# Electromagnetic Compatibility

Jacopo Ferretti

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# Contents

# 1 Class 1 - 22/02/21

Today's lesson was a recap on Maxwell equation.
First of all we have seen the Faraday equation:

$$\oint_l \overline{E} \cdot \hat{l}\, dl = -\frac{d}{dt} \int_s \overline{B} \cdot \hat{n}\, ds = -\frac{d\Phi_B}{dt} \tag{1.1}$$

And this second equation:

$$\oint_l \overline{H} \cdot \hat{l}\, dl = \int_s \left( \frac{d\overline{D}}{dt} + \overline{J} \right) \cdot \hat{n}\, ds \tag{1.2}$$

Where those symbol are:

- $E$ = intensity of electric field $[\frac{V}{m}]$
- $B$ = magnetic induction vector $[T]$
- $H$ = intensity magnetic field $[A]$
- $D$ = electronic displacement $[\frac{A}{m}]$
- $J$ = intensity of electric current $[\frac{A}{m^2}]$

Now, we can think $E - H$ as the element measuring the real electromagnetic field, and $D - B$ as something that measure the effect of the EMF.
We also remember that:

$$\frac{d}{dt} \int_V \overline{\rho}\, dV = -\int_s \overline{J} \cdot \hat{n}\, ds \tag{1.3}$$

Where $\rho$ is the density of charge (volumetric) and $s$ is the boundary of the volume $V$ in the integral on the left.
On the right of eq. (1.3) we can find a minus sign, because if my electron goes from out to in in our volume, it means that the current is going from in to out (the right integral is positive because of the conversion for the versor $\hat{n}$ that goes from in to out). If the the electron are going in as we already said, we are accumulating a negative charge, for this reason we need to put this negative sign.

## Recap of operator nabla

### Gradient $\nabla T$

Usually we refer to a variation of a scalar quantity with the derivative (for example $\frac{dT}{dt}$).
If the scalar $T$ variates in more than one component, we want have a look at the derivative of everything: $\frac{dT}{dx}, \frac{dT}{dy}, \frac{dT}{dz}$.

If we are in space, our scalar field vary his value over we have 3 components, so we introduce the *gradient* of the scalar field T (temperature) as:

$$\left(\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}\right) = \nabla T \tag{1.4}$$

Where $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ is the nabla operator.
Now from the scalar $T$ we obtained a vector and that is okay, but if our field is not scalar but vectorial?
The electric field $\overline{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$ is a vectorial field. What we can do now is using the dot or cross product between our vector E and the $\nabla$ operator.

### Scalar product (divergence) $\nabla \cdot \overline{E}$

The scalar product (dot) between nabla and a vectorial field is named *divergence*

$$\nabla \cdot \overline{E} = \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{1.5}$$

Physically speaking the divergence of the vector $E$ is *how $E_x$ is changing over the $x$ direction etc...*
Another way to look at this divergence is:

$$\nabla \cdot \overline{E} = \cdots = \lim_{\Delta v \to 0} \frac{\oint_s \overline{E} \cdot \hat{n}\, ds}{\Delta v} \tag{1.6}$$

### Divergence theorem

This theorem is very useful to transform an integral over a surface to an integral over a volume.

$$\int_v \nabla \cdot \overline{E} = \int_s \overline{E} \cdot \hat{n}\, ds \tag{1.7}$$

So the integral of the divergence over a volume is equal to the flux across the boundary surface of the volume.

### Cross product (curl) $\nabla \times \overline{E}$

The cross product of nabla and our vectorial field is named *curl*

$$\begin{aligned}\nabla \times \overline{E} = det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = \\ = \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k}\end{aligned} \tag{1.8}$$

The curl (as the name in italian *rotore* says) explains the behavior of $E_x$ (for example).

