

# K Nearest Neighbors Regression

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## Introduction: Non-parametric Models

Non-parametric model is a statistical model that does not make any assumptions about the underlying data distributions, meaning it does not require specifying functional form for the relationships between variables, instead learning directly from the data points without pre-defined parameters.

## $K$ –Nearest Neighbors (KNN) Algorithm

K-Nearest Neighbors (KNN) is one of the simplest yet effective algorithms used in supervised learning for both classification and regression problems. It's a **lazy learner**—meaning it does not perform any specific training of a model but memorizes the training dataset and makes predictions based on proximity in feature space.

We are given a set of data points  $(\bar{x}_i, y_i)$  with  $\bar{x}_i \in \mathbb{R}^d$  and  $y_i \in \mathbb{R}$

1. Choose the number of neighbors  $K$
2. Compute the distance between the new data point and all the training samples
3. Select the  $K$  nearest neighbors based on distance.
4. For **classification**, the output is the most common class among the  $K$  neighbors.
5. For **regression**, the output is the average of the target values of  $K$  neighbors

## **$K$ –Nearest Neighbors Classification**

The KNN classification algorithm can be summarized with the following steps:

Given:

- $X_{train} = [x_1, x_2, \dots, x_n]$  (the training data features)
- $y_{train} = [y_1, y_2, \dots, y_n]$  (the training data labels)
- $x_{test}$  (the new data point for which we want to predict the class)

### **Steps**

**1. Compute Distance:** For each training point  $x_i$ , calculate the distance  $d(x_i, x_{test})$  using a distance metric like **Euclidean distance**:

$$d(x_i, x_{test}) = \sqrt{\sum_{j=1}^m (x_{i,j} - x_{test,j})^2}$$

where  $m$  is the number of features.

**2. Find K Nearest Neighbors:** Sort the distances and pick the **K** closest points.

**3. Majority Voting:** Look at the labels  $y_i$  of the **K** nearest neighbors. The predicted label for  $x_{test}$  is the most frequent label (majority vote) among the neighbors.

For example, let's say our data looks like this

Table 1: Training Data

area	bedroom	bathroom	price	condition
7420	4	2	1300000	1
7520	3	3	1450000	1
6420	2	1	1110000	0
5423	3	2	1363400	0
5423	3	1	1263400	1

Table 2: Test Data

area	bedroom	bathroom	price	condition
5420	3	2.5	1302000	1
7120	5	4	1453000	0

For the data points  $x_i$  from the training set and a single test data point  $xt = [5420, 3, 2.5, 1302000]$

$$\begin{aligned}
d(x_1, xt) &= \sqrt{(x_{11} - xt_1)^2 + (x_{12} - xt_2)^2 + (x_{13} - xt_3)^2 + (x_{14} - xt_4)^2} \\
&= \sqrt{(7420 - 5420)^2 + (4 - 5)^2 + (2 - 2.5)^2 + (1300000 - 1302000)^2} \approx 2828.43 \\
d(x_2, xt) &= \sqrt{(x_{21} - xt_1)^2 + (x_{22} - xt_2)^2 + (x_{23} - xt_3)^2 + (x_{24} - xt_4)^2} \\
&= \sqrt{(7520 - 5420)^2 + (3 - 5)^2 + (3 - 2.5)^2 + (1450000 - 1302000)^2} \approx 14805.92 \\
d(x_3, xt) &= \sqrt{(x_{31} - xt_1)^2 + (x_{32} - xt_2)^2 + (x_{33} - xt_3)^2 + (x_{34} - xt_4)^2} \\
&= \sqrt{(6420 - 5420)^2 + (2 - 5)^2 + (1 - 2.5)^2 + (1110000 - 1302000)^2} \approx 19209.38 \\
d(x_4, xt) &= \sqrt{(x_{41} - xt_1)^2 + (x_{42} - xt_2)^2 + (x_{43} - xt_3)^2 + (x_{44} - xt_4)^2} \\
&= \sqrt{(6420 - 5420)^2 + (2 - 5)^2 + (1 - 2.5)^2 + (1110000 - 1302000)^2} \approx 19209.38 \\
d(x_5, xt) &= \sqrt{(x_{51} - xt_1)^2 + (x_{52} - xt_2)^2 + (x_{53} - xt_3)^2 + (x_{54} - xt_4)^2} \\
&= \sqrt{(5423 - 5420)^2 + (3 - 5)^2 + (1 - 2.5)^2 + (1263400 - 1302000)^2} \approx 38602.95
\end{aligned}$$

