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DEPARTMENT OF MATHEMATICS AND STATISTICS
Master's Thesis BS-MS Dual Degree Program

Introduction to Complex Dynamics and the Mandelbrot Set

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Chapter 1

Introduction

1.1 Equicontinuity and Normality

1.2 Completely Invariant Components

A domain D is called:

- *forward invariant* under the map f if $f(D) = D$.
- *backward invariant* under the map f if $f^{-1}(D) = D$.
- *completely invariant* under the map f if it is both forward and backward invariant under f i.e. $f(D) = D$ and $f^{-1}(D) = D$.

Definition 1.2.1 (Connectivity). *The connectivity of a domain $D \subset \hat{\mathbb{C}}$ is defined as the number of components of ∂D .*

Theorem 1.2.1. *The following are equivalent for a domain $D \subset \hat{\mathbb{C}}$:*

1. D is simply connected.
2. D^c is connected.
3. ∂D is connected or $c(D) = 1$.

Theorem 1.2.2. *If R is a rational function, with $\deg(R) \geq 2$, and F_0 is a completely invariant Fatou component of R , then:*

1. $\partial F_0 = J$.
2. F_0 is simply connected or infinitely connected.
3. All other components of F are simply connected.

4. F_0 is simply connected $\iff J$ is connected.

Theorem 1.2.3. $\partial R(U) \subset R(\partial U)$

Lemma 1.2.4. For a rational map R , if F_1 and F_2 are two Fatou components and R maps a point of F_1 to a point of F_2 , then $R(F_1) = F_2$.

Theorem 1.2.5. The unbounded Fatou component of a polynomial P , i.e. the Fatou component containing ∞ is a completely invariant Fatou component. It is denoted by $F_\infty(P)$ or simply F_∞ when the context is clear.

Theorem 1.2.6 (Vitali's Theorem).

Lemma 1.2.7. If α is a (super)-attracting fixed point of a rational map R and F_α is the Fatou component containing α then $R^n(z) \rightarrow \alpha$ locally uniformly in F_α .

Theorem 1.2.8 (Riemann-Hurwitz Formula). Let F_0 and F_1 be components of the Fatou set F of a rational map R and R maps F_0 into F_1 . Then, for some integer m , R is an m -fold map of F_0 onto F_1 and

$$\chi(F_0) + \delta_R(F_0) = m\chi(F_1).$$

1.3 Some properties of the Julia Sets

Let J denote the Julia set for a rational map R with $\deg(R) \geq 2$. Then we have the following properties:

Theorem 1.3.1. J is infinite.

Theorem 1.3.2 (Minimality of J).

Theorem 1.3.3. J is a perfect set, and hence, uncountable.

Chapter 2

Behaviour of analytic functions near fixed points

2.1 Behaviour near parabolic fixed points

A point p is called a parabolic fixed point of f if $f(p) = p$ and $f'(p) = e^{2\pi it}$, where t is a rational number.

Lemma 2.1.1. *Suppose f is analytic and satisfies*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{p+2})$$

in some neighbourhood N of the origin. Let $\omega_1, \dots, \omega_p$ be the p -th roots of unity and let η_1, \dots, η_p be the p -th roots of -1 . Then for sufficiently small r_0 and θ_0 ,

1. $|f(z)| < |z|$ on each sector

$$S_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\omega_j)| < \theta_0\}.$$

2. $|f(z)| > |z|$ on each sector

$$\Sigma_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\eta_j)| < \theta_0\}.$$

Proof. We have,

$$f(z)/z = 1 - z^p + \mathcal{O}(z^{p+1}).$$

□

Before stating the *Petal Theorem*, which discusses the behaviour of analytic functions near parabolic fixed points, we first define the notions of *petals*.

Definition 2.1.1 (Petals). Let $p \in \mathbb{N}$. For each $k \in \{1, \dots, p-1\}$, define the sets as a function of a parameter $t > 0$ as follows,

$$\Pi_k(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)); |\theta - 2k\pi/p| < \pi/p\}.$$

The sets $\Pi_k(t)$ are known as *Petals*.

We have shown a diagram of the petals $\Pi_k(t)$ in [Figure 2.1](#) for $p = 6$. Note that all the petals are pairwise disjoint and each petal subtends an angle of $2\pi/p$ at the origin.

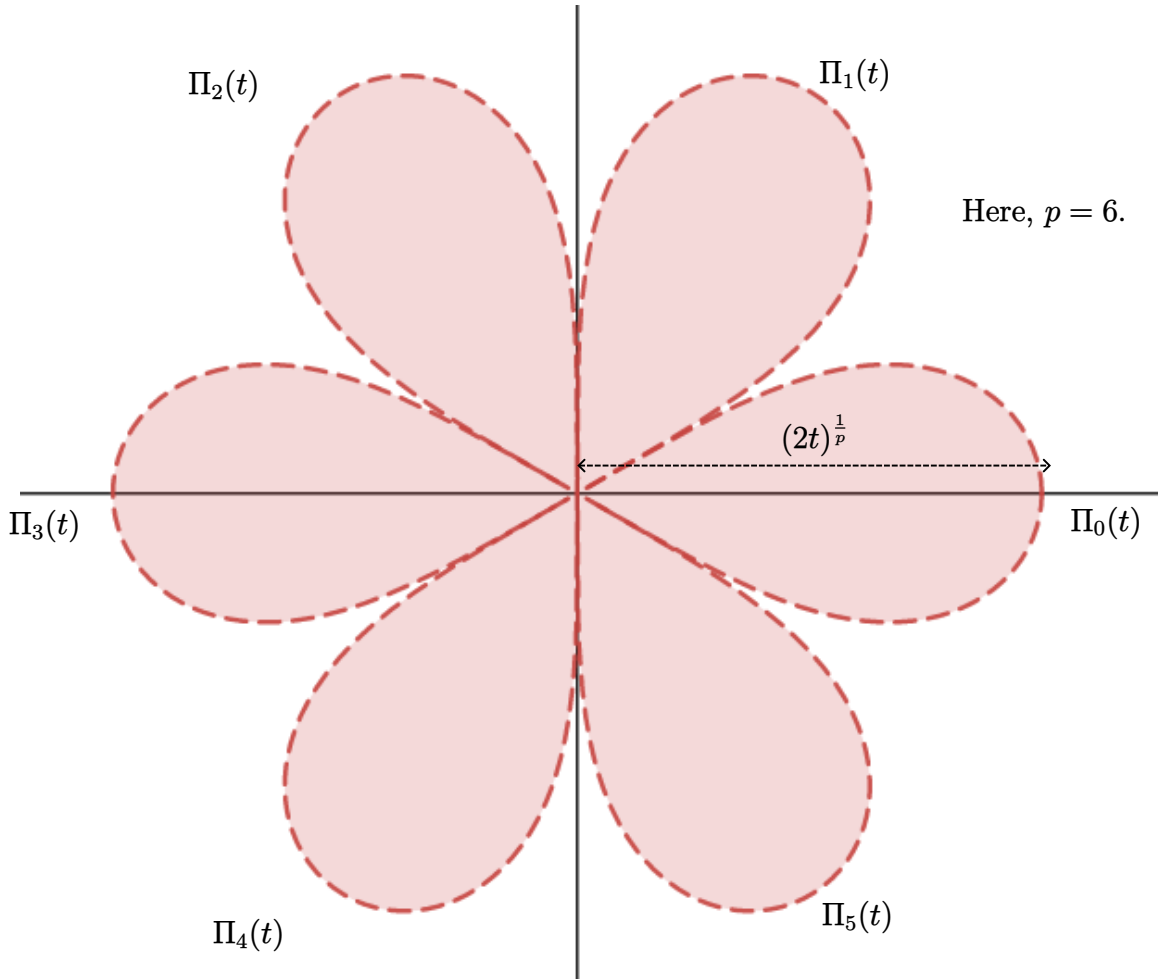


Figure 2.1: Six petals at the origin for $p = 6$.

