Indian Institute of Science Education and Research, Kolkata
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Introduction to Complex Dynamics and the Mandelbrot Set

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Chapter 1

Introduction

1.1 Spherical and Chordal metric

1.2 Definition of Fatou and Julia sets in terms of equicontinuity

Definition 1.2.1 (Fatou and Julia Sets). *Let R be a non-constant rational function. The Fatou set of R denoted by $F(R)$ is the maximal open subset of $\hat{\mathbb{C}}$ on which $\{R^{\circ n}\}$ is equicontinuous. The Julia set of R, denoted by $J(R)$ is the complement of $F(R)$ in $\hat{\mathbb{C}}$.*

By definition, $F(R)$ is open and $J(R)$ is compact.
They are denoted by simply F or J when the context is clear.

1.3 Completely Invariant Components

If $f : X \to X$, then a subset $D \subset X$ is:

- *forward invariant* under the map f if $f(D) = D$.
- *backward invariant* under the map f if $f^{-1}(D) = D$.
- *completely invariant* under the map f if it is both forward and backward invariant under f i.e. $f(D) = D$ and $f^{-1}(D) = D$.

Note that if f is surjective, i.e. $f(X) = X$, then backward invariance implies complete invariance. This is because, $f(f^{-1}(D)) = D$ if f is surjective. Hence, if $f^{-1}(D) = D$, we have $f(D) = D$ i.e. forward invariance.

Theorem 1.3.1. *If $f : X \to X$ be a continuous, open and surjective map of a topological space X onto itself. If $D \subset X$ is completely invariant under f, then so are the complement $X \backslash D$, the interior D^0, the boundary ∂D and the closure $\overline{D}$.*

Proof. Firstly, note that it is enough to prove backward invariance since f is surjective. It is trivial to see that $X \backslash D$ is completely invariant.

Now, since f is a continuous map, $f^{-1}(D^0)$ is an open subset of $f^{-1}(D) = D$. Hence, $f^{-1}(D^0) \subset D^0$. Now, since f is an open map, $f(D^0)$ is an open subset of $f(D) = D$. Hence, $f(D^0) \subset D^0 \implies D^0 \subset f^{-1}(f(D^0)) \subset f^{-1}(D^0)$. Hence, $f^{-1}(D^0) = D^0$ and hence, D^0 is completely invariant.

From the general fact for continuous maps, $\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$. Hence, $\overline{D} \subset f^{-1}(\overline{D})$. Now, let $x \in f^{-1}(\overline{D})$ (or $f(x) \in \overline{D}$). If $x \notin \overline{D}$, then there exists and open set around x, say U such that $U \cap D = \phi$. Since f is an open map, $f(U)$ is an open set containing $f(x)$. Since, $f(x) \in \overline{D}$, $f(U) \cap D \neq \phi$. But since, $f^{-1}(D) = D$, $f^{-1}(f(U) \cap D) \subset D$. But, $f^{-1}(f(U) \cap D) \cap U \neq \phi \implies D \cap U \neq \phi$, which is a contradiction. Hence, $\overline{D} = f^{-1}(\overline{D})$. Hence, $\overline{D}$ is also completely invariant.

Consequently, $\partial D = \overline{D} \backslash D^0$ is also completely invariant. □

Theorem 1.3.2. *For any rational function R, the Fatou and Julia sets of R i.e. $F(R)$ and $J(R)$ are completely invariant.*

Proof. First note that it is enough to prove only backward invariance because R is surjective. Also, we will only prove the complete invariance of $F(R)$, the complete invariance of $J(R)$ then follows from above theorem. We will use F to denote $F(R)$.

Let $z_0 \in R^{-1}(F)$ and let $w_0 = R(z_0) \in F$. By equicontinuity, for any $\epsilon > 0$, $\exists \delta > 0$ such that if $\sigma(z, z_0) < \delta$, then for all $n \in \mathbb{N}$, $\sigma(R^{\circ n}(w), R^{\circ n}(w_0)) < \epsilon$. By continuity of R, there exists $\delta' > w_0$ such that if $\sigma(z, w_0) < \delta'$, then $\sigma(R(z), w_0) < \delta$ and hence, $\sigma(R^{\circ n+1}(z), R^{\circ n+1}(z_0)) < \epsilon$ for all $n \in \mathbb{N}$. Hence, $\{R^{\circ n+1} : n \in \mathbb{N}\}$ is equicontinuous at z_0 and hence, so is $\{R^{\circ n} : n \in \mathbb{N}\}$. Therefore, $z_0 \in F$ and $R^{-1}(F) \subset F$.

Now, let $z_0 \in F$. To prove that $z_0 \in R^{-1}(F)$, we need to prove that $R(z_0) \in F$. Let $w_0 = R(z_0)$. We have by equicontinuity, that for any $\epsilon > 0$, $\exists \delta > 0$ such that for all $n \in \mathbb{N}$, if $\sigma(z, z_0) < \delta$, then $\sigma(R^{\circ n+1}(z), R^{\circ n+1}(z_0)) < \epsilon$. Now, $N = \{z : \sigma(z, z_0) < \delta\}$ is an open set containing z_0 and hence, $R(N)$ is an open set containing w_0. Now, if $w \in R(N)$ then $w = R(z)$ for some $z \in N$. Hence,

$$\sigma(R^{\circ n}(w), R^{\circ n}(w_0)) = \sigma(R^{\circ n+1}(z), R^{\circ n+1}(z_0)) < \epsilon.$$

Hence, $z_0 \in R^{-1}(F)$ and $F \subset R^{-1}(F)$.

Therefore, $R^{-1}(F) = F$ and $F(R)$ is completely invariant. □

Lemma 1.3.3. *For any rational map R and a domain $U \subset \hat{\mathbb{C}}$, $\partial R(U) \subset R(\partial U)$.*

Proof. Let $w_0 \in \partial R(U)$ such that it is approximated by $R(z_n)$ for $(z_n)_{n=1}^{\infty} \subset U$. Now, assume $z_n \to z_0$ (after taking a subsequence). Now, z_0 cannot lie in U, otherwise $R(z_0) =$

$w_0 \in R(U)$. Since, R is an open map, $R(U)$ is an open set and is disjoint from $\partial R(U)$. Hence, $z_0 \in \partial U$ and $R(z_0) = w_0 \in R(\partial U)$. Therefore, $\partial R(U) \subset R(\partial U)$. $\square$

Lemma 1.3.4. *For a rational map R, if F_1 and F_2 are two Fatou components and R maps a point of F_1 to a point of F_2, then $R(F_1) = F_2$.*

Proof. Clearly, $R(F_1) \subset F_2$ because of forward invariance of F under R and since F_1 and F_2 are connected components of F. If $R(F_1) \neq F_2$, then $\exists z \in \partial F_1$ such that $R(z) \in F_2$ and this is not possible as $z \in \partial F_1 \implies z \in J$ and J is completely invariant. Hence, $R(F_1) = F_2$. $\square$

Theorem 1.3.5. *The unbounded Fatou component of a polynomial P, i.e. the Fatou component containing ∞ is a completely invariant Fatou component. It is denoted by $F_\infty(P)$ or simply F_∞ when the context is clear.*

Proof. First note that since $P(\infty) = \infty$, we have $P(F_\infty) = F_\infty$ by the above lemma. $\square$

1.4 Equicontinuity and Normality

Definition 1.4.1 (Connectivity). *The connectivity of a domain $D \subset \hat{\mathbb{C}}$ is defined as the number of components of ∂D.*

Theorem 1.4.1. *The following are equivalent for a domain $D \subset \hat{\mathbb{C}}$:*

1. *D is simply connected.*
2. *D^c is connected.*
3. *∂D is connected or $c(D) = 1$.*

Theorem 1.4.2. *If R is a rational function, with $\deg(R) \geq 2$, and F_0 is a completely invariant Fatou component of R, then:*

1. *$\partial F_0 = J$.*
2. *F_0 is simply connected or infinitely connected.*
3. *All other components of F are simply connected.*
4. *F_0 is simply connected $\iff$ J is connected.*

Theorem 1.4.3 (Vitali's Theorem).

Lemma 1.4.4. *If α is a (super)-attracting fixed point of a rational map R and F_α is the Fatou component containing α then $R^{\circ n}(z) \to \alpha$ locally uniformly in F_α.*

Theorem 1.4.5 (**Riemann-Hurwitz Formula**). *Let F_0 and F_1 be components of the Fatou set F of a rational map R and R maps F_0 into F_1. Then, for some integer m, R is an m-fold map of F_0 onto F_1 and*

$$\chi(F_0) + \delta_R(F_0) = m\chi(F_1).$$

1.5 Some properties of the Julia Sets

Let J denote the Julia set for a rational map R with $\deg(R) \geq 2$. Then we have the following properties:

Theorem 1.5.1. *J is infinite.*

Theorem 1.5.2 (**Minimality of J**).

Theorem 1.5.3. *J is a perfect set, and hence, uncountable.*

Chapter 2

Petal Theorem

A point p is called a parabolic fixed point of f if $f(p) = p$ and $f'(p) = e^{2\pi i t}$, where t is a rational number.

Lemma 2.0.1. *Suppose f is analytic and satisfies*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{p+2})$$

in some neighbourhood N of the origin. Let $\omega_1, \ldots, \omega_p$ be the p-th roots of unity and let $\eta_1, \ldots, \eta_p$ be the p-th roots of -1. Then for sufficiently small r_0 and θ_0,

1. *$|f(z)| < |z|$ on each sector*

$$S_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\omega_j)| < \theta_0\}.$$

2. *$|f(z)| > |z|$ on each sector*

$$\Sigma_j = \{re^{i\theta} : 0 < r < r_0, |\theta - \arg(\eta_j)| < \theta_0\}.$$

Proof. We have,

$$f(z)/z = 1 - z^p + \mathcal{O}(z^{p+1}) = 1 - z^p(1 + g(z)),$$

where g is analytic in N with $g(0) = 0$.
Now, consider the sector,

$$S = \{z \in \mathbb{C} : |z| < \frac{1}{2}; |\arg(z)| < \pi/4\}.$$

For small r_0 and θ_0, $z \in S_j \implies z^p(1 + g(z)) \in S$ and $z \in \Sigma_j \implies -z^p(1 + g(z)) \in S$. This is because for small enough r_0 and θ_0, $z \mapsto z^p$ maps S_j onto the set

$$S_0 = \{z \in \mathbb{C} : |z| < r_0^p; |\arg(z)| < p\theta_0\} \subset S.$$

And, $|z^p - z^p(1 + g(z))| = |z|^p|g(z)| \leq M|z|^{p+1} = M(|z|^p)^{1+\frac{1}{p}}$. Hence, for any $w \in S_0$, the perturbation of any point is $\leq M|w|^{1+1/p}$.

$\square$

Before stating the *Petal Theorem*, which discusses the behaviour of analytic functions near parabolic fixed points, we first define the notions of *petals*.

Definition 2.0.1 (Petals). *Let $p \in \mathbb{N}$. For each $k \in \{0, 1, \ldots, p-1\}$, define the sets as a function of a parameter $t > 0$ as follows,*

$$\Pi_k(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)); |\theta - 2k\pi/p| < \pi/p\}.$$

The sets $\Pi_k(t)$ are known as Petals.

We have shown a diagram of the petals $\Pi_k(t)$ in Figure 2.1 for $p = 6$.
Note that all the petals are pairwise disjoint and each petal subtends an angle of $2\pi/p$ at the origin.

