Electromagnetic Compatibility

Jacopo Ferretti

2021

Contents

Introduction

I share with you my work, it is not perfect, but i think it could be helpful for anyone.
You can share it, modify or do whatever you want, i enjoy to write in LaTeX, so i'm totally fine with this.
Please consider to help me to find any mistakes or misunderstanding, contact me or just modify the code from Github.
If you want to be added to my private Github repository just give me your email and I'll add you with no problem.

Contents

1 Class 1 - 22/02/21

Today's lesson was a recap on Maxwell equation.
First of all we have seen the Faraday equation:

$$\oint_l \overline{E} \cdot \hat{l}\, dl = -\frac{d}{dt}\int_s \overline{B} \cdot \hat{n}\, ds = -\frac{d\Phi_B}{dt} \tag{1.1}$$

And this second equation:

$$\oint_l \overline{H} \cdot \hat{l}\, dl = \int_s \left(\frac{d\overline{D}}{dt} + \overline{J}\right) \cdot \hat{n}\, ds \tag{1.2}$$

Where those symbol are:

- E = intensity of electric field $[\frac{V}{m}]$
- B = magnetic induction vector $[T]$
- H = intensity magnetic field $[A]$
- D = electronic displacement $[\frac{A}{m}]$
- J = intensity of electric current $[\frac{A}{m^2}]$

Now, we can think $E - H$ as the element measuring the real electromagnetic field, and $D - B$ as something that measure the effect of the EMF.
We also remember that:

$$\frac{d}{dt}\int_V \overline{\rho}\, dV = -\int_s \overline{J} \cdot \hat{n}\, ds \tag{1.3}$$

Where ρ is the density of charge (volumetric) and s is the boundary of the volume V in the integral on the left.
On the right of eq. (1.3) we can find a minus sign, because if my electron goes from out to in in our volume, it means that the current is going from in to out (the right integral is positive because of the conversion for the versor $\hat{n}$ that goes from in to out). If the the electron are going in as we already said, we are accumulating a negative charge, for this reason we need to put this negative sign.

Recap of operator nabla

Gradient ∇T

Usually we refer to a variation of a scalar quantity with the derivative (for example $\frac{dT}{dt}$).
If the scalar T variates in more than one component, we want have a look at the derivative of everything: $\frac{dT}{dx}, \frac{dT}{dy}, \frac{dT}{dz}$.

If we are in space, our scalar field vary his value over we have 3 components, so we introduce the *gradient* of the scalar field T (temperature) as:

$$\left(\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}\right) = \nabla T \tag{1.4}$$

Where $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ is the nabla operator.
Now from the scalar T we obtained a vector and that is okay, but if our field is not scalar but vectorial?
The electric field $\overline{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$ is a vectorial field. What we can do now is using the dot or cross product between our vector E and the ∇ operator.

Scalar product (divergence) $\nabla \cdot \overline{E}$

The scalar product (dot) between nabla and a vectorial field is named *divergence*

$$\nabla \cdot \overline{E} = \frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{1.5}$$

Physically speaking the divergence of the vector E is *how E_x is changing over the x direction etc...*
Another way to look at this divergence is:

$$\nabla \cdot \overline{E} = \cdots = \lim_{\Delta v \to 0} \frac{\oint_s \overline{E} \cdot \hat{n}\, ds}{\Delta v} \tag{1.6}$$

Divergence theorem

This theorem is very useful to transform an integral over a surface to an integral over a volume.

$$\int_v \nabla \cdot \overline{E} = \int_s \overline{E} \cdot \hat{n}\, ds \tag{1.7}$$

So the integral of the divergence over a volume is equal to the flux across the boundary surface of the volume.

Cross product (curl) $\nabla \times \overline{E}$

The cross product of nabla and our vectorial field is named *curl*

$$\begin{aligned} \nabla \times \overline{E} = det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = \\ = \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k} \end{aligned} \tag{1.8}$$

The curl (as the name in italian *rotore* says) explains the behavior of E_x (for example).

Again, another way to look at the curl is:

$$\nabla \times \overline{E} = \lim_{\Delta s \to 0} \frac{\hat{n} \cdot \oint_l \overline{E} \cdot \hat{l}\, dl}{\delta s} \tag{1.9}$$

The $\hat{n}$ is there to maximize the value of $\oint_l \overline{E} \cdot \hat{l}\, dl$. I don 't really know why, PLEASE GIVE IT A LOOK.
Note that we are not using the vector symbol over the nabla because we are not multiplying two vector, instead nabla is an *operator* as $\cdot$, sin or$\oplus$, so we don't write $\cancel{\overline{\nabla} \cdot \overline{E}}$

Stokes Theorem

This theorem is similar to divergence theorem, but instead of surface and volume, we deal with surface and line:

$$\int_s \nabla \times \overline{E} \cdot \hat{n}\, ds = \oint_l \overline{E} \cdot \hat{l}\, dl \tag{1.10}$$

It means that the flux over a surface of the curl of a vectorial field, is the integral of that field on the line that is bounding s.

