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EPG-based signal modeling and sequence optimization

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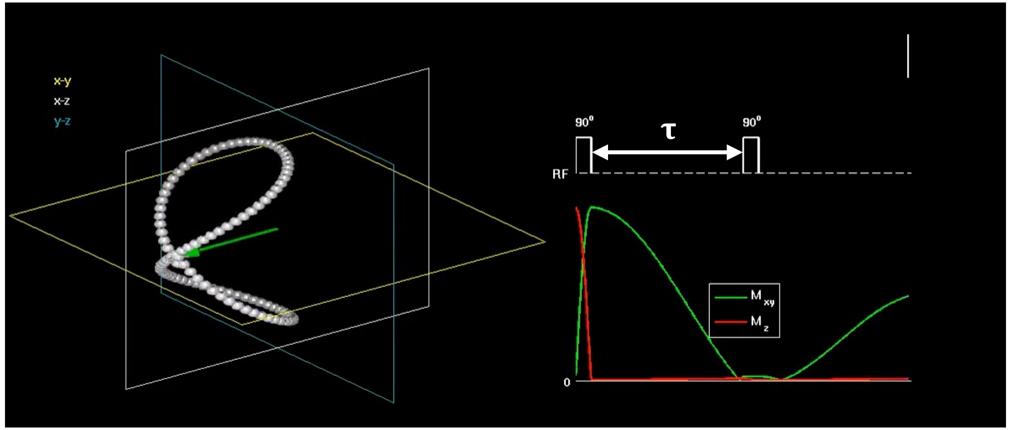
EPG-based signal modeling and sequence optimization

Outline

- History: computing echo amplitudes *without* explicit isochromat averaging (Signal Eq.)
  - Woessner decomposition
  - Phase graphs and branching rules
- Hennig's Extended Phase Graphs (EPGs)
  - The EPG algorithm
  - Turbo-spin echo sequences
  - Steady-state sequences
- Examples: solving optimization problems with EPGs
  - Signal modeling $\Rightarrow$ MR parameter estimation (quantitative MRI)
  - Signal shaping $\Rightarrow$ flip angle computation (sequence optimization)

"eight-ball" spin echo [90° - 2π – 90°- 2π]: TE=2$\boldsymbol{\tau}$, A = ?



Calculating the spin echo amplitude by solving Bloch *and* signal equation
(don’t do this at home)

ing analysis. First, the effect of an $\alpha_{y'}$-pulse is calculated using the following transform:

$$\begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix} \xrightarrow{\alpha_{y'}} \begin{bmatrix} M_{x'}\cos\alpha - M_{z'}\sin\alpha \\ M_{y'} \\ M_{x'}\sin\alpha + M_{z'}\cos\alpha \end{bmatrix} \tag{4.22}$$

Second, the effect of a τ delay is described by

$$\begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix} \xrightarrow{\tau} \begin{bmatrix} (M_{x'}\cos\omega\tau + M_{y'}\sin\omega\tau)e^{-\tau/T_2} \\ (-M_{x'}\sin\omega\tau + M_{y'}\cos\omega\tau)e^{-\tau/T_2} \\ M_z^0(1-e^{-\tau/T_1}) + M_{z'}e^{-\tau/T_1} \end{bmatrix} \tag{4.23}$$

where ω is the precessional frequency in the rotating frame.

Consider an arbitrary isochromat of frequency ω initially at the thermal equilibrium state. We have after the α_1 pulse

$$\begin{aligned} M_{x'}(\omega, 0_+) &= -M_z^0(\omega)\sin\alpha_1 \\ M_{y'}(\omega, 0_+) &= 0 \\ M_{z'}(\omega, 0_+) &= M_z^0(\omega)\cos\alpha_1 \end{aligned}$$

After the τ delay, the magnetization components take the following set of values:

$$M_{x'}(\omega,\tau) = -M_z^0(\omega)\sin\alpha_1\cos\omega\tau e^{-\tau/T_2} \tag{4.24a}$$

$$M_{y'}(\omega,\tau) = M_z^0(\omega)\sin\alpha_1\sin\omega\tau e^{-\tau/T_2} \tag{4.24b}$$

$$\begin{aligned} M_{z'}(\omega,\tau) &= M_z^0(\omega)(1-e^{-\tau/T_1}) + M_z^0(\omega)\cos\alpha_1 e^{-\tau/T_1} \\ &= M_z^0(\omega)[1-(1-\cos\alpha_1)e^{-\tau/T_1}] \end{aligned} \tag{4.24c}$$

The α_2 has no effect on the y'-component but transforms the x'- and z'-components to the following set of values:

$$\begin{aligned} M_{x'}(\omega,\tau_+) &= -M_z^0(\omega)\sin\alpha_1\cos\alpha_2\cos\omega\tau e^{-\tau/T_2} \\ &\quad -M_z^0(\omega)[1-(1-\cos\alpha_1)e^{-\tau/T_1}]\sin\alpha_2 \\ M_{z'}(\omega,\tau_+) &= M_z^0(\omega)[1-(1-\cos\alpha_1)e^{-\tau/T_1}]\cos\alpha_2 \\ &\quad -M_z^0(\omega)\sin\alpha_1\sin\alpha_2\cos\omega\tau e^{-\tau/T_2} \end{aligned} \tag{4}$$

The x'-component in Eq. (4.25a) can be rewritten as

$$\begin{aligned} M_{x'}(\omega,\tau_+) = &-M_z^0(\omega)\sin\alpha_1\cos^2\frac{\alpha_2}{2}\cos\omega\tau e^{-\tau/T_2} \\ &+M_z^0(\omega)\sin\alpha_1\sin^2\frac{\alpha_2}{2}\cos\omega\tau e^{-\tau/T_2} \\ &-M_z^0(\omega)[1-(1-\cos\alpha_1)e^{-\tau/T_1}]\sin\alpha_2 \end{aligned} \tag{4.26}$$

Similarly, the y'-component in Eq. (4.24b) can be rewritten as

$$\begin{aligned} M_{y'}(\omega,\tau_+) = &M_z^0(\omega)\sin\alpha_1\cos^2\frac{\alpha_2}{2}\sin\omega\tau e^{-\tau/T_2} \\ &+M_z^0(\omega)\sin\alpha_1\sin^2\frac{\alpha_2}{2}\sin\omega\tau e^{-\tau/T_2} \end{aligned} \tag{4.27}$$

Consequently, the transverse magnetization immediately after the second pulse can be written as

$$\begin{aligned} M_{x'y'}(\omega,\tau_+) = &M_z^0(\omega)\sin\alpha_1\left(\sin^2\frac{\alpha_2}{2}e^{-i\omega\tau} - \cos^2\frac{\alpha_2}{2}e^{i\omega\tau}\right)e^{-\tau/T_2} \\ &-M_z^0(\omega)[1-(1-\cos\alpha_1)e^{-\tau/T_1}]\sin\alpha_2 \end{aligned} \tag{4.28}$$

Free precession of this vector about the z'-axis after the pulse is described by

$$\begin{aligned} M_{x'y'}(\omega,t) &= M_{x'y'}(\omega,\tau_+)e^{-(t-\tau)/T_2}e^{-i\omega(t-\tau)} \\ &= M_z^0(\omega)\sin\alpha_1\left(-\cos^2\frac{\alpha_2}{2}e^{-i\omega\tau} + \sin^2\frac{\alpha_2}{2}e^{i\omega\tau}\right)e^{-t/T_2}e^{-i\omega(t-\tau)} \\ &\quad -M_z^0(\omega)[1-(1-\cos\alpha_1)e^{-\tau/T_1}]\sin\alpha_2 e^{-(t-\tau)/T_2}e^{-i\omega(t-\tau)} \\ &= M_z^0(\omega)\sin\alpha_1\sin^2\frac{\alpha_2}{2}e^{-t/T_2}e^{-i\omega(t-2\tau)} \\ &\quad -M_z^0(\omega)\sin\alpha_1\cos^2\frac{\alpha_2}{2}e^{-t/T_2}e^{-i\omega t} \\ &\quad -M_z^0(\omega)[1-(1-\cos\alpha_1)e^{-\tau/T_1}]\sin\alpha_2 e^{-(t-\tau)/T_2}e^{-i\omega(t-\tau)} \end{aligned} \tag{4.29}$$

for $t > \tau$. For an ensemble of a large number of isochromats, the second and ... Eq. (4.29) represent a purely dephasing component because the ...he isochromats becomes larger as time progresses. These ...te to the FID signal formed from the second pulse. On ...st term are rephasing gradually and achieve complete ... Therefore, this component will produce an echo ...d as

$$A_E = \sin\alpha_1\sin^2\frac{\alpha_2}{2}$$

$$\ldots\frac{\alpha_2}{2}\int_{-\infty}^{\infty}\rho(\omega)e^{-t/T_2(\omega)}e^{-i\omega(t-T_E)}d\omega \tag{4.30}$$

.../2, where

$T_E = 2\tau$:	echo time
T_{acq} :	data acquisition interval

It is clear from Eq. (4.30) that the echo peaks at $t = T_E$, and its value is given by

$$A_E = \sin\alpha_1\sin^2\frac{\alpha_2}{2}\int_{-\infty}^{\infty}\rho(\omega)e^{-T_E/T_2(\omega)}d\omega \tag{4.31}$$

Zhi-Pei Liang, Paul C. Lauterbur. Principles of Magnetic Resonance Imaging: A Signal Processing Perspective. IEEE Press series in biomedical engineering. 2000.

