

# Option pricing techniques: A performance-based comparative study of the randomized quasi-Monte Carlo method and Fourier cosine method

Mohammad Rafiqul Islam<sup>a,1,\*</sup>

<sup>a</sup>*Department of Mathematics 1017 Academic Way Tallahassee FL 32304*

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## Abstract

Pricing financial derivatives such as options with desired accuracy can be hard due to the nature of the functions and complicated integrals required by the pricing techniques. In this paper we investigate the pricing methodology of the European style options using two advanced numerical methods, namely, Quasi-Monte Carlo and Fourier Cosine (COS). For the RQMC method, we use the random-start Halton sequence. We use the Black-Scholes-Merton model to measure the pricing quality of both of the methods. For the numerical results we compute the option price of the call option and we found a few reasons to prefer the RQMC method over the COS method to approximate the European style options.

*Keywords:* Option Pricing, Quasi Monte Carlo, Fourier Cosine (COS)

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## 1. Introduction

Option pricing is still an active research area in the field of financial mathematics. The price of an option highly rely on the volatility of the underlying asset and the market. There are many models in the option pricing literature but many of them do not match with the real-world situation. The very first model goes to Louis Bachelier (Sullivan and Weithers [10]), who introduced the Brownian motion and used it to model the stock price in 1900. Bachelier's model was based on the assumption that the stock price process follows normal distribution. Later it was discovered that stock price process follows log-normal distribution, rather normal (Antoniou et al. [1]). Based on this the most famous option pricing model was given by the Black-Scholes model. Again, due the nature of some real-world complicated options BS model is not sufficient (Janková [4]). Therefore, alternative methods have been developed, including the Randomized Quasi Monte Carlo (RQMC) method and the Fourier COS method.

The RQMC method is a Monte Carlo-based technique that produces more precise estimates of the option value by employing a low-discrepancy sequence of random numbers. By increasing the approximation's accuracy, this method, as compared to traditional Monte Carlo simulations, reduces error. In contrast, the Fourier COS method decomposes the option value into a sum of cosine functions using the Fourier transform. For specific types of options, such as European options, this strategy has proven to be particularly effective and precise.

The performance of the RQMC and Fourier COS methods for pricing various types of options will be compared and analyzed in this term paper. We will start by going through the Black-Scholes model and the theory underlying option pricing. The RQMC and Fourier COS approaches will next be thoroughly covered, along with their benefits and drawbacks. The accuracy and effectiveness of these methods compared to the traditional Black Scholes method for pricing options with various characteristics will next be demonstrated

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\*Corresponding author

Email address: [mislam@math.fsu.com](mailto:mislam@math.fsu.com) (Mohammad Rafiqul Islam)

numerically.

Overall, in this term paper, we provide a comprehensive analysis of the RQMC and Fourier COS methods for option pricing, with a focus on their performance in comparison to the Black-Scholes model. Our findings contribute to the growing body of literature on alternative methods for option pricing and can help inform investment decisions in real-world scenarios.

## 2. Mathematical Models and Theoretical Framework

Let  $(\Omega, \mathcal{F}, \mathbb{Q})$  be a probability space where the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  contains all the price information of the underlying asset  $S_t$  up to and including time  $t$ ,  $\mathbb{Q}$  is the risk-neutral probability measure. Under this probability measure, the Black-Scholes-Merton model follows the following SDE

$$dS_t = rS_t dt + S_t \sigma dW_t^{\mathbb{Q}} \quad (1)$$

where,  $r$  and  $\sigma$  are the drift and diffusion terms but here in this case it is the risk-free interest rate and standard deviation, respectively, and  $W$  is the standard Brownian motion. The solution to (1) can be obtained applying Itô's formula to the function  $Z_t = f(t, S_t) = \ln S_t$ .