Laplacian

If i use the operator curl twice we get the laplacian.
Consider the gradient of a scalar field T: $\nabla T = \left(\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}\right)$ now:

$$\nabla \cdot (\nabla T) = \nabla^2 T = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{1.11}$$

If we consider a vectorial field $\overline{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k}$

$$\nabla^2 \cdot \overline{E} = \nabla^2 E_x\, \hat{\imath} + \nabla^2 E_y\, \hat{\jmath} + \nabla^2 E_z\, \hat{k} \tag{1.12}$$

Actually we can write $\nabla^2 E_x\, \hat{\imath} + \nabla^2 E_y\, \hat{\jmath} + \nabla^2 E_z\, \hat{k}$ because we are dealing with different Laplacian for each E component, that is scalar.
The same can be said to $\nabla^2 T$ on eq. (1.11)

Going back to the Stokes Theorem

With stokes theorem we can write something similar to the first Maxwell equation:
Starting from the first Maxwell equation seen in eq. (1.1)

$$\oint_l \overline{E} \cdot \hat{l}\, dl = -\frac{d}{dt}\int_s \overline{B} \cdot \hat{n}\, ds \tag{1.13}$$

And with the Stokes theorem eq. (1.10), we can write:

$$\int_s (\nabla \times \overline{E}) \cdot \hat{n}\, ds = \frac{d}{dt} \int_s \overline{B} \cdot \hat{n}\, ds \tag{1.14}$$

We notice that we have two integral over the same surface s, so:

$$\int_s \left(\nabla \times \overline{E} + \frac{d\overline{B}}{dt} \right) \cdot \hat{n}\, ds = 0 \tag{1.15}$$

Finally we know that this integral is equal to zero if the equation inside the brackets is also equal to zero, so:

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt} \tag{1.16}$$

This eq. (1.16) can be considered as the first Maxwell equation but for a point. We can do the same for the second Maxwell equation:

$$\nabla \times \overline{H} = \frac{d\overline{D}}{dt} + \overline{J} \tag{1.17}$$

eq. (1.16) and eq. (1.17)are also named as the local formulation of the Maxwell equation.
To be more precise we can actually split the intensity of current J in two contribution:

- J_σ: current generated by the electromagnetic field on iron (σ actually is the conductivity).
- J_i: current that generates the electromagnetic field, given by for example a battery of the phone.

We have seen J_σ as the metallic behavior of the receiver and J_i as the source of the EMF, and we can also see at the component $\frac{d\overline{D}}{dt}$ as he behavior of the dielectric material due to the displacement of the charge.
eq. (1.17) becomes:

$$\nabla \times \overline{H} = \frac{d\overline{D}}{dt} + \overline{J}_\sigma + \overline{J}_i \tag{1.18}$$

Step forward: solution of the Maxwell equation

The solutions of the Maxwell equations are not always simple to obtain analytically, we need to have a sort of classification for the equation to be solved. Those can be classified as:

- **Linear** and **non linear**
- **isotropic** and **anisotropic**
- **stationary** and **non stationary**

- **dispersive** and **non dispersive**
  - in time
  - in space

Linear means that the equation is composed only and only only by the sum of each variable x_i multiplied by their own coefficient a_i: $\boldsymbol{a_1 x_2 + a_1 x_2 \cdots + a_n x_n + b = 0}$, this is an important class of equation because we can use a lot of useful properties.
Stationary means that the results will not change with the time, so if i do the experiment now or 10 years later I'll be sure that nothing will change.
We need this little recap to neglect all those non optimal behavior, and for the sake of simplicity we use those relation:

- $\overline{D} = \varepsilon \overline{E}$ $\rightarrow \varepsilon$ is the dielectric permittivity
- $\overline{B} = \mu \overline{H}$ $\rightarrow \mu$ is the electric permeability
- $\overline{J_\sigma} = \sigma \overline{E}$ $\rightarrow \sigma$ is the conductivity (is there when we have a metallic object)

Those are very oversimplified, but can be useful to study our EMF.
We remember that we have obtained the $\varepsilon\,\mu\,\sigma$ value in vacuum, that are:

- $\varepsilon_0 = \frac{1}{36\pi}\,10^{-9}\left[\frac{F}{m}\right]$
- $\mu_0 = 4\pi\,10^{-7}\left[\frac{H}{m}\right]$
- $\sigma_0 = 0\left[\frac{S}{m}\right]$

When we are dealing with linear material, we can not consider the numeric value of these constant over vacuum, but instead with some approximation we can consider:

- $\varepsilon = \varepsilon_0\,\varepsilon_r$
- $\mu = \mu_0\,\mu_r$
- $\sigma = \frac{1}{\rho}$

Now, the first two local Maxwell equation which we have already seen are:

$$\begin{cases} \nabla \times \overline{E} = -\frac{d\overline{B}}{dt} \\ \nabla \times \overline{H} = \frac{d\overline{D}}{dt} + \overline{J}_\sigma + \overline{J}_i \end{cases} \tag{1.19}$$

We try to make this system solvable by substitution:

$$\begin{cases} \nabla \times \overline{E} = -\mu\frac{d\overline{H}}{dt} \\ \nabla \times \overline{H} = \varepsilon\frac{d\overline{E}}{dt} + \sigma\,\overline{E} + \overline{J}_i \end{cases} \tag{1.20}$$

As can be seen, the system in eq. (1.20) is a systems of equations in two unknown variable (E and H), we can solve that but it would be very complicated.

