Hurwitz formula, (in the next section)

$$k(d-1) \leq 2d - 2$$

and hence, $k \leq 2$. □

1.5 Valency and the Riemann-Hurwitz formula

Let f be a holomorphic map on the complex plane. Then, near a point z_0, f has the Taylor expansion,

$$f(z) = f(z_0) + a_k(z - z_0)^k + \dots,$$

where $a_k \neq 0$ and $k \geq 1$. Then, we define the valency of f at z_0, $v_f(z_0) = k$.

For $f : X \to Y$ where X and Y are Riemann surfaces, we have local analytic co-ordinates near z_0 and $f(z_0)$ such that f has the form $f(z) = a_k z^k + \ldots$, ($a_k \neq 0$ and $k \geq 1$) then again we define the valency of f at z_0 as $v_f(z_0) = k$. The valency is independent of the choice of co-ordinates.

Definition 1.5.1 (Deficiency). *We define the deficiency of f over a set A as,*

$$\delta_f(A) = \sum_{z \in A} (v_f(z) - 1).$$

Theorem 1.5.1 (Generalized Riemann-Hurwitz formula). *Let X and Y be Riemann surfaces and $f : X \to Y$ be a complex analytic map of degree d. Then,*

$$\chi(X) + \delta_f(X) = d\chi(Y),$$

where $\chi(X)$ denotes the Euler characteristic of X.
For a compact, connected and orientable surface S, the Euler characteristic $\chi(S) = 2-2g$, where g is the genus of S. Hence, if X and Y are compact Riemann surfaces, we get the following formula (after multiplying both sides by -1),

$$2g(X) - 2 = d(2g(Y) - 2) + \delta_f(X).$$

Theorem 1.5.2 (Riemann-Hurwitz Formula (version 1)**).** *Now, genus of a sphere is zero and hence, $g(\mathbb{C}_\infty) = 0$. For a rational map, which is a d-fold map of the complex sphere onto itself, we have,*

$$\begin{aligned}
&\implies 2g(\mathbb{C}_\infty) - 2 = d(2g(\mathbb{C}_\infty) - 2) + \delta_R(\mathbb{C}_\infty) \\
&\implies \delta_R(\mathbb{C}_\infty) = 2d - 2 \\
&\implies \sum_{z \in \mathbb{C}_\infty} (v_R(z) - 1) = 2d - 2.
\end{aligned}$$

Theorem 1.5.3 (Riemann-Hurwitz Formula (version 2)**).** *Let F_0 and F_1 be components of the Fatou set F of a rational map R and R maps F_0 into F_1. Then, for some integer m, R is an m-fold map of F_0 onto F_1 and*

$$\chi(F_0) + \delta_R(F_0) = m\chi(F_1).$$

Proof. We only need to prove that if R maps F_0 into F_1, then it is a m-fold map of F_0 onto F_1 for some $m \in \mathbb{Z}$. □

1.6 Equicontinuity and Normality

There is another criterion that we use more in practice to define Fatou sets.

Definition 1.6.1 (Normal Families). *A family of maps, $\mathcal{F}$ of maps from metric space (X_1, d_1) to (X_2, d_2) is said to be a normal family in X_1, if every infinite sequence of function in $\mathcal{F}$ has a subsequence which converges locally uniformly on X_1.*

The Arzela-Ascoli theorem connects equicontinuity and normality. This is one of the many forms of the theorem, which is suitable for our use.

Arzela-Ascoli Theorem. *Let D be a domain on the complex sphere and let $\mathcal{F}$ be a family of continuous maps defined on D. Then, $\mathcal{F}$ is equicontinuous in D if and only if it is a normal family in D.*

Thus, we can redefine the Fatou set of R as the maximal open set of $\mathbb{C}_\infty$ on which the family $\{R^{\circ n}\}$ is normal.

Theorem 1.6.1 (Vitali's Theorem). *Let D be a subdomain of complex sphere. Suppose $\{f_n\}_{n\in\mathbb{N}}$ be a family of analytic maps normal in D. Also suppose, $\{f_n\}$ converges pointwise on some subset $W \subset D$ such that W contains a limit point in D. Then, $f(z) := \lim_{n\to\infty} f_n(z), z \in W$ extends to an analytic function F on D and $f_n \to F$ locally uniformly in D.*

Proof. As $\{f_n\}$ is a normal family, there is a subsequence of (f_n) which converges locally uniformly in D to some analytic function F and $F = f$ on W.

Now, assume (f_n) fails to converge locally uniformly to F on D. Then, there is some subsequence (g_n) of (f_n) and $\epsilon > 0$ such that for all n and all $z \in K$,

$$\sigma(g_n(z), F(z)) \geq \epsilon.$$

But again by normality, there is a subsequence (h_n) of (g_n) which converges locally uniformly in D to some analytic function h. Clearly, $h = F = f$ on W and since W has a limit point in D, $h = F$ throughout D by the Identity theorem. It follows that,

$$\sigma(h_n(z), F(z)) \to 0$$

uniformly in K. This is a contradiction as (h_n) is a subsequence of (g_n). Hence, (f_n) converges locally uniformly to $F = f$ on D. □

Corollary 1.6.1.1. *If α is a (super)-attracting fixed point of a rational map R and F_α is the Fatou component containing α then $R^{\circ n}(z) \to \alpha$ locally uniformly in F_α.*

We now state one of the most important theorem for normal families, the *Montel's Fundamental Normality Criterion.* It provides us with a very easy way to check if some family of maps is normal on a domain of the complex sphere.

Theorem 1.6.2 (Montel's Fundamental Normality Criterion). *Let $\mathcal{F}$ be a family of maps, each analytic in a domain D of the complex sphere. Suppose $\exists m > 0$ and for each $f \in \mathcal{F}$, three distinct points a_f, b_f and c_f such that,*

1. *$f(D)$ does not contain a_f, b_f and c_f,*
2. *$\min\{\sigma(a_f, b_f), \sigma(b_f, c_f), \sigma(c_f, a_f)\} \geq m$,*

then $\mathcal{F}$ is normal in D.

1.7 Exceptional points and minimality of the Julia set

Theorem 1.7.1 (Minimality of J). *Let R be a rational map with $\deg(R) \geq 2$ and suppose that E is a closed, completely invariant subset of the complex sphere. Then either,*

1. *E has atmost two elements and $E \subset E(R) \subset F(R)$.*
2. *E is infinite and $J(R) \subset E$.*

Proof. We know that either E has atmost two points or it is infinte. If E is finite, then E contains only exceptional points which lie in $F(R)$. Now, suppose E is infinite. As E is completely invariant, so is its complement, say G. Hence, $R^{\circ n}$ maps G into itself for each $n \in \mathbb{N}$. Hence, applying Montel's Fundamental Normality Criterion, while choosing a_f, b_f and c_f to be any three points in E, we see that $\{R^{\circ n}\}$ is a normal family in G. Hence, $G \subset F \implies J \subset G^c = E$. □

This result can also be stated as:
J is the smallest, closed completely invariant set with atleast three points.

Corollary 1.7.1.1. *If R is a rational function, with $\deg(R) \geq 2$, and F_0 is a completely invariant Fatou component of R, then, $\partial F_0 = J$.*

Proof. As F_0 is completely invariant, so is $\overline{F_0}$. By minimality of J, $J \subset \overline{F_0}$. As J is disjoint from F_0, $J = \partial F_0$. □

1.8 Connectivity

Definition 1.8.1 (Connectivity). *The connectivity of a domain $D \subset \mathbb{C}_\infty$ is defined as the number of components of ∂D.*

Theorem 1.8.1. *The following are equivalent for a domain $D \subset \mathbb{C}_\infty$:*

1. *D is simply connected.*

2. *D^c is connected.*

3. *∂D is connected or $c(D) = 1$.*

Proof. D^c being connected can be taken as the definition of D being simply connected, as it is done by Ahlfors [Ahl79]. Later it is this definition of simply connected, which is used to prove the Riemann mapping theorem, which proves biholomorphism between simply connected sets and the unit disc which is actually simply connected.

We will prove the equivalence of *1* and *3*.

If D is not simply connected, then there is a simple closed curve γ in D which separtes the complement of D, proving that ∂D is disconnected.

Now, suppose that ∂D is disconnected. Then there is a simple closed curve γ which separtes ∂D into two disjoint subsets A and B. Since, D is path-connected and D is arbitrarily close to A and B, D intersects γ. By construction, γ does not intersect ∂D and hence, γ lies in D. Thus, A and B lie in different components of the complement of D and hence, D cannot be simply connected. □

Note: If D is simply connected, we have $c(D) = 1$ and $\chi(D) = 1$.
More generally, $\chi(D) = 2 - c(D)$.

2 Petal Theorem

A point p is called a parabolic fixed point of f if $f(p) = p$ and $f'(p) = e^{2\pi it}$, where t is a rational number.

Lemma 2.0.1. *Suppose f is analytic and satisfies*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{p+2})$$

in some neighbourhood N of the origin. Let $\omega_1, \dots, \omega_p$ be the p-th roots of unity and let $\eta_1, \dots, \eta_p$ be the p-th roots of -1. Then for sufficiently small r_0 and θ_0,

1. *$|f(z)| < |z|$ on each sector*

$$S_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\omega_j)| < \theta_0\}.$$

2. *$|f(z)| > |z|$ on each sector*

$$\Sigma_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\eta_j)| < \theta_0\}.$$

Proof. We have,

$$f(z)/z = 1 - z^p + \mathcal{O}(z^{p+1}) = 1 - z^p(1 + g(z)),$$

where g is analytic in N with $g(0) = 0$.
Now, consider the sector,

$$S = \{z \in \mathbb{C} : |z| < \frac{1}{2}; |\arg(z)| < \pi/4\}.$$

For small r_0 and θ_0, $z \in S_j \implies z^p(1 + g(z)) \in S$ and $z \in \Sigma_j \implies -z^p(1 + g(z)) \in S$. This is because for small enough r_0 and θ_0, $z \mapsto z^p$ maps S_j onto the set

$$S_0 = \{z \in \mathbb{C} : |z| < r_0^p; |\arg(z)| < p\theta_0\} \subset S.$$

And, $|z^p - z^p(1+g(z))| = |z|^p|g(z)| \leq M|z|^{p+1} = M(|z|^p)^{1+\frac{1}{p}}$. Hence, for any $w \in S_0$, the perturbation of any point is $\leq M|w|^{1+1/p}$.

□

Before stating the *Petal Theorem*, which discusses the behaviour of analytic functions near parabolic fixed points, we first define the notions of *petals*.

Definition 2.0.1 (Petals). *Let $p \in \mathbb{N}$. For each $k \in \{0, 1, \ldots, p-1\}$, define the sets as a function of a parameter $t > 0$ as follows,*

$$\Pi_k(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)); |\theta - 2k\pi/p| < \pi/p\}.$$

The sets $\Pi_k(t)$ are known as Petals.

We have shown a diagram of the petals $\Pi_k(t)$ in Figure 2.1 for $p = 6$. Note that all the petals are pairwise disjoint and each petal subtends an angle of $2\pi/p$ at the origin.