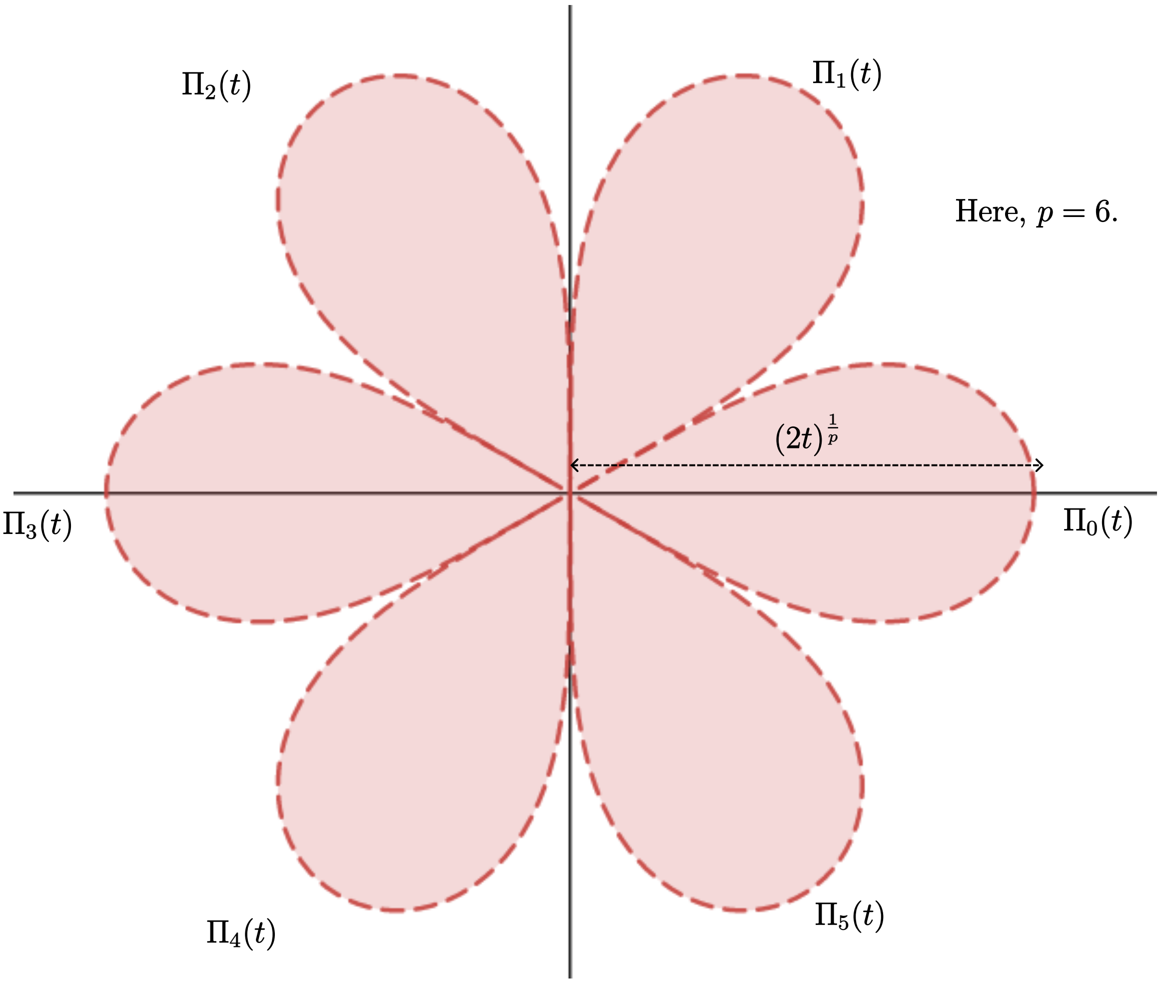

Figure 2.1: Six petals at the origin for $p = 6$.

Theorem 2.0.2 (The Petal Theorem). *Suppose that an analytic map f has the form:*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1})$$

near the origin. Then for sufficiently small t,

1. f *maps each* $\Pi_k(t)$ *into itself;*

2. $f^{\circ n}(z) \to 0$ *uniformly on each petal;*

3. $\arg(f^{\circ n}(z)) \to 2k\pi/p$ *locally uniformly on each petal;*

4. $f : \Pi_k(t) \to \Pi_k(t)$ *is conjugate to a translation.*

5. $|f(z)| < |z|$ *on a neighbourhood of the axis of each petal;*

Proof. For $0 < r_0 < 1$, define the sector S_0,

$$S_0 = \{re^{i\theta} : 0 < r < r_0, |\theta| < \pi/p\}$$

and the region W,

$$W = \{re^{i\theta} : r > \frac{1}{r_0^p}, |\theta| > \pi\}.$$

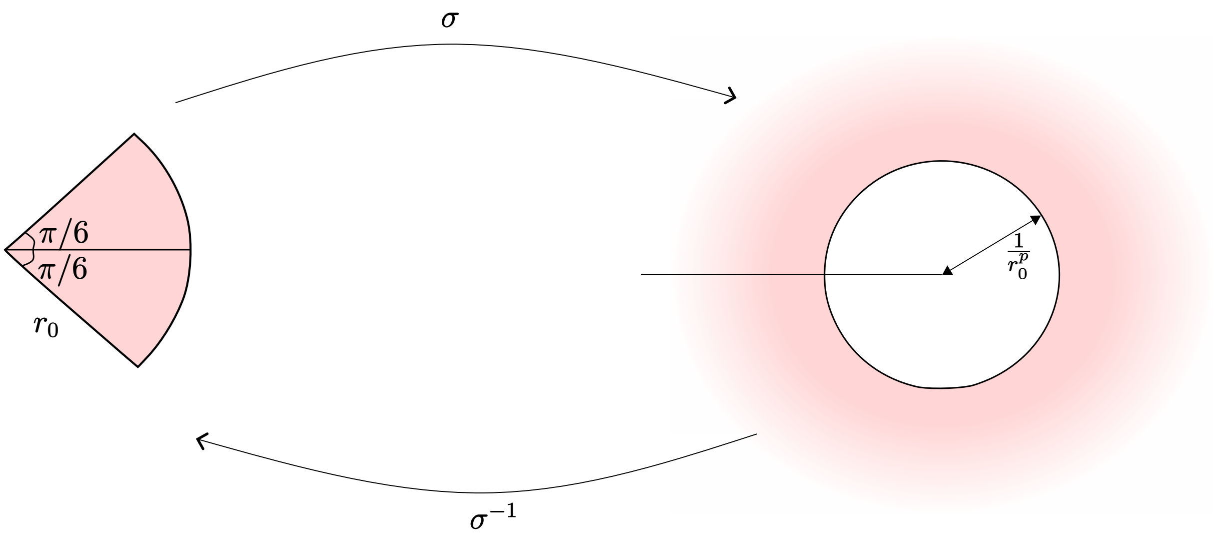

Figure 2.2: σ is a biholomorphism from S_0 onto W.

It is clear that the map $\sigma : z \mapsto \frac{1}{z^p}$ is a biholomorphism of S_0 onto W with $\sigma^{-1} : W \to S_0$ given by $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$. Actually, σ is a biholomorphism of each sector $S_k = \{0 < r < r_0, |\theta - 2k\pi/p| < \pi/p\}$ onto W. The branch of $\sigma^{-1}(w) = 1/w^{\frac{1}{p}}$ that we choose determines which sector the inverse map maps to.

Now, the conjugate map of f on W is given by,

$$g(w) = \sigma f \sigma^{-1}(w) = f(w^{-\frac{1}{p}})^{-p}.$$

This just replaces the action of f on S_0 by g on W, and we have the following commutative diagram:

Now, we will use the power series expansion of f near the origin to get information about g.
First let us try to get a estimate of the power series expansion of $f(z)^{-p}$. We have,

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}) = z(1 - z^p + \mathcal{O}(z^{2p})) = z(1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + - \dots).$$

So,

$$\frac{1}{f(z)^p} = z^{-p}\left(\frac{1}{1 - z^p - a_0 z^{2p} - a_1 z^{2p+1} + - \dots}\right)^p.$$

Now, let $\alpha(z) = z^p + a_0 z^{2p} + a_1 z^{2p+1} + \dots$, then for r_0 small enough such that $|\alpha(z)| < 1$ on $\{|z| < r_0\}$, we can write,

$$\frac{1}{1 - \alpha(z)} = 1 + \alpha(z) + \alpha(z)^2 + \dots.$$

Therefore,

$$\begin{aligned}\frac{1}{f(z)^p} &= z^{-p}(1 + \alpha(z) + \alpha(z)^2 + \dots)^p \\ &= z^{-p}(1 + pz^p + Az^{2p} + A_1 z^{2p+1} + \dots) \\ &= \frac{1}{z^p} + p + Az^p + v(z),\end{aligned}$$

where A is some constant and $v(z)$ is holomorphic on $\{|z| < r_0\}$, and for some small $r_0 > 0$, it satisfies $|v(z)| \leq B|z|^{p+1}$, $B > 0$.

Now, if $w \in W$, then $\sigma^{-1}(w) \in S$. Hence, by substituting $z = \sigma^{-1}(w) = w^{-1/p}$, we have,

$$\begin{aligned}g(w) &= \sigma f \sigma^{-1}(w) \\ &= \frac{1}{f(w^{-1/p})^p} \\ &= w + p + A/w + \theta(w),\end{aligned}$$

where $|\theta(w)| = |v(w^{-1/p})| \leq B|w^{-1/p}|^{p+1} = B/|w|^{1+\frac{1}{p}}$.
Hence, we have the following estimates for g which will be crucial in everything that will follow:

$$g(w) = w + p + A/w + \theta(w), \text{ where } A \text{ is a constant and} \tag{2.1}$$
$$|\theta(w)| \leq B/|w|^{1+\frac{1}{p}}, B > 0. \tag{2.2}$$

