Simple Linear Regression

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A simple linear regression in multiple predictors/input variables/features/independent variables/ explanatory variables/regressors/ covariates (many names) often takes the form

$$y = f(\mathbf{x}) + \epsilon = \beta\mathbf{x} + \epsilon$$

where $\beta \in \mathbb{R}^d$ are regression parameters or constant values that we aim to estimate and $\epsilon \sim \mathcal{N}(0, 1)$ is a normally distributed error term independent of x or also called the white noise.

In this case, the model:

$$y = f(x) + \epsilon = \beta_0 + \beta_1 x + \epsilon$$

Therefore, in our model we need to estimate the parameters β_0, β_1. The true relationship between the explanatory variables and the dependent variable is $y = f(x)$. But our model is $y = f(x) + \epsilon$. Here, this $f(x)$ is the working model with the data. In other words, $\hat{y} = f(x) = \hat{\beta}_0 + \hat{\beta}_1 x$. Therefore, there should be some error in the model prediction which we are calling $\epsilon = \|y - \hat{y}\|$ where y is the true value and $\hat{y}$ is the predicted value. This error term is normally distributed with mean 0 and variance 1. To get the best estimate of the parameters

β_0, β_1 we can minimize the error term as much as possible. So, we define the residual sum of squares (RSS) as:

$$\begin{aligned} RSS &= \epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_{10}^2 \qquad (1) \\ &= \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \qquad (2) \\ \hat{\updownarrow}(\bar{\beta}) &= \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \qquad (3) \\ & \qquad (4) \end{aligned}$$

Using multivariate calculus we see

$$\begin{aligned} \frac{\partial l}{\partial \beta_0} &= \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-1) \qquad (5) \\ \frac{\partial l}{\partial \beta_1} &= \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) \qquad (6) \end{aligned}$$

Setting the partial derivatives to zero we solve for $\hat{\beta}_0, \hat{\beta}_1$ as follows

$$\begin{aligned} \frac{\partial l}{\partial \beta_0} &= 0 \\ \implies \sum_{i=1}^{10} y_i - 10\hat{\beta}_0 - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i \right) &= 0 \\ \implies \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \end{aligned}$$

and,

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_1} &= 0 \\
\implies \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) &= 0 \\
\implies \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(x_i) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \hat{\beta}_0 \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \left(\bar{y} - \hat{\beta}_1 \bar{x} \right) \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) + \hat{\beta}_1 \bar{x} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - \bar{x} \sum_{i=1}^{10} x_i \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - 10\bar{x}^2 \right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - 2 \times 10 \times \bar{x}^2 + 10\bar{x}^2 \right) &= 0 \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - 10\bar{x}\bar{y}}{\sum_{i=1}^{10} x_i^2 - 2 \times 10 \times \bar{x}^2 + 10\bar{x}^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - 10\bar{x}\bar{y} - 10\bar{x}\bar{y} + 10\bar{x}\bar{y}}{\sum_{i=1}^{10} x_i^2 - 2\bar{x} \times 10 \times \frac{1}{10} \sum_{i=1}^{10} x_i + 10\bar{x}^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i \right) - \bar{x} \left(\sum_{i=1}^{10} y_i \right) + 10\bar{x}\bar{y}}{\sum_{i=1}^{10} (x_i - \bar{x})^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} \left(x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y} \right)}{\sum_{i=1}^{10} (x_i - \bar{x})^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{10} (x_i - \bar{x})^2}
\end{aligned}
$$

Therefore, we have the following

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{10}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{10}(x_i - \bar{x})^2}$$

Simple Linear Regression `slr` is applicable for a single feature data set with contineous response variable.

```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.linear_model import LinearRegression
```

Assumptions of Linear Regressions

- **Linearity:** The relationship between the feature set and the target variable has to be linear.
- **Homoscedasticity:** The variance of the residuals has to be constant.
- **Independence:** All the observations are independent of each other.
- **Normality:** The distribution of the dependent variable y has to be normal.

