Simple Linear Regression

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Table of contents

Simple Linear Regression **1**
Assumptions of Linear Regressions 4
Synthetic Data 4
Model 5

Simple Linear Regression

A simple linear regression in multiple predictors/input variables/features/independent variables/ explanatory variables/regressors/ covariates (many names) often takes the form

$$y = f(\mathbf{x}) + \epsilon = \beta\mathbf{x} + \epsilon$$

where $\beta \in \mathbb{R}^d$ are regression parameters or constant values that we aim to estimate and $\epsilon \sim \mathcal{N}(0, 1)$ is a normally distributed error term independent of x or also called the white noise.

In this case, the model:

$$y = f(x) + \epsilon = \beta_0 + \beta_1 x + \epsilon$$

Therefore, in our model we need to estimate the parameters β_0, β_1. The true relationship between the explanatory variables and the dependent variable is $y = f(x)$. But our model is $y = f(x) + \epsilon$. Here, this $f(x)$ is the working model with the data. In other words, $\hat{y} = f(x) = \hat{\beta}_0 + \hat{\beta}_1 x$. Therefore, there should be some error in the model prediction which we are calling $\epsilon = \|y - \hat{y}\|$ where y is the true value and $\hat{y}$ is the predicted value. This error term is normally distributed with mean 0 and variance 1. To get the best estimate of the parameters

β_0, β_1 we can minimize the error term as much as possible. So, we define the residual sum of squares (RSS) as:

$$RSS = \epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_{10}^2 \tag{1}$$

$$= \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \tag{2}$$

$$\hat{\updownarrow}(\bar{\beta}) = \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \tag{3}$$

$$\tag{4}$$

Using multivariate calculus we see

$$\frac{\partial l}{\partial \beta_0} = \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-1) \tag{5}$$

$$\frac{\partial l}{\partial \beta_1} = \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) \tag{6}$$

Setting the partial derivatives to zero we solve for $\hat{\beta}_0, \hat{\beta}_1$ as follows

$$\begin{aligned} & \frac{\partial l}{\partial \beta_0} = 0 \\ \implies & \sum_{i=1}^{10} y_i - 10\hat{\beta}_0 - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i \right) = 0 \\ & \implies \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \end{aligned}$$

and,

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_1} &= 0 \\
\implies \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) &= 0 \\
\implies \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(x_i) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \hat{\beta}_0 \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \left(\bar{y} - \hat{\beta}_1 \bar{x} \right) \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) + \hat{\beta}_1 \bar{x} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - \bar{x} \sum_{i=1}^{10} x_i \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - 10\bar{x}^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - 2 \times 10 \times \bar{x}^2 + 10\bar{x}^2 \right) &= 0 \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - 10\bar{x}\bar{y}}{\sum_{i=1}^{10} x_i^2 - 2 \times 10 \times \bar{x}^2 + 10\bar{x}^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - 10\bar{x}\bar{y} - 10\bar{x}\bar{y} + 10\bar{x}\bar{y}}{\sum_{i=1}^{10} x_i^2 - 2\bar{x} \times 10 \times \frac{1}{10} \sum_{i=1}^{10} x_i + 10\bar{x}^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \bar{x} \left(\sum_{i=1}^{10} y_i \right) + 10\bar{x}\bar{y}}{\sum_{i=1}^{10} (x_i - \bar{x})^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} (x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y})}{\sum_{i=1}^{10} (x_i - \bar{x})^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{10} (x_i - \bar{x})^2}
\end{aligned}
$$

Therefore, we have the following

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{10}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{10}(x_i - \bar{x})^2}$$

Simple Linear Regression `slr` is applicable for a single feature data set with contineous response variable.

```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.linear_model import LinearRegression
```

Assumptions of Linear Regressions

- **Linearity:** The relationship between the feature set and the target variable has to be linear.
- **Homoscedasticity:** The variance of the residuals has to be constant.
- **Independence:** All the observations are independent of each other.
- **Normality:** The distribution of the dependent variable y has to be normal.