Theorem 2.1.2 (The Petal Theorem). Suppose that an analytic map f has the form:

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1})$$

near the origin. Then for sufficiently small t ,

1. f maps each $\Pi_k(t)$ into itself;
2. $f^{\circ n}(z) \rightarrow 0$ uniformly on each petal;
3. $\arg(f^{\circ n}(z)) \rightarrow 2k\pi/p$ locally uniformly on each petal;
4. $f : \Pi_k(t) \rightarrow \Pi_k(t)$ is conjugate to a translation.
5. $|f(z)| < |z|$ on a neighbourhood of the axis of each petal;

Proof. For $0 < r_0 < 1$, define the sector S_0 ,

$$S_0 = \{re^{i\theta} : 0 < r < r_0, |\theta| < \pi/p\}$$

and the region W ,

$$W = \{re^{i\theta} : r > \frac{1}{r_0^p}, |\theta| > \pi\}.$$

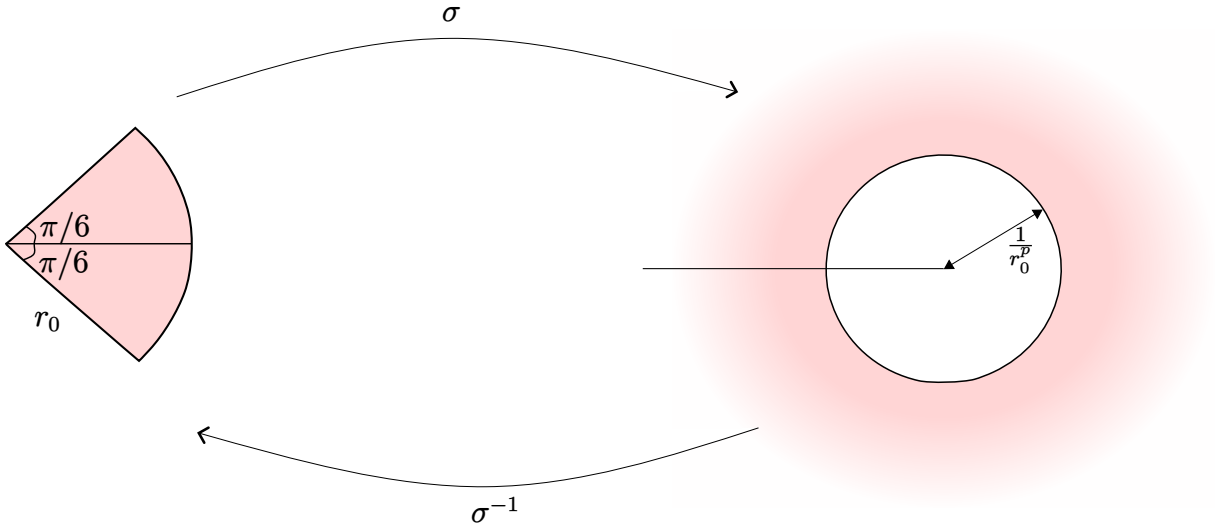


Figure 2.2: σ is a biholomorphism from S_0 onto W .

It is clear that the map $\sigma : z \mapsto \frac{1}{z^p}$ is a biholomorphism of S_0 onto W with $\sigma^{-1} : W \rightarrow S_0$ given by $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$. The branch of p -th root that we select determines which sector of width $2\pi/p$, the inverse map maps to. (The other sectors being $S_k = \{0 < r < r_0, |\theta - 2k\pi/p| < \pi/p\}$.)

Now, the conjugate map of f on W is given by,

$$g(w) = \sigma f \sigma^{-1}(w) = f(w^{-\frac{1}{p}})^{-p}.$$

This just replaces the action of f on S_0 by g on W , and we have the following commutative diagram:

Now, we will use the power series expansion of f near the origin to get information about g .

First let us try to get a estimate of the power series expansion of $f(z)^{-p}$. We have,

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}) = z(1 - z^p + \mathcal{O}(z^{2p})) = z(1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + \dots).$$

So,

$$\frac{1}{f(z)^p} = z^{-p} \left(\frac{1}{1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + \dots} \right)^p.$$

Now, let $\alpha(z) = z^p + a_0 z^{2p} + a_1 z^{2p+1} + \dots$, then for r_0 small enough such that $|\alpha(z)| < 1$ on $\{|z| < r_0\}$, we can write,

$$\frac{1}{1 - \alpha(z)} = 1 + \alpha(z) + \alpha(z)^2 + \dots$$

Therefore,

$$\begin{aligned} \frac{1}{f(z)^p} &= z^{-p} (1 + \alpha(z) + \alpha(z)^2 + \dots)^p \\ &= z^{-p} (1 + p\alpha(z) + A\alpha(z)^2 + \dots) \\ &= \frac{1}{z^p} + p + Az^p + v(z), \end{aligned}$$

where A is some constant and $v(z)$ is holomorphic on $\{|z| < r_0\}$, and for some small $r_0 > 0$, it satisfies $|v(z)| \leq B|z|^{p+1}$, $B > 0$.

Now, if $w \in W$, then $\sigma^{-1}(w) \in S$. Hence, by substituting $z = \sigma^{-1}(w) = w^{-1/p}$, we have,

$$\begin{aligned} g(w) &= \sigma f \sigma^{-1}(w) \\ &= \frac{1}{f(w^{-1/p})^p} \\ &= w + p + A/w + \theta(w), \end{aligned}$$

where $|\theta(w)| = |v(w^{-1/p})| \leq B|w^{-1/p}|^{p+1} = B/|w|^{1+\frac{1}{p}}$.

Hence, we have the following estimates for g which will be crucial in everything that will follow:

$$g(w) = w + p + A/w + \theta(w), \text{ where } A \text{ is a constant and} \quad (2.1)$$

$$|\theta(w)| \leq B/|w|^{1+\frac{1}{p}}, B > 0. \quad (2.2)$$

Choose any K satisfying

$$K > \max\{1/r_0^p, 3(|A| + B)\} > 1 \text{ (as } r_0 < 1)$$

and let,

$$\Pi = \{x + iy : y^2 > 4K(K - x)\}.$$

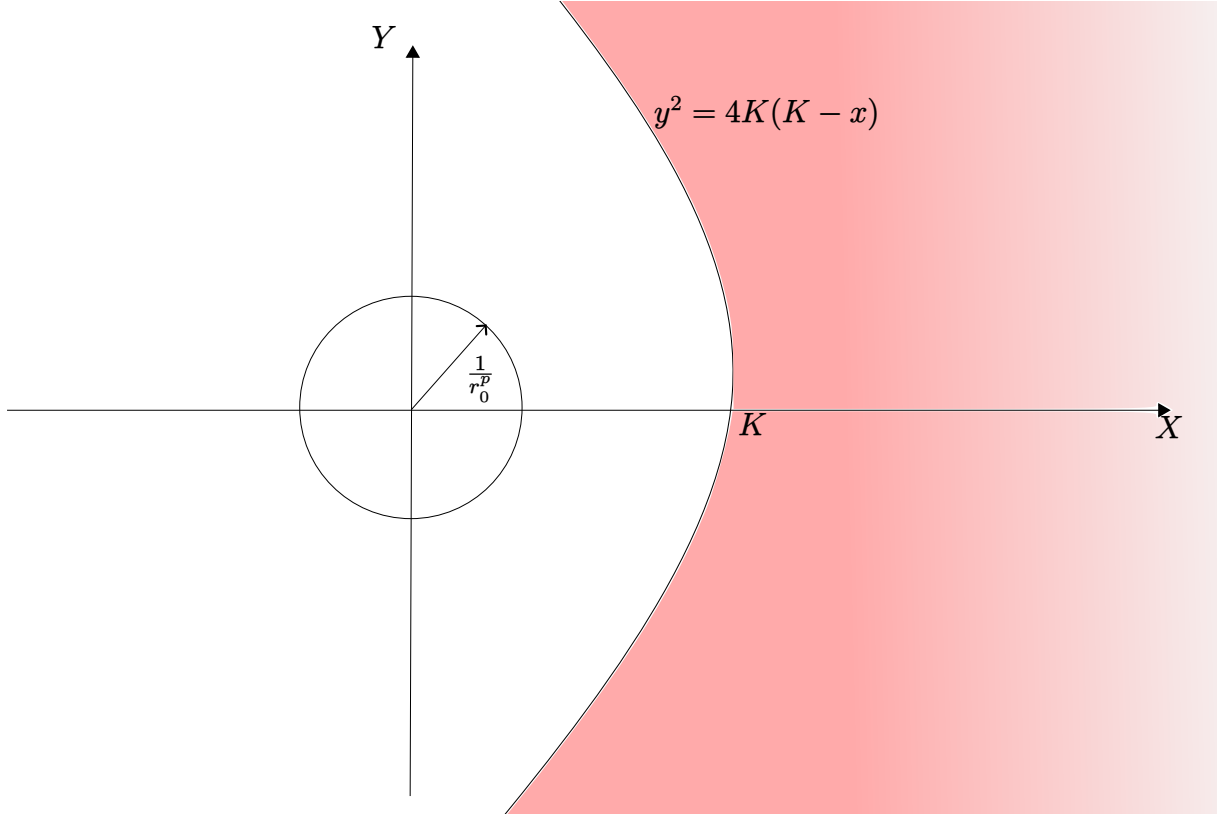


Figure 2.3: $\Pi = \{(x, y) : y^2 > 4K(K - x)\}$.