Woessner Decomposition [1]

1. Short RF pulse with arbitrary flip angle
(instantaneous rotation)

$$\vec{M}^{+}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \vec{M}(\theta)$$

2. In terms of complex notation: $F = M_x + \mathrm{i}M_y$ $(Z = M_z)$

$$F^{+}(\theta) = \cos^2\left(\frac{\alpha}{2}\right)F(\theta) + \sin^2\left(\frac{\alpha}{2}\right)F^{*}(\theta) - \mathrm{i}\sin(\alpha)Z(\theta)$$

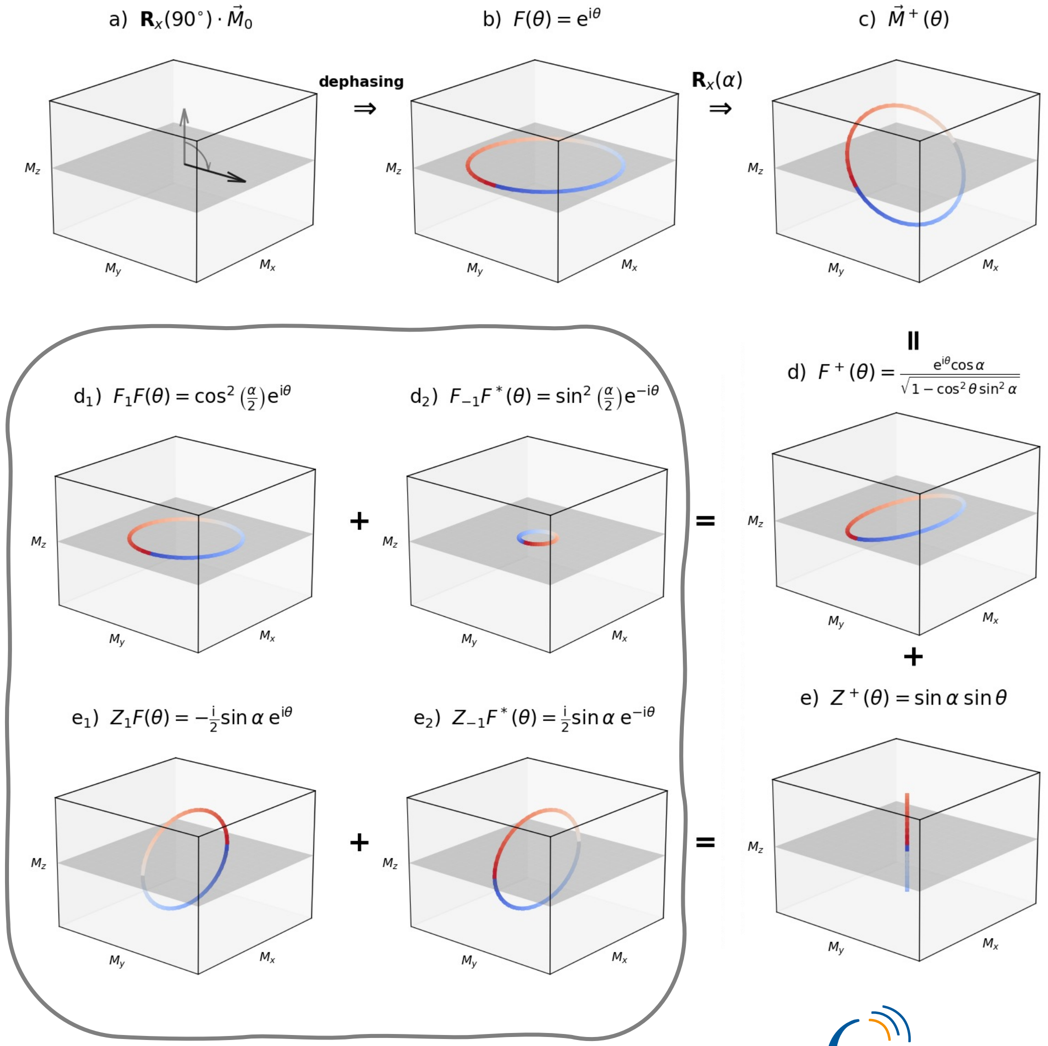
$$Z^{+}(\theta) = -\frac{\mathrm{i}}{2}\sin(\alpha)(F(\theta) - F^{*}(\theta)) + \cos(\alpha)Z(\theta)$$

3. Interpretation of terms

α **= 0°** $\Rightarrow$ $F^{+} = F(\theta)$ **no phase change!**

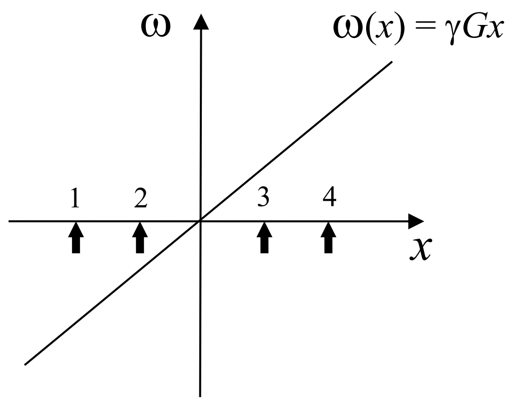
α **= 180°** $\Rightarrow$ $F^{+} = F^{*}(\theta)$ **inverted phase** ($e^{i\theta} \longrightarrow e^{-i\theta}$)
$\Rightarrow$ **spin echo!**

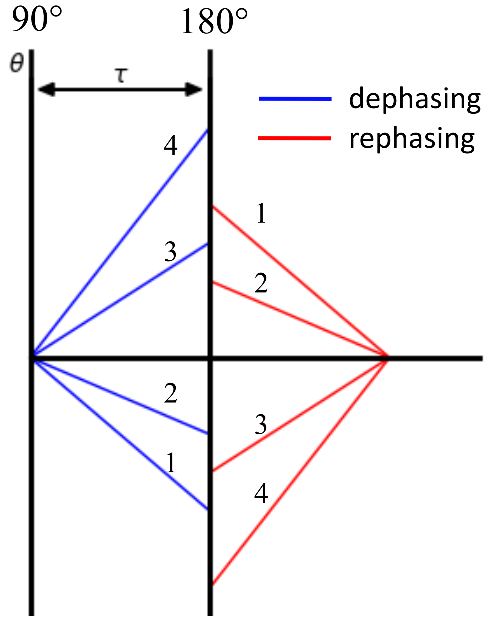
Decomposition of magnetization into canonical
circularly dephased states



[1] D.E. Woessner, Effects of Diffusion in Nuclear Magnetic Resonance Spin-Echo Experiments, The Journal of Chemical Physics 34 (1961) 2057.

Phase graphs and branching rules





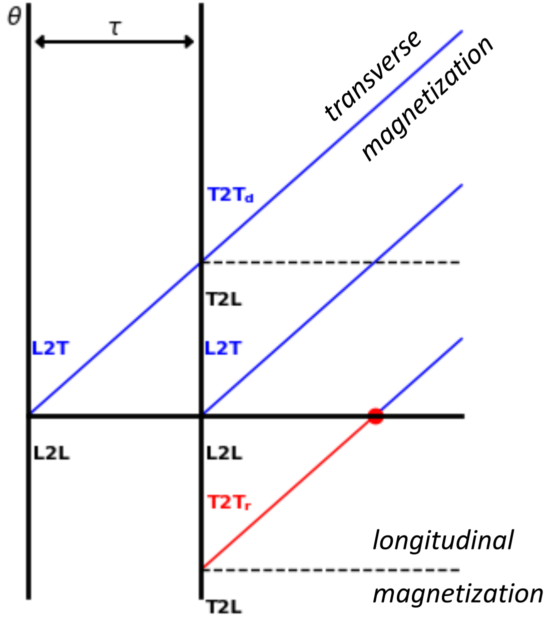
- time-constant spin dephasing (e.g. by gradient field)
- plot the spins' phase as a function of time [1]
- 180° pulse inverts phase sign ($e^{i\theta} \rightarrow e^{-i\theta}$)
- zero crossing in phase graph = spin echo

[1] J.R. Singer, NMR diffusion and flow measurements and an introduction to spin phase graphing, J. Phys. E: Sci. Instrum. 11 (1978) 281–291

[2] J. Hennig, Multiecho imaging sequences with low refocusing flip angles, Journal of Magnetic Resonance 78 (1988) 397–407.

Hennig's Phase Graphs [2]

- Draw a *single* spin representing the entire dephased state
- Account for all possible state conversions ("branching rules")



$$F^{+}(\theta) = +\cos^2\left(\frac{\alpha}{2}\right)F(\theta) + \sin^2\left(\frac{\alpha}{2}\right)F^{*}(\theta) - \mathrm{i}\sin(\alpha)Z(\theta)$$

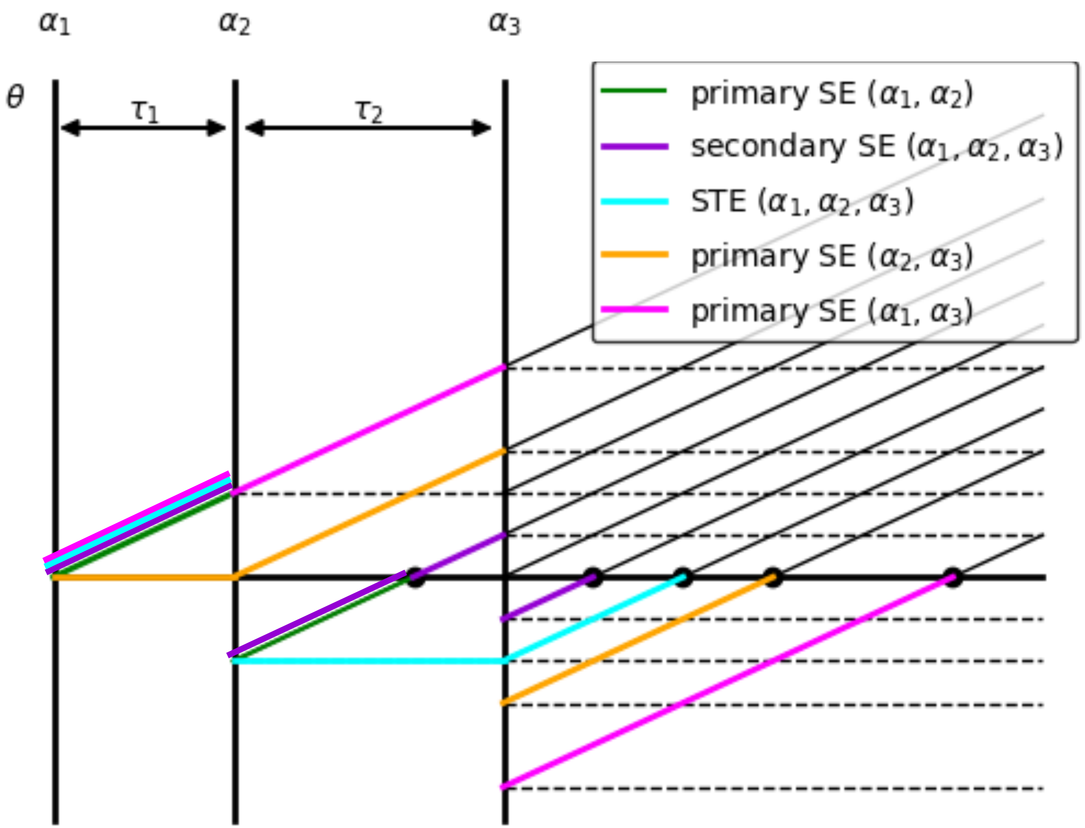
$$Z^{+}(\theta) = -\frac{\mathrm{i}}{2}\sin(\alpha)F(\theta) + \frac{\mathrm{i}}{2}\sin(\alpha)F^{*}(\theta) + \cos(\alpha)Z(\theta)$$