Choose any K satisfying

$$K > \max\{1/r_0^p, 3(|A| + B)\} > 1 \text{ (as } r_0 < 1)$$

and let,

$$\Pi = \{x + iy : y^2 > 4K(K - x)\}.$$

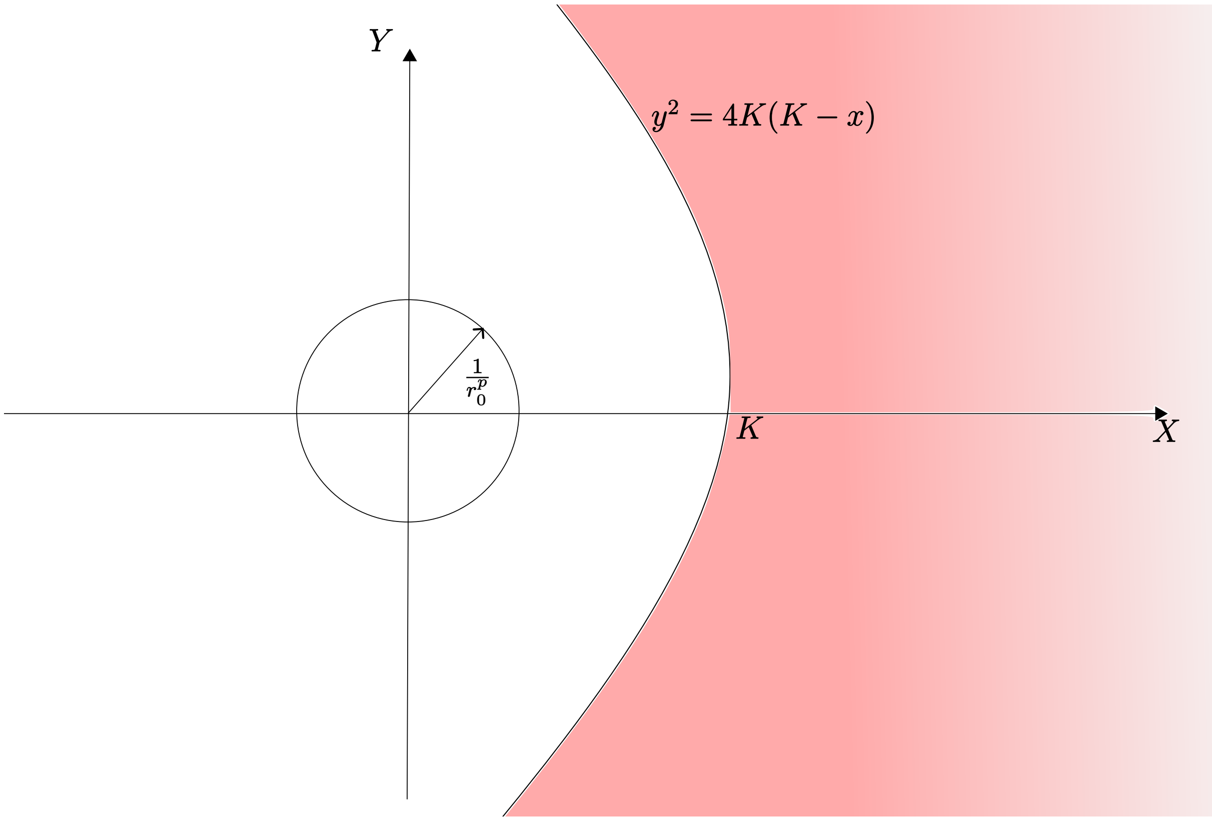

Figure 2.3: $\Pi = \{(x, y) : y^2 > 4K(K - x)\}$.

Clearly, Π is bounded by a parabola and $\Pi \subset W$ (See Figure 2.3).

We have chosen this subset $\Pi \subset W$ because we will show that Π is nothing but the conformal image of $\Pi_0(t)$ under σ (for a suitable t) and g satisfies all the corresponding conditions that f should satisfy on $\Pi_0(t)$ according to the theorem.

Claim. Π is the conformal image of $\Pi_0(t)$ under σ for a suitable t.
The easiest way to see this is using polar coordinates. We write, $z = re^{i\theta}$ for $z \in S_0$ and $w = \rho e^{i\phi}$ for $w \in W$. Then, $\rho = \frac{1}{r^p}$ and $\phi = -p\theta$.

Now, we need to express Π in polar co-ordinates. To do so, we notice that points on the parabola are given by

$$\rho \text{ (distance from focus i.e. 0)} = 2K - \rho \cos \phi \text{ (distance from directrix i.e. } y = 2K).$$

(See Figure 2.4). Therefore, points on Π are given by

$$\rho > 2K - \rho \cos \phi.$$

Hence,

$$\Pi = \{\rho e^{i\phi} : 2K < \rho(1 + \cos \phi)\}.$$

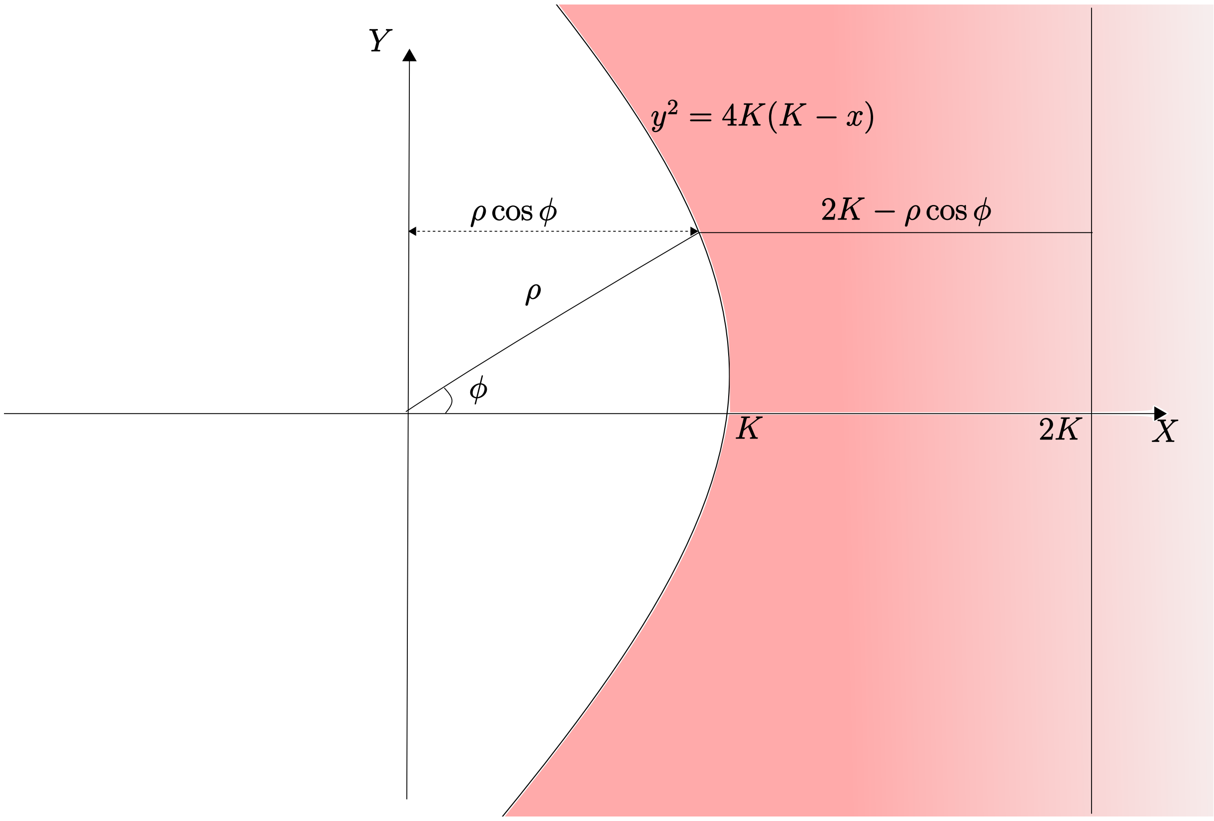

Figure 2.4: $\Pi = \{\rho e^{i\phi} : \rho > 2K - \rho\cos\phi\}$.

Now, let $\Omega = \sigma^{-1}(\Pi)$. Then, Ω is given by

$$\Omega = \{re^{i\theta} : 2Kr^p < 1 + \cos(p\theta)\}.$$

Hence, $\Omega = \Pi_0\left(\frac{1}{2K}\right)$.

Lemma 2.0.3. *g satisfies the following properties on Π:*

1. *Π is forward invariant under g.*
2. *$g^{\circ n}(w) \to \infty$ uniformly on Π.*
3. *$\arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π.*
4. *$g : \Pi \to \Pi$ is conjugate to a translation.*

Proof.
1. We write,

$$w = x + iy,\ g(w) = X + iY,\ A/w + \theta(w) = a + ib.$$

From Equation (2.1), we obtain,

$$X + iY = (x + iy) + p + (a + ib)$$
$$\implies X = x + p + a \text{ and } Y = y + b.$$

Now, if $w \in \Pi$,

$$\begin{aligned} Y^2 - 4K(K - X) &= (y + b)^2 - 4K(K - x - p - a) \\ &= [y^2 - 4K(K - x)] + b^2 + 2yb + 4K(a + p) \\ &> 4Kp + (2yb + 4Ka). \end{aligned}$$

Now, for $w \in \Pi$, $|w| > K > 1$. (It is clear for $\mathrm{Re}(w) > K$. For $\mathrm{Re}(w) \leq K$, we use the polar description $\rho > 2K - \rho \cos \phi$ to get $|w| > 2K - \mathrm{Re}(w) \geq K$). Hence we get,

$$|w||A/w + \theta(w)| \leq |w|(|A|/|w| + B/|w|^{1+\frac{1}{p}}) = |A| + B/|w|^{\frac{1}{p}} < |A| + B \tag{2.3}$$

(since for $|w| > 1$, $|w|^{\frac{1}{p}} > 1$). Therefore,

$$\begin{aligned} |2yb + 4Ka| &\leq 2|yb| + 4K|a| \\ &\leq 2|y||b| + 4K|a| \\ &\leq 2|w||a + ib| + 4|w||a + ib| \\ &= 6|w||a + ib| \\ &< 2K \leq 2Kp. \end{aligned}$$

Therefore, we see that $Y^2 - 4K(K - X) > 0$ and hence, $g(w) \in \Pi$ for $w \in \Pi$. Hence, Π is forward invariant under g.

2. Now, we will prove a stronger statement that for any $t > 0$ g maps $\Pi + t$ into $\Pi + t + p/2$. This is simply because, for $w \in \Pi + t$, we have, $y^2 - 4K(K + t - x) > 0$. Hence,

$$\begin{aligned} Y^2 - 4K(K + t + p/2 - X) &= [y^2 - 4K(K + t - x)] + b^2 + 2yb + 4K(a - p/2) \\ &> 2Kp + (2yb + 4Ka) \\ &> 0. \end{aligned}$$

Therefore, if $w \in \Pi$, $g^{\circ n}(w) \in \Pi + np/2$. Hence, $|g^{\circ n}(w)| > \sqrt{n}$. This is simply because, if $x + iy \in \Pi + np/2$, we have

$$x^2 + y^2 - n > x^2 + 4K(K + np/2 - x) - n = x^2 - 4Kx + (4K^2 + 2npK - n).$$

The discriminant of this quadratic equation in x is

$$16K^2 - 4(4K^2 + 2npK - n) = 4n(1 - 2pK) < 0$$

. Thus, $x^2 + y^2 - n > 0$ for all $x + iy \in \Pi + np/2$.

Hence, $g^{\circ n}(w) \to \infty$ uniformly on Π.

3. Firstly, we see that,

$$g^{\circ k}(w) = g(g^{\circ k}(w)) = g^{\circ k}(w) + p + A/g^{\circ k}(w) + \theta(g^{\circ k}(w)).$$

By expanding the first term again and again, we obtain,

$$g^{\circ n}(w) = w + np + \sum_{k=0}^{n-1}\left(\frac{A}{g^{\circ k}(w)} + \theta(g^{\circ k}(w))\right). \tag{2.4}$$

Also note that form Equation (2.3), we have,

$$|A/w + \theta(w)| < (|A| + B)/K < \frac{1}{3}.$$

Let Q be a compact subset of Π. From now, we will assume that $w \in Q$ and we will use $C_1, C_2, C_2, \ldots$ to denote positive constants which will be dependent on Q.