Clearly, Π is bounded by a parabola and $\Pi \subset W$ (See [Figure 2.3](#)).

We have chosen this subset $\Pi \subset W$ because we will show that Π is nothing but the conformal image of $\Pi_0(t)$ under σ (for a suitable t) and g satisfies all the corresponding conditions that f should satisfy on $\Pi_0(t)$ according to the theorem.

Claim. Π is the conformal image of $\Pi_0(t)$ under σ for a suitable t .

The easiest way to see this is using polar coordinates. We write, $z = re^{i\theta}$ for $z \in S_0$ and $w = \rho e^{i\phi}$ for $w \in W$. Then, $\rho = \frac{1}{r^p}$ and $\phi = -p\theta$.

Now, we need to express Π in polar co-ordinates. To do so, we notice that points on the parabola are given by

$$\rho \text{ (distance from focus i.e. } 0) = 2K - \rho \cos \phi \text{ (distance from directrix i.e. } y = 2K).$$

(See [Figure 2.4](#)). Therefore, points on Π are given by

$$\rho > 2K - \rho \cos \phi.$$

Hence,

$$\Pi = \{\rho e^{i\phi} : 2K < \rho(1 + \cos \phi)\}.$$

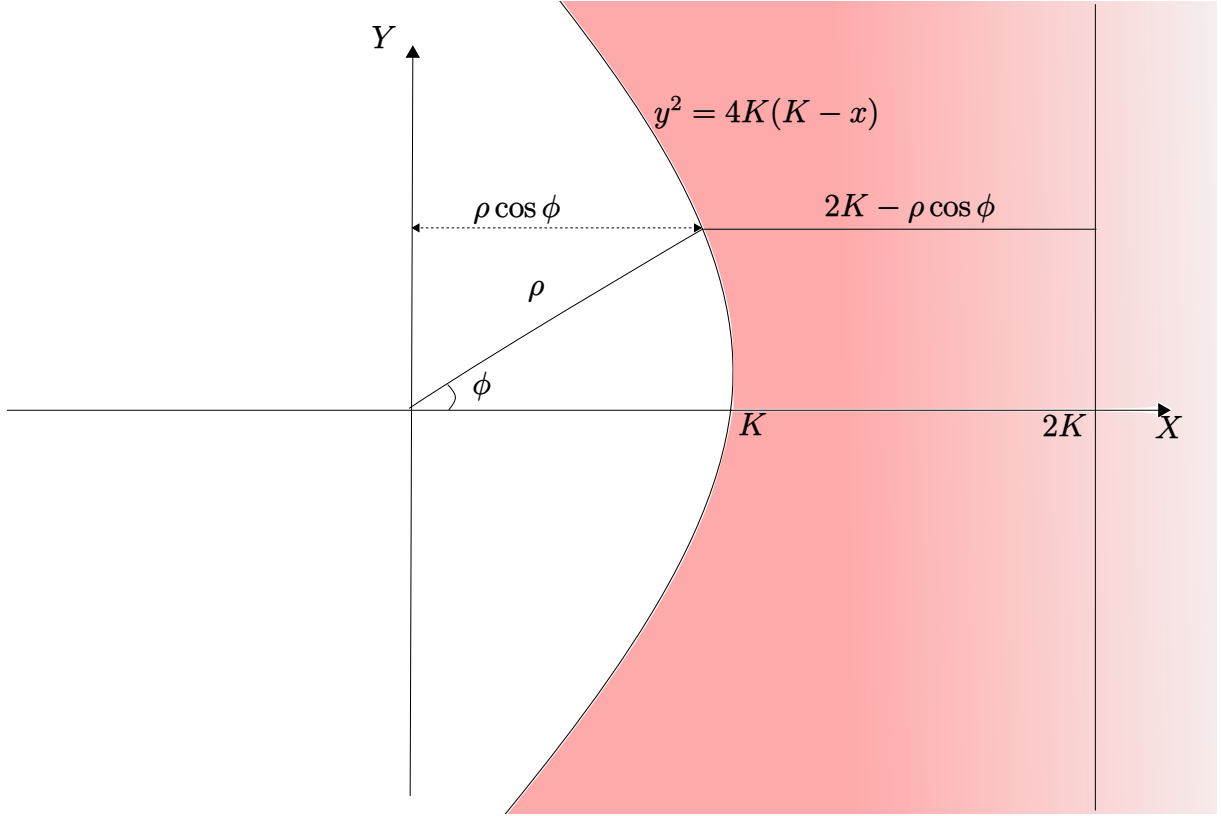


Figure 2.4: $\Pi = \{\rho e^{i\phi} : \rho > 2K - \rho \cos \phi\}$.

Now, let $\Omega = \sigma^{-1}(\Pi)$. Then, Ω is given by

$$\Omega = \{re^{i\theta} : 2Kr^p < 1 + \cos(p\theta)\}.$$

Hence, $\Omega = \Pi_0\left(\frac{1}{2K}\right)$.

Lemma 2.1.3. *g satisfies the following properties on Π :*

1. Π is forward invariant under g .
2. $g^{\circ n}(w) \rightarrow \infty$ uniformly on Π .
3. $\arg(g^{\circ n}(w)) \rightarrow 0$ locally uniformly on Π .
4. $g : \Pi \rightarrow \Pi$ is conjugate to a translation.

Proof.

1. We write,

$$w = x + iy, \quad g(w) = X + iY, \quad A/w + \theta(w) = a + ib.$$

From Equation (2.1), we obtain,

$$\begin{aligned} X + iY &= (x + iy) + p + (a + ib) \\ \implies X &= x + p + a \text{ and } Y = y + b. \end{aligned}$$

Now, if $w \in \Pi$,

$$\begin{aligned}
Y^2 - 4K(K - X) &= (y + b)^2 - 4K(K - x - p - a) \\
&= [y^2 - 4K(K - x)] + b^2 + 2yb + 4K(a + p) \\
&> 4Kp + (2yb + 4Ka) \\
&\geq |4Kp - |2yb + 4Ka||.
\end{aligned}$$

Now, for $w \in \Pi$, $|w| > K > 1$. Hence we get,

$$|w||A/w + \theta(w)| \leq |w|(|A|/|w| + B/|w|^{1+\frac{1}{p}}) = |A| + B/|w|^{\frac{1}{p}} < |A| + B \quad (2.3)$$

(since for $|w| > 1$, $|w|^{\frac{1}{p}} > 1$). Therefore,

$$\begin{aligned}
|2yb + 4Ka| &\leq 2|yb| + 4K|a| \\
&\leq 2|y||b| + 4K|a| \\
&\leq 2|w||a + ib| + 4|w||a + ib| \\
&= 6|w||a + ib| \\
&< 2K < 2Kp.
\end{aligned}$$

Therefore, we see that $Y^2 - 4K(K - X) > 0$ and hence, $g(w) \in \Pi$ for $w \in \Pi$. Hence, Π is forward invariant under g .

2. Now, we will prove a stronger statement that for any $t > 0$ g maps $\Pi + t$ into $\Pi + t + p/2$. This is simply because, for $w \in \Pi + t$, we have, $y^2 - 4K(K + t - x) > 0$. Hence,

$$\begin{aligned}
Y^2 - 4K(K + t + p/2 - X) &= [y^2 - 4K(K + t - x)] + b^2 + 2yb + 4K(a - p/2) \\
&> 2Kp + (2yb + 4Ka) \\
&\geq |2Kp - |2yb + 4Ka|| \\
&> 0.
\end{aligned}$$

Therefore, if $w \in \Pi$, $g^{on}(w) \in \Pi + np/2$. Hence, $|g^{on}(w)| > \sqrt{n}$. This is simply because, $K + np/2 > 1 + n/2 > \sqrt{n}$ and hence $\Pi + np/2$ is disjoint from the disc $\{|z| \leq \sqrt{n}\}$.

Hence, $g^{on}(w) \rightarrow \infty$ uniformly on Π .

3. Firstly, we see that,

$$g^{\circ k}(w) = g(g^{\circ k}(w)) = g^{\circ k}(w) + p + A/g^{\circ k}(w) + \theta(g^{\circ k}(w)).$$

By expanding the first term again and again, we obtain,

$$g^{on}(w) = w + np + \sum_{k=0}^{n-1} \left(\frac{A}{g^{\circ k}(w)} + \theta(g^{\circ k}(w)) \right). \quad (2.4)$$

Also note that from [Equation \(2.3\)](#), we have,

$$|A/w + \theta(w)| < (|A| + B)/K < \frac{1}{3}.$$

Let Q be a compact subset of Π . From now, we will assume that $w \in Q$ and we will use C_1, C_2, C_3, \dots to denote positive constants which will be dependent on Q .