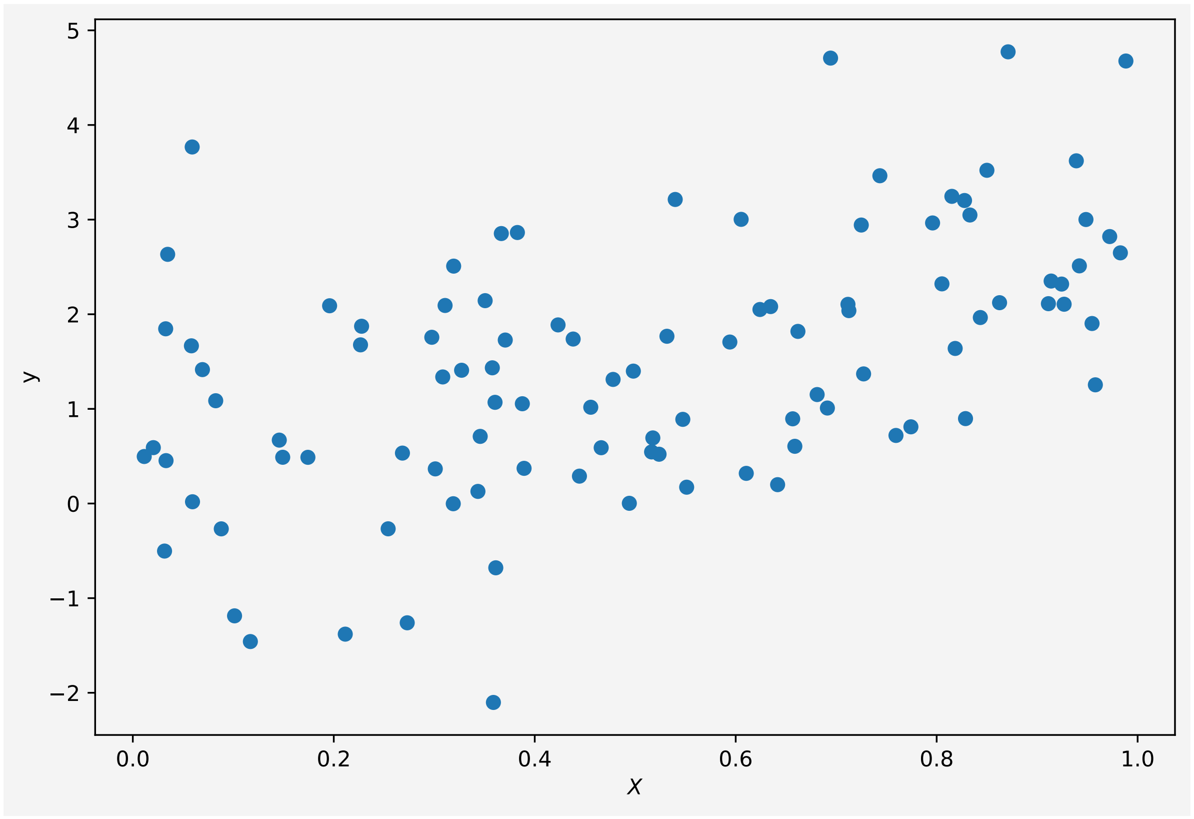
Synthetic Data

To implement the algorithm, we need some synthetic data. To generate the synthetic data we use the linear equation $y(x) = 2x + \frac{1}{2} + \xi$ where $\xi \sim \mathbf{N}(0, 1)$

```
X=np.random.random(100)
y=2*X+0.5+np.random.randn(100)
```

Note that we used two random number generators, `np.random.random(n)` and `np.random.randn(n)`. The first one generates n random numbers of values from the range (0,1) and the second one generates values from the standard normal distribution with mean 0 and variance or standard deviation 1.

```
plt.figure(figsize=(9,6))
plt.scatter(X,y)
plt.xlabel('$X$')
plt.ylabel('y')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



Model

We want to fit a simple linear regression to the above data.

```
slr=LinearRegression()
```

Now to fit our data X and y we need to reshape the input variable. Because if we look at X,

```
X
```

```
array([0.05830442, 0.49814365, 0.81513633, 0.25412215, 0.35064737,
       0.3577926 , 0.52372607, 0.65885353, 0.65672333, 0.21143074,
       0.02041635, 0.38766446, 0.03270907, 0.74348235, 0.43823623,
       0.93898915, 0.1169925 , 0.32721359, 0.36674013, 0.87109664,
       0.51752825, 0.19585288, 0.4779813 , 0.62412175, 0.75950264,
       0.55119021, 0.95464152, 0.22661323, 0.22767299, 0.9112042 ,
       0.71263296, 0.3084487 , 0.31077314, 0.31931019, 0.61051865,
       0.37066403, 0.72498311, 0.53980915, 0.08804413, 0.82871726,
       0.8500045 , 0.26834099, 0.5162873 , 0.69441085, 0.08250853,
       0.45576097, 0.77436392, 0.63475968, 0.91391659, 0.72727953,
       0.79593505, 0.44448607, 0.34572978, 0.49414085, 0.6810519 ,
       0.94857226, 0.59417387, 0.30102217, 0.17424938, 0.86260318,
       0.60541926, 0.54736915, 0.64171575, 0.69124747, 0.03306703,
       0.53156362, 0.42320644, 0.92685442, 0.2731017 , 0.3588733 ,
       0.94202416, 0.82773277, 0.8184295 , 0.38939723, 0.03469281,
       0.31891852, 0.98832794, 0.71171317, 0.03158087, 0.98283516,
       0.95793961, 0.10124961, 0.46612808, 0.80525141, 0.84345153,
       0.38274352, 0.06929041, 0.9722184 , 0.05941343, 0.01137009,
       0.34346217, 0.36123625, 0.29756055, 0.14576759, 0.14918338,
       0.05909317, 0.66192138, 0.9243533 , 0.83310358, 0.36049446])
```

It is a one-dimensional array/vector but the `slr` object accepts input variable as matrix or two-dimensional format.

