Exercise List: Proving convergence of the Stochastic Gradient Descent and Coordinate Descent on the Ridge Regression Problem.

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1 Introduction

Consider the task of learning a rule that maps the *feature vector* $x \in \mathbb{R}^d$ to outputs $y \in \mathbb{R}$. Furthermore you are given a set of labelled observations (x_i, y_i) for $i = 1, \ldots, n$. We restrict ourselves to linear mappings. That is, we need to find $w \in \mathbb{R}^d$ such that

$$x_i^\top w \approx y_i, \quad \text{for } i = 1, \ldots, n. \tag{1}$$

That is the *hypothesis function* is parametrized by w and is given by $h_w : x \mapsto w^\top x$.[1] To choose a w such that each $x_i^\top w$ is close to y_i, we use the squared loss $\ell(y) = y^2/2$ and the squared regularizor. That is, we minimize

$$w^* = \arg\min_w \frac{1}{n}\sum_{i=1}^n \frac{1}{2}(x_i^\top w - y_i)^2 + \frac{\lambda}{2}\|w\|_2^2, \tag{2}$$

where $\lambda > 0$ is the regularization parameter. We now have a complete training problem (2)[2].

Using the matrix notation

$$X \overset{\text{def}}{=} [x_1, \ldots, x_n] \in \mathbb{R}^{d\times n}, \quad \text{and} \quad y = [y_1, \ldots, y_n] \in \mathbb{R}^n, \tag{3}$$

we can re-write the objective function in (2) as

$$f(w) \overset{\text{def}}{=} \frac{1}{2n}\|X^\top w - y\|_2^2 + \frac{\lambda}{2}\|w\|_2^2. \tag{4}$$

First we introduce some necessary notation.

[1]We need only consider a linear mapping as opposed to the more general *affine* mapping $x_i \mapsto w^\top x_i + \beta$, because the zero order term $\beta \in \mathbb{R}$ can be incorporated by defining a new feature vectors $\hat{x}_i = [x_1, 1]$ and new variable $\hat{w} = [w, \beta]$ so that $\hat{x}_i^\top \hat{w} = x_i^\top w + \beta$

[2]Excluding the issue of selection λ using something like crossvalidation `https://en.wikipedia.org/wiki/Cross-validation_(statistics)`

Notation: For every $x, w, \in \mathbb{R}^d$ let $\langle x, w\rangle \stackrel{\text{def}}{=} x^\top y$ and let $\|x\|_2 = \sqrt{\langle x, x\rangle}$. Let $A \in \mathbb{R}^{d\times d}$ be a matrix and let $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ be the smallest and largest singular values of A defined by

$$\sigma_{\min}(A) \stackrel{\text{def}}{=} \min_{x\in\mathbb{R}^d,\, x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad \text{and} \quad \sigma_{\max}(A) \stackrel{\text{def}}{=} \max_{x\in\mathbb{R}^d,\, x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}. \tag{5}$$

Finally, a result you will need, if A is a symmetric positive semi-definite matrix the largest singular value of A can be defined instead as

$$\sigma_{\max}(A) = \max_{x\in\mathbb{R}^d,\, x\neq 0} \frac{\langle Ax, x\rangle_2}{\|x\|_2^2} = \max_{x\in\mathbb{R}^d,\, x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}. \tag{6}$$

Therefore

$$\frac{\langle Ax, x\rangle}{\|x\|_2^2} \leq \sigma_{\max}(A), \quad \forall x \in \mathbb{R}^d. \tag{7}$$

and

$$\frac{\|Ax\|_2}{\|x\|_2} \leq \sigma_{\max}(A), \quad \forall x \in \mathbb{R}^d. \tag{8}$$

We will now solve the following ridge regression problem

$$w^* = \arg\min_{w\in\mathbb{R}^d} \left(\frac{1}{2n}\|X^\top w - y\|_2^2 + \frac{\lambda}{2}\|w\|_2^2 \stackrel{\text{def}}{=} f(w) \right), \tag{9}$$

using stochastic gradient descent and stochastic coordinate descent.

2 Stochastic Gradient Descent

Some more notation: Let $\|A\|_F^2 \stackrel{\text{def}}{=} \operatorname{Tr}\left(A^\top A\right)$ denote the Frobenius norm of A. Let

$$A \stackrel{\text{def}}{=} \tfrac{1}{n} X X^\top + \lambda I \in \mathbb{R}^{d\times d} \quad \text{and} \quad b \stackrel{\text{def}}{=} \tfrac{1}{n} X y. \tag{10}$$

Ex. 1 — We can exploit the separability of the objective function (2) to design a *stochastic* gradient method. For this, first we re-write the problem $Aw = b$ as different linear least squares problem

$$\hat{w}^* = \min_w \tfrac{1}{2}\|Aw - b\|_2^2 \quad = \quad \min_w \sum_{i=1}^d \tfrac{1}{2}(A_{i:}w - b_i)^2 \quad \stackrel{\text{def}}{=} \quad \min_w \sum_{i=1}^d p_i f_i(w), \tag{11}$$

where $f_i(w) = \frac{1}{2p_i}(A_{i:}w - b_i)^2$, $A_{i:}$ denotes the ith row of A, b_i denotes the ith element of b and $p_i = \frac{\|A_{i:}\|_2^2}{\|A\|_F^2}$ for $i = 1, \ldots, d$. Note that $\sum_{i=1}^d p_i = 1$ thus the p_i's are probabilities.