(Woessner decomposition)

#	Conversion Type	trigonometric factor
1	transverse → transverse dephasing ($T2T_d$)	$\cos^2(\alpha/2)$
2	transverse → transverse rephasing ($T2T_r$)	$\sin^2(\alpha/2)$
3	transverse → longitudinal (T2L)	$\frac{1}{2}\sin(\alpha)$
4	longitudinal → transverse (L2T)	$\sin(\alpha)$
5	longitudinal → longitudinal (L2L)	$\cos(\alpha)$

Phase graph of Three RF pulses

- Detect echo times directly from the phase graph
- "Calculate" (collect) amplitudes by multiplying the trigonometric factors along the coherent pathways
- Include relaxation factors by accounting for the magnetization state (transverse/longitudinal)



#	Echo Type	TE	$\lvert A\rvert$
1	primary spin echo (α_1, α_2)	$2\tau_1$	$M_0\ \sin\alpha_1\ \sin^2\left(\frac{\alpha_2}{2}\right)\ e^{-TE/T_2}$
2	secondary spin echo $(\alpha_1, \alpha_2, \alpha_3)$	$2\tau_2$	$M_0\ \sin\alpha_1\ \sin^2\left(\frac{\alpha_2}{2}\right)\ \sin^2\left(\frac{\alpha_3}{2}\right)\ e^{-TE/T_2}$
3	stimulated echo $(\alpha_1, \alpha_2, \alpha_3)$	$2\tau_1 + \tau_2$	$M_0\ \sin\alpha_1\ \frac{1}{2}\sin\alpha_2\ \sin\alpha_3\ e^{-2\tau_1/T_2}\ e^{-\tau_2/T_1}$
4	primary spin echo (α_2, α_3)	$\tau_1 + 2\tau_2$	$Z_0(\tau_1)\ \sin\alpha_2\ \sin^2\left(\frac{\alpha_3}{2}\right)\ e^{-TE/T_2}$, where $Z_0(\tau_1) = M_0(1-(1-\cos\alpha_1)e^{-\tau_1/T_1})$
5	primary spin echo (α_1, α_3)	$2(\tau_1 + \tau_2)$	$M_0\ \sin\alpha_1\ \cos^2\left(\frac{\alpha_2}{2}\right)\ \sin^2\left(\frac{\alpha_3}{2}\right)\ e^{-TE/T_2}$

#	Conversion Type	trigonometric factor
1	transverse $\rightarrow$ transverse dephasing ($T2T_d$)	$\cos^2(\alpha/2)$
2	transverse $\rightarrow$ transverse rephasing ($T2T_r$)	$\sin^2(\alpha/2)$
3	transverse $\rightarrow$ longitudinal (T2L)	$\frac{1}{2}\sin(\alpha)$
4	longitudinal $\rightarrow$ transverse (L2T)	$\sin(\alpha)$
5	longitudinal $\rightarrow$ longitudinal (L2L)	$\cos(\alpha)$

Phase graph of Three RF pulses

- Detect echo times directly from the phase graph
- "Calculate" (collect) amplitudes by multiplying the trigonometric factors along the <u>coherent pathways</u>
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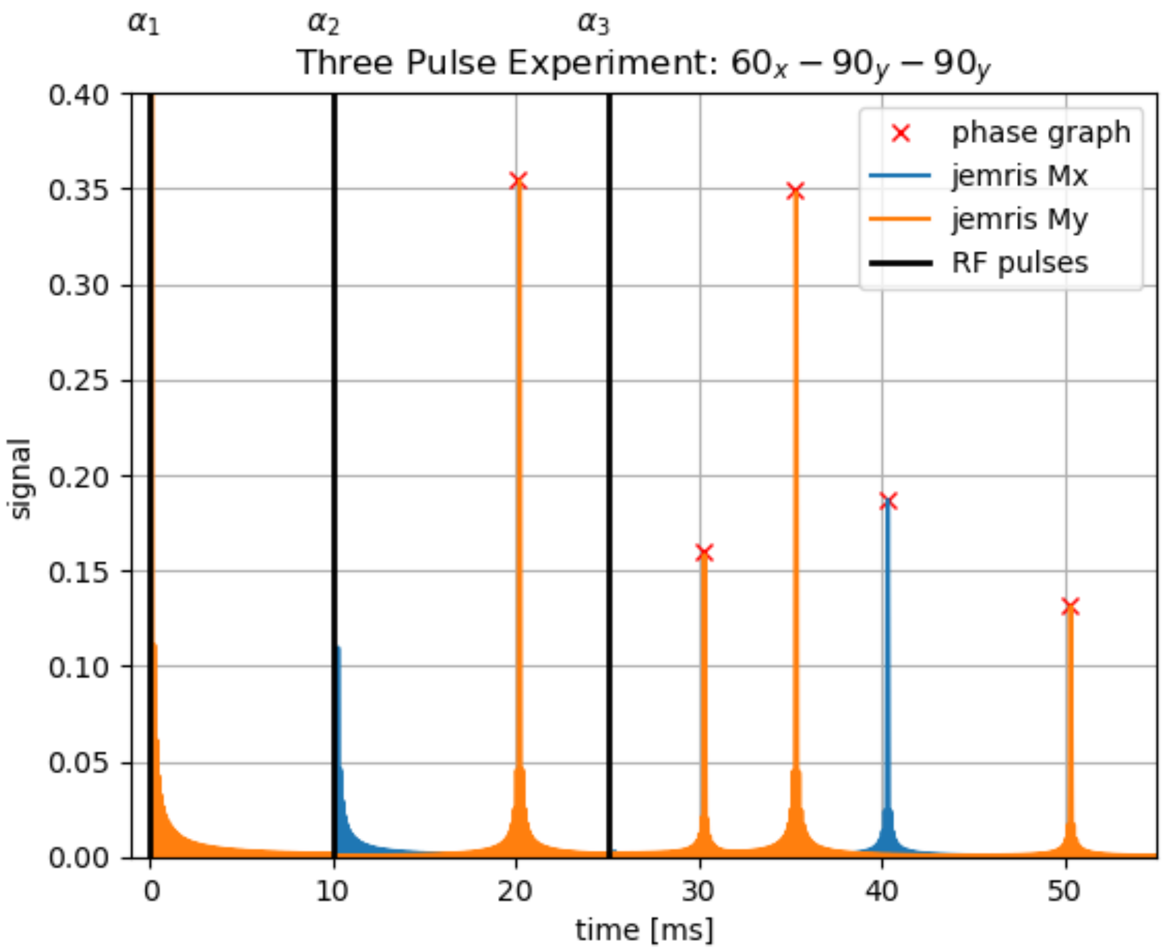
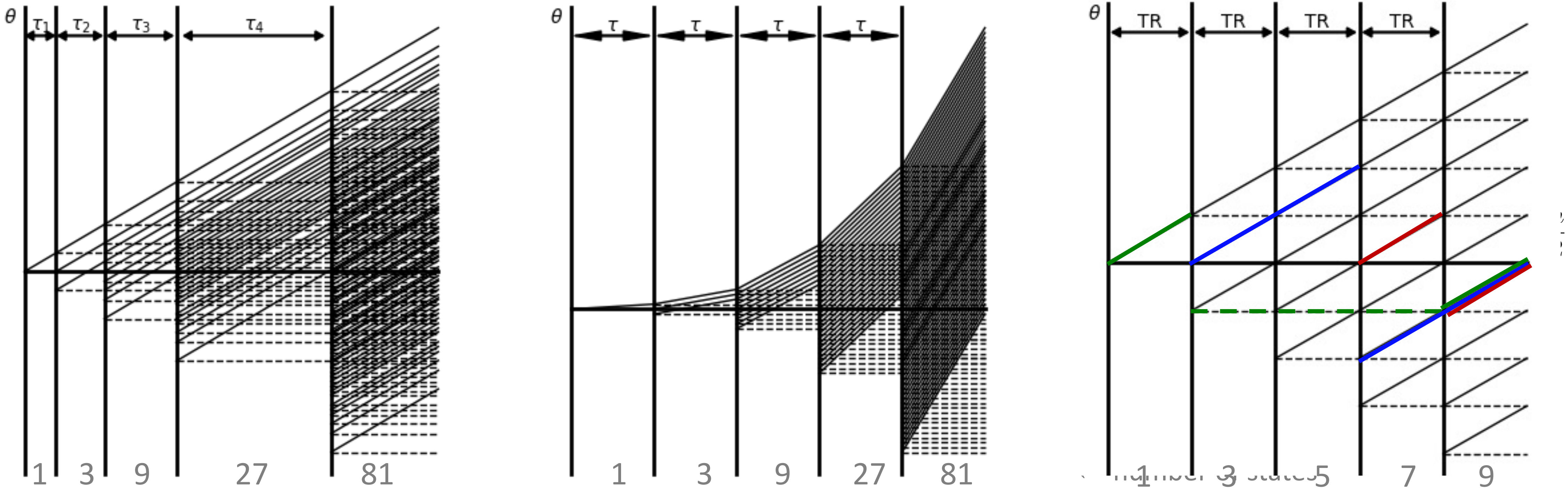


