Lecture 6

Generating Sets and Bases

Let V be the vector space $\mathbb{R}^2$ and consider the vectors $(1,0),(0,1)$. Then, every vector $(x,y) \in \mathbb{R}^2$ can be written as a combination of those vectors. That is:

$$(x,y) = x(1,0) + y(0,1).$$

Similarly, the two vectors $(1,1)$ and $(1,2)$ do not belong to the same line, and every vector in $\mathbb{R}^2$ can be written as a combination of those two vectors.

6.1 Introduction

In particular:

$$(x,y) = a(1,1) + (1,2)$$

gives us two equations

$$a + b = x \quad \text{and} \quad a + 2b = y$$

Thus, by substituting the first equation to the second, we get

$$b = -x + y$$

Inserting this into the first equation we get

$$a = 2x - y$$

Take for example the point $(4,3)$. Then:

$$\begin{aligned} (4,3) &= 5(1,1) + (-1)(1,2) \\ &= 5(1,1) - (1,2) \end{aligned}$$

We have similar situation for $\mathbb{R}^3$ and all of the spaces $\mathbb{R}^n$.

In the case of $\mathbb{R}^3$, for example, every vector can be written as combinations of $(1,0,0),(0,1,0)$ and $(0,0,1)$, i.e.,

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Or, as a combination of $(1,-1,0),(1,1,1)$ and $(0,1,-1)$, that is:

$$(x,y,z) = a(1,-1,0) + b(1,1,1) + c(0,1,-1).$$

The latter gives three equation:

$$\begin{array}{rcll} a+b &=& x & (1) \\ -a+b+c &=& y & (2) \\ b-c &=& z & (3). \end{array}$$

$(2)+(3)$ gives:

$$-a+2b = y+z \quad (4)$$

$(4)+(1)$ gives:

$$3b = x+y+z \text{or} b = \frac{x+y+z}{3}.$$

Then (1) gives:

$$\begin{aligned} a &= x-b \\ &= x - \frac{x+y+z}{3} \\ &= \frac{2x-y-z}{3}. \end{aligned}$$

Finally, (3) gives:

$$c = b - z = \frac{x+y-2z}{3}$$

Hence, we get:

$$(x,y,z) = \frac{2x-y-z}{3}(1,-1,0) + \frac{x+y+z}{3}(1,1,1) + \frac{x+y-2z}{3}(0,1,-1).$$

6.2 In general

Notice that we get only one solution, so there is only one way that we can write a vector in $\mathbb{R}^3$ as a combination of those vectors. In general, if we have k vectors in $\mathbb{R}^3$, then the equation:

$$x = (x_1, x_2, ..., x_n) = c_1 v_1 + c_2 v_2 + ... + c_k v_k \quad (*)$$

gives n-equations involving the n-coordinates of $v_1, v_2, ..., v_k$ and the unknowns $c_1, c_2, ..., c_k$. There are three possibilities:

6.3 Possibilities

A The equation(*) has no solution. Thus, x can not be written as a combination of the vectors $v_1, v_2, ..., v_k$.

B The equation (*) has only one solution, so x can be written in exactly one way as a combination of $v_1, v_2, ..., v_k$.

C The system of equations has infinitely many solutions, so there are more than one way to write x as a combination of $v_1, v_2, ..., v_k$.

6.3.1 Case C

Let us look at the last case a little closer. If we write x in two different ways:

$$\begin{aligned} x &= c_1 v_1 + c_2 v_2 + ... + c_k v_k \\ x &= d_1 v_1 + d_2 v_2 + ... + d_k v_k \end{aligned}$$

Then, by subtracting, we get:

$$0 = (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + ... + (c_k - d_k) v_k$$

where some of the numbers $c_i - d_i$ are non-zero.

Similarly, since we can write:

$$0 = a_1 v_1 + a_2 v_2 + ... + a_k v_k$$

and

$$x = c_1 v_1 + c_2 v_2 + ... + c_k v_k$$

then we also have:

$$x = (c_1 + a_1) v_1 + (c_2 + a_2) v_2 + ... + (c_k + a_k) v_k.$$

Thus, we can write x as a combination of the vectors $v_1, v_2, ..., v_k$ in several different ways (in fact ∞-many ways).