Hence,

$$\begin{aligned}|g(w)| = |w + p + A/w + \theta(w)| &\geq ||w + p| - |A/w + \theta(w)|| \\ &= |w + p| - |A/w + \theta(w)| \\ &\geq |w| + p - \frac{1}{3}.\end{aligned}$$

Therefore, we obtain,

$$|g^{\circ n}(w)| \geq |w| + n(p - 1/3) \geq C_1 + C_2 n.$$

(Here, $C_1 = \min\{|w| : w \in Q\} > 0$ and $C_2 = p - \frac{1}{3} > 0$.)
Hence,

$$|g^{\circ n}(w)| \geq C_2 n. \tag{2.5}$$

Next, with Equation (2.2), and the above inequality, we get,

$$|\theta(g^{\circ n}(w)| \leq B/|g^{\circ n}(w)|^{1+\frac{1}{p}} \leq C_3/n^{1+\frac{1}{p}}. \tag{2.6}$$

Finally, combining the above two inequalities and Equation (2.4), we obtain,

$$\begin{aligned}|g^{\circ n}(w) - np| &\leq |w| + |A/w + \theta(w)| + \frac{|A|}{C_2}\sum_{k=1}^{n-1}\frac{1}{k} + C_3\sum_{k=1}^{n-1}\frac{1}{n^{1+\frac{1}{p}}} \\ &< C_4 + C_5\sum_{k=1}^{n}\frac{1}{k}.\end{aligned}$$

(Here, $C_4 = \max\{|w| : w \in Q\} + \frac{1}{3} + C_3\sum_{n=1}^{\infty} 1/n^{1+\frac{1}{p}}$ and $C_5 = |A|/C_2$.)
We can select C_6 large enough such that

$$|g^{\circ n}(w) - np| < C_6 \log n. \tag{2.7}$$

Remark. The above inequality follows from the fact that, if $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then $H_n - \log n \to \gamma$. (γ is known as the Euler's constant). So, we have that

$$\begin{aligned} P + QH_n &= P + Q(\log n + \gamma + \epsilon_n), \text{where } \epsilon_n \to 0 \\ &\leq Q \log n + (P + Q \max\{\epsilon_n\} + Q\gamma) \\ &= Q \log n + R \\ &< S \log n \end{aligned}$$

for S large enough.

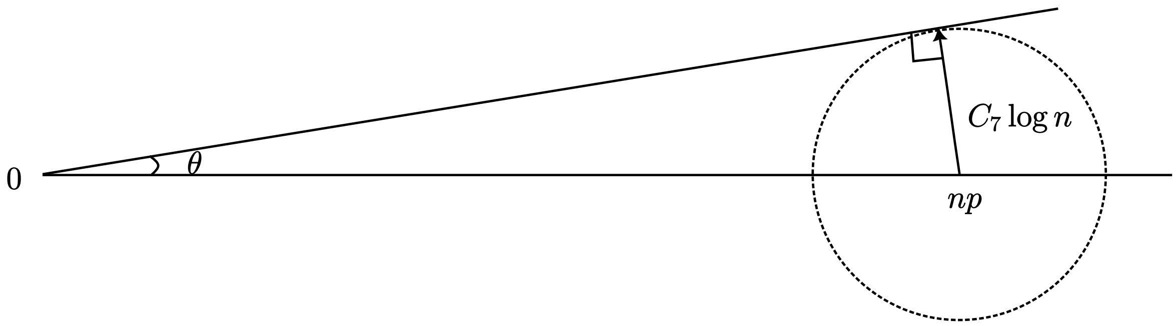

Figure 2.5: $|\arg(g^{\circ n}(w))| \leq \sin^{-1}(\frac{C_6 \log n}{np})$.

From, $|g^{\circ n}(w) - np| < C_6 \log n$, it follows that $|\arg(g^{\circ n}(w)| < \sin^{-1}\left(\frac{C_6 \log n}{np}\right)$ for n large enough. Hence, $\arg(g^{\circ n}(w)) \to 0$ uniformly on Q, and consequently, locally uniformly on Π.

4. Define,

$$u_n(w) = g^{\circ n}(w) - np - (A/p) \log n.$$

Claim. $u_n(w)$ converges locally uniformly on Π to a holomorphic function u, that is one-to-one on Π.

$$u_{n+1}(w) - u_n(w) = [g^{\circ n+1}(w) - g^{\circ n}(w)] - p - (A/p) \log\left(\frac{n+1}{n}\right).$$

From Equation (2.2), we obtain,

$$\begin{aligned} u_{n+1}(w) - u_n(w) &= [g^{\circ n}(w) + p + A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - g^{\circ n}(w)] \\ &\qquad - p - (A/p) \log(1 + 1/n) \\ &= A/g^{\circ n}(w) + \theta(g^{\circ n}(w)) - (A/p) \log(1 + 1/n) \\ &= A(1/g^{\circ n}(w) - 1/np) + \theta(g^{\circ n}(w)) + (A/p)(1/n - \log(1 + 1/n)). \end{aligned}$$

Now, let Q is a compact subset of Π and $w \in Q$. We need to prove that u_n converges uniformly in Q. From the above equation, to prove that u_n converges uniformly in Q, we need to show that each of the following series converges uniformly in Q:

$$\sum_n |1/g^{\circ n}(w) - 1/np|, \ \sum_n |\theta(g^{\circ n}(w)), \ \sum_n |1/n - \log(1+1/n)|.$$

Let us look at the first series. We have, (using Equations (2.5) and (2.7))

$$|1/g^{\circ n}(w) - 1/np| = \frac{|g^{\circ n}(w) - np|}{|g^{\circ n}(w)|np} \leq \frac{C_6 \log n}{C_2 n^2 p} = C_7 \log n/n^2.$$

(Here $C_7 = C_6/(pC_2)$).

From Equation (2.6), it is clear that $\sum_n |\theta(g^{\circ n}(w))|$ converges.

Now, $0 < x - \log(1+x) \leq x^2$ for $x > 0$.

This is because, it is zero at $x = 0$ and $\frac{d}{dx}(x - \log(1+x)) = 1 - \frac{1}{1+x} > 0$ for $x > 0$.

Also, $x^2 - x + \log(1+x)$ is zero at $x = 0$ and $\frac{d}{dx}(x^2 - x + \log(1+x)) = 2x - 1 + \frac{1}{1+x} > 0$ for $x > 0$.

Putting $x = \frac{1}{n}$, we get,

$$|1/n - \log(1+1/n)| < 1/n^2.$$

Therefore, u_n converges locally uniformly to some holomorphic function u on Π.

Now, from $u_n(w) = g^{\circ n}(w) - np - (A/p)\log n$, we get that,

$$\begin{aligned}
(n+1)p + (A/p)\log(n+1) + u_{n+1}(w) &= g^{\circ n+1}(w) \\
&= g^{\circ n}(g(w)) \\
&= np + (A/p)\log n + u_n(g(w)) \\
\implies p + (A/p)\log(1+1/n) + u_{n+1}(w) &= u_n(g(w)).
\end{aligned}$$

Taking limit $n \to \infty$, we get,

$$p + u(w) = u(g(w)).$$

Since f is injective near the origin, g is injective on Π, (if K is chosen large enough). Therefore, $g^{\circ n}$ is injective on Π and hence, so is u_n. By Hurwitz Theorem, u is either injective or constant, but it is clearly not a constant since it satisfies the above equation.

This shows that $g : \Pi \to \Pi$ is conjugate to the map $z \mapsto z + p$ of $u(\Pi)$ into itself. $\square$

Coming back to our original theorem, we see that our original theorem is also proved as we had just replaced the action of f on Π_0 by the action of its conjugate g on Π and we just proved all the parts of the theorem that the conjugate of f, i.e. g must satisfy.

From, $g = \sigma f \sigma^{-1}$, we get, $g^{\circ n} = \sigma f^{\circ n} \sigma^{-1} \implies g^{\circ n}\sigma = \sigma f^{\circ n}$. Writing, $\sigma(z) = w$, we have,

$$g^{\circ n}(w) = \frac{1}{f^{\circ n}(z)^p} \implies g^{\circ n}(w)(f^{\circ n}(z))^p = 1. \tag{2.8}$$

1. Since, g maps Π into itself, f maps Π_0 into itself.

2. Now, since $|g^{\circ n}(w)| > \sqrt{n}$, $|f^{\circ n}(z)| < \frac{1}{n^{1/2p}}$ from Equation (2.8). Hence, $f^{\circ n}(z) \to 0$ uniformly on Π_0.

3. Also, $\arg(f^{\circ n}(z)) = \left(-\frac{1}{p}\right) \arg(g^{\circ n}(w))$ from Equation (2.8). Since, $\arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π, $f^{\circ n}(z) = \left(-\frac{1}{p}\right) \arg(g^{\circ n}(w)) \to 0$ locally uniformly on Π_0.

4. Since, $g : \Pi \to \Pi$ is conjugate to a translation, and g is conjugate to f, $f : \Pi_0 \to \Pi_0$ is also conjugate to a translation.

5. It is immediate from Lemma 2.0.1 that $|f(z)| < |z|$ on the axis of Π_0.

$\square$

Theorem 2.0.4. *Suppose that f has the power series expansion near 0 as,*

$$f(z) = z + az^{p+1} + \mathcal{O}(z^{p+2}).$$

Then, f is conjugate near 0 to a function

$$F(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}).$$

Proof. First, we conjugate f by the map $z \mapsto \lambda z$, where $\lambda^p = a$. Then, we get that f is conjugate to the map,

$$\tilde{f} = \lambda f(z/\lambda) = \lambda z/\lambda + \lambda a z^{p+1}/\lambda^{p+1} + \mathcal{O}(z^{p+2}) = z + z^{p+1} + \mathcal{O}(z^{p+2}).$$

We will now proceed via induction over a finite number of steps. Let,

$$f_k(z) = z + z^{p+1} + bz^{p+k+1} + \ldots, b \neq 0.$$

Here $k \geq 1$. Also if $k \geq p$, our theorem is proved. Hence, we assume, $1 \leq k < p$.
Now, define the polynomial,

$$\sigma(z) = z + \alpha z^{k+1},$$

where $\alpha = \frac{b}{p-k}$ and let σ^{-1} be its inverse near 0 (We can do this because $\sigma'(0) = 1$).

Now, we will show that we obtain f_r (for some $r \geq k+1$) by conjugating f_k with σ. Hence, let

$$g = \sigma f_k \sigma^{-1}$$

and we need to show that $g = f_r$ (for some $r \geq k+1$). Since, $g'(0) = f_k'(0) = 1$, we let,

$$g(z) = z + \sum_{m=2}^{\infty} a_m z^m.$$

Now, we will use the identity, $g\sigma = \sigma f_k$.

$$\begin{aligned}\sigma f_k(z) &= (z + z^{p+1} + bz^{p+k+1} + \dots) + \alpha(z + z^{p+1} + bz^{p+k+1})^{k+1}\\ &= z + \alpha z^{k+1} + z^{p+1} + (b + \alpha(k+1))z^{p+k+1} + \mathcal{O}(z^{p+k+2})\\ &= z + \alpha z^{k+1} + z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

The last equality follows because,

$$\alpha(p-k) = b \implies \alpha(p+1) - \alpha(k+1) = b \implies \alpha(p+1) = b + \alpha(k+1).$$

Now,

$$\begin{aligned}g\sigma(z) &= (z + \alpha z^{k+1}) + \sum_{m=2}^{\infty} a_m(z + \alpha z^{k+1})^m\\ &= z + \alpha z^{k+1} + \sum_{m=2}^{p+k+1} a_m(z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

Now, equating $\sigma f_k(z) = g\sigma(z)$, we get,

$$z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) = \sum_{m=2}^{p+k+1} a_m(z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2}).$$

Firstly, we see that on the right hand side, the coefficient of z^2 will be a_2, the coefficient of z^3 will be some linear combination of a_2 and a_3, the coefficient of z^4 will be some linear combination of a_2, a_3 and a_4 and so on upto the coefficient of z^p will be some linear combination of $a_2, a_3, \dots a_p$. Since, the coefficient of $z^2, \dots, z^p$ is zero on the left hand side, it follows that $a_2 = a_3 = \dots, a_p = 0$. (This argument follows assuming $p \geq 2$, but if $p = 1$ the coefficient of $z^p = z$ i.e. a_1 is automatically 0).