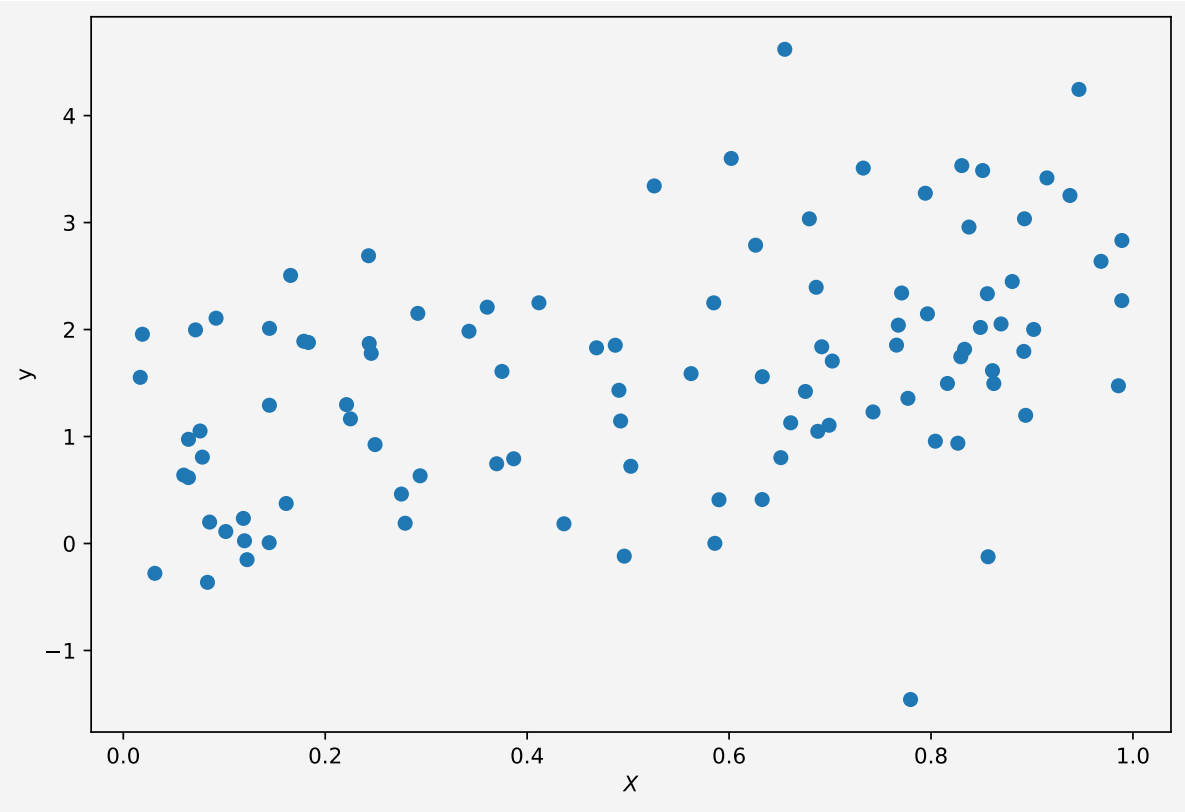
Synthetic Data

To implement the algorithm, we need some synthetic data. To generate the synthetic data we use the linear equation $y(x) = 2x + \frac{1}{2} + \xi$ where $\xi \sim \mathbf{N}(0, 1)$

```
X=np.random.random(100)
y=2*X+0.5+np.random.randn(100)
```

Note that we used two random number generators, `np.random.random(n)` and `np.random.randn(n)`. The first one generates n random numbers of values from the range (0,1) and the second one generates values from the standard normal distribution with mean 0 and variance or standard deviation 1.

```
plt.figure(figsize=(9,6))
plt.scatter(X,y)
plt.xlabel('$X$')
plt.ylabel('y')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



Model

We want to fit a simple linear regression to the above data.

```
slr=LinearRegression()
```

Now to fit our data X and y we need to reshape the input variable. Because if we look at X,

```
X
```

```
array([0.83050101, 0.62626315, 0.79437209, 0.81627154, 0.07830052,
       0.86936106, 0.16133112, 0.01891194, 0.77089663, 0.07152251,
       0.90168674, 0.60212015, 0.50265725, 0.70222296, 0.03127112,
       0.76761931, 0.24555273, 0.82949437, 0.76583182, 0.82659372,
       0.06453054, 0.29171388, 0.65518725, 0.22108747, 0.91483667,
       0.9465445 , 0.07601811, 0.11892256, 0.29397958, 0.85578244,
       0.98893608, 0.67561352, 0.77713032, 0.46872298, 0.89257687,
       0.8042952 , 0.58593213, 0.14480479, 0.12256498, 0.05988074,
       0.22495519, 0.49618183, 0.69904358, 0.68784499, 0.73288011,
       0.36981779, 0.06456356, 0.14448993, 0.86088644, 0.43644406,
       0.52587997, 0.63276179, 0.48728927, 0.98562299, 0.8565424 ,
       0.69178047, 0.08337981, 0.24357665, 0.93761474, 0.24293057,
       0.09188404, 0.16569639, 0.01677713, 0.88042867, 0.65127343,
       0.96839696, 0.10149401, 0.77973108, 0.08549146, 0.79645468,
       0.41165669, 0.67949679, 0.27915976, 0.83324678, 0.98906336,
       0.89191865, 0.58481574, 0.4909031 , 0.59000285, 0.84889002,
       0.27546194, 0.18335474, 0.14473934, 0.89375806, 0.38666234,
       0.74255536, 0.63290159, 0.8376054 , 0.37505356, 0.56235946,
       0.12004034, 0.24936629, 0.68627895, 0.85114565, 0.66100709,
       0.49255299, 0.36043815, 0.17880856, 0.34239837, 0.86222399])
```

It is a one-dimensional array/vector but the `slr` object accepts input variable as matrix or two-dimensional format.