#	Echo Type	TE	$\|A\|$
1	primary spin echo (α_1, α_2)	$2\tau_1$	$M_0 \sin\alpha_1 \sin^2\left(\frac{\alpha_2}{2}\right) e^{-TE/T_2}$
2	secondary spin echo $(\alpha_1, \alpha_2, \alpha_3)$	$2\tau_2$	$M_0 \sin\alpha_1 \sin^2\left(\frac{\alpha_2}{2}\right) \sin^2\left(\frac{\alpha_3}{2}\right) e^{-TE/T_2}$
3	stimulated echo $(\alpha_1, \alpha_2, \alpha_3)$	$2\tau_1 + \tau_2$	$M_0 \sin\alpha_1 \frac{1}{2}\sin\alpha_2 \sin\alpha_3 e^{-2\tau_1/T_2} e^{-\tau_2/T_1}$
4	primary spin echo (α_2, α_3)	$\tau_1 + 2\tau_2$	$Z_0(\tau_1) \sin\alpha_2 \sin^2\left(\frac{\alpha_3}{2}\right) e^{-TE/T_2}$, where $Z_0(\tau_1) = M_0(1-(1-\cos\alpha_1)e^{-\tau_1/T_1})$
5	primary spin echo (α_1, α_3)	$2(\tau_1 + \tau_2)$	$M_0 \sin\alpha_1 \cos^2\left(\frac{\alpha_2}{2}\right) \sin^2\left(\frac{\alpha_3}{2}\right) e^{-TE/T_2}$

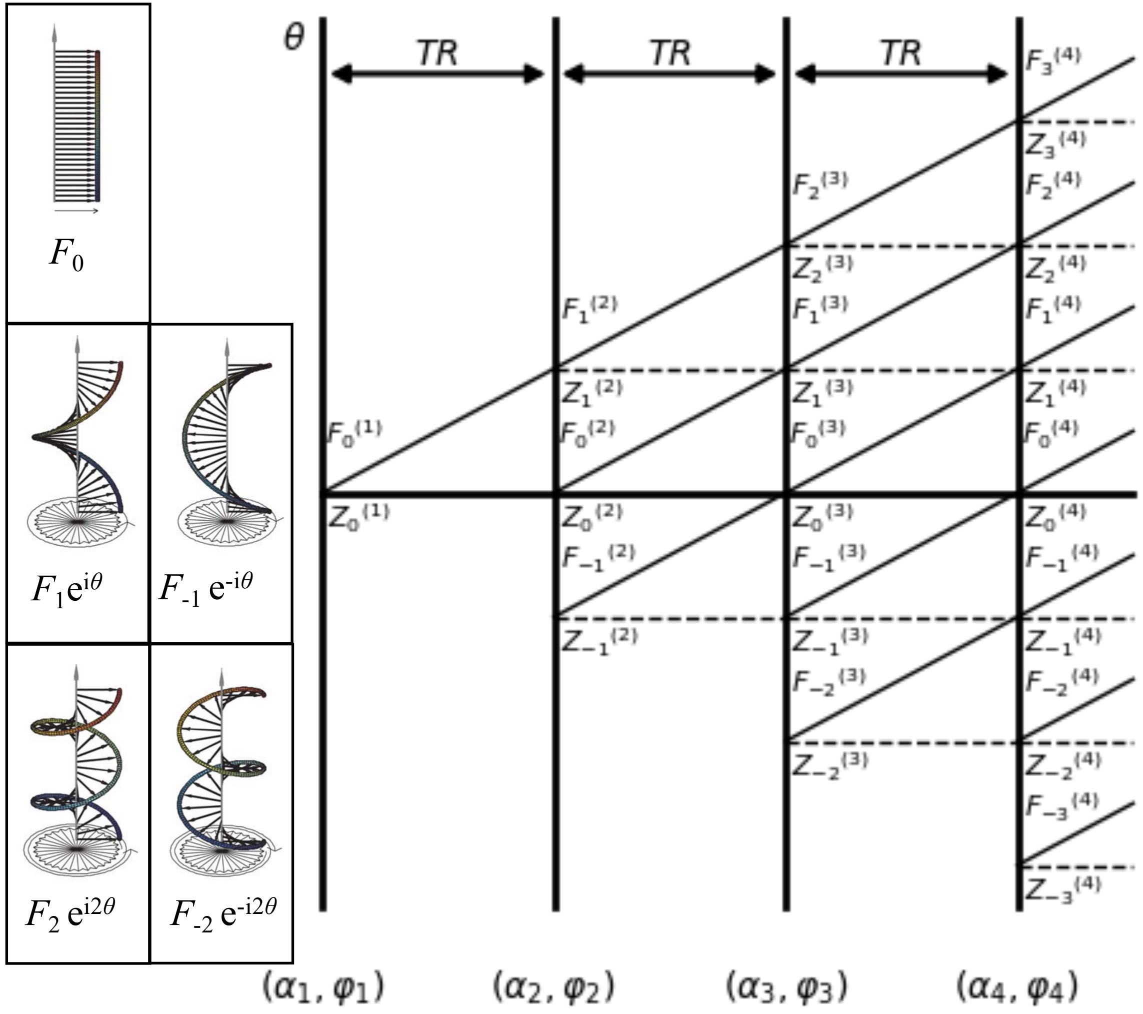
1	transverse $\rightarrow$ transverse dephasing ($T2T_d$)	$\cos^2(\alpha/2)$
2	transverse $\rightarrow$ transverse rephasing ($T2T_r$)	$\sin^2(\alpha/2)$
3	transverse $\rightarrow$ longitudinal (T2L)	$\frac{1}{2}\sin(\alpha)$
4	longitudinal $\rightarrow$ transverse (L2T)	$\sin(\alpha)$
5	longitudinal $\rightarrow$ longitudinal (L2L)	$\cos(\alpha)$

Exponential grow reduces to linear grow, if θ is constant $\forall$ TR $\Rightarrow$ EPG !



- In general, each state gives rise to three new states $\Rightarrow$ 3^{n-1} states after the n-th pulse! (transverse *and* longitudinal, respectively)
- Incomputable for hundreds of pulses as in an MRI sequence (and mostly useless due to many unwanted echoes – to be discussed later …)
- If there is constant dephasing between RF pulses, then the state number grows only linearly
- However, the number of coherent pathways is still the same $\Rightarrow$ additive complex signals !

Hennig's Extended Phase Graph



The MR signal after the n-th pulse *must be* periodic with respect to θ
$\Rightarrow$ Fourier Series representation[1]

$$F(\theta) = \sum_{k=-n}^{n} F_k \, e^{ik\theta}$$

$$Z(\theta) = \sum_{k=-n}^{n} Z_k \, e^{ik\theta} \;\; (\Rightarrow Z_k = Z_{-k}^*)$$

By construction, the complex Fourier coefficients correspond to the EPG state amplitudes! [2,3] They can be efficiently calculated:

1. Rotation (mixing) of states: RF pulse with flip angle α_n and phase φ_n

$$\begin{pmatrix} F_k \\ F_{-k}^* \\ Z_k \end{pmatrix}^+ = \begin{pmatrix} \cos^2 \frac{\alpha_n}{2} & e^{i2\varphi_n} \sin^2 \frac{\alpha_n}{2} & -ie^{i\varphi_n} \sin \alpha_n \\ e^{-i2\varphi_n} \sin^2 \frac{\alpha_n}{2} & \cos^2 \frac{\alpha_n}{2} & ie^{-i\varphi_n} \sin \alpha_n \\ \frac{-i}{2} e^{-i\varphi_n} \sin \alpha_n & \frac{i}{2} e^{i\varphi_n} \sin \alpha_n & \cos \alpha_n \end{pmatrix} \cdot \begin{pmatrix} F_k \\ F_{-k}^* \\ Z_k \end{pmatrix}^-$$

2. Time evolution of states: dephasing and relaxation in TR

$e^{ik\theta} \longrightarrow e^{i(k+1)\theta}$ $\qquad E_{1/2} = \exp[-TR/T_{1/2}]$

$$F_{k+1}^-(n+1) = E_2 \, F_k^+(n) \;\forall\; k \in \mathbb{N}$$
$$Z_k^-(n+1) = E_1 \, Z_k^+(n) \;\forall\; k \neq 0$$
$$Z_0^-(n+1) = E_1 \, Z_0^+(n) + M_0(1 - E_1)$$

[1] R. Kaiser, E. Bartholdi, R. Ernst, Diffusion and field-gradient effects in NMR Fourier spectroscopy, The Journal of Chemical Physics 60 (1974) 2966.
[2] J. Hennig, Echoes - how to generate, recognize, use or avoid them in MR-imaging sequences. Part I, Concepts Magn. Reson. 3 (1991) 125–143.
[3] J. Hennig, Echoes - how to generate, recognize, use or avoid them in MR-imaging sequences. Part II, Concepts Magn. Reson. 3 (1991) 179–192.

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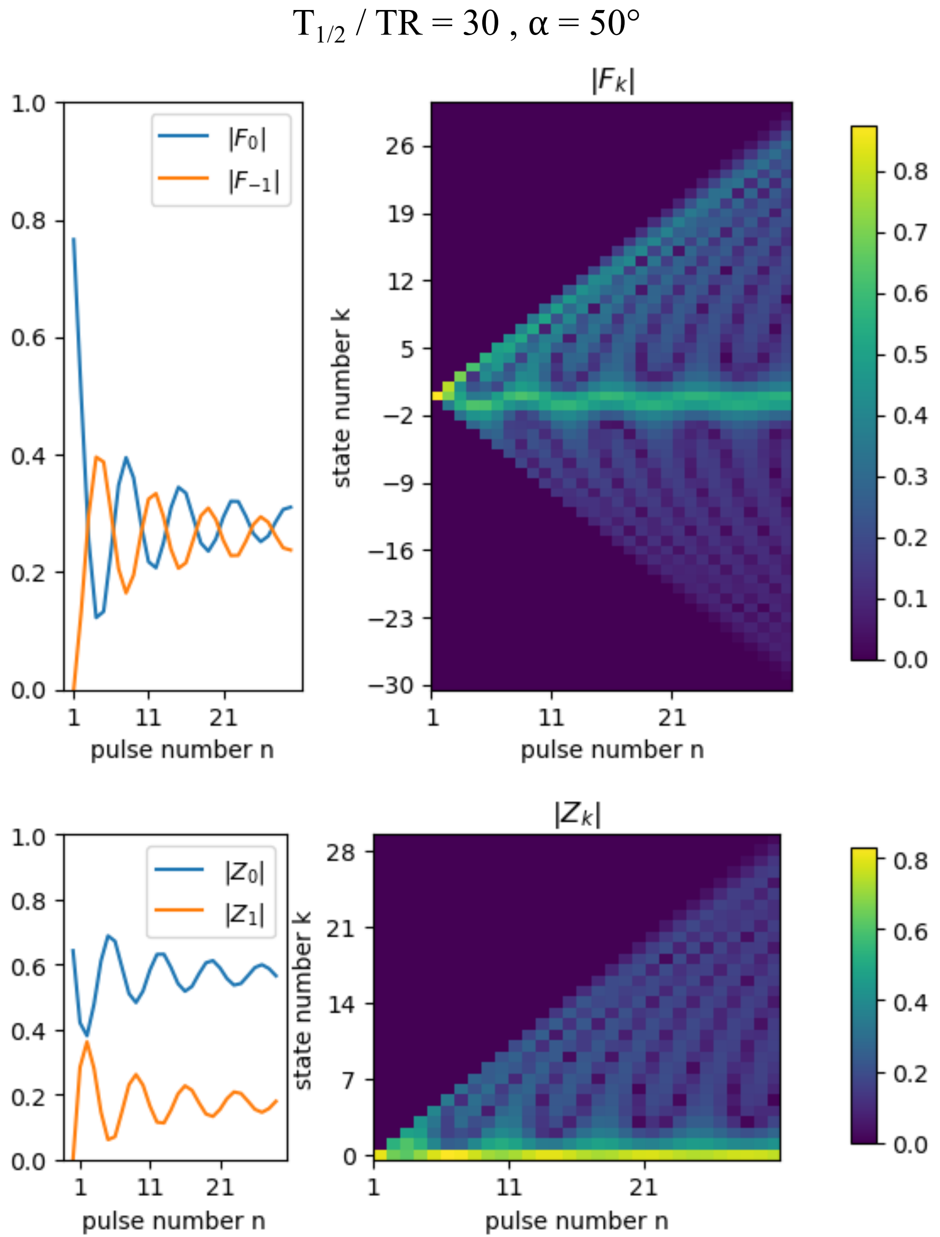
Extended Phase Graph Algorithm