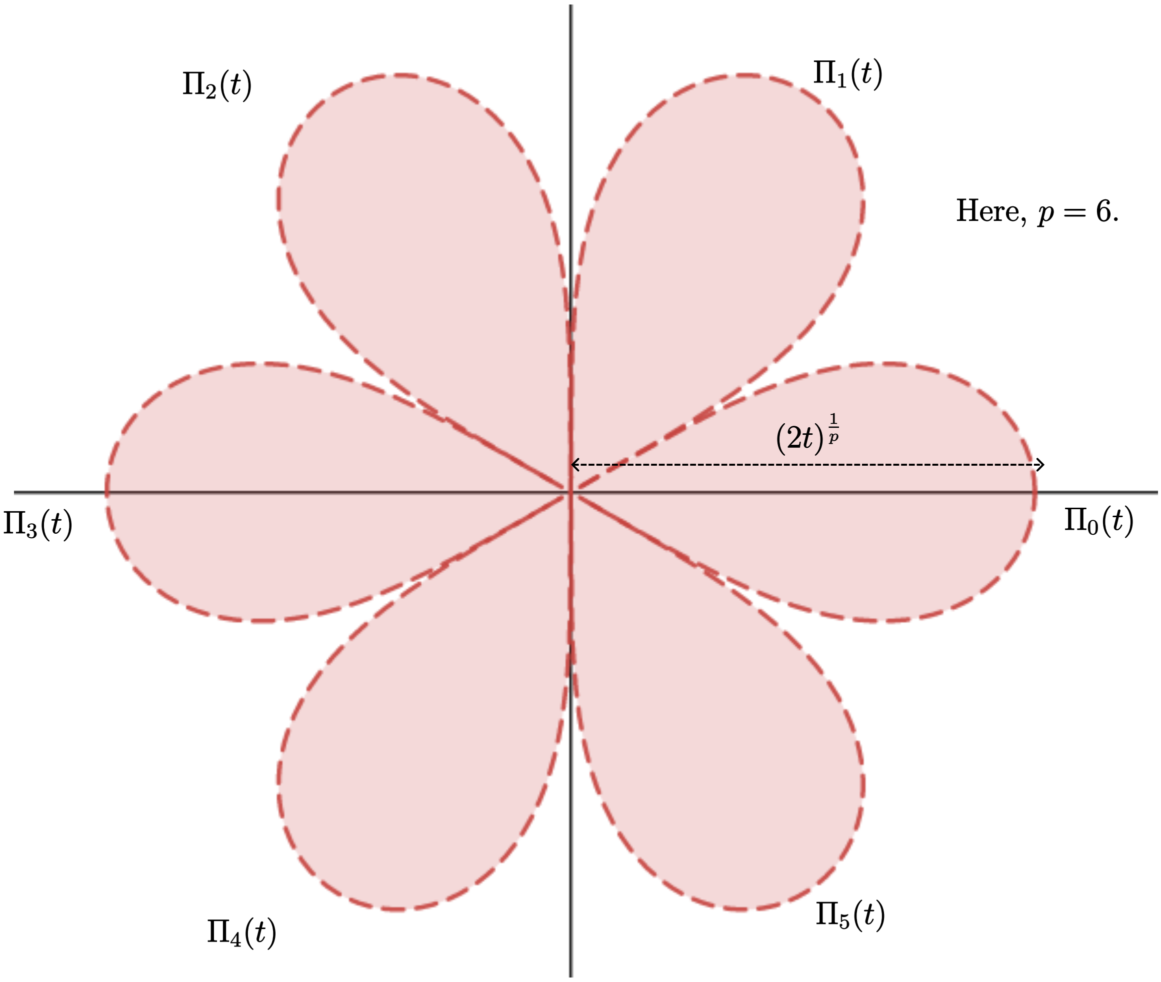


Figure 2.1: Six petals at the origin for $p = 6$.

Theorem 2.0.2 (The Petal Theorem). *Suppose that an analytic map f has the form:*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1})$$

near the origin. Then for sufficiently small t,

1. *f maps each $\Pi_k(t)$ into itself;*

2. *$f^{\circ n}(z) \to 0$ uniformly on each petal;*

3. *$\arg(f^{\circ n}(z)) \to 2k\pi/p$ locally uniformly on each petal;*

4. *$f : \Pi_k(t) \to \Pi_k(t)$ is conjugate to a translation.*

5. *$|f(z)| < |z|$ on a neighbourhood of the axis of each petal;*

Proof. For $0 < r_0 < 1$, define the sector S_0,

$$S_0 = \{re^{i\theta} : 0 < r < r_0, |\theta| < \pi/p\}$$

and the region W,

$$W = \{re^{i\theta} : r > \frac{1}{r_0^p}, |\theta| > \pi\}.$$

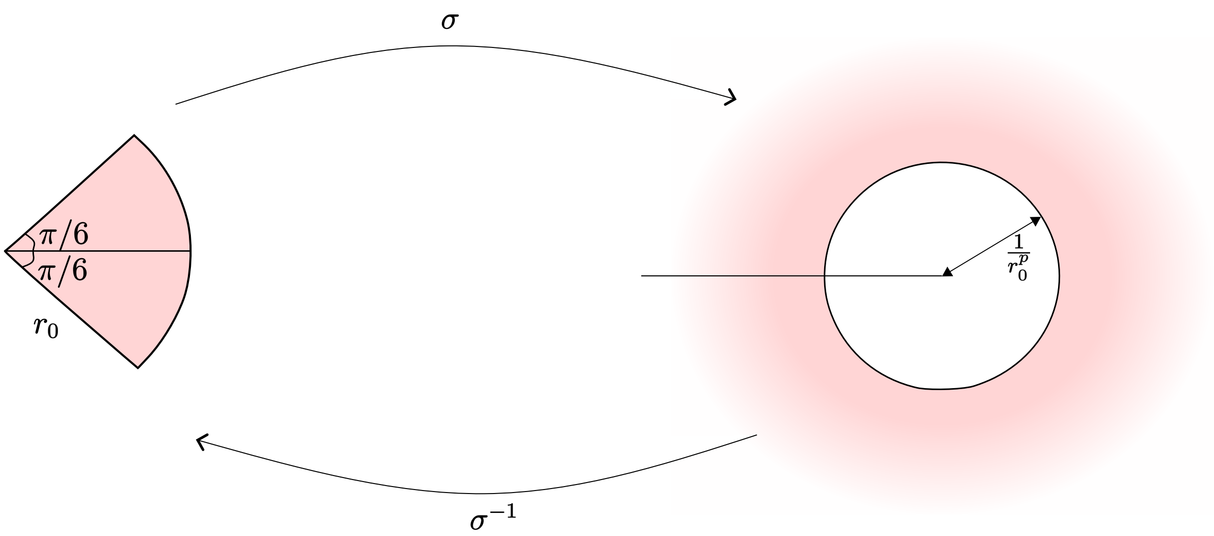


Figure 2.2: σ is a biholomorphism from S_0 onto W.

It is clear that the map $\sigma : z \mapsto \frac{1}{z^p}$ is a biholomorphism of S_0 onto W with $\sigma^{-1} : W \to S_0$ given by $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$. Actually, σ is a biholomorphism of each sector $S_k = \{0 < r < r_0, |\theta - 2k\pi/p| < \pi/p\}$ onto W. The branch of $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$ that we choose determines which sector the inverse map maps to.

Now, the conjugate map of f on W is given by,

$$g(w) = \sigma f \sigma^{-1}(w) = f(w^{-\frac{1}{p}})^{-p}.$$

This just replaces the action of f on S_0 by g on W, and we have the following commutative diagram:

Now, we will use the power series expansion of f near the origin to get information about g.
First let us try to get a estimate of the power series expansion of $f(z)^{-p}$. We have,

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}) = z(1 - z^p + \mathcal{O}(z^{2p})) = z(1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + - \ldots).$$

So,

$$\frac{1}{f(z)^p} = z^{-p}\left(\frac{1}{1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + - \ldots}\right)^p.$$

Now, let $\alpha(z) = z^p + a_0 z^{2p} + a_1 z^{2p+1} + \ldots$, then for r_0 small enough such that $|\alpha(z)| < 1$ on $\{|z| < r_0\}$, we can write,

$$\frac{1}{1-\alpha(z)} = 1 + \alpha(z) + \alpha(z)^2 + \ldots.$$

Therefore,

$$\begin{aligned}
\frac{1}{f(z)^p} &= z^{-p}(1 + \alpha(z) + \alpha(z)^2 + \ldots)^p \\
&= z^{-p}(1 + pz^p + Az^{2p} + A_1 z^{2p+1} + \ldots) \\
&= \frac{1}{z^p} + p + Az^p + v(z),
\end{aligned}$$

where A is some constant and $v(z)$ is holomorphic on $\{|z| < r_0\}$, and for some small $r_0 > 0$, it satisfies $|v(z)| \leq B|z|^{p+1}$, $B > 0$.

Now, if $w \in W$, then $\sigma^{-1}(w) \in S$. Hence, by substituting $z = \sigma^{-1}(w) = w^{-1/p}$, we have,

$$\begin{aligned}
g(w) &= \sigma f \sigma^{-1}(w) \\
&= \frac{1}{f(w^{-1/p})^p} \\
&= w + p + A/w + \theta(w),
\end{aligned}$$

where $|\theta(w)| = |v(w^{-1/p})| \leq B|w^{-1/p}|^{p+1} = B/|w|^{1+\frac{1}{p}}$.
Hence, we have the following estimates for g which will be crucial in everything that will follow:

$$g(w) = w + p + A/w + \theta(w), \text{ where } A \text{ is a constant and} \tag{2.1}$$
$$|\theta(w)| \leq B/|w|^{1+\frac{1}{p}}, B > 0. \tag{2.2}$$

Choose any K satisfying

$$K > \max\{1/r_0^p, 3(|A| + B)\} > 1 \text{ (as } r_0 < 1)$$

and let,

$$\Pi = \{x + iy : y^2 > 4K(K - x)\}.$$

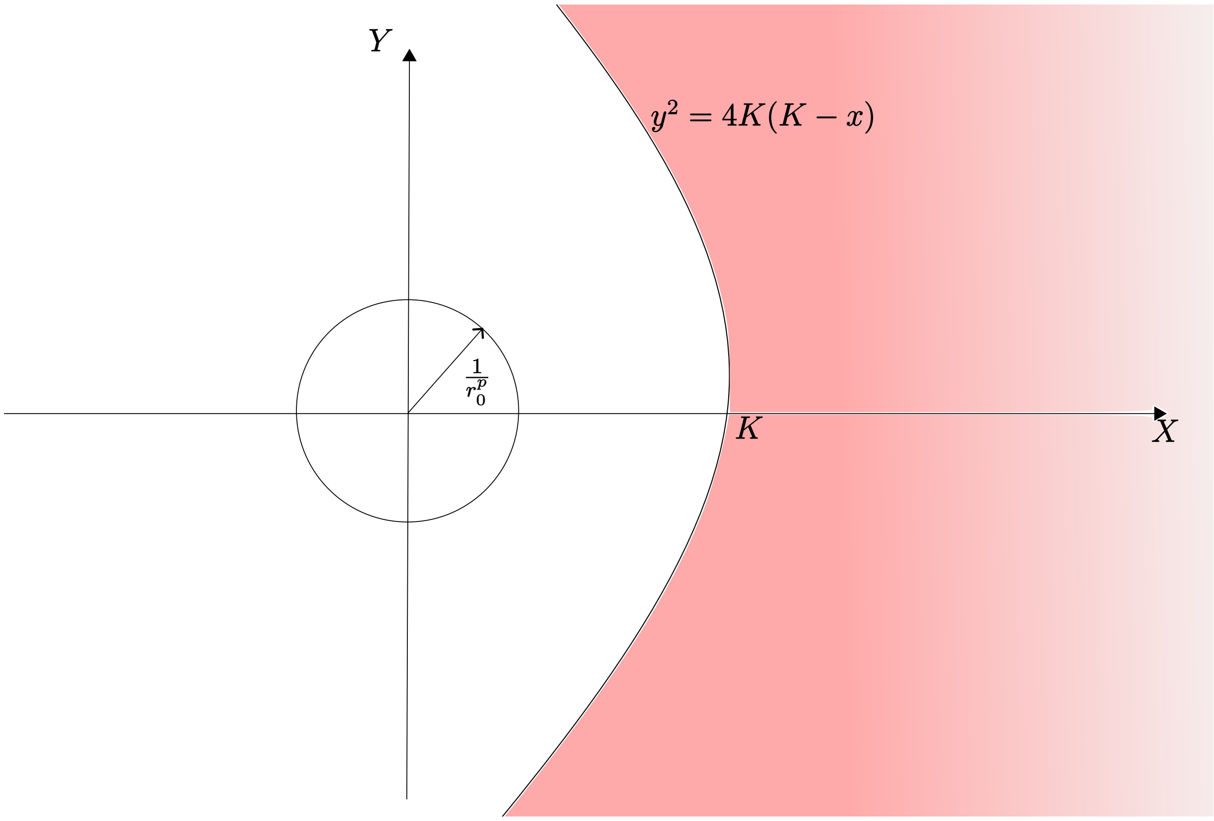


Figure 2.3: $\Pi = \{(x, y) : y^2 > 4K(K - x)\}$.

