Electromagnetic Compatibility

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Contents

Introduction

I share with you my work, it is not perfect, but i think it could be helpful for anyone.
You can share it, modify or do whatever you want, i enjoy to write in LaTeX, so i'm totally fine with this.
Please consider to help me to find any mistakes or misunderstanding, contact me or just modify the code from Github.
If you want to be added to my private Github repository just give me your email and I'll add you with no problem.
I want to suggest you to give a look at the book *High Speed Digital Design*[1], in the introduction it is explained in very simple and intuitive way the first part of our course.
You can find it easily in 1lib.eu but i didn't told you nothing :x

1 Class 1 - 22/02/21

Today's lesson was a recap on Maxwell equation.
First of all we have seen the Faraday equation:

$$\oint_l \overline{E} \cdot \hat{l}\, dl = -\frac{d}{dt}\int_s \overline{B} \cdot \hat{n}\, ds = -\frac{d\Phi_B}{dt} \tag{1.1}$$

And this second equation:

$$\oint_l \overline{H} \cdot \hat{l}\, dl = \int_s \left(\frac{d\overline{D}}{dt} + \overline{J}\right) \cdot \hat{n}\, ds \tag{1.2}$$

Where those symbol are:

- E = intensity of electric field $[\frac{V}{m}]$
- B = magnetic induction vector $[T]$
- H = intensity magnetic field $[A]$
- D = electronic displacement $[\frac{A}{m}]$
- J = intensity of electric current $[\frac{A}{m^2}]$

Now, we can think $E - H$ as the element measuring the real electromagnetic field, and $D - B$ as something that measure the effect of the EMF.
We also remember that:

$$\frac{d}{dt}\int_V \overline{\rho}\, dV = -\int_s \overline{J} \cdot \hat{n}\, ds \tag{1.3}$$

Where ρ is the density of charge (volumetric) and s is the boundary of the volume V in the integral on the left.
On the right of eq. (1.3) we can find a minus sign, because if my electron goes from out to in in our volume, it means that the current is going from in to out (the right integral is positive because of the conversion for the versor $\hat{n}$ that goes from in to out). If the the electron are going in as we already said, we are accumulating a negative charge, for this reason we need to put this negative sign.

Recap of operator nabla

Gradient ∇T

Usually we refer to a variation of a scalar quantity with the derivative (for example $\frac{dT}{dt}$).
If the scalar T variates in more than one component, we want have a look at the derivative of everything: $\frac{dT}{dx}, \frac{dT}{dy}, \frac{dT}{dz}$.

If we are in space, our scalar field vary his value over we have 3 components, so we introduce the *gradient* of the scalar field T (temperature) as:

$$\left(\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}\right) = \nabla T \tag{1.4}$$

Where $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ is the nabla operator.
Now from the scalar T we obtained a vector and that is okay, but if our field is not scalar but vectorial?
The electric field $\overline{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$ is a vectorial field. What we can do now is using the dot or cross product between our vector E and the ∇ operator.

Scalar product (divergence) $\nabla \cdot \overline{E}$

The scalar product (dot) between nabla and a vectorial field is named *divergence*

$$\nabla \cdot \overline{E} = \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{1.5}$$

Physically speaking the divergence of the vector E is *how E_x is changing over the x direction etc...*
Another way to look at this divergence is:

$$\nabla \cdot \overline{E} = \cdots = \lim_{\Delta v \to 0} \frac{\oint_s \overline{E} \cdot \hat{n}\, ds}{\Delta v} \tag{1.6}$$

Divergence theorem

This theorem is very useful to transform an integral over a surface to an integral over a volume.

$$\int_v \nabla \cdot \overline{E} = \int_s \overline{E} \cdot \hat{n}\, ds \tag{1.7}$$

So the integral of the divergence over a volume is equal to the flux across the boundary surface of the volume.

Cross product (curl) $\nabla \times \overline{E}$

The cross product of nabla and our vectorial field is named *curl*

$$\begin{aligned}\nabla \times \overline{E} = det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = \\ = \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k}\end{aligned} \tag{1.8}$$

The curl (as the name in italian *rotore* says) explains the behavior of E_x (for example).

Again, another way to look at the curl is:

$$\nabla \times \overline{E} = \lim_{\Delta s \to 0} \frac{\hat{n} \cdot \oint_l \overline{E} \cdot \hat{l}\, dl}{\delta s} \tag{1.9}$$

The $\hat{n}$ is there to maximize the value of $\oint_l \overline{E} \cdot \hat{l}\, dl$. I don 't really know why, PLEASE GIVE IT A LOOK.
Note that we are not using the vector symbol over the nabla because we are not multiplying two vector, instead nabla is an *operator* as $\cdot$, sin or$\oplus$, so we don't write $\cancel{\overline{\nabla} \cdot \overline{E}}$

Stokes Theorem

This theorem is similar to divergence theorem, but instead of surface and volume, we deal with surface and line:

$$\int_s \nabla \times \overline{E} \cdot \hat{n}\, ds = \oint_l \overline{E} \cdot \hat{l}\, dl \tag{1.10}$$

It means that the flux over a surface of the curl of a vectorial field, is the integral of that field on the line that is bounding s.

Laplacian

If i use the operator curl twice we get the laplacian.
Consider the gradient of a scalar field T: $\nabla T = \left(\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k} \right)$ now:

$$\nabla \cdot (\nabla T) = \nabla^2 T = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \tag{1.11}$$

If we consider a vectorial field $\overline{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$

$$\nabla^2 \cdot \overline{E} = \nabla^2 E_x\, \hat{\imath} + \nabla^2 E_y\, \hat{\jmath} + \nabla^2 E_z\, \hat{k} \tag{1.12}$$

Actually we can write $\nabla^2 E_x\, \hat{\imath} + \nabla^2 E_y\, \hat{\jmath} + \nabla^2 E_z\, \hat{k}$ because we are dealing with different Laplacian for each E component, that is scalar.
The same can be said to $\nabla^2 T$ on eq. (1.11)

Going back to the Stokes Theorem

With stokes theorem we can write something similar to the first Maxwell equation:
Starting from the first Maxwell equation seen in eq. (1.1)

$$\oint_l \overline{E} \cdot \hat{l}\, dl = -\frac{d}{dt} \int_s \overline{B} \cdot \hat{n}\, ds \tag{1.13}$$

And with the Stokes theorem eq. (1.10), we can write:

$$\int_s (\nabla \times \overline{E}) \cdot \hat{n}\, ds = \frac{d}{dt} \int_s \overline{B} \cdot \hat{n}\, ds \tag{1.14}$$

We notice that we have two integral over the same surface s, so:

$$\int_s \left(\nabla \times \overline{E} + \frac{d\overline{B}}{dt} \right) \cdot \hat{n}\, ds = 0 \tag{1.15}$$

Finally we know that this integral is equal to zero if the equation inside the brackets is also equal to zero, so:

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt} \tag{1.16}$$

This eq. (1.16) can be considered as the first Maxwell equation but for a point. We can do the same for the second Maxwell equation:

$$\nabla \times \overline{H} = \frac{d\overline{D}}{dt} + \overline{J} \tag{1.17}$$

eq. (1.16) and eq. (1.17)are also named as the local formulation of the Maxwell equation.
To be more precise we can actually split the intensity of current J in two contribution:

- J_σ: current generated by the electromagnetic field on iron (σ actually is the conductivity).
- J_i: current that generates the electromagnetic field, given by for example a battery of the phone.

We have seen J_σ as the metallic behavior of the receiver and J_i as the source of the EMF, and we can also see at the component $\frac{d\overline{D}}{dt}$ as he behavior of the dielectric material due to the displacement of the charge.
eq. (1.17) becomes:

$$\nabla \times \overline{H} = \frac{d\overline{D}}{dt} + \overline{J}_\sigma + \overline{J}_i \tag{1.18}$$

Step forward: solution of the Maxwell equation

The solutions of the Maxwell equations are not always simple to obtain analytically, we need to have a sort of classification for the equation to be solved. Those can be classified as:

- **Linear** and **non linear**
- **isotropic** and **anisotropic**
- **stationary** and **non stationary**

- **dispersive** and **non dispersive**
  - in time
  - in space

Linear means that the equation is composed only and only only by the sum of each variable x_i multiplied by their own coefficient a_i: $\boldsymbol{a_1x_2 + a_1x_2 \cdots + a_nx_n + b = 0}$, this is an important class of equation because we can use a lot of useful properties.
Stationary means that the results will not change with the time, so if i do the experiment now or 10 years later I'll be sure that nothing will change.
We need this little recap to neglect all those non optimal behavior, and for the sake of simplicity we use those relation:

- $\overline{D} = \varepsilon\overline{E}\ \rightarrow \varepsilon$ is the dielectric permittivity
- $\overline{B} = \mu\overline{H}\ \rightarrow \mu$ is the electric permeability
- $\overline{J_\sigma} = \sigma\overline{E}\ \rightarrow \sigma$ is the conductivity (is there when we have a metallic object)

Those are very oversimplified, but can be useful to study our EMF.
We remember that we have obtained the $\varepsilon\,\mu\,\sigma$ value in vacuum, that are:

- $\varepsilon_0 = \frac{1}{36\pi}\,10^{-9}\left[\frac{F}{m}\right]$
- $\mu_0 = 4\pi\,10^{-7}\left[\frac{H}{m}\right]$
- $\sigma_0 = 0\left[\frac{S}{m}\right]$

When we are dealing with linear material, we can not consider the numeric value of these constant over vacuum, but instead with some approximation we can consider:

- $\varepsilon = \varepsilon_0\,\varepsilon_r$
- $\mu = \mu_0\,\mu_r$
- $\sigma = \frac{1}{\rho}$

Now, the first two local Maxwell equation which we have already seen are:

$$\begin{cases} \nabla\times\overline{E} = -\frac{d\overline{B}}{dt} \\ \nabla\times\overline{H} = \frac{d\overline{D}}{dt} + \overline{J}_\sigma + \overline{J}_i \end{cases} \tag{1.19}$$

We try to make this system solvable by substitution:

$$\begin{cases} \nabla\times\overline{E} = -\mu\frac{d\overline{H}}{dt} \\ \nabla\times\overline{H} = \varepsilon\frac{d\overline{E}}{dt} + \sigma\,\overline{E} + \overline{J}_i \end{cases} \tag{1.20}$$

As can be seen, the system in eq. (1.20) is a systems of equations in two unknown variable (E and H), we can solve that but it would be very complicated.

Another useful equation

similarly to what we have done in eq. (1.16), we can write this equation exploiting the Stokes Theorem (eq. (1.10)):

$$\begin{aligned} &\frac{d}{dt}\int_v \overline{\rho}\, dv = -\int_s \overline{J}\cdot\hat{n}ds = -\int_v \nabla\cdot\overline{J}\, dv \\ &\int_v \left(\frac{d\overline{\rho}}{dt} + \nabla\cdot\overline{J}\right) dv = 0 \end{aligned} \tag{1.21}$$

Then we obtain:

$$\nabla\cdot\overline{J} = -\frac{d\overline{\rho}}{dt} \tag{1.22}$$

Third maxwell equation in local formulation

First of all we try to do the divergence of $\nabla\times\overline{E}$.
Note that the result of the divergence of a curl is equal to zero because the result of the curl will be perpendicular to the nabla operator, so $\nabla\cdot(\nabla\times\overline{E}) = 0$

$$\begin{aligned} &\nabla\cdot\left[(\nabla\times\overline{E}) = -\frac{d\overline{B}}{dt}\right] = \\ &\nabla\cdot\left(-\frac{d\overline{B}}{dt}\right) = 0 \\ &\frac{d}{dt}(\nabla\cdot\overline{B}) = 0 \end{aligned} \tag{1.23}$$

We obtained the *Third maxwell equation in local formulation*:

$$\nabla\cdot\overline{B} = 0 \tag{1.24}$$

We can say that eq. (1.24) ist true because we are considering that at the starting time t_0 our EMF was turned off ($\nabla\cdot\overline{B} = 0$ at t_0). If μ is constant:

$$\nabla\cdot\overline{B} = \nabla\cdot(\mu\overline{H}) = \nabla\cdot\overline{H} = 0 \tag{1.25}$$

Forth maxwell equation in local formulation

Using the same passages as before, we can obtain the forth maxwell equation in local formulation by doing the divergence of the curl of H

$$
\begin{aligned}
&\nabla \cdot \left[(\nabla \times \overline{H}) = -\frac{d\overline{D}}{dt} + \overline{J} \right] = \\
&\nabla \cdot \left(-\frac{d\overline{D}}{dt} + \overline{J} \right) = 0 \\
&\frac{d}{dt}(\nabla \cdot \overline{D}) + \nabla \cdot \overline{J} = 0 \\
&\frac{d}{dt}(\nabla \cdot \overline{D}) + \nabla \cdot \frac{d\overline{\rho}}{dt} = 0 \\
&\frac{d}{dt}\left(\nabla \cdot \overline{D} - \overline{\rho}\right) = 0
\end{aligned}
\tag{1.26}
$$

At the end, the Forth maxwell equation in local formulation is:

$$\nabla \cdot \overline{D} = \overline{\rho} \tag{1.27}$$

If ε is constant we obtain

$$
\begin{aligned}
&\nabla \cdot (\varepsilon \overline{E}) = \overline{\rho} \\
&\nabla \cdot \overline{E} = \frac{\overline{\rho}}{\epsilon}
\end{aligned}
\tag{1.28}
$$

Those equation does not say anything more than the maxwell equation. To summarize all the equation we have obtained (for some reason):

$$
\begin{array}{cc}
\nabla \cdot \overline{E} = \frac{\overline{\rho}}{\epsilon} & \nabla \cdot \overline{J} = -\frac{d\overline{\rho}}{dt} \\
\nabla \cdot \overline{B} = 0 & \nabla \cdot \overline{D} = \rho
\end{array}
$$

Today the Daft Punk duo broke up, this page is in memory of the 2 best DJs and composers of the last 3 decades, who came from space.

2 Class 2 - 26/02/21

Vector wave equation

Here we go again, today from eq. (1.20) we want to obtain the wave equation of the EMF, we will make a lot of simplification.
First of all we consider:

- No accumulated charge (source free, no current that generates the EMF) $\boldsymbol{\overline{J_i} = 0}$
- No losses (nonconducting medium) $\boldsymbol{\overline{J_i}\sigma \cdot \overline{E}}$

Then from eq. (1.20) we now have:

$$\begin{cases} \nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \\ \nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} \end{cases} \tag{2.1}$$

We continue to try to find the wave equation by using some tricks, the next passage is to apply the curl on the first equation from eq. (2.1).

$$\begin{aligned} \nabla \times \nabla \times \overline{E} = & - \nabla \times \mu \frac{\partial \overline{H}}{\partial t} = -\mu \frac{\partial}{\partial t} \nabla \times \overline{H} \\ & = -\mu \frac{\partial}{\partial t} \varepsilon \frac{\partial \overline{E}}{\partial t} = \\ & = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \end{aligned} \tag{2.2}$$

We remember a cool property of the curl of a curl of a vector field:

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2 \, \overline{E} \tag{2.3}$$

Please note that we have already discussed the ∇^2 symbol before in section 1 as the Laplacian. Now let's write:

$$\nabla(\nabla \cdot \overline{E}) - \nabla^2 \, \overline{E} = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \tag{2.4}$$

We simplify a little bit more, and we consider that we don't have free charge: $\nabla(\varepsilon \overline{E} = 0)$ and we obtain:

$$\nabla^2 \, \overline{E} - \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} = 0 \tag{2.5}$$

WE FINALLY OBTAINED THE *WAVE EQUATION*. Here we define a constant that will be helpful next:

$$c = \frac{1}{\sqrt{\mu\varepsilon}} \tag{2.6}$$

Then eq. (2.5) becomes:

$$\nabla^2 \overline{E} - \frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} = 0 \tag{2.7}$$

Pay attention to one thing: if you look at the maxwell equation eq. (2.1) it is a differential equation of the first order, but now we have a second order differential equation (eq. (2.7)) and the set of solution is increased. We don't address this problem now, but keep in mind that later we will be able to select the good solution by using the divergence equation (*i don't really know what does it mean*).
If instead use the second equation from eq. (2.7), and we do all the passages as before, we obtain nearly the same equation, but with $\overline{H}$

$$\nabla^2 \overline{H} - \frac{1}{c^2}\frac{\partial^2 \overline{H}}{\partial t^2} = 0 \tag{2.8}$$

Scalar wave equation

Now let's try to write our equation with scalars and not vectors, it should simplify a bit.
If we consider the vector E to be completely parallel to the z axes, it means:

$$\overline{E}(x,y,z) = \cancel{E_x \hat{\imath}} + \cancel{E_y \hat{\jmath}} + E_z \hat{k} \;\rightarrow\; \overline{E}(x,y,z) = E(z) \tag{2.9}$$

But actually E is also dependent on time, so we deal with: $E(z,t)$
Now eq. (2.7) with scalar E becomes:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{2.10}$$

Solution of the scalar wave equation

Searching for the solution of eq. (2.10) is a bit time consuming, but it should have the shape of:

$$E(z,t) = f_1\left(t - \frac{z}{c}\right) + f_2\left(t + \frac{z}{c}\right) \tag{2.11}$$

So without going into tedious calculation, we look at 2 possible solutions for f_1 and f_2:

$$\begin{aligned} E_1(z,t) &= E_0 \cos\left[\omega\left(t - \frac{z}{c}\right)\right] \\ E_2(z,t) &= E_0 \cos\left[\omega\left(t + \frac{z}{c}\right)\right] \end{aligned} \tag{2.12}$$

Does that satisfy the maxwell equation? Let's consider the first solution of eq. (2.12) and verify that it actually satisfy eq. (2.10):

$$\begin{aligned}
\frac{\partial^2 E}{\partial z^2} =& \frac{\partial}{\partial z}\left\{E_0\left[-\left(-\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{z}{c}\right)\right]\right]\right\} = \\
=& E_0\,\frac{\omega}{c}\left\{\left(-\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = \\
=& -E_0\,\frac{\omega^2}{c^2}\cos\left[\omega\left(t-\frac{z}{c}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 E}{\partial t^2} =& \frac{\partial}{\partial t}\left\{E_0\left[-\omega\,\sin\left[\omega\left(t-\frac{z}{c}\right)\right]\right]\right\} = \\
=& -E_0\,\omega\left\{\omega\,\cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = \\
=& -E_0\,\omega^2\,\cos\left[\omega\left(t-\frac{z}{c}\right)\right]
\end{aligned}$$

Then it is simple to verify eq. (2.10)

$$\begin{aligned}
&\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \\
=& -E_0\,\frac{\omega^2}{c^2}\cos\left[\omega\left(t-\frac{z}{c}\right)\right] - \frac{1}{c^2}\left\{-E_0\,\omega^2\,\cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = 0
\end{aligned}$$

We see that this solution is dependent both in space and time, but what else we can say?