6.4 Definitions

We will now use this as a motivation for the following definitions.

Definition. Let V be a vector space and $v_1, v_2, ..., v_n \in V$.

1. Let $W \subseteq V$ be a subspace. We say that W is spanned by the vectors $v_1, v_2, ..., v_n$ if every vector in W can be written as a linear combination of $v_1, v_2, ..., v_n$. Thus, if $w \in W$, then there exist numbers $c_1, c_2, ..., c_n \in \mathbb{R}$ such that $w = c_1v_1 + c_2v_2 + ... + c_nv_n$.

2. The set of vectors $v_1, v_2, ..., v_n$ is linearly dependent if there exist $c_1, c_2, ..., c_n$, not all equal to zero, such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$.

Definition.

1. The set of vectors $v_1, v_2, ..., v_n$ is linearly independent if the set is not linearly dependent(if and only if we can only write $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ with all $c_i = 0$).

2. The set of vectors $v_1, v_2, ..., v_n$ is a basis for W, if $v_1, v_2, ..., v_n$ is linearly independent and spans W.

Before we show some examples, let us make the following observations:

Lemma. Let V be a vector space with an inner product $(.,.)$. Assume that $v_1, v_2, ..., v_n$ is an orthogonal subset of vectors in V (thus $(v_i, v_j) = 0$ if $i \neq j$). If $v = c_1v_1 + c_2v_2 + ... + c_nv_n$, then $c_i = \frac{(v,v_i)}{\|v_i\|^2}$, $i = 1, ..., n$.

Proof. Assume that $v = c_1v_1 + c_2v_2 + ... + c_nv_n$. Take the inner product with v_1 in both sides of the equation. The LHS is (v, v_1). The RHS is:

$$\begin{aligned}(c_1v_1 + c_2v_2 + ... + c_nv_n, v_1) &= c_1(v_1, v_1) + c_2(v_2, v_1) \\ &\quad +... + c_n(v_n, v_1) \\ &= c_1(v_1, v_1) \\ &= c_1 \|v_1\|^2 .\end{aligned}$$

Thus, $(v, v_1) = c_1 \|v_1\|^2$, or $c_1 = \frac{(v,v_1)}{\|v_1\|^2}$. Repeat this for $v_2, ..., v_n$. □

Corollary. If the vectors $v_1, v_2, ..., v_n$ are orthogonal, then they are linearly independent.

6.5 Examples

Example. Let $V = \mathbb{R}^2$. The vectors $(1,2)$ and $(-2,-4)$ are linearly dependent because:

$$(-2)(1,2) + 1(-2,-4) = 0.$$

The vectors $(1,2),(1,1)$ are linearly independent. In fact, $(1,2),(1,1)$ is a basis for $\mathbb{R}^2$.

Indeed, let $(x,y) \in \mathbb{R}^2$. Then,

$$\begin{aligned} (x,y) &= c_1(1,2) + c_2(1,1) \\ &= (c_1 + c_2, 2c_1 + c_2). \end{aligned}$$

Thus,

$$\begin{aligned} x &= c_1 + c_2 \\ y &= 2c_1 + c_2. \end{aligned}$$

Subtracting we get: $x - y = -c_1$, or $c_1 = y - x$. Plugging this into the first equation we get:

$$c_2 = x - c_1 = x - (y - x) = 2x - y.$$

Thus, we can write any vector in $\mathbb{R}^2$ as a combination of those two. In particular, for $(0,0)$ we get $c_1 = c_2 = 0$. The vectors $(1,2),(-2,1)$ are orthogonal and hence linearly independent, and in fact a basis. Hence,

$$(x,y) = c_1(1,2) + c_2(-2,1).$$

Taking the inner product we get: $c_1 = \frac{x+2y}{\|v_1\|^2} = \frac{x+2y}{5}$ and $c_2 = \frac{-2x+y}{5}$.