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} (dS_t)^2 \\ &= \frac{1}{S_t} (rS_t dt + S_t \sigma dW_t^{\mathbb{Q}}) + \frac{1}{2} \left( -\frac{1}{S_t^2} \right) \sigma^2 S_t^2 dt \\ &= \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t^{\mathbb{Q}} \end{aligned} \quad (2)$$

For  $Z_0 = \ln S_0$  we have

$$S_t = S_0 \cdot \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t^{\mathbb{Q}} \right) \quad (3)$$

Since forward start options are path-dependent exotic options, therefore, equation (3) would be very useful for both of the methods used to price these options.

### 2.1. Fourier Cosine Method (COS)

Any function supported on  $[0, \pi]$  has the cosine expansion in the following form

$$f(\eta) = \sum_{k=1}^{\infty} \alpha_k \cdot \cos(k\eta) \quad (4)$$

where

$$\alpha_k = \frac{2}{\pi} \int_0^{\pi} f(\eta) \cos(k\eta) d\eta$$

where the notation  $\sum'$  implies that the first term in the summation is weighted by one-half (Fang and Oosterlee [3]). If the support of the function is different than  $[0, \pi]$ , say  $[a, b]$  then we use the following scale variable

$$\eta = \frac{x-a}{b-a} \pi, \quad x = \frac{b-a}{\pi} \eta + a$$

Then we have the following form

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \cdot \cos\left(k\pi \frac{x-a}{b-a}\right)$$

$$\alpha_k = \frac{2}{b-a} \int_a^b f(x) \cos\left(k\pi \frac{x-a}{b-a}\right) dx \quad (5)$$

Expectations or densities are defined on infinite support but when we express in terms of the cosine expansion we truncate the infinite support to a finite support yet we do not lose accuracy significantly due to the conditions for the existence of a Fourier transform, the integrands in densities have to decay to zero at  $\pm\infty$ . Now suppose  $[a, b] \in \mathbb{R}$  is an interval such that the truncated integral approximates the infinite integral

$$\hat{\phi}_X(\omega) = \int_a^b e^{i\omega y} f_X(y) dy \approx \int_{\mathbb{R}} e^{i\omega y} f_X(y) dy = \phi_X(\omega) \quad (6)$$

Now we use Euler formula to simplify further

$$e^{i\omega} = \cos \omega + i \sin \omega$$

and the real part of  $e^{i\omega}$  is  $\text{Re}(e^{i\omega}) = \cos \omega$ . Therefore, for any random variable  $X$ , and a constant  $a \in \mathbb{R}$  we have

$$\begin{aligned} \phi_X(\omega) e^{ia} &= \mathbb{E} \left[ e^{i(\omega X + a)} \right] \\ &= \int_{-\infty}^{\infty} e^{i(\omega y + a)} f_X(y) dy \\ \implies \text{Re}(\phi_X(\omega) e^{ia}) &= \text{Re} \left( \int_{-\infty}^{\infty} e^{i(\omega y + a)} f_X(y) dy \right) \\ &= \int_{-\infty}^{\infty} \cos(\omega y + a) f_X(y) dy \end{aligned}$$

Now we use the Fourier argument  $\omega = \frac{k\pi}{b-a}$  and multiply the characteristic function in (6) by  $e^{-i \frac{ka\pi}{b-a}}$  we obtain

$$\begin{aligned} \hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot e^{-i \frac{ka\pi}{b-a}} &= \int_a^b e^{i\left(y \frac{k\pi}{b-a} - \frac{ka\pi}{b-a}\right)} f_X(y) dy \\ \implies \text{Re} \left( \hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot e^{-i \frac{ka\pi}{b-a}} \right) &= \text{Re} \left( \int_a^b e^{i\left(y \frac{k\pi}{b-a} - \frac{ka\pi}{b-a}\right)} f_X(y) dy \right) \\ &= \int_a^b \cos\left(y \frac{k\pi}{b-a} - \frac{ka\pi}{b-a}\right) f_X(y) dy \\ &= \int_a^b \cos\left(k\pi \frac{y-a}{b-a}\right) f_X(y) dy \end{aligned}$$