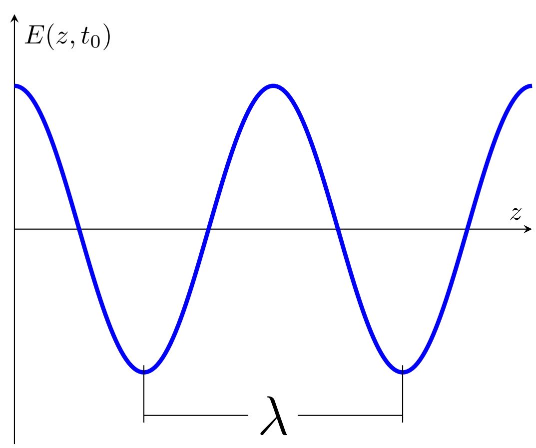


Figure 2.1: Plot of E in space

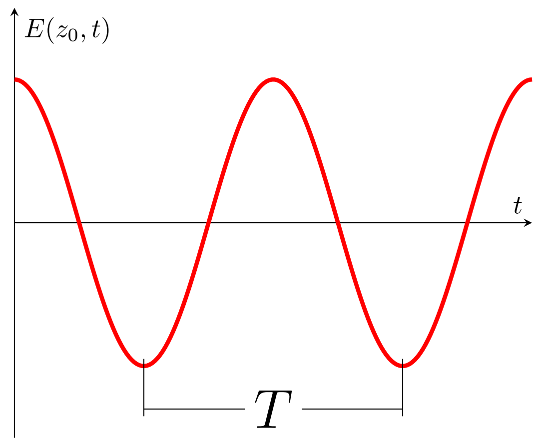


Figure 2.2: Plot of E in time

We can plot the solution in eq. (2.12) by considering one of the two variable constant.

- **fig. 2.1**: if we assume constant time (like if you would take a photograph to the wave) we are evaluating the propagation in space, and so we can obtain the wavelength λ
- **fig. 2.2**: if we assume constant space (like if you look the wave from a fixed position) we are evaluating the propagation in space, and so we can obtain the wavelength T

Speed of the wave

As we said if we plot E in constant time (fig. 2.1) it is like to take a picture of the wave. If we evaluate the same plot, but in another time point, we can notice that the points of the wave has changed position fig. 2.3.
From the variation of the space in time we can evaluate the speed of the E wave.

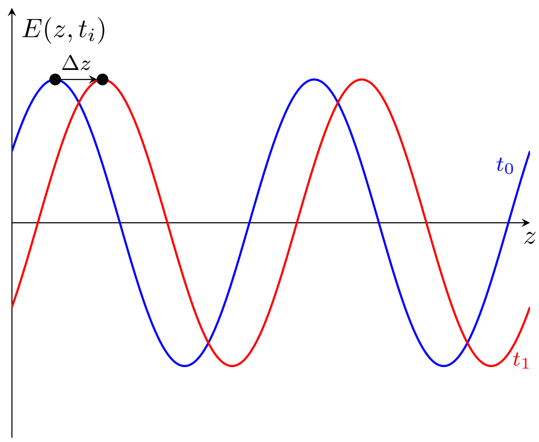


Figure 2.3: Plot of E in two different time

If we look at fig. 2.3, we can assumed that a point of the wave as moved from z_1 to z_2 from the instant t_1 to t_2. The function in the two points (z_1, t_1) and (z_2, t_2) has the same relative position (consider to "sit in the wave", you would feel like not moving, but the world around you is moving with a certain speed). So we can write:

$$\begin{aligned}
&\omega\left(t_1 - \frac{z_1}{c}\right) = \omega\left(t_1 - \frac{z_1}{c}\right) \\
&t_2 - t_1 = \frac{z_2 - z_1}{c} \\
&\Delta t = \frac{\Delta z}{c} \\
&c = \frac{\Delta z}{\Delta t} = \frac{\lambda}{T}
\end{aligned} \tag{2.13}$$

We obtained the propagation speed of the wave:

$$c = \frac{\partial z}{\partial t} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{2.14}$$

Note that the propagation speed is dependent of μ and ε, so we can calculate the speed in the vacuum:

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\frac{1}{36\pi}\cdot 10^{-9}\cdot 4\,\pi\cdot 10^{-7}}} \approx 3\cdot 10^8 [m/s] \tag{2.15}$$

What we have found is a forward speed because Δz is positive and c positive, if we would have used the second equation from eq. (2.12) the space path Δz need to be negative, or we would not be able to have a solution if we try to calculate the propagation speed:

$$\omega\left(t_1+\frac{z_1}{c}\right)=\omega\left(t_1+\frac{z_1}{c}\right) \tag{2.16}$$

Going back to the EMF

Some more consideration of the EMF

$$E(z,t)=E_0\,\cos(\omega t-\frac{\omega\,z}{c}) \tag{2.17}$$

Now we give some notation for:

- $\boldsymbol{E_0}$ is the amplification of the field.
- $\boldsymbol{\omega=2\,\pi\,\gamma}$ is the angular frequency of the EMF
- $\boldsymbol{\gamma=\frac{1}{T}}$ is the frequency of the EMF

We can also introduce the phase constant $\boldsymbol{\beta=\frac{\omega}{c}}$, and now the wave equation becomes:

$$E(z,t)=E_0\cos(\omega\,t-\beta\,z) \tag{2.18}$$

Those two parameters ω and β are useful to obtain the propagation speed.
As we have done before we evaluate this propagation speed by considering $(\omega\,t-\beta\,z)$ to be constant, then:

$$\frac{\partial z}{\partial t}=\frac{\omega}{\beta} \tag{2.19}$$

Generalization of the EMF

We can generalize a bit the EMF equation by adding an attenuation constant α and a reference phase φ

$$E(z,t)=E_0\,e^{-\alpha\,z}\cos(\omega\,t-\beta\,z+\varphi) \tag{2.20}$$

α is used to show how the wave is attenuated during his path on the medium.

EMF over a general direction

Usually we consider $\hat{k}$ as the direction of the EMF, but sometimes we need to generalize this direction over all the axes.
Consider the forward equation of the EMF over the 3 direction:

$$\begin{aligned}
E(x,t)&=E_0\,\cos(\omega\,t-\beta_x\,x)\\
E(y,t)&=E_0\,\cos(\omega\,t-\beta_y\,y)\\
E(z,t)&=E_0\,\cos(\omega\,t-\beta_z\,z)
\end{aligned} \tag{2.21}$$

What we do now is to find a way to merge those equation and describe the EMF that goes in a general direction $\overline{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$.
In order to do this we introduce the vector $\overline{k} = k_x\,\hat{\imath} + k_y\,\hat{\jmath} + k_z\,\hat{k}$ that generalize the β factor, so we can write the wave equation with a general direction:

$$E(\overline{E}, t) = \overline{E}_0\ \cos(\omega\, t - \overline{k} \cdot \overline{r}) \tag{2.22}$$

Keep in mind that the direction is not $\overline{r}$ but $\overline{k}$, because $\overline{r}$ represent the variables, and $\overline{k}$ the three weights that defines the direction of the wave. This is also called the plane wave equation, why?
Because we consider $\varphi = 0$ and $\omega\, t = 0$ (we take a photo of the wave in $t = 0$), so the wave fronts (where E is constant) can be obtained with:

$$\cancel{\omega t} - \overline{k} \cdot \overline{r} + \cancel{\varphi} = -\overline{k} \cdot \overline{r} = k_x\,\hat{\imath} + k_y\,\hat{\jmath} + k_z\,\hat{k} = constant \tag{2.23}$$

Then the last part of the equation is the equation of a plane.
Another thing that we can say, is that if $\overline{k} \cdot \overline{r}$ is constant, the front of the wave have the shape of a sphere.

3 Class 3 - 1/03/21

During this class we will talk about the passage from the time domain to frequency. We do that because in frequency domain we do not need anymore to do the derivate in time, and thus the calculations simplify a lot. Another reason is that is more important to look at the signal on the corner frequency other than all over the spectrum

EMF in phasor domain

Now we take a look at the electrical field propagating on the z axe: $\overline{E}(z,t) = E_x\,\hat{\jmath} + E_y\,\hat{\imath} + E_z\,\hat{k}$, and we explore all the element:

$$\begin{aligned} E_x(z,t) &= E_{x_0}\,\cos(\omega\,t - \beta\,z + \varphi_x) \\ E_y(z,t) &= E_{y_0}\,\cos(\omega\,t - \beta\,z + \varphi_y) \\ E_z(z,t) &= E_{z_0}\,\cos(\omega\,t - \beta\,z + \varphi_z) \end{aligned} \tag{3.1}$$

Note that here ω and β does not change, but φ does, this is not very important, but it is just a note.
From eq. (3.1), and exploiting the cos property:

$$\cos(\alpha + \beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$$

By considering $\alpha = \omega\,t$ and $\beta = \beta z + \varphi + z$ we obtain the total equation of the EMF on z direction:

$$\begin{aligned} &\overline{E}(z,t) = E_x\,\hat{\jmath} + E_y\,\hat{\imath} + E_z\,\hat{k} = \\ &= \cos(\omega\,t)[E_{x_0}\cos(\varphi - \beta\,z)\hat{\imath} + E_{y_0}\cos(\varphi_y - \beta\,z)\hat{\jmath} + E_{z_0}\cos(\varphi_z - \beta\,z)\hat{k}] + \\ &- \sin(\omega\,t)[E_{x_0}\sin(\varphi - \beta\,z)\hat{\imath} + E_{y_0}\sin(\varphi_y - \beta\,z)\hat{\jmath} + E_{z_0}\sin(\varphi_z - \beta\,z)\hat{k}]) = \\ &= \overline{E}_1\,\cos(\omega\,t) + \overline{E}_2\,\sin(\omega\,t) \end{aligned} \tag{3.2}$$

First of all we notice that $\overline{E}_1$ and $\overline{E}_2$ are vectors that are only in function of space (and that is great and useful).
Then we can write:

$$\overline{E}(z,t) = \mathrm{Re}\left\{(\overline{E}_1 + j\overline{E}_2)[\cos(\omega t) + j\sin(\omega t)]\right\} = \mathrm{Re}\left\{(\overline{E}_1 + j\overline{E}_2)e^{j\omega t}\right\} \tag{3.3}$$

With this notation we can introduce the phasor

$$\overset{\circ\!\!\rightarrow}{E}(z) = \overline{E}_1 + j\overline{E}_2 \tag{3.4}$$

And we will use this notation to describe the EMF in complex notation.

$$\overline{E}(z,t) = \mathrm{Re}\left\{\overset{\circ\!\!\rightarrow}{E}(z)e^{j\omega t}\right\} \tag{3.5}$$

If we want to go from time domain to phasor, we need to find E_1 and E_2. To do the opposite we need to use eq. (3.5).
Now let's give a look at the derivative of the field:

$$\frac{\partial \overline{E}(z,t)}{\partial t} = \text{Re}\left\{\frac{\partial}{\partial t}\overset{\circ\!\!\rightarrow}{E} e^{j\omega t}\right\} = \text{Re}\left\{j\omega \overset{\circ\!\!\rightarrow}{E} e^{j\omega t}\right\} \tag{3.6}$$

With this trick we can use the derivative by only multiplying $j\omega$.
That being said, we can give the definition of electric or magnetic field that propagates in a general direction:

$$\begin{aligned} \overline{E}(\overline{r},t) &= \text{Re}\left\{\overset{\circ\!\!\rightarrow}{E}(\overline{r}) e^{j\omega t}\right\} \\ \overline{H}(\overline{r},t) &= \text{Re}\left\{\overset{\circ\!\!\rightarrow}{H}(\overline{r}) e^{j\omega t}\right\} \end{aligned} \tag{3.7}$$

Now in phasor domain, we can write the maxwell equations that we have already seen in eq. (1.19), but with a simpler notation.

$$\begin{aligned} \nabla \times \overset{\circ\!\!\rightarrow}{E} &= j\omega \overset{\circ\!\!\rightarrow}{B} \\ \nabla \times \overset{\circ\!\!\rightarrow}{H} &= j\omega \overset{\circ\!\!\rightarrow}{D} + \overset{\circ\!\!\rightarrow}{J}_\sigma + \overset{\circ\!\!\rightarrow}{J}_i \end{aligned} \tag{3.8}$$

Obviously this is true for any general direction $\overline{r}$, evn if we didn't mentioned that for a better notation elegance.
Another important thing that we need to stress is the relation of the other vectors that can be represented in the phasor space:

$$\overset{\circ\!\!\rightarrow}{D} = -\varepsilon \overset{\circ\!\!\rightarrow}{E} \qquad \overset{\circ\!\!\rightarrow}{B} = \mu \overset{\circ\!\!\rightarrow}{H} \qquad \overset{\circ\!\!\rightarrow}{J}_\sigma = \sigma \overset{\circ\!\!\rightarrow}{E}$$

Note that ε,μ and σ in this case are dependent on the position and frequency (not in time as before).

$$\begin{aligned} \varepsilon &= \varepsilon(\overline{r},\omega) \\ \mu &= \mu(\overline{r},\omega) \\ \sigma &= \sigma(\overline{r},\omega) \end{aligned} \tag{3.9}$$

Note on the refraction index

The refraction index can become useful next, here we only introduce it and say what does it mean.
First of all we define refraction index n as the square root of ε_r

$$n = \sqrt{\varepsilon_r} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \tag{3.10}$$

Actually this is a simplified relation, it is better to define n as:

$$n = \sqrt{\varepsilon_r \, \mu_r} = \sqrt{\frac{\varepsilon \, \mu}{\varepsilon_0 \, \mu_0}} \tag{3.11}$$

We already seen before that $c = \frac{1}{\sqrt{\varepsilon_0 \, \mu_0}}$ and $v_p = \frac{1}{\sqrt{\varepsilon \, \mu}}$, so we the refraction index is very useful to describe the speed of an EMF through a medium:

$$n = \frac{c}{v_0} \tag{3.12}$$

Wave equation in phasor domain

Given the system of equation in eq. (3.8), and applying some substitution we can obtain:

$$\begin{aligned} \nabla \times \overset{\circ\!\!\rightarrow}{E} &= -j\omega\mu \, \overset{\circ\!\!\rightarrow}{H} \\ \nabla \times \overset{\circ\!\!\rightarrow}{H} &= j\omega\varepsilon \, \overset{\circ\!\!\rightarrow}{E} + \sigma \, \overset{\circ\!\!\rightarrow}{E}_{\sigma} + \overline{J}_i \end{aligned} \tag{3.13}$$

It is evident that without derivative it is much more simple to find the wave equation.
For the sake of simplicity we will not make all the calculation to arrive to the final value, instead we use the old wave equation in time domain (eq. (2.7)) and we obtain the new wave equation in phasor domain:

$$\nabla^2 \, \overset{\circ\!\!\rightarrow}{E} - \frac{\omega^2}{c^2} \, \overset{\circ\!\!\rightarrow}{E} = 0 \tag{3.14}$$

If we assume E as a scalar field, then:

$$\frac{\partial^2 \overset{\circ\!\!\rightarrow}{E}(z)}{\partial z^2} \nabla^2 - \frac{\omega^2}{c^2} \, \overset{\circ\!\!\rightarrow}{E}(z) = 0 \tag{3.15}$$

eq. (3.14) and eq. (3.15) are also known as the Helmholtz equation (strange name, i know).