Hence,

$$\begin{aligned} |g(w)| &= |w + p + A/w + \theta(w)| \geq ||w + p| - |A/w + \theta(w)|| \\ &= |w + p| - |A/w + \theta(w)| \\ &\geq |w| + p - \frac{1}{3}. \end{aligned}$$

Therefore, we obtain,

$$|g^{on}(w)| \geq |w| + n(p - 1/3) \geq C_1 + C_2 n.$$

(Here, $C_1 = \min\{|w| : w \in Q\} > 0$ and $C_2 = p - \frac{1}{3} > 0$.)

We can select C_3 large enough such that

$$|g^{on}(w)| \geq C_3 n. \tag{2.5}$$

Next, with [Equation \(2.2\)](#), and the above inequality, we get,

$$|\theta(g^{on}(w))| \leq B/|g^{on}(w)|^{1+\frac{1}{p}} \leq C_4/n^{1+\frac{1}{p}}. \tag{2.6}$$

Finally, combining the above two inequalities and [Equation \(2.4\)](#), we obtain,

$$\begin{aligned} |g^{on}(w) - np| &\leq |w| + |A/w + \theta(w)| + \frac{|A|}{C_3} \sum_{k=1}^{n-1} \frac{1}{k} + C_4 \sum_{k=1}^{n-1} \frac{1}{n^{1+\frac{1}{p}}} \\ &< C_5 + C_6 \sum_{k=1}^n \frac{1}{k}. \end{aligned}$$

(Here, $C_5 = \max\{|w| : w \in Q\} + \frac{1}{3} + C_4 \sum_{n=1}^{\infty} 1/n^{1+\frac{1}{p}}$ and $C_6 = |A|/C_3$.)

We can select C_7 large enough such that

$$|g^{on}(w) - np| < C_7 \log n. \tag{2.7}$$

Remark. The above inequality follows from the fact that, if $H_n = \sum_{k=1}^n \frac{1}{k}$, then $H_n - \log n \rightarrow \gamma$. (γ is known as the Euler's constant). So, we have that

$$\begin{aligned} P + QH_n &= P + Q(\log n + \gamma + \epsilon_n), \text{ where } \epsilon_n \rightarrow 0 \\ &\leq Q \log n + (P + Q \max\{\epsilon_n\} + Q\gamma) \\ &= Q \log n + R \\ &< S \log n \end{aligned}$$

for S large enough.

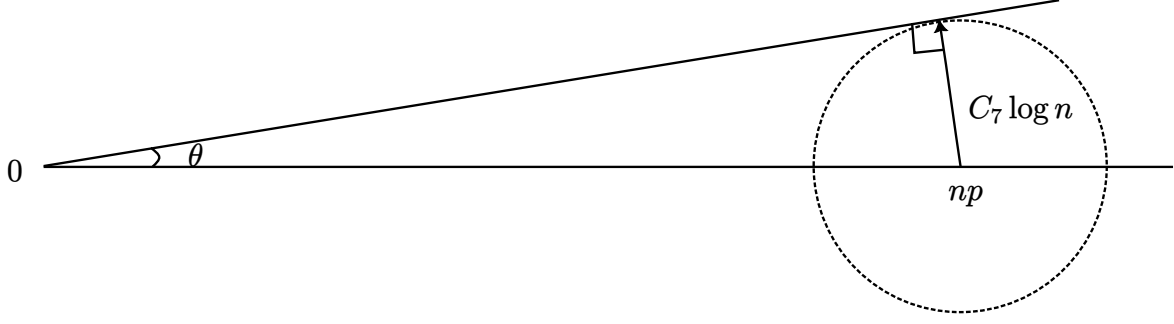


Figure 2.5: $|\arg(g^{\circ n}(w))| \leq \sin^{-1}\left(\frac{C_7 \log n}{np}\right)$.

From, $|g^{\circ n}(w) - np| < C_7 \log n$, it follows that $|\arg(g^{\circ n}(w))| < \sin^{-1}\left(\frac{C_7 \log n}{np}\right)$ for n large enough. Hence, $\arg(g^{\circ n}(w)) \rightarrow 0$ uniformly on Q , and consequently, locally uniformly on Π .

4. Define,

$$u_n(w) = g^{\circ n}(w) - np - (A/p) \log n.$$

Claim. $u_n(w)$ converges locally uniformly on Π to a holomorphic function u , that is one-to-one on Π .

$$u_{n+1}(w) - u_n(w) = [g^{\circ n+1}(w) - g^{\circ n}(w)] - p - (A/p) \log\left(\frac{n+1}{n}\right).$$

From [Equation \(2.2\)](#), we obtain,

$$\begin{aligned} u_{n+1}(w) - u_n(w) &= [g^{\circ n}(w) + p + A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - g^{\circ n}(w)] \\ &\quad - p - (A/p) \log(1 + 1/n) \\ &= A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - (A/p) \log(1 + 1/n) \\ &= A(1/g^{\circ n}(w) - 1/np) + \theta(g^{\circ n}(w)) + (A/p)(1/n - \log(1 + 1/n)). \end{aligned}$$

Now, let Q is a compact subset of Π and $w \in Q$. We need to prove that u_n converges uniformly in Q . From the above equation, to prove that u_n converges uniformly in Q , we need to show that each of the following series converges uniformly in Q :

$$\sum_n |1/g^{\circ n}(w) - 1/np|, \sum_n |\theta(g^{\circ n}(w))|, \sum_n |1/n - \log(1 + 1/n)|.$$

Let us look at the first series. We have, (using [Equations \(2.5\)](#) and [\(2.7\)](#))

$$|1/g^{\circ n}(w) - 1/np| = \frac{|g^{\circ n}(w) - np|}{|g^{\circ n}(w)|np} \leq \frac{C_7 \log n}{C_3 n^2 p} = C_8 \log n / n^2.$$

(Here $C_8 = C_7/(pC_3)$).

From Equation (2.6), it is clear that $\sum_n |\theta(g^{on}(w))|$ converges.

Now, $0 < x - \log(1+x) \leq x^2$ for $x > 0$.

This is because, it is zero at $x = 0$ and $\frac{d}{dx}(x - \log(1+x)) = 1 - \frac{1}{1+x} > 0$ for $x > 0$.

Also, $x^2 - x + \log(1+x)$ is zero at $x = 0$ and $\frac{d}{dx}(x^2 - x + \log(1+x)) = 2x - 1 + \frac{1}{1+x} > 0$ for $x > 0$.

Putting $x = \frac{1}{n}$, we get,

$$|1/n - \log(1 + 1/n)| < 1/n^2.$$

Therefore, u_n converges locally uniformly to some holomorphic function u on Π .

Now, from $u_n(w) = g^{on}(w) - np - (A/p) \log n$, we get that,

$$\begin{aligned} (n+1)p + (A/p) \log(n+1) + u_{n+1}(w) &= g^{on+1}(w) \\ &= g^{on}(g(w)) \\ &= np + (A/p) \log n + u_n(g(w)) \\ \implies p + (A/p) \log(1 + 1/n) + u_{n+1}(w) &= u_n(g(w)). \end{aligned}$$

Taking limit $n \rightarrow \infty$, we get,

$$p + u(w) = u(g(w)).$$

Since f is injective near the origin, g is injective on Π , (if K is chosen large enough). Therefore, g^{on} is injective on Π and hence, so is u_n . By Hurwitz Theorem, u is either injective or constant, but it is clearly not a constant since it satisfies the above equation.

This shows that $g : \Pi \rightarrow \Pi$ is conjugate to the map $z \mapsto z + p$ of $u(\Pi)$ into itself. \square

Coming back to our original theorem, we see that our original theorem is also proved as we had just replaced the action of f on Π_0 by the action of its conjugate g on Π and we just proved all the parts of the theorem that the conjugate of f , i.e. g must satisfy.

From, $g = \sigma f \sigma^{-1}$, we get, $g^{on} = \sigma f^{on} \sigma^{-1} \implies g^{on} \sigma = \sigma f^{on}$. Writing, $\sigma(z) = w$, we have,

$$g^{on}(w) = \frac{1}{f^{on}(z)^p} \implies g^{on}(w)(f^{on}(z))^p = 1. \quad (2.8)$$

1. Since, g maps Π into itself, f maps Π_0 into itself.
2. Now, since $|g^{on}(w)| > \sqrt{n}$, $|f^{on}(z)| < \frac{1}{n^{1/2p}}$ from Equation (2.8). Hence, $f^{on}(z) \rightarrow 0$ uniformly on Π_0 .
3. Also, $\arg(f^{on}(z)) = \left(-\frac{1}{p}\right) \arg(g^{on}(w))$ from Equation (2.8). Since, $\arg(g^{on}(w)) \rightarrow 0$ locally uniformly on Π , $f^{on}(z) = \left(-\frac{1}{p}\right) \arg(g^{on}(w)) \rightarrow 0$ locally uniformly on Π_0 .
4. Since, $g : \Pi \rightarrow \Pi$ is conjugate to a translation, and g is conjugate to f , $f : \Pi_0 \rightarrow \Pi_0$ is also conjugate to a translation.