Therefore, matching with equation (5)

$$\alpha_k = \frac{2}{b-a} \int_a^b \cos\left(k\pi \frac{y-a}{b-a}\right) f_X(y) dy = \frac{2}{b-a} \text{Re} \left( \hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot e^{-i \frac{ka\pi}{b-a}} \right) \quad (7)$$

So from equation (6) we have  $\alpha_k \approx F_k$  where

$$F_k = \frac{2}{b-a} \text{Re} \left( \phi_X\left(\frac{k\pi}{b-a}\right) \cdot e^{-i \frac{ka\pi}{b-a}} \right) \quad (8)$$

By replacing  $\alpha_k$  by  $F_k$  we have

$$\hat{f}_X(y) \approx \sum_{k=0}^{\infty} F_k \cos\left(k\pi \frac{y-a}{b-a}\right) \quad (9)$$

For the truncated version of equation (9) we will have

$$\hat{f}_X(y) \approx \sum_{k=0}^{N-1} F_k \cos\left(k\pi \frac{y-a}{b-a}\right) \quad (10)$$

### 2.1.1. Pricing European Options Under COS Method

We denote the random variable  $X$  as the log of the asset price, i.e.,  $X_t = \log S_t$  and consider two points  $t$  and  $T$  in  $[t, T]$  where  $t$  is the beginning of the interval and  $T$  is the maturity of the options and set the corresponding random variable as  $X_t = \log S_t = x$  and  $X_T = \log S_T = y$ . Then the risk-neutral price of the option at time  $t$  can be given by the following formula

$$P(t, x) = e^{-r(T-t)} \mathbb{E}^{\mathbb{R}}[V(T, y) | \mathcal{F}_t] = e^{-r(T-t)} \int_{\mathbb{R}} V(T, y) f_{X_T}(y) dy \quad (11)$$

where  $f_{X_T}(y)$  is the probability density function of the asset price process  $X_t$  and  $X_t$  normally distributed with parameters  $((r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ . Therefore, we can truncate the integral as follows

$$\hat{P}(t, x) = e^{-r(T-t)} \int_a^b V(T, y) f_{X_T}(y) dy \quad (12)$$

as  $f_{X_T}(y) \rightarrow 0$  when  $y \rightarrow 0$ . Now we represent the integral in equation (12) in terms of Fourier Cosine (COS) method as follow

$$\begin{aligned} \hat{P}(t, x) &= e^{-r(T-t)} \int_a^b V(T, y) \sum_{k=0}^{\infty} \alpha_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \frac{b-a}{2} e^{-r(T-t)} \sum_{k=0}^{\infty} \alpha_k(x) H_k \end{aligned} \quad (13)$$

where,

$$H_k = \frac{2}{b-a} e^{-r(T-t)} \int_a^b V(T, y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \quad (14)$$

In a similar fashion in equation (8) we can approximate  $\alpha_k$  by the Fourier coefficient  $F_k$  and we get the option price as below

$$P(t, x) \approx \hat{P}(t, x) = e^{-r(T-t)} \sum_{k=0}^{N-1} \text{Re} \left( \phi_X \left( \frac{k\pi}{b-a} \right) e^{-ik\pi \frac{a}{b-a}} \right) H_k \quad (15)$$

where,  $\phi_X(x)$  is the characteristic function of  $X$ . The next thing is to find a closed form expression of the payoff function  $H_k$  to approximate the price. As Fang and Oosterlee [3] showed in their paper, we can find that form as follows: The payoff function  $V(T, y)$  can be expressed as

$$V(T, y) = \begin{cases} K(e^y - 1) & \text{for a call option} \\ K(1 - e^y) & \text{for a put option} \end{cases}$$

Now we need to use the following two results to deal with  $H_k$ :