Hence, now we have,

$$\begin{aligned}z^{p+1} + \alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}) &= \sum_{m=p+1}^{p+k+1} a_m(z + \alpha z^{k+1})^m + \mathcal{O}(z^{p+k+2})\\ &= a_{p+1}z^{p+1} + \dots + a_{p+k+1}z^{p+k+1} +\\ &\quad a_{p+1}\alpha(p+1)z^{p+k+1} + \mathcal{O}(z^{p+k+2}).\end{aligned}$$

Therefore, we obtain

$$a_{p+1} = 1, a_{p+2} = \dots = a_{p+k} = 0 \text{ and } a_{p+k+1} + a_{p+1}\alpha(p+1) = \alpha(p+1).$$

Hence, $a_{p+k+1} = 0$. This gives that f_k is conjugate to the map

$$g(z) = z + z^{p+1} + \mathcal{O}(z^{p+k+2}).$$

Thus, $g = f_r$ for some $r \geq k+1$. Continuing the induction process, we get that f is conjugate near 0 to a map

$$z \mapsto z + z^{p+1} + \mathcal{O}(z^{2p+1}).$$

Now, we an again conjugate this map with the map, $z \mapsto \lambda z$, where $\lambda^p = -1$ to get that f is conjugate to a map,

$$F(z) = z - z^{p+1} + \mathcal{O}(z^{2p+1}).$$

$\square$

At the end we consider the most general situation, i.e. when R is a rational function and

$$R(z) = az + bz^{p+1} + \dots, \text{ where } a \neq 1 \text{ but } a = e^{2\pi i p/q}.$$

In this case, $R^{\circ q}(z)$ is of the form,

$$R^{\circ q}(z) = z + cz^{r+1} + \dots, \text{ for some } r \in \mathbb{N}.$$

Then $R^{\circ q}$ has r petals at the origin. Now, $F(R^{\circ q}) = F(R)$.

Chapter 3

Bottcher's Theorem and its extension

3.1 Bottcher's Coordinates

A fixed point p is called a super-attracting fixed point of f if $f'(p) = 0$.
If p is a super-attracting fixed point for f, we can conjugate the map such that $z = 0$ becomes our super-attracting fixed point.

Thus, our map takes the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$$

in a neighbourhood of 0 , with $n \geq 2$ and $a_n \neq 0$. Here the integer n is called the *local degree* of f at 0.

Theorem 3.1.1 (Bottcher's Theorem). *With f as above, $\exists$ a local holomorphic change of coordinates $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to $w \mapsto w^n$ throughout some neighbourhood of 0 .*
Furthermore, ϕ is unique upto multiplication by an $(n-1)$ th root of unity.

Proof. **Existence.** Let $c \in \mathbb{C}$ be such that $c^{n-1} = a_n$. Then, the linearly conjugate map $cf(z/c)$ will have leading coefficient $+1$. Thus, without loss of generality, we will assume that our map f has the form

$$f(z) = z^n(1 + b_1 z + b_2 z^2 + \dots) = z^n(1 + \eta(z)), \text{ where } \eta(z) = \left(1 + b_1 z + b_2 z^2 + \dots\right).$$

Choose $r \in \left(0, \frac{1}{2}\right)$ such that $|\eta(z)| < \frac{1}{2} \forall z \in \mathbb{D}_r$. This can be done since $\eta(0) = 0$ and η is continuous.
On this disc, we have two properties of f :

1. f maps this disc into itself: We have, $|f(z)| = |z^n|\,|1+\eta(z)| \leq |z|^n(1+|\eta(z)|) < \frac{3}{2}|z|^n \leq \frac{3}{2^n}|z| \leq \frac{3}{4}|z| \forall z \in \mathbb{D}_r$. Here we are using the fact that $n \geq 2, |z| < \frac{1}{2}$ and $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$.

2. $f(z) \neq 0 \; \forall z \in \mathbb{D}_r \backslash \{0\}$. This is simply because $|f(z)| = |z|^n|1+\eta(z)|$ and since $|\eta(z)| < \frac{1}{2}$ on $\mathbb{D}_r$, we can't have $\eta(z) = -1$.

The k-th iterate of f i.e. $f^{\circ k}$ also maps the $\mathbb{D}_r$ into itself and $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$. Inductively, it can be shown that it has the form $f^{\circ k}(z) = z^{n^k}\left(1 + n^{k-1}b_1 z + \ldots\right)$.

The idea of the proof is to set,

$$\phi_k(z) = \left(f^{\circ k}(z)\right)^{\frac{1}{n^k}} = z\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}}$$

We choose z as our branch of holomorphic n^k th root of z^{n^k}.

Now, we can choose a holomorphic branch of $\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}}$ on $\mathbb{D}_r$ since $\mathbb{D}_r$ is simply connected and $\left(1 + n^{k-1}b_1 z + \ldots\right) \neq 0$ on $\mathbb{D}_r$ since $f^{\circ k}(z) \neq 0$ on $\mathbb{D}_r \backslash \{0\}$. Therefore we set,

$$\phi_k(z) = z\left(1 + n^{k-1}b_1 z + \ldots\right)^{\frac{1}{n^k}} = z\left(1 + \frac{b_1}{n}z + \ldots\right)$$

where the expression on the right provides us an explicit choice of n^k th root.

We will show that the functions ϕ_k converge uniformly to a limit function ϕ on $\mathbb{D}_r$. To prove the convergence, we make the substitution $z = e^u$ where u ranges over the left half plane $\mathbb{H}_r := \{u : \mathrm{Re}(u) < \log r\}$. The exponential map maps $\mathbb{H}_r$ onto $\mathbb{D}_r \backslash \{0\}$.

The map f from $\mathbb{D}_r$ into itself corresponds to a map from $\mathbb{H}_r$ into itself given by $F(u) = \log f\left(e^u\right)$. We can select a holomorphic branch of the logarithm of $f\left(e^u\right)$ because $\mathbb{H}_r$ is simply connected and $f\left(e^u\right) \neq 0$ on $\mathbb{H}_r$.

Set $\eta = \eta\left(e^u\right) = b_1 e^u + b_2 e^{2u} + \ldots$, then since $|\eta| < \frac{1}{2}$, we see that F can be written as

$$F(u) = \log\left(e^{nu}(1+\eta)\right) = nu + \log(1+\eta) = nu + \left(\eta - \frac{\eta^2}{2} + \frac{\eta^3}{3} - + \ldots\right)$$

where the final expression provides us an explicit choice of which branch of logarithm we are using. Clearly, $F : \mathbb{H}_r \to \mathbb{H}_r$ is a well-defined holomorphic function.

Similarly, the map ϕ_k corresponds to a map, $\Phi_k(u) = \log \phi_k\left(e^u\right)$.

$$\Phi_k(u) = \log \phi_k\left(e^u\right) = \log f^{\circ k}\left(e^u\right)^{\frac{1}{n^k}} = \frac{1}{n^k}\log f^{\circ k}\left(e^u\right).$$

Since we have already fixed the branch of logarithm that we are using, we see that,

$$\log f^{\circ k}\left(e^u\right) = \log f\left(f^{\circ k-1}\left(e^u\right)\right) = \log f\left(e^{\log f^{\circ k-1}\left(e^u\right)}\right) = F\left(\log f^{\circ k-1}\left(e^u\right)\right)$$

Hence, inductively we can see that $\log f^{\circ k}(e^u) = F^{\circ k}(u)$.

Therefore, $\Phi_k(u) = F^{\circ k}(u)/n^k$. It is clear from this expression that $\Phi_k : \mathbb{H}_r \to \mathbb{H}$.

Now since $|\eta| < \frac{1}{2}$, we have

$$|F(u) - nu| = |\log(1+\eta)| < \log 2 < 1$$

Hence,

$$|\Phi_{k+1}(u) - \Phi_k(u)| = \frac{1}{n^{k+1}} \left|F^{\circ k+1}(u) - nF^{\circ k}(u)\right| < \frac{1}{n^{k+1}}$$

by the above inequality.

We have, $\phi_k(e^u) = e^{\Phi_k(u)}$. Since, the exponential map, $e^{\square} : \mathbb{H} \to \mathbb{D}$ from the left half plane to the unit disc decreases distance, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r \backslash \{0\}.$$

Since $\phi_k(0) = 0$ for all k, we have

$$|\phi_{k+1}(z) - \phi_k(z)| < \frac{1}{n^{k+1}} \forall z \in \mathbb{D}_r$$

Hence, the maps ϕ_k converge uniformly to some limit function ϕ on $\mathbb{D}_r$ by the Cauchy criterion for uniform convergence.

Clearly, $\phi(0) = 0$ and ϕ is holomorphic on $\mathbb{D}_r$ by Weierstrass convergence theorem.

It is clear that each $\phi_k : \mathbb{D}_r \to \mathbb{D}$. This is because $\phi_k(e^u) = e^{\Phi_k(u)}$ and $\Phi_k : \mathbb{H}_r \to \mathbb{H}$ and $e^{\square} : \mathbb{H} \to \mathbb{D}\backslash\{0\}$. Hence, $\phi : \mathbb{D}_r \to \mathbb{D}$. (Clearly $\mathrm{Im}(\phi)$ cannot contain points from $\partial\mathbb{D}$ because ϕ is holomorphic, hence it is an open map).

Now, it can be easily seen that, $\phi_k(f(z)) = \phi_{k+1}(z)^n$.

Hence, $\lim_{k\to\infty} \phi_k(f(z)) = \lim_{k\to\infty} \phi_{k+1}(z)^n \implies \phi(f(z)) = \phi(z)^n$ by continuity of nth power map.

Also, since $\phi_k'(0) = 1 \forall k \in \mathbb{N}$ (from the power series of ϕ_k), we have $\phi'(0) = 1$. Hence, ϕ is invertible in some neighbourhood of 0 .

Therefore, we have a holomorphic change of coordinates in some neighbourhood of 0 which conjugates f to the nth power map. In this neighbourhood, ϕ is one-to-one, $f(z) \neq 0$ for $z \neq 0$ (i.e. no other point maps to the super-attracting fixed point via f) and f maps this neighbourhood into itself.

Uniqueness. It suffices to study the special case $f(z) = z^n$. If we can prove that any map which conjugates $z \mapsto z^n$ to itself is just multiplication by $(n-1)$ th root of unity, then for any general map $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, if we have two maps ϕ and ψ which conjugate it to $z \mapsto z^n$, then $\phi \circ \psi^{-1}$ is a map which conjugates $z \mapsto z^n$ to itself.

Hence, $\phi \circ \psi^{-1} = cz$, where $c^{n-1} = 1$. Therefore, $\phi = c\psi$, where c is a $(n-1)$ th root of unity.