So the distances

- $d_1 = d(x_1, xt) \approx 2828.43$
- $d_2 = d(x_2, xt) \approx 14805.92$
- $d_3 = d(x_3, xt) \approx 19209.38$
- $d_4 = d(x_4, xt) \approx 19209.38$
- $d_5 = d(x_5, xt) \approx 38602.95$

If we sort the above distances, we get  $d_1 < d_2 < d_3 < d_5 < d_4$  and if we choose  $K = 3$  nearest neighbors, then  $d_1 < d_2 < d_3$  and

- Data point  $x_1$  has class label **condition= 1**
- Data point  $x_2$  has class label **condition= 1**
- Data point  $x_3$  has class label **condition= 0**

We can clearly see that the majority class (2 out of 3) is **condition= 1**. Therefore, for the given test data, the label would be also **condition= 1**.

## KNN Classifier Using Python

Here's how to implement KNN for classification in Python from scratch:

```
import numpy as np
import pandas as pd
from collections import Counter

class CustomKNNclassifier:

    def __init__(self, k=3):
        self.k = k

    def fit(self, X, Y):
        self.X = X
        self.Y = Y

    def predict(self, X):
        predictions = [self._predict(x) for x in X.to_numpy()]
        return np.array(predictions)

    def _predict(self, x):
        # Compute the Euclidean distances
        distances = [np.linalg.norm(x - X_train) for X_train in self.X.to_numpy()]

        # Get the indices of the k nearest neighbors
        k_indices = np.argsort(distances)[:self.k]

        # Get the labels of k nearest neighbors
        k_nearest_neighbors = [self.Y[i] for i in k_indices]

        # Return the most common label
        common_label = Counter(k_nearest_neighbors).most_common(1)[0][0]
        return common_label

# Example usage
train_data = pd.DataFrame(
    {
        'area': [7420, 7520, 6420, 5423, 5423],
        'bedroom': [4, 3, 2, 3, 3],
        'bathroom': [2, 3, 1, 2, 1],
        'price': [1300000, 1450000, 1110000, 1363400, 1263400],
        'condition': [1, 1, 0, 0, 1]
```

```

    }
)
test_data = pd.DataFrame(
    {
        'area': [5420, 7120],
        'bedroom': [3, 5],
        'bathroom': [2.5, 4],
        'price': [1302000, 1453000]
    }
)

X_train = train_data.drop('condition', axis=1)
y_train = train_data['condition']

X_test = test_data

# Initialize and train the KNN model
classifier = CustomKNNclassifier(k=3)
classifier.fit(X_train, y_train)

# Predict on test data
predictions = classifier.predict(X_test)
print(predictions)

```

[1 1]

So the complete test set would be

area	bedroom	bathroom	price	condition
5420	3	2.5	1302000	1
7120	5	4	1453000	1

Note: We did not scale the data before applying the classifier. If we scaled, the result might have been different (?). In practice, we need to scale the data before applying KNN algorithm. Because computing a large number of distances with big numbers may get us wrong order and also time consuming.

## K-Nearest Neighbors Regression

KNN regression is slightly different from classification. Instead of taking a majority vote, we predict the output by averaging the values of the **K** nearest neighbors.