Again, another way to look at the curl is:

$$\nabla \times \overline{E} = \lim_{\Delta s \to 0} \frac{\hat{n} \cdot \oint_l \overline{E} \cdot \hat{l}\, dl}{\delta s} \tag{1.9}$$

The $\hat{n}$ is there to maximize the value of $\oint_l \overline{E} \cdot \hat{l}\, dl$. I don 't really know why, PLEASE GIVE IT A LOOK.
Note that we are not using the vector symbol over the nabla because we are not multiplying two vector, instead nabla is an *operator* as $\cdot$, sin or$\oplus$, so we don't write $\cancel{\overline{\nabla} \cdot \overline{E}}$

## Stokes Theorem

This theorem is similar to divergence theorem, but instead of surface and volume, we deal with surface and line:

$$\int_s \nabla \times \overline{E} \cdot \hat{n}\, ds = \oint_l \overline{E} \cdot \hat{l}\, dl \tag{1.10}$$

It means that the flux over a surface of the curl of a vectorial field, is the integral of that field on the line that is bounding s.

### Laplacian

If i use the operator curl twice we get the laplacian.
Consider the gradient of a scalar field $T$: $\nabla T = \left( \frac{\partial T}{\partial x} \hat{\imath} + \frac{\partial T}{\partial y} \hat{\jmath} + \frac{\partial T}{\partial z} \hat{k} \right)$ now:

$$\nabla \cdot (\nabla T) = \nabla^2 T = \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \tag{1.11}$$

If we consider a vectorial field $\overline{E} = E_x \hat{\imath} + E_y \hat{\jmath} + E_z \hat{k}$

$$\nabla^2 \cdot \overline{E} = \nabla^2 E_x\, \hat{\imath} + \nabla^2 E_y\, \hat{\jmath} + \nabla^2 E_z\, \hat{k} \tag{1.12}$$

Actually we can write $\nabla^2 E_x\, \hat{\imath} + \nabla^2 E_y\, \hat{\jmath} + \nabla^2 E_z\, \hat{k}$ because we are dealing with different Laplacian for each $E$ component, that is scalar.
The same can be said to $\nabla^2 T$ on eq. (1.11)

### Going back to the Stokes Theorem

With stokes theorem we can write something similar to the first Maxwell equation:
Starting from the first Maxwell equation seen in eq. (1.1)

$$\oint_l \overline{E} \cdot \hat{l}\, dl = -\frac{d}{dt} \int_s \overline{B} \cdot \hat{n}\, ds \tag{1.13}$$

And with the Stokes theorem eq. (1.10), we can write:

$$\int_s (\nabla \times \overline{E}) \cdot \hat{n}\, ds = \frac{d}{dt} \int_s \overline{B} \cdot \hat{n}\, ds \tag{1.14}$$

We notice that we have two integral over the same surface $s$, so:

$$\int_s \left( \nabla \times \overline{E} + \frac{d\overline{B}}{dt} \right) \cdot \hat{n}\, ds = 0 \tag{1.15}$$

Finally we know that this integral is equal to zero if the equation inside the brackets is also equal to zero, so:

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt} \tag{1.16}$$

This eq. (1.16) can be considered as the first Maxwell equation but for a point. We can do the same for the second Maxwell equation:

$$\nabla \times \overline{H} = \frac{d\overline{D}}{dt} + \overline{J} \tag{1.17}$$

eq. (1.16) and eq. (1.17)are also named as the local formulation of the Maxwell equation.
To be more precise we can actually split the intensity of current $J$ in two contribution:

- $J_\sigma$: current generated by the electromagnetic field on iron ($\sigma$ actually is the conductivity).
- $J_i$: current that generates the electromagnetic field, given by for example a battery of the phone.