Clearly, Π is bounded by a parabola and $\Pi \subset W$ (See Figure 2.3).

We have chosen this subset $\Pi \subset W$ because we will show that Π is nothing but the conformal image of $\Pi_0(t)$ under σ (for a suitable t) and g satisfies all the corresponding conditions that f should satisfy on $\Pi_0(t)$ according to the theorem.

Claim. Π is the conformal image of $\Pi_0(t)$ under σ for a suitable t.
The easiest way to see this is using polar coordinates. We write, $z = re^{i\theta}$ for $z \in S_0$ and $w = \rho e^{i\phi}$ for $w \in W$. Then, $\rho = \frac{1}{r^p}$ and $\phi = -p\theta$.

Now, we need to express Π in polar co-ordinates. To do so, we notice that points on the parabola are given by

$$\rho \text{ (distance from focus i.e. } 0) = 2K - \rho \cos \phi \text{ (distance from directrix i.e. } y = 2K).$$

(See Figure 2.4). Therefore, points on Π are given by

$$\rho > 2K - \rho \cos \phi.$$

Hence,

$$\Pi = \{\rho e^{i\phi} : 2K < \rho(1 + \cos \phi)\}.$$

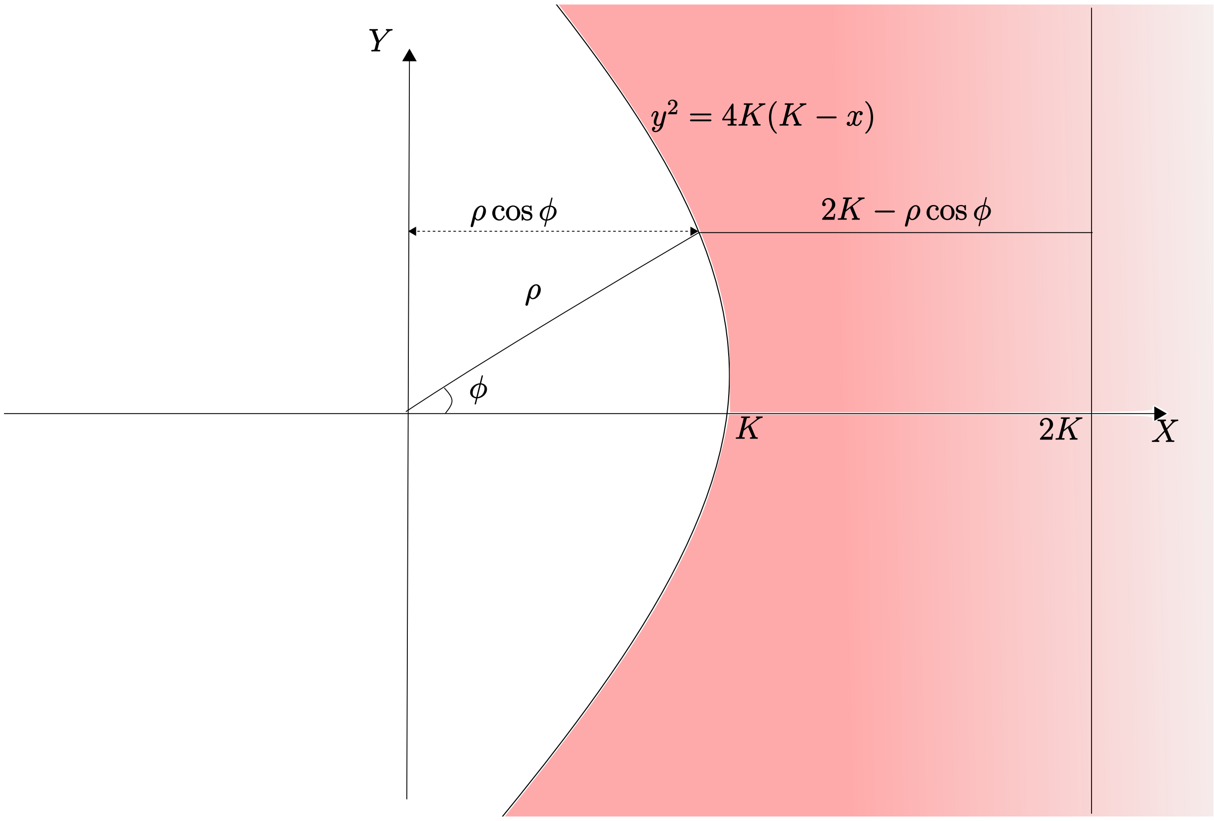


Figure 2.4: $\Pi = \{\rho e^{i\phi} : \rho > 2K - \rho\cos\phi\}$.

Now, let $\Omega = \sigma^{-1}(\Pi)$. Then, Ω is given by

$$\Omega = \{re^{i\theta} : 2Kr^p < 1 + \cos(p\theta)\}.$$

Hence, $\Omega = \Pi_0\left(\frac{1}{2K}\right)$.

Lemma 2.0.3. *g satisfies the following properties on Π:*

1. *Π is forward invariant under g.*

2. *$g^{\circ n}(w) \to \infty$ uniformly on Π.*

3. *$\arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π.*

4. *$g : \Pi \to \Pi$ is conjugate to a translation.*

Proof.

1. We write,

$$w = x + iy,\ g(w) = X + iY,\ A/w + \theta(w) = a + ib.$$

From Equation (2.1), we obtain,

$$X + iY = (x + iy) + p + (a + ib)$$
$$\implies X = x + p + a \text{ and } Y = y + b.$$

Now, if $w \in \Pi$,

$$\begin{aligned} Y^2 - 4K(K - X) &= (y + b)^2 - 4K(K - x - p - a) \\ &= [y^2 - 4K(K - x)] + b^2 + 2yb + 4K(a + p) \\ &> 4Kp + (2yb + 4Ka). \end{aligned}$$

Now, for $w \in \Pi$, $|w| > K > 1$. (It is clear for $\mathrm{Re}(w) > K$. For $\mathrm{Re}(w) \leq K$, we use the polar description $\rho > 2K - \rho\cos\phi$ to get $|w| > 2K - \mathrm{Re}(w) \geq K$). Hence we get,

$$|w||A/w + \theta(w)| \leq |w|(|A|/|w| + B/|w|^{1+\frac{1}{p}}) = |A| + B/|w|^{\frac{1}{p}} < |A| + B \tag{2.3}$$

(since for $|w| > 1$, $|w|^{\frac{1}{p}} > 1$). Therefore,

$$\begin{aligned} |2yb + 4Ka| &\leq 2|yb| + 4K|a| \\ &\leq 2|y||b| + 4K|a| \\ &\leq 2|w||a + ib| + 4|w||a + ib| \\ &= 6|w||a + ib| \\ &< 2K \leq 2Kp. \end{aligned}$$

Therefore, we see that $Y^2 - 4K(K - X) > 0$ and hence, $g(w) \in \Pi$ for $w \in \Pi$. Hence, Π is forward invariant under g.

2. Now, we will prove a stronger statement that for any $t > 0$ g maps $\Pi + t$ into $\Pi + t + p/2$. This is simply because, for $w \in \Pi + t$, we have, $y^2 - 4K(K + t - x) > 0$. Hence,

$$\begin{aligned} Y^2 - 4K(K + t + p/2 - X) &= [y^2 - 4K(K + t - x)] + b^2 + 2yb + 4K(a - p/2) \\ &> 2Kp + (2yb + 4Ka) \\ &> 0. \end{aligned}$$

Therefore, if $w \in \Pi$, $g^{\circ n}(w) \in \Pi + np/2$. Hence, $|g^{\circ n}(w)| > \sqrt{n}$. This is simply because, if $x + iy \in \Pi + np/2$, we have

$$x^2 + y^2 - n > x^2 + 4K(K + np/2 - x) - n = x^2 - 4Kx + (4K^2 + 2npK - n).$$

The discriminant of this quadratic equation in x is

$$16K^2 - 4(4K^2 + 2npK - n) = 4n(1 - 2pK) < 0$$

. Thus, $x^2 + y^2 - n > 0$ for all $x + iy \in \Pi + np/2$.

Hence, $g^{\circ n}(w) \to \infty$ uniformly on Π.

3. Firstly, we see that,

$$g^{\circ k}(w) = g(g^{\circ k}(w)) = g^{\circ k}(w) + p + A/g^{\circ k}(w) + \theta(g^{\circ k}(w)).$$

By expanding the first term again and again, we obtain,

$$g^{\circ n}(w) = w + np + \sum_{k=0}^{n-1}\left(\frac{A}{g^{\circ k}(w)} + \theta(g^{\circ k}(w))\right). \tag{2.4}$$

Also note that form Equation (2.3), we have,

$$|A/w + \theta(w)| < (|A| + B)/K < \frac{1}{3}.$$

Let Q be a compact subset of Π. From now, we will assume that $w \in Q$ and we will use $C_1, C_2, C_2, \ldots$ to denote positive constants which will be dependent on Q.

Hence,

$$\begin{aligned} |g(w)| = |w + p + A/w + \theta(w)| &\geq ||w + p| - |A/w + \theta(w)|| \\ &= |w + p| - |A/w + \theta(w)| \\ &\geq |w| + p - \frac{1}{3}. \end{aligned}$$

Therefore, we obtain,

$$|g^{\circ n}(w)| \geq |w| + n(p - 1/3) \geq C_1 + C_2 n.$$

(Here, $C_1 = \min\{|w| : w \in Q\} > 0$ and $C_2 = p - \frac{1}{3} > 0$.)
Hence,

$$|g^{\circ n}(w)| \geq C_2 n. \tag{2.5}$$

Next, with Equation (2.2), and the above inequality, we get,

$$|\theta(g^{\circ n}(w)| \leq B/|g^{\circ n}(w)|^{1+\frac{1}{p}} \leq C_3/n^{1+\frac{1}{p}}. \tag{2.6}$$

Finally, combining the above two inequalities and Equation (2.4), we obtain,

$$\begin{aligned} |g^{\circ n}(w) - np| &\leq |w| + |A/w + \theta(w)| + \frac{|A|}{C_2}\sum_{k=1}^{n-1}\frac{1}{k} + C_3\sum_{k=1}^{n-1}\frac{1}{n^{1+\frac{1}{p}}} \\ &< C_4 + C_5\sum_{k=1}^{n}\frac{1}{k}. \end{aligned}$$

(Here, $C_4 = \max\{|w| : w \in Q\} + \frac{1}{3} + C_3 \sum_{n=1}^{\infty} 1/n^{1+\frac{1}{p}}$ and $C_5 = |A|/C_2$.)
We can select C_6 large enough such that

$$|g^{\circ n}(w) - np| < C_6 \log n. \tag{2.7}$$

Remark. The above inequality follows from the fact that, if $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then $H_n - \log n \to \gamma$. (γ is known as the Euler's constant). So, we have that

$$\begin{aligned} P + QH_n &= P + Q(\log n + \gamma + \epsilon_n), \text{where } \epsilon_n \to 0 \\ &\leq Q \log n + (P + Q \max\{\epsilon_n\} + Q\gamma) \\ &= Q \log n + R \\ &< S \log n \end{aligned}$$

for S large enough.