So, let $\phi(z) = c_1 z + c_k z^k + \ldots, (c_1 \neq 0)$ be a map which conjugates $z \mapsto z^n$ to itself. Then, we should have $\phi\left(z^n\right) = \phi(z)^n$. Now,

$$\phi\left(z^n\right) = c_1 z^n + c_k z^{nk} + \ldots$$

and

$$\phi(z)^n = c_1^n z^n + n c_1^{n-1} c_k z^{n+k-1} + \ldots$$

Comparing coefficients, we get $c_1^n = c_1$ and $n c_1^{n-1} c_k = 0$ since $nk > n+k-1$ for $k \geq 2$. Therefore, we get $c_1^{n-1} = 1$ and $c_k = 0$. The form $\phi(z) = c_1 z + c_k z^k + \ldots$ can be modified to any $k \geq 2$ to get $c_k = 0$ by the same process.

Therefore, $\phi(z) = cz$, where c is a $(n-1)$ th root of unity.

□

3.2 Extension of Bottcher's coordinates

We might hope to extend the Bottcher's coordinates to the whole of the immediate basin of attraction of the super-attracting fixed point. But this is not always possible. This is because, it requires computing expressions of the form $z \mapsto \left(\phi\left(f^{\circ k}(z)\right)\right)^{\frac{1}{n^k}}$, which is not always possible. For example, the basin may not be simply connected, or some point may map directly onto the super-attracting fixed point, in which case we can't take n^k-th roots, because $\phi\left(f^{\circ k}(z)\right)$ must be zero at those points.

Theorem 3.2.1 (Extension of $|\phi|$). *If f has a super-attracting fixed point p, with immediate basin of attraction $\mathcal{A}$, then the function $z \mapsto |\phi(z)|$ of the above theorem extends uniquely to a continuous map $|\phi| : \mathcal{A} \to [0,1)$ which satisfies $|\phi|(f(z)) = |\phi|(z)^n$.*

Furthermore, $|\phi|$ is real analytic except at the iterated preimages of p, where it takes the value 0 .

Proof. Set $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{1}{n^k}}$ for large enough k for each $z \in \mathcal{A}$. ϕ is only defined in a some small neighbourhood of p. But since, $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, after k many iterates for some large k, $f^{\circ k}(z)$ belongs to the domain of definition of ϕ, which we shall call $\hat{U}$.

It is independent of the value of k (if k is large enough). Note that, if $f^{\circ k}(z) \in \hat{U}$, then so does $f^{\circ k+1}(z)$, since f maps $\hat{U}$ into itself.

Suppose we choose k minimal such that $f^{\circ k}(z) \in \hat{U}$. Then,

$$| \phi \left(f^{\circ k+1}(z) \right|^{\frac{1}{n^{k+1}}} = \left|\phi \left(f \left(f^{\circ k}(z) \right)\right)\right|^{\frac{1}{n^{k+1}}} = \left|\phi \left(f^{\circ k}(z) \right)^n\right|^{\frac{1}{n^{k+1}}} = \left|\phi \left(f^{\circ k}(z) \right)\right|^{\frac{1}{n^{k}}} = |\phi|(z).$$

In the proof of the Bottcher's theorem, we saw that $\phi(z) \in \mathbb{D} \forall z \in \hat{U}$ Hence, $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right| < 1 \forall z \in \mathcal{A}$. Therefore, $|\phi| : \mathcal{A} \to [0, 1)$.

Also,

$$\begin{aligned}
|\phi|(f(z)) &= \left|\phi\left(f^{\circ k}(f(z))\right)\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f\left(f^{\circ k}(z)\right)\right)\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f^{\circ k}(z)\right)^n\right|^{\frac{1}{n^k}} \\
&= \left|\phi\left(f^{\circ k}(z)\right)\right|^{\frac{n}{n^k}} \\
&= |\phi|(z)^n.
\end{aligned}$$

It is also clear that $|\phi| = 0$ only at p and its iterated preimages.

If q is an iterated preimage of p, say $f^{\circ k}(q) = p$, then we have $|\phi|(q) = \left|\phi\left(f^{\circ k}(q)\right)\right|^{\frac{1}{n^k}} = |\phi(p)|^{\frac{1}{n^k}} = 0$.

Now, Suppose $|\phi|(z) = 0$ for some z. Then, $|\phi|(z)^{n^k} = 0 \forall k \implies |\phi|\left(f^{\circ k}(z)\right) = 0 \forall k$. But for some large $k, f^{\circ k}(z)$ belongs to the domain of definition of ϕ. But that means, $f^{\circ k}(z) = p$, since no other point in that domain is mapped to zero by ϕ. Hence, z is an an iterated preimage of p.

Now, since $f^{\circ k} \to p$ locally uniformly in $\mathcal{A}$, for each $a \in \mathcal{A}$, we have a neighbourhood W_a and a constant $k \in \mathbb{N}$ such that $f^{\circ k}(z) \in \hat{U} \forall z \in W_a$.

Hence, for $z \in W_a$, we can define $|\phi|(z) = \left|\phi\left(f^{\circ k}(z)\right)\right| = |g(z)|$, where $g = \phi \circ f^{\circ k}\big|_{W_a}$. Therefore, $|\phi|_{W_a} = |g|$, where g is some holomorphic function defined on W_a.

It is clear from this that $|\phi|$ is continuous in $\mathcal{A}$.

Now, if h is any holomorphic function, then $|h(z)|$ is real-analytic everywhere in its domain except at those z, where $h(z) = 0$.

Since, $|g| = |\phi|_{W_a}$ is zero only at the iterated preimages of f in $W_a, |\phi|_{W_a}$ is real analytic everywhere in W_a except at the iterated preimages of p.

Therefore, $|\phi|$ is real analytic everywhere in $\mathcal{A}$ except at the iterated preimages of p. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational map with a super-attracting fixed point p. Then the associated Bottcher map ϕ carries a neighbourhood of p biholomorphically onto a neighbourhood of zero, conjugating f to the nth power map, where n is the local degree of f near p. ϕ has a local inverse ψ_ϵ which maps the ϵ-disc around zero to a neighbourhood of p. $\square$

Theorem 3.2.2 (Extending ψ_ϵ). *There exists a unique open disc of maximal radius $0 < r \leq 1$ such that ψ_ϵ extends holomorphically to a map $\psi : \mathbb{D}_r \to \mathcal{A}$, where $\mathcal{A}$ is the immediate basin of attraction of p.*

1. *If $r = 1$, then ψ maps the open unit disc $\mathbb{D}$ onto $\mathcal{A}$ biholomorphically.*

2. *If $0 < r < 1$, then ψ maps $\mathbb{D}_r$ onto its image biholomorphically and there exists atleast one other critical point in $\mathcal{A}$ on the boundary of $\psi(\mathbb{D}_r)$.*

If ψ_ϵ is extended biholomorphically in this way to the map ψ defined on $\mathbb{D}_r$, then the inverse map $\psi^{-1} : \psi(\mathbb{D}_r) \to \mathbb{D}_r$ must be the extension of ϕ from some neighbourhood of p to $\psi(\mathbb{D}_r)$ (since ψ^{-1} agrees with ϕ on some neighbourhood of p).

Proof. Let us try to extend ψ_ϵ along radial lines by analytic continuation. Then, we can't extend it indefinitely as it would yeild a holomorphic map ψ from the entire complex plane onto an open set $\psi(\mathbb{C}) \subset \mathcal{A} \subsetneq \mathbb{C}_\infty$. ($\mathcal{A}$ cannot be the whole of $\mathbb{C}_\infty$ since the Julia set of f cannot be empty as $\deg(f) \geq 2$). We can conjugate f such that $\infty \notin \mathcal{A}$. Then the corresponding map ψ will map the whole of the complex plane into $\mathcal{A} \subset \mathbb{C}$. By Louiville's theorem, since the map ψ cannot be a constant, $\psi(\mathbb{C}) = \mathbb{C} = \mathcal{A}$. Therefore, $\mathbb{C}_\infty \backslash \mathcal{A} = \{\infty\}$. This too is not possible since the Julia set of f must be an infinite set since $\deg(f) \geq 2$.

Thus, there must be some largest radius r so that ψ_ϵ extends analytically throughout the open disc $\mathbb{D}_r$.

Also, $|\phi|(\psi(w)) = |\phi(\psi(w))| = |w|$ near 0 , hence for all $w \in \mathbb{D}_r$ by analytic continuation.???

Since, $|\phi| : \mathcal{A} \to [0, 1)$, this proves that for any $w \in \mathbb{D}_r, |\phi|(\psi(w)) = |w| < 1$. Therefore, ψ can be defined only on $\mathbb{D}_r$ for $r \leq 1$.

We will now show that ψ is actually one-to-one on $\mathbb{D}_r$. Suppose $\psi(w_1) = \psi(w_2)$. Applying $|\phi|$, we see that $|w_1| = |w_2|$. Choose such a pair such that $\psi(w_1) = \psi(w_2)$ $(w_1 \neq w_2)$ with $|w_1| = |w_2|$ minimal. A minimal pair exists because for $|w| < \epsilon, \psi = \psi_\epsilon$ which is one-to-one as it is invertible.

Now, ψ is an open mapping. Choose a sufficiently small neighbourhood U_{w_2} of w_2. Then, $\psi(U_{w_2})$ is a small neighbourhood of $\psi(w_1) = \psi(w_2)$. Hence, for any w_1' sufficiently close to $w_1, \psi(w_1') \in \psi(U_{w_2})$. Hence, we can find w_2' sufficiently close to w_2 such that $\psi(w_1') = \psi(w_2')$. Choosing $|w_1'| < |w_1|$, we get a contradiction.

Hence, ψ maps $\mathbb{D}_r$ onto its image biholomorphically.

In case when $r = 1, U = \psi(\mathbb{D}) = \mathcal{A}$. If not then we would have some boundary point of U, say $z_0 \in \mathcal{A}$. We can approximate z_0 by points of $\psi(w_j)$, where $|w_j| \to 1$.

Now, $\lim_{j \to \infty} \psi(w_j) = z_0$. Hence,

$$\lim_{j\to\infty} |\phi| (\psi (w_j)) = |\phi| (z_0) \implies \lim_{j\to\infty} |w_j| = |\phi| (z_0) \implies |\phi| (z_0) = 1$$

which is impossible.

Now, let $0 < r < 1$. We need to prove that ∂U, where $U = \psi (\mathbb{D}_r)$ must contain a critical point of f. Suppose, $w_0 \in \partial \mathbb{D}_r$ and let $(w_j)_{j=1}^{\infty} \subset \mathbb{D}_r$ such that $w_j \to w_0$. Let $\psi (w_j) \to z_0$. Then $z_0 \in \partial U$ because ψ maps $\mathbb{D}_r$ onto U biholomorphically.

If z_0 is not a critical point of f, then f maps a neighbourhood of z_0, say A onto a neighbourhood of $f(z_0)$, say B biholomorphically.