5. It is immediate from [Lemma 2.1.1](#) that $|f(z)| < |z|$ on the axis of Π_0 .

Theorem 2.1.4. *Suppose that f has the power series expansion near 0 as,*

$$f(z) = z + az^{p+1} + \mathcal{O}(z^{p+2}).$$

Then, f is conjugate near 0 to a function

$$F(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}).$$

Proof. First, we conjugate f by the map $z \mapsto \lambda z$, where $\lambda^p = a$. Then, we get that f is conjugate to the map,

$$\tilde{f} = \lambda f(z/\lambda) = \lambda z/\lambda + \lambda a z^{p+1}/\lambda^{p+1} + \mathcal{O}(z^{p+2}) = z + z^{p+1} + \mathcal{O}(z^{p+2}).$$

We will now proceed via induction over a finite number of steps. Let,

$$f_k(z) = z + z^{p+1} + bz^{p+k+1} + \dots, b \neq 0.$$

Here $k \geq 1$. Also if $k \geq p$, our theorem is proved. Hence, we assume, $1 \leq k < p$.

Now, define the polynomial,

$$\sigma(z) = z + \alpha z^{k+1},$$

where $\alpha = \frac{b}{p-k}$ and let σ^{-1} be its inverse near 0 (We can do this because $\sigma'(0) = 1$).

Now, we will show that we obtain f_r (for some $r \geq k+1$) by conjugating f_k with σ . Hence, let

$$g = \sigma f_k \sigma^{-1}$$

and we need to show that $g = f_r$ (for some $r \geq k+1$). Since, $g'(0) = f'_k(0) = 1$, we let,

$$g(z) = z + \sum_{m=2}^{\infty} a_m z^m.$$

Now, we will use the identity, $g\sigma = \sigma f_k$.

$$\begin{aligned} \sigma f_k(z) &= (z + z^{p+1} + bz^{p+k+1} + \dots) + \alpha(z + z^{p+1} + bz^{p+k+1})^{k+1} \\ &= z + \alpha z^{k+1} + z^{p+1} + (b + \alpha(k+1))z^{p+k+1} + \mathcal{O}(z^{p+k+2}) \\ &= z + \alpha z^{k+1} + z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}). \end{aligned}$$

The last equality follows because,

$$\alpha(p-k) = b \implies \alpha(p+1) - \alpha(k+1) = b \implies \alpha(p+1) = b + \alpha(k+1).$$

Now,

$$\begin{aligned} g\sigma(z) &= (z + \alpha z^{k+1}) + \sum_{m=2}^{\infty} a_m (z + \alpha z^{k+1})^m \\ &= z + \alpha z^{k+1} + \sum_{m=2}^{p+k+1} a_m (z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}). \end{aligned}$$

Now, equating $\sigma f_k(z) = g\sigma(z)$, we get,

$$z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) = \sum_{m=2}^{p+k+1} a_m(z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}).$$

Firstly, we see that on the right hand side, the coefficient of z^2 will be a_2 , the coefficient of z^3 will be some linear combination of a_2 and a_3 , the coefficient of z^4 will be some linear combination of a_2, a_3 and a_4 and so on upto the coefficient of z^p will be some linear combination of a_2, a_3, \dots, a_p . Since, the coefficient of z^2, \dots, z^p is zero on the left hand side, it follows that $a_2 = a_3 = \dots, a_p = 0$. (This argument follows assuming $p \geq 2$, but if $p = 1$ the coefficient of $z^p = z$ i.e. a_1 is automatically 0).

Hence, now we have,

$$\begin{aligned} z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) &= \sum_{m=p+1}^{p+k+1} a_m(z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}) \\ &= a_{p+1}z^{p+1} + \dots + a_{p+k+1}z^{p+k+1} + \\ &\quad a_{p+1}\alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}). \end{aligned}$$

Therefore, we obtain

$$a_{p+1} = 1, a_{p+2} = \dots = a_{p+k} = 0 \text{ and } a_{p+k+1} + a_{p+1}\alpha(p+1) = \alpha(p+1).$$

Hence, $a_{p+k+1} = 0$. This gives that f_k is conjugate to the map

$$g(z) = z + z^{p+1} + \mathcal{O}(z^{p+k+2}).$$

Thus, $g = f_r$ for some $r \geq k+1$. Continuing the induction process, we get that f is conjugate near 0 to a map

$$z \mapsto z + z^{p+1} + \mathcal{O}(z^{2p+1}).$$

□

2.2 Behaviour near attracting fixed points

2.3 Behaviour near super-attracting fixed points

We will study the behaviour of analytic maps near super-attracting fixed points in the next chapter under Bottcher's theorem.

□

Chapter 3

Bottcher's Theorem and its extension

3.1 Bottcher's Coordinates

A fixed point z_0 is called a super-attracting fixed point of f if $f'(z_0) = 0$.

If z_0 is a super-attracting fixed point for f , we can conjugate the map such that $z = 0$ becomes our super-attracting fixed point.

Thus, our map takes the form $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$ in a neighbourhood of 0, with $n \geq 2$ and $a_n \neq 0$, where the integer n is called the local degree.

Theorem 3.1.1 (Bottcher's Theorem). *With f as above, \exists a local holomorphic change of coordinates $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to $w \mapsto w^n$ throughout some neighbourhood of 0.*

Furthermore, ϕ is unique upto multiplication by an $(n-1)$ th root of unity.

Proof. Existence. Let $c \in \mathbb{C}$ be such that $c^{n-1} = a_n$. Then, the linearly conjugate map $cf(z/c)$ will have leading coefficient $+1$. Thus, without loss of generality, we will assume that our map f has the form $f(z) = z^n(1 + b_1 z + b_2 z^2 + \dots) = z^n(1 + \eta(z))$, where $\eta(z) = (1 + b_1 z + b_2 z^2 + \dots)$.

Choose $r \in (0, \frac{1}{2})$ such that $|\eta(z)| < \frac{1}{2} \forall z \in \mathbb{D}_r$. This can be done since $\eta(0) = 0$ and η is continuous.

On this disc, we have two properties of f :

1. f maps this disc into itself:

We have, $|f(z)| = |z^n| |1 + \eta(z)| \leq |z|^n (1 + |\eta(z)|) < \frac{3}{2} |z|^n \leq \frac{3}{2^n} |z| \leq \frac{3}{4} |z| \forall z \in \mathbb{D}_r$. Here we are using the fact that $n \geq 2$, $|z| < \frac{1}{2}$ and $|\eta(z)| < \frac{1}{2}$ on \mathbb{D}_r .

2. $f(z) \neq 0 \forall z \in \mathbb{D}_r \setminus \{0\}$.

This is simply because $|f(z)| = |z|^n |1 + \eta(z)|$ and since $|\eta(z)| < \frac{1}{2}$ on \mathbb{D}_r , we can't have $\eta(z) = -1$.

The k -th iterate of f i.e. $f^{\circ k}$ also maps the \mathbb{D}_r into itself and $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \setminus \{0\}$. Inductively, it can be shown that it has the form $f^{\circ k}(z) = z^{n^k} (1 + n^{k-1}b_1z + \dots)$.

The idea of the proof is to set,

$$\phi_k(z) = (f^{\circ k}(z))^{\frac{1}{n^k}} = z (1 + n^{k-1}b_1z + \dots)^{\frac{1}{n^k}}$$

We choose z as our branch of holomorphic n^k th root of z^{n^k} .

Now, we can choose a holomorphic branch of $(1 + n^{k-1}b_1z + \dots)^{\frac{1}{n^k}}$ on \mathbb{D}_r since \mathbb{D}_r is simply connected and $(1 + n^{k-1}b_1z + \dots) \neq 0$ on \mathbb{D}_r since $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \setminus \{0\}$. Therefore we set,

$$\phi_k(z) = z (1 + n^{k-1}b_1z + \dots)^{\frac{1}{n^k}} = z \left(1 + \frac{b_1}{n}z + \dots\right)$$

where the expression on the right provides us an explicit choice of n^k th root.

We will show that the functions ϕ_k converge uniformly to a limit function ϕ on \mathbb{D}_r . To prove the convergence, we make the substitution $z = e^u$ where u ranges over the left half plane $\mathbb{H}_r := \{u : \operatorname{Re}(u) < \log r\}$. The exponential map maps \mathbb{H}_r onto $\mathbb{D}_r \setminus \{0\}$.