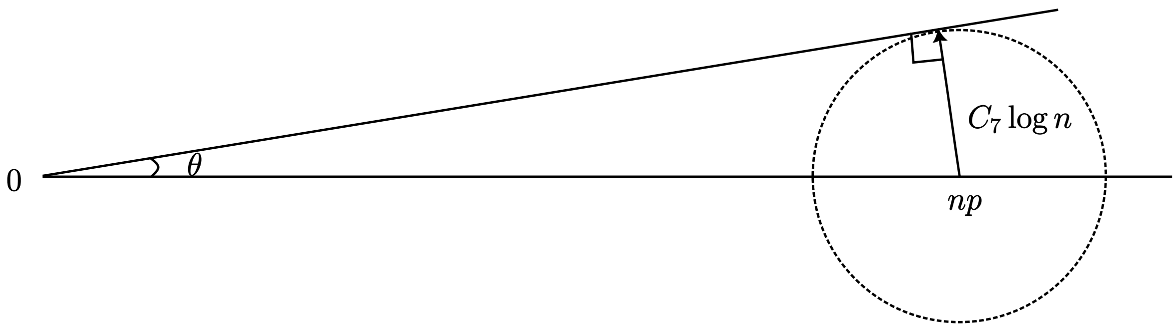


Figure 2.5: $|\arg(g^{\circ n}(w))| \leq \sin^{-1}(\frac{C_6 \log n}{np})$.

From, $|g^{\circ n}(w) - np| < C_6 \log n$, it follows that $|\arg(g^{\circ n}(w)| < \sin^{-1}\left(\frac{C_6 \log n}{np}\right)$ for n large enough. Hence, $\arg(g^{\circ n}(w)) \to 0$ uniformly on Q, and consequently, locally uniformly on Π.

4. Define,

$$u_n(w) = g^{\circ n}(w) - np - (A/p) \log n.$$

Claim. $u_n(w)$ converges locally uniformly on Π to a holomorphic function u, that is one-to-one on Π.

$$u_{n+1}(w) - u_n(w) = [g^{\circ n+1}(w) - g^{\circ n}(w)] - p - (A/p) \log\left(\frac{n+1}{n}\right).$$

From Equation (2.2), we obtain,

$$\begin{aligned} u_{n+1}(w) - u_n(w) &= [g^{\circ n}(w) + p + A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - g^{\circ n}(w)] \\ &\qquad - p - (A/p)\log(1 + 1/n) \\ &= A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - (A/p)\log(1 + 1/n) \\ &= A(1/g^{\circ n}(w) - 1/np) + \theta(g^{\circ n}(w)) + (A/p)(1/n - \log(1 + 1/n)). \end{aligned}$$

Now, let Q is a compact subset of Π and $w \in Q$. We need to prove that u_n converges uniformly in Q. From the above equation, to prove that u_n converges uniformly in Q, we need to show that each of the following series converges uniformly in Q:

$$\sum_n |1/g^{\circ n}(w) - 1/np|, \ \sum_n |\theta(g^{\circ n}(w)), \ \sum_n |1/n - \log(1+1/n)|.$$

Let us look at the first series. We have, (using Equations (2.5) and (2.7))

$$|1/g^{\circ n}(w) - 1/np| = \frac{|g^{\circ n}(w) - np|}{|g^{\circ n}(w)|np} \leq \frac{C_6 \log n}{C_2 n^2 p} = C_7 \log n/n^2.$$

(Here $C_7 = C_6/(pC_2)$).

From Equation (2.6), it is clear that $\sum_n |\theta(g^{\circ n}(w))|$ converges.

Now, $0 < x - \log(1+x) \leq x^2$ for $x > 0$.

This is because, it is zero at $x = 0$ and $\frac{d}{dx}(x - \log(1+x)) = 1 - \frac{1}{1+x} > 0$ for $x > 0$.

Also, $x^2 - x + \log(1+x)$ is zero at $x = 0$ and $\frac{d}{dx}(x^2 - x + \log(1+x)) = 2x - 1 + \frac{1}{1+x} > 0$ for $x > 0$.

Putting $x = \frac{1}{n}$, we get,

$$|1/n - \log(1+1/n)| < 1/n^2.$$

Therefore, u_n converges locally uniformly to some holomorphic function u on Π.

Now, from $u_n(w) = g^{\circ n}(w) - np - (A/p)\log n$, we get that,

$$\begin{aligned}
(n+1)p + (A/p)\log(n+1) + u_{n+1}(w) &= g^{\circ n+1}(w) \\
&= g^{\circ n}(g(w)) \\
&= np + (A/p)\log n + u_n(g(w)) \\
\implies p + (A/p)\log(1+1/n) + u_{n+1}(w) &= u_n(g(w)).
\end{aligned}$$

Taking limit $n \to \infty$, we get,

$$p + u(w) = u(g(w)).$$

Since f is injective near the origin, g is injective on Π, (if K is chosen large enough). Therefore, $g^{\circ n}$ is injective on Π and hence, so is u_n. By Hurwitz Theorem, u is either injective or constant, but it is clearly not a constant since it satisfies the above equation.

This shows that $g : \Pi \to \Pi$ is conjugate to the map $z \mapsto z + p$ of $u(\Pi)$ into itself. $\square$

Coming back to our original theorem, we see that our original theorem is also proved as we had just replaced the action of f on Π_0 by the action of its conjugate g on Π and we just proved all the parts of the theorem that the conjugate of f, i.e. g must satisfy.

From, $g = \sigma f \sigma^{-1}$, we get, $g^{\circ n} = \sigma f^{\circ n} \sigma^{-1} \implies g^{\circ n}\sigma = \sigma f^{\circ n}$. Writing, $\sigma(z) = w$, we have,

$$g^{\circ n}(w) = \frac{1}{f^{\circ n}(z)^p} \implies g^{\circ n}(w)(f^{\circ n}(z))^p = 1. \tag{2.8}$$

1. Since, g maps Π into itself, f maps Π_0 into itself.

2. Now, since $|g^{\circ n}(w)| > \sqrt{n}$, $|f^{\circ n}(z)| < \frac{1}{n^{1/2p}}$ from Equation (2.8). Hence, $f^{\circ n}(z) \to 0$ uniformly on Π_0.

3. Also, $\arg(f^{\circ n}(z)) = \left(-\frac{1}{p}\right)\arg(g^{\circ n}(w))$ from Equation (2.8). Since, $\arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π, $f^{\circ n}(z) = \left(-\frac{1}{p}\right)\arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π_0.

4. Since, $g : \Pi \to \Pi$ is conjugate to a translation, and g is conjugate to f, $f : \Pi_0 \to \Pi_0$ is also conjugate to a translation.

5. It is immediate from Lemma 2.0.1 that $|f(z)| < |z|$ on the axis of Π_0.

□

Theorem 2.0.4. *Suppose that f has the power series expansion near 0 as,*

$$f(z) = z + az^{p+1} + \mathcal{O}(z^{p+2}).$$

Then, f is conjugate near 0 to a function

$$F(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}).$$

Proof. First, we conjugate f by the map $z \mapsto \lambda z$, where $\lambda^p = a$. Then, we get that f is conjugate to the map,

$$\tilde{f} = \lambda f(z/\lambda) = \lambda z/\lambda + \lambda a z^{p+1}/\lambda^{p+1} + \mathcal{O}(z^{p+2}) = z + z^{p+1} + \mathcal{O}(z^{p+2}).$$

We will now proceed via induction over a finite number of steps. Let,

$$f_k(z) = z + z^{p+1} + bz^{p+k+1} + \ldots, b \neq 0.$$

Here $k \geq 1$. Also if $k \geq p$, our theorem is proved. Hence, we assume, $1 \leq k < p$.
Now, define the polynomial,

$$\sigma(z) = z + \alpha z^{k+1},$$

where $\alpha = \frac{b}{p-k}$ and let σ^{-1} be its inverse near 0 (We can do this because $\sigma'(0) = 1$).

Now, we will show that we obtain f_r (for some $r \geq k+1$) by conjugating f_k with σ. Hence, let

$$g = \sigma f_k \sigma^{-1}$$

and we need to show that $g = f_r$ (for some $r \geq k+1$). Since, $g'(0) = f_k'(0) = 1$, we let,

$$g(z) = z + \sum_{m=2}^{\infty} a_m z^m.$$

Now, we will use the identity, $g\sigma = \sigma f_k$.

$$\begin{aligned}\sigma f_k(z) &= (z + z^{p+1} + bz^{p+k+1} + \dots) + \alpha(z + z^{p+1} + bz^{p+k+1})^{k+1} \\ &= z + \alpha z^{k+1} + z^{p+1} + (b + \alpha(k+1))z^{p+k+1} + \mathcal{O}(z^{p+k+2}) \\ &= z + \alpha z^{k+1} + z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

The last equality follows because,

$$\alpha(p-k) = b \implies \alpha(p+1) - \alpha(k+1) = b \implies \alpha(p+1) = b + \alpha(k+1).$$

Now,

$$\begin{aligned}g\sigma(z) &= (z + \alpha z^{k+1}) + \sum_{m=2}^{\infty} a_m (z + \alpha z^{k+1})^m \\ &= z + \alpha z^{k+1} + \sum_{m=2}^{p+k+1} a_m (z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

Now, equating $\sigma f_k(z) = g\sigma(z)$, we get,

$$z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) = \sum_{m=2}^{p+k+1} a_m (z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}).$$

Firstly, we see that on the right hand side, the coefficient of z^2 will be a_2, the coefficient of z^3 will be some linear combination of a_2 and a_3, the coefficient of z^4 will be some linear combination of a_2, a_3 and a_4 and so on upto the coefficient of z^p will be some linear combination of $a_2, a_3, \dots a_p$. Since, the coefficient of $z^2, \dots, z^p$ is zero on the left hand side, it follows that $a_2 = a_3 = \dots, a_p = 0$. (This argument follows assuming $p \geq 2$, but if $p = 1$ the coefficient of $z^p = z$ i.e. a_1 is automatically 0).

Hence, now we have,

$$\begin{aligned}z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) &= \sum_{m=p+1}^{p+k+1} a_m (z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}) \\ &= a_{p+1}z^{p+1} + \dots + a_{p+k+1}z^{p+k+1} + \\ &\quad a_{p+1}\alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

Therefore, we obtain

$$a_{p+1} = 1, a_{p+2} = \dots = a_{p+k} = 0 \text{ and } a_{p+k+1} + a_{p+1}\alpha(p+1) = \alpha(p+1).$$

Hence, $a_{p+k+1} = 0$. This gives that f_k is conjugate to the map

$$g(z) = z + z^{p+1} + \mathcal{O}(z^{p+k+2}).$$

Thus, $g = f_r$ for some $r \geq k+1$. Continuing the induction process, we get that f is conjugate near 0 to a map

$$z \mapsto z + z^{p+1} + \mathcal{O}(z^{2p+1}).$$

Now, we an again conjugate this map with the map, $z \mapsto \lambda z$, where $\lambda^p = -1$ to get that f is conjugate to a map,

$$F(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}).$$

$\square$

3 Bottcher's Theorem and its extension

3.1 Bottcher's Coordinates

A fixed point p is called a super-attracting fixed point of f if $f'(p) = 0$.
If p is a super-attracting fixed point for f, we can conjugate the map such that $z = 0$ becomes our super-attracting fixed point.

