Special Distributions

Course: CSEG 2036P | *School of Computing Sciences, UPES*

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Note: ***Revise*** *all given references for the various distributions.* ***Remember*** *the support for the probability mass/density functions for the different distributions.*

1 Bernoulli Distribution

The probability mass function for a Bernoulli random variable $X \sim \text{Bernoulli}(p)$ is given by

$$P(X;p) = \begin{cases} 1-p \text{ if X} = 0 \\ p \text{ if X} = 1 \end{cases}$$

The expectation value is $E[X] = p$ and variance is $\text{Var}(X) = p(1-p)$.

Ref: Lecture 8, Slide 3

2 Binomial Distribution

The probability mass function for a Binomial random variable $X \sim Bin(n,p)$ is given by

$$P(X = x) =^n C_x p^x (1-p)^{n-x}$$

where the combinatorial term ${}^nC_x = \frac{n!}{x!(n-x)!}$.

2.1 Expectation Value of a Binomial Random Variable

The expectation value of a Binomial random variable can be found as follows:

$$E[X] = \sum_{x=0}^{n} x \times^n C_x p^x (1-p)^{n-x}$$

We now have the identity: $k \times^n C_k = k \times \frac{n!}{k!(n-k)!} = n \times \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} = n \times^{n-1} C_{k-1}$, which makes the expression for the expectation value be

$$E[X] = \sum_{x=1}^{n} n \times^{n-1} C_{x-1} p^x (1-p)^{n-x} = np \times \sum_{x=1}^{n} {}^{n-1}C_{x-1} p^{x-1} (1-p)^{n-1-(x-1)} = np$$

since $\sum_{x=1}^{n} {}^{n-1}C_{x-1} p^{x-1} (1-p)^{n-1-(x-1)} = (p + (1-p)^{n-1} = 1$.

2.2 Variance of a Binomial Random Variable

For finding the variance of a Binomial random variable, let us first find

$$E[X^2] = \sum_{x=0}^{n} x^2 \times^n C_x p^x (1-p)^{n-x}$$

We have the identity: $k^2 \times^n C_k = k^2 \times \frac{n!}{k!(n-k)!} = k \times n \times \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} = nk \times^{n-1} C_{k-1}$, which makes the expression for the variance be

$$E[X^2] = \sum_{x=1}^{n} nx \times^{n-1} C_{x-1} p^x (1-p)^{n-x} = np \times \sum_{x=1}^{n} x \times^{n-1} C_{x-1} p^{x-1} (1-p)^{n-1-(x-1)}$$

Let us consider $n-1 = m$ and $x-1 = j$, and obtain the expression

$$\begin{aligned}\sum_{x=1}^{n} x \times^{n-1} C_{x-1} p^{x-1} (1-p)^{n-1-(x-1)} &= \sum_{j=0}^{m} (j+1).^m C_j p^j (1-p)^{m-j} \\ &= \sum_{j=0}^{m} j.^m C_j p^j (1-p)^{m-j} + \sum_{j=0}^{m} {}^m C_j p^j (1-p)^{m-j} = mp + 1 = (n-1)p + 1\end{aligned}$$

Putting this result back into the sum for $E[X^2]$,

$$E[X^2] = np[(n-1)p + 1] = n^2p^2 - np^2 + np$$

The variance is given by

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = (np)^2 - np^2 + np - (np)^2 = np(1-p)$$

3 Geometric Distribution

If we consider $X \sim \mathrm{Geo}(p)$ where the random variable X represents the number of unsuccessful trials before the first success, then the probability distribution is given by

$$P(X = n) = (1-p)^n \times p$$

The expectation value and variance of the Geometric Distribution is given by

$$E[X] = \frac{q}{p}, \mathrm{Var}(X) = \frac{q}{p^2}$$

Ref: Lecture 8, Slides 12-13

4 Poisson Distribution

A Poisson random variable $X \sim \mathrm{Po}(\lambda)$ has the probability mass function:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

4.1 Expectation Value of Poisson Distribution

The expectation value of a Poisson distribution is given by

$$E[X] = \sum_{k=0}^{\infty} k.\frac{\lambda^k e^{-\lambda}}{k!} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \times e^{\lambda} = \lambda$$

where we have considered the expansion of the exponential series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

4.2 Variance of Poisson Distribution

The variance of a Poisson distribution can be found by first evaluating

$$E[X^2] = \sum_{k=0}^{\infty} k^2.\frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} k.\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \left[\sum_{k=1}^{\infty} (k-1).\frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right]$$

This can be simplified as

$$E[X^2] = \lambda e^{-\lambda} \left[\lambda \times e^{\lambda} + e^{\lambda} \right] = \lambda^2 + \lambda$$

Then the variance is given by

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

5 Normal/Gaussian Distribution

A Gaussian distribution $X \sim \mathcal{N}(\mu, \sigma)$ has the probability density function

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right]$$

The expectation value and variance for this distribution is μ and σ^2 respectively. Points of Inflection are at $\mu \pm \sigma$ (check **Proof** on Lecture 9, Slide 17).

Ref: Lecture 9, Slides 15-16

Revise the concept of Standardization.

6 Gamma Distribution

Revise Gamma Functions and their properties. A Gamma random variable $X \sim \mathrm{Gamma}(\alpha, \beta)$ has the probability density function given by

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, x > 0 \\ 0, x \leq 0 \end{cases}$$

The expectation value and variance of the Gamma Distribution is given by $\alpha\beta$ and $\alpha\beta^2$ respectively.

Ref: Lecture 10, Slide 5

Revise first 3 properties of Gamma distribution from Lecture 10, Slide 6 and first three reasons for calling α the shape parameter from Lecture 10, Slide 7.

Revise the derivation of cumulative distribution function for the Gamma distribution from Lecture 10, Slide 10.

7 Beta Distributions

Revise Beta Functions and their properties. A Beta random variable $X \sim \text{Beta}(\alpha, \beta)$ has the probability density function

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

The mean and variance of the Beta distribution is given by $\frac{\alpha}{\alpha+\beta}$ and $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ respectively.

Ref: Lecture 10, Slide 19

Revise the first three properties of the Beta Distribution, as given on Lecture 10, Slide 20.