Boosting Algorithm: Adaptive Boosting Method (AdaBoost)

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Table of contents

Introduction

Boosting is a powerful ensemble learning technique that focuses on improving the performance of weak learners to build a robust predictive model. Now the question is what the heck is weak learner? Well, roughly speaking, a statistical learning algorithm is called a weak learner if it is slightly better than just random guess. In contrast, a statistical learning algorithm is called a strong learner if it can be made arbitrarily close to the true value. Unlike bagging (bootstrap aggregating, e.g. random forest), which builds models independently, boosting builds models sequentially, where each new model corrects the errors of its predecessors. This approach ensures that the ensemble concentrates on the difficult-to-predict instances, making boosting highly effective for both classification and regression problems.

Key Characteristics of Boosting:

1. **Sequential Model Building:** Boosting builds one model at a time, with each model improving upon the errors of the previous one.

2. **Weight Assignment:** It assigns weights to instances, emphasizing misclassified or poorly predicted ones in subsequent iterations.
3. **Weak to Strong Learners:** The goal of boosting is to combine multiple weak learners (models slightly better than random guessing) into a strong learner.

Mathematical Visualization

Before writing the formal algorithm, let's do some math by hand. Say, we have a toy dataset:

x_1	x_2	y
1	2	1
2	1	1
3	2	-1
4	3	-1

Here:

- x_1 and x_2 are features.
- y is the target label, with values $+1$ or -1.

Now, let's apply the AdaBoost algorithm step-by-step using this dataset.

Iteration 1

Step 1: Initialize Weights
Initially, all data points are assigned equal weights:

$$w_i^{(1)} = \frac{1}{N} = \frac{1}{4} = 0.25$$

Weights: $w = [0.25, 0.25, 0.25, 0.25]$.

Step 2: Train Weak Learner

Suppose we use a decision stump (a simple decision rule) as the weak learner. The first decision stump might split on x_1 as:

- Predict $+1$ if $x_1 \leq 1.5$, otherwise -1.

$$h_1(x) = \begin{cases} +1 & \text{if } x_1 \leq 1.5 \\ -1 & \text{otherwise} \end{cases}$$

Note, that even though we are deciding based on the feature x_1, however, for $h_1(x)$ learner, x is the row from the data set, i.e. $x = [x_1, x_2]$. Therefore, for $h_1(x_1)$ would mean that, we are feeding first row to the learner h at iteration 1.

Step 3: Evaluate Weak Learner

Predictions for the dataset:

$$h_1(x) = [1, -1, -1, -1]$$

But our true labels are $[1, 1, -1, -1]$. So the error

$$\epsilon_1 = \frac{\sum_{i=1}^{4} w_i^1 \nVdash (y_i \neq h_1(x_i))}{\sum_{i=1}^{4} w_i^1}$$

where, $\nVdash$ is an indicator function that equals 1 when the prediction is incorrect and 0 otherwise. Therefore, in iteration 1:

$$\epsilon_1 = \frac{0.25(0 + 1 + 0 + 0)}{1} = 0.25$$

Step 4: Calculate α_1

$$\alpha_1 = \ln\left(\frac{1 - \epsilon_1}{\epsilon_1}\right) = 1.0986$$

Step 5: Update Weights:

For each instance:

$$w_i^{(1)} = w_i^{(1)} \cdot \exp\left(\alpha_1 \cdot y_i \cdot h_1(x_i)\right)$$

Now you may wonder how and from where we came up with this updating rule? We will explain this update process in the next post, but for now let's just focus on the update.

$$
\begin{aligned}
w_1^1 &= w_1^1 e^{\alpha_1 \mathbb{1}(y_1 \neq h_1(x_1))} = 0.25e^{1.0986\times 0} = 0.25\\
w_2^1 &= w_2^1 e^{\alpha_1 \mathbb{1}(y_1 \neq h_1(x_2))} = 0.25e^{1.0986\times 1} = 0.75\\
w_3^1 &= w_3^1 e^{\alpha_1 \mathbb{1}(y_1 \neq h_1(x_3))} = 0.25e^{1.0986\times 0} = 0.25\\
w_4^1 &= w_4^1 e^{\alpha_1 \mathbb{1}(y_1 \neq h_1(x_4))} = 0.25e^{1.0986\times 0} = 0.25
\end{aligned}
$$

Updated weights (before normalization):

$$[0.25, 0.75, 0.25, 0.25]$$

Normalize to ensure the weights sum to 1:

$$w_i^{(1)} = \frac{w_i^{(1)}}{\sum w_i^{(1)}}$$

Final normalized weights: $w = [0.17, 0.5, 0.17, 0.17]$. Notice that, for the incorrect prediction, the weight increased and for the correct prediction the weights decreased.