Solution of the wave equation in phasor domain

Given eq. (3.15), it is very simple to obtain its solution.
Before to do that we introduce the parameter γ such that:

$$\gamma^2 = -\frac{\omega^2}{c^2} = -\omega \, \mu \, \varepsilon \tag{3.16}$$

Substituting $-\frac{\omega^2}{c^2}$ with *gamma* from eq. (3.15) we are able to get a possible solution:

$$\overset{\circ\!\!\rightarrow}{E} = \overline{E}_{0_1} \, e^{\gamma z} + \overline{E}_{0_2} \, e^{-\gamma z} \tag{3.17}$$

We know that γ is a complex number because both μ and ε are so:

$$\gamma = \alpha + j\beta \tag{3.18}$$

So we can write eq. (3.17) as:

$$\overset{\circ\!\rightarrow}{E} = \overline{E}_{0_1}\, e^{\alpha z}\, e^{j\beta z} + \overline{E}_{0_2}\, e^{-\alpha z}\, e^{-j\beta z} = \tag{3.19}$$

We will focus on the forward wave equation $\overset{\circ\!\rightarrow}{E} = \overline{E}_0\, e^{-\alpha z}\, e^{-j\beta z}$.

Going back from phasor to time domain

If we want to go back to time domain from eq. (3.19) we can just apply the relation from eq. (3.5):

$$\begin{aligned}
&\overline{E}(z,t) = \mathrm{Re}\left\{\overset{\circ\!\rightarrow}{E}(z)e^{j\omega t}\right\} = \\
&= \mathrm{Re}\left\{\overline{E}_{0_2}\, e^{-\alpha z}\, e^{-j\beta z} e^{j\omega t}\right\} = \\
&= \mathrm{Re}\left\{|E_{0_2}|\, e^{j\varphi_0}\, e^{-\alpha z}\, e^{-j\beta z} e^{j\omega t}\right\} = \\
&= |E_{0_2}| e^{-\alpha z}\, \mathrm{Re}\left\{e^{j(\omega t - \beta z + \varphi_0)}\right\} = \\
&= |E_{0_2}| e^{-\alpha z}\, \mathrm{Re}\left\{\cos(\omega t - \beta z + \varphi_0) + j\sin(\omega t - \beta z + \varphi_0)\right\} = \\
&= |E_{0_2}| e^{-\alpha z}\, \cos(\omega t - \beta z + \varphi_0)
\end{aligned} \tag{3.20}$$

We obtained the forward equation of an EMF that propagates over the z direction, that we have already seen in eq. (2.20).

Why φ is a complex number?

Consider the Maxwell equation in eq. (3.13), but without the $\overline{J}_i$ term (no current that is generating the EMF).
I'll not write again the eq. (3.13) because you can find it simply clicking at the reference number, that being said we can do something with the second relation:

$$\begin{aligned}
\nabla \times \overset{\circ\!\rightarrow}{H} &= j\omega\varepsilon \left(1 + \frac{\sigma}{j\omega\epsilon}\right) \overset{\circ\!\rightarrow}{E} = \\
\nabla \times \overset{\circ\!\rightarrow}{H} &= j\omega\varepsilon \left(1 - j\frac{\sigma}{\omega\epsilon}\right) \overset{\circ\!\rightarrow}{E} = \\
\nabla \times \overset{\circ\!\rightarrow}{H} &= j\omega \underbrace{\varepsilon \left(1 + \frac{\sigma}{j\omega\epsilon}\right)}_{\varepsilon_c} \overset{\circ\!\rightarrow}{E} = \\
\nabla \times \overset{\circ\!\rightarrow}{H} &= j\omega\varepsilon_c\, \overset{\circ\!\rightarrow}{E}
\end{aligned} \tag{3.21}$$

ε_c is the complex permittivity, with this the maxwell equation becomes very similar to eq. (2.1):

$$\begin{cases} \nabla \times \overset{\circ\!\!\to}{E} = -j\omega\,\mu\,\overset{\circ\!\!\to}{H} \\ \nabla \times \overset{\circ\!\!\to}{H} = \quad j\omega\,\varepsilon_c\,\overset{\circ\!\!\to}{E} \end{cases} \tag{3.22}$$

So eq. (3.16) is not totally correct because we need to consider the complex permittivity ε_c: $\gamma^2 = -\omega\,\mu\,\varepsilon_c$.
An interesting thing to note is that the imaginary part of $\varepsilon_c = \varepsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)$ are the losses during the propagation of the EMF through a medium

- $\boldsymbol{\sigma}$: metallic medium loss
- $\boldsymbol{\omega\epsilon}$: dielectric medium loss

We can also use $\frac{\sigma}{\omega\epsilon}$ to know the property of our medium:

- $\boldsymbol{\frac{\sigma}{\omega\epsilon} > 1}$: metallic medium
- $\boldsymbol{\frac{\sigma}{\omega\epsilon} < 1}$: dielectric medium loss

If we suppose no metallic loss: $\sigma = 0$, then:

$$\theta = 0 \to \varepsilon_c = \varepsilon \to \alpha = 0 \to \gamma = j\beta$$

With this simplification the forward magnetic field becomes:

$$\overset{\circ\!\!\to}{E} = \overline{E}_0\, e^{-j\beta z} \tag{3.23}$$

EMF in frequency domain

Until now we have seen a pure sinusoidal EMF that propagates, what if this EMF is not pure?
We can say that our signal is not a pure sinusoid if we have more than 1 component other than the fundamental harmonic, this mean that we deal with noisy signal.
Similarly to eq. (3.5), we can define the transformation from time domain to frequency domain as:

$$\overline{E}(\overline{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{E}(\overline{r}, w) e^{j\omega t} \tag{3.24}$$

That transformation is actually the same as the one for the complex domain, but we now can consider more than 1 harmonic.
Again here we don't deal anymore with derivatives in time, so our job simplify a lot!:

$$\frac{\partial \overline{E}(\overline{r}, t)}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega\, \overline{E}(\overline{r}, w) e^{j\omega t} \tag{3.25}$$

Like in eq. (3.22) we can have a look at the Maxwell equation in phasor domain again without the $\overline{J}_i$ term (no current that is generating the EMF).
You can notice that they are actually the same, but in frequency the field E and H are dependent in time and also in frequency.

$$
\begin{aligned}
\nabla \times \overline{E}(t,\omega) &= -j\omega\mu\,\overline{H}(t,\omega) \\
\nabla \times \overline{H}(t,\omega) &= \quad j\omega\varepsilon\,\overline{E}(t,\omega)
\end{aligned}
\tag{3.26}
$$

And the wave equation becomes:

$$
\nabla^2\,\overline{E}(t,\omega) - \gamma^2\,\overline{E}(t,\omega) = 0 \tag{3.27}
$$

A little exercise

The prof said it was little... but i'm too tired to copy all the numbers, but here i reported the passages.
The request was to find the expression of the EMF (E and H equation) given γ, the direction z and supposing no losses ($\alpha = 0$). The field of E is on his peak E_0 wen $t = 0$ and $z = 50$
First of all the peak of the field E_0 is obtained when $\cos(\omega t - \beta z + \phi_0) = 1$, so when $(\omega t - \beta z + \phi_0) = 0$.
We can simplify saying that $\omega t = 0$ (we can assume the initial time $t_0 = 0$ because of the given data).
From eq. (3.16) we can obtain ω from γ ($\omega^2 = c^2 - \omega^2$).
From eq. (2.18) we also know that $\beta = \frac{\omega}{c}$.
Then we obtain φ_0 from $-\frac{\omega}{c} + \phi_0 = 0$.
We have all we need to write down the equation for E in time

$$
E(z,t) = E_0\,\cos(-\beta z + \phi_0)\,\hat{\imath}
$$

What about H?

$$
\nabla \times \overline{E} = \frac{\partial E_x}{\partial z}\hat{\jmath} = -\mu\frac{\partial \overline{H}}{\partial t}\hat{\jmath} \tag{3.28}
$$

Doing some strange calculation we can obtain the H equation, where the argument of *cos* are the same, but H_0 changes:

$$
H(z,t) = H_0\,\cos(-\beta z + \phi_0)\,\hat{\jmath}
$$

Frequency domain

In frequency it is much more simple:

$$
\overset{\circ\!\rightarrow}{E}(z) = \overline{E_0}\,\cancelto{0}{e^{-\alpha z}}\;e^{-j\beta z} = |E_0|\,e^{-j\beta z}\,e^{j\varphi_0}
$$

We already have E_0, β and φ_0 can be simply calculated as before.
Now $\overset{\circ\!\!\to}{H}(z)$??

$$
\begin{aligned}
\frac{\partial \overset{\circ\!\!\to}{E_x}}{\partial z} &= -j\omega\mu \overset{\circ\!\!\to}{H} \\
\overset{\circ\!\!\to}{H} &= \frac{1}{-j\omega\mu}\frac{\partial \overset{\circ\!\!\to}{E_x}}{\partial z} = \\
&= \frac{j}{\omega\beta}\overline{E_0}e^{-j\beta z} = \\
&= \frac{\beta}{\mu\omega}|E_0|\, e^{j\varphi_0}e^{-j\beta z}
\end{aligned}
\tag{3.29}
$$

We know that $\beta = \frac{\omega}{c}$, then we obtain the intrinsic impedance η:

$$
\frac{\omega}{c}\frac{1}{\omega\mu} = \frac{\sqrt{\mu\varepsilon}}{\mu} = \sqrt{\frac{\epsilon}{\mu}} = \eta \tag{3.30}
$$

So we obtain a very useful equation:

$$
\overset{\circ\!\!\to}{H} = \frac{1}{\eta}\,\overline{E_0}e^{-j\beta z} \tag{3.31}
$$

And we can also say that if the field propagates along z:

$$
\begin{aligned}
\overset{\circ\!\!\to}{H} &= \frac{1}{\eta}\,\hat{k} \times \overset{\circ\!\!\to}{E} \\
\overset{\circ\!\!\to}{E} &= \eta\, \overset{\circ\!\!\to}{H} \times \hat{k}
\end{aligned}
\tag{3.32}
$$

Note that the second equation in eq. (3.32) is very similar to the first hom law because we have:

$$
\left[\frac{V}{m}\right] = \eta\, \left[\frac{A}{m}\right]
$$

Just like $[V] = \Omega\ [A]$

One more thing: Poynting vector

The Poynting vector $\overline{S}$ describe the density of power of our radiation, and it is defined by:

$$
\overline{S} = \overline{E} \times \overline{H} \rightarrow \left[\frac{w}{m}\right] \tag{3.33}
$$

To obtain $\overline{S}$ is not very simple, but in phasor domain it is *na crema* (italian way to say "very beautiful"):

$$
\overline{S} = \frac{\overset{\circ\!\!\to}{E} \times \overset{\circ\!\!\to}{H}^{*}}{2} = \frac{1}{2}\,\overline{E}_x \cdot \overline{H}_y^{*}\,\hat{k} \tag{3.34}
$$

4 Class 4 - 5/03/21

Transmission line

When we have a signal to be transmitted over a cable, the problem to connect the source of the signal to the load is not as simple as cabling a dc power source. In our case the voltage level to be transmitted can be considered as a wave, and his behavior over the transmission line can vary with the length, the frequency of the signal or also with the geometry of the cable.

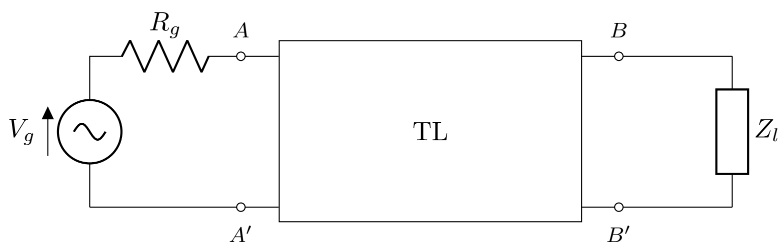


Figure 4.1: Transmission line example

In fig. 4.1 a very simplified view of a transmission line, where the voltage at the input is the one over AA', and BB' on the output.
If we do not consider the resistance R_g, then the voltage in input of the TL can be seen as:

$$V_{AA'}(t) = V_g(t) = V_0 \cos(\omega t) \tag{4.1}$$

Now for the output voltage equation we need to consider the time delay of signal travelling from A to B. So we need to consider the same signal $V_{AA'}$ but delayed by $\frac{l}{c}$ (l is the length of the TL, and c is the speed of the signal, the same speed of light).

$$\begin{aligned} V_{BB'}(t) = V_{AA'}\left(t - \frac{l}{c}\right) = \\ = V_0 \cos\left[\omega\left(t - \frac{l}{c}\right)\right] = \\ = V_0 \cos\left(\omega t - \frac{\omega}{c}\, l\right) = \\ = V_0 \cos\left(\omega t - \beta\, l\right) = \end{aligned} \tag{4.2}$$

Actually in eq. (4.2) we can see that our signal over the transmission line is actually a wave dependent on the frequency and the length of the cables.

Example on different transmission line length

As we can see in eq. (4.2), we need to face a new problem: how the strength of our signal change accordingly to the length of my transmission line?
For example, on the input of the TL we have a signal with amplitude A_0 and a frequency of $f = 1\,\text{kHz}$.
Suppose the length of the TL $l = 5\,\text{cm}$, then the signal on the output at $t = 0$ will be:

$$V_{BB'}(t) = V_0\,\cos(\omega t - \beta\,l) = \cos\left(\omega t - \frac{\omega}{c}\,l\right) =$$

$$= V_0\,\cos\left(2\,\pi\,10^3 \cdot 0 + \frac{2\,\pi\,10^3}{3 \cdot 10^8} \cdot 0.05\right) =$$

$$= V_0\,\cos(0,105 \cdot 10^{-5}) \approx V_0 \cdot 0.999 \approx V_0$$

As we can see, we have a very short length of the cable compared to the wavelength, the attenuation over the line due is totally negligible.
If we suppose instead the length of the TL: $l = 100\,\text{km}$, then the signal on the output at $t = 0$ will be:

$$V_{BB'}(t) = V_0\,\cos\left(2\,\pi\,10^3 \cdot 0 + \frac{2\,\pi\,10^3}{3 \cdot 10^8} \cdot 100 \cdot 10^3\right) =$$

$$= V_0\,\cos(2.09) \approx V_0 \cdot -0.49 \approx -\frac{1}{2}\,V_0$$

It is very important to notice that now the amplitude of the signal is half as before, just by moving away from the source, and without considering the attenuation of the line.
This behavior can be explained by looking at the argument of the *cos* on eq. (4.2):

- $\boldsymbol{(\omega t)}$: is responsible to the information propagation (otherwise we could not detect the wave).
- $\boldsymbol{(\beta\,l)}$: is the one dependent on the length that we need to investigate.

What we can notice is that the effect of that attenuation depends a lot on the wavelength:

$$\beta\,l = \frac{\omega}{c}\,l = \frac{2\pi}{c} \cdot f \cdot l = 2\pi \cdot \frac{l}{\lambda}$$

The cosine operator is periodic over 2π, this mean that when the length of the TL l is a multiple of the wavelength λ, the output signal won't be attenuated: $\cos(k\,2\pi) = 1$.
But when $l = (\frac{n}{2} + 1)\,\lambda$ for $n = 0, 1, \cdots$ at $t = 0$, on the output of the TL we will not see any signal because $\cos(k\,\frac{\pi}{2}) = 1$
Now you could say: *"hey!, that sounds like a simple phase shift, we can just wait $\frac{\pi}{2}$ and then the signal amplitude will be again A_0"*, that is true my friend,

but by this example it is very simple to understand how a signal can change if we move over a transmission line.
In the next section we will talk about a real attenuation of the signal along the TL (for example we will be able to choose the right distance from the source to guarantee the maximum power transfer).
Another thing that we will take in consideration is the losses over the line, and the attenuation of the reflected signal.

How can we model our transmission line?

To propagate an electric signal we need 2 conductors, it is not important if they are twisted pairs or coaxial (for now).
Then we can divide our line in different sectors of length Δz, and each segment can be modelled as a circuit:

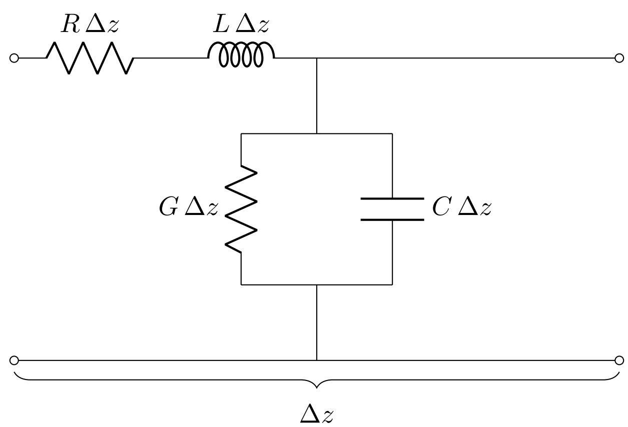


Figure 4.2: Circuital representation of a TL section

Now from fig. 4.2 we can see different parameters:

- $\boldsymbol{R\,\Delta z}$: Resistance representing the losses on both conductors along the line.
- $\boldsymbol{L\,\Delta z}$: Inductive behavior of the cable.
- $\boldsymbol{G\,\Delta z}$: Conductance representing the losses due to the separator.
- $\boldsymbol{C\,\Delta z}$: Capacitive behavior due to the two conductors.