**Theorem 2.1.** The cosine series coefficients,  $\chi_k$ , of  $h(y) = e^y$  on an integration interval  $[c, d] \subset [a, b]$ ,

$$\begin{aligned}\chi_k(c, d) &= \int_c^d e^y \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \frac{1}{1 + \left(\frac{k\pi}{b-a}\right)^2} \left[ \cos\left(k\pi \frac{d-a}{b-a}\right) e^d - \cos\left(k\pi \frac{c-a}{b-a}\right) e^c \right. \\ &\quad \left. + \frac{k\pi}{b-a} \sin\left(k\pi \frac{d-a}{b-a}\right) e^d - \frac{k\pi}{b-a} \sin\left(k\pi \frac{c-a}{b-a}\right) e^c \right]\end{aligned}\tag{16}$$

and the cosine series coefficients,  $\psi_k$ , of  $h(y) = 1$  on an integration interval  $[c, d] \subset [a, b]$ ,

$$\begin{aligned}\psi_k(c, d) &= \int_c^d \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \begin{cases} \left[ \sin\left(k\pi \frac{d-a}{b-a}\right) - \sin\left(k\pi \frac{c-a}{b-a}\right) \right] & k \neq 0 \\ (d-c) & k = 0 \end{cases}\end{aligned}\tag{17}$$

So for a call option we have

$$\begin{aligned}H_k^{\text{call}} &= \frac{2}{b-a} \int_0^b K(e^y - 1) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a} K(\chi_k(0, b) - \psi_k(0, b))\end{aligned}\tag{18}$$

Similarly, for a put option we have

$$\begin{aligned}H_k^{\text{put}} &= \frac{2}{b-a} \int_0^b K(1 - e^y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a} K(-\chi_k(0, b) + \psi_k(0, b))\end{aligned}\tag{19}$$

The last thing we need to compute the price of an European call or put option in this COS method is the choice of integration range: on what interval we integrate so that the truncation error is not too big and the computational cost is not too much. As proposed by Fang and Oosterlee [3], the efficient interval can be given as

$$[a, b] = [(x + \zeta_1) - L\sqrt{\zeta_2 + \sqrt{\zeta_4}}, (x + \zeta_1) + L\sqrt{\zeta_2 + \sqrt{\zeta_4}}]\tag{20}$$

where,  $L \in [6, 12]$  is a number depending on a user-defined tolerance level and the variables  $\zeta_1, \dots, \zeta_4$  are the specific cumulants of the underlying stochastic process,  $x = \frac{\log S_t}{K}$ . An alternative form of the integration range was proposed by Oosterlee and Grzelak [8] as below

$$[a, b] = [-L\sqrt{T-t}, L\sqrt{T-t}]\tag{21}$$

In the comparative analysis section we apply this COS method to compare the pricing accuracy with the randomized quasi-Monte Carlo method along with the error analysis.

## 2.2. Higher Dimensional Randomized Quasi Monte Carlo Method

Pricing of a security derivative involves the computation of the discounted expected value of the payoff function. In continuous time setting, it is evaluating the integrals for the expectation. Suppose we want to get an estimate  $\theta$  for the expected payoff of a derivative. Then we can express this as follows

$$\mathbb{E}[\text{payoff}] = \theta = \int_{I^s} f(x) dx$$

where  $I^s$  is the  $s$ -dimensional unit cube. The primary benefit of Monte Carlo and quasi-Monte Carlo techniques is that they allow us to assess integrals using sums and to utilize statistical techniques to analyze the error in the context of uniformly distributed modulo 1 sequences (Ökten and Eastman [7]). That is, we can express the integrals as sums of the form

$$\int_{I^s} f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(q_n)$$

where,  $\{q_n\}_{n \geq 1}$  is  $s$ -dimensional low discrepancy sequence. The quasi-Monte Carlo method differs with regular Monte Carlo in the sense that  $q_n$  is a  $s$ -dimensional random vector for the MC method and on the contrary, it is the  $n$ th term of a  $s$ -dimensional low-discrepancy sequence for the QMC method.