The map f from \mathbb{D}_r into itself corresponds to a map from \mathbb{H}_r into itself given by $F(u) = \log f(e^u)$. We can select a holomorphic branch of the logarithm of $f(e^u)$ because \mathbb{H}_r is simply connected and $f(e^u) \neq 0$ on \mathbb{H}_r .

Set $\eta = \eta(e^u) = b_1e^u + b_2e^{2u} + \dots$, then since $|\eta| < \frac{1}{2}$, we see that F can be written as

$$F(u) = \log(e^{nu}(1 + \eta)) = nu + \log(1 + \eta) = nu + \left(\eta - \frac{\eta^2}{2} + \frac{\eta^3}{3} - \dots\right)$$

where the final expression provides us an explicit choice of which branch of logarithm we are using. Clearly, $F : \mathbb{H}_r \rightarrow \mathbb{H}_r$ is a well-defined holomorphic function.

Similarly, the map ϕ_k corresponds to a map, $\Phi_k(u) = \log \phi_k(e^u)$.

$$\Phi_k(u) = \log \phi_k(e^u) = \log f^{\circ k}(e^u)^{\frac{1}{n^k}} = \frac{1}{n^k} \log f^{\circ k}(e^u).$$

Since we have already fixed the branch of logarithm that we are using, we see that,

$$\log f^{\circ k}(e^u) = \log f(f^{\circ k-1}(e^u)) = \log f(e^{\log f^{\circ k-1}(e^u)}) = F(\log f^{\circ k-1}(e^u))$$

Hence, inductively we can see that $\log f^{\circ k}(e^u) = F^{\circ k}(u)$.

Therefore, $\Phi_k(u) = F^{\circ k}(u)/n^k$. It is clear from this expression that $\Phi_k : \mathbb{H}_r \rightarrow \mathbb{H}$.

Now since $|\eta| < \frac{1}{2}$, we have

$$|F(u) - nu| = |\log(1 + \eta)| < \log 2 < 1$$

Hence,

$$|\Phi_{k+1}(u) - \Phi_k(u)| = \frac{1}{n^{k+1}} |F^{\circ k+1}(u) - nF^{\circ k}(u)| < \frac{1}{n^{k+1}}$$

by the above inequality.

We have, $\phi_k(e^u) = e^{\Phi_k(u)}$. Since, the exponential map, $e^\square : \mathbb{H} \rightarrow \mathbb{D}$ from the left half plane to the unit disc decreases distance, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r \setminus \{0\}.$$

Since $\phi_k(0) = 0$ for all k , we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r.$$

Hence, the maps ϕ_k converge uniformly to some limit function ϕ on \mathbb{D}_r by the Cauchy criterion for uniform convergence.

Clearly, $\phi(0) = 0$ and ϕ is holomorphic on \mathbb{D}_r by Weierstrass convergence theorem.

It is clear that each $\phi_k : \mathbb{D}_r \rightarrow \mathbb{D}$. This is because $\phi_k(e^u) = e^{\Phi_k(u)}$ and $\Phi_k : \mathbb{H}_r \rightarrow \mathbb{H}$ and $e^\square : \mathbb{H} \rightarrow \mathbb{D} \setminus \{0\}$. Hence, $\phi : \mathbb{D}_r \rightarrow \mathbb{D}$. (Clearly $\text{Im}(\phi)$ cannot contain points from $\partial\mathbb{D}$ because ϕ is holomorphic, hence it is an open map).

Now, it can be easily seen that, $\phi_k(f(z)) = \phi_{k+1}(z)^n$.

Hence, $\lim_{k \rightarrow \infty} \phi_k(f(z)) = \lim_{k \rightarrow \infty} \phi_{k+1}(z)^n \implies \phi(f(z)) = \phi(z)^n$ by continuity of n th power map.

Also, since $\phi'_k(0) = 1 \forall k \in \mathbb{N}$ (from the power series of ϕ_k), we have $\phi'(0) = 1$. Hence, ϕ is invertible in some neighbourhood of 0.

Therefore, we have a holomorphic change of coordinates in some neighbourhood of 0 which conjugates f to the n th power map. In this neighbourhood, ϕ is one-to-one, $f(z) \neq 0$ for $z \neq 0$ (i.e. no other point maps to the super-attracting fixed point via f) and f maps this neighbourhood into itself.

Uniqueness. It suffices to study the special case $f(z) = z^n$. If we can prove that any map which conjugates $z \mapsto z^n$ to itself is just multiplication by $(n-1)$ th root of unity, then for any general map $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$, if we have two maps ϕ and ψ which conjugate it to $z \mapsto z^n$, then $\phi \circ \psi^{-1}$ is a map which conjugates $z \mapsto z^n$ to itself. Hence, $\phi \circ \psi^{-1} = cz$, where $c^{n-1} = 1$. Therefore, $\phi = c\psi$, where c is a $(n-1)$ th root of unity.

So, let $\phi(z) = c_1z + c_kz^k + \dots, (c_1 \neq 0)$ be a map which conjugates $z \mapsto z^n$ to itself. Then, we should have $\phi(z^n) = \phi(z)^n$. Now,

$$\phi(z^n) = c_1z^n + c_kz^{nk} + \dots$$

and

$$\phi(z)^n = c_1^n z^n + nc_1^{n-1}c_k z^{n+k-1} + \dots$$

Comparing coefficients, we get $c_1^n = c_1$ and $nc_1^{n-1}c_k = 0$ since $nk > n+k-1$ for $k \geq 2$. Therefore, we get $c_1^{n-1} = 1$ and $c_k = 0$. The form $\phi(z) = c_1z + c_kz^k + \dots$ can be modified to any $k \geq 2$ to get $c_k = 0$ by the same process.

Therefore, $\phi(z) = cz$, where c is a $(n-1)$ th root of unity. □

3.2 Extension of Bottcher's coordinates

We might hope to extend the Bottcher's coordinates to the whole of the immediate basin of attraction of the super-attracting fixed point. But this is not always possible. This is because, it requires computing expressions of the form $z \mapsto \left(\phi(f^{\circ k}(z))\right)^{\frac{1}{n^k}}$, which is not always possible. For example, the basin may not be simply connected, or some point may map directly onto the super-attracting fixed point, in which case we can't take n^k -th roots, because $\phi(f^{\circ k}(z))$ must be zero at those points.

Theorem 3.2.1 (Extension of $|\phi|$). *If f has a super-attracting fixed point p , with immediate basin of attraction \mathcal{A} , then the function $z \mapsto |\phi(z)|$ of the above theorem extends uniquely to a continuous map $|\phi| : \mathcal{A} \rightarrow [0, 1)$ which satisfies $|\phi|(f(z)) = |\phi|(z)^n$.*

Furthermore, $|\phi|$ is real analytic except at the iterated preimages of p , where it takes the value 0.

Proof. Set $|\phi|(z) = \left|\phi(f^{\circ k}(z))\right|^{\frac{1}{n^k}}$ for large enough k for each $z \in \mathcal{A}$. ϕ is only defined in a some small neighbourhood of p . But since, $f^{\circ k} \rightarrow p$ locally uniformly in \mathcal{A} , after k many iterates for some large k , $f^{\circ k}(z)$ belongs to the domain of definition of ϕ , which we shall call \hat{U} .

It is independent of the value of k (if k is large enough). Note that, if $f^{\circ k}(z) \in \hat{U}$, then so does $f^{\circ k+1}(z)$, since f maps \hat{U} into itself.

Suppose we choose k minimal such that $f^{\circ k}(z) \in \hat{U}$. Then,

$$\left|\phi(f^{\circ k+1}(z))\right|^{\frac{1}{n^{k+1}}} = \left|\phi(f(f^{\circ k}(z)))\right|^{\frac{1}{n^{k+1}}} = \left|\phi(f^{\circ k}(z))^n\right|^{\frac{1}{n^{k+1}}} = \left|\phi(f^{\circ k}(z))\right|^{\frac{1}{n^k}} = |\phi|(z).$$

In the proof of the Bottcher's theorem, we saw that $\phi(z) \in \mathbb{D} \forall z \in \hat{U}$. Hence, $|\phi|(z) = |\phi(f^{\circ k}(z))| < 1 \forall z \in \mathcal{A}$. Therefore, $|\phi| : \mathcal{A} \rightarrow [0, 1)$.

Also,

$$\begin{aligned} |\phi|(f(z)) &= |\phi(f^{\circ k}(f(z)))|^{\frac{1}{n^k}} \\ &= |\phi(f(f^{\circ k}(z)))|^{\frac{1}{n^k}} \\ &= |\phi(f^{\circ k}(z))^n|^{\frac{1}{n^k}} \\ &= |\phi(f^{\circ k}(z))|^{\frac{n}{n^k}} \\ &= |\phi|(z)^n. \end{aligned}$$

It is also clear that $|\phi| = 0$ only at p and its iterated preimages.