Example of coaxial conductor

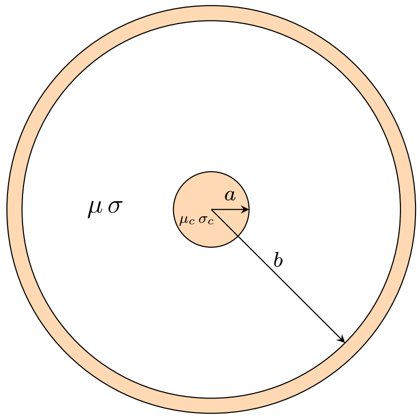


Figure 4.3: Simplified coaxial section

The most common example of transmission line is the coaxial cable:
in our example we suppose the inner conductor radius a and outer conductor radius b, then we also need to know the electric characteristic of the conductor (μ_c and σ_c) and the one of the separator (μ and σ). We can write now the characteristic value of the transmission line:

$$R = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) \quad \left[\frac{\Omega}{\mathrm{m}}\right]$$

$$L = \frac{\mu}{2\pi} ln\left(\frac{b}{a}\right) \quad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

$$G = \frac{2\pi\sigma}{ln\left(\frac{b}{a}\right)} \quad \left[\frac{\mathrm{S}}{\mathrm{m}}\right]$$

$$C = \frac{2\pi\epsilon}{ln\left(\frac{b}{a}\right)} \quad \left[\frac{\mathrm{F}}{\mathrm{m}}\right]$$

R_s is the resistance on the surface of the conductor and $R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$.
Now let's make some consideration on L and C:

$$L \cdot C = \frac{\mu}{\cancel{2\pi}} \cancel{ln\left(\frac{b}{a}\right)} \frac{\cancel{2\pi}\,\varepsilon}{\cancel{ln\left(\frac{b}{a}\right)}} = \mu \cdot \varepsilon \tag{4.3}$$

This relation $L \cdot C = \mu \cdot \varepsilon$ is very interesting for us, because we have already sen that $c = \frac{1}{\sqrt{\mu\,\varepsilon}}$ and $\beta = \omega\sqrt{\mu\,\varepsilon}$.
So what does the eq. (4.3) mean? The answer is that when the EMF is forced to move inside a cable we will not use μ and ε, but L and C, and that is very useful and cool for us.

Telegraph equations

To obtain some cool relation for our TL we can use the Kirchhoff law as usual.

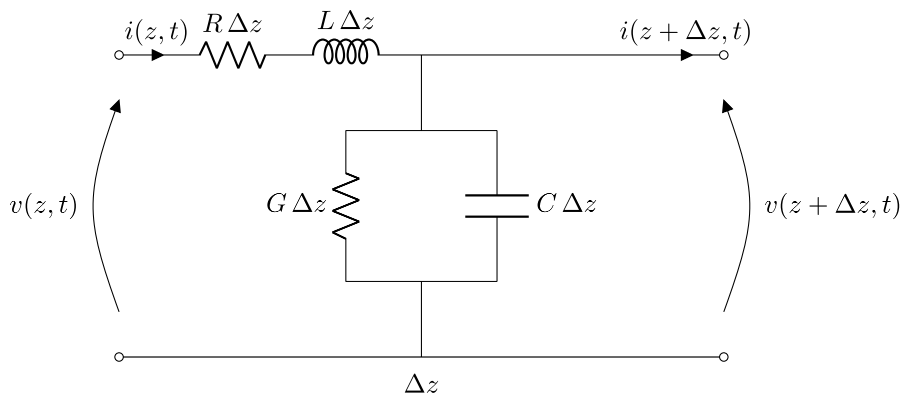


Figure 4.4: Circuital representation of a TL section with Kirchhoff

Now we can analyze the circuit in fig. 4.4 using the Kirchhoff law:

$$v(z,t) - R\,\Delta z\, i(z,t) - L\,\Delta z\,\frac{\partial i(z,t)}{\partial t} = v(z+\Delta z,t)$$

Divide all for Δz

$$-\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R\, i(z,t) + L\,\frac{\partial i(z,t)}{\partial t}$$

Now let shrink the interval $\Delta z \to 0$

$$\begin{cases} -\frac{\partial v(z,t)}{\partial z} = R\, i(z,t) + L\frac{\partial i(z,t)}{\partial t} \\ -\frac{\partial i(z,t)}{\partial z} = G\, v(z,t) + C\frac{\partial v(z,t)}{\partial t} \end{cases} \tag{4.4}$$

In eq. (4.4) we obtained the so called *telegraph equations*.
The second equation in eq. (4.4) can be obtained by doing the same calculation that we have already done for the first one.
If we do not consider losses, those equations becomes even prettier:

$$\begin{cases} -\frac{\partial v(z,t)}{\partial z} = L\frac{\partial i(z,t)}{\partial t} \\ -\frac{\partial i(z,t)}{\partial z} = C\frac{\partial v(z,t)}{\partial t} \end{cases} \tag{4.5}$$

Wave equation of the signal on the TL

To obtain the wave equation, we can use the derivative over the space on the first equation of eq. (4.5):

$$
\begin{aligned}
-\frac{\partial^2 v(z,t)}{\partial z^2} &= L\frac{\partial}{\partial t}\frac{\partial i(z,t)}{\partial z} \\
\frac{\partial^2 v(z,t)}{\partial z^2} &- L\,C\frac{\partial^2 v(z,t)}{\partial t^2} = 0 \\
\frac{\partial^2 v(z,t)}{\partial z^2} &- \frac{1}{c^2}\frac{\partial^2 v(z,t)}{\partial t^2} = 0
\end{aligned}
\tag{4.6}
$$

The wave equation obtained in eq. (4.6) is very similar to what we have fond before, from here we can also obtain the solution of that equation:

$$
v(z,t) = V_{+}\cos\left(\omega t - \frac{\omega}{c}z\right) + V_{-}\cos\left(\omega t + \frac{\omega}{c}z\right) \tag{4.7}
$$

Telegraph and wave equation in phasor domain

Starting from eq. (4.5) we can obtain the telegraph equation in phasor domain:

$$
\begin{cases}
-\frac{\partial \overset{\circ\rightarrow}{V}}{\partial z} = (R + j\omega L)\overset{\circ\rightarrow}{I} \\
-\frac{\partial \overset{\circ\rightarrow}{I}}{\partial z} = (G + j\omega C)\overset{\circ\rightarrow}{V}
\end{cases}
\tag{4.8}
$$

Then the wave equation becomes:

$$
\begin{aligned}
-\frac{\partial^2 \overset{\circ\rightarrow}{V}}{\partial z^2} &= (R + j\omega L)\frac{\partial \overset{\circ\rightarrow}{I}}{\partial z} = \\
&= (R + j\omega L)(G + j\omega C)\overset{\circ\rightarrow}{V}
\end{aligned}
\tag{4.9}
$$

We introduce $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$, then we obtain the telephone equation (another name for the wave equation for TL):

$$
\begin{cases}
\frac{\partial^2 \overset{\circ\rightarrow}{V}}{\partial z^2} - \gamma^2\,\overset{\circ\rightarrow}{I} = 0 \\
\frac{\partial^2 \overset{\circ\rightarrow}{I}}{\partial z^2} - \gamma^2\,\overset{\circ\rightarrow}{V} = 0
\end{cases}
\tag{4.10}
$$

We can also write a beautiful solution from eq. (4.10):

$$
\overset{\circ\rightarrow}{V}(z) = V_{+}\,e^{-\gamma z} + V_{-}\,e^{\gamma z} \tag{4.11}
$$

We also know that γ is a complex number, so what we can do is to write the same equation, but dividing $\gamma = \alpha + j\beta$ and looking only at the forward:

$$
\overset{\circ\rightarrow}{V}(z) = V_{+}\,e^{-\alpha z}\,e^{-j\beta z} \tag{4.12}
$$

If i go back to time domain (see eq. (3.20)):

$$v(z,t) = V_{+} e^{\alpha z} \cos(\omega t - \beta z) \tag{4.13}$$

Inside γ we have R,L,G and C, so we have all the geometry and physical characteristic of the TL.

What could happen if we assume no losses?

If we assume no losses over the line, then $\alpha = 0$, and so $\gamma = j\beta$, then the wave propagation will follow eq. (4.14):

$$\begin{cases} \overset{\circ\rightarrow}{V}(z) = V_{+} e^{-j\beta z} + V_{-} e^{j\beta z} = \overset{\circ\rightarrow}{V_p}(z) + \overset{\circ\rightarrow}{V_r}(z) \\ \overset{\circ\rightarrow}{I}(z) = I_{+} e^{-j\beta z} + I_{-} e^{j\beta z} = \overset{\circ\rightarrow}{I_p}(z) + \overset{\circ\rightarrow}{I_r}(z) \end{cases} \tag{4.14}$$

So what are the parameters that give losses? We obtain the α and β value from the real and imaginary value of:

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} = \\ &= \sqrt{R\,G + j\omega L\,G + j\omega C\,R - \omega^2 L\,C} = \\ &= \sqrt{j\omega(L\,G + C\,R) + R\,G - \omega^2 L\,C} \end{aligned} \tag{4.15}$$

I will not go further because i don't know how to do that, but the imaginary part is:

$$\beta = \omega\sqrt{L\,C} \tag{4.16}$$

5 Class 5 - 08/03/21

characteristic impedance

We can start this lecture by computing the derivative over the space of the voltage equation in phasor domain in eq. (4.11).

$$\frac{\partial \overset{\circ\!\!\rightarrow}{V}(z)}{\partial z} = \gamma \cdot \left(-Z_i(l)\, e^{-\gamma z} + V_{-}\, e^{\gamma z}\right) \tag{5.1}$$

Then from the first equation in eq. (4.8) we can obtain another way to express the current in phasor domain:

$$\overset{\circ\!\!\rightarrow}{I}(z) = \overbrace{\frac{\gamma}{(R + j\omega L)}}^{\frac{1}{Z_0}} \left[V_{+}\, e^{-\gamma z} - V_{-}\, e^{\gamma z}\right] \tag{5.2}$$

If we take a look on that equation, we can notice a very interesting pattern, that look like the ohm's law: $I = G\,V$ with G the conductivity of the circuit, or the inverse of the resistivity.
If you look closely, also $\frac{\gamma}{(R+j\omega L)}$ is a conductance:

$$\frac{\gamma}{(R + j\omega L)} = \frac{\sqrt{(R + j\omega L)\,(G + j\omega C)}}{(R + j\omega L)} = \sqrt{\frac{G + j\omega L}{R + j\omega C}} \tag{5.3}$$

The inverse of that term is an impedance, and we call that *characteristic impedance* (lol, what a fantasy, you should give pokemon's name):

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}\ [\Omega] \tag{5.4}$$

Without losses, the same equation becomes:

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \tag{5.5}$$

If you look closely, eq. (5.5) is very similar to the intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ (eq. (3.30)).
The current and current expression becomes:

$$\begin{cases} \overset{\circ\!\!\rightarrow}{V}(z) = V_{+}\, e^{-j\beta z} + V_{-}\, e^{j\beta z} \\ \overset{\circ\!\!\rightarrow}{I}(z) = \frac{1}{Z_0}\left[V_{+}\, e^{-\gamma z} - V_{-}\, e^{\gamma z}\right] \end{cases} \tag{5.6}$$

From now on, sometimes i could miss some phasor notation over V and I, this was very important to emphasize the difference between the different domains,

but no the next pages if you see $V(l)$ or $I(l)$ just remember that those wave expression does not contain the time t, so they are in phasor domain.
That being said, we can also write:

$$\begin{aligned} V_- &= -Z_0\, I_- \\ V_+ &= Z_0\, I_+ \end{aligned} \tag{5.7}$$

Input impedance

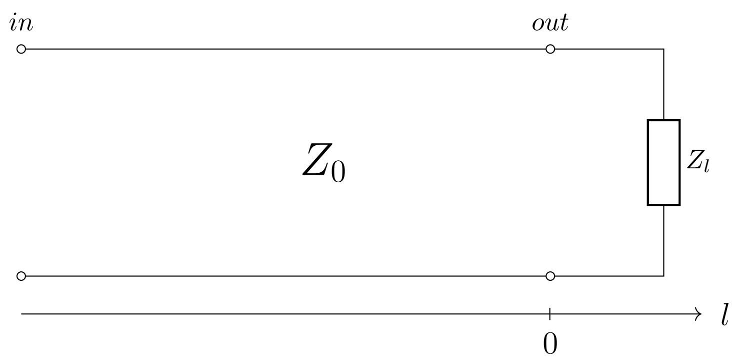


Figure 5.1: Transmission line with reference system

We assume that our transmission line is like in fig. 5.1, here you can see that we introduced a reference system on the length of the line. The new coordinate l is increasing on the right, and the zero point is set on the output coordinate of the line. Starting from the equation of the signal over the line (with no losses) in eq. (4.14), we can obtain the *input impedance* of the line dependent on the position l:

$$\begin{aligned} Z_i(l) &= \frac{\overset{\circ\!\!\to}{V}(l)}{\overset{\circ\!\!\to}{I}(l)} = \frac{V_+\, e^{-j\beta z} + V_-\, e^{j\beta z}}{I_+\, e^{-j\beta z} + I_-\, e^{j\beta z}} \\ &= Z_0\, \frac{V_+\, e^{-j\beta z} + V_-\, e^{j\beta z}}{V_+\, e^{-j\beta z} - V_-\, e^{j\beta z}} \end{aligned} \tag{5.8}$$

Note that $Z_i(l)$ and Z_0 are very different, the first one can depend on the position over the line. But we can notice an important relation between the two when we are in $l = 0$:

$$Z_i(l=0) = Z_L = \frac{V_L}{I_L} = Z_0\, \frac{V_+ + V_-}{V_+ - V_-} \tag{5.9}$$

To not make confusion!

All this notation is very confusing in my opinion, so a fast clarification:

- Z_0 is the characteristic impedance, it is the same for all the line and can be considered like the impedance of the tiny tiny section of the TL.
- $Z_i(l)$ is the impedance that you would measure if you take a very special oscilloscope in a specific position l on your line.
If you are in $l = 0$ you are measuring the load (Z_L).
If you are in $l = L$ you are measuring the impedance on the input side of the TL (also named input impedance), so what impedance your generator see (Z_{in}).
- $\overset{\circ\!\rightarrow}{V}(l)$ is the voltage that you would measure if you take a special oscilloscope in a specific position l on your line (in phasor domain).
- V_L is the voltage that the load see, that is also $\overset{\circ\!\rightarrow}{V}(0)$.
- V_+ is the progressive component of the voltage over the line, think at it like the module of the progressive voltage wave. It is a part of the wave equation solution.
- $\overset{\circ\!\rightarrow}{V_p}(z) = V_+ \, e^{-j\beta z}$ is the progressive (or incident) voltage wave equation over the line in phasor domain.
- $v(z,t)$ is the real voltage over the line and over the time, we are not using it because it is very difficult to manipulate, so we prefer to go in complex domain, then going back to time domain with simpler calculations

Still very confusing, i know (>_<).

Input impedance in trigonometric form

As we can see from the final form of the input impedance of eq. (5.8), Z_i is a complex number, so it could be useful to express it in trigonometric form (you should remember $a\, e^{j\, b} = a(\cos\, b + j \sin\, b)$).
Now the calculation can seems scary, but they are very simple but long:

$$
\begin{aligned}
Z_i(l) &= Z_0 \, \frac{V_+ \, e^{-j\beta l} + V_- \, e^{j\beta l}}{V_+ \, e^{-j\beta l} - V_- \, e^{j\beta l}} = \\
&= Z_0 \frac{V_+[\cos(\beta\, l) - j\ \sin(\beta\, l)] + V_-[\cos(\beta\, l) + j\ \sin(\beta\, l)]}{V_+[\cos(\beta\, l) - j\ \sin(\beta\, l)] - V_-[\cos(\beta\, l) + j\ \sin(\beta\, l)]} = \\
&= Z_0 \, \frac{(V_+ + V_-)\, \cos(\beta\, l) - j(V_+ - V_-)\, \sin(\beta\, l)}{(V_+ - V_-)\, \cos(\beta\, l) - j(V_+ + V_-)\, \sin(\beta\, l)} \cdot \frac{\frac{1}{(V_+ - V_-)}}{\frac{1}{(V_+ - V_-)}} = \\
&= Z_0 \frac{\frac{Z_L}{Z_0} \cos(\beta\, l) - j\ \sin(\beta\, l)}{\cos(\beta\, l) - j\, \frac{Z_L}{Z_0} \sin(\beta l)} = \\
&= Z_0 \frac{Z_L\ \cos(\beta\, l) - j\, Z_0\ \sin(\beta\, l)}{Z_0\ \cos(\beta\, l) - j\, Z_L\ \sin(\beta\, l)}
\end{aligned}
\tag{5.10}
$$

Now for the sake of simplicity we say that l is not the coordinate of the point where we are looking over the line, but the distance from the 0 point.
In this case, the $Z_i(l)$ equation of the input impedance becomes a bit simpler (it only changes a couple of sign):

$$Z_i(l) = Z_0 \frac{Z_L \cos(\beta l) + j Z_0 \sin(\beta l)}{Z_0 \cos(\beta l) + j Z_L \sin(\beta l)} \tag{5.11}$$

In terms of admittance, the eq. (5.11) becomes:

$$Y_i(l) = Y_0 \frac{Y_L \cos(\beta l) + j Y_0 \sin(\beta l)}{Y_0 \cos(\beta l) + j Y_L \sin(\beta l)} \tag{5.12}$$

Reflection coefficient

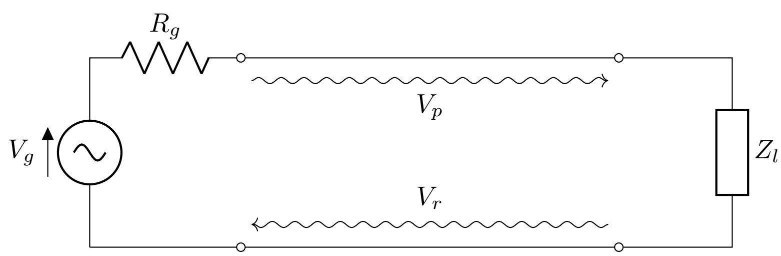


Figure 5.2: Forward and backward wave signal in a TL

In fig. 5.2 you can see a very common example of how the signal propagates in a TL when a load Z_l is connected: we have both a forward and backward wave inside the same medium (ex. coaxial cable).
The forward signal V_+ is the one that we like, the backward V_- is not helpful for us, and it cause unwanted interference (by adding to the forward one, and making a mess).
To describe a bit better this situation, we can introduce the *reflection coefficient* $\rho(l)$, that define the reflected wave with respect to the incident wave:

$$\rho(l) = \frac{V_- e^{-j\beta l}}{V_+ e^{j\beta l}} = \frac{V_-}{V_+} e^{j2\beta l} \tag{5.13}$$

$$\rho(l) = \frac{Z_i(l) - Z_0}{Z_i(l) + Z_0} \tag{5.14}$$

Note that the amplitude of $\rho(l)$ is always fixed along l, and only the exponential part $e^{j2\beta l}$ is dependent on l (I don't really know why this is important ¯_(ツ)_/¯).
Keep in mind that this reflection coefficient relates on the position over the line.