Given:

- $X_{train} = [x_1, x_2, \dots, x_n]$  (the training data features)
- $y_{train} = [y_1, y_2, \dots, y_n]$  (the continuous target values)
- $x_{test}$  (the new data point for which we want to predict the value)

### Step-by-Step:

- 1. Compute Distance:** Calculate the Euclidean distance between  $x_{test}$  and each training point  $x_i$ .
- 2. Find K Nearest Neighbors:** Sort the distances and select the **K** nearest points.
- 3. Averaging:** The predicted value for  $x_{test}$  is the average of the target values  $y_i$  of the **K** nearest neighbors:

$$\hat{y}_{test} = \frac{1}{K} \sum_{i=1}^K y_i$$

## KNN Regressor Using Python

Now we use the same training data and test data for this regression. But this time, our target variable is **price** and test data looks like this

area	bedroom	bathroom	Condition	price
5420	3	2.5	1	
7120	5	4	1	

After scaling the data looks like this

Table 5: Training Data

area	bedroom	bathroom	condition	price
1.213	1.414	0.267	0.730	1300000
1.336	0.000	1.603	0.730	1450000
-0.026	-1.414	-1.336	-1.095	1110000
-1.261	0.000	0.267	-1.095	1363400
-1.261	0.000	-1.336	0.730	1263400

Table 6: Test Data

area	bedroom	bathroom	condition	price
0.000	0.000	0.803	0.730	
2.828	3.876	0.730		

Now we see that

$$\begin{aligned}d_1 &= d(x_1, x_t) = \sqrt{(1.213 - (-1.266))^2 + (1.414 - 0)^2 + (0.267 - 0.803)^2 + (0.730 - 0.730)^2} \approx 2.904 \\d_2 &= d(x_2, x_t) = \sqrt{(1.336 - (-1.266))^2 + (0.000 - 0)^2 + (1.603 - 0.803)^2 + (0.730 - 0.730)^2} \approx 2.721 \\d_3 &= d(x_3, x_t) = \sqrt{(-0.026 - (-1.266))^2 + (-1.414 - 0)^2 + (-1.336 - 0.803)^2 + (-1.095 - 0.730)^2} \approx 3.382 \\d_4 &= d(x_4, x_t) = \sqrt{(-1.261 - (-1.266))^2 + (0.000 - 0)^2 + (0.267 - 0.803)^2 + (-1.095 - 0.730)^2} \approx 1.902 \\d_5 &= d(x_5, x_t) = \sqrt{(-1.261 - (-1.266))^2 + (0.000 - 0)^2 + (-1.336 - 0.803)^2 + (0.730 - 0.730)^2} \approx 2.140\end{aligned}$$

But this time, the order is  $d_4 < d_5 < d_2 < d_1 < d_3$  and for  $k = 3$  we have  $d_4 < d_5 < d_2$ . The price for this distances

- For data point  $x_4$ , the price= 1363400
- For data point  $x_5$ , the price= 1263400
- For data point  $x_2$ , the price= 1450000

So the predicted price should be the average of this three prices, that for  $xt = [5420, 3, 2.5, 1]$  the price we expect

$$price = \frac{1363400 + 1263400 + 1450000}{3} = 1358933.33$$

Here's how to implement KNN for regression in Python from scratch and we see if we get the same as the hand calculation.

```
from sklearn.preprocessing import StandardScaler

class CustomKNNRegressor:
    def __init__(self, k=3):
        self.k = k

    def fit(self, X_train, y_train):
        self.X_train = X_train
        self.y_train = y_train.to_numpy()

    def predict(self, X_test):
        predictions = [self._predict(x) for x in X_test]
        return np.array(predictions)
```

```

def _predict(self, x):
    distances = [np.linalg.norm(x-x_train) for x_train in self.X_train]
    k_indices = np.argsort(distances)[:self.k]
    k_nearest_values = [self.y_train[i] for i in k_indices]
    return np.mean(k_nearest_values)

X_train = train_data.drop('price', axis=1)
y_train = train_data['price']

test_data = pd.DataFrame(
    {
        'area': [5420, 7120],
        'bedroom': [3, 5],
        'bathroom': [2.5, 4],
        'condition': [1, 1]
    }
)

X_test = test_data

scaler = StandardScaler()

X_train_sc = scaler.fit_transform(X_train)
X_test_sc = scaler.transform(X_test)

# Initialize and train the KNN regressor
regressor = CustomKNNRegressor(k=3)
regressor.fit(X_train_sc, y_train)

# Predict on test data
predictions = regressor.predict(X_test_sc)
print(np.round(predictions,2))