We have seen $J_\sigma$ as the metallic behavior of the receiver and $J_i$ as the source of the EMF, and we can also see at the component $\frac{d\overline{D}}{dt}$ as he behavior of the dielectric material due to the displacement of the charge.
eq. (1.17) becomes:

$$\nabla \times \overline{H} = \frac{d\overline{D}}{dt} + \overline{J}_\sigma + \overline{J}_i \tag{1.18}$$

## Step forward: solution of the Maxwell equation

The solutions of the Maxwell equations are not always simple to obtain analytically, we need to have a sort of classification for the equation to be solved. Those can be classified as:

- **Linear** and **non linear**
- **isotropic** and **anisotropic**
- **stationary** and **non stationary**

- **dispersive** and **non dispersive**
  - in time
  - in space

**Linear** means that the equation is composed only and only only by the sum of each variable $x_i$ multiplied by their own coefficient $a_i$: $\boldsymbol{a_1 x_2 + a_1 x_2 \cdots + a_n x_n + b = 0}$, this is an important class of equation because we can use a lot of useful properties.
**Stationary** means that the results will not change with the time, so if i do the experiment now or 10 years later I'll be sure that nothing will change.
We need this little recap to neglect all those non optimal behavior, and for the sake of simplicity we use those relation:

- $\overline{D} = \varepsilon \overline{E}\ \rightarrow \varepsilon$ is the dielectric permittivity
- $\overline{B} = \mu \overline{H}\ \rightarrow \mu$ is the electric permeability
- $\overline{J_\sigma} = \sigma \overline{E}\ \rightarrow \sigma$ is the conductivity (is there when we have a metallic object)

Those are very oversimplified, but can be useful to study our EMF.
We remember that we have obtained the $\varepsilon\, \mu\, \sigma$ value in vacuum, that are:

- $\varepsilon_0 = \frac{1}{36\pi}\, 10^{-9} \left[\frac{F}{m}\right]$
- $\mu_0 = 4\pi\, 10^{-7} \left[\frac{H}{m}\right]$
- $\sigma_0 = 0 \left[\frac{S}{m}\right]$

When we are dealing with linear material, we can not consider the numeric value of these constant over vacuum, but instead with some approximation we can consider:

- $\varepsilon = \varepsilon_0\, \varepsilon_r$
- $\mu = \mu_0\, \mu_r$
- $\sigma = \frac{1}{\rho}$

With this kept in mind we can write the first two local Maxwell equation as:

$$\begin{cases} \nabla \times \overline{E} = -\mu \frac{d\overline{H}}{dt} \\ \nabla \times \overline{H} = \varepsilon \frac{d\overline{E}}{dt} + \sigma\, \overline{E} + \overline{J}_i \end{cases} \tag{1.19}$$

Ad can be seen, the system in eq. (1.19) is a systems of equations in two unknown variable ($E$ and $H$), we can solve that but it will be very complicated.

## Another useful equation

similarly to what we have done in eq. (1.16), we can write this equation exploiting the Stokes Theorem (eq. (1.10)):

$$\begin{aligned} &\frac{d}{dt}\int_v \overline{\rho}\, dv = -\int_s \overline{J}\cdot\hat{n} ds = -\int_v \nabla\cdot\overline{J}\, dv \\ &\int_v \left(\frac{d\overline{\rho}}{dt} + \nabla\cdot\overline{J}\right) dv = 0 \end{aligned} \tag{1.20}$$

Then we obtain:

$$\nabla\cdot\overline{J} = -\frac{d\overline{\rho}}{dt} \tag{1.21}$$

## Third maxwell equation in local formulation

First of all we try to do the divergence of $\nabla\times\overline{E}$.
Note that the result of the divergence of a curl is equal to zero because the result of the curl will be perpendicular to the nabla operator, so $\nabla\cdot(\nabla\times\overline{E}) = 0$

$$\begin{aligned} &\nabla\cdot\left[(\nabla\times\overline{E}) = -\frac{d\overline{B}}{dt}\right] = \\ &\nabla\cdot\left(-\frac{d\overline{B}}{dt}\right) = 0 \\ &\frac{d}{dt}(\nabla\cdot\overline{B}) = 0 \end{aligned} \tag{1.22}$$