Iteration 2

Similarly, we proceed with second iteration with the following weak learner:

$$h_2(x) = \begin{cases} +1 & \text{if} x_2 \leq 1.5 \\ -1 & \text{otherwise} \end{cases}$$

For this learner, the prediction

$$h_2(x) = [-1, 1, -1, -1]$$

where as the actual labels are $[1, 1, -1, 1]$. So, the error

$$\epsilon_2 = \frac{0.17 \times 1 + 0.5 \times 0 + 0.17 \times 0 + 0.17 \times 0}{1} = 0.17$$

and

$$\alpha_2 = \ln\left(\frac{0.756}{0.244}\right) = 1.586$$

Next, we update the weights

$$\begin{aligned}
w_1^2 &= w_1^2 e^{\alpha_2 \mathbb{1}(y_1 \neq h_2(x_1))} = 0.17e^{1.586\times 1} = 0.83 \\
w_2^2 &= w_2^2 e^{\alpha_2 \mathbb{1}(y_1 \neq h_2(x_2))} = 0.5e^{1.1308\times 0} = 0.5 \\
w_3^2 &= w_3^2 e^{\alpha_2 \mathbb{1}(y_1 \neq h_2(x_1))} = 0.17e^{1.586\times 0} = 0.17 \\
w_4^2 &= w_4^2 e^{\alpha_2 \mathbb{1}(y_1 \neq h_2(x_1))} = 0.17e^{1.586\times 0} = 0.17
\end{aligned}$$

So, $w = [0.83, 0.5, 0.17, 0.17]$ and after normalizing $w = [0.50, 0.3, 0.10, 0.10]$. The final ensemble model combines the weak learners using their weights (α):

$$F(x) = \text{sign}\left(\alpha_1 \cdot h_1(x) + \alpha_2 \cdot h_2(x)\right)$$

For the toy dataset:

1. $\alpha_1 = 1.0986$, $h_1(x) = [1, -1, -1, -1]$
2. $\alpha_2 = 1.586$, $h_2(x) = [-1, 1, -1, -1]$

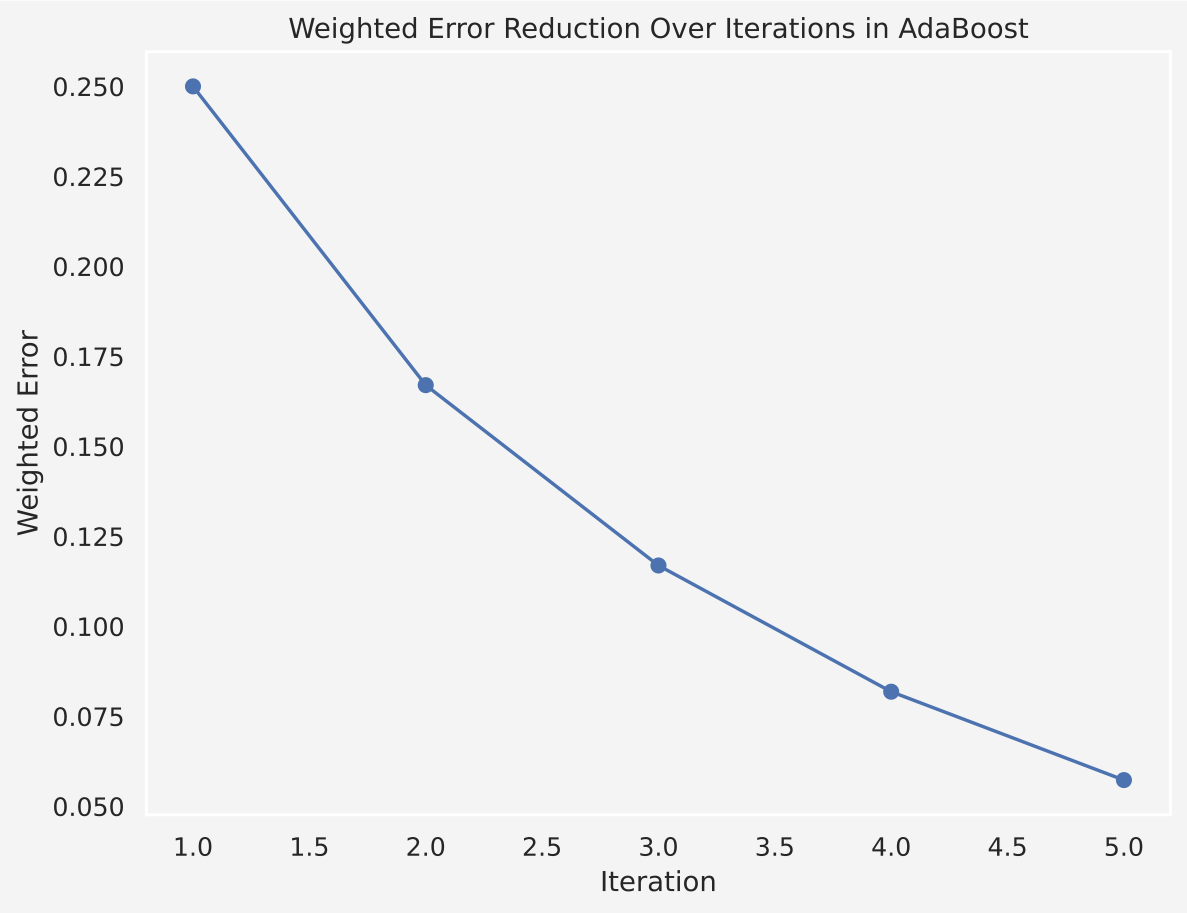
Weighted predictions:

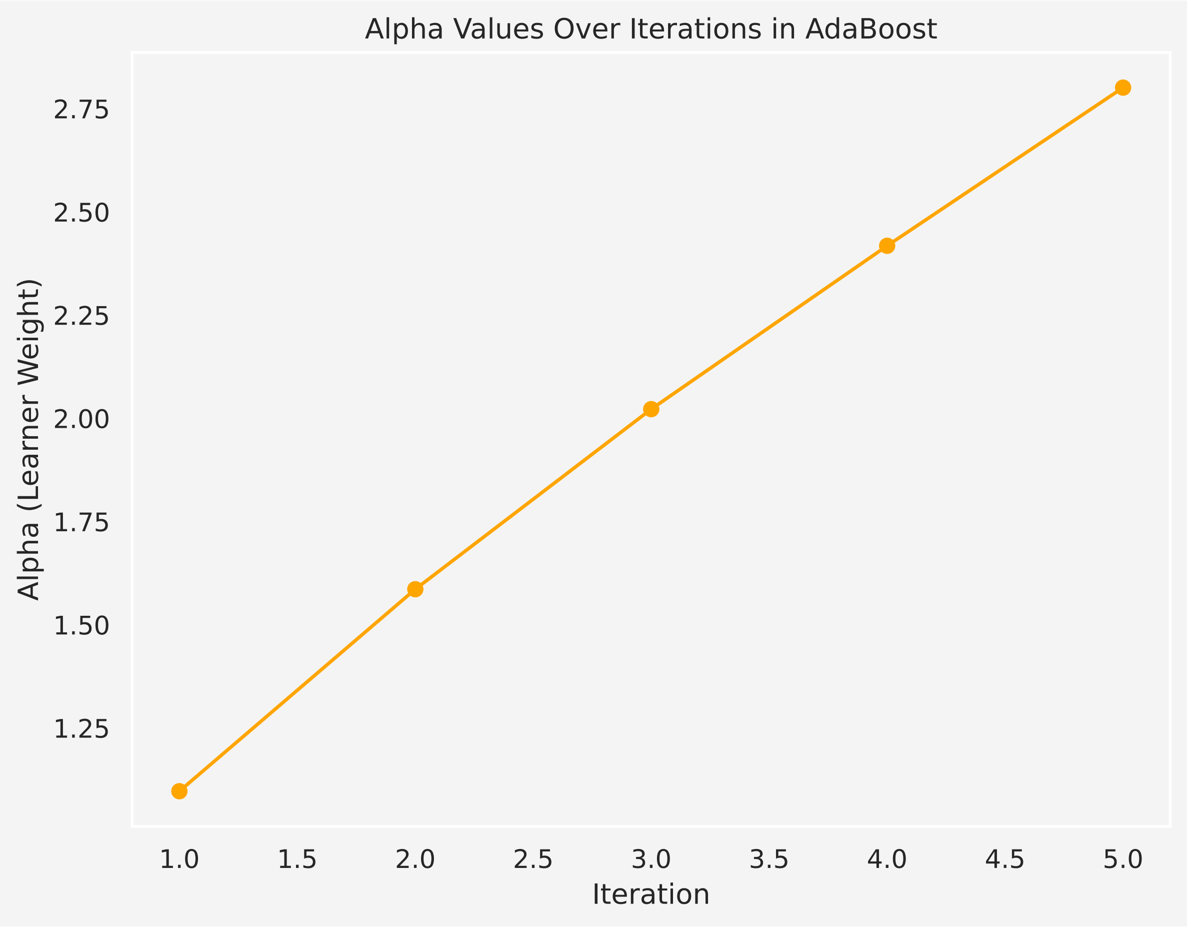
$$\begin{aligned}
F(x) &= (\alpha_1 \cdot h_1(x) + \alpha_2 \cdot h_2(x)) \\
&= [1.0986 - 1.586, -1.0986 + 1.586, -1.0986 - 1.586, -1.0986 - 1.586] \\
&= [-1, 1, -1, -1]
\end{aligned}$$