Thus, our map takes the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$$

in a neighbourhood of 0 , with $n \geq 2$ and $a_n \neq 0$. Here the integer n is the *local degree* or *valency* of f at 0.

Theorem 3.1.1 (Bottcher's Theorem). *With f as above, $\exists$ a local holomorphic change of coordinates $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to $w \mapsto w^n$ throughout some neighbourhood of 0 .*
Furthermore, ϕ is unique upto multiplication by an $(n-1)$ th root of unity.

Proof. **Existence.** Let $c \in \mathbb{C}$ be such that $c^{n-1} = a_n$. Then, the linearly conjugate map $cf(z/c)$ will have leading coefficient $+1$. Thus, without loss of generality, we will assume that our map f has the form,

$$f(z) = z^n(1 + b_1 z + b_2 z^2 + \dots) = z^n(1 + \eta(z)), \text{ where } \eta(z) = \left(1 + b_1 z + b_2 z^2 + \dots\right).$$

Choose $r \in \left(0, \frac{1}{2}\right)$ such that $|\eta(z)| < \frac{1}{2}\ \forall z \in \mathbb{D}_r$. This can be done since $\eta(0) = 0$ and η is continuous.
On this disc, we have two properties of f :

1. f maps this disc into itself: We have, $|f(z)| = |z^n|\,|1 + \eta(z)| \leq |z|^n(1 + |\eta(z)|) < \frac{3}{2}|z|^n \leq \frac{3}{2^n}|z| \leq \frac{3}{4}|z|\ \forall z \in \mathbb{D}_r$. Here we are using the fact that $n \geq 2, |z| < \frac{1}{2}$ and $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$.

2. $f(z) \neq 0\ \forall z \in \mathbb{D}_r \backslash \{0\}$. This is simply because $|f(z)| = |z|^n |1 + \eta(z)|$ and since $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$, we can't have $\eta(z) = -1$.

The k-th iterate of f i.e. $f^{\circ k}$ also maps the $\mathbb{D}_r$ into itself and $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$. Inductively, it can be shown that it has the form $f^{\circ k}(z) = z^{n^k}\left(1 + n^{k-1}b_1 z + \ldots\right)$.

The idea of the proof is to set,

$$\phi_k(z) = \left(f^{\circ k}(z)\right)^{\frac{1}{n^k}} = z\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}}$$

We choose z as our branch of holomorphic n^k th root of z^{n^k}.

Now, we can choose a holomorphic branch of $\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}}$ on $\mathbb{D}_r$ since $\mathbb{D}_r$ is simply connected and $\left(1 + n^{k-1}b_1 z + \ldots\right) \neq 0$ on $\mathbb{D}_r$ since $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$. Therefore we set,

$$\phi_k(z) = z\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}} = z\left(1 + \frac{b_1}{n}z + \ldots\right)$$

where the expression on the right provides us an explicit choice of n^k th root.

We will show that the functions ϕ_k converge uniformly to a limit function ϕ on $\mathbb{D}_r$. To prove the convergence, we make the substitution $z = e^u$ where u ranges over the left half plane $\mathbb{H}_r := \{u : \mathrm{Re}(u) < \log r\}$. The exponential map maps $\mathbb{H}_r$ onto $\mathbb{D}_r \backslash \{0\}$.

The map f from $\mathbb{D}_r$ into itself corresponds to a map from $\mathbb{H}_r$ into itself given by $F(u) = \log f\left(e^u\right)$. We can select a holomorphic branch of the logarithm of $f\left(e^u\right)$ because $\mathbb{H}_r$ is simply connected and $f\left(e^u\right) \neq 0$ on $\mathbb{H}_r$.

Set $\eta = \eta\left(e^u\right) = b_1 e^u + b_2 e^{2u} + \ldots$, then since $|\eta| < \frac{1}{2}$, we see that F can be written as

$$F(u) = \log\left(e^{nu}(1+\eta)\right) = nu + \log(1+\eta) = nu + \left(\eta - \frac{\eta^2}{2} + \frac{\eta^3}{3} - + \ldots\right)$$

where the final expression provides us an explicit choice of which branch of logarithm we are using. Clearly, $F : \mathbb{H}_r \to \mathbb{H}_r$ is a well-defined holomorphic function.

Similarly, the map ϕ_k corresponds to a map, $\Phi_k(u) = \log \phi_k\left(e^u\right)$.

$$\Phi_k(u) = \log \phi_k\left(e^u\right) = \log f^{\circ k}\left(e^u\right)^{\frac{1}{n^k}} = \frac{1}{n^k}\log f^{\circ k}\left(e^u\right).$$

Since we have already fixed the branch of logarithm that we are using, we see that,

$$\log f^{\circ k}\left(e^u\right) = \log f\left(f^{\circ k-1}\left(e^u\right)\right) = \log f\left(e^{\log f^{\circ k-1}\left(e^u\right)}\right) = F\left(\log f^{\circ k-1}\left(e^u\right)\right)$$

Hence, inductively we can see that $\log f^{\circ k}\left(e^u\right) = F^{\circ k}(u)$.

Therefore, $\Phi_k(u) = F^{\circ k}(u)/n^k$. It is clear from this expression that $\Phi_k : \mathbb{H}_r \to \mathbb{H}$.

Now since $|\eta| < \frac{1}{2}$, we have

$$|F(u) - nu| = |\log(1+\eta)| < \log 2 < 1.$$

Hence,

$$|\Phi_{k+1}(u) - \Phi_k(u)| = \frac{1}{n^{k+1}}\left|F^{\circ k+1}(u) - nF^{\circ k}(u)\right| < \frac{1}{n^{k+1}},$$

by the above inequality.

We have, $\phi_k(e^u) = e^{\Phi_k(u)}$. Since, the exponential map, $e^{\square} : \mathbb{H} \to \mathbb{D}$ from the left half plane to the unit disc decreases distance, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \ \forall z \in \mathbb{D}_r \backslash \{0\}.$$

Since $\phi_k(0) = 0$ for all k, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \ \forall z \in \mathbb{D}_r.$$

Hence, the maps ϕ_k converge uniformly to some limit function ϕ on $\mathbb{D}_r$ by the Cauchy criterion for uniform convergence.

Clearly, $\phi(0) = 0$ and ϕ is holomorphic on $\mathbb{D}_r$ by Weierstrass convergence theorem.

It is clear that each $\phi_k : \mathbb{D}_r \to \mathbb{D}$. This is because $\phi_k(e^u) = e^{\Phi_k(u)}$ and $\Phi_k : \mathbb{H}_r \to \mathbb{H}$ and $e^{\square} : \mathbb{H} \to \mathbb{D}\backslash\{0\}$. Hence, $\phi : \mathbb{D}_r \to \mathbb{D}$. (Clearly $\mathrm{Im}(\phi)$ cannot contain points from $\partial\mathbb{D}$ because ϕ is holomorphic, hence it is an open map).

Now, it can be easily seen that, $\phi_k(f(z)) = \phi_{k+1}(z)^n$.

Hence, $\lim_{k\to\infty} \phi_k(f(z)) = \lim_{k\to\infty} \phi_{k+1}(z)^n \implies \phi(f(z)) = \phi(z)^n$ by continuity of nth power map.

Also, since $\phi_k'(0) = 1 \ \forall k \in \mathbb{N}$ (from the power series of ϕ_k), we have $\phi'(0) = 1$. Hence, ϕ is invertible in some neighbourhood of 0 .

Therefore, we have a holomorphic change of coordinates in some neighbourhood of 0 which conjugates f to the nth power map. In this neighbourhood, ϕ is one-to-one, $f(z) \neq 0$ for $z \neq 0$ (i.e. no other point maps to the super-attracting fixed point via f) and f maps this neighbourhood into itself.

Uniqueness. It suffices to study the special case $f(z) = z^n$. If we can prove that any map which conjugates $z \mapsto z^n$ to itself is just multiplication by $(n-1)$ th root of unity, then for any general map $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, if we have two maps ϕ and ψ which conjugate it to $z \mapsto z^n$, then $\phi \circ \psi^{-1}$ is a map which conjugates $z \mapsto z^n$ to itself. Hence, $\phi \circ \psi^{-1} = cz$, where $c^{n-1} = 1$. Therefore, $\phi = c\psi$, where c is a $(n-1)$ th root of unity.

So, let $\phi(z) = c_1 z + c_k z^k + \ldots, (c_1 \neq 0)$ be a map which conjugates $z \mapsto z^n$ to itself. Then, we should have $\phi(z^n) = \phi(z)^n$. Now,

$$\phi(z^n) = c_1 z^n + c_k z^{nk} + \ldots$$

and

$$\phi(z)^n = c_1^n z^n + n c_1^{n-1} c_k z^{n+k-1} + \ldots$$

Comparing coefficients, we get $c_1^n = c_1$ and $nc_1^{n-1}c_k = 0$ since $nk > n+k-1$ for $k \geq 2$. Therefore, we get $c_1^{n-1} = 1$ and $c_k = 0$. The form $\phi(z) = c_1 z + c_k z^k + \dots$ can be modified to any $k \geq 2$ to get $c_k = 0$ by the same process.

Therefore, $\phi(z) = cz$, where c is a $(n-1)$ th root of unity. $\square$

3.2 Extension of Bottcher's coordinates

We might hope to extend the Bottcher's coordinates to the whole of the immediate basin of attraction of the super-attracting fixed point. But this is not always possible. This is because, it requires computing expressions of the form $z \mapsto \phi\left(f^{\circ k}(z)\right)^{\frac{1}{n^k}}$, which is not always possible. For example, the basin may not be simply connected, or some point may map directly onto the super-attracting fixed point, in which case we can't take n^k-th roots, because $\phi\left(f^{\circ k}(z)\right)$ must be zero at those points.

Theorem 3.2.1 (Extension of $|\phi|$). *If f has a super-attracting fixed point p, with immediate basin of attraction $\mathcal{A}$, then the function $z \mapsto |\phi(z)|$ of the above theorem extends uniquely to a continuous map $|\phi| : \mathcal{A} \to [0,1)$ which satisfies $|\phi|(f(z)) = |\phi|(z)^n$.*

Furthermore, $|\phi|$ is real analytic except at the iterated preimages of p, where it takes the value 0 .

Proof. Set $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{1}{n^k}}$ for large enough k for each $z \in \mathcal{A}$. ϕ is only defined in a some small neighbourhood of p. But since, $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, after k many iterates for some large k, $f^{\circ k}(z)$ belongs to the domain of definition of ϕ, which we shall call $\hat{U}$.

It is independent of the value of k (if k is large enough). Note that, if $f^{\circ k}(z) \in \hat{U}$, then so does $f^{\circ k+1}(z)$, since f maps $\hat{U}$ into itself.

Suppose we choose k minimal such that $f^{\circ k}(z) \in \hat{U}$. Then,

$$| \phi\left(f^{\circ k+1}(z)\right|^{\frac{1}{n^{k+1}}} = \left|\phi\left(f\left(f^{\circ k}(z)\right)\right)\right|^{\frac{1}{n^{k+1}}} = \left|\phi\left(f^{\circ k}(z)\right)^n\right|^{\frac{1}{n^{k+1}}} = \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{1}{n^k}} = |\phi|(z).$$

In the proof of the Bottcher's theorem, we saw that $\phi(z) \in \mathbb{D}\ \forall z \in \hat{U}$. Hence, $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right| < 1\ \forall z \in \mathcal{A}$. Therefore, $|\phi| : \mathcal{A} \to [0,1)$.