Reflection coefficient at the load

If you place a load Z_L at the output of the transmission line, and you measure the voltage $V(l=0)$ and the current $I(l=0)$ at the end of that TL ($l=0$), you can have the *reflection coefficient at the load* ρ_L:

$$\rho(l=0) = \rho_L = \frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{5.15}$$

From eq. (5.13) we can write:

$$\rho(l) = \rho_L\, e^{j2\beta l} \tag{5.16}$$

This is a parameter that is used to define the reflected wave with respect to the incident wave on the load, and i think that it is much more easy to understand than $\rho(l)$. If you look closely at eq. (5.15), it is possible to have $\rho_L = 0$ when $Z_L = Z_0$. The explanation that the prof gave to us is a little confusing for me, but it works like the wave is always seeing Z_0 while it is travelling along all the tiny portion of TL; when the wave arrives to Z_L, it sees no difference, so for him it is like the TL is infinite.
Note that sometimes the reflection coefficient ρ_L is written like only the symbol ρ. This is a little confusing, but it simplify a lot the notation.

Little exercise!

We have a transmission line with:

- characteristic impedance $Z_0 = 100\Omega$
- Load resistance $R_L = 50\Omega$
- Load capacitance $C_L = 10\text{pF}$
- Frequency of the generator $f = 10\text{MHz}$

The question is: **What is the value of $\rho(l)$ and ρ_L?** First of all we can calculate the load impedance in phasor domain Z_l

$$Z_l = R_L + \frac{1}{j\omega C_L} = R_L - j\,\frac{1}{\omega C_L} = \cdots = 50 - j\,159\;\Omega$$

Then it is simple to obtain ρ_L:

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \cdots = -0.76\, e^{j119^\circ}$$

In this case the 76% of the signal is reflected back to the emitter, very bad! :(
now let's calculate $\rho(l)$:

$$\rho(l) = \rho_L\, e^{j2\beta l} = \rho_L\, e^{j2\frac{\omega}{c}l} = -0.76\, e^{j119^\circ}\; e^{j2\pi 33.3 l}$$

Normalized impedance and reflection coefficient

Sometimes it could be useful to use *normalized input impedance*:

$$\mathcal{Z}_i = \frac{Z_i}{Z_0} \tag{5.17}$$

And also the *normalized load impedance*

$$\mathcal{Z}_L = \frac{Z_L}{Z_0} \tag{5.18}$$

Then the reflection coefficient becomes:

$$\rho(l) = \tfrac{\mathcal{Z}_i(l)-1}{\mathcal{Z}_i(l)+1} \quad \text{and} \quad \rho_L = \tfrac{\mathcal{Z}_L-1}{\mathcal{Z}_L+1}$$

Transmission line wth a short circuit

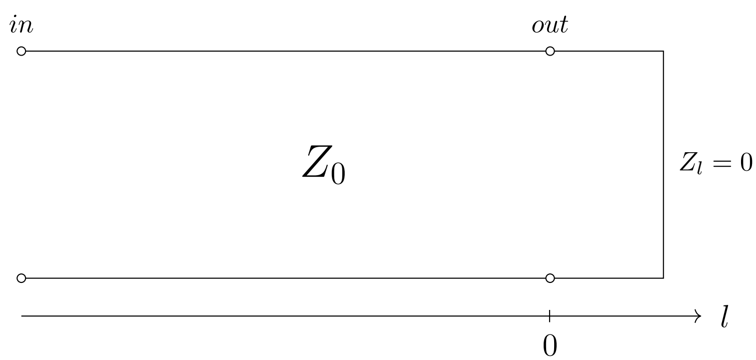


Figure 5.3: Transmission line with short circuit load

If we put a short circuit in the load side as in fig. 5.3, we can notice some useful property.
First of all we know that the voltage on the load is 0V, so:

$$V(l=0) = V_L = 0 = V_+ + V_- \ \rightarrow \ V_+ = -V_- \tag{5.19}$$

This mean that the the reflection coefficient at the load is:

$$\rho_L = \frac{V_-}{V_+} = \frac{-V_+}{V_+} = -1 \tag{5.20}$$

This means that:

$$|\rho_L| = 1 \quad \text{and} \quad \angle \rho_L = 180°$$

In other words, if you place a short circuit as load of your TL, the incident wave will be 100% reflected and it will also be shifted by π (fig. 5.4)

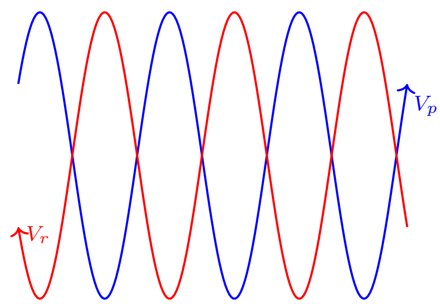


Figure 5.4: Incident and reflected signal when $Z_L = 0$

The voltage and the current over the line is:

$$\begin{cases} V(l) & = V_+ e^{-j\beta l} - V_- e^{j\beta l} = V_+(e^{-j\beta l} + e^{j\beta l}) = -2\, jV_+ \sin(\beta l) \\ I(l) & = \frac{V_+}{Z_0} e^{-j\beta l} + \frac{V_+}{Z_0} e^{j\beta l} = 2\frac{V_+}{Z_0}\cos(\beta l) \end{cases} \tag{5.21}$$

So knowing only that $V_+ = -V_-$ we can derive the voltage and current wave.
• If we go back in time domain:

$$\begin{aligned} V(l,t) &= \mathrm{Re}\left\{V(l)\, e^{j\omega t}\right\} = \mathrm{Re}\left\{j2V_+ \sin(\beta l)\, e^{j\omega t}\right\} = \\ &= \mathrm{Re}\left\{j2V_+ \sin(\omega t)\,(\cos(\omega t) + j\sin(\omega t))\right\} = \\ &= -2\,|V_+|\, \sin(\beta l)\sin(\omega t) \end{aligned} \tag{5.22}$$

Note that this is the equation of a *stationary wave*! this mean that the voltage is moving across the line, but no information is transmitted from te generator to the load.
We can identify the stationary wave when βl and ωt are in different cos (or sin) operator.
• Power of this stationary wave:

$$\begin{aligned} P &= \frac{1}{2} V \cdot I^* = \frac{1}{2}\left(-2jV_+ \sin(\beta l)\right)\left(2\frac{V_+}{Z_0}\cos(\beta l)\right)^* \\ &= -j2\frac{|V_+|^2}{Z_0}\sin(\beta l)\cos(\beta l) \end{aligned} \tag{5.23}$$

Notice that the power is completely imaginary, this mean that we are transmitting only reactive power.
• The input impedance:

$$Z_i(l) = Z_0 \frac{\cancelto{0}{Z_L}\cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0\cos(\beta l) + j\cancelto{0}{Z_L}\sin(\beta l)} = jZ_0 \tan(\beta l) \tag{5.24}$$

• The input admittance:

$$Y_i(l) = \frac{1}{Z_i(l)} = \frac{1}{jZ_0 \tan(\beta l)} = -Y_0 \cot(\beta l) \tag{5.25}$$

We again see that this input impedance is completely imaginary, so could be seen by the input port as a capacitor or an inductor depending on the sign of this imaginary number.
If we plot the behavior of Z_i over $\frac{l}{\lambda}$ we have a graphical way to see this behavior (fig. 5.5):

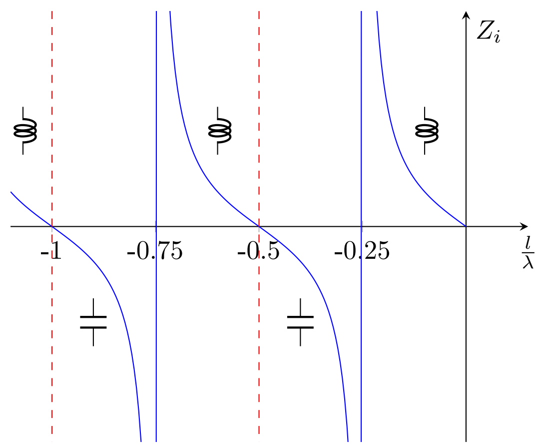


Figure 5.5: Behavior of Z_i a TL if the load is a short

Another little exercise

We have the usual information of our TL, but we know that the load is a short and on the behavior of the line is inductive; we need to find the length of the TL in order to have that behavior.
Data:

- $v_p = 2.07 \cdot 10^8 \, \frac{\mathrm{m}}{\mathrm{s}}$ speed of the signal.
- $L = 15\mathrm{nH}$ behavior of the TL.
- $f = 3\mathrm{GHz}$ frequency of the signal.
- $Z_0 = 50\Omega$ characteristic impedance of the TL.

Find length l.
• First of all we know that the load is a short, so:

$$Z_i(l) = jZ_0 \, \tan(\beta l)$$

We also know that:

$$Z_i(l) = j\omega L$$

We go on to find the value of l:

$$Z_0 \tan(\beta l) = \omega l$$

$$\beta l = \arctan(\frac{\omega l}{Z_0})$$

$$l = \frac{\lambda}{2\pi} \arctan(\frac{\omega l}{Z_0}) = 1.53\text{m}$$

This mean that if my TL described by the data i gave before measure 1.53m, at the eye of the transmitter the TL is an inductor of $L = 15$nH.
• If we change $f = 4$GHz? Nothing to worry about:

$$\frac{l}{\lambda} = \frac{l \cdot f}{v_p} = 0.3 \geq 0.25$$

Assuming that fig. 5.5 is right, now the circuit should behave like a capacitor
Then we calculate Z_i:

$$Z_i = jZ_0 \tan(\beta l) = jZ_0 \tan(\frac{\omega}{v_p} l) = -j167.4$$

We obtained a negative imaginary impedance, so this is a capacitor!

$$-j\frac{1}{\omega C} = -j167.4 \rightarrow C = 0.238\text{pF}$$

Transmission line wth an open circuit

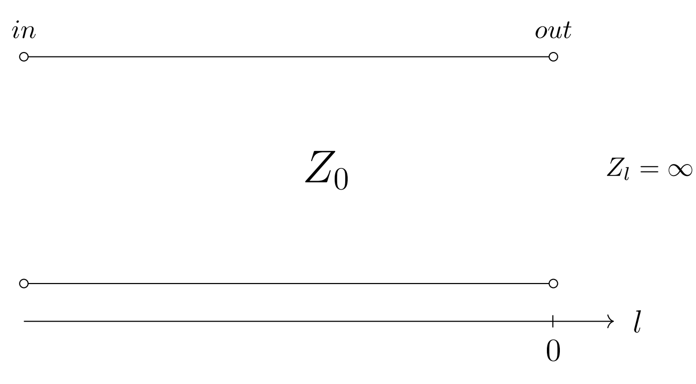


Figure 5.6: Transmission line with open circuit load

In this case we do the opposite as before, we put an open circuit at the output of the TL, in this case $Z_L = \infty$.

Instead to have the voltage equal to 0 at the load, we have the current equal to zero:

$$I(l) = \frac{V_+}{Z_0}\, e^{-j\omega l} - \frac{V_-}{Z_0}\, e^{-j\omega l} \tag{5.26}$$

$$I(l=0) = I_L = 0 = I_+ + I_- = \frac{1}{Z_0}\, [V_+ - V_-] \;\rightarrow\; V_+ = V_- \tag{5.27}$$

• For the reflection coefficient this mean:

$$\rho_L = \frac{V_-}{V_+} = \frac{V_+}{V_+} = 1 \tag{5.28}$$

And this mean that:

$$|\rho_L| = 1 \quad \text{and} \quad \underline{/\rho_L} = 0°$$

• We do the same calculation we already with the open circuit, then wu obtain the voltage and the current over the line:

$$\begin{cases} V(l) & = V_+\, e^{-j\beta l} - V_-\, e^{j\beta l} = V_+(e^{-j\beta l} - e^{j\beta l}) = 2\, jV_+ \sin(\beta l) \\ I(l) & = \frac{V_+}{Z_0} e^{-j\beta l} - \frac{V_+}{Z_0} e^{j\beta l} = -2\frac{V_+}{Z_0}\cos(\beta l) \end{cases} \tag{5.29}$$

Again if we go back to time domain we will notice that we are looking at a stationary wave.
• The calculation for the input impedance is a bit different, but at the end we obtain:

$$Z_i(l) = \lim_{Z_0\to\infty} Z_0 \frac{Z_L\cos(\beta l) + jZ_0\sin(\beta l)}{Z_0\cos(\beta l) + jZ_L\sin(\beta l)} = jZ_0\cot(\beta l) \tag{5.30}$$

• The input admittance:

$$Y_i(l) = \frac{1}{Z_i(l)} = \frac{1}{jZ_0\cot(\beta l)} = -Y_0\tan(\beta l) \tag{5.31}$$

• We can plot again the behavior of the TL, that is very similar as before, but mirrored at the x axe:

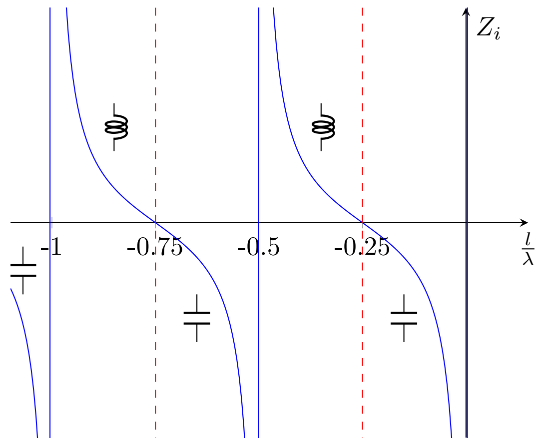


Figure 5.7: Behavior of Z_i a TL if the load is an open

Last little curiosity if the signal is an EMF

Until now we have seen a signal over a TL, but can we consider all those thing in the free space with the EMF? It depend, let's see how:

Short circuit in EMF

It is possible to use a metallic conductor material at the output to impose the electric field on the surface equal to zero. It is like to send the EMF to a wall made of metal.

Open circuit in EMF

As we have done before, we should send the EMF to a wall made by a material that let the magnetic material equal to zero... but this does not exist, or at least we didn't find it yet.

6 Class 6 - 15/03/21

Transmission line with general load Z_0

In the previous lectures we saw the transmission line in short circuit or open circuit, but those case are not the most common one, today we will see the most general situation when the load is a complex number $Z_L = a + jb$, like in fig. 6.1.

Figure 6.1: Forward and backward wave signal in a TL

Now, what we want to do is to attempt some magic trick to modify the wave equation in eq. (4.14), maybe we find a way to evaluate better how the reflected signal is affecting the line.

$$V_+ = (V_+ - V_-) + V_- \tag{6.1}$$

Then we can write the voltage over the line as:

$$\begin{aligned} V(l) &= [(V_+ - V_-) + V_-]\, e^{-j\beta l} + V_-\, e^{j\beta l} = \\ &= (V_+ - V_-) e^{j\beta l} + V_-\, e^{-j\beta l} + V_-\, e^{j\beta l} = \\ &= (V_+ - V_-) e^{j\beta l} + V_- \left(e^{-j\beta l} + e^{j\beta l}\right) = \\ &= (V_+ - V_-) e^{j\beta l} + 2\, V_-\, \cos(\beta l) \end{aligned} \tag{6.2}$$

We skipped some passages, but that should only be the application of the trigonometric form of a complex number.
We can notice from eq. (6.2) that the voltage over a TL is composed by two terms:

- $\boldsymbol{(V_+ - V_-) e^{j\beta l}}$: This is the equation of a direct wave with amplitude $(V_+ - V_-)$. The amplitude is not V_+ because part of the wave is reflected backward
- $\boldsymbol{2\, V_-\, \cos(\beta l)}$: This is the expression of a stationary wave. This part is "eating" part of the forward wave, this mean that if i fall in some "unlucky" point inside my TL the signal will be distorted.