From a given $w^0 \in \mathbb{R}^d$, consider the iterates

$$w^{t+1} = w^t - \alpha \nabla f_j(w^t), \tag{12}$$

where

$$\alpha = \frac{1}{\|A\|_F^2}, \tag{13}$$

and j is a random index chosen from $\{1, \ldots, d\}$ sampled with probability p_j. In other words, $\mathbb{P}(j = i) = p_i = \frac{\|A_{i:}\|_2^2}{\|A\|_F^2}$ for all $i \in \{1, \ldots, d\}$.

Part I

Show that the solution $\hat{w}^*$ to (11) and the solution to w^* to (9) are equal.

Part II

Show that

$$\nabla f_j(w) = \frac{1}{p_j} A_{j:}^\top A_{j:}(w - w^*)$$

and that

$$\mathbb{E}_{j \sim p}\left[\nabla f_j(w)\right] \stackrel{\text{def}}{=} \sum_{i=1}^{d} p_i \nabla f_i(w) = A^\top (Aw - b) = A^\top A(w - w^*),$$

thus $\nabla f_j(w)$ is an unbiased estimator of the full gradient of the objective function in (11). This justifies applying the stochastic gradient method.

$$\Pi_j \stackrel{\text{def}}{=} \frac{A_{j:}^\top A_{j:}}{\|A_{j:}\|_2^2}, \tag{14}$$

show that

$$\Pi_j \Pi_j = \Pi_j \quad \text{and} \quad (I - \Pi_j)(I - \Pi_j) = I - \Pi_j. \tag{15}$$

In other words, Π_j is a projection operator which projects orthogonally onto $\textbf{Range}\left(A_{j:}\right)$. Furthermore, if $j \sim p_j$ verify that

$$\mathbb{E}\left[\Pi_j\right] = \sum_{i=1}^{d} p_i \Pi_i = \frac{A^\top A}{\|A\|_F^2}. \tag{16}$$

Part III

Using analogous techniques from the previous exercise, show that the iterates (12) converge according to

$$\mathbb{E}\left[\|w^{t+1} - w^*\|_2^2\right] \quad \leq \quad \left(1 - \frac{\sigma_{\min}(A)^2}{\|A\|_F^2}\right) \mathbb{E}\left[\|w^t - w^*\|_2^2\right]. \tag{17}$$

This is an amazing and recent result [2], since it shows that SGD converges exponentially fast despite the fact that the iterates (12) only require access to a single row of A at a time! This result can be extended to solving any linear system $Aw = b$, including the case where A rank deficient. Indeed, so long as there exists a solution to $Aw = b$, the iterates (12) converge to the solution of least norm and at rate of $\left(1 - \frac{\sigma_{\min}^{+}(A)^2}{\|A\|_F^2}\right)$ where $\sigma_{\min}^{+}(A)$ is the smallest nonzero singular value of A [1]. Thus this method can solve any linear system.

Part IV

When is this stochastic gradient method (12) *faster* than the stochastic coordinate descent method of gradient descent? Note that the each iteration of SGD and CD costs $O(d)$ floating point operations while an iteration of the GD method costs $O(d^2)$ floating point operations (assuming that A has been previously calculated and stored). What happens if d is very big? What if $\|A\|_F^2$ is very large? Discuss this.

3 Stochastic Coordinate Descent

Ex. 2 — Consider the minimization problem

$$w^* = \arg\min_{x\in\mathbb{R}^d} \left(f(w) \stackrel{\text{def}}{=} \frac{1}{2} w^\top A w - w^\top b \right), \tag{20}$$

where $A \in \mathbb{R}^{d\times d}$ is a symmetric positive definite matrix, and $w, b \in \mathbb{R}^d$.

Part I

First show that using the notation (10) that solving (20) is equivalent to solving (9).

Part II

Show that

$$\frac{\partial f(w)}{\partial w_i} = A_{i:} w - b_i, \tag{21}$$

where $A_{i:}$ is the ith row of A. Furthermore note that $w^* = A^{-1} b$ thus

$$\frac{\partial f(w)}{\partial w_i} = e_i^\top (Aw - b) = e_i^\top A(w - w^*). \tag{22}$$

Part III

Consider a step of the stochastic coordinate descent method

$$w^{k+1} = w^k - \alpha_i \frac{\partial f(w^k)}{\partial x_i} e_i, \tag{23}$$

where $e_i \in \mathbb{R}^d$ is the ith unit coordinate vector, $\alpha_i = \frac{1}{A_{ii}}$, and $i \in \{1, \dots, d\}$ is sampled i.i.d at each step according to $i \sim p_i$ where $p_i = \frac{A_{ii}}{\mathrm{Tr}\,(A)}$. Let $\|x\|_A^2 \overset{\text{def}}{=} x^\top A x$.
Prove that

$$\mathbb{E}\left[\|w^{k+1} - w^*\|_A^2\right] \quad \leq \quad \left(1 - \frac{\lambda_{\min}(A)}{\mathrm{Tr}\,(A)}\right) \mathbb{E}\left[\|w^k - w^*\|_A^2\right] \tag{24}$$

thus (23) converges to the solution.
Hint: Since A is symmetric positive definite you can use that

$$\lambda_{\min}(A) = \inf_{w \in \mathbb{R}^n} \frac{w^\top A w}{\|w\|_2^2}.$$

You will need to use that $x^\top A x \geq \lambda_{\min}(A) \|x\|_2^2$ at some point.

References

[1] R. M. Gower and P. Richtárik. “Stochastic Dual Ascent for Solving Linear Systems”. In: *arXiv:1512.06890* (2015).

[2] T. Strohmer and R. Vershynin. “A Randomized Kaczmarz Algorithm with Exponential Convergence”. In: *Journal of Fourier Analysis and Applications* 15.2 (2009), pp. 262–278.