- Many open-source packages available
- **`pyepg`**: fast cpp implementation with python wrapper
  https://github.com/mrphysics-bonn/EPGpp

					F^+_4, Z^+_4
				F^+_3, Z^+_3	F^+_3, Z^+_3
			F^+_2, Z^+_2	F^+_2, Z^+_2	F^+_2, Z^+_2
		F^+_1, Z^+_1	F^+_1, Z^+_1	F^+_1, Z^+_1	F^+_1, Z^+_1
$Z^-_0 = M_0$	F^+_0, Z^+_0	F^+_0, Z^+_0	F^+_0, Z^+_0	F^+_0, Z^+_0	F^+_0, Z^+_0
		F^+_{-1}, Z^+_{-1}	F^+_{-1}, Z^+_{-1}	F^+_{-1}, Z^+_{-1}	F^+_{-1}, Z^+_{-1}
			F^+_{-2}, Z^+_{-2}	F^+_{-2}, Z^+_{-2}	F^+_{-2}, Z^+_{-2}
				F^+_{-3}, Z^+_{-3}	F^+_{-3}, Z^+_{-3}
					F^+_{-4}, Z^+_{-4}
n=0	n=1	n=2	n=3	n=4	n=5

`pyepg.Step()` ...



Extended Phase Graph Algorithm

- Many open-source packages available
- **`pyepg`**: fast cpp implementation with python wrapper
  https://github.com/mrphysics-bonn/EPGpp

					F_4^+,Z_4^+
				F_3^+,Z_3^+	F_3^+,Z_3^+
			F_2^+,Z_2^+	F_2^+,Z_2^+	F_2^+,Z_2^+
		F_1^+,Z_1^+	F_1^+,Z_1^+	F_1^+,Z_1^+	F_1^+,Z_1^+
$Z_0^- = M_0$	F_0^+,Z_0^+	F_0^+,Z_0^+	F_0^+,Z_0^+	F_0^+,Z_0^+	F_0^+,Z_0^+
		F_{-1}^+,Z_{-1}^+	F_{-1}^+,Z_{-1}^+	F_{-1}^+,Z_{-1}^+	F_{-1}^+,Z_{-1}^+
			F_{-2}^+,Z_{-2}^+	F_{-2}^+,Z_{-2}^+	F_{-2}^+,Z_{-2}^+
				F_{-3}^+,Z_{-3}^+	F_{-3}^+,Z_{-3}^+
					F_{-4}^+,Z_{-4}^+
n=0	n=1	n=2	n=3	n=4	n=5

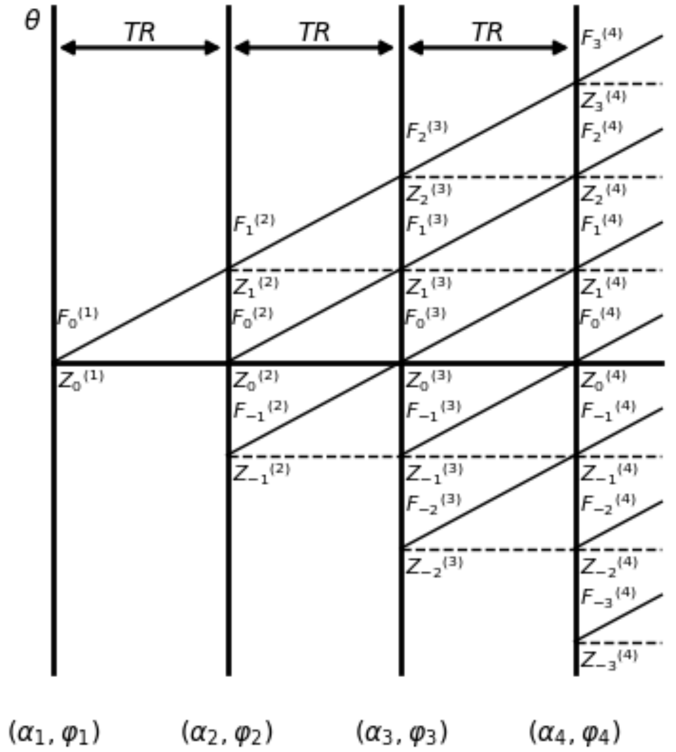
`pyepg.Step()` ...

```python
import pyepg
M0 = 1.0 ; T1 = 300.0 ; T2 = 300.0 ; TR = 10.0

EPG = pyepg.PyEPG(M0,T1,T2,TR)  # instantiate pyepg object
print('\nEquilibrium')
print('Z^-( 0) = %5.2f + i * %5.2f' % (EPG.GetReZb(0) , EPG.GetImZb(0)) )

EPG.Step(fa=90,ph=90) # apply first step of EPG (RF pulse & time evolution)
print('\nafter 1st pulse')
print('Z^+( 0) = %5.2f + i * %5.2f' % (EPG.GetReZa(0) , EPG.GetImZa(0)) )
print('F^+( 0) = %5.2f + i * %5.2f' % (EPG.GetReFa(0) , EPG.GetImFa(0)) )

EPG.Step(fa=90,ph=0)  # apply next step of EPG (RF pulse & time evolution)
print('\nafter 2nd pulse')
print('Z^+( 0) = %5.2f + i * %5.2f' % (EPG.GetReZa(0) , EPG.GetImZa(0)) )
print('Z^+( 1) = %5.2f + i * %5.2f' % (EPG.GetReZa(1) , EPG.GetImZa(1)) )
print('F^+( 0) = %5.2f + i * %5.2f' % (EPG.GetReFa(0) , EPG.GetImFa(0)) )
print('F^+( 1) = %5.2f + i * %5.2f' % (EPG.GetReFa(1) , EPG.GetImFa(1)) )
print('F^+(-1) = %5.2f + i * %5.2f' % (EPG.GetReFa(-1), EPG.GetImFa(-1)))
```

✓ 0.0s Python

```
Equilibrium
Z^-( 0) =  1.00 + i *  0.00

after 1st pulse
Z^+( 0) =  0.00 + i *  0.00
F^+( 0) =  1.00 + i * -0.00

after 2nd pulse
Z^+( 0) =  0.00 + i *  0.00
Z^+( 1) = -0.00 + i *  0.48
F^+( 0) =  0.00 + i * -0.03
F^+( 1) =  0.48 + i * -0.00
F^+(-1) =  0.48 + i *  0.00
```

EPG of the TSE sequence (RARE[1,2])

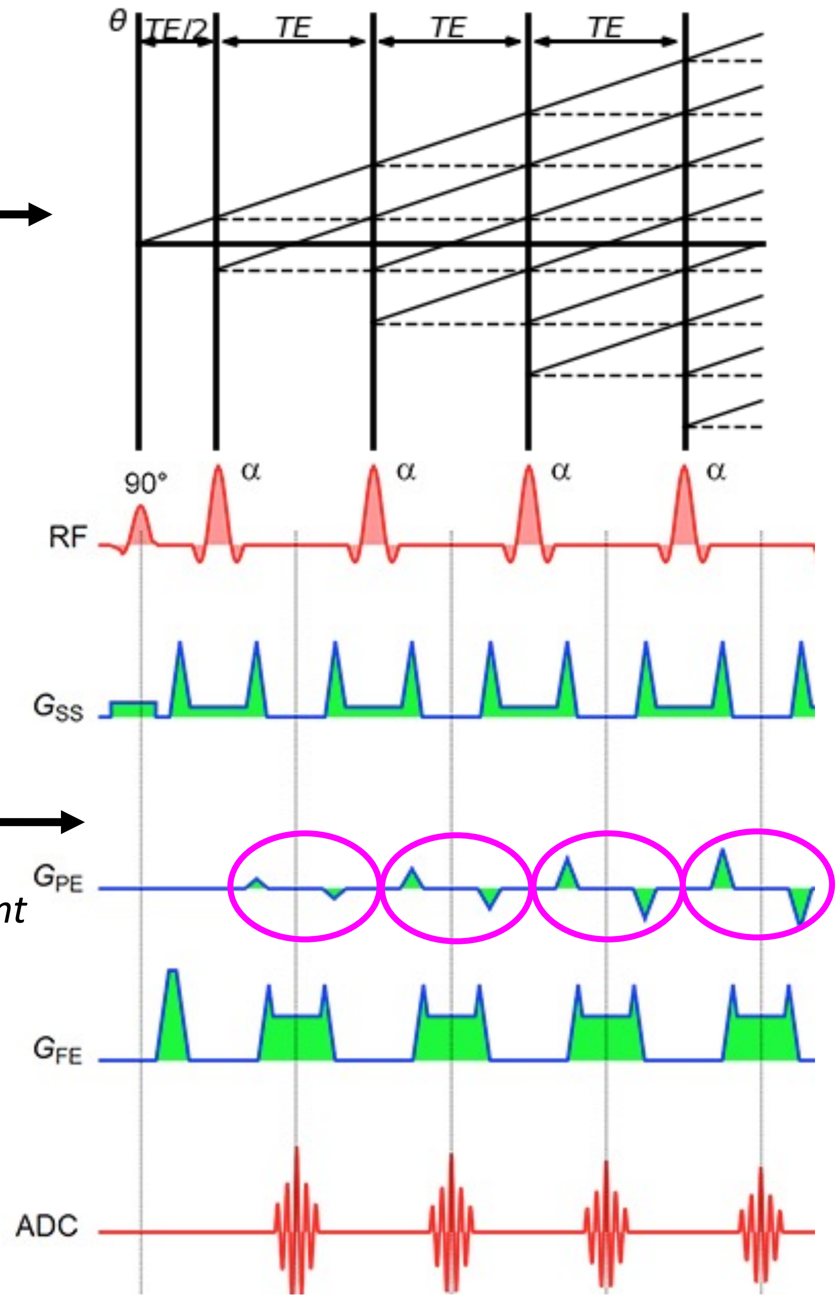


special case of the general EPG, where

- TR = TE/2
- α_1 = 90° (excitaiton)
- α_{2n} = α (refocusing)
- α_{2n+1}= 0° (echoes)