Example. Let $V = \mathbb{R}^3$. One vector can only generate a line, two vectors can at most span a plane, so we need at least three vectors to span $\mathbb{R}^3$. The vectors $(1,2,1),(1,-1,1)$ are orthogonal but <u>not</u> a basis. In fact, those two vectors span the plane:

$$W = (x,y,z) \in \mathbb{R}^3 : x - z = 0$$

(explain why).

On the other hand, the vectors: $(1,2,1),(1,-1,1)$ and $(1,0,-1)$ are orthogonal, and hence a basis.

We have, for example:

$$(4,3,1) = c_1(1,2,1) + c_2(1,-1,1) + c_3(1,0,-1)$$

with

$$c_1 = \frac{4+6+1}{1+4+1}$$
$$c_2 = \frac{4-3+1}{3}$$
$$c_3 = \frac{4-1}{2}$$

In general, we have:

$$(x,y,z) = \frac{x+2y+z}{6}(1,2,1) + \frac{x-y+z}{3}(1,-1,1) + \frac{x-z}{2}(1,0,-1)$$

Let us now discuss some spaces of functions:

a) Let $v_0(x) = 1, v_1(x) = x$ and $v_2(x) = x^2$. Then, v_o, v_1 and v_2 are linearly independent.

$$\begin{aligned} 0 &= c_0v_0(x) + c_1v_1(x) + c_2v_2(x) \text{ for all } x \\ &= c_0 + c_1x + c_2x^2 \end{aligned}$$

Take $x = 0$, then we get $c_0 = 0$

Differentiate both sides to get:

$$0 = c_1 + 2c_2x$$

Take again $x = 0$ to find $c_1 = 0$. Differentiate one more time to get that $c_2 = 0$. Notice that the span of v_0, v_1, v_2 is in the space of polynomials of degree ≤ 2. Hence, the functions $1, x, x^2$ form a basis for this space. Notice that the functions $1+x, 1-2x, x^2$ are also a basis.

b) Are the functions $v_0(x) = x, v_1(x) = xe^x$ linearly independent/dependent on $\mathbb{R}$? Answer: No.

Assume that $0 = c_0x + c_1xe^x$. It does not help to put $x = 0$ now, but let us first differentiate both sides and get:

$$0 = c_0 + c_1e^x + c_1xe^x$$

Now, $x = 0$ gives:

$$0 = c_0 + c_1 \quad (1)$$

Differentiating again, we get: $0 = c_1 e^x + c_1 e^x + c_1 x e^x$. Now, $x = 0$ gives $0 = 2c_1$, or $c_1 = 0$. Hence, (1) gives $c_1 = 0$.

c) The functions $\chi_{[0,\frac{1}{2})}, \chi_{[\frac{1}{2},1)}$ are orthogonal and hence linearly independent. Let us show this directly. Assume that

$$0 = c_1 \chi_{[0,\frac{1}{2})} + c_2 \chi_{[\frac{1}{2},1)}.$$

An equation like this means that every x we put into the function on the RHS, the result is always 0.

Let us take $x = \frac{1}{4}$. Then: $\chi_{[0,\frac{1}{2})}(\frac{1}{4}) = 1$, but $\chi_{[\frac{1}{2},1)}(\frac{1}{4}) = 0$. Hence, $0 = c_1 \cdot 1 + c_2 \cdot 0$, or $c_1 = 0$. Taking $x = \frac{3}{4}$ shows that $c_2 = 0$.

d) The functions $\chi_{[0,\frac{1}{2})}, \chi_{[0,1)}$ are not orthogonal, but linearly independent.

$$0 = c_1 \chi_{[0,1)} + c_2 \chi_{[0,\frac{1}{2})}.$$

Take x so that $\chi_{[0,\frac{1}{2})}(x) = 0$ but $\chi_{[0,1)}(x) = 1$. Thus, any $x \in [0,1) \setminus [0,\frac{1}{2}) = [\frac{1}{2},1)$ will do the job. So, take $x = \frac{3}{4}$. Then, we see that:

$$0 = c_1 \cdot 1 + c_2 \cdot 0, \mathrm{or} c_1 = 0.$$

Then take $x = \frac{1}{4}$ to see that $c_2 = 0$.