Simple Linear Regression

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A simple linear regression in multiple predictors/input variables/features/independent variables/ explanatory variables/regressors/ covariates (many names) often takes the form

$$y = f(\mathbf{x}) + \epsilon = \beta\mathbf{x} + \epsilon$$

where $\beta \in \mathbb{R}^d$ are regression parameters or constant values that we aim to estimate and $\epsilon \sim \mathcal{N}(0,1)$ is a normally distributed error term independent of x or also called the white noise.

In this case, the model:

$$y = f(x) + \epsilon = \beta_0 + \beta_1 x + \epsilon$$

Therefore, in our model we need to estimate the parameters β_0, β_1. The true relationship between the explanatory variables and the dependent variable is $y = f(x)$. But our model is $y = f(x) + \epsilon$. Here, this $f(x)$ is the working model with the data. In other words, $\hat{y} = f(x) = \hat{\beta}_0 + \hat{\beta}_1 x$. Therefore, there should be some error in the model prediction which we are calling $\epsilon = \|y - \hat{y}\|$ where y is the true value and $\hat{y}$ is the predicted value. This error term is normally distributed with mean 0 and variance 1. To get the best estimate of the parameters

β_0, β_1 we can minimize the error term as much as possible. So, we define the residual sum of squares (RSS) as:

$$
\begin{aligned}
RSS &= \epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_{10}^2 && (1) \\
&= \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 && (2) \\
\hat{\updownarrow}(\bar{\beta}) &= \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 && (3) \\
& && (4)
\end{aligned}
$$

Using multivariate calculus we see

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_0} &= \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-1) && (5) \\
\frac{\partial l}{\partial \beta_1} &= \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) && (6)
\end{aligned}
$$

Setting the partial derivatives to zero we solve for $\hat{\beta}_0, \hat{\beta}_1$ as follows

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_0} &= 0 \\
\implies \sum_{i=1}^{10} y_i - 10\hat{\beta}_0 - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i \right) &= 0 \\
\implies \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}
\end{aligned}
$$

and,

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_1} &= 0 \\
\implies \sum_{i=1}^{10} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) &= 0 \\
\implies \sum_{i=1}^{10} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(x_i) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \hat{\beta}_0 \left(\sum_{i=1}^{10} x_i\right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2\right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \left(\bar{y} - \hat{\beta}_1 \bar{x}\right) \left(\sum_{i=1}^{10} x_i\right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2\right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i\right) + \hat{\beta}_1 \bar{x} \left(\sum_{i=1}^{10} x_i\right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2\right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i\right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - \bar{x} \sum_{i=1}^{10} x_i\right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i\right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - 10\bar{x}^2\right) &= 0 \\
\implies \sum_{i=1}^{10} x_i y_i - \bar{y} \left(\sum_{i=1}^{10} x_i\right) - \hat{\beta}_1 \left(\sum_{i=1}^{10} x_i^2 - 2 \times 10 \times \bar{x}^2 + 10\bar{x}^2\right) &= 0 \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - 10\bar{x}\bar{y}}{\sum_{i=1}^{10} x_i^2 - 2 \times 10 \times \bar{x}^2 + 10\bar{x}^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - 10\bar{x}\bar{y} - 10\bar{x}\bar{y} + 10\bar{x}\bar{y}}{\sum_{i=1}^{10} x_i^2 - 2\bar{x} \times 10 \times \frac{1}{10}\sum_{i=1}^{10} x_i + 10\bar{x}^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} x_i y_i - \bar{y}\left(\sum_{i=1}^{10} x_i\right) - \bar{x}\left(\sum_{i=1}^{10} y_i\right) + 10\bar{x}\bar{y}}{\sum_{i=1}^{10} (x_i - \bar{x})^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} \left(x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y}\right)}{\sum_{i=1}^{10} (x_i - \bar{x})^2} \\
\implies \hat{\beta}_1 &= \frac{\sum_{i=1}^{10} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{10} (x_i - \bar{x})^2}
\end{aligned}
$$

Therefore, we have the following

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{10}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{10}(x_i - \bar{x})^2}$$

Simple Linear Regression `slr` is applicable for a single feature data set with contineous response variable.

```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.linear_model import LinearRegression
```

Assumptions of Linear Regressions

- **Linearity:** The relationship between the feature set and the target variable has to be linear.
- **Homoscedasticity:** The variance of the residuals has to be constant.
- **Independence:** All the observations are independent of each other.
- **Normality:** The distribution of the dependent variable y has to be normal.