We obtained the *Third maxwell equation in local formulation*:

$$\nabla\cdot\overline{B} = 0 \tag{1.23}$$

We can say that eq. (1.23) ist true because we are considering that at the starting time $t_0$ our EMF was turned off ($\nabla\cdot\overline{B} = 0$ at $t_0$). If $\mu$ is constant:

$$\nabla\cdot\overline{B} = \nabla\cdot(\mu\overline{H}) = \nabla\cdot\overline{H} = 0 \tag{1.24}$$

## Forth maxwell equation in local formulation

Using the same passages as before, we can obtain the forth maxwell equation in local formulation by doing the divergence of the curl of $H$

$$\begin{aligned}
&\nabla \cdot \left[ (\nabla \times \overline{H}) = -\frac{d\overline{D}}{dt} + \overline{J} \right] = \\
&\nabla \cdot \left( -\frac{d\overline{D}}{dt} + \overline{J} \right) = 0 \\
&\frac{d}{dt}(\nabla \cdot \overline{D}) + \nabla \cdot \overline{J} = 0 \\
&\frac{d}{dt}(\nabla \cdot \overline{D}) + \nabla \cdot \frac{d\overline{\rho}}{dt} = 0 \\
&\frac{d}{dt}\left(\nabla \cdot \overline{D} - \overline{\rho}\right) = 0
\end{aligned} \tag{1.25}$$

At the end, the Forth maxwell equation in local formulation is:

$$\nabla \cdot \overline{D} = \overline{\rho} \tag{1.26}$$

If $\varepsilon$ is constant we obtain

$$\begin{aligned}
&\nabla \cdot (\varepsilon \overline{E}) = \overline{\rho} \\
&\nabla \cdot \overline{E} = \frac{\overline{\rho}}{\epsilon}
\end{aligned} \tag{1.27}$$

Those equation does not say anything more than the maxwell equation. To summarize all the equation we have obtained (for some reason):

$$\begin{array}{ll}
\nabla \cdot \overline{E} = \frac{\overline{\rho}}{\epsilon} & \nabla \cdot \overline{J} = -\frac{d\overline{\rho}}{dt} \\
\nabla \cdot \overline{B} = 0 & \nabla \cdot \overline{D} = \rho
\end{array}$$

## 2 Class 2 - 26/02/21

### Vector wave equation

Here we go again, today from eq. (1.19) we want to obtain the wave equation of the EMF, we will make a lot of simplification.
First of all we consider:

- No accumulated charge (source free, no current that generates the EMF) $\boldsymbol{\overline{J_i} = 0}$
- No losses (nonconducting medium) $\boldsymbol{\overline{J_i}\sigma \cdot \overline{E}}$

Then from eq. (1.19) we now have:

$$\begin{cases} \nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \\ \nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} \end{cases} \tag{2.1}$$

We continue to try to find the wave equation by using some tricks, the next passage is to apply the curl on the first equation from eq. (2.1).

$$\begin{aligned} \nabla \times \nabla \times \overline{E} = & - \nabla \times \mu \frac{\partial \overline{H}}{\partial t} = -\mu \frac{\partial}{\partial t} \nabla \times \overline{H} \\ & = -\mu \frac{\partial}{\partial t} \varepsilon \frac{\partial \overline{E}}{\partial t} = \\ & = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \end{aligned} \tag{2.2}$$

We remember a cool property of the curl of a curl of a vector field:

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2 \cdot \overline{E} \tag{2.3}$$

Please note that we have already discussed the $\nabla^2$ symbol before in section 1 as the Laplacian. Now let's write:

$$\nabla(\nabla \cdot \overline{E}) - \nabla^2 \cdot \overline{E} = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \tag{2.4}$$

We simplify a little bit more, and we consider that we don't have free charge: $\nabla(\varepsilon\overline{E} = 0)$ and we obtain:

$$\nabla^2 \cdot \overline{E} - \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} = 0 \tag{2.5}$$