We start developing the theoretical background with the definition of a uniformly distributed sequence of modulo 1

**Definition 2.1.** A sequence  $\{x_n\}_{n \geq 1} \in I = [0, 1)$  is said to be uniformly distributed modulo 1 (u.d. mod 1) if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N 1_{[a,b)}(x_n) = b - a$$

for any  $0 \leq a < b \leq 1$

We can extend the definition from one dimension to a finite dimension by the following theorem

**Theorem 2.2.** A sequence  $\{x_n\}_{n \geq 0} \in I^s = [0, 1)^s$  is u.d. mod 1 iff

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N f(x_i) = \int_{I^s} f(x) dx$$

for all Riemann integrable functions  $f$  on  $I^s$

The next thing is to know what we mean by a low discrepancy sequence. If we have a sequence  $\{x_n\}_{n \geq 1} \in I^s$  its extreme discrepancy is defined as

$$D_N(x_n) = \sup_{[a,b)} \left| \frac{1}{N} \sum 1_{[a,b)}(x_n) - \lambda_s([a,b)) \right|$$

If the supremum is taken over all the intervals of the form  $[0, a)$  then the extreme discrepancy is said to be star-discrepancy, i.e.,

$$D_N^* = \sup_{[0,a)} \left| \frac{1}{N} \sum 1_{[0,a)}(x_n) - \lambda_s([0,a)) \right|$$

The following theorem relates the star-discrepancy with u.d. mod 1 sequence.

**Theorem 2.3.**  $D_N(q_n) \rightarrow 0$  iff  $\{q_n\}_{n \geq 1}$  is a u.d. mod 1 sequence.

Now we are ready to introduce the  $s$ -dimensional randomized quasi-Monte Carlo method to evaluate integrals by the quadrature rule

$$I \approx Q(q_u) = \frac{1}{N} \sum_{n=1}^N f(q_u^{(n)}) \quad (22)$$

where  $u$  is the randomization index. Our goal is to find the expected value of  $Q(q_u)$  and reduce the variance of  $Q(q_u)$  as much as possible by generating low discrepancy sequences. By Ökten and Eastman [7], we want the discrepancy to be as small as possible, since the accuracy of our quadrature rule (22) depends on it by the Koksma-Hlawka inequality

$$|Q(q_u) - I| \leq V_{HK}(f) D_N^*(q_u) \quad (23)$$

where,  $V_{HK}(f)$  is the variation of  $f$  in the sense of Hardy and Krause (Niederreiter [6]). The inequality in (23) tells us that the more smaller variation of  $f$  and star-discrepancy the better accuracy.

There are a quite good number of low-discrepancy sequences in the literature but for our forward start option we apply mainly two low-discrepancy sequences, random-start Halton sequence and scrambled Faure sequences.

### 2.2.1. Random-start Halton Sequence

Random-start Halton sequence is based on the van der Corput and Halton sequence. For any base  $b$ , any positive integer  $n$  can be expressed as

$$n = (a_k, a_{k-1}, \dots, a_1, a_0)b = a_0 + a_1b + a_2b^2 + \dots + a_kb^k$$

then the van der Corput sequence in base  $b$  is defined as

$$\begin{aligned}\phi_b(n) &= (.a_0, a_1, \dots, a_{k-1}, a_k)b \\ &= \sum_{i=0}^k \frac{a_i}{b^{i+1}}\end{aligned}\tag{24}$$

The Halton sequence is a generalization of the van der Corput sequence to the higher dimension.

**Definition 2.2.** The  $s$ -dimensional Halton sequence in bases  $b_1, b_2, \dots, b_s$  is defined as

$$(\phi_{b_1}(n), \phi_{b_2}(n), \dots, \phi_{b_s}(n))_{n=1}^{\infty}$$

**Theorem 2.4.** The Halton sequence is u.d. mod 1 if its bases  $b_1, b_2, \dots, b_s$  are relatively prime.

The random start idea was introduced by Struckmeier [9] that relies on an alternative derivation of the Halton sequence which we call *generalized Halton Sequence* given by Lambert [5] based on von Neuman-Kakutani transformation.