```
X=X.reshape(-1,1)
X[:10]
```

```
array([[0.05830442],
       [0.49814365],
       [0.81513633],
       [0.25412215],
       [0.35064737],
       [0.3577926 ],
       [0.52372607],
       [0.65885353],
       [0.65672333],
       [0.21143074]])
```

Now we fit the data to our model

```
slr.fit(X,y)
slr.predict([[2],[3]])
```

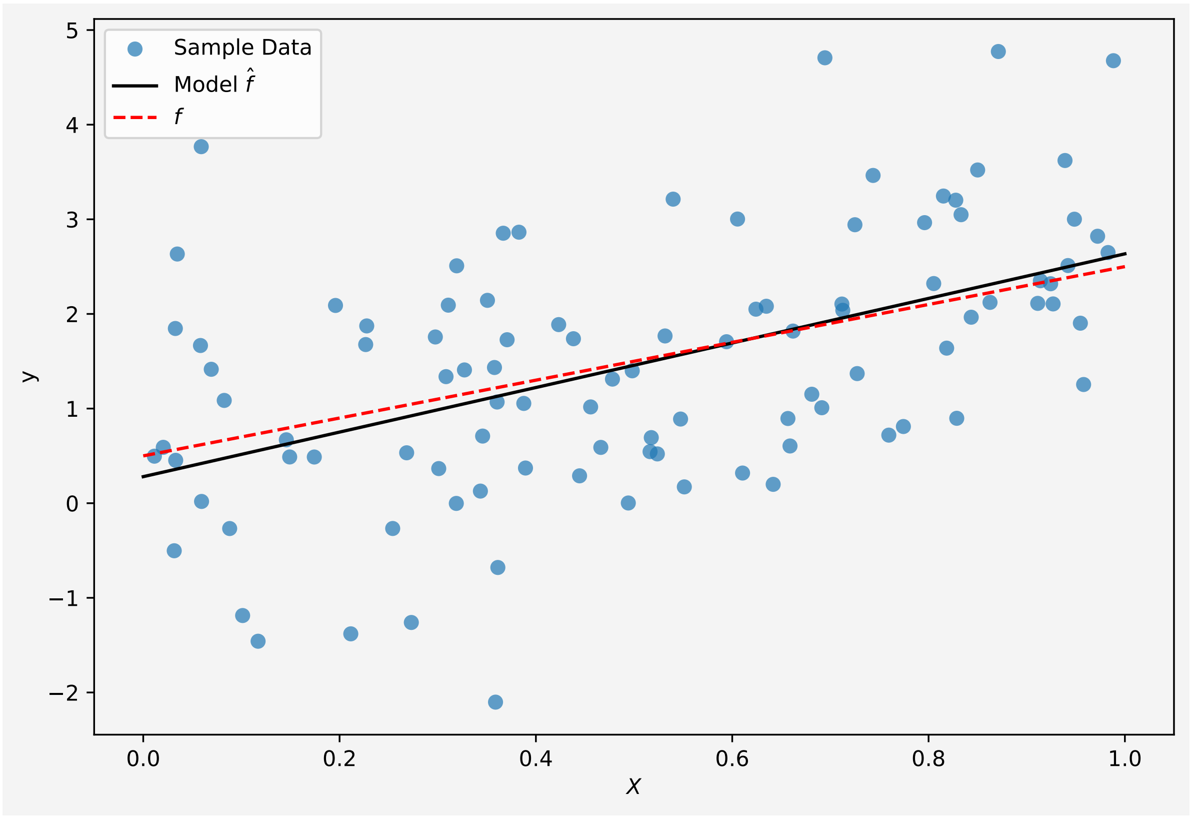
```
array([4.98947731, 7.34395157])
```

We have our $X = 2, 3$ and the corresponding y values are from the above cell output, which are pretty close to the model $y = 2x + \frac{1}{2}$.

```
intercept = round(slr.intercept_,4)
slope = slr.coef_
```

Now our model parameters are: intercept $\beta_0 = 0.2805$ and slope $\beta_1 = \text{array}([2.35447427])$.

```
plt.figure(figsize=(9,6))
plt.scatter(X,y, alpha=0.7,label="Sample Data")
plt.plot(np.linspace(0,1,100),
    slr.predict(np.linspace(0,1,100).reshape(-1,1)),
    'k',
    label='Model $\hat{f}$'
)
plt.plot(np.linspace(0,1,100),
    2*np.linspace(0,1,100)+0.5,
    'r--',
    label='$f$'
)
plt.xlabel('$X$')
plt.ylabel('y')
plt.legend(fontsize=10)
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



So the model fits the data almost perfectly.

Up next multiple linear regression.

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