WE FINALLY OBTAINED THE *WAVE EQUATION*. Here we define a constant that will be helpful next:

$$c = \frac{1}{\sqrt{\mu\varepsilon}} \tag{2.6}$$

Then eq. (2.5) becomes:

$$\nabla^2 \cdot \overline{E} - \frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} = 0 \tag{2.7}$$

Pay attention to one thing: if you look at the maxwell equation eq. (2.1) it is a differential equation of the first order, but now we have a second order differential equation (eq. (2.7)) and the set of solution is increased. We don't address this problem now, but keep in mind that later we will be able to select the good solution by using the divergence equation (*i don't really know what does it mean*).
If instead use the second equation from eq. (2.7), and we do all the passages as before, we obtain nearly the same equation, but with $\overline{H}$

$$\nabla^2 \cdot \overline{H} - \frac{1}{c^2}\frac{\partial^2 \overline{H}}{\partial t^2} = 0 \tag{2.8}$$

## Scalar wave equation

Now let's try to write our equation with scalars and not vectors, it should simplify a bit.
If we consider the vector $E$ to be completely parallel to the $z$ axes, it means:

$$\overline{E}(x,y,z) = \cancel{E_x\hat{i}} + \cancel{E_y\hat{j}} + E_z\hat{k} \ \rightarrow \ \overline{E}(x,y,z) = E(z) \tag{2.9}$$

But actually $E$ is also dependent on time, so we deal with: $E(z,t)$
Now eq. (2.7) with scalar $E$ becomes:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{2.10}$$

### Solution of the scalar wave equation

Searching for the solution of eq. (2.10) is a bit time consuming, but it should have the shape of:

$$E(z,t) = f_1\left(t - \frac{z}{c}\right) + f_2\left(t + \frac{z}{c}\right) \tag{2.11}$$

So without going into tedious calculation, we look at 2 possible solutions for $f_1$ and $f_2$:

$$\begin{aligned} E_1(z,t) &= E_0 cos\left[\omega\left(t - \frac{z}{c}\right)\right] \\ E_2(z,t) &= E_0 cos\left[\omega\left(t + \frac{z}{c}\right)\right] \end{aligned} \tag{2.12}$$

Does that satisfy the maxwell equation? Let's consider the first solution of eq. (2.12) and verify that it actually satisfy eq. (2.10):

$$\begin{aligned}\frac{\partial^2 E}{\partial z^2} =& \frac{\partial}{\partial z}\left\{E_0\left[-\left(-\frac{\omega}{c}\right) sen\left[\omega\left(t-\frac{z}{c}\right)\right]\right]\right\} = \\ =& E_0\,\frac{\omega}{c}\left\{\left(-\frac{\omega}{c}\right) cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = \\ =& - E_0\,\frac{\omega^2}{c^2} cos\left[\omega\left(t-\frac{z}{c}\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 E}{\partial t^2} =& \frac{\partial}{\partial t}\left\{E_0\left[-\omega\, sen\left[\omega\left(t-\frac{z}{c}\right)\right]\right]\right\} = \\ =& - E_0\,\omega\left\{\omega\, cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = \\ =& - E_0\,\omega^2\, cos\left[\omega\left(t-\frac{z}{c}\right)\right]\end{aligned}$$

Then it is simple to verify eq. (2.10)

$$\begin{aligned}&\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \\ =& - E_0\,\frac{\omega^2}{c^2} cos\left[\omega\left(t-\frac{z}{c}\right)\right] - \frac{1}{c^2}\left\{-E_0\,\omega^2\, cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = 0\end{aligned}$$

We see that this solution is dependent both in space and time, but what else we can say?