It should be noted that B can be chosen such that $B \subset U$. This is because $f(z_0) \in U$. We have, $\lim_{j\to\infty} \psi (w_j) = z_0 \implies \lim_{j\to\infty} f (\psi (w_j)) = f (z_0) \implies \lim_{j\to\infty} \psi \left(w_j^n\right) = f (z_0) \implies \psi (w_0^n) = f (z_0)$. Since, $|w_0| = r < 1, |w_0|^n < r^n < r$. Hence, $w_0 \in \mathbb{D}_r$. Therefore, $\psi (w_0^n) = f (z_0) \in U$

Let g be the local inverse of f near $f(z_0)$. Then, ψ can be extended throughout a neighbourhood of w_0 by

$$w \mapsto g (\psi (w^n))$$

We have, $\psi (w_0^n) = f (z_0) \implies w_0^n = \phi (f (z_0))$. Since, $\phi(B)$ is a neighbourhood of $\phi (f (z_0))$ lying inside $\mathbb{D}_r$, choose a small enough neighbourhood of w_0, say C such that $w^n \in \phi(B)$, for all $w \in C$. In this neighbourhood, C our newly defined map agrees with ψ on $C \cap \mathbb{D}_r$. This is because, for $w \in C \cap \mathbb{D}_r, f(\psi(w)) = \psi (w^n) \in B$. Therefore, $g (\psi (w^n))$ can be defined and $\psi(w) = g (\psi (w^n)) \in A$. Hence, our new map is an analytic continuation of ψ on the neighbourhood C.

Now, if none of the $z_0 \in \partial U$ are critical points, we can extend ψ to a neighbourhood of $w_0 \forall w_0 \in \partial \mathbb{D}_r$. Clearly, these continutations would patch together to define ψ in a strictly greater disc than $\mathbb{D}_r$, which is a contradiction. $\square$

Chapter 4

Introduction to the Mandelbrot Set

We consider the set of quadratic polynomials $\{f_c(z) = z^2 + c : c \in \mathbb{C}\}$. It is enough to consider this set because every quadratic polynomial is linearly conjugate to a quadratic polynomial of the type $f_c(z)$ for some unique $c \in \mathbb{C}$.

To prove this, let $f(z) = pz^2 + qz + r$, $p \neq 0$. Consider the conjugation with the map, $\sigma(z) = az + b$ with $a = p$ and $b = q/2$. Then, we have

$$\begin{aligned}
\sigma f \sigma^{-1}(z) &= a\left(p\left(\frac{z-b}{a}\right)^2 + q\left(\frac{z-b}{a}\right) + r\right) + b \\
&= (z-b)^2 + q(z-b) + ar + b \\
&= z^2 + b^2 - qb + ar + b \\
&= z^2 + q^2/4 - q^2/2 + pr + q/2 \\
&= z^2 + (pr + q/2 - q^2/4).
\end{aligned}$$

This is of the form $z^2 + c$ for $c = pr + q/2 - q^2/4$.

4.1 Definition of the Mandelbrot Set

First, we define as new type of set, known as the *Filled-in Julia Set* for a polynomial P.

Definition 4.1.1 (Filled-in Julia Set). *The Filled-in Julia Set of a polynomial P is defined as $K(P) = \hat{\mathbb{C}} \backslash F_\infty(P)$. It is the union of the Julia set and the bounded Fatou components. It is denoted by $K(P)$ or simply K when the context is clear.*

By Lemma 1.4.4, K can also be defined as follows:

$$K = \{z \in \mathbb{C} : P^{\circ n}(z) \text{ is bounded}\}.$$

Notation. We will use F_c, J_c and K_c for the $F_\infty(f_c)$, $J(f_c)$ and $K(f_c)$ respectively.

Definition 4.1.2 (Mandelbrot Set). *The Mandelbrot Set is defined as*

$$M = \{c \in \mathbb{C} : K_c \textit{ is connected}\}.$$

Now, we have the following two theorems for polynomials:

- For polynomials, since F_∞ is a completely invariant Fatou component (by Theorem 1.3.5), $\partial F_\infty = J$ (by Theorem 1.4.2).
- And, from Theorem 1.4.1, we have that F_∞ is simply connected $\iff$ $\hat{\mathbb{C}} \backslash F_\infty$ is connected $\iff$ ∂F_∞ is connected.

Thus, for a polynomial,

$$F_\infty \text{ is simply connected } \iff K \text{ is connected } \iff J \text{ is connected.}$$

Hence, we have the following equivalent descriptions for the Mandelbrot Set:

$$\begin{aligned} M &= \{c \in \mathbb{C} : K_c \text{ is connected}\} \\ &= \{c \in \mathbb{C} : F_c \text{ is simply connected}\} \\ &= \{c \in \mathbb{C} : J_c \text{ is connected}\}. \end{aligned}$$

4.2 The Fundamental Dichotomy

Theorem 4.2.1. *For a polynomial P, the following are equivalent:*

1. *F_∞ is simply connected $\iff$ J is connected $\iff$ K is connected.*
2. *There are no finite critical points of P in F_∞.*

Proof. First assume that F_∞ is simply connected $\implies$ $c(F_\infty) = 1$ and hence, $\chi(F_\infty) = 2 - c(F_\infty) = 1$. Now, since F_∞ is completely invariant and P is a polynomial of degree d (say), P is a d-fold map of F_∞ onto itself. Applying the Riemann-Hurwitz relation to the map P of F_∞ onto itself, we obtain,

$$\begin{aligned} &\chi(F_\infty) + \delta_P(F_\infty) = d\,\chi(F_\infty) \\ \implies &1 + \delta_P(F_\infty) = d \\ \implies &\delta_P(F_\infty) = d - 1. \end{aligned}$$

Now, $\delta_P(\infty) = d - 1$ and therefore, P does not have any finite critical points in F_∞.

For the converse part, assume there are no critical points of P in F_∞. Then, the Bottcher's map ϕ which conjugates P to the map, $z \mapsto z^d$ can be extended to the whole of F_∞ and $\phi : F_\infty \to \mathbb{D}$ is a biholomorphism. Hence, F_∞ is simply connected. ☐

Now, quadratic maps have only one finite critical point and f_c have the critical point at 0 for all $c \in \mathbb{C}$. Hence, by the Fundamental Dichotomy, F_c is simply connected $\iff$ $0 \notin F_c$ or $0 \in K_c$. Using, c_n to denote $f_c^{\circ n}(0)$, we get,

$$\begin{aligned} M &= \{c \in \mathbb{C} : 0 \in K_c\} \\ &= \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}. \end{aligned}$$

Note that $c_0 = 0$ and $c_1 = f_c(0) = c$. So, (c_n) is also the forward orbit of c. Hence, in other words, the Mandelbrot Set consists of $c \in \mathbb{C}$ such that its forward orbit under the map f_c remains bounded.

4.3 The other end of the Dichotomy Theorem

By the Dichotomy Theorem, we can say that if even one finite critical point of P lies in F_∞, then K cannot be connected. But this theorem states that if all finite critical points of P lie in F_∞, then K is not only disconnected, but totally disconnected.

Definition 4.3.1 (Cantor set). *A subset $X \subset \hat{\mathbb{C}}$ is called a Cantor set if it is non-empty, closed, perfect and totally disconnected.*

Theorem 4.3.1. *Let R be a rational map with $\deg(R) \geq 2$. Let α be a super-attracting fixed point of R. If the Fatou component of R containing α, say F_α, contains all the critical points of R, then $J(R)$ is a Cantor set.*

Corollary 4.3.1.1. *If $c \notin M$, then J_c is a Cantor set.*

Proof. Since 0 is the only critical points of f_c (apart from ∞), if it belongs to $F_c = F_\infty(f_c)$, i.e. if $0 \notin K_c \iff c \notin M$, then J_c is a Cantor set. □

4.4 Some properties of the Mandelbrot Set

We know, $M = \{c \in \mathbb{C} : (c_n) \text{ is bounded}\}$. This description for M can be strengthened significantly by the following theorem:

Theorem 4.4.1. $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Proof. Obviously, $\{c \in \mathbb{C} : |c_n| \leq 2\} \subseteq M$.

Now, Suppose that $c \in M$. We need to prove that $|f_c^{\circ n}(c)| = |c_n| \leq 2$ for all $n \in \mathbb{N}$. Consider the set $W_c = \{z \in \mathbb{C} : |z| \geq |c|, |z| > 2\}$. For $z \in W_c$,

$$|f_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| \geq |z|(|z| - 1) = |z|(1 + \epsilon)$$

for some $\epsilon > 0$ (as $|z| > 2$). Clearly, $|f_c^{\circ n}(z)| \geq |z|(1+\epsilon)^n \implies z \notin K_c$.
This implies $|c| \leq 2$. Consequently, $|f_c^{\circ n}(c)| \leq 2$ for all $n \in \mathbb{N}$.
Hence, $M \subseteq \{c \in \mathbb{C} : |c_n| \leq 2\}$.

Therefore, $M = \{c \in \mathbb{C} : |c_n| \leq 2\}$. □

As $c_1 = c$, we have that $|c| \leq 2$ for all $c \in M$ i.e $M \subseteq \{c \in \mathbb{C} : |c| \leq 2\}$. This turns out to be the strongest bound possible for $|c|$ as $-2 \in M$. The orbit of 0 under the map $z \mapsto z^2 - 2$ is:

$$0 \mapsto -2 \mapsto 2 \mapsto 2$$

and hence is bounded.

Theorem 4.4.2. *The Mandelbrot set M is compact and $\hat{\mathbb{C}} \backslash M$ is open and connected.*

Proof. Let, $c_n = f_c^{\circ n}(c) = Q_n(c)$ be a polynomial in c. Clearly, from Theorem 4.4.1

$$M = \cap_{n=1}^{\infty} Q_n^{-1}(\overline{\mathbb{D}_2}),$$

where $\overline{\mathbb{D}_2} = \{z \in \mathbb{C} : |z| \leq 2\}$. Thus, M is closed. It is already known that it is bounded. Hence, it is compact.

Now,

$$\hat{\mathbb{C}} \backslash M = \cup_{n=1}^{\infty} Q_n^{-1}(E)$$

where $E = \hat{\mathbb{C}} \backslash \overline{\mathbb{D}_2}$. Now, E is open and connected and since, Q_n are non-constant polynomials, $Q_n^{-1}(E)$ is open and connected for all $n \in \mathbb{N}$. Also, each one of them contains ∞ and hence, their union is also open and connected.
Therefore, $\hat{\mathbb{C}} \backslash M$ is open and connected.

□

4.5 Plotting the Mandelbrot Set

Theorem 4.4.1 is also used to plot the Mandelbrot Set. A simple code in python would

Chapter 5

Connectedness of the Mandelbrot Set

In the previous chapter, we proved that the Mandelbrot set is compact and $\mathbb{C}\backslash M$ is open and connected. In this chapter, we will prove that $\hat{\mathbb{C}}\backslash M$ is biholomorphic to $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$, proving that $\hat{\mathbb{C}}\backslash M$ is simply connected, thus implying that M is connected by Theorem 1.4.1.

For $f_c(z) = z^2 + c$, Bottcher's theorem and its extension guarantees the existence of a unique biholomorphic function ϕ_c defined on a simply-connected neighbourhood of infinity, say $U_c \subseteq F_c$ (where F_c is the basin of attraction of the super-attracting fixed point ∞), which conjugates f_c to the map $z \mapsto z^2$ and $\phi_c(z)/z \to 1$ as $z \to \infty$.
Furthermore, if $c \in M$, then $U_c = F_c$ and $\phi_c(U_c) = \mathbb{C}\backslash\overline{\mathbb{D}}$.
If $c \notin M$, then $U_c \subsetneq F_c$, ∂U_c contains the critical point 0 and $\phi_c(U_c) = \mathbb{C}\backslash\overline{\mathbb{D}}_r$, where $r > 1$.