phase encode rewinder gradients after readout

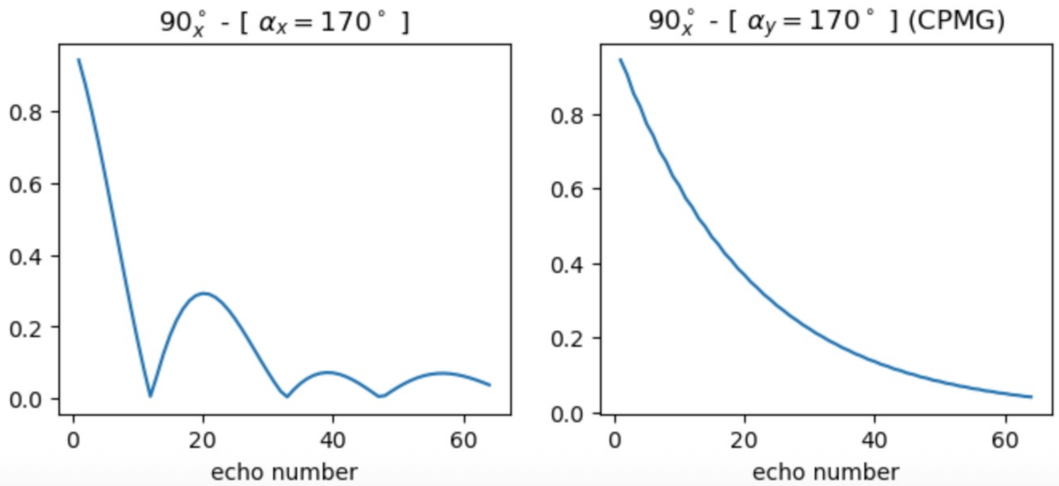
- “EPG condition”: *constant dephasing per TR*
- avoids spurious echoes (example on next slide)

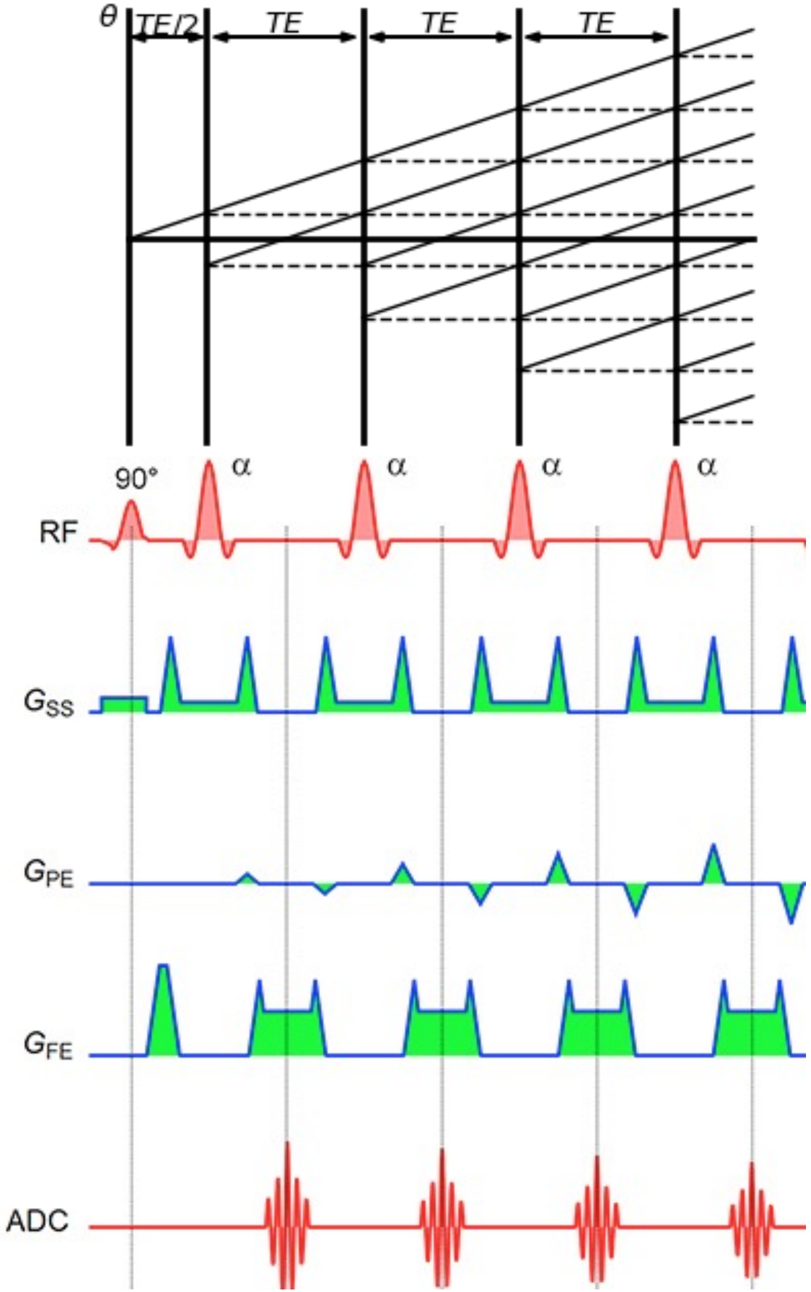


[1] J. Hennig et al, Magnetic Resonance in Medicine 3 (1986) 823–33.
[2] J. Hennig, Journal of Magnetic Resonance 78 (1988) 397–407.

```python
def tse(M0, T1, T2, alpha, phi, TE, N):
    EPG = pyepg.PyEPG(M0, T1, T2, TE/2)
    signal = np.zeros((N,))
    EPG.Step(fa=90,ph=0)            # excitation around x-axis and TE/2 evolution
    for i in range(N):              # TSE train
        EPG.Step(fa=alpha, ph=phi)  # refocusing around phi-axis and TE/2 evolution
        EPG.Step(fa=0    , ph=0)    # another TE/2 evolution
        signal[i]=EPG.GetMagFa(0)   # spin echo directly after the last "pulse" (zero flip-angle)
    return signal

def plot_tse_signal(signal,axis,alpha,subplot):…

# MR parameters
M0 = 1 ; T1 = 1000; T2 = 100

# Sequence parameters: echo time; flip angle; number of echoes
TE = 5 ; alpha = 170; N = 64

# TSE with a)  0 degree refocusing axis (same axis as for the excitation)
#          b) 90 degree refocusing axis (CPMG = Carr-Purcell-Meiboom-Gill)
PHI = [0,90]
fig, ax = plt.subplots(1,2,figsize=(8,3))
for i in range(2):
    signal = tse(M0, T1, T2, alpha, PHI[i], TE, N)
    plot_tse_signal(signal,ax,alpha,i)
plt.show()
```

✓ 0.0s   Python





DZNE

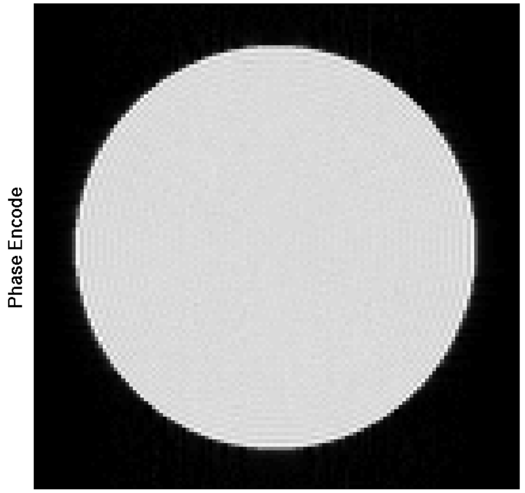
EPG of steady-state sequences

MRI sequences with short TR ($\lesssim T_2$) and <u>constant flip angle</u>

- Drive the magnetisation into a dynamic steady state
- Constant dephasing per TR needed (**PE rewinders**)
- Coherent pathway selection depends on gradient scheme [1]
- Generally, complex contrast properties (T_1 and T_2 dependence)
- EPGs are well-suited to study (the mysteries of) RF spoiling [2]