```

[1358933.33 1371133.33]

---

## Choosing the Value of K

The value of **K** significantly affects the performance of the KNN algorithm:



- **Small K:** If **K** is too small, the model is sensitive to noise, and the predictions can be unstable.
- **Large K:** If **K** is too large, the model becomes more biased, and the predictions may be overly smoothed.

A typical way to choose **K** is by trying different values and using cross-validation to see which value yields the best performance.

---

## Distance Metrics

The default metric for KNN is **Euclidean distance**, but depending on the dataset, other metrics like **Manhattan distance** or **Minkowski distance** might be more suitable.

- **Euclidean Distance** (L2 Norm):

$$d(x_i, x_j) = \sqrt{\sum_{k=1}^m (x_{i,k} - x_{j,k})^2}$$

- **Manhattan Distance** (L1 Norm):

$$d(x_i, x_j) = \sum_{k=1}^m |x_{i,k} - x_{j,k}|$$

## KNN Implementation

In this section we use KNN regression for **Boston Housing** dataset and find the optimal  $K$  using the **KFold** cross-validation.

```
df = pd.read_csv('HousingData.csv')
```

Next we see if there is any missing values. If we have any, we will skip those observations.

```
print(df.isnull().sum())
df.dropna(axis=1,inplace=True)
df.head()
```

```

CRIM      20
ZN        20
INDUS     20
CHAS      20
NOX       0
RM        0
AGE       20
DIS       0
RAD       0
TAX       0
PTRATIO   0
B         0
LSTAT     20
MEDV      0
dtype: int64

```

	NOX	RM	DIS	RAD	TAX	PTRATIO	B	MEDV
0	0.538	6.575	4.0900	1	296	15.3	396.90	24.0
1	0.469	6.421	4.9671	2	242	17.8	396.90	21.6
2	0.469	7.185	4.9671	2	242	17.8	392.83	34.7
3	0.458	6.998	6.0622	3	222	18.7	394.63	33.4
4	0.458	7.147	6.0622	3	222	18.7	396.90	36.2

The data looks clean and ready to implement to the `KNNRegressor`. Note that, for predictive modeling we need a lot of things, such as exploratory data analysis (EDA), feature engineering, preprocessing and others. However, we will simply apply the `KNNRegressor` that we built from scratch and built-in library function from `scikit-learn` to explore the algorithm and find the optimal  $K$ .

```

from sklearn.model_selection import KFold, train_test_split
from sklearn.metrics import mean_squared_error, r2_score
import matplotlib.pyplot as plt

X = df.drop('MEDV',axis=1)
y = df['MEDV']

X_train, X_test, y_train, y_test = train_test_split(
    X,y, test_size=0.30, random_state=123
)
scaler = StandardScaler()