If i want to go back to the time domain with the forward wave using the usual formula in eq. (4.13):

$$V(l, t) = (V_+ - V_-) \cos(\omega t - \beta l) + 2 V_- \cos(\omega t) \cos(\beta l) \tag{6.3}$$

We find again a stationary wave (just as expected).

Module of the voltage wave in phasor

Before to introduce the next topic (SWR), we need to introduce how we can calculate the module of the voltage wave in phasor domain.
In fig. 6.2 you can see the graphical representation of the two components that build up the voltage wave in complex domain that we have already seen in eq. (5.6).

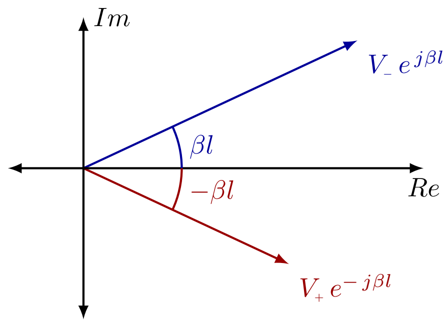


Figure 6.2: Forward and backward Voltage wave in phasor domain

What we are doing now is to find a proper way to describe the module of $\overset{\circ\rightarrow}{V}(l)$, that is NOT the real voltage value in the TL (remember that we are in phasor domain).
If you remember how we defined ρ[1] in eq. (5.15), there is no doubt that we can write:

$$\rho = \frac{V_{-}}{V_{+}}$$

So the voltage equation in fig. 6.2 will become:

$$\begin{aligned} \overset{\circ\rightarrow}{V}(l) &= V_{+}\, e^{-j\beta l} + \rho\, V_{+}\, e^{j\beta l} = \\ &= V_{+} \left(e^{-j\beta l} + \rho\, e^{j\beta l} \right) \end{aligned} \tag{6.4}$$

Now, the best way to compute the module is to obtain it from $|\overset{\circ\rightarrow}{V}(l)|^2$:

$$\begin{aligned} |\overset{\circ\rightarrow}{V}(l)|^2 &= \overset{\circ\rightarrow}{V}(l)\, \overset{\circ\rightarrow}{V}^{*}(l) = \\ &= |V_{+}|^2 \left(e^{-j\beta l} + \rho\, e^{j\beta l} \right) \left(e^{-j\beta l} + \rho\, e^{j\beta l} \right)^{*} = \\ &= |V_{+}|^2 \left(e^{-j\beta l} + \rho\, e^{j\beta l} \right) \left(e^{j\beta l} + \rho^{*}\, e^{-j\beta l} \right) = \\ &= |V_{+}|^2 \left(1 + |\rho|^2 + \rho\, e^{j2\beta l} + \rho^{*}\, e^{-j2\beta l} \right) \end{aligned} \tag{6.5}$$

[1] remember that ρ and ρ_L are the same thing with different name

Let's assume φ is the phase of the complex number $\rho = |\rho| e^{j\varphi}$:

$$
\begin{aligned}
|\overset{\circ\rightarrow}{V}(l)|^2 &= |V_+|^2 \left(1 + |\rho|^2 + |\rho|\, e^{j(2\beta l + \varphi)} + |\rho|\, e^{-j(2\beta l+\varphi)}\right) = \\
&= |V_+|^2 \left(1 + |\rho|^2 + |\rho| \left[e^{j(2\beta l+\varphi)} + e^{-j(2\beta l+\varphi)}\right]\right) = \\
&= |V_+|^2 \left[1 + |\rho|^2 + |\rho| \cos(2\beta l + \varphi)\right]
\end{aligned}
\tag{6.6}
$$

From here we can easily obtain the expression for $|\overset{\circ\rightarrow}{V}(l)| = \sqrt{|\overset{\circ\rightarrow}{V}(l)|^2}$ and also for $|\overset{\circ\rightarrow}{I}(l)|$ (the calculation are very similar):

$$
\begin{cases}
|\overset{\circ\rightarrow}{V}(l)| = |V_+| \sqrt{1 + |\rho|^2 + |\rho| \cos(2\beta l + \varphi)} \\
|\overset{\circ\rightarrow}{I}(l)| = \frac{|V_+|}{Z_0} \sqrt{1 + |\rho|^2 - |\rho| \cos(2\beta l + \varphi)}
\end{cases}
\tag{6.7}
$$

From what we can see in eq. (6.7), the module of both voltage and current in phasor domain increase and decrease periodically along l because $\cos \in [-1, 1]$

Obtaining V_{max} and V_{min}

We have already seen the refraction ratio ρ, but we can introduce a better parameter to describe how distorted is my signal. To do so we need to have a closer look at the module of the voltage in phasor domain that we have seen in the last section.
As we said, the cos inside the expression of $|\overset{\circ\rightarrow}{V}(l)|$ is periodic thanks to the cos inside it, so we can know the maximum and minimum value of $|\overset{\circ\rightarrow}{V}(l)|$ and $|\overset{\circ\rightarrow}{I}(l)|$ moving along l.
From eq. (6.7) we can simply find those values by replacing cos with respectively $+1$ and -1:

$$
\begin{cases}
V_{max} = |V_+| \sqrt{1 + |\rho|^2 + 2|\rho|} = |V_+|(1 + |\rho|) \\
V_{min} = |V_+| \sqrt{1 + |\rho|^2 - 2|\rho|} = |V_+|(1 - |\rho|)
\end{cases}
\tag{6.8}
$$

From here, we assume that we have found a certain l_0 value where we can find the maximum value of $|V(l_0)|$:

$$
\begin{aligned}
|V(l_0)| &= |V_+|(1 + |\rho|) = \\
&= |V_+| + |V_-| = V_{max}
\end{aligned}
\tag{6.9}
$$

If i move from that section of the line, the phase will not be equal anymore and we will not see V_{max} anymore.

Just a little example if i move by $\Delta L = \frac{\lambda}{4}$ from V_{max}, we will reach the minimum:

$$\begin{aligned} \left| V\left(l_0 + \frac{\lambda}{4}\right)\right| &= |V_+|(1-|\rho|) = \\ &= |V_+| - |V_-| = V_{min} \end{aligned} \tag{6.10}$$

Why this? Look at eq. (6.7), here you can see that to reach $\cos(\cdots) = -1$ we need to move its argument from 0 to π, and $\frac{\lambda}{4}$ fits this purpose:

$$\begin{aligned} &\left(2\beta\left(l_0 + \frac{\lambda}{4}\right) + \varphi\right) = \\ &= \left(\cancelto{0}{2\beta l_0 + \varphi} + 2\beta\frac{\lambda}{4}\right) \\ &= \left(2\,\frac{2\pi}{\lambda}\,\frac{\lambda}{4}\right) = (\pi) \end{aligned} \tag{6.11}$$

A little recap:

$$\begin{cases} V_{max} = (|V_+| + |V_-|) \\ V_{min} = (|V_+| - |V_-|) \end{cases} \tag{6.12}$$

As we can see in eq. (6.10), we reached the minimum value V_{min} by moving from V_{max} by $\Delta L = \frac{\lambda}{4}$, so it was not just a little example but it is very important! Again if i move by another $\Delta L = \frac{\lambda}{4}$ from V_{min} we reach another maximum and so on ...

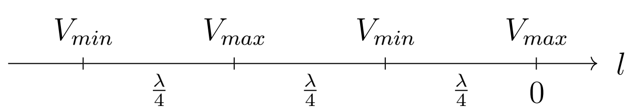


Figure 6.3: Voltage behavior over the line

Obtaining I_{max} and I_{min}

Obviously, everything that we said until now can be said also for the current. Starting from eq. (6.7) we can again replace the cos with -1 and $+1$ and obtain the maximum and minimum value of $|\overset{\circ\!\!\rightarrow}{I}(l)|$

$$\begin{cases} I_{max} = \frac{|V_+|}{Z_0}\sqrt{1 + |\rho|^2 - 2|\rho|} = \frac{|V_+|}{Z_0}(1 - |\rho|) \\ I_{min} = \frac{|V_+|}{Z_0}\sqrt{1 + |\rho|^2 + 2|\rho|} = \frac{|V_+|}{Z_0}(1 + |\rho|) \end{cases} \tag{6.13}$$

The real differences between the voltage and current equation are $\frac{1}{Z_0}$ and the fact that the minus is swapped.

If we do the same calculation that we have done for the voltage we can obtain:

$$\begin{cases} I_{max} = (|I_+| + |I_-|) \\ I_{min} = (|I_+| - |I_-|) \end{cases} \tag{6.14}$$

We can notice that the periodic behavior remains the same, but I_{max} is shifted from V_{max} by $\frac{\lambda}{4}$, this also mean that I_{max} and V_{min} share the same position on the line:

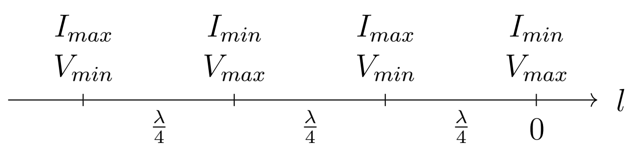


Figure 6.4: Voltage behavior over the line

Obtaining Z_{max} and Z_{min}

With that being said, we can also introduce the maximum and minimum impedance:

$$\begin{cases} Z_{max} = \frac{V_{max}}{I_{max}} \\ Z_{min} = \frac{V_{min}}{I_{min}} \end{cases} \tag{6.15}$$

And:

$$Z_{max} = \frac{V_{max}}{I_{max}} = \frac{|V_+| + |V_-|}{|V_+| - |V_-|} Z_0 \tag{6.16}$$

Standing wave ratio (SWR)

Finally we can introduce this famous value of Standing wave ratio (SWR). This is very useful to describe the quality of our signal when we have a forward and backward wave.

$$\text{SWR} = \frac{V_{max}}{V_{min}} = \frac{1 + |\rho|}{1 - |\rho|} \tag{6.17}$$

Another interesting relation is the one with the maximum and minimum impedance.
For Z_{max}:

$$\begin{cases} Z_{max} = \text{SWR}\, Z_0 \\ Z_{min} = \frac{Z_0}{\text{SWR}} \end{cases} \tag{6.18}$$

We can also refer to those values as normalized to Z_0, this will help a bit the calculations during exercises.

$$\begin{cases} \mathcal{Z}_{max} = \frac{Z_{max}}{Z_0} = \text{SWR} \\ \mathcal{Z}_{min} = \frac{Z_{min}}{Z_0} = \frac{1}{\text{SWR}} \end{cases} \tag{6.19}$$

This last part is a bit different from what explained the prof during lectures, but i think that during this class he was not very clear with math, so i decided to consult the world wide web, and i've actually found a really interesting free book available at [2].

Little exercise #1

Now we will do some simple exercise useful to better comprehend what we are saying.
In the first exercise we have a transmission line in where we need to find the SWR.

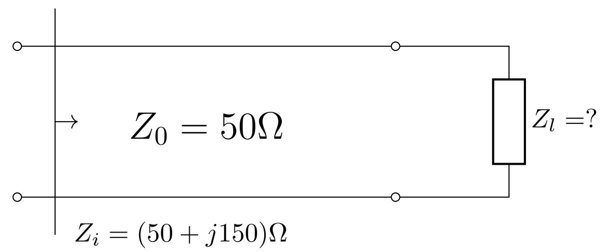


Figure 6.5: Transmission line of exercise 2

First of all we normalize the input impedance

$$\mathcal{Z}_i = \frac{Z_i}{Z_0} = (1 - j3)$$

Then we need the reflection coefficient on the load:

$$\rho_L = \frac{Z_i - Z_0}{Z_i + Z_0} = \frac{\mathcal{Z}_i - 1}{\mathcal{Z}_i + 1} = \frac{1 - j3 - 1}{1 - j3 + 1} = 0.7 - j0.46$$

Note that we didn't specified the position l because the data already give us the value of the input impedance.
The module of the reflection coefficient is: $|\rho(l)| = 0.83$ Now it is very simple to obtain the SWR:

$$\text{SWR} = \frac{1 + |\rho|}{1 - |\rho|} = \frac{1 + 0.83}{1 - 0.83} = 10.76$$

The SWR value is higher than 10, and it is not a very good value :/

Little exercise #2

In this second exercise we have a UHF antenna in an airplane (don't ask me why all those strange details) working in a range of frequency $f = 225 - 400\text{MHz}$.
Now, inside the airplane there is a transmission line with $Z_0 = 50\Omega$ where signal

is transmitted from the cabinet to the antenna. We know that for different frequencies the antenna behave like a different load Z_L:

frequency	input impedance Z_L
225MHz	$22.5 - j51\ \Omega$
300MHz	$35 - j16\ \Omega$
400MHz	$45 - j2.5\ \Omega$

Now we need to find ρ_L and SWR.
First of all we use the normalized impedance so simplify things:

$$\mathcal{Z}_L = \frac{Z_L}{Z_0}$$

Then it is very simple to obtain ρ_L:

$$\rho_L = \frac{\mathcal{Z}_L - 1}{\mathcal{Z}_L + 1}$$

As we have done before, we can calculate the module of ρ then the SWR is:

$$\text{SWR} = \frac{1 + |\rho|}{1 - |\rho|}$$

Little exercise #3

We consider a transmission line with a length of 75cm

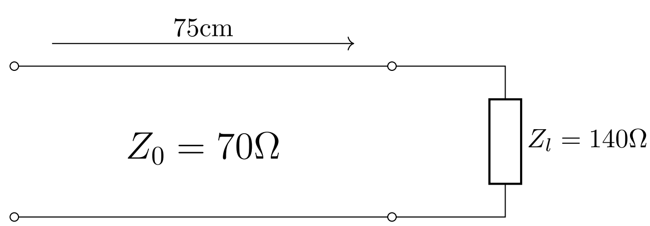


Figure 6.6: Transmission line of exercise 3

In this transmission line we can have signals with different frequencies:

frequency
50MHz
100MHz
150MHz
200MHz

We want to calculate the input impedance Z_i. We just remind that

$$\beta l = \frac{2\pi}{\lambda} = \frac{2\pi}{c} fl$$

We calculate β for each frequencies, then we can calculate Z_i:

$$Z_i = Z_0 \frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)}$$

I won't do all the calculations here, but the results are:

- $\boldsymbol{f = 50}$MHz $\rightarrow \boldsymbol{Z_i = 56 - j42}\Omega$.
- $\boldsymbol{f = 100}$MHz $\rightarrow \boldsymbol{Z_i = 35}\Omega$: That is very interesting!
  Note that $l = 0.75$m and $\lambda = 3$m, and $l = \frac{\lambda}{4}$ so we have reached a minimum or a maximum.
  We know that this is a minimum because we are at $l = \frac{\lambda}{4}$ from $l = 0$ where we have the actual impedance.
  If we calculate the normalized impedance for the maximum and minimum value: $\mathcal{Z}_{min} = \frac{140}{70} = 2$ and $\mathcal{Z}_{min} = \frac{35}{70} = \frac{1}{2}$.
  Then we notice that $\mathcal{Z}_{min} = \frac{1}{\mathcal{Z}_{max}}$
- $\boldsymbol{f = 150}$MHz $\rightarrow \boldsymbol{Z_i = 56 + j42}\Omega$.
-
- $\boldsymbol{f = 200}$MHz $\rightarrow \boldsymbol{Z_i = 140}\Omega$: This is Z_{max}

Smith chart

All those numbers and complex numbers are quite confusing... But what if i tell you that there is a graphical way to describe the transmission line and all the useful coefficient.
We need to prepare ourself a bit, first of all we need to define a bit better the normalized impedance (already seen in eq. (6.19)):

$$\mathcal{Z}(l) = \frac{Z_i(l)}{Z_0} = \mathcal{R} + j\mathcal{X} \tag{6.20}$$

We also remember the reflection coefficient ρ:

$$\rho(l) = \rho_L e^{j2\beta l} = u + jv \tag{6.21}$$

Those two parameters are dependent on l, but now we consider that we have chosen a l^* value so $\mathcal{Z}$ and ρ are fixed (very flexible notation, we can also use it for the load impedance when $l = 0$).