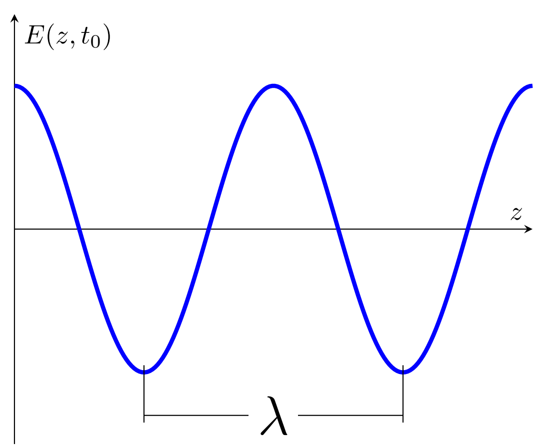


Figure 1: Plot of E in space

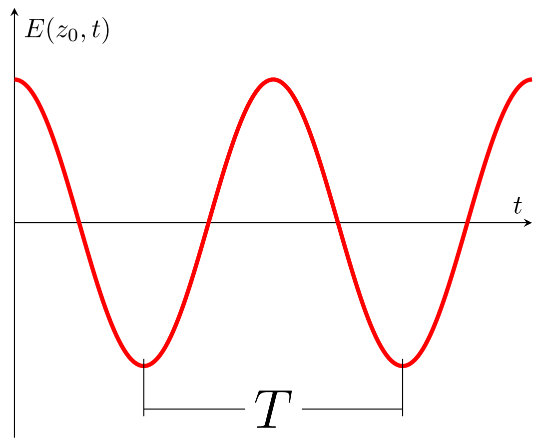


Figure 2: Plot of E in time

We can plot the solution in eq. (2.12) by considering one of the two variable constant.

- **fig. 1**: if we assume constant time (like if you would take a photograph to the wave) we are evaluating the propagation in space, and so we can obtain the wavelength $\lambda$
- **fig. 2**: if we assume constant space (like if you look the wave from a fixed position) we are evaluating the propagation in space, and so we can obtain the wavelength $T$

### Speed of the wave

As we said if we plot $E$ in constant time (fig. 1) it is like to take a picture of the wave. If we evaluate the same plot, but in another time point, we can notice that the points of the wave has changed position fig. 3.
From the variation of the space in time we can evaluate the speed of the $E$ wave.

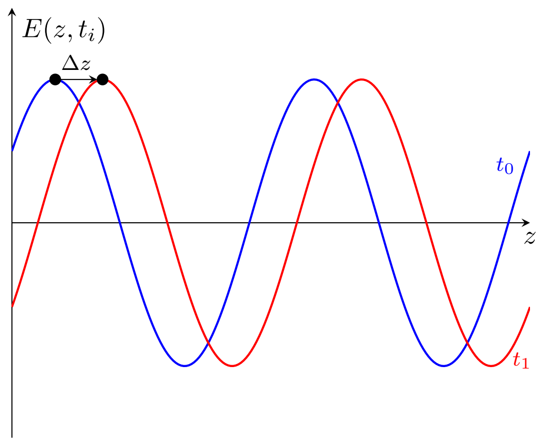


Figure 3: Plot of E in two different time

If we look at fig. 3, we can assumed that a point of the wave as moved from $z_1$ to $z_2$ from the instant $t_1$ to $t_2$. The function in the two points $(z_1, t_1)$ and $(z_2, t_2)$ has the same relative position (consider to "sit in the wave", you would feel like not moving, but the world around you is moving with a certain speed). So we can write:

$$\begin{aligned} &\omega\left(t_1 - \frac{z_1}{c}\right) = \omega\left(t_1 - \frac{z_1}{c}\right) \\ &t_2 - t_1 = \frac{z_2 - z_1}{c} \\ &\Delta t = \frac{\Delta z}{c} \\ &c = \frac{\Delta z}{\Delta t} = \frac{\lambda}{T} \end{aligned} \tag{2.13}$$

We obtained the propagation speed of the wave:

$$c = \frac{\partial z}{\partial t} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{2.14}$$