```

```

X_train_sc = scaler.fit_transform(X_train)
X_test_sc = scaler.transform(X_test)

k_values = [5,15,30,40]

kfold = KFold(n_splits=7, shuffle=True, random_state=123)
mses = np.zeros((7,4))

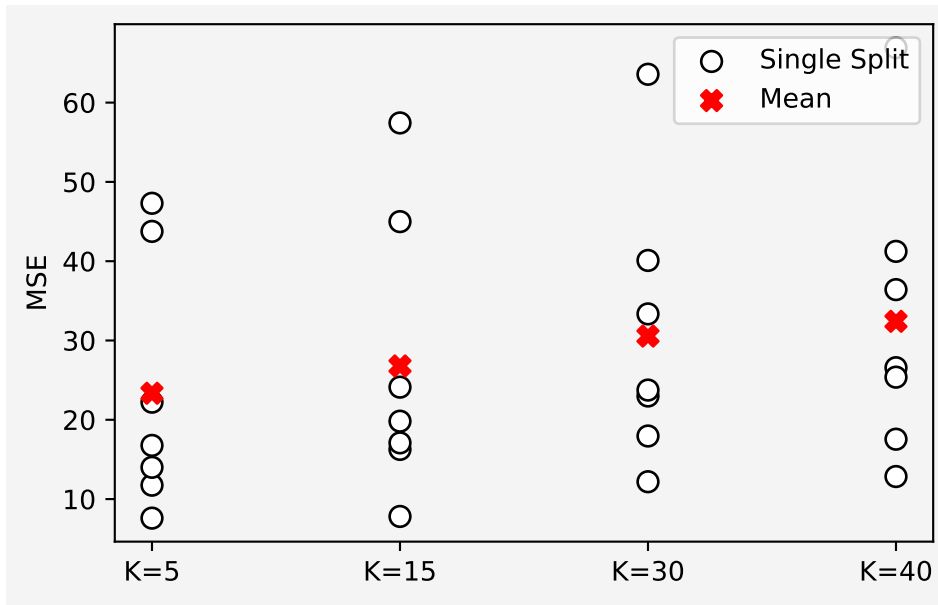
for i,(train_index,test_index) in enumerate(kfold.split(X_train_sc)):
    X_train_train = X_train_sc[train_index]
    X_train_holdout = X_train_sc[test_index]

    y_train_train = y_train.iloc[train_index]
    y_train_holdout = y_train.iloc[test_index]

    for j,k in enumerate(k_values):
        regressor1 = CustomKNNRegressor(k=k)
        regressor1.fit(X_train_train, y_train_train)
        preds = regressor1.predict(X_train_holdout)
        mses[i,j] = mean_squared_error(preds, y_train_holdout)

plt.scatter(np.zeros(7),mses[:,0], s=60, c='white', edgecolors='black', label='Single Split')
plt.scatter(np.ones(7),mses[:,1],s=60, c='white', edgecolors='black')
plt.scatter(2*np.ones(7),mses[:,2],s=60, c='white', edgecolors='black')
plt.scatter(3*np.ones(7),mses[:,3],s=60, c='white', edgecolors='black')
plt.scatter([0,1,2,3], np.mean(mses, axis=0), s=60,c='r', marker='X', label='Mean')
plt.legend(loc='upper right')
plt.xticks([0,1,2,3],['K=5','K=15','K=30','K=40'])
plt.ylabel('MSE')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()