Also,

$$\begin{aligned}
|\phi|(f(z)) &= \left|\phi\left(f^{\circ k}(f(z))\right)\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f\left(f^{\circ k}(z)\right)\right)\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f^{\circ k}(z)\right)^n\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{n}{n^k}} \\
&= |\phi|(z)^n.
\end{aligned}$$

It is also clear that $|\phi| = 0$ only at p and its iterated preimages.

If q is an iterated preimage of p, say $f^{\circ k}(q) = p$, then we have $|\phi|(q) = \left|\phi\left(f^{\circ k}(q)\right)\right|^{\frac{1}{n^k}} = |\phi(p)|^{\frac{1}{n^k}} = 0$.

Now, Suppose $|\phi|(z) = 0$ for some z. Then, $|\phi|(z)^{n^k} = 0 \; \forall k \implies |\phi|\left(f^{\circ k}(z)\right) = 0 \; \forall k$. But for some large k, $f^{\circ k}(z)$ belongs to the domain of definition of ϕ. But that means, $f^{\circ k}(z) = p$, since no other point in that domain is mapped to zero by ϕ. Hence, z is an an iterated preimage of p.

Now, since $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, for each $a \in \mathcal{A}$, we have a neighbourhood W_a and a constant $k \in \mathbb{N}$ such that $f^{\circ k}(z) \in \hat{U}, \; \forall z \in W_a$.

Hence, for $z \in W_a$, we can define $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right| = |g(z)|$, where $g = \phi \circ f^{\circ k}\big|_{W_a}$. Therefore, $|\phi|_{W_a} = |g|$, where g is some holomorphic function defined on W_a.

It is clear from this that $|\phi|$ is continuous in $\mathcal{A}$.

Now, if h is any holomorphic function, then $|h(z)|$ is real-analytic everywhere in its domain except at those z, where $h(z) = 0$.

Since, $|g| = |\phi|_{W_a}$ is zero only at the iterated preimages of f in W_a, $|\phi|_{W_a}$ is real analytic everywhere in W_a except at the iterated preimages of p.

Therefore, $|\phi|$ is real analytic everywhere in $\mathcal{A}$ except at the iterated preimages of p. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with a super-attracting fixed point p. Then the associated Bottcher map ϕ carries a neighbourhood of p biholomorphically onto a neighbourhood of zero, conjugating f to the nth power map, where n is the local degree of f near p. ϕ has a local inverse ψ_ϵ which maps the ϵ-disc around zero to a neighbourhood of p. □

Theorem 3.2.2 (Extending ψ_ϵ). *There exists a unique open disc of maximal radius $0 < r \leq 1$ such that ψ_ϵ extends holomorphically to a map $\psi : \mathbb{D}_r \to \mathcal{A}$, where $\mathcal{A}$ is the immediate basin of attraction of p.*

1. *If $r = 1$, then ψ maps the open unit disc $\mathbb{D}$ onto $\mathcal{A}$ biholomorphically.*

2. *If $0 < r < 1$, then ψ maps $\mathbb{D}_r$ onto its image biholomorphically and there exists atleast one other critical point in $\mathcal{A}$ on the boundary of $\psi(\mathbb{D}_r)$.*

Proof. Let us try to extend ψ_ϵ along radial lines by analytic continuation. Then, we can't extend it indefinitely. We proceed by contradiction. If it can be extended indefinitely, it would yeild a holomorphic map ψ from the entire complex plane onto an open set $\psi(\mathbb{C}) \subset \mathcal{A} \subsetneq \mathbb{C}_\infty$. Later we have independently proved that ψ must be one-to-one on its domain of definition. Therefore, ψ is a one-to-one entire function. Hence, $\psi(\mathbb{C}) = \mathbb{C}_\infty \backslash \{p\}$ for some $p \in \mathbb{C}_\infty$. But this would imply that the Julia set is finite with cardinality atmost 2, which is impossible.

Thus, there must be some largest radius r so that ψ_ϵ extends analytically throughout the open disc $\mathbb{D}_r$.

Also, $|\phi|(\psi(w)) = |\phi(\psi(w))| = |w|$ near 0 , hence for all $w \in \mathbb{D}_r$ by analytic continuation.

Since, $|\phi| : \mathcal{A} \to [0,1)$, this proves that for any $w \in \mathbb{D}_r$, $|\phi|(\psi(w)) = |w| < 1$. Therefore, ψ can be defined only on $\mathbb{D}_r$ for $r \leq 1$.

We will now show that ψ is actually one-to-one on $\mathbb{D}_r$. Suppose $\psi(w_1) = \psi(w_2)$. Applying $|\phi|$, we see that $|w_1| = |w_2|$. Choose such a pair such that $\psi(w_1) = \psi(w_2)$ $(w_1 \neq w_2)$ with $|w_1| = |w_2|$ minimal. A minimal pair exists because for $|w| < \epsilon, \psi = \psi_\epsilon$ which is one-to-one as it is invertible.

Now, ψ is an open mapping. Choose a sufficiently small neighbourhood U_{w_2} of w_2. Then, $\psi(U_{w_2})$ is a small neighbourhood of $\psi(w_1) = \psi(w_2)$. Hence, for any w_1' sufficiently close to $w_1, \psi(w_1') \in \psi(U_{w_2})$. Hence, we can find w_2' sufficiently close to w_2 such that $\psi(w_1') = \psi(w_2')$. Choosing $|w_1'| < |w_1|$, we get a contradiction.

Hence, ψ maps $\mathbb{D}_r$ onto its image biholomorphically.

In case when $r = 1, U = \psi(\mathbb{D}) = \mathcal{A}$. If not then we would have some boundary point of U, say $z_0 \in \mathcal{A}$. We can approximate z_0 by points of $\psi(w_j)$, where $|w_j| \to 1$.

Now, $\lim_{j\to\infty} \psi(w_j) = z_0$. Hence,

$$\lim_{j\to\infty} |\phi|(\psi(w_j)) = |\phi|(z_0) \implies \lim_{j\to\infty} |w_j| = |\phi|(z_0) \implies |\phi|(z_0) = 1$$

which is impossible.

Now, let $0 < r < 1$. We need to prove that ∂U, where $U = \psi(\mathbb{D}_r)$ must contain a critical point of f. Suppose, $w_0 \in \partial \mathbb{D}_r$ and let $(w_j)_{j=1}^{\infty} \subset \mathbb{D}_r$ such that $w_j \to w_0$. Let $\psi(w_j) \to z_0$. Then $z_0 \in \partial U$ because ψ maps $\mathbb{D}_r$ onto U biholomorphically.

If z_0 is not a critical point of f, then f maps a neighbourhood of z_0, say A onto a neighbourhood of $f(z_0)$, say B biholomorphically.

It should be noted that A and B can be chosen such that $B \subset U$. This is because $f(z_0) \in U$. Indeed we have,

$$\begin{aligned}
&\lim_{j\to\infty} \psi(w_j) = z_0 \\
\implies &\lim_{j\to\infty} f(\psi(w_j)) = f(z_0) \\
\implies &\lim_{j\to\infty} \psi\left(w_j^n\right) = f(z_0) \\
\implies &\psi(w_0^n) = f(z_0).
\end{aligned}$$

Since, $|w_0| = r < 1, |w_0|^n < r^n < r$. Hence, $w_0^n \in \mathbb{D}_r$. Therefore, $\psi(w_0^n) = f(z_0) \in U$.

Let g be the local inverse of f near $f(z_0)$. Then, ψ can be extended throughout a neighbourhood of w_0 by

$$w \mapsto g(\psi(w^n))$$

We have, $\psi\left(w_0^n\right) = f\left(z_0\right) \implies w_0^n = \phi\left(f\left(z_0\right)\right)$. Since, $\phi(B)$ is a neighbourhood of $\phi\left(f\left(z_0\right)\right)$ lying inside $\mathbb{D}_r$, choose a small enough neighbourhood of w_0, say C such that $w^n \in \phi(B)$, for all $w \in C$. In this neighbourhood, C our newly defined map agrees with ψ on $C \cap \mathbb{D}_r$. This is because, for $w \in C \cap \mathbb{D}_r, f(\psi(w)) = \psi\left(w^n\right) \in B$. Therefore, $g\left(\psi\left(w^n\right)\right)$ can be defined and $\psi(w) = g\left(\psi\left(w^n\right)\right) \in A$. Hence, our new map is an analytic continuation of ψ on the neighbourhood C.

Now, if none of the $z_0 \in \partial U$ are critical points, we can extend ψ to a neighbourhood of $w_0 \; \forall \, w_0 \in \partial \mathbb{D}_r$. Clearly, these continutations would patch together to define ψ in a strictly greater disc than $\mathbb{D}_r$, which is a contradiction. $\square$

If ψ_ϵ is extended biholomorphically in this way to the map ψ defined on $\mathbb{D}_r$, then the inverse map $\psi^{-1} : \psi\left(\mathbb{D}_r\right) \rightarrow \mathbb{D}_r$ must be the extension of ϕ from some neighbourhood of p to $\psi\left(\mathbb{D}_r\right)$ (since ψ^{-1} agrees with ϕ on some neighbourhood of p).

4 Introduction to the Mandelbrot Set

We consider the set of quadratic polynomials $\{f_c(z) = z^2 + c : c \in \mathbb{C}\}$. It is enough to consider this set because every quadratic polynomial is linearly conjugate to a quadratic polynomial of the type $f_c(z)$ for some unique $c \in \mathbb{C}$.

To prove this, let $f(z) = pz^2 + qz + r$, $p \neq 0$. Consider the conjugation with the map, $\sigma(z) = az + b$ with $a = p$ and $b = q/2$. Then, we have

$$\begin{aligned}
\sigma f \sigma^{-1}(z) &= a\left(p\left(\frac{z-b}{a}\right)^2 + q\left(\frac{z-b}{a}\right) + r\right) + b \\
&= (z-b)^2 + q(z-b) + ar + b \\
&= z^2 + b^2 - qb + ar + b \\
&= z^2 + q^2/4 - q^2/2 + pr + q/2 \\
&= z^2 + (pr + q/2 - q^2/4).
\end{aligned}$$

This is of the form $z^2 + c$ for $c = pr + q/2 - q^2/4$.

4.1 Definition of the Mandelbrot Set

First, we define as new type of set, known as the *Filled-in Julia Set* for a polynomial P.

Definition 4.1.1 (Filled-in Julia Set). *The Filled-in Julia Set of a polynomial P is defined as $K(P) = \mathbb{C}_\infty \backslash F_\infty(P)$. It is the union of the Julia set and the bounded Fatou components. It is denoted by $K(P)$ or simply K when the context is clear.*

By Corollary 1.6.1.1, K can also be defined as follows:

$$K = \{z \in \mathbb{C} : P^{\circ n}(z) \text{ is bounded}\}.$$

Notation. We will use F_c, J_c and K_c for the $F_\infty(f_c)$, $J(f_c)$ and $K(f_c)$ respectively.

Definition 4.1.2 (**Mandelbrot Set**). *The Mandelbrot Set is defined as*

$$M = \{c \in \mathbb{C} : K_c \textit{ is connected}\}.$$

Now, we have the following two theorems for polynomials:

- For polynomials, since F_∞ is a completely invariant Fatou component (by Theorem 1.4.5), $\partial F_\infty = J$ (by Corollary 1.7.1.1).
- And, from Theorem 1.8.1, we have that F_∞ is simply connected $\iff$ $\mathbb{C}_\infty \backslash F_\infty$ is connected $\iff$ ∂F_∞ is connected.