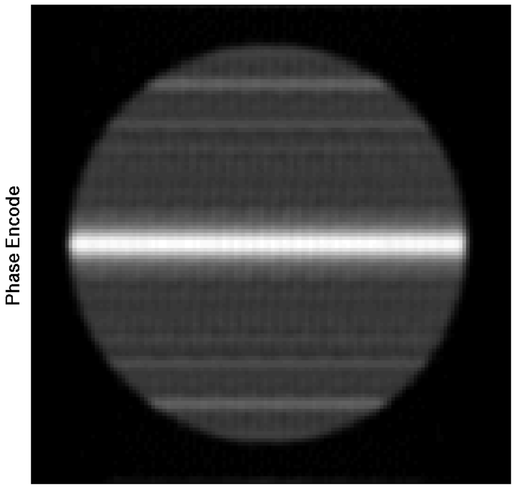
$T_2 = 10 \cdot TR$

with PE rewinder

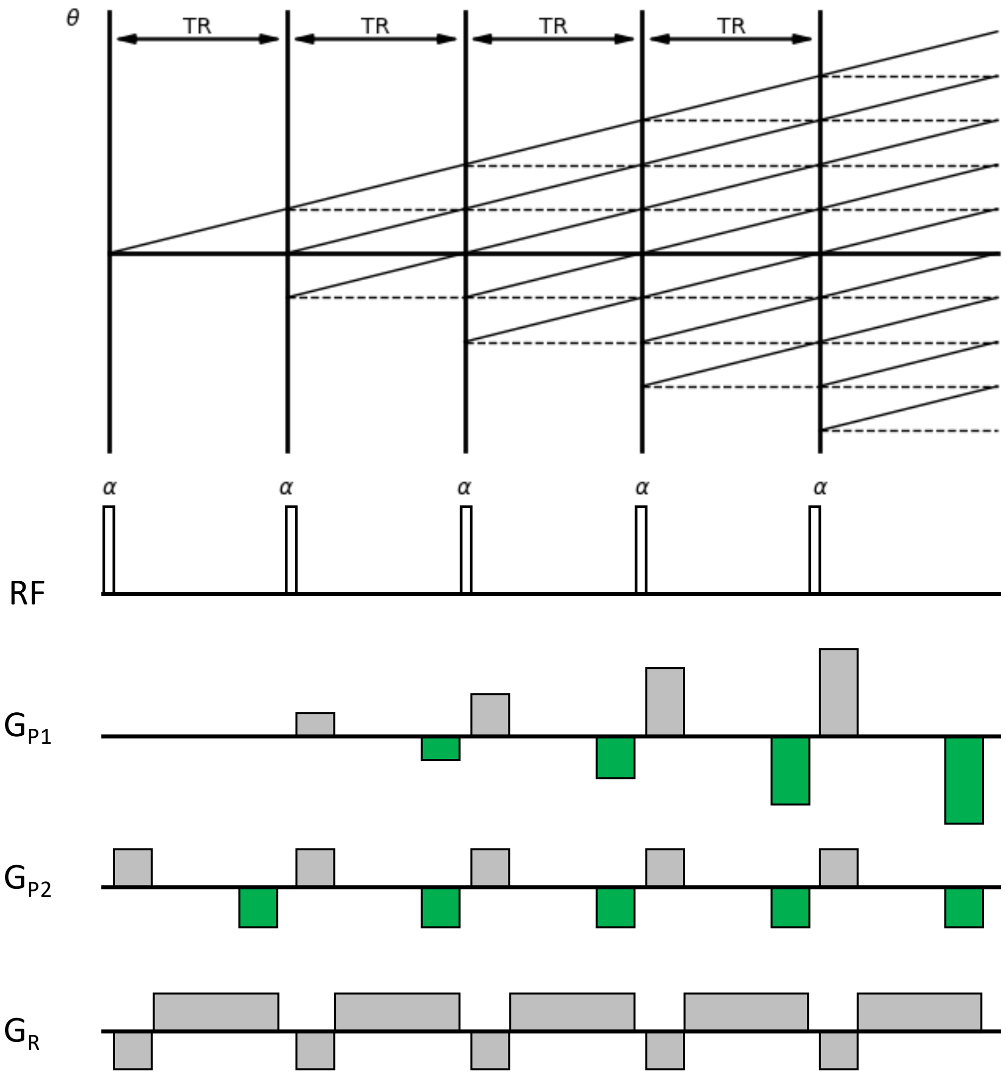


Readout (Freq. Enc.)

no PE rewinder



Readout (Freq. Enc.)



[1] K. Scheffler. Concepts in MR 11 (1999) 291–304.
[2] W.T. Sobol, D.M. Gauntt, J. JMRI 6 (1996) 384–98.

DZNE

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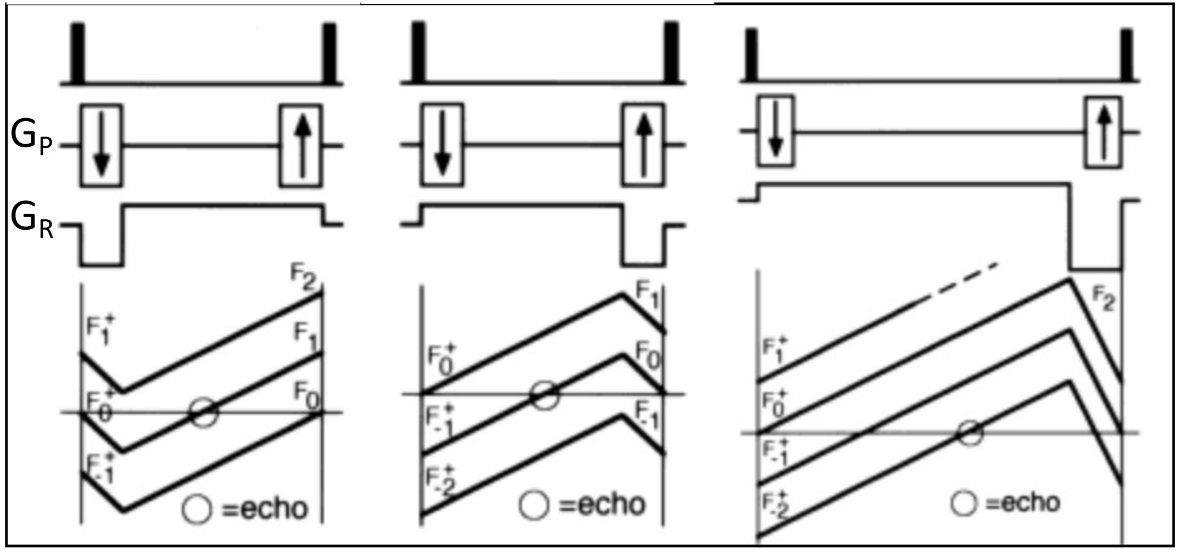
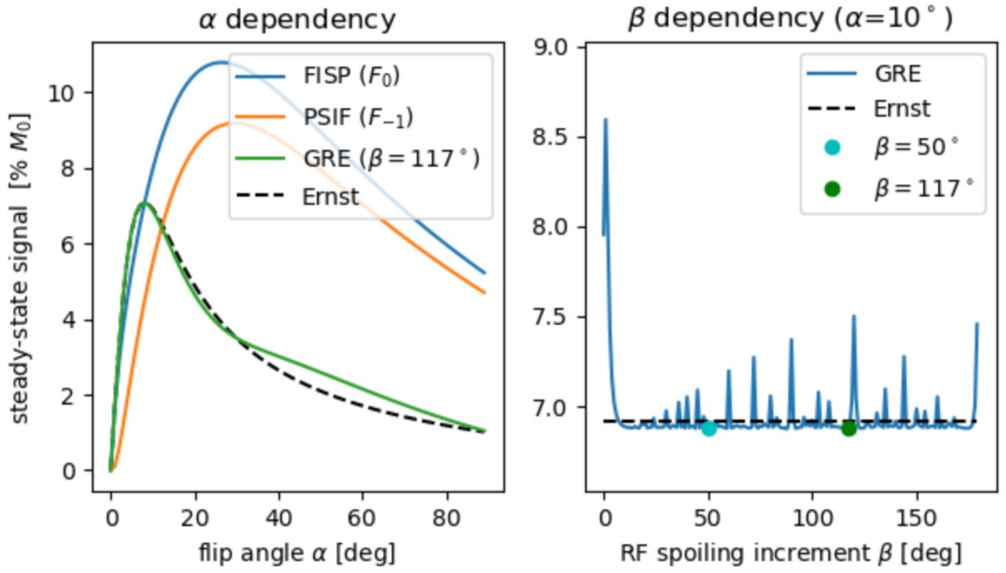
F_0 imaging (FID):	F_{-1} imaging (ECHO):
GRE, FLASH, FISP, GRASS, FAST, FFE, …	PSIF, T2-FAST, FFE-T2, …



[1] K. Scheffler. Concepts in MR 11 (1999) 291–304.
[2] W.T. Sobol, D.M. Gauntt, J. JMRI 6 (1996) 384–98.


```python
# fixed parameters
M0  = 1 ; T1  = 1000 ; T2  = 100 ; TR  = 10 ; E1  = np.exp(-TR/T1)
#Ernst Equation for comparison
ernst = lambda a : M0*np.sin(np.deg2rad(a))*(1-E1)/(1-np.cos(np.deg2rad(a))*E1)
# function to calculate steady state signal with pyepg class
EPG = pyepg.PyEPG(M0, T1, T2, TR)
def epg_ss(alpha, beta, num=0, tol = 1e-10 ):
    EPG.Equilibrium()
    EPG.StepsToSS(fa=alpha, Qph=beta, tol=tol)
    return EPG.GetMagFa(num)
#Experiment 1: flip angle dependency
N = 90 ; alpha  = np.linspace(0,90,N,endpoint=False) ; v = np.zeros((N,))
SEQS = {'FISP':[0,0,v.copy()],'PSIF':[0,-1,v.copy()], 'GRE':[117,0,v.copy()]}
for i in range(N):
    for seq,val in SEQS.items():
        val[2][i] = epg_ss(alpha[i], val[0], val[1])
#Experiment 2: RF spoiling increment dependency in GRE (constant alpha=10)
N = 180 ; RFspGRE = np.zeros((N,)); beta = np.linspace(0,180,N,endpoint=False)
for i in range(N):
    RFspGRE[i] = epg_ss(10, beta[i] )
#plot results
if True: ...
```