If  $\mathbf{b} = (b_1, b_2, \dots, b_s)$  be a vector of positive integers that are pairwise relatively prime and we have a vector  $\mathbf{x} \in I^s$  then the  $s$ -dimensional von Neuman-Kakutani transformation is given by

$$\mathbf{T}_{\mathbf{b}}(\mathbf{x}) = (T_{b_1}(x_1), T_{b_2}(x_2), \dots, T_{b_s}(x_s))\tag{25}$$

For  $b = \frac{1}{2}$ ,  $\mathbf{T}_{\mathbf{b}}(\mathbf{x})$  is the Halton sequence in base  $b_1, b_2, \dots, b_s$  if the orbit vector  $\mathbf{x}$  is set to  $\mathbf{0}$ . Let  $q_u$  be the orbit if a randomly selected “starting values”  $\mathbf{u} \in U(0, 1)^s$  is used under  $\mathbf{T}_{\mathbf{u}}$ . Then the sequence we get will be the random-start permuted Halton sequence. So we redefine the quadrature equation (22) as follows

$$Q(q_{\mathbf{u}}) = \frac{1}{N} \sum_{i=0}^{N-1} f(\mathbf{T}_{\mathbf{b}}^i(\mathbf{u}))\tag{26}$$

By Wang and Hickernell [11], the quadrature rule is provided by the following theorem.

**Theorem 2.5.** The quadrature rules based on random-start Halton sequences satisfy

1.  $\mathbb{E}[Q(q_{\mathbf{u}})] = I$
2.  $Var(Q(q_{\mathbf{u}})) = \mathcal{O}\left(\frac{(\log N)^{2s}}{N^2}\right)$

The discrepancy for this  $q_{\mathbf{u}}^{(n)}$  sequences is given by Bouleau and Lepingue [2] (p. 77)

$$D_N(q_{\mathbf{u}}) \leq \left(1 + \prod_{i=1}^k (b_i - 1) \frac{\log(b_i N)}{\log b_i}\right)$$

### 3. Comparative Analysis

In this section we provide the performance of the both method. We show our results for only for the European style call options. However, both of the methods are designed to price both put or call options. We consider a hypothetical non-dividend paying stock with the initial price  $S_0 = 100$ , risk-free interest rate  $r = 0.1$ , volatility  $\sigma = 0.25$ , and time to maturity  $\tau = T - t = 0.1$ .

#### 3.1. COS Method Implementation

First we set the integration interval as mentioned in equation (21) and choose the parameter  $L = 10$  as mentioned in that section. We present the results for three different strike prices  $K = 80, 100$ , and  $120$ .

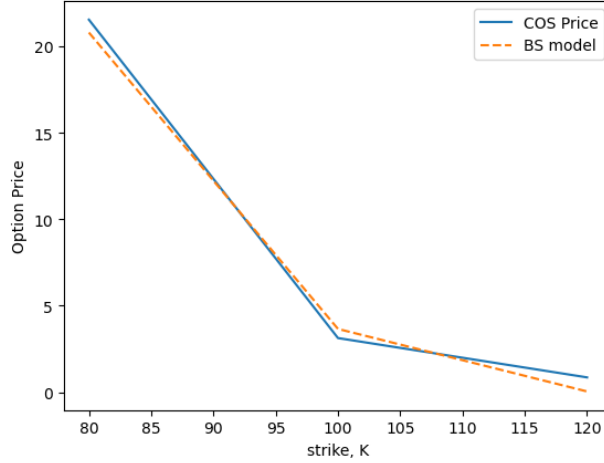


Figure 1: COS method prices are shown against the BS prices.

The figure above (1) depicts the price computed for three different strikes  $K = 80, 100$ , and  $120$  where the blue solid line represents the price computed by the BS model and the red dashed line represents the prices computed by the COS method. Next we represent the absolute error and CPU time usage for the COS method as the measure of the performance.

Table 1: Absolute error and CPU time

N	N16	N32	N64	N128
msec.	3.66e-01	3.710000e-01	4.164400e-01	4.241371e-01
abs. err. K=80	3.66e+00	7.441700e-01	1.922192e-02	1.305295e-07
abs. err. K=100	3.88e+00	5.279164e-01	1.515078e-02	3.868614e-07
abs. err. K=120	3.17e+00	8.132994e-01	2.135178e-02	3.496943e-07

As we can see from table (1), the number of point estimation  $N$  does not increase significantly. Also the errors reduces as we increase  $N$ .