Another useful equation

similarly to what we have done in eq. (1.16), we can write this equation exploiting the Stokes Theorem (eq. (1.10)):

$$
\begin{aligned}
&\frac{d}{dt}\int_v \overline{\rho}\, dv = -\int_s \overline{J}\cdot\hat{n} ds = -\int_v \nabla\cdot\overline{J}\, dv \\
&\int_v \left(\frac{d\overline{\rho}}{dt} + \nabla\cdot\overline{J}\right) dv = 0
\end{aligned}
\tag{1.21}
$$

Then we obtain:

$$\nabla\cdot\overline{J} = -\frac{d\overline{\rho}}{dt} \tag{1.22}$$

Third maxwell equation in local formulation

First of all we try to do the divergence of $\nabla\times\overline{E}$.
Note that the result of the divergence of a curl is equal to zero because the result of the curl will be perpendicular to the nabla operator, so $\nabla\cdot(\nabla\times\overline{E}) = 0$

$$
\begin{aligned}
&\nabla\cdot\left[(\nabla\times\overline{E}) = -\frac{d\overline{B}}{dt}\right] = \\
&\nabla\cdot\left(-\frac{d\overline{B}}{dt}\right) = 0 \\
&\frac{d}{dt}(\nabla\cdot\overline{B}) = 0
\end{aligned}
\tag{1.23}
$$

We obtained the *Third maxwell equation in local formulation*:

$$\nabla\cdot\overline{B} = 0 \tag{1.24}$$

We can say that eq. (1.24) ist true because we are considering that at the starting time t_0 our EMF was turned off ($\nabla\cdot\overline{B} = 0$ at t_0). If μ is constant:

$$\nabla\cdot\overline{B} = \nabla\cdot(\mu\overline{H}) = \nabla\cdot\overline{H} = 0 \tag{1.25}$$

Forth maxwell equation in local formulation

Using the same passages as before, we can obtain the forth maxwell equation in local formulation by doing the divergence of the curl of H

$$
\begin{aligned}
&\nabla \cdot \left[(\nabla \times \overline{H}) = -\frac{d\overline{D}}{dt} + \overline{J} \right] = \\
&\nabla \cdot \left(-\frac{d\overline{D}}{dt} + \overline{J} \right) = 0 \\
&\frac{d}{dt}(\nabla \cdot \overline{D}) + \nabla \cdot \overline{J} = 0 \\
&\frac{d}{dt}(\nabla \cdot \overline{D}) + \nabla \cdot \frac{d\overline{\rho}}{dt} = 0 \\
&\frac{d}{dt}\left(\nabla \cdot \overline{D} - \overline{\rho}\right) = 0
\end{aligned}
\tag{1.26}
$$

At the end, the Forth maxwell equation in local formulation is:

$$\nabla \cdot \overline{D} = \overline{\rho} \tag{1.27}$$

If ε is constant we obtain

$$
\begin{aligned}
&\nabla \cdot (\varepsilon\, \overline{E}) = \overline{\rho} \\
&\nabla \cdot \overline{E} = \frac{\overline{\rho}}{\epsilon}
\end{aligned}
\tag{1.28}
$$

Those equation does not say anything more than the maxwell equation. To summarize all the equation we have obtained (for some reason):

$$
\begin{array}{cc}
\nabla \cdot \overline{E} = \frac{\overline{\rho}}{\epsilon} & \nabla \cdot \overline{J} = -\frac{d\overline{\rho}}{dt} \\
\nabla \cdot \overline{B} = 0 & \nabla \cdot \overline{D} = \rho
\end{array}
$$

2 Class 2 - 26/02/21

Vector wave equation

Here we go again, today from eq. (1.20) we want to obtain the wave equation of the EMF, we will make a lot of simplification.
First of all we consider:

- No accumulated charge (source free, no current that generates the EMF) $\boldsymbol{\overline{J_i} = 0}$
- No losses (nonconducting medium) $\boldsymbol{\overline{J_i}\sigma \cdot \overline{E}}$

Then from eq. (1.20) we now have:

$$\begin{cases} \nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \\ \nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} \end{cases} \tag{2.1}$$

We continue to try to find the wave equation by using some tricks, the next passage is to apply the curl on the first equation from eq. (2.1).

$$\begin{aligned} \nabla \times \nabla \times \overline{E} = & - \nabla \times \mu \frac{\partial \overline{H}}{\partial t} = -\mu \frac{\partial}{\partial t} \nabla \times \overline{H} \\ & = -\mu \frac{\partial}{\partial t} \varepsilon \frac{\partial \overline{E}}{\partial t} = \\ & = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \end{aligned} \tag{2.2}$$

We remember a cool property of the curl of a curl of a vector field:

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2 \overline{E} \tag{2.3}$$

Please note that we have already discussed the ∇^2 symbol before in section 1 as the Laplacian. Now let's write:

$$\nabla(\nabla \cdot \overline{E}) - \nabla^2 \overline{E} = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \tag{2.4}$$

We simplify a little bit more, and we consider that we don't have free charge: $\nabla(\varepsilon\overline{E} = 0)$ and we obtain:

$$\nabla^2 \overline{E} - \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} = 0 \tag{2.5}$$

