
Assignment (Mode 1): Probability and Statistics

Course: CSEG 2036P | *School of Computing Sciences, UPES*

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Uncertainty is everywhere! Let us use the conceptual tools developed in our *Probability and Statistics* course (CSEG 2036P) to explore this further! Even if we had the best apparatus to measure properties of physical objects, like the mass or velocity of a ball, there may still remain some uncertainty. This is especially so in the realm of quantum physics - the world of the smallest of particles!

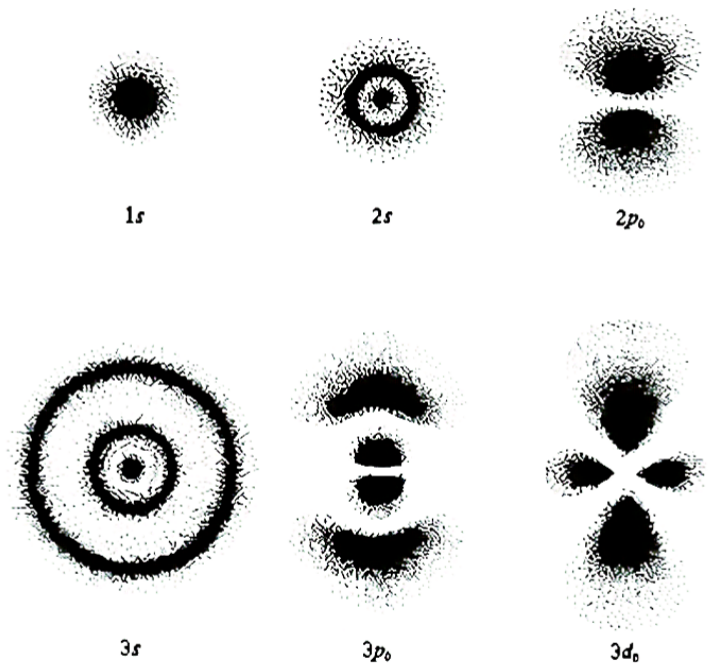

Figure 1: Uncertainty in the position of an electron in an atom is seen with the concept of an orbital, as shown here.

The uncertainty principle in quantum physics is commonly viewed as a fundamental limit on how precisely we can measure incompatible observables. However, our understanding has predominantly focused on the lowest-order approximation, where the incompatibility between these observables is seen as statistical correlation. Can we generalize this in any way?

Let us explore this a little more in this assignment!

1. For a random variable X, we have the moment-generating function given by

$$M(t) = E[e^{tX}]$$

Show that

$$M_X(t) = \sum_{n=0}^{\infty} E[X^n] \frac{t^n}{n!}$$

2. Show that the k^{th} moment of the random variable is given by

$$E[X^k] = \frac{d^k}{dt^k} M_X(t)|_{t=0}$$

3. We can define what is known as a cumulant-generating function given by

$$K(tX) = \log(E[e^{tX}]) = \sum_{n=0}^{\infty} \kappa_n \frac{t^n}{n!} \implies \kappa_n = \frac{d^n}{dt^n} \log(E[e^{tX}])|_{t=0}$$

where κ_m is known as the cumulant of order m. Using the implied result above as well as the result from part 2, show that

$$\kappa_1 = E[X]$$

$$\kappa_2 = E[X^2] - E[X]^2$$

and

$$\kappa_3 = E[X^3] - 3E[X^2][X] + 2E[X]^3$$

4. For two random variables X_1 and X_2, we can define a joint cumulant-generating function

$$K_X(t_1 X_1, t_2 X_2) = \log(E[e^{t_1 X_1 + t_2 X_2}]) = \sum_{m+n=1}^{\infty} \kappa_{mn} \frac{t_1^m t_2^n}{m!n!}$$

where κ_{mn} are called *Cross-Cumulants*.