Consider the impedance for a given position on the TL and we can now write:

$$\begin{aligned}
\mathcal{Z}(l^*) &= \frac{Z(l^*)}{Z_0} \\
&= \frac{1}{Z_0} \cdot \frac{V(l^*)}{I(l^*)} \\
&= \frac{1}{\cancel{Z_0}} \cdot \cancel{Z_0} \frac{V_+ + V_-}{V_+ - V_-} \\
&= \frac{V_+ + \rho V_+}{V_+ - \rho V_+} = \frac{1+\rho}{1-\rho}
\end{aligned} \tag{6.22}$$

We simplify a bit the notation and we continue:

$$\begin{aligned}
\mathcal{Z} = \frac{1+\rho}{1-\rho} &= \frac{1+u+jv}{1-u-jv} \cdot \frac{1-u+jv}{1-u+jv} = \mathcal{R} + j\mathcal{X} \\
\mathcal{R} + j\mathcal{X} &= \frac{(1-u^2-v^2)+j2v}{(1-u)^2+v^2}
\end{aligned} \tag{6.23}$$

Then if we split the real from the imaginary part:

$$\mathcal{R} = \tfrac{1-(u^2+v^2)}{(1-u)^2+v^2} \qquad \mathcal{X} = \tfrac{2v}{1-u^2+v^2}$$

We do some more calcultations with $\mathcal{R}$:

$$\begin{aligned}
&[(1-u)^2+v^2]\,\mathcal{R} = 1-(u^2+v^2) \\
&\mathcal{R}\,u^2 - 2u\,\mathcal{R} + \mathcal{R} + \mathcal{R}v^2 + u^2 + v^2 = 1 \\
&(1+\mathcal{R})u^2 - 2u\,\mathcal{R} + \mathcal{R} + (1+\mathcal{R})v^2 = 1 \\
&u^2 - 2\frac{\mathcal{R}}{1+\mathcal{R}}u + \frac{\mathcal{R}}{1+\mathcal{R}} + v^2 = \frac{1}{1+\mathcal{R}}
\end{aligned} \tag{6.24}$$

Then we add on left and on right the same number: $\left(\frac{\mathcal{R}}{1+\mathcal{R}}\right)^2$
And we can write:

$$\left(u - \frac{\mathcal{R}}{1+\mathcal{R}}\right)^2 + v^2 = \frac{1}{1+\mathcal{R}} - \frac{\mathcal{R}}{1+\mathcal{R}} + \left(\frac{\mathcal{R}}{1+\mathcal{R}}\right)^2 \tag{6.25}$$

Some math passage later...

$$\begin{cases}
\left(u - \frac{\mathcal{R}}{1+\mathcal{R}}\right)^2 + v^2 = \frac{1}{(1+\mathcal{R})^2} \\
(u-1)^2 + \left(v - \frac{1}{\mathcal{X}}\right)^2 = \frac{1}{\mathcal{X}^2}
\end{cases} \tag{6.26}$$

FINALLY, so what is happening here?
We notice that the first equation is the equation of a circle centered in $\left(\frac{\mathcal{R}}{1+\mathcal{R}}, 0\right)$

and radius $\frac{1}{\mathcal{R}+1}$.
In fig. 6.7 you can see some plots of this circle with different value of $\mathcal{R}$

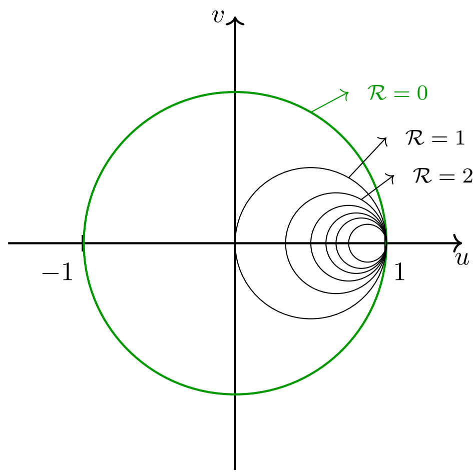


Figure 6.7: Plots of different circles from the first equation

All the circumference will collapse in $(1, 0)$ when $\mathcal{R} \to \infty$
. For $\mathcal{R} = 0$ we find the limit circumference (green one in fig. 6.7), this is the limit because we can not have negative real impedance.
Good news for the second equation in eq. (6.19), because also with this we can find some circumferences with center in $(1, \frac{1}{\mathcal{X}})$ and radius $\frac{1}{\mathcal{X}}$:

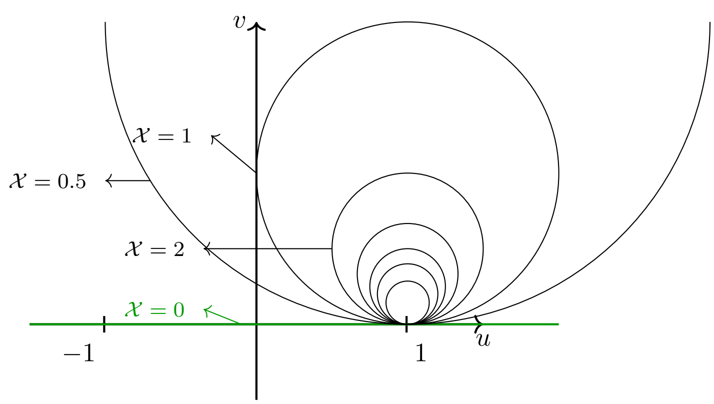


Figure 6.8: Plots of different circles from the second equation

In fig. 6.8 you can have a look at the circles made by changing the $\mathcal{R}$ parameter. When $\mathcal{R} \to \infty$ those circles collapse in $(1, 0)$, then by decreasing the

$\mathcal{R}$ the circumference radius will increase, until we reach the limit $\mathcal{R} = 0$ where the circumference is lying in the horizontal u axe (in green). You can also go in the negative part of the plot with $\mathcal{R} < 0$
Hooray! We found a very intuitive way to represent our normalized impedance $\mathcal{Z}(l) = \mathcal{R} + j\mathcal{X}$ by choosing the right circles in my new plot that we will call *Smith Chart* (Smith Will for friends). With that we can obtain the value of v and u of the reflection coefficient by looking at the position of the point on the plot.

A simple example

In this example, we want to represent $\mathcal{Z}_i = 0.2 + j0.5\Omega$ in our new plot (fig. 6.9):

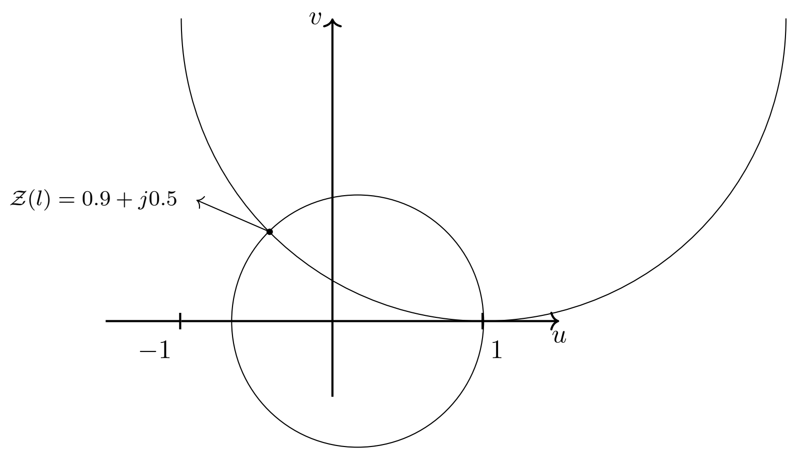


Figure 6.9: Smith Chart of $\mathcal{Z}(l) = 0.9 + j0.5$

By looking at the plot in fig. 6.9, we obtain the reflection coefficient:

$$\rho(l) = u + jv = |\rho| e^{j2\beta l} \tag{6.27}$$

Second example

Consider the transmission line in fig. 6.10 and its value of $Z_0 = 50$ and $Z_L = 100 + j150\Omega$

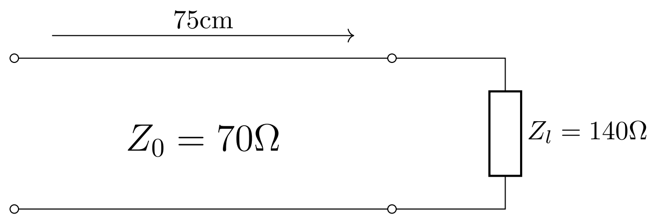


Figure 6.10: Transmission line of this second example

The first thing to do is to obtain the normalized impedance:

$$\mathcal{Z}_L = \frac{Z_L}{Z_0} = 2 + j3$$

We draw $\mathcal{Z}_L = \mathcal{Z}(l = 0)$ in the Smith Chart:

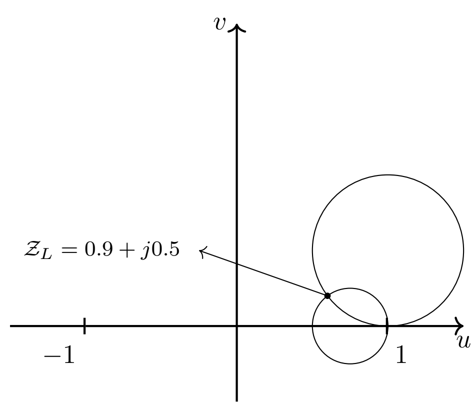


Figure 6.11: Smith Chart of $\mathcal{Z} = 2 + j3$

Then what we can do is to look at the reflection coefficient ρ, not only by looking ad u and v, but obtaining the module and the phase of the complex vector ρ. You can see the example in fig. 6.12

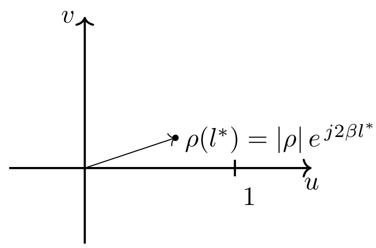


Figure 6.12: Smith Chart of $\mathcal{Z} = 2 + j3$

According to eq. (5.16), the module of the ρ vector that we have found in the Smith chart is ρ_L and it's phase angle is $2\beta l$.
This phase angle is very important, this is why you can find it already printed in the first outer diameter of the Smith chart, usually under the name of *angle of reflection coefficient in degrees.*
If we move in the transmission line changing the l value, in the Smith chart we will see that we are rotating the ρ complex vector, but we don't change its module, like in fig. 6.13

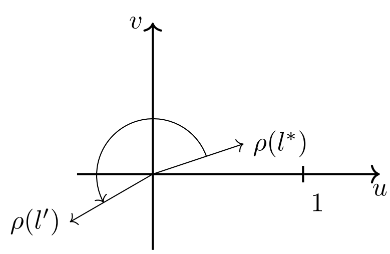


Figure 6.13: Smith Chart when we move on the line

We can exploit this cool behavior of the chart because whenever we move along the line, ρ will rotate, and then we move in another point where we can interpolate the famous circles and we can obtain the value of the new $\mathcal{Z}$ without going troughs all the ugly formulas that we have already seen before.
Another interesting characteristic of this chart is that when you rotate the vector, and touch the horizontal u axe, the impedance will be completely real.

If from my point i do a complete rotation, this mean that my phase variation is 2π, this mean that:

$$\begin{gathered} 2\beta l = 2\pi \\ 2\frac{\cancel{2\pi}}{\lambda} l = \cancel{2\pi} \end{gathered} \tag{6.28}$$

We can see in eq. (6.28) that by completing a rotation we are moving over the

line by $l = \frac{\lambda}{2}$. This is true for every rotation, so in the outer diameters of the Smith chart .
It is interesting to see that if we move by $l = \frac{\lambda}{4}$ we are rotating by 180°, and if we look at the impedance we notice that $\mathcal{Z}_a = \frac{1}{\mathcal{Z}_b}$ as expected.
In the outer diameter of the Smith chart we can see also that is marked the how far we are moving from or towards the load normalized by the λ. We know also the direction because if we are moving towards the load mean that the l value is increasing, and thus the angle of ρ, so we are moving anticlockwise in the chart.

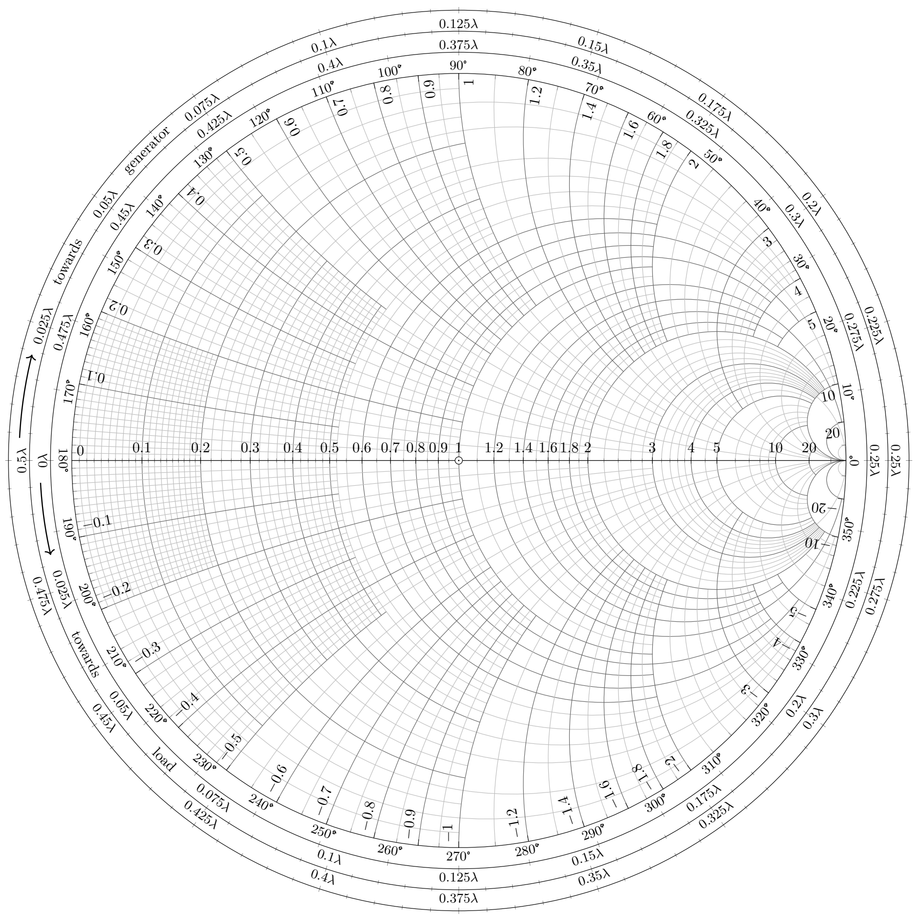


Figure 6.14: Smith chart

7 Class 7 - 19/03/21

Today we start very quickly talking about the Smith chart, and some new cool property about the SWR.
From eq. (6.17) we can obtain the value of $|\rho|$ as:

$$|\rho| = \frac{\text{SWR} - 1}{\text{SWR} + 1} \tag{7.1}$$

Looking back at eq. (5.14) we can write $\rho(l)$ as:

$$\rho(l) = \frac{Z(l) - Z_0}{Z(l) + Z_0} = \frac{\mathcal{Z}(l) - 1}{\mathcal{Z} + 1} \tag{7.2}$$

WOW, $\frac{\mathcal{Z}(l)-1}{\mathcal{Z}(l)+1}$ and $\frac{\text{SWR}-1}{\text{SWR}+1}$ looks very similar, let's see if we can find something interesting.
If we take in consideration a completely real impedance $\mathcal{Z}(l^*)$ (in l^*), this mean also that $\rho(l^*) \in \text{Re}$

$$\rho(l^*) = |\rho(l^*)| = \frac{\text{SWR} - 1}{\text{SWR} + 1} = \frac{\mathcal{Z}(l^*) - 1}{\mathcal{Z}(l^*) + 1} \tag{7.3}$$

This mean that in this situation $\text{SWR} = \mathcal{Z}(l^*)$, so in the Smith chart when my impedance lay on the horizontal axe:

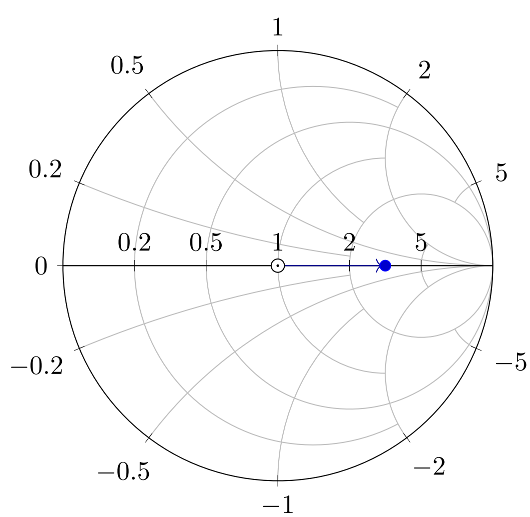


Figure 7.1: Smith Chart of $\mathcal{Z} = 3$

So, what we can do is that we can obtain the SWR of any impedance by rotating in the chart until we reach the horizontal axe, for example in fig. 7.1 the standing wave ratio is $\text{SWR} = \mathcal{Z} = 3$.