WE FINALLY OBTAINED THE *WAVE EQUATION*. Here we define a constant that will be helpful next:

$$c = \frac{1}{\sqrt{\mu\varepsilon}} \tag{2.6}$$

Then eq. (2.5) becomes:

$$\nabla^2 \overline{E} - \frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} = 0 \tag{2.7}$$

Pay attention to one thing: if you look at the maxwell equation eq. (2.1) it is a differential equation of the first order, but now we have a second order differential equation (eq. (2.7)) and the set of solution is increased. We don't address this problem now, but keep in mind that later we will be able to select the good solution by using the divergence equation (*i don't really know what does it mean*).
If instead use the second equation from eq. (2.7), and we do all the passages as before, we obtain nearly the same equation, but with $\overline{H}$

$$\nabla^2 \overline{H} - \frac{1}{c^2}\frac{\partial^2 \overline{H}}{\partial t^2} = 0 \tag{2.8}$$

Scalar wave equation

Now let's try to write our equation with scalars and not vectors, it should simplify a bit.
If we consider the vector E to be completely parallel to the z axes, it means:

$$\overline{E}(x,y,z) = \cancel{E_x \hat{\imath}} + \cancel{E_y \hat{\jmath}} + E_z \hat{k} \;\rightarrow\; \overline{E}(x,y,z) = E(z) \tag{2.9}$$

But actually E is also dependent on time, so we deal with: $E(z,t)$
Now eq. (2.7) with scalar E becomes:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{2.10}$$

Solution of the scalar wave equation

Searching for the solution of eq. (2.10) is a bit time consuming, but it should have the shape of:

$$E(z,t) = f_1\left(t - \frac{z}{c}\right) + f_2\left(t + \frac{z}{c}\right) \tag{2.11}$$

So without going into tedious calculation, we look at 2 possible solutions for f_1 and f_2:

$$\begin{aligned} E_1(z,t) &= E_0 cos\left[\omega\left(t - \frac{z}{c}\right)\right] \\ E_2(z,t) &= E_0 cos\left[\omega\left(t + \frac{z}{c}\right)\right] \end{aligned} \tag{2.12}$$

Does that satisfy the maxwell equation? Let's consider the first solution of eq. (2.12) and verify that it actually satisfy eq. (2.10):

$$
\begin{aligned}
\frac{\partial^2 E}{\partial z^2} =& \frac{\partial}{\partial z}\left\{E_0\left[-\left(-\frac{\omega}{c}\right) sen\left[\omega\left(t-\frac{z}{c}\right)\right]\right]\right\} = \\
=& E_0\,\frac{\omega}{c}\left\{\left(-\frac{\omega}{c}\right) cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = \\
=& -E_0\,\frac{\omega^2}{c^2} cos\left[\omega\left(t-\frac{z}{c}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 E}{\partial t^2} =& \frac{\partial}{\partial t}\left\{E_0\left[-\omega\, sen\left[\omega\left(t-\frac{z}{c}\right)\right]\right]\right\} = \\
=& -E_0\,\omega\left\{\omega\, cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = \\
=& -E_0\,\omega^2\, cos\left[\omega\left(t-\frac{z}{c}\right)\right]
\end{aligned}
$$

Then it is simple to verify eq. (2.10)

$$
\begin{aligned}
&\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \\
=& -E_0\,\frac{\omega^2}{c^2} cos\left[\omega\left(t-\frac{z}{c}\right)\right] - \frac{1}{c^2}\left\{-E_0\,\omega^2\, cos\left[\omega\left(t-\frac{z}{c}\right)\right]\right\} = 0
\end{aligned}
$$

We see that this solution is dependent both in space and time, but what else we can say?

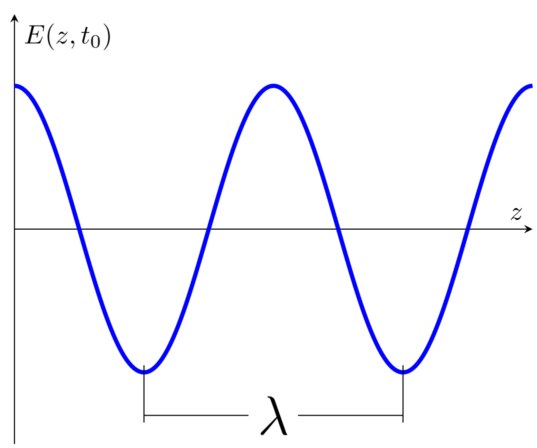

Figure 1: Plot of E in space

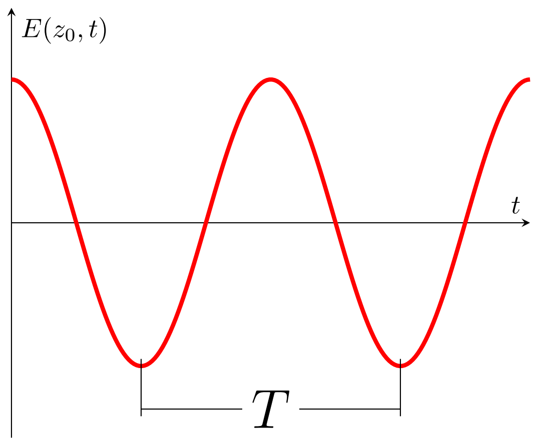

Figure 2: Plot of E in time

We can plot the solution in eq. (2.12) by considering one of the two variable constant.