#### 3.2. RQMC Method Implementation

In this section we present a similar numerical results that we provided for the COS method. For the inverse transformations, we use the box muller transformation. Here we generate  $N = 10,000$  asset price paths and estimate 40 of the option prcies for  $K = 100$  and we obtain the following figure

The mean of the 40 estimates is very close to the true mean by Black-Scholes option price for case  $K = 100$ . We run few more simulations changing  $K$  values and obtained similar close approximations. However, run-time is significantly higher for the RQMC approximations.



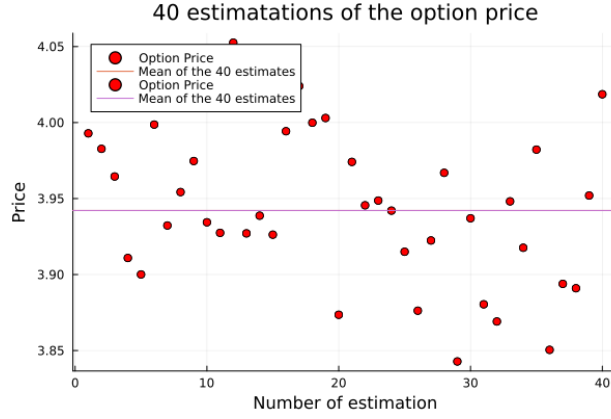


Figure 2: 40 Estimates of the option price for K=100 by RQMC method

### 3.3. Comparision

Now we present the combined summary of the numerical experiments we had so far. In table (2) and (3) we see that both of the methods works very well. However, run time is higher for the randomized quasi-Monte Carlo method, but absolute error is less than that of the COS method.

Table 2: Option Pricing Summary by COS method

K	BS	COS	COS.Abs.Err	COS.RT.msec
80	20.80	21.54	7.4e-01	0.36600
100	3.66	3.13	5.3e-01	0.37100
120	0.04	0.85	8.1e-01	0.41644

Table 3: Option Pricing Summary by RQMC method

K	BS	RQMC	RQMC.Abs.Err	RQMC.RT.sec
80	20.80	20.81	1.0e-02	8.32
100	3.66	3.73	7.0e-02	8.38
120	0.04	0.18	1.4e-01	8.69

In figure (3) we see the combined option price estimations for those three strike prices. We notice that the RQMC gives more closer approximation to the BS estimate.

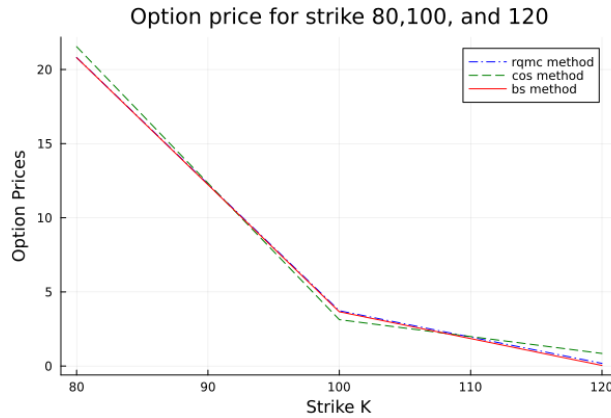


Figure 3: Price comparision of BS method, COS method, and RQMC method

## 4. Conclusion

In conclusion, we conclude that RQMC provides a better approximation for estimating the European option price. Even though it takes a bit longer to converge, still RQMC may be preferred over the COS method. Since using a large number of point estimations in the COS method provides overfitting and a too-small interval of the estimation, therefore, RQMC method may be chosen over the COS method. Moreover, the COS method requires more parameters to estimate the option price, and choosing the truncation interval parameter  $L$  may cause an increase in the error estimation.

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