```



So,  $K = 5$  seems optimal based on our custom built regressor. Now if we do the same thing using the `scikit-learn` library

```
from sklearn.neighbors import KNeighborsRegressor

mses = np.zeros((7,4))

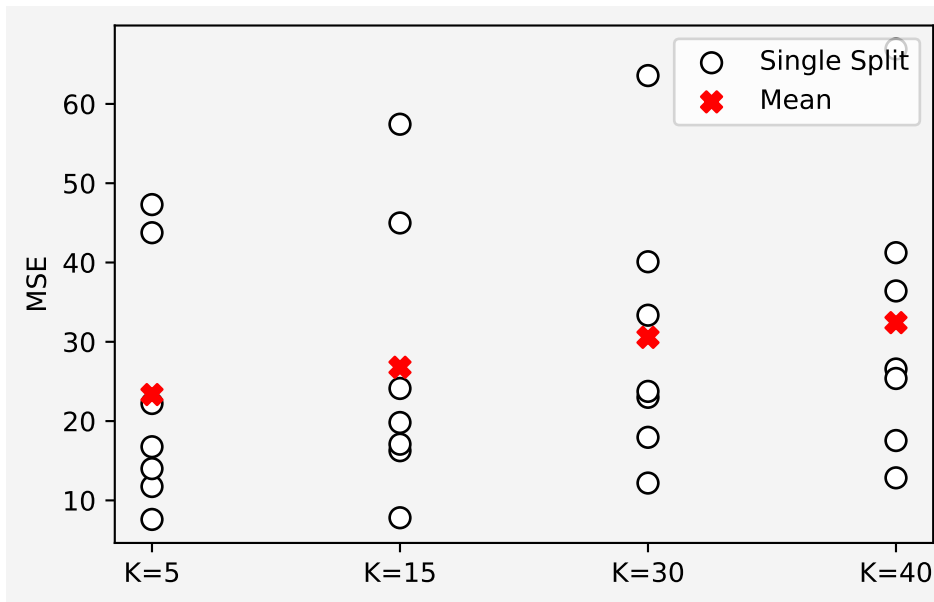
for i,(train_index,test_index) in enumerate(kfold.split(X_train_sc)):
    X_train_train = X_train_sc[train_index]
    X_train_holdout = X_train_sc[test_index]

    y_train_train = y_train.iloc[train_index]
    y_train_holdout = y_train.iloc[test_index]

    for j,k in enumerate(k_values):
        regressor2 = KNeighborsRegressor(k)
        regressor2.fit(X_train_train, y_train_train)
        preds = regressor2.predict(X_train_holdout)
        mses[i,j] = mean_squared_error(preds, y_train_holdout)

plt.scatter(np.zeros(7),mses[:,0], s=60, c='white', edgecolors='black', label='Single Split')
plt.scatter(np.ones(7),mses[:,1],s=60, c='white', edgecolors='black')
plt.scatter(2*np.ones(7),mses[:,2],s=60, c='white', edgecolors='black')
plt.scatter(3*np.ones(7),mses[:,3],s=60, c='white', edgecolors='black')
```

```
plt.scatter([0,1,2,3], np.mean(mses, axis=0), s=60, c='r', marker='X', label='Mean')
plt.legend(loc='upper right')
plt.xticks([0,1,2,3], ['K=5', 'K=15', 'K=30', 'K=40'])
plt.ylabel('MSE')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



In both method, we got  $K = 5$  is the optimal number of neighbors for KNN regression. Let's apply this in our test dataset

```
regressor = CustomKNNRegressor(k=5)
regressor.fit(X_train_sc, y_train)

predictions = regressor.predict(X_test_sc)

mse = mean_squared_error(predictions, y_test)
rsquared = r2_score(predictions, y_test)
print('MSE = {}'.format(np.round(mse, 2)), ' and R-square = {}'.format(np.round(rsquared, 2)))
```

MSE = 41.26 and R-square = 0.23

## Conclusion

K-Nearest Neighbors is a simple, intuitive algorithm that can be highly effective in both classification and regression problems. Its simplicity comes from the fact that it doesn't make any assumptions about the underlying data distribution (it's non-parametric). However, its performance can be sensitive to the choice of  $K$  and the distance metric. Although it's easy to implement, KNN can become computationally expensive for large datasets, as it requires calculating distances between the test point and all training samples. If you need an efficient version, it's always possible to use optimized libraries like scikit-learn, but writing the algorithm from scratch helps build a solid understanding.

## When to Use KNN Over Linear Regression?

We would consider using KNN regression over linear regression in the following situations:

- **Non-linear relationships:** When the data shows non-linear patterns or complex relationships between features and target variables that cannot be captured by a straight line.
- **Local behavior:** When data has local patterns or clusters, and you believe that predictions should rely on the nearest data points.
- **Minimal assumptions:** If you do not want to assume a specific relationship between the features and target, KNN's non-parametric nature might be more appropriate.
- **Smaller datasets:** KNN works well with smaller datasets and lower-dimensional data where calculating distances is feasible and efficient.

However, KNN becomes less efficient and struggles in high dimensions or when the dataset is large. In those cases, linear regression or other more scalable models may be more appropriate

---

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