- **fig. 1**: if we assume constant time (like if you would take a photograph to the wave) we are evaluating the propagation in space, and so we can obtain the wavelength λ
- **fig. 2**: if we assume constant space (like if you look the wave from a fixed position) we are evaluating the propagation in space, and so we can obtain the wavelength T

Speed of the wave

As we said if we plot E in constant time (fig. 1) it is like to take a picture of the wave. If we evaluate the same plot, but in another time point, we can notice that the points of the wave has changed position fig. 3.
From the variation of the space in time we can evaluate the speed of the E wave.

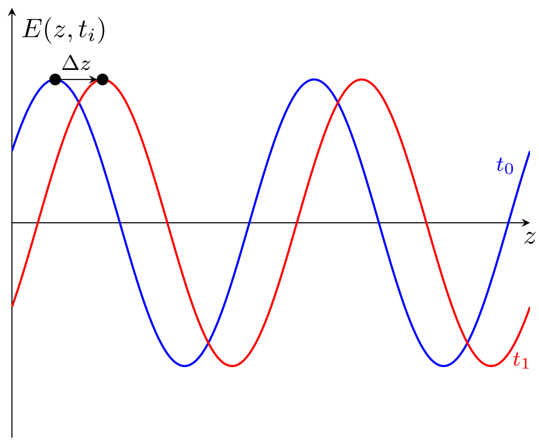

Figure 3: Plot of E in two different time

If we look at fig. 3, we can assumed that a point of the wave as moved from z_1 to z_2 from the instant t_1 to t_2. The function in the two points (z_1, t_1) and (z_2, t_2) has the same relative position (consider to "sit in the wave", you would feel like not moving, but the world around you is moving with a certain speed). So we can write:

$$\begin{aligned}
&\omega\left(t_1 - \frac{z_1}{c}\right) = \omega\left(t_1 - \frac{z_1}{c}\right) \\
&t_2 - t_1 = \frac{z_2 - z_1}{c} \\
&\Delta t = \frac{\Delta z}{c} \\
&c = \frac{\Delta z}{\Delta t} = \frac{\lambda}{T}
\end{aligned} \tag{2.13}$$

We obtained the propagation speed of the wave:

$$c = \frac{\partial z}{\partial t} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{2.14}$$

Note that the propagation speed is dependent of μ and ε, so we can calculate the speed in the vacuum:

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\frac{1}{36\pi} \cdot 10^{-9} \cdot 4\,\pi \cdot 10^{-7}}} \approx 3 \cdot 10^8 [m/s] \tag{2.15}$$

What we have found is a forward speed because Δz is positive and c positive, if we would have used the second equation from eq. (2.12) the space path Δz need to be negative, or we would not be able to have a solution if we try to calculate the propagation speed:

$$\omega\left(t_1 + \frac{z_1}{c}\right) = \omega\left(t_1 + \frac{z_1}{c}\right) \tag{2.16}$$

Going back to the EMF

Some more consideration of the EMF

$$E(z,t) = E_0\, cos(\omega t - \frac{\omega\, z}{c}) \tag{2.17}$$

Now we give some notation for:

- $\boldsymbol{E_0}$ is the amplification of the field.
- $\boldsymbol{\omega = 2\,\pi\,\gamma}$ is the angular frequency of the EMF
- $\boldsymbol{\gamma = \frac{1}{T}}$ is the frequency of the EMF

We can also introduce the phase constant $\boldsymbol{\beta = \frac{\omega}{c}}$, and now the wave equation becomes:

$$E(z,t) = E_0 cos(\omega\, t - \beta\, z) \tag{2.18}$$

Those two parameters ω and β are useful to obtain the propagation speed.
As we have done before we evaluate this propagation speed by considering $(\omega\, t - \beta\, z)$ to be constant, then:

$$\frac{\partial z}{\partial t} = \frac{\omega}{\beta} \tag{2.19}$$

Generalization of the EMF

We can generalize a bit the EMF equation by adding an attenuation constant α and a reference phase φ

$$E(z,t) = E_0\, e^{-\alpha\, z} cos(\omega\, t - \beta\, z + \varphi) \tag{2.20}$$

α is used to show how the wave is attenuated during his path on the medium.

EMF over a general direction

Usually we consider $\hat{k}$ as the direction of the EMF, but sometimes we need to generalize this direction over all the axes.
Consider the forward equation of the EMF over the 3 direction:

$$\begin{aligned} E(x,t) &= E_0\, cos(\omega\, t - \beta_x\, x) \\ E(y,t) &= E_0\, cos(\omega\, t - \beta_y\, y) \\ E(z,t) &= E_0\, cos(\omega\, t - \beta_z\, z) \end{aligned} \tag{2.21}$$

What we do now is to find a way to merge those equation and describe the EMF that goes in a general direction $\overline{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$.
In order to do this we introduce the vector $\overline{k} = k_x\,\hat{\imath} + k_y\,\hat{\jmath} + k_z\,\hat{k}$ that generalize the β factor, so we can write the wave equation with a general direction:

$$E(\overline{E}, t) = \overline{E}_0\,cos(\omega\,t - \overline{k}\cdot\overline{r}) \tag{2.22}$$