If we assume a continuous joint probability distribution function $f(X_1, X_2)$, show that

$$E[X_1^{\alpha_1} X_2^{\alpha_2}] = \left[\frac{\partial^{\alpha_1+\alpha_2}}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2}} E[e^{t_1 X_1 + t_2 X_2}] \right]_{t_i=0}$$

This gives us a *Joint Moment*.

Going back to the joint cumulant-generating function, show that

$$\text{Cov}(X_1, X_2) = \left[\frac{\partial^2 K_X}{\partial t_1 \partial t_2} \right]_{t_i=0}$$

5. In Quantum Physics, the state of a system is defined by a *Wavefunction*, often denoted by $|\psi\rangle$. A *Quantum Operator* O, representative of a physical observable, is a transformation that takes a system from an initial state $|\psi_i\rangle$ to a final state $|\psi_f\rangle$: $O|\psi_i\rangle \to |\psi_f\rangle$.

For two physical obervables X and Y, let us consider state vectors $e^{s^*X}|\psi\rangle$ and $e^{tY}|\psi\rangle$, and if expectation values for quantum operators are given by

$$E[O] = \langle\psi|O|\psi\rangle$$

and the *Cauchy-Schwarz inequality* for quantum physics given the state vectors $e^{s^*X}|\psi\rangle$ and $e^{tY}|\psi\rangle$ is

$$E[e^{sX}e^{s^*X}]E[e^{sX}e^{s^*X}] \geq E[e^{sX}e^{tY}]E[e^{sX}e^{tY}]^*$$

where z^* is the complex conjugate of the complex number z.

What is the Cauchy-Schwarz inequality, in probability theory?

Using the *Cauchy-Schwarz inequality for Quantum Physics* above, prove the generalized uncertainty principle

$$K[(s+s^*)X] + K[(t+t^*)Y] \geq K[Z_{st}] + c.c. \forall s, t \in \mathbb{C}$$

Here K signifies the generating function of cumulants to all orders; "*c.c.*" means the complex conjugation, $Z_{st} := \{Z_{10}, Z_{01}, Z_{11}, ...\}$ is a set of operators defined as

$$\log(e^{sX}e^{tY}) = sX + tY + \frac{1}{2}[sX, tY] + ... = Z_{10} + Z_{01} + Z_{11} + ...$$

in light of what is known as the well-known *Baker-Campbell-Hausdorff formula.*

This uncertainty principle gives a fundamental relationship where the combination of statistical properties arising from two observables, representing the left-hand side, is constrained by their statistical interdependence, represented by $K(Z_{st})$ on the right-hand side in terms of cross cumulants. This equation offers a critical advantage as it distinguishes between quantum and classical correlations, particularly evident in higher-order terms within Z_{st} derived from the noncommutative elements in the Baker-Campbell-Hausdorff formula.

6. If we consider the various order of terms in this uncertainty principle, we can get some interesting relations. Show that

$$(s+s^*)\kappa_1(X) + (t+t^*)\kappa_1(Y) = E[sX + tY] + c.c.$$

and

$$|s|^2\kappa_2(X) + |t|^2\kappa_2(Y) \geq \kappa_{11}(sX, tY) + E\left[\frac{1}{2}[sX, tY]\right] + c.c. \forall s, t \in \mathbb{C}$$

This is in contradiction to the classical case, where

$$|s|^2\kappa_2(X) + |t|^2\kappa_2(Y) \geq \kappa_{11}(sX, tY) + c.c. \forall s, t \in \mathbb{C}$$

In instances where the cross cumulant (κ_{11}) equals zero, classical theory suggests there is no limitation on variances. However, in quantum mechanics, even when $\kappa_{11} = 0$, a constraint persists due to Z_{11}. Furthermore, if both $\kappa_{11} = 0$ and the expected value of Z_{11} also equals zero, the traditional Heisenberg uncertainty principle loses significance, while the generalized uncertainty relation maintains its relevance and meaning.