Note that the propagation speed is dependent of $\mu$ and $\varepsilon$, so we can calculate the speed in the vacuum:

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\frac{1}{36\pi} \cdot 10^{-9} \cdot 4\,\pi \cdot 10^{-7}}} \approx 3 \cdot 10^8 [m/s] \tag{2.15}$$

What we have found is a forward speed because $\Delta z$ is positive and $c$ positive, if we would have used the second equation from eq. (2.12) the space path $\Delta z$ need to be negative, or we would not be able to have a solution if we try to calculate the propagation speed:

$$\omega\left(t_1 + \frac{z_1}{c}\right) = \omega\left(t_1 + \frac{z_1}{c}\right) \tag{2.16}$$

**Going back to the EMF**

Some more consideration of the EMF

$$E(z,t) = E_0\, cos(\omega t - \frac{\omega\, z}{c}) \tag{2.17}$$

Now we give some notation for:

- $\boldsymbol{E_0}$ is the amplification of the field.
- $\boldsymbol{\omega = 2\,\pi\,\gamma}$ is the angular frequency of the EMF
- $\boldsymbol{\gamma = \frac{1}{T}}$ is the frequency of the EMF

We can also introduce the phase constant $\boldsymbol{\beta = \frac{\omega}{c}}$, and now the wave equation becomes:

$$E(z,t) = E_0 cos(\omega\, t - \beta\, z) \tag{2.18}$$

Those two parameters $\omega$ and $\beta$ are useful to obtain the propagation speed.
As we have done before we evaluate this propagation speed by considering $(\omega\, t - \beta\, z)$ to be constant, then:

$$\frac{\partial z}{\partial t} = \frac{\omega}{\beta} \tag{2.19}$$

**Generalization of the EMF**

We can generalize a bit the EMF equation by adding an attenuation constant $\alpha$ and a reference phase $\varphi$

$$E(z,t) = E_0\, e^{-\alpha\, z} cos(\omega\, t - \beta\, z + \varphi) \tag{2.20}$$

$\alpha$ is used to show how the wave is attenuated during his path on the medium.

**EMF over a general direction**

Usually we consider $\hat{k}$ as the direction of the EMF, but sometimes we need to generalize this direction over all the axes.
Consider the forward equation of the EMF over the 3 direction:

$$\begin{aligned} E(x,t) &= E_0\, cos(\omega\, t - \beta_x\, x) \\ E(y,t) &= E_0\, cos(\omega\, t - \beta_y\, y) \\ E(z,t) &= E_0\, cos(\omega\, t - \beta_z\, z) \end{aligned} \tag{2.21}$$

What we do now is to find a way to merge those equation and describe the EMF that goes in a general direction $\overline{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$.
In order to do this we introduce the vector $\overline{k} = k_x\,\hat{\imath} + k_y\,\hat{\jmath} + k_z\,\hat{k}$ that generalize the $\beta$ factor, so we can write the wave equation with a general direction:

$$E(\overline{E}, t) = \overline{E}_0\, cos(\omega\, t - \overline{k} \cdot \overline{r}) \tag{2.22}$$

Keep in mind that the direction is not $\overline{r}$ but $\overline{k}$, because $\overline{r}$ represent the variables, and $\overline{k}$ the three weights that defines the direction of the wave. This is also called the plane wave equation, why?
Because we consider $\varphi = 0$ and $\omega\, t = 0$ (we take a photo of the wave in $t = 0$), so the wave fronts (where E is constant) can be obtained with:

$$\cancel{\omega t} - \overline{k} \cdot \overline{r} + \cancel{\varphi} = -\overline{k} \cdot \overline{r} = k_x\,\hat{\imath} + k_y\,\hat{\jmath} + k_z\,\hat{k} = constant \tag{2.23}$$

Then the last part of the equation is the equation of a plane.
Another thing that we can say, is that if $\overline{k} \cdot \overline{r}$ is constant, the front of the wave have the shape of a sphere.