If we keep iterating this way, we will have

```
import matplotlib.pyplot as plt
import numpy as np
from mywebstyle import plot_style
plot_style('#f4f4f4')

# Data points for visualization
iterations = [1, 2]
errors = [0.25, 0.167]  # Errors from the two iterations
alphas = [1.0968, 1.586]  # Alpha values for the weak learners

# Extend to further iterations
# Simulating error reduction and alpha calculation for a few more iterations
```

```
for i in range(3, 6):  # Iterations 3 to 5
    new_error = errors[-1] * 0.7  # Simulating decreasing errors
    errors.append(new_error)
    alphas.append( np.log((1 - new_error) / new_error))
    iterations.append(i)

# Plot weighted errors over iterations
plt.figure(figsize=(8, 6))
plt.plot(iterations, errors, marker='o')
plt.title("Weighted Error Reduction Over Iterations in AdaBoost")
plt.xlabel("Iteration")
plt.ylabel("Weighted Error")
plt.grid()
plt.show()

# Plot alpha values (importance of weak learners)
plt.figure(figsize=(8, 6))
plt.plot(iterations, alphas, marker='o', color='orange')
plt.title("Alpha Values Over Iterations in AdaBoost")
plt.xlabel("Iteration")
plt.ylabel("Alpha (Learner Weight)")
plt.grid()
plt.show()
```


Weighted Error Reduction Over Iterations in AdaBoost
Weighted Error
0.250
0.225
0.200
0.175
0.150
0.125
0.100
0.075
0.050
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
Iteration



Adaptive Boosting (AdaBoost) Algorithm

Now it's time to write the formal algorithm for Adaptive Boosting or AdaBoost method. It is one of the earliest and most widely used boosting algorithms. It was introduced by Freund and Schapire in 1996. AdaBoost combines weak learners, typically decision stumps (single-level decision trees), to form a strong learner.

Algorithm: AdaBoost

1. Initialize the observation weights $w_i = \frac{1}{N}$ for $i = 1, 2, \cdots, N$ 2. For $m = 1$ to M: (a) Fit a classifier $G_m(x)$ to the training data using weights w_i (b) Compute $err_m = \frac{\sum_{i=1}^{N} w_i \nVdash (y_i \neq G_m(x_i))}{\sum_{i=1}^{N} w_i}$ (c) Compute $\alpha_m = \log\left(\frac{1-err_m}{err_m}\right)$ (d) Set $w_i \to w_i \cdot \exp\left[\alpha_m \cdot \nVdash (y_i \neq G_m(x_i))\right], \; i = 1, 2, \cdots, N$ 3. Output $G(x) = \text{sign}\left[\sum_{m=1}^{M} \alpha_m G_m(x)\right]$

In the next posts, we will continue discussing on this algorithm, specially the loss function, optimization techniques, advantages and limitations of AdaBoost, and many other facts about this algorithm.

Thanks for reading this.

Reference

- Hastie, Trevor, Robert Tibshirani, and Jerome Friedman. "The elements of statistical learning: data mining, inference, and prediction." (2017).

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