5.1 The Green's Function

Definition 5.1.1 (Green's Function). *A continuous function $G : \mathbb{C} \to \mathbb{R}$ is called the potential theoretical Green's function of a compact set $K \subset \mathbb{C}$, if G is harmonic outside K, vanishing on K and has the property that $G(z)/\log|z| \to 1$ as $|z| \to \infty$.*

We know that $z \mapsto |\phi_c(z)|$ can be extended to a continuous function $|\phi_c| : F_c \to (1, \infty)$. (Note that since for polynomials, $P^{-1}(\infty) = \{\infty\}$, $|\phi_c|$ is finite everywhere on F_c).
In practice, it is customary to work with the logarithm of $|\phi_c|$. Hence define,

$$G_c(z) = \begin{cases} \log|\phi_c|(z) & \text{if } z \in F_c \\ 0 & \text{if } z \in K_c. \end{cases}$$

Clearly, $G_c : \mathbb{C} \to [0, \infty)$ and $G_c(z) > 0$ on F_c as $|\phi_c| > 1$ on F_c. Also, note that G satisfies the functional equation, $G_c(f(z)) = 2G_c(z)$. Also, it can be proven that G_c is

harmonic on F_c and hence, G_c is indeed the Green's function for K_c.
Now,

- If $c \in M$, then $U_c = F_c$. Since $G_c(z) > 0$ for all $z \in F_c$ and $G_c(0) = 0$ as $0 \in K_c$, we can say that $U_c = F_c = \{z \in \mathbb{C} : G_c(z) > G_c(0) = 0\}$.

- If $c \notin M$, the $U_c \subsetneq F_c$. From the maximum principle, it is easy to see that minimum of $|\phi_c|$ on $\overline{U_c}$ lies of on ∂U_c. But, $|\phi_c|(\partial U_c) = r =$ constant and since, $0 \in \partial U_c$, we have $|\phi_c|(z) > |\phi_c|(0)$ for all $z \in U_c$. Therefore, $U_c = \{z \in \mathbb{C} : G_c(z) > G_c(0)\}$.

Therefore, $U_c = \{z \in \mathbb{C} : G_c(z) > G_c(0)\}$.

Lemma 5.1.1. *The map $(z, c) \mapsto G_c(z)$ is jointly continuous in two variables on $\mathbb{C}^2$.*

Lemma 5.1.2. *The map $\Phi(z, c) = \phi_c(z)$ is holomorphic in two variables on the set $S = \{(z, c) : c \in \mathbb{C} \backslash M, G_c(z) > G_c(0)\}$ and $\Phi(c, c)/c \to 1$ as $c \to \infty$.*

Proof. First we note that S is an open set in $\mathbb{C}^2$. This is due to the map $(z, c) \mapsto G_c(z)$ being jointly continuous. For a map to be holomorphic in two variables, it should be holomorphic in each variable when the other variable is kept fixed.

Now, we already know that for a fixed $c \in \mathbb{C} \backslash M$, the map $z \mapsto \phi_c(z)$ is holomorophic on $\{z \in \mathbb{C} : G_c(z) > G_c(0)\}$. But, we still need to prove that for a fixed z, the map $c \mapsto \phi_c(z)$ is holomorphic on the "z-slice" of S, i.e. $S_z = \{c \in \mathbb{C} \backslash M : G_c(z) > G_c(0)\}$.
We define, $\Phi_n(z, c) = \phi_{c,n}(z) = (f_c^{\circ n}(z))^{\frac{1}{2^n}}$, where $\phi_{c,n}$ are as defined in the proof of the Bottcher's theorem.
(In the proof of Bottcher's theorem, we had defined $\phi_n(z) = (f^{\circ n}(z))^{\frac{1}{2^n}}$ in a neighbourhood of the super-attracting fixed point. Here, we are defining an analytic 2^n-th root $f_c^{\circ n}$ throughout U_c (which is simply connected), which agrees with ϕ_n defined on the neighbourhood of the super-attracting fixed point.)
We write,

$$\Phi_n(z, c) = z \prod_{k=0}^{n-1} \frac{\Phi_{k+1}(z, c)}{\Phi_k(z, c)}.$$

Now,

$$\frac{\Phi_{n+1}(z, c)}{\Phi_n(z, c)} = \left(\frac{f_c^{\circ n+1}(z)}{(f_c^{\circ n}(z))^2} \right)^{\frac{1}{2^{n+1}}} = \left(1 + \frac{c}{(f_c^{\circ n}(z))^2} \right)^{\frac{1}{2^{n+1}}}.$$

and we write Φ as the infinite product,

$$\Phi(z, c) = z \prod_{n=0}^{\infty} \frac{\Phi_{n+1}(z, c)}{\Phi_n(z, c)} = z \prod_{n=0}^{\infty} \left(1 + \frac{c}{(f_c^{\circ n}(z))^2} \right)^{\frac{1}{2^{n+1}}}.$$

By Weierstrass Factorization Theorem (Conway Pg. 167), if $(f_n)_{n=1}^{\infty}$ is a sequence of analytic functions on $G \subset \mathbb{C}$ then $\prod_{n=1}^{\infty} f_n(z)$ is analytic if $\sum(f_n(z) - 1)$ converges absolutely and uniformly on compact subsets of G. Hence, let

$$\left(1+\frac{c}{(f_c^{\circ n}(z))^2}\right)^{\frac{1}{2^{n+1}}} = 1+\theta_n(z,c).$$

Select a fixed z_0 ($G_{c_0}(z_0) > G_{c_0}(0)$ for some $c_0 \in \mathbb{C}\backslash M$).
In order to prove that $c \mapsto \Phi(z_0, c)$ is analytic on S_{z_0}, we need to prove that $\theta_n(z_0,\cdot)$ converges uniformly and absolutely on compact subsets of S_{z_0}. Let K be a compact subset of S_{z_0}.

Claim. *There exists $N \in \mathbb{N}$ large enough so that for all $c \in K$, $|f_c^{\circ n}(z_0)|^2 > 2|c|$ for all $n \geq N$.*

Proof. For each $c \in K$, we have $n_c \in \mathbb{N}$ such that $|f_c^{\circ n}(z_0)|^2 > 2|c|$ for all $n \geq n_c$. (Note that if $|z| > 2$ and $|z|^2 > 2|c|$, then $|f_c^{\circ n}(z)|^2 > 2|c|$ for all $n \in \mathbb{N}$. Take n_c large enough such that $|f_c^{\circ n_c}(z_0)| > 2$ and $|f_c^{\circ n_c}(z_0)|^2 > 2|c|$.) Suppose, we have such a $\hat{c} \in K$ and correspondingly $n_{\hat{c}} \in \mathbb{N}$. Then there exists a neighbourhood of $\hat{c}$, say $B_{\hat{c}}$, such that for all $c \in B_{\hat{c}}$, $|f_c^{\circ n}(z_0)|^2 > 2|c|$ for all $n \geq n_{\hat{c}}$. This is because, $f_c^{\circ n_{\hat{c}}}(z_0)$ is a continuous function in c and so is $(f_c^{\circ n_{\hat{c}}}(z_0))^2/c$. Now, cover the compact set K by all such neighbourhoods, take a finite subcover and take N as the maximum of all such n_c obtained from this finite subcover. □

Claim. *For $|w| < \frac{1}{2}$, $|(1+w)^{\frac{1}{k}} - 1| \leq 2|w|/k$.*

Proof. □

Now, for $c \in K$ and $n \geq N$, $\frac{|c|}{|f_c^{\circ n}(z_0)|^2} < \frac{1}{2}$. Hence, from the above inequality,

$$|\theta_n(z_0,c)| \leq \frac{2|c|}{2^{n+1}|f_c^{\circ n}(z_0)|^2} < \frac{1}{2^{n+1}}.$$

Hence, $\sum_{n=1}^{\infty} \theta_n(z_0, c)$ converges absolutely and uniformly on K. Therefore, $\Phi(z_0, c)$ is analytic on S_{z_0}. Therefore, $\Phi(z, c)$ is analytic in two variables on the set $S = \{(z,c) : c \in \mathbb{C}\backslash M, G_c(z) > G_c(0)\}$.

Now, for the second part, recall that for $c \in M$, $|c| \leq 2$. Thus, if $|c| > 2$, $c \in \mathbb{C}\backslash M$. We have,

$$\Phi(c,c)/c = \prod_{n=0}^{\infty}\left(1+\frac{c}{(f_c^{\circ n}(c))^2}\right)^{\frac{1}{2^{n+1}}}.$$

□

5.2 The Isomorphism by Douady and Hubbard

Douady and Hubbard proved that the Mandelbrot set is connected by defining an isomorphism $\Psi : \mathbb{C}\backslash M \to \mathbb{C}\backslash\overline{\mathbb{D}}$ given by,

$$\Psi(c) = \Phi(c, c) = \phi_c(c).$$

To prove that it is indeed an isomorphism, we will proceed in the following steps:

1. Ψ is a well defined map: For $c \in \mathbb{C}\backslash M$, $G_c(c) = 2G_c(0) > G_c(0)$. Thus, $c \in S_c$ and $\phi_c(c)$ can be defined. Also, $|\phi_c(c)| > 1$.

2. Ψ is analytic: We already proved that $(z, c) \mapsto \Phi(z, c)$ is analytic in two variables. Hence the map $c \mapsto (c, c) \mapsto \Phi(c, c)$ is analytic.

3. $|\Psi(c_n)| \to 1$ as $c_n \to M$: This is due to the continuity of $(z, c) \mapsto G_c(z)$ in two variables. Hence, $c \mapsto (c, c) \mapsto G_c(c)$ is continuous. Hence, as $c_n \to c_0 \in M$, $G_{c_n}(c_n) \to G_{c_0}(c_0) = 0$. Hence, $\log|\phi_{c_n}|(c_n) = \log|\phi_{c_n}(c_n)| \to 0 \implies |\phi_{c_n}(c_n)| \to 1$.

4. Ψ can be extended to an analytic map $\Psi : \hat{\mathbb{C}}\backslash M \to \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ by defining $\Psi(\infty) = \infty$: This is due to the fact that $\Psi(c)/c = \Phi(c, c)/c \to 1$ as $c \to \infty$.

5. This extension is a proper map: Let K be a compact subset of $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$. Clearly, $\Psi^{-1}(K)$ is a closed subset of $\hat{\mathbb{C}}\backslash M$. If $\Psi^{-1}(K)$ is not compact, there is a sequence, $(c_n)_{n=1}^{\infty} \subset \Psi^{-1}(K)$ such that $c_n \to M$. This implies $|\Psi(c_n)| \to 1$ by point 3. This is not possible as K being a compact subset of $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ is at a positive distance from $\mathbb{D}$.

6. Now, Ψ being a proper holomorphic map, it is a branched covering of some degree d. As $\Psi^{-1}(\infty) = \{\infty\}$ with multiplicity 1 (because $\Psi(c)/c \to 1$ as $c \to \infty$), $d = 1$.

Therefore, $\Psi : \hat{\mathbb{C}}\backslash M \to \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ is an isomorphism and M is connected. Consecutively, $\Psi : \mathbb{C}\backslash M \to \mathbb{C}\backslash\overline{\mathbb{D}}$ is also an isomorphism.