If q is an iterated preimage of p , say $f^{\circ k}(q) = p$, then we have $|\phi|(q) = |\phi(f^{\circ k}(q))^{\frac{1}{n^k}}| = |\phi(p)|^{\frac{1}{n^k}} = 0$.

Now, Suppose $|\phi|(z) = 0$ for some z . Then, $|\phi|(z)^{n^k} = 0 \forall k \implies |\phi|(f^{\circ k}(z)) = 0 \forall k$. But for some large k , $f^{\circ k}(z)$ belongs to the domain of definition of ϕ . But that means, $f^{\circ k}(z) = p$, since no other point in that domain is mapped to zero by ϕ . Hence, z is an iterated preimage of p .

Now, since $f^{\circ k} \rightarrow p$ locally uniformly in \mathcal{A} , for each $a \in \mathcal{A}$, we have a neighbourhood W_a and a constant $k \in \mathbb{N}$ such that $f^{\circ k}(z) \in \hat{U} \forall z \in W_a$.

Hence, for $z \in W_a$, we can define $|\phi|(z) = |\phi(f^{\circ k}(z))|^{\frac{1}{n^k}} = |g(z)|$, where $g = \phi \circ f^{\circ k}|_{W_a}$. Therefore, $|\phi|_{W_a} = |g|$, where g is some holomorphic function defined on W_a .

It is clear from this that $|\phi|$ is continuous in \mathcal{A} .

Now, if h is any holomorphic function, then $|h(z)|$ is real-analytic everywhere in its domain except at those z , where $h(z) = 0$.

Since, $|g| = |\phi|_{W_a}$ is zero only at the iterated preimages of f in W_a , $|\phi|_{W_a}$ is real analytic everywhere in W_a except at the iterated preimages of p .

Therefore, $|\phi|$ is real analytic everywhere in \mathcal{A} except at the iterated preimages of p . Let $f : \mathbb{C}_{\infty} \rightarrow \mathbb{C}_{\infty}$ be a rational map with a super-attracting fixed point p . Then the associated Bottcher map ϕ carries a neighbourhood of p biholomorphically onto a neighbourhood of zero, conjugating f to the n th power map, where n is the local degree of f near p . ϕ has a local inverse ψ_{ϵ} which maps the ϵ -disc around zero to a neighbourhood of p . \square

Theorem 3.2.2 (Extending ψ_{ϵ}). *There exists a unique open disc of maximal radius $0 < r \leq 1$ such that ψ_{ϵ} extends holomorphically to a map $\psi : \mathbb{D}_r \rightarrow \mathcal{A}$, where \mathcal{A} is the immediate basin of attraction of p .*

1. If $r = 1$, then ψ maps the open unit disc \mathbb{D} onto \mathcal{A} biholomorphically.

2. If $0 < r < 1$, then ψ maps \mathbb{D}_r onto its image biholomorphically and there exists atleast one other critical point in \mathcal{A} on the boundary of $\psi(\mathbb{D}_r)$.

If ψ_ϵ is extended biholomorphically in this way to the map ψ defined on \mathbb{D}_r , then the inverse map $\psi^{-1} : \psi(\mathbb{D}_r) \rightarrow \mathbb{D}_r$ must be the extension of ϕ from some neighbourhood of p to $\psi(\mathbb{D}_r)$ (since ψ^{-1} agrees with ϕ on some neighbourhood of p).

Proof. Let us try to extend ψ_ϵ along radial lines by analytic continuation. Then, we can't extend it indefinitely as it would yeild a holomorphic map ψ from the entire complex plane onto an open set $\psi(\mathbb{C}) \subset \mathcal{A} \subsetneq \mathbb{C}_\infty$. (\mathcal{A} cannot be the whole of \mathbb{C}_∞ since the Julia set of f cannot be empty as $\deg(f) \geq 2$). We can conjugate f such that $\infty \notin \mathcal{A}$. Then the corresponding map ψ will map the whole of the complex plane into $\mathcal{A} \subset \mathbb{C}$. By Louiville's theorem, since the map ψ cannot be a constant, $\psi(\mathbb{C}) = \mathbb{C} = \mathcal{A}$. Therefore, $\mathbb{C}_\infty \setminus \mathcal{A} = \{\infty\}$. This too is not possible since the Julia set of f must be an infinite set since $\deg(f) \geq 2$.

Thus, there must be some largest radius r so that ψ_ϵ extends analytically throughout the open disc \mathbb{D}_r .

Also, $|\phi|(\psi(w)) = |\phi(\psi(w))| = |w|$ near 0, hence for all $w \in \mathbb{D}_r$ by analytic continuation.???

Since, $|\phi| : \mathcal{A} \rightarrow [0, 1)$, this proves that for any $w \in \mathbb{D}_r$, $|\phi|(\psi(w)) = |w| < 1$. Therefore, ψ can be defined only on \mathbb{D}_r for $r \leq 1$.

We will now show that ψ is actually one-to-one on \mathbb{D}_r . Suppose $\psi(w_1) = \psi(w_2)$. Applying $|\phi|$, we see that $|w_1| = |w_2|$. Choose such a pair such that $\psi(w_1) = \psi(w_2)$ ($w_1 \neq w_2$) with $|w_1| = |w_2|$ minimal. A minimal pair exists because for $|w| < \epsilon$, $\psi = \psi_\epsilon$ which is one-to-one as it is invertible.

Now, ψ is an open mapping. Choose a sufficiently small neighbourhood U_{w_2} of w_2 . Then, $\psi(U_{w_2})$ is a small neighbourhood of $\psi(w_1) = \psi(w_2)$. Hence, for any w'_1 sufficiently close to w_1 , $\psi(w'_1) \in \psi(U_{w_2})$. Hence, we can find w'_2 sufficiently close to w_2 such that $\psi(w'_1) = \psi(w'_2)$. Choosing $|w'_1| < |w_1|$, we get a contradiction.

Hence, ψ maps \mathbb{D}_r onto its image biholomorphically.

In case when $r = 1$, $U = \psi(\mathbb{D}) = \mathcal{A}$. If not then we would have some boundary point of U , say $z_0 \in \mathcal{A}$. We can approximate z_0 by points of $\psi(w_j)$, where $|w_j| \rightarrow 1$.

Now, $\lim_{j \rightarrow \infty} \psi(w_j) = z_0$. Hence,

$$\lim_{j \rightarrow \infty} |\phi|(\psi(w_j)) = |\phi|(z_0) \implies \lim_{j \rightarrow \infty} |w_j| = |\phi|(z_0) \implies |\phi|(z_0) = 1$$

which is impossible.

Now, let $0 < r < 1$. We need to prove that ∂U , where $U = \psi(\mathbb{D}_r)$ must contain a critical point of f . Suppose, $w_0 \in \partial \mathbb{D}_r$ and let $(w_j)_{j=1}^\infty \subset \mathbb{D}_r$ such that $w_j \rightarrow w_0$. Let

$\psi(w_j) \rightarrow z_0$. Then $z_0 \in \partial U$ because ψ maps \mathbb{D}_r onto U biholomorphically.

If z_0 is not a critical point of f , then f maps a neighbourhood of z_0 , say A onto a neighbourhood of $f(z_0)$, say B biholomorphically.

It should be noted that B can be chosen such that $B \subset U$. This is because $f(z_0) \in U$. We have, $\lim_{j \rightarrow \infty} \psi(w_j) = z_0 \implies \lim_{j \rightarrow \infty} f(\psi(w_j)) = f(z_0) \implies \lim_{j \rightarrow \infty} \psi(w_j^n) = f(z_0) \implies \psi(w_0^n) = f(z_0)$. Since, $|w_0| = r < 1, |w_0|^n < r^n < r$. Hence, $w_0 \in \mathbb{D}_r$. Therefore, $\psi(w_0^n) = f(z_0) \in U$

Let g be the local inverse of f near $f(z_0)$. Then, ψ can be extended throughout a neighbourhood of w_0 by

$$w \mapsto g(\psi(w^n))$$

We have, $\psi(w_0^n) = f(z_0) \implies w_0^n = \phi(f(z_0))$. Since, $\phi(B)$ is a neighbourhood of $\phi(f(z_0))$ lying inside \mathbb{D}_r , choose a small enough neighbourhood of w_0 , say C such that $w^n \in \phi(B)$, for all $w \in C$. In this neighbourhood, C our newly defined map agrees with ψ on $C \cap \mathbb{D}_r$. This is because, for $w \in C \cap \mathbb{D}_r$, $f(\psi(w)) = \psi(w^n) \in B$. Therefore, $g(\psi(w^n))$ can be defined and $\psi(w) = g(\psi(w^n)) \in A$. Hence, our new map is an analytic continuation of ψ on the neighbourhood C .