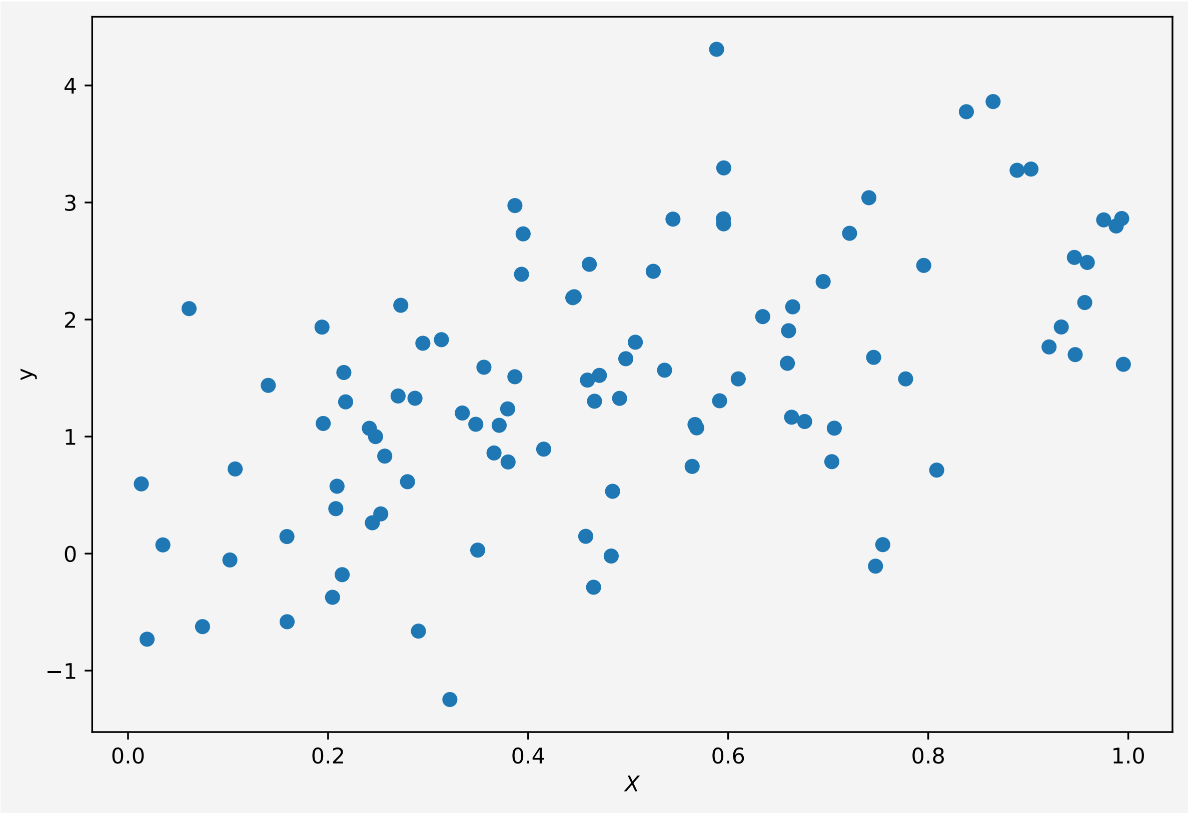
Synthetic Data

To implement the algorithm, we need some synthetic data. To generate the synthetic data we use the linear equation $y(x) = 2x + \frac{1}{2} + \xi$ where $\xi \sim \mathbf{N}(0, 1)$

```
X=np.random.random(100)
y=2*X+0.5+np.random.randn(100)
```

Note that we used two random number generators, `np.random.random(n)` and `np.random.randn(n)`. The first one generates n random numbers of values from the range (0,1) and the second one generates values from the standard normal distribution with mean 0 and variance or standard deviation 1.

```
plt.figure(figsize=(9,6))
plt.scatter(X,y)
plt.xlabel('$X$')
plt.ylabel('y')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



Model

We want to fit a simple linear regression to the above data.

```
slr=LinearRegression()
```

Now to fit our data X and y we need to reshape the input variable. Because if we look at X,

```
X
```

```
array([0.46117607, 0.75455554, 0.48306664, 0.03483697, 0.4712601 ,
       0.10182266, 0.80849425, 0.3950062 , 0.92081028, 0.06103222,
       0.27939739, 0.72138751, 0.5668466 , 0.29475273, 0.31332705,
       0.14018554, 0.20771773, 0.37952775, 0.21575998, 0.36588038,
       0.01326162, 0.07447165, 0.29033272, 0.7774143 , 0.44604124,
       0.46545788, 0.45757166, 0.20442791, 0.56854434, 0.38674415,
       0.15885487, 0.26999379, 0.25266019, 0.28688145, 0.10710911,
       0.41555347, 0.74730646, 0.70361246, 0.37105588, 0.59144141,
       0.45930625, 0.66340595, 0.35580185, 0.15905723, 0.54479886,
       0.52509775, 0.65922071, 0.48443201, 0.56399568, 0.95649528,
       0.94695669, 0.46638276, 0.88884066, 0.38679252, 0.90273486,
       0.50718033, 0.19392084, 0.24127133, 0.34958718, 0.59557858,
       0.86484624, 0.63454026, 0.19512754, 0.49143848, 0.70612115,
       0.34767385, 0.66450324, 0.74544994, 0.27267999, 0.74067695,
       0.6604034 , 0.6764827 , 0.24745505, 0.95902902, 0.61008131,
       0.94612257, 0.44465716, 0.49763196, 0.69496503, 0.93303078,
       0.5954627 , 0.97543079, 0.24426492, 0.98791035, 0.83816917,
       0.33420108, 0.21410478, 0.01904663, 0.59517759, 0.25664138,
       0.99345018, 0.21754063, 0.39339675, 0.58844316, 0.20895607,
       0.9951077 , 0.37997327, 0.32175479, 0.53640329, 0.79554033])
```

It is a one-dimensional array/vector but the `slr` object accepts input variable as matrix or two-dimensional format.