Keep in mind that the direction is not $\overline{r}$ but $\overline{k}$, because $\overline{r}$ represent the variables, and $\overline{k}$ the three weights that defines the direction of the wave. This is also called the plane wave equation, why?
Because we consider $\varphi = 0$ and $\omega\,t = 0$ (we take a photo of the wave in $t = 0$), so the wave fronts (where E is constant) can be obtained with:

$$\cancel{\omega t} - \overline{k}\cdot\overline{r} + \cancel{\varphi} = -\overline{k}\cdot\overline{r} = k_x\,\hat{\imath} + k_y\,\hat{\jmath} + k_z\,\hat{k} = constant \tag{2.23}$$

Then the last part of the equation is the equation of a plane.
Another thing that we can say, is that if $\overline{k}\cdot\overline{r}$ is constant, the front of the wave have the shape of a sphere.

3 Class 3 - 1/03/21

During this class we will talk about the passage from the time domain to frequency. We do that because in frequency domain we do not need anymore to do the derivate in time, and thus the calculations simplify a lot. Another reason is that is more important to look at the signal on the corner frequency other than all over the spectrum

EMF in phasor domain

Now we take a look at the electrical field propagating on the z axe: $\overline{E}(z,t) = E_x\,\hat{\jmath} + E_y\,\hat{\imath} + E_z\,\hat{k}$, and we explore all the element:

$$\begin{aligned} E_x(z,t) &= E_{x_0}\,cos(\omega\,t - \beta\,z + \varphi_x) \\ E_y(z,t) &= E_{y_0}\,cos(\omega\,t - \beta\,z + \varphi_y) \\ E_z(z,t) &= E_{z_0}\,cos(\omega\,t - \beta\,z + \varphi_z) \end{aligned} \tag{3.1}$$

Note that here ω and β does not change, but φ does, this is not very important, but it is just a note.
From eq. (3.1), and exploiting the cos property:

$$cos(\alpha + \beta) = cos\alpha\,cos\beta - sin\alpha\,sin\beta$$

By considering $\alpha = \omega\,t$ and $\beta = \beta z + \varphi + z$ we obtain the total equation of the EMF on z direction:

$$\begin{aligned} &\overline{E}(z,t) = E_x\,\hat{\jmath} + E_y\,\hat{\imath} + E_z\,\hat{k} = \\ &= cos(\omega\,t)[E_{x_0}cos(\varphi - \beta\,z)\hat{\imath} + E_{y_0}cos(\varphi_y - \beta\,z)\hat{\jmath} + E_{z_0}cos(\varphi_z - \beta\,z)\hat{k}] + \\ &- sin(\omega\,t)[E_{x_0}sin(\varphi - \beta\,z)\hat{\imath} + E_{y_0}sin(\varphi_y - \beta\,z)\hat{\jmath} + E_{z_0}sin(\varphi_z - \beta\,z)\hat{k}]) = \\ &= \overline{E}_1\,cos(\omega\,t) + \overline{E}_2\,sen(\omega\,t) \end{aligned} \tag{3.2}$$

First of all we notice that $\overline{E}_1$ and $\overline{E}_2$ are vectors that are only in function of space (and that is great and useful).
Then we can write:

$$\overline{E}(z,t) = \text{Re}\left\{(\overline{E}_1 + j\overline{E}_2)[cos(\omega t) + j\,sin(\omega t)]\right\} = \text{Re}\left\{(\overline{E}_1 + j\overline{E}_2)e^{j\omega t}\right\} \tag{3.3}$$

With this notation we can introduce the phasor

$$\overset{\circ\!\!\rightarrow}{E}(z) = \overline{E}_1 + j\overline{E}_2 \tag{3.4}$$

And we will use this notation to describe the EMF in complex notation.

$$\overline{E}(z,t) = \text{Re}\left\{\overset{\circ\!\!\rightarrow}{E}(z)e^{j\omega t}\right\} \tag{3.5}$$

If we want to go from time domain to phasor, we need to find E_1 and E_2. To do the opposite we need to use eq. (3.5).
Now let's give a look at the derivative of the field:

$$\frac{\partial \overline{E}(z,t)}{\partial t} = \text{Re}\left\{\frac{\partial}{\partial t}\overset{\circ\!\!\rightarrow}{E}e^{j\omega t}\right\} = \text{Re}\left\{j\omega\overset{\circ\!\!\rightarrow}{E}e^{j\omega t}\right\} \tag{3.6}$$

With this trick we can use the derivative by only multiplying $j\omega$.
That being said, we can give the definition of electric or magnetic field that propagates in a general direction:

$$\begin{aligned}\overline{E}(\overline{r},t) &= \text{Re}\left\{\overset{\circ\!\!\rightarrow}{E}(\overline{r})e^{j\omega t}\right\}\\ \overline{H}(\overline{r},t) &= \text{Re}\left\{\overset{\circ\!\!\rightarrow}{H}(\overline{r})e^{j\omega t}\right\}\end{aligned} \tag{3.7}$$

Now in phasor domain, we can write the maxwell equations that we have already seen in eq. (1.19), but with a simpler notation.