Thus, for a polynomial,

$$F_\infty \text{ is simply connected } \iff K \text{ is connected } \iff J \text{ is connected.}$$

Hence, we have the following equivalent descriptions for the Mandelbrot Set:

$$\begin{aligned} M &= \{c \in \mathbb{C} : K_c \text{ is connected}\} \\ &= \{c \in \mathbb{C} : F_c \text{ is simply connected}\} \\ &= \{c \in \mathbb{C} : J_c \text{ is connected}\}. \end{aligned}$$

4.2 The Fundamental Dichotomy

Theorem 4.2.1. *For a polynomial P, the following are equivalent:*

1. *F_∞ is simply connected $\iff$ J is connected $\iff$ K is connected.*
2. *There are no finite critical points of P in F_∞.*

Proof. First assume that F_∞ is simply connected $\implies$ $c(F_\infty) = 1$ and hence, $\chi(F_\infty) = 2 - c(F_\infty) = 1$. Now, since F_∞ is completely invariant and P is a polynomial of degree d (say), P is a d-fold map of F_∞ onto itself. Applying the Riemann-Hurwitz relation to the map P of F_∞ onto itself, we obtain,

$$\begin{aligned} &\chi(F_\infty) + \delta_P(F_\infty) = d\,\chi(F_\infty) \\ \implies &1 + \delta_P(F_\infty) = d \\ \implies &\delta_P(F_\infty) = d - 1. \end{aligned}$$

Now, $\delta_P(\infty) = d - 1$ and therefore, P does not have any finite critical points in F_∞.

For the converse part, assume there are no critical points of P in F_∞. Then, the Bottcher's map ϕ which conjugates P to the map, $z \mapsto z^d$ can be extended to the whole of F_∞ and $\phi : F_\infty \to \mathbb{D}$ is a biholomorphism. Hence, F_∞ is simply connected. □

Now, quadratic maps have only one finite critical point and f_c have the critical point at 0 for all $c \in \mathbb{C}$. Hence, by the Fundamental Dichotomy, F_c is simply connected $\iff$ $0 \notin F_c$ or $0 \in K_c$. Using, c_n to denote $f_c^{\circ n}(0)$, we get,

$$\begin{aligned} M &= \{c \in \mathbb{C} : 0 \in K_c\} \\ &= \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}. \end{aligned}$$

Note that $c_0 = 0$ and $c_1 = f_c(0) = c$. So, (c_n) is also the forward orbit of c. Hence, in other words, the Mandelbrot Set consists of $c \in \mathbb{C}$ such that its forward orbit under the map f_c remains bounded.

4.3 The other end of the Dichotomy Theorem

By the Dichotomy Theorem, we can say that if even one finite critical point of P lies in F_∞, then K cannot be connected. But this theorem states that if all finite critical points of P lie in F_∞, then K is not only disconnected, but totally disconnected.

Definition 4.3.1 (Cantor set)**.** *A subset $X \subset \mathbb{C}_\infty$ is called a Cantor set if it is non-empty, closed, perfect and totally disconnected.*

Theorem 4.3.1. *Let R be a rational map with $\deg(R) \geq 2$. Let α be a super-attracting fixed point of R. If the Fatou component of R containing α, say F_α, contains all the critical points of R, then $J(R)$ is a Cantor set.*

Corollary 4.3.1.1. *If $c \notin M$, then J_c is a Cantor set.*

Proof. Since 0 is the only critical points of f_c (apart from ∞), if it belongs to $F_c = F_\infty(f_c)$, i.e. if $0 \notin K_c \iff c \notin M$, then J_c is a Cantor set. □

4.4 Some properties of the Mandelbrot Set

We know, $M = \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}$. This description for M can be strengthened significantly by the following theorem:

Theorem 4.4.1. $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Proof. Obviously, $\{c \in \mathbb{C} : |c_n| \leq 2\} \subseteq M$.

Now, Suppose that $c \in M$. We need to prove that $|f_c^{\circ n}(c)| = |c_n| \leq 2$ for all $n \in \mathbb{N}$. Consider the set $W_c = \{z \in \mathbb{C} : |z| \geq |c|, |z| > 2\}$. For $z \in W_c$,

$$|f_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| \geq |z|(|z| - 1) = |z|(1 + \epsilon)$$

for some $\epsilon > 0$ (as $|z| > 2$). Clearly, $|f_c^{\circ n}(z)| \geq |z|(1 + \epsilon)^n \implies z \notin K_c$.
This implies $|c| \leq 2$. Consequently, $|f_c^{\circ n}(c)| \leq 2$ for all $n \in \mathbb{N}$.
Hence, $M \subseteq \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Therefore, $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$. □

As $c_1 = c$, we have that $|c| \leq 2$ for all $c \in M$ i.e $M \subseteq \{c \in \mathbb{C} : |c| \leq 2\}$. This turns out to be the strongest bound possible for $|c|$ as $-2 \in M$. The orbit of 0 under the map $z \mapsto z^2 - 2$ is:

$$0 \mapsto -2 \mapsto 2 \mapsto 2$$

and hence is bounded.

Theorem 4.4.2. *The Mandelbrot set M is compact and $\mathbb{C}_\infty \backslash M$ is open and connected.*

Proof. Let, $c_n = f_c^{\circ n}(c) = Q_n(c)$ be a polynomial in c. Clearly, from Theorem 4.4.1

$$M = \cap_{n=1}^{\infty} Q_n^{-1}(\overline{\mathbb{D}_2}),$$

where $\overline{\mathbb{D}_2} = \{z \in \mathbb{C} : |z| \leq 2\}$. Thus, M is closed. It is already known that it is bounded. Hence, it is compact.

Now,

$$\mathbb{C}_\infty \backslash M = \cup_{n=1}^{\infty} Q_n^{-1}(E)$$

where $E = \mathbb{C}_\infty \backslash \overline{\mathbb{D}_2}$. Now, E is open and connected and since, Q_n are non-constant polynomials, $Q_n^{-1}(E)$ is open and connected for all $n \in \mathbb{N}$. Also, each one of them contains ∞ and hence, their union is also open and connected.
Therefore, $\mathbb{C}_\infty \backslash M$ is open and connected.

□

4.5 Plotting the Mandelbrot Set

Theorem 4.4.1 is also used to plot the Mandelbrot Set. A simple code would involve going through each pixel (x, y) in a $N \times N$ square (where $N > 2$) and seeing if the iterates of 0 under the map, $f_c(z) = z^2 + c$, where $c = x + iy$, become larger than of modulus 2 after a large number of iterations.

If the modulus of iterates of 0 do not cross the value 2 after a large number of iterations, $c = x + iy$ probably lies in the Mandelbrot set and we color it black. If not, then we leave it uncolored.

A simple code in Python generates the following picture:

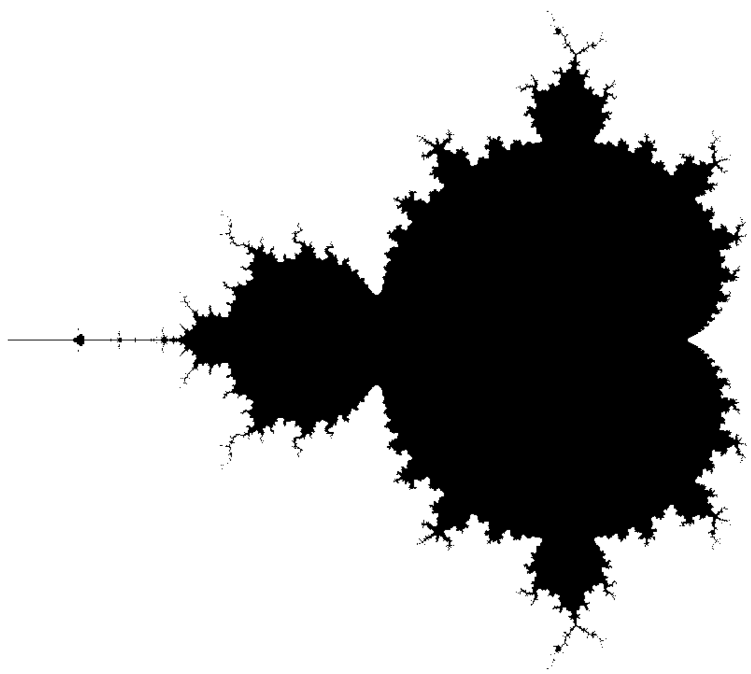


Figure 4.1: The Mandelbrot Set

5 Connectedness of the Mandelbrot Set

In the previous chapter, we proved that the Mandelbrot set is compact and $\mathbb{C}\backslash M$ is open and connected. In this chapter, we will prove that $\mathbb{C}_\infty\backslash M$ is biholomorphic to $\mathbb{C}_\infty\backslash\overline{\mathbb{D}}$, proving that $\mathbb{C}_\infty\backslash M$ is simply connected, thus implying that M is connected by Theorem 1.8.1.

For $f_c(z) = z^2 + c$, Bottcher's theorem and its extension guarantees the existence of a unique biholomorphic function ϕ_c defined on a simply-connected neighbourhood of infinity, say $U_c \subseteq F_c$ (where F_c is the basin of attraction of the super-attracting fixed point ∞), which conjugates f_c to the map $z \mapsto z^2$ and $\phi_c(z)/z \to 1$ as $z \to \infty$.
Furthermore, if $c \in M$, then $U_c = F_c$ and $\phi_c(U_c) = \mathbb{C}\backslash\overline{\mathbb{D}}$.
If $c \notin M$, then $U_c \subsetneq F_c$, ∂U_c contains the critical point 0 and $\phi_c(U_c) = \mathbb{C}\backslash\overline{\mathbb{D}}_r$, where $r > 1$.

5.1 The Green's Function

Definition 5.1.1 (Green's Function). *A continuous function $G : \mathbb{C} \to \mathbb{R}$ is called the potential theoretical Green's function of a compact set $K \subset \mathbb{C}$, if G is harmonic outside K, vanishing on K and has the property that $G(z)/\log|z| \to 1$ as $|z| \to \infty$.*

We know that $z \mapsto |\phi_c(z)|$ can be extended to a continuous function $|\phi_c| : F_c \to (1, \infty)$. (Note that since for polynomials, $P^{-1}(\infty) = \{\infty\}$, $|\phi_c|$ is finite everywhere on F_c).
In practice, it is customary to work with the logarithm of $|\phi_c|$. Hence define,

$$G_c(z) = \begin{cases} \log|\phi_c|(z) & \text{if } z \in F_c \\ 0 & \text{if } z \in K_c. \end{cases}$$

Clearly, $G_c : \mathbb{C} \to [0, \infty)$ and $G_c(z) > 0$ on F_c as $|\phi_c| > 1$ on F_c. Also, note that G satisfies the functional equation, $G_c(f(z)) = 2G_c(z)$. Also, it can be proven that G_c is harmonic on F_c and hence, G_c is indeed the Green's function for K_c.
Now,

- If $c \in M$, then $U_c = F_c$. Since $G_c(z) > 0$ for all $z \in F_c$ and $G_c(0) = 0$ as $0 \in K_c$, we can say that $U_c = F_c = \{z \in \mathbb{C} : G_c(z) > G_c(0) = 0\}$.

- If $c \notin M$, the $U_c \subsetneq F_c$. From the maximum principle, it is easy to see that minimum of $|\phi_c|$ on $\overline{U_c}$ lies of on ∂U_c. But, $|\phi_c|(\partial U_c) = r =$ constant and since, $0 \in \partial U_c$, we have $|\phi_c|(z) > |\phi_c|(0)$ for all $z \in U_c$. Therefore, $U_c = \{z \in \mathbb{C} : G_c(z) > G_c(0)\}$.