```
X=X.reshape(-1,1)
X[:10]
```

```
array([[0.46117607],
       [0.75455554],
       [0.48306664],
       [0.03483697],
       [0.4712601 ],
       [0.10182266],
       [0.80849425],
       [0.3950062 ],
       [0.92081028],
       [0.06103222]])
```

Now we fit the data to our model

```
slr.fit(X,y)
slr.predict([[2],[3]])
```

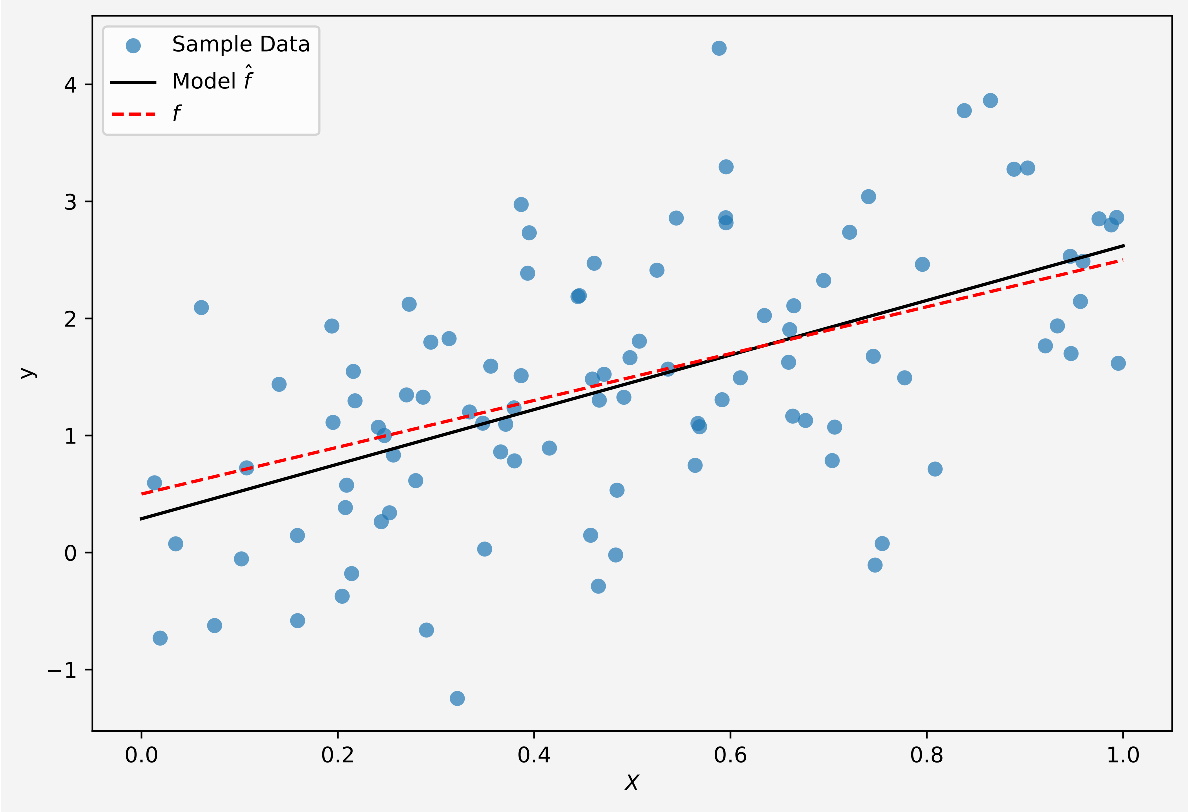
```
array([4.95212089, 7.28364311])
```

We have our $X = 2, 3$ and the corresponding y values are from the above cell output, which are pretty close to the model $y = 2x + \frac{1}{2}$.

```
intercept = round(slr.intercept_,4)
slope = slr.coef_
```

Now our model parameters are: intercept $\beta_0 = 0.2891$ and slope $\beta_1 = \text{array}([2.33152221])$.

```
plt.figure(figsize=(9,6))
plt.scatter(X,y, alpha=0.7,label="Sample Data")
plt.plot(np.linspace(0,1,100),
    slr.predict(np.linspace(0,1,100).reshape(-1,1)),
    'k',
    label='Model $\hat{f}$'
)
plt.plot(np.linspace(0,1,100),
    2*np.linspace(0,1,100)+0.5,
    'r--',
    label='$f$'
)
plt.xlabel('$X$')
plt.ylabel('y')
plt.legend(fontsize=10)
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



So the model fits the data almost perfectly.

Up next multiple linear regression.

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