$$\begin{aligned}\nabla\times\overset{\circ\!\!\rightarrow}{E} &= j\omega\overset{\circ\!\!\rightarrow}{B}\\ \nabla\times\overset{\circ\!\!\rightarrow}{H} &= j\omega\overset{\circ\!\!\rightarrow}{D} + \overset{\circ\!\!\rightarrow}{J}_\sigma + \overset{\circ\!\!\rightarrow}{J}_i\end{aligned} \tag{3.8}$$

Obviously this is true for any general direction $\overline{r}$, evn if we didn't mentioned that for a better notation elegance.
Another important thing that we need to stress is the relation of the other vectors that can be represented in the phasor space:

$$\overset{\circ\!\!\rightarrow}{D} = -\varepsilon\overset{\circ\!\!\rightarrow}{E} \qquad \overset{\circ\!\!\rightarrow}{B} = \mu\overset{\circ\!\!\rightarrow}{H} \qquad \overset{\circ\!\!\rightarrow}{J}_\sigma = \sigma\overset{\circ\!\!\rightarrow}{E}$$

Note that ε,μ and σ in this case are dependent on the position and frequency (not in time as before).

$$\begin{aligned}\varepsilon &= \varepsilon(\overline{r},\omega)\\ \mu &= \mu(\overline{r},\omega)\\ \sigma &= \sigma(\overline{r},\omega)\end{aligned} \tag{3.9}$$

Note on the refraction index

The refraction index can become useful next, here we only introduce it and say what does it mean.
First of all we define refraction index n as the square root of ε_r

$$n = \sqrt{\varepsilon_r} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \tag{3.10}$$

Actually this is a simplified relation, it is better to define n as:

$$n = \sqrt{\varepsilon_r \, \mu_r} = \sqrt{\frac{\varepsilon \, \mu}{\varepsilon_0 \, \mu_0}} \tag{3.11}$$

We already seen before that $c = \frac{1}{\sqrt{\varepsilon_0 \, \mu_0}}$ and $v_p = \frac{1}{\sqrt{\varepsilon \, \mu}}$, so we the refraction index is very useful to describe the speed of an EMF through a medium:

$$n = \frac{c}{v_0} \tag{3.12}$$

Wave equation in phasor domain

Given the system of equation in eq. (3.8), and applying some substitution we can obtain:

$$\begin{aligned} \nabla \times \overset{\circ\rightarrow}{E} &= -j\omega\mu \, \overset{\circ\rightarrow}{H} \\ \nabla \times \overset{\circ\rightarrow}{H} &= j\omega\varepsilon \, \overset{\circ\rightarrow}{E} + \sigma \, \overset{\circ\rightarrow}{E}_{\sigma} + \overline{J}_i \end{aligned} \tag{3.13}$$

It is evident that without derivative it is much more simple to find the wave equation.
For the sake of simplicity we will not make all the calculation to arrive to the final value, instead we use the old wave equation in time domain (eq. (2.7)) and we obtain the new wave equation in phasor domain:

$$\nabla^2 \, \overset{\circ\rightarrow}{E} - \frac{\omega^2}{c^2} \, \overset{\circ\rightarrow}{E} = 0 \tag{3.14}$$

If we assume E as a scalar field, then:

$$\frac{\partial^2 \overset{\circ\rightarrow}{E}(z)}{\partial z^2} \nabla^2 - \frac{\omega^2}{c^2} \, \overset{\circ\rightarrow}{E}(z) = 0 \tag{3.15}$$

eq. (3.14) and eq. (3.15) are also known as the Helmholtz equation (strange name, i know).

Solution of the wave equation in phasor domain

Given eq. (3.15), it is very simple to obtain its solution.
Before to do that we introduce the parameter γ such that:

$$\gamma^2 = -\frac{\omega^2}{c^2} = -\omega \, \mu \, \varepsilon \tag{3.16}$$

Substituting $-\frac{\omega^2}{c^2}$ with *gamma* from eq. (3.15) we are able to get a possible solution:

$$\overset{\circ\rightarrow}{E} = \overline{E}_{0_1} \, e^{\gamma z} + \overline{E}_{0_2} \, e^{-\gamma z} \tag{3.17}$$

We know that γ is a complex number because both μ and ε are so:

$$\gamma = \alpha + j\beta \tag{3.18}$$

So we can write eq. (3.17) as:

$$\overset{\circ\!\!\to}{E} = \overline{E}_{0_1}\, e^{\alpha z}\, e^{j\beta z} + \overline{E}_{0_2}\, e^{-\alpha z}\, e^{-j\beta z} = \tag{3.19}$$

We will focus on the forward wave equation $\overset{\circ\!\!\to}{E} = \overline{E}_0\, e^{-\alpha z}\, e^{-j\beta z}$.