✓ 0.9s Python

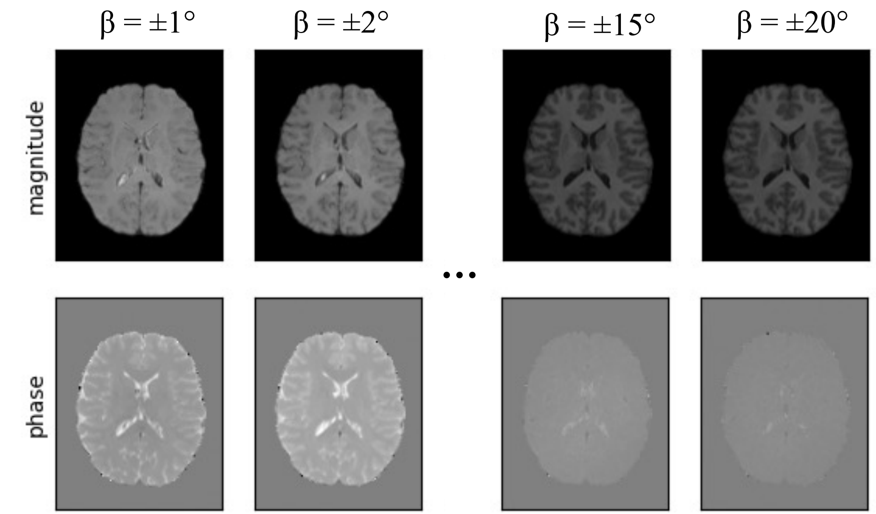


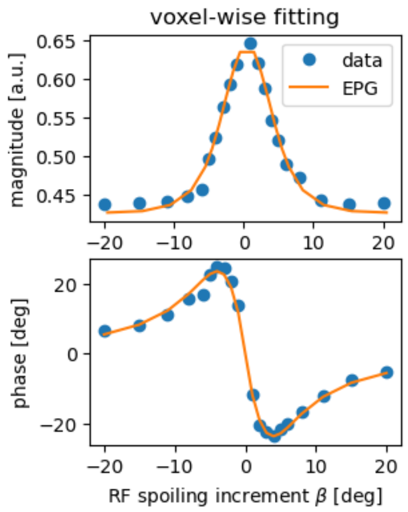
EPG for signal modeling: parameter estimation

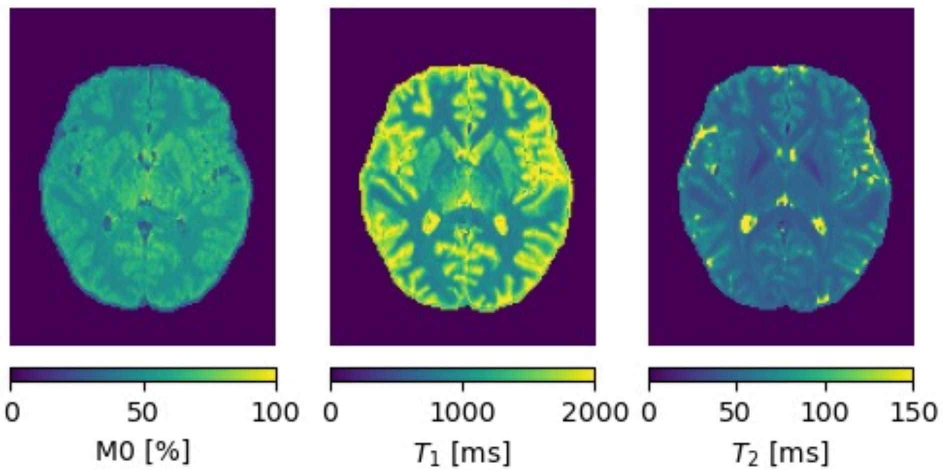
$$\min_{M_0,T_1,T_2} \|\mathbf{S}_{EPG}(M_0,T_1,T_2;P) - \mathbf{S}_{aq}(P)\| \qquad P = (TR, TE, TI, \alpha, \ldots)$$

- Voxel-wise modeling of the acquired signal $\mathbf{S}_{aq}$
- Multiple images with different contrast e.g. through varying sequence parameters (TE, TI, ...): $\mathbf{S} = [S_1, S_2, S_3, \ldots.]$
- Signal needs to be sensitive to parameter changes $(\Delta T_1, \Delta T_2, \ldots)$
- The optimizer is crucial – NNLS is often a good choice. (e.g. `scipy.optimize.least_squares`)
- Many different approaches exist such as MRF (Fingerprinting)

Example: (M_0, T_1, T_2) estimation from GRE with small RF spoiling increments [1]







[1] D. Wang, R. Stirnberg, T. Stöcker, Magn Reson Med. 2024;92:2328–2342

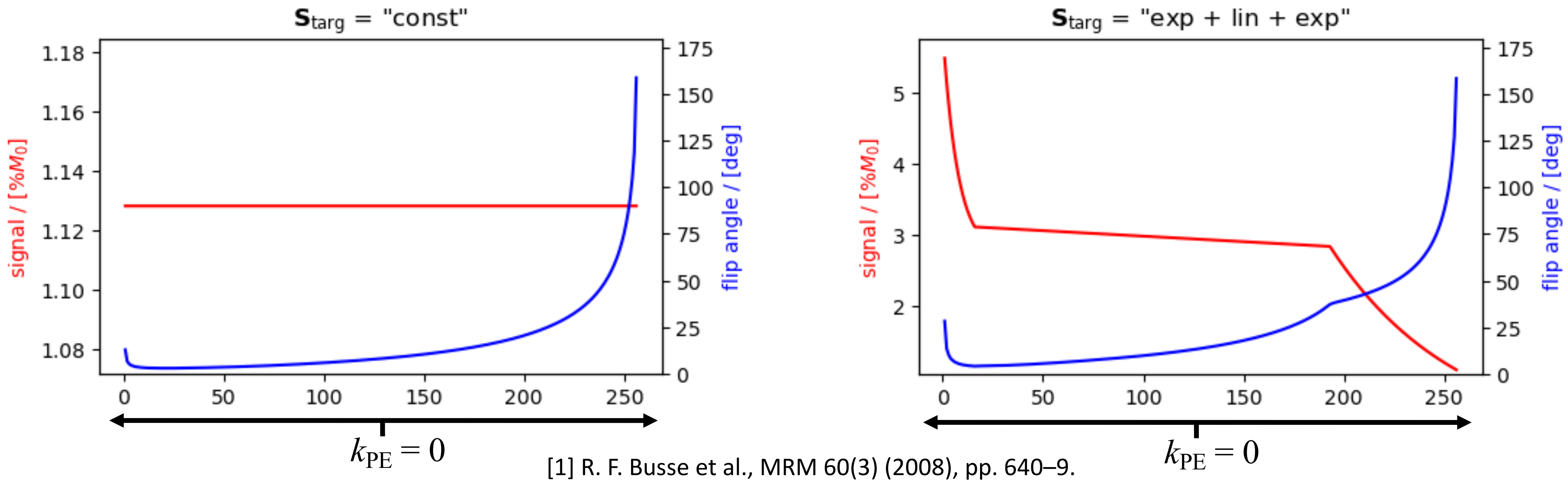
EPG for signal shaping: sequence optimization

$$\max_P \|\mathbf{S}_{EPG}(P)\| \quad \text{subject to} \quad \min_P \left\| \frac{\mathbf{S}_{EPG}(P)}{\|\mathbf{S}_{EPG}(P)\|} - \mathbf{S}_{\text{targ}} \right\| \quad \text{with} \quad \|\mathbf{S}_{\text{targ}}\| = 1$$

- Find sequence parameters, P, which maximize the signal vector constrained to a target signal shape, $\boldsymbol{S}_{\text{targ}}$
- Usually, the signal vector is a transient echo train $\mathbf{S} = [S_1, S_2, S_3, \ldots]$
- Often used to control the PSF while optimizing SNR

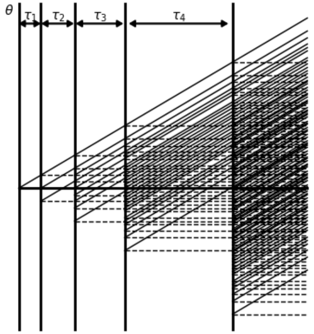
<u>Example in demo session</u>: find variable flip-angles for the TSE sequence with linear phase encode order [1]

- constant signal shape $\rightarrow$ optimal PSF but low SNR
- “exp+lin+exp” shape $\rightarrow$ acceptable PSF and higher SNR



[1] R. F. Busse et al., MRM 60(3) (2008), pp. 640–9.

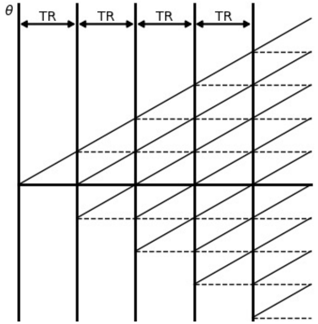
Summary: (Extended) Phase Graphs



- General approach, few assumptions (e.g. instantaneous RF)
- *Echo amplitudes* = analytic solutions of the *Signal Equation*
- Magnetization decomposition (canonical *circularly dephased states*)
- *Phase graphs:* plot the states' phase evolution vs. time, incl. state splitting
- 1 state ⟶ RF pulse ⟶ 3 new states ⟹ *exponential state growth*

$$F^+(\theta) = +\cos^2\left(\frac{\alpha}{2}\right)F(\theta) + \sin^2\left(\frac{\alpha}{2}\right)F^*(\theta) - \mathrm{i}\sin(\alpha)Z(\theta)$$

$$Z^+(\theta) = -\frac{\mathrm{i}}{2}\sin(\alpha)F(\theta) + \frac{\mathrm{i}}{2}\sin(\alpha)F^*(\theta) + \cos(\alpha)Z(\theta)$$



- *Extended phase graphs:* const. dephasing between RF pulses ⟹ EPG algorithm
  - *Linear state growth* & Fourier series expansion of the magnetization
  - Iterative closed-form solution for state amplitudes = Fourier coefficients!
  - Fast signal calculation for most MRI sequences

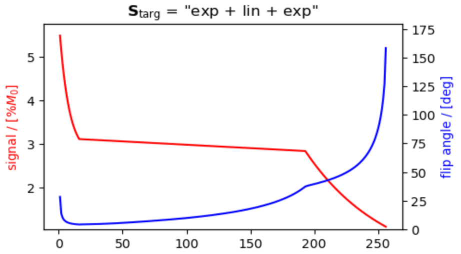
(But: EPG is *not* an imaging theory, does not require gradients, any stationary off-resonance is covered.)

$$F(\theta) = \sum_{k=-n}^{n} F_k\, \mathrm{e}^{\mathrm{i}k\theta}$$

$$Z(\theta) = \sum_{k=-n}^{n} Z_k\, \mathrm{e}^{\mathrm{i}k\theta}$$

EPGs are well-suited forward solvers for many MR inverse problems, e.g.

- Sequence parameter optimization
- MR parameter estimation (qMRI)



Several extensions, e.g.

- Diffusion: straight-forward extension — cf. e.g. M. Weigel, MRM 41 (2015) 266–295.
- Magnetization transfer ⟶ "EPG-X" — S.J. Malik et al, MRM 80 (2018) 767–779.
- Full signal simulation ⟶ "PDGs" — J. Endres et al, MRM 92 (2024) 1189–1204.

DZNE

Thank You