Little exercise

Now we will do a little exercise to get the hands on this topic, let's consider a coaxial cable with the following specs:

- $\varepsilon_r = 2.25$
- $\mu_r = 1$
- $\sigma = 0 \; \rightarrow$ no losses
- $l = 10\text{m}$
- $f = 34\text{MHz}$
- $Z_L = 50 + j100\Omega$
- $Z_0 = 50\Omega$

We are asked to find the input impedance Z_i.
We can start by calculating some useful stuff:

$$\boldsymbol{v_p} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \cdot \frac{1}{\sqrt{\mu_r\varepsilon_r}} = c\,\frac{1}{\sqrt{\mu_r\varepsilon_r}} = 2 \cdot 10^8\,\frac{\text{m}}{\text{s}}$$

$$\boldsymbol{\lambda} = \frac{v_p}{f} = 5.882\text{m}$$

$$\frac{\boldsymbol{l}}{\boldsymbol{\lambda}} = 1.17$$

$$\boldsymbol{\beta\, l} = \frac{2\pi}{\lambda} l = 3.4\pi$$

Now with a bit of patience we can obtain Z_i with the usual equation that we have used until now (eq. (5.11)).
But we have a new friend by our side: the Smith chart! let's draw the normalized load impedance on it: $\mathcal{Z}_{\mathcal{L}} = \frac{Z_L}{Z_0} = 1 + j2$.
Now to find the normalized input impedance $\mathcal{Z}_{\mathcal{L}}$ is a very easy task, what we are going to do is to rotate around the central point by a certain angle.
From what we have seen in the past lecture and in eq. (6.28) the impedance rotate on the Smith chart by 360° every $\frac{\lambda}{l} = \frac{1}{2}$, this mean that $\mathcal{Z}_{\mathcal{L}}$ will rotate around the center of the chart $\frac{1.7}{1/2} = 3.4$ times.
This mean that we need to do 3 full rotation, plus an angle of $0.4 \cdot 360° = 144°$ in clockwise direction (we are moving away from the load).
In fig. 7.2 you can admire the Smith chart of this exercise. At the end we obtained the normalized input impedance $\mathcal{Z}_i \approx 0.45 - j0.8$ and thus the input impedance $Z_i = \mathcal{Z} \cdot Z_0 \approx 22.5 + j40\Omega$

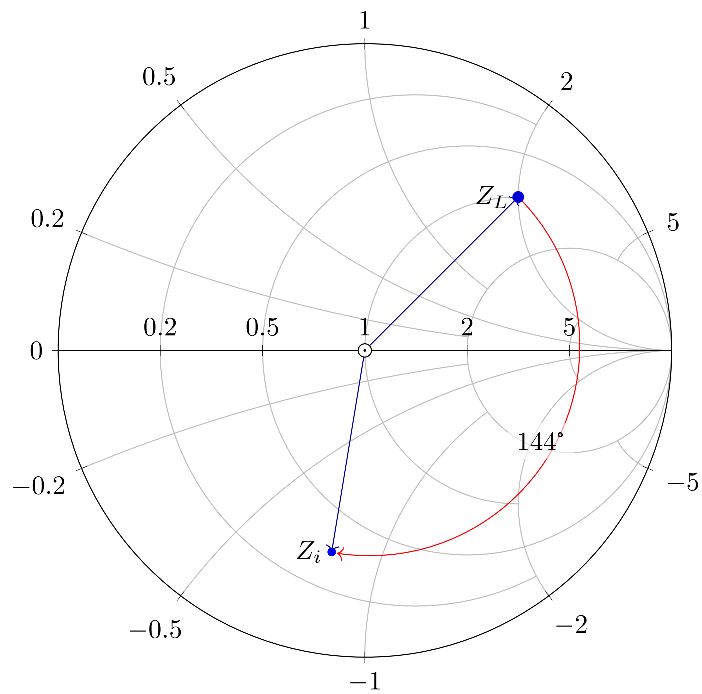


Figure 7.2: Smith Chart of $\mathcal{Z_L} = 1 + j2$ and $\mathcal{Z}_i \approx 0.45 - j0.8$

From fig. 7.2 we can also obtain the SWR by moving the normalized impedance until it lay on the vertical axe, then:

$$\text{SWR} = \mathcal{Z}(l^*) \approx 6$$

From that we can also obtain the module of the reflection coefficient

$$|\rho| = \frac{\text{SWR} - 1}{\text{SWR} + 1} = \frac{5}{7} = 0.71$$

Looking at the results that we have here (SWR $= 6$, $|\rho| = 0.71$ and $Z_0 \neq Z_L$) this line is pretty bad... What we can do to make it greater?

Maximization of the power transmission

When we use a transmission line, we can have a lot of troubles and strange things as we have already seen, an that is also true for the power. Take in consideration a geneal transmission line:

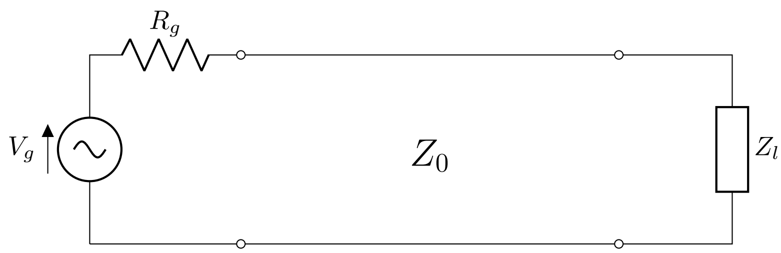


Figure 7.3: General TL

We want to know what is the power that the generator emits to the TL. This sound pretty easy because in phasor domain the power is defined by the voltage and the current in input on the line (the generator is connected to the input of the line):

$$\overset{\circ\!\!\rightarrow}{P} = \frac{1}{2}\overset{\circ\!\!\rightarrow}{V}_i \overset{\circ\!\!\rightarrow}{I}_i^* \tag{7.4}$$

For the voltage there is no problem, we already know it when we buy that generator, so what about the current?
We are able to represent both the load and the characteristic impedance with Z_i because that is how the generator sees the transmission line.

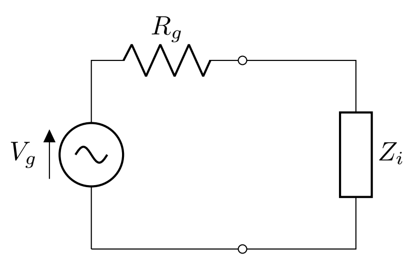


Figure 7.4: TL from the generator point of view

We can apply the kirchhoff law from fig. 7.4 (to simplify the notation i'll omit the phasor symbol, but keep in mind that those are complex number)

$$V_i = \frac{Z_i}{Z_i + Z_g} V_g = \frac{R_i + jX_i}{R_i + jX_i + R_g + jX_g} V_g \tag{7.5}$$

With $Z_i = R_i + jX_i$ and $Z_g = R_g + jX_g$, the power equation becomes:

$$
\begin{aligned}
P_i &= \frac{1}{2}\, V_i \left(\frac{V_i}{Z_i}\right)^* = \\
&= \frac{1}{2}\, \frac{Z_i}{Z_i + Z_g} V_g \left(\frac{\cancel{Z_i}}{\cancel{Z_i}(Z_i + Z_g)} V_g\right)^* = \\
&= \frac{1}{2}\, \frac{Z_i}{(Z_i + Z_g)\,(Z_i + Z_g)^*}\, V_g\, V_g^* \\
&= \frac{1}{2}\, \frac{R_i + jX_i}{|R_i + R_g + j(X_i + X_g)|^2} |V_g|^2
\end{aligned}
$$

What we want to maximize is the active power, so we will care only about the real part of P_i

$$
\mathrm{Re}(P_i) = \frac{1}{2}\, \frac{R_i}{|R_i + R_g + j(X_i + X_g)|^2} |V_g|^2 \tag{7.6}
$$

Let's say that we have already bought the generator, so Z_g and X_g are given, then to maximize eq. (7.6) we can use a few cool tricks:
first of all we can set $X_i = -X_g$, so the remaining equation become:

$$
\mathrm{Re}(P_i) = \frac{1}{2}\, \frac{R_i}{|R_i + R_g|^2} |V_g|^2 = \frac{1}{2}\, \frac{R_i}{|R_i^2 + R_g^2 + 2R_iR_g|} |V_g|^2 \tag{7.7}
$$

In order to maximize eq. (7.7) we can only change R_i , so we can use the derivative of P

$$
\begin{aligned}
\frac{d\,\mathrm{Re}(P_i)}{dR_i} &= 0 \\
\frac{(R_g - R_i)}{(R_g + R_i)^3} &= 0 \qquad (7.8)\\
R_i &= R_g
\end{aligned}
$$

According to what we have said until now, to maximize the active power we need to design a transmission line which has $R_i = R_g$ and $X_i = -X_g$.
In other terms, to maximize the active power, we will need to design a TL which input impedance is the complex conjugate of the generator series impedance:

$$
Z_i = Z_g^* \tag{7.9}
$$

Then the maximum active power will be:

$$
P_{max} = \frac{1}{8}\frac{|V_g|}{R_g} \tag{7.10}
$$

Why it is so important to have a good TL?

We have quite a bit reason to have a good transmission line with no or few reflection (SWR=1 and $\rho = 0$):

1. The energy wasted or not used by a non optimal power transfer

2. Think about the power value where $V(l) = V_{max}$:

$$P = \frac{1}{2}\frac{|V_{max}^2|}{Z_{max}} = \frac{1}{2}\frac{|V_{max}^2|}{Z_0 \cdot \mathrm{SWR}}$$

   and so:

$$|V_{max}| = \sqrt{2\,Z_0\,\mathrm{SWR}\,P} \tag{7.11}$$

   This mean that for the same power value, when SWR is higher than 1, the maximum voltage could increase and make a few problems (if the voltage is too high it can also burn the line medium)

3. It is very important in communication tho have zero or at least very few distortion: if the signal becomes reflected we introduce unwanted noise

Adaption of the Transmission Line

Going back to the main topic, how it is possible to build a well designed transmission line? there are some tricks, and they consist in the addition of a special "box" in the middle of the transmission line in order to adapt it to the load (fig. 7.5). It is very useful because wi will not need to change the entire TL if we deal to a different load, but we can simply adjust it.

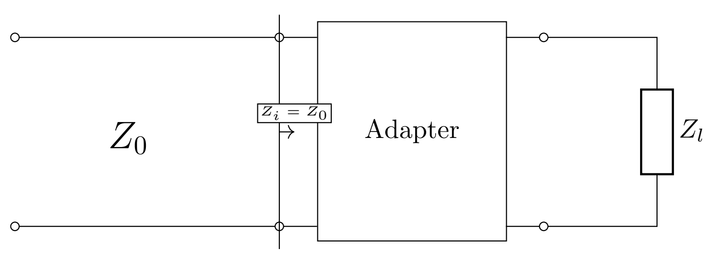


Figure 7.5: TL adapted by a general "black box"

As you can see from fig. 7.5, what we want from the adapter line is to modify the impedance value in that point to Z_0, if so there would be no reflected signals. There are different approach to accomplish this adaption:

1) $\lambda/4$ line adapter

The first adapter that we will see here is the $\lambda/4$ line adapter, it consist in a short piece of line with the length of $l = \lambda/4\,\mathrm{m}$ as in fig. 7.6:

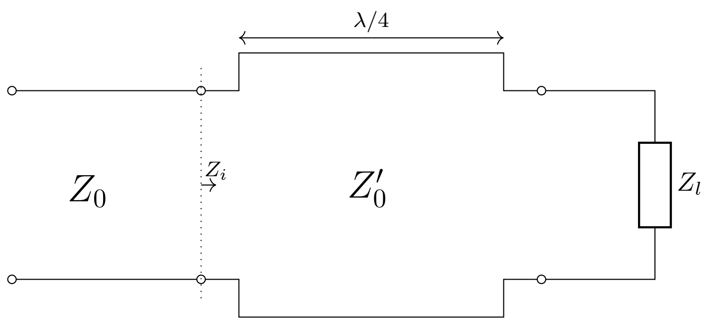


Figure 7.6: TL adapted by a general "black box"

From the transmission line in fig. 7.6 we want that the ingress impedance on the adaption line is equal to the characteristic impedance of the line ($Z_i = Z_0$). We start by evaluating Z_i from eq. (5.11):

$$\begin{aligned} Z_i &= Z_0' \frac{Z_L \cancelto{0}{\cos(\beta \frac{\lambda}{4})} + j\, Z_0' \cancelto{1}{\sin(\beta \frac{\lambda}{4})}}{Z_0' \cancelto{0}{\cos(\beta \frac{\lambda}{4})} + j\, Z_L \cancelto{1}{\sin(\beta \frac{\lambda}{4})}} = \\ &= \frac{Z_0'\, Z_0'}{Z_L} \end{aligned} \tag{7.12}$$

As we have already said we want that $Z_i = Z_0$, so if my $\lambda/4$ line, and now it is petty simple to do this changing the characteristic impedance of the adapter line:

$$\frac{Z_0'\, Z_0'}{Z_L} = Z_0 \;\rightarrow\; Z_0' = \sqrt{Z_L\, Z_0} \tag{7.13}$$

To recap:
We will need a transmission line in series at the used one with characteristic impedance $Z_0' = \sqrt{Z_L\, Z_0}$ and length $\lambda/4$.
Note:
Usually this technique is used to match only the real value of the load impedance, because as we have seen in eq. (5.5), the characteristic impedance of the transmission line is usually considered real to neglect the power loss. To match the imaginary part of my load, it is possible to use this method, but we need to add another layer of adaption that could be another piece of transmission line in series, or better a STUB.

Closed circuit Stub

Another possibility we have is to modify directly the characteristic impedance of the line by inserting something in parallel, like a short circuit. In this way wi will change the impedance of the entire TL.
From what we have already seen in eq. (5.24), a short circuit could behave like

an impedance or capacitor by changing his length. This is crucial in order to adapt the TL, because with that we can match the imaginary part of the load impedance.
What we are going to do is so insert a special short circuit (STUB) in parallel to the transmission line, near the load position, like you can see in fig. 7.7:

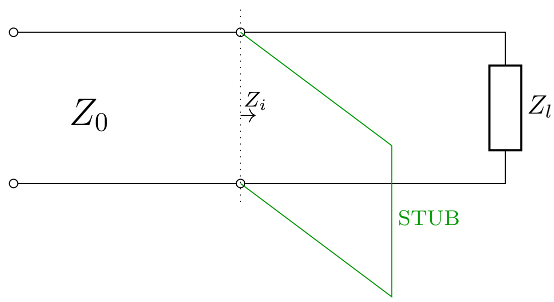


Figure 7.7: Stub implementation example

The working principe of this Stub is very simple: if the load impedance has some reactive element, i can compensate it by applying a stub with the right length. We placed this short circuit stub in parallel to the load, so it is more simple to work in terms of admittance.
From the widely famous Hom law, when we have two admittance placed in parallel, the total admittance will be the sum of the two: $Y_{tot} = Y_a + Y_b$.
So, if the imaginary part of the stub admittance has the same value but opposite sign compared to the one of the load, we have cancelled his reactive part.

$$\frac{1}{Z_L} = Y_L = G_L + jB_L \rightarrow Y_{stub} = -jB \tag{7.14}$$

We can chose the stub to have both negative or positive admittance.

Open circuit Stub

There is another type of stub, which is similar to the first one, but it utilize an open circuit in series on the transmission line.
As we can see from eq. (5.30), the open circuit can behave like both as an impedance or capacitor.
A simple representation of the application of that type of stub is shown in fig. 7.8.

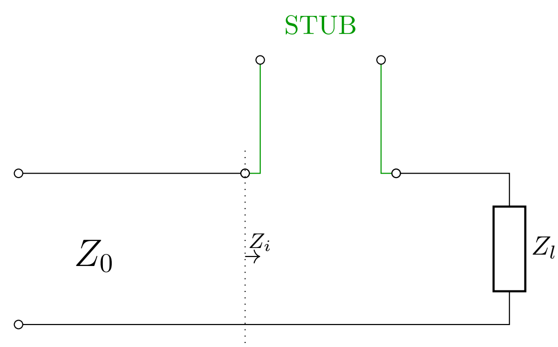


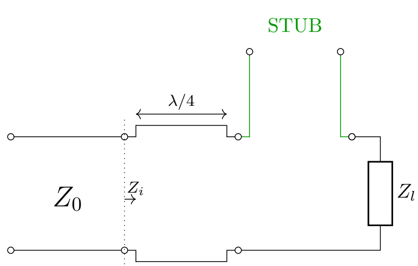
Figure 7.8: Stub implementation example

According to the Ohm law, the total impedance of a series of impedance is the sum of all the impedances ($Z_{tot} = Z_a + Z_b$).
So, adjusting the stub length we can obtain a positive or negative impedance value to compensate the load impedance.

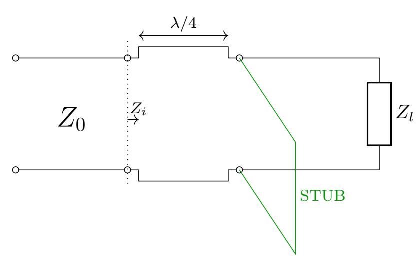
$$Z_L = A_L + jX_L \;\rightarrow\; Y_{stub} = -jX \tag{7.15}$$

General consideration on the adaption of a TL

As you may have already understood, in order to adapt the real part of the load impedance we use the $\lambda/4$ line adapter. If my load has both a real and imaginary part different from Z_0, we can combine the $\lambda/4$ line adapter and a stub.



(a) Open stub + $\lambda/4$ line adapter example



(b) Stub + $\lambda/4$ line adapter example

Those 2 stubs (short and open circuit) seems to do the exactly same thing... This is true! We can choose to use one or another, the only difference is the production in the factory. So with some kind of application could be useful to use one instead of another, or you can just use the one that easier to use in order to solve the exercise.

References

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[2] Steven W. Ellingson. Book: Electromagnetics I (Ellingson), 5 2020. [Online; accessed 2021-03-17].