Now, if none of the $z_0 \in \partial U$ are critical points, we can extend ψ to a neighbourhood of $w_0 \forall w_0 \in \partial \mathbb{D}_r$. Clearly, these continuations would patch together to define ψ in a strictly greater disc than \mathbb{D}_r , which is a contradiction. \square

Chapter 4

Introduction to the Mandelbrot Set

We consider the set of quadratic polynomials $\{f_c(z) = z^2 + c : c \in \mathbb{C}\}$. It is enough to consider this set because every quadratic polynomial is conjugate to a quadratic polynomial of the type $f_c(z)$ for some unique $c \in \mathbb{C}$.

To prove this, let $f(z) = az^2 + bz + c$, $a \neq 0$. And consider the conjugation, $\sigma(z) =$

4.1 Definition of the Mandelbrot Set

First, we define as new type of set, known as the *Filled-in Julia Set* for a polynomial P .

Definition 4.1.1 (Filled-in Julia Set). *The Filled-in Julia Set of a polynomial P is defined as $K(P) = \hat{\mathbb{C}} \setminus F_\infty(P)$. It is the union of the Julia set and the bounded Fatou components. It is denoted by $K(P)$ or simply K when the context is clear.*

By [Lemma 1.2.7](#), K can also be defined as follows:

$$K = \{z \in \mathbb{C} : P^{\circ n}(z) \text{ is bounded}\}.$$

Notation. We will use F_c , J_c and K_c for the $F_\infty(f_c)$, $J(f_c)$ and $K(f_c)$ respectively.

Definition 4.1.2 (Mandelbrot Set). *The Mandelbrot Set is defined as*

$$M = \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$

Now, we have the following two theorems for polynomials:

- For polynomials, since F_∞ is a completely invariant Fatou component (by [Theorem 1.2.5](#)), $\partial F_\infty = J$ (by [Theorem 1.2.2](#)).
- And, from [Theorem 1.2.1](#), we have that F_∞ is simply connected $\iff \hat{\mathbb{C}} \setminus F_\infty$ is connected $\iff \partial F_\infty$ is connected.

Thus, for a polynomial,

$$F_\infty \text{ is simply connected} \iff K \text{ is connected} \iff J \text{ is connected.}$$

Hence, we have the following equivalent descriptions for the Mandelbrot Set:

$$\begin{aligned} M &= \{c \in \mathbb{C} : K_c \text{ is connected}\} \\ &= \{c \in \mathbb{C} : F_c \text{ is simply connected}\} \\ &= \{c \in \mathbb{C} : J_c \text{ is connected}\}. \end{aligned}$$

4.2 The Fundamental Dichotomy

Theorem 4.2.1. *For a polynomial P , the following are equivalent:*

1. F_∞ is simply connected $\iff J$ is connected $\iff K$ is connected.
2. There are no finite critical points of P in F_∞ .

Proof. First assume that F_∞ is simply connected $\implies c(F_\infty) = 1$ and hence, $\chi(F_\infty) = 2 - c(F_\infty) = 1$. Now, since F_∞ is completely invariant and P is a polynomial of degree d (say), P is a d -fold map of F_∞ onto itself. Applying the Riemann-Hurwitz relation to the map P of F_∞ onto itself, we obtain,

$$\begin{aligned} \chi(F_\infty) + \delta_P(F_\infty) &= d \chi(F_\infty) \\ \implies 1 + \delta_P(F_\infty) &= d \\ \implies \delta_P(F_\infty) &= d - 1. \end{aligned}$$

Now, $\delta_P(\infty) = d - 1$ and therefore, P does not have any finite critical points in F_∞ .

For the converse part, assume there are no critical points of P in F_∞ . Then, the Bottcher's map ϕ which conjugates P to the map, $z \mapsto z^d$ can be extended to the whole of F_∞ and $\phi : F_\infty \rightarrow \mathbb{D}$ is a biholomorphism. Hence, F_∞ is simply connected. \square

Now, quadratic maps have only one finite critical point and f_c have the critical point at 0 for all $c \in \mathbb{C}$. Hence, by the Fundamental Dichotomy, F_c is simply connected $\iff 0 \notin F_c$ or $0 \in K_c$. Using, c_n to denote $f_c^{on}(0)$, we get,

$$\begin{aligned} M &= \{c \in \mathbb{C} : 0 \in K_c\} \\ &= \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}. \end{aligned}$$

Note that $c_0 = 0$ and $c_1 = f_c(0) = c$. So, (c_n) is also the forward orbit of c . Hence, in other words, the Mandelbrot Set consists of $c \in \mathbb{C}$ such that its forward orbit under the map f_c remains bounded.

4.3 The other end of the Dichotomy Theorem

By the Dichotomy Theorem, we can say that if even one finite critical point of P lies in F_∞ , then K cannot be connected. But this theorem states that if all finite critical points of P lie in F_∞ , then K is not only disconnected, but totally disconnected.

Definition 4.3.1 (Cantor set). *A subset $X \subset \hat{\mathbb{C}}$ is called a Cantor set if it is non-empty, closed, perfect and totally disconnected.*

Theorem 4.3.1. *Let R be a rational map with $\deg(R) \geq 2$. Let α be a super-attracting fixed point of R . If the Fatou component of R containing α , say F_α , contains all the critical points of R , then $J(R)$ is a Cantor set.*

Corollary 4.3.1.1. *If $c \notin M$, then J_c is a Cantor set.*

Proof. Since 0 is the only critical points of f_c (apart from ∞), if it belongs to $F_c = F_\infty(f_c)$, i.e. if $0 \notin K_c \iff c \notin M$, then J_c is a Cantor set. \square

4.4 Some properties of the Mandelbrot Set

We know, $M = \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}$. This description for M can be strengthened significantly by the following theorem:

Theorem 4.4.1. $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Proof. Obviously, $\{c \in \mathbb{C} : |c_n| \leq 2\} \subseteq M$.

Now, Suppose that $c \in M$. We need to prove that $|f_c^{on}(c)| = |c_n| \leq 2$ for all $n \in \mathbb{N}$. Consider the set $W_c = \{z \in \mathbb{C} : |z| \geq |c|, |z| > 2\}$. For $z \in W_c$,

$$|f_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| \geq |z|(|z| - 1) = |z|(1 + \epsilon)$$

for some $\epsilon > 0$ (as $|z| > 2$). Clearly, $|f_c^{on}(z)| \geq |z|(1 + \epsilon)^n \implies z \notin K_c$.

This implies $|c| \leq 2$. Consequently, $|f_c^{on}(c)| \leq 2$ for all $n \in \mathbb{N}$.

Hence, $M \subseteq \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Therefore, $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$. \square

As $c_1 = c$, we have that $|c| \leq 2$ for all $c \in M$ i.e $M \subseteq \{c \in \mathbb{C} : |c| \leq 2\}$. This turns out to be the strongest bound possible for $|c|$ as $-2 \in M$. The orbit of 0 under the map $z \mapsto z^2 - 2$ is:

$$0 \mapsto -2 \mapsto 2 \mapsto 2$$

and hence is bounded.

Theorem 4.4.2. *The Mandelbrot set, M is compact and $\hat{\mathbb{C}} \setminus M$ is open and connected.*

Proof. Let, $c_n = f_c^{on}(c) = Q_n(c)$ be a polynomial in c . Clearly, from [Theorem 4.4.1](#)

$$M = \cap_{n=1}^{\infty} Q_n^{-1}(\overline{\mathbb{D}_2}),$$

where $\overline{\mathbb{D}_2} = \{z \in \mathbb{C} : |z| \leq 2\}$. Thus, M is closed. It is already known that it is bounded. Hence, it is compact.

Now,

$$\hat{\mathbb{C}} \setminus M = \cup_{n=1}^{\infty} Q_n^{-1}(E)$$

where $E = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}_2}$. Now, E is open and connected and since, Q_n are non-constant polynomials, $Q_n^{-1}(E)$ is open and connected for all $n \in \mathbb{N}$. Also, each one of them contains ∞ and hence, their union is also open and connected.

Therefore, $\hat{\mathbb{C}} \setminus M$ is open and connected.

□

4.5 Plotting the Mandelbrot Set

[Theorem 4.4.1](#) is also used to plot the Mandelbrot Set. A simple code in python would

Chapter 5

Connectedness of the Mandelbrot Set

In the previous chapter, we proved that the Mandelbrot set is compact and $\mathbb{C} \setminus M$ is open and connected. In this chapter, we will prove that $\hat{\mathbb{C}} \setminus M$ is biholomorphic to the open unit disc, proving that it is simply connected, thus implying that M is connected by [Theorem 1.2.1](#).

5.1 The Green's Function