Therefore, $U_c = \{z \in \mathbb{C} : G_c(z) > G_c(0)\}$.

Lemma 5.1.1. *The map $\Phi(z, c) = \phi_c(z)$ is holomorphic in two variables on the set $S = \{(z, c) : c \in \mathbb{C}\backslash M, G_c(z) > G_c(0)\}$ and $\Phi(c, c)/c \to 1$ as $c \to \infty$.*

Proof. For a map to be holomorphic in two variables, it should be holomorphic in each variable when the other variable is kept fixed.

Now, we already know that for a fixed $c \in \mathbb{C}\backslash M$, the map $z \mapsto \phi_c(z)$ is holomorophic on $\{z \in \mathbb{C} : G_c(z) > G_c(0)\}$. But, we still need to prove that for a fixed z, the map $c \mapsto \phi_c(z)$ is holomorphic on the "z-slice" of S, i.e. $S_z = \{c \in \mathbb{C}\backslash M : G_c(z) > G_c(0)\}$.
We define, $\Phi_n(z, c) = \phi_{c,n}(z) = (f_c^{\circ n}(z))^{\frac{1}{2^n}}$, where $\phi_{c,n}$ are as defined in the proof of the Bottcher's theorem.
(In the proof of Bottcher's theorem, we had defined $\phi_n(z) = (f^{\circ n}(z))^{\frac{1}{2^n}}$ in a neighbourhood of the super-attracting fixed point. Here, we are defining an analytic 2^n-th root $f_c^{\circ n}$ throughout U_c (which is simply connected), which agrees with ϕ_n defined on the neighbourhood of the super-attracting fixed point.)
We write,

$$\Phi_n(z, c) = z \prod_{k=0}^{n-1} \frac{\Phi_{k+1}(z, c)}{\Phi_k(z, c)}.$$

Now,

$$\frac{\Phi_{n+1}(z, c)}{\Phi_n(z, c)} = \left(\frac{f_c^{\circ n+1}(z)}{(f_c^{\circ n}(z))^2}\right)^{\frac{1}{2^{n+1}}} = \left(1 + \frac{c}{(f_c^{\circ n}(z))^2}\right)^{\frac{1}{2^{n+1}}}.$$

and we write Φ as the infinite product,

$$\Phi(z, c) = z \prod_{n=0}^{\infty} \frac{\Phi_{n+1}(z, c)}{\Phi_n(z, c)} = z \prod_{n=0}^{\infty} \left(1 + \frac{c}{(f_c^{\circ n}(z))^2}\right)^{\frac{1}{2^{n+1}}}.$$

By Weierstrass Factorization Theorem [See Con94, pg 167], if $(f_n)_{n=1}^{\infty}$ is a sequence of analytic functions on $G \subset \mathbb{C}$ then $\prod_{n=1}^{\infty} f_n(z)$ is analytic if $\sum(f_n(z) - 1)$ converges absolutely and uniformly on compact subsets of G. Hence, let

$$\left(1 + \frac{c}{(f_c^{\circ n}(z))^2}\right)^{\frac{1}{2^{n+1}}} = 1 + \theta_n(z, c).$$

Select a fixed z_0 ($G_{c_0}(z_0) > G_{c_0}(0)$ for some $c_0 \in \mathbb{C}\backslash M$).
In order to prove that $c \mapsto \Phi(z_0, c)$ is analytic on S_{z_0}, we need to prove that $\theta_n(z_0, \cdot)$ converges uniformly and absolutely on compact subsets of S_{z_0}. Let K be a compact subset of S_{z_0}.

Claim. *There exists $N \in \mathbb{N}$ large enough so that for all $c \in K$, $|f_c^{\circ n}(z_0)|^2 > 2|c|$ for all $n \geq N$.*

Proof. For each $c \in K$, we have $n_c \in \mathbb{N}$ such that $|f_c^{\circ n}(z_0)|^2 > 2|c|$ for all $n \geq n_c$. (Note that if $|z| > 2$ and $|z|^2 > 2|c|$, then $|f_c^{\circ n}(z)|^2 > 2|c|$ for all $n \in \mathbb{N}$. Take n_c large enough such that $|f_c^{\circ n_c}(z_0)| > 2$ and $|f_c^{\circ n_c}(z_0)|^2 > 2|c|$.) Suppose, we have such a $\hat{c} \in K$ and correspondingly $n_{\hat{c}} \in \mathbb{N}$. Then there exists a neighbourhood of $\hat{c}$, say $B_{\hat{c}}$, such that for all $c \in B_{\hat{c}}$, $|f_c^{\circ n}(z_0)|^2 > 2|c|$ for all $n \geq n_{\hat{c}}$. This is because, $f_c^{\circ n_{\hat{c}}}(z_0)$ is a continuous function in c and so is $(f_c^{\circ n_{\hat{c}}}(z_0))^2/c$. Now, cover the compact set K by all such neighbourhoods, take a finite subcover and take N as the maximum of all such n_c obtained from this finite subcover. $\square$

Claim. *For $|w| < \frac{1}{2}$, $|(1+w)^{\frac{1}{k}} - 1| \leq 2|w|/k$.*

Proof. We integrate the derivative of $(1+w)^{1/k}$ along the radial segment from 0 to w.

$$\begin{aligned}
\int_0^w \frac{d}{dw}(1+w)^{1/k} dw &= \int_0^w \frac{1}{k}(1+w)^{1/k-1} dw \\
\implies (1+w)^{1/k} - 1 &= \frac{1}{k}\int_0^w (1+w)^{1/k-1} dw \\
\implies |(1+w)^{1/k} - 1| &\leq \frac{|w|}{k} \sup(|1+w|^{1/k-1}) \leq 2|w|/k.
\end{aligned}$$

For $|w| < \frac{1}{2}$, $\frac{1}{2} < |1+w| < \frac{3}{2} \implies (\frac{1}{2})^{1/k-1} \geq |1+w|^{1/k-1} \geq (\frac{3}{2})^{1/k-1}$.
As k increases, $(\frac{1}{2})^{1/k-1}$ increases and approaches 2 as $k \to \infty$.
Hence, $|1+w|^{1/k-1} \leq (\frac{1}{2})^{1/k-1} < 2$. $\square$

Now, for $c \in K$ and $n \geq N$, $\frac{|c|}{|f_c^{\circ n}(z_0)|^2} < \frac{1}{2}$. Hence, from the above inequality,

$$|\theta_n(z_0, c)| \leq \frac{2|c|}{2^{n+1}|f_c^{\circ n}(z_0)|^2} < \frac{1}{2^{n+1}}.$$

Hence, $\sum_{n=1}^{\infty} \theta_n(z_0, c)$ converges absolutely and uniformly on K. Therefore, $\Phi(z_0, c)$ is analytic on S_{z_0}. Therefore, $\Phi(z, c)$ is analytic in two variables on the set $S = \{(z, c) : c \in \mathbb{C} \backslash M, G_c(z) > G_c(0)\}$.

Now, for the second part, recall that for $c \in M$, $|c| \leq 2$. Thus, if $|c| > 2$, $c \in \mathbb{C} \backslash M$. Firstly, we observe that if $|c| > 2$ and $|z|^2 > 2|c|$,

$$|f_c(z)| \geq |z|^2 - |c| \geq |c| \implies |f_c(z)|^2 \geq |c|^2 > 2|c|.$$

Hence, if $|c| > 2$, $|f_c^{\circ n}(c)|^2 > 2|c|$ for all n.
We have,

$$\Phi(c, c)/c = \prod_{n=0}^{\infty} \left(1 + \frac{c}{(f_c^{\circ n}(c))^2}\right)^{\frac{1}{2^{n+1}}} = \prod_{n=0}^{\infty}(1 + \theta_n(c, c)).$$

Also by $\frac{|c|}{|f_c^{\circ n}(c)|^2} < \frac{1}{2}$, for all $n \in \mathbb{N}$,

$$|\theta_n(c,c)| < \frac{1}{2^{n+1}}.$$

Hence, $\Phi(c,c)/c$ converges uniformly on $|c| > 2$.

We see that each term of the product,

$$\Phi(c,c)/c = \left(1 + \frac{c}{c^2}\right)^{1/2} \cdot \left(1 + \frac{c}{(c^2+c)^2}\right)^{1/4} \cdots \left(1 + \frac{c}{(f_c^{\circ n}(c))^2}\right)^{1/2^{n+1}} \cdots$$

converges to 1. Hence, $\Phi(c,c)/c \to 1$ as $c \to \infty$. □

5.2 The Isomorphism by Douady and Hubbard

Douady and Hubbard proved that the Mandelbrot set is connected by defining an isomorphism $\Psi : \mathbb{C}\backslash M \to \mathbb{C}\backslash\overline{\mathbb{D}}$ given by,

$$\Psi(c) = \Phi(c,c) = \phi_c(c).$$

To prove that it is indeed an isomorphism, we will proceed in the following steps:

1. Ψ is a well defined map: For $c \in \mathbb{C}\backslash M$, $G_c(c) = 2G_c(0) > G_c(0)$. Thus, $c \in S_c$ and $\phi_c(c)$ can be defined. Also, $|\phi_c(c)| > 1$.

2. Ψ is analytic: We already proved that $(z,c) \mapsto \Phi(z,c)$ is analytic in two variables. Hence the map $c \mapsto (c,c) \mapsto \Phi(c,c)$ is analytic.

3. $|\Psi(c_n)| \to 1$ as $c_n \to M$: This is due to the continuity of $(z,c) \mapsto G_c(z)$ in two variables. Hence, $c \mapsto (c,c) \mapsto G_c(c)$ is continuous. Hence, as $c_n \to c_0 \in M$, $G_{c_n}(c_n) \to G_{c_0}(c_0) = 0$. Hence, $\log|\phi_{c_n}|(c_n) = \log|\phi_{c_n}(c_n)| \to 0 \implies |\phi_{c_n}(c_n)| \to 1$.

4. Ψ can be extended to an analytic map $\Psi : \mathbb{C}_\infty\backslash M \to \mathbb{C}_\infty\backslash\overline{\mathbb{D}}$ by defining $\Psi(\infty) = \infty$: This is due to the fact that $\Psi(c)/c = \Phi(c,c)/c \to 1$ as $c \to \infty$.

5. This extension is a proper map: Let K be a compact subset of $\mathbb{C}_\infty\backslash\overline{\mathbb{D}}$. Clearly, $\Psi^{-1}(K)$ is a closed subset of $\mathbb{C}_\infty\backslash M$. If $\Psi^{-1}(K)$ is not compact, there is a sequence, $(c_n)_{n=1}^\infty \subset \Psi^{-1}(K)$ such that $c_n \to M$. This implies $|\Psi(c_n)| \to 1$ by point 3. This is not possible as K being a compact subset of $\mathbb{C}_\infty\backslash\overline{\mathbb{D}}$ is at a positive distance from $\mathbb{D}$.

6. Now, Ψ being a proper holomorphic map, it is a branched covering of some degree d. As $\Psi^{-1}(\infty) = \{\infty\}$ with multiplicity 1 (because $\Psi(c)/c \to 1$ as $c \to \infty$), $d = 1$.

Therefore, $\Psi : \mathbb{C}_\infty\backslash M \to \mathbb{C}_\infty\backslash\overline{\mathbb{D}}$ is an isomorphism and M is connected. Consecutively, $\Psi : \mathbb{C}\backslash M \to \mathbb{C}\backslash\overline{\mathbb{D}}$ is also an isomorphism.

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