```
X=X.reshape(-1,1)
X[:10]
```

```
array([[0.83050101],
       [0.62626315],
       [0.79437209],
       [0.81627154],
       [0.07830052],
       [0.86936106],
       [0.16133112],
       [0.01891194],
       [0.77089663],
       [0.07152251]])
```

Now we fit the data to our model

```
slr.fit(X,y)
slr.predict([[2],[3]])
```

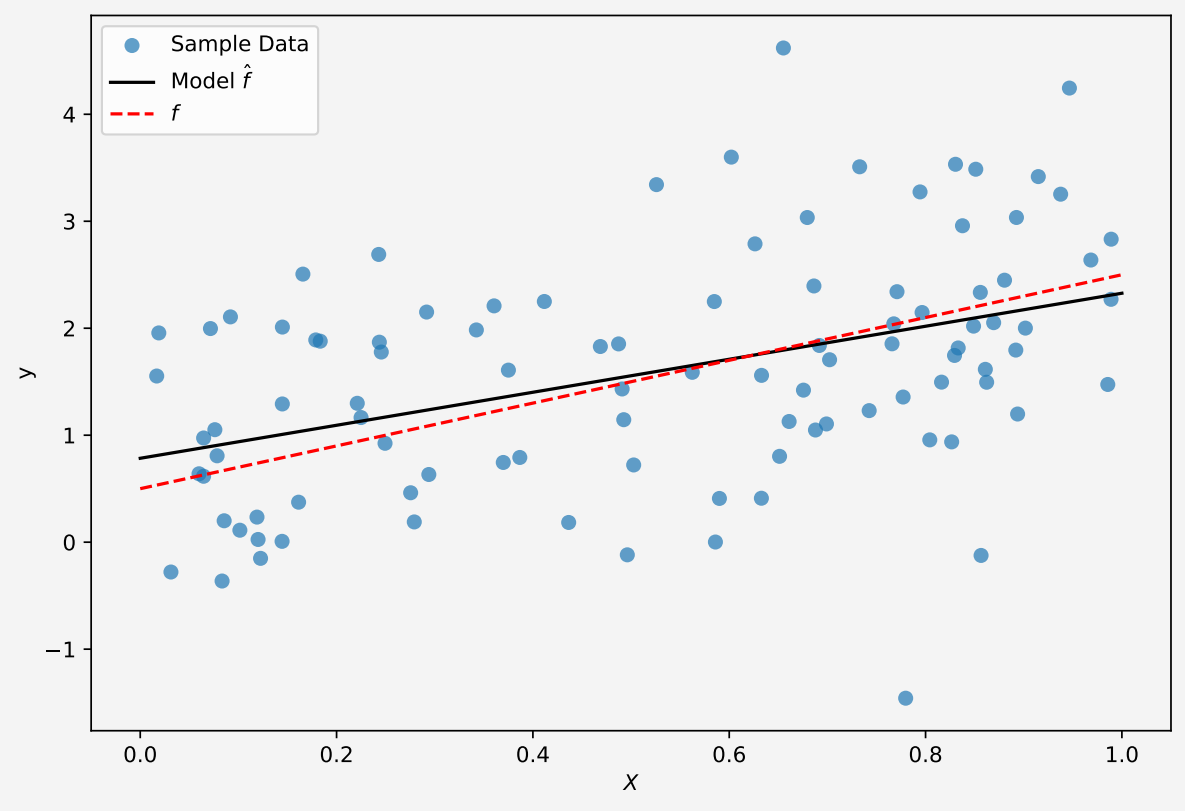
```
array([3.86993576, 5.41275763])
```

We have our $X = 2, 3$ and the corresponding y values are from the above cell output, which are pretty close to the model $y = 2x + \frac{1}{2}$.

```
intercept = round(slr.intercept_,4)
slope = slr.coef_
```

Now our model parameters are: intercept β_0 = np.float64(0.7843) and slope β_1 = array([1.54282188]).

```
plt.figure(figsize=(9,6))
plt.scatter(X,y, alpha=0.7,label="Sample Data")
plt.plot(np.linspace(0,1,100),
    slr.predict(np.linspace(0,1,100).reshape(-1,1)),
    'k',
    label='Model $\hat{f}$'
)
plt.plot(np.linspace(0,1,100),
    2*np.linspace(0,1,100)+0.5,
    'r--',
    label='$f$'
)
plt.xlabel('$X$')
plt.ylabel('y')
plt.legend(fontsize=10)
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



So the model fits the data almost perfectly.

Up next multiple linear regression.

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