Going back from phasor to time domain

If we want to go back to time domain from eq. (3.19) we can just apply the relation from eq. (3.5):

$$\begin{aligned}
&\overline{E}(z,t) = \mathrm{Re}\left\{\overset{\circ\!\!\to}{E}(z)e^{j\omega t}\right\} = \\
&= \mathrm{Re}\left\{\overline{E}_{0_2}\, e^{-\alpha z}\, e^{-j\beta z} e^{j\omega t}\right\} = \\
&= \mathrm{Re}\left\{|E_{0_2}|\, e^{j\varphi_0}\, e^{-\alpha z}\, e^{-j\beta z} e^{j\omega t}\right\} = \\
&= |E_{0_2}|e^{-\alpha z}\, \mathrm{Re}\left\{e^{j(\omega t - \beta z + \varphi_0)}\right\} = \\
&= |E_{0_2}|e^{-\alpha z}\, \mathrm{Re}\left\{cos(\omega t - \beta z + \varphi_0) + j sin(\omega t - \beta z + \varphi_0)\right\} = \\
&= |E_{0_2}|e^{-\alpha z}\, cos(\omega t - \beta z + \varphi_0)
\end{aligned} \tag{3.20}$$

We obtained the forward equation of an EMF that propagates over the z direction, that we have already seen in eq. (2.20).

Why φ is a complex number?

Consider the Maxwell equation in eq. (3.13), but without the $\overline{J}_i$ term (no current that is generating the EMF).
I'll not write again the eq. (3.13) because you can find it simply clicking at the reference number, that being said we can do something with the second relation:

$$\begin{aligned}
&\nabla \times \overset{\circ\!\!\to}{H} = j\omega\varepsilon \left(1 + \frac{\sigma}{j\omega\epsilon}\right) \overset{\circ\!\!\to}{E} = \\
&\nabla \times \overset{\circ\!\!\to}{H} = j\omega\varepsilon \left(1 - j\frac{\sigma}{\omega\epsilon}\right) \overset{\circ\!\!\to}{E} = \\
&\nabla \times \overset{\circ\!\!\to}{H} = j\omega \underbrace{\varepsilon \left(1 + \frac{\sigma}{j\omega\epsilon}\right)}_{\varepsilon_c} \overset{\circ\!\!\to}{E} = \\
&\nabla \times \overset{\circ\!\!\to}{H} = j\omega\varepsilon_c \overset{\circ\!\!\to}{E}
\end{aligned} \tag{3.21}$$

ε_c is the complex permittivity, with this the maxwell equation becomes very similar to eq. (2.1):

$$\begin{cases} \nabla \times \overset{\circ\!\!\rightarrow}{E} = -j\omega\,\mu \overset{\circ\!\!\rightarrow}{H} \\ \nabla \times \overset{\circ\!\!\rightarrow}{H} = \quad j\omega\,\varepsilon_c \overset{\circ\!\!\rightarrow}{E} \end{cases} \tag{3.22}$$

So eq. (3.16) is not totally correct because we need to consider the complex permittivity ε_c: $\gamma^2 = -\omega\,\mu\,\varepsilon_c$.
An interesting thing to note is that the imaginary part of $\varepsilon_c = \varepsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)$ are the losses during the propagation of the EMF through a medium

- $\boldsymbol{\sigma}$: metallic medium loss
- $\boldsymbol{\omega\epsilon}$: dielectric medium loss

We can also use $\frac{\sigma}{\omega\epsilon}$ to know the property of our medium:

- $\frac{\sigma}{\omega\epsilon} > 1$: metallic medium
- $\boldsymbol{\frac{\sigma}{\omega\epsilon} < 1}$: dielectric medium loss

If we suppose no metallic loss: $\sigma = 0$, then:

$$\theta = 0 \rightarrow \varepsilon_c = \varepsilon \rightarrow \alpha = 0 \rightarrow \gamma = j\beta$$

With this simplification the forward magnetic field becomes:

$$\overset{\circ\!\!\rightarrow}{E} = \overline{E}_0\, e^{-j\beta z} \tag{3.23}$$

EMF in frequency domain

Until now we have seen a pure sinusoidal EMF that propagates, what if this EMF is not pure?
We can say that our signal is not a pure sinusoid if we have more than 1 component other than the fundamental harmonic, this mean that we deal with noisy signal.
Similarly to eq. (3.5), we can define the transformation from time domain to frequency domain as:

$$\overline{E}(\overline{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{E}(\overline{r}, w) e^{j\omega t} \tag{3.24}$$

That transformation is actually the same as the one for the complex domain, but we now can consider more than 1 harmonic.
Again here we don't deal anymore with derivatives in time, so our job simplify a lot!:

$$\frac{\partial \overline{E}(\overline{r}, t)}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega\,\overline{E}(\overline{r}, w) e^{j\omega t} \tag{3.25}$$

Like in eq. (3.22) we can have a look at the Maxwell equation in phasor domain again without the $\overline{J}_i$ term (no current that is generating the EMF).
You can notice that they are actually the same, but in frequency the field E and H are dependent in time and also in frequency.

$$\begin{aligned} \nabla \times \overline{E}(t,\omega) &= -j\omega\mu\,\overline{H}(t,\omega) \\ \nabla \times \overline{H}(t,\omega) &= \quad j\omega\varepsilon\,\overline{E}(t,\omega) \end{aligned} \tag{3.26}$$

And the wave equation becomes:

$$\nabla^2\,\overline{E}(t,\omega) - \gamma^2\,\overline{E}(t,\omega) = 0 \tag{3.27}$$

A little exercise

The prof said it was little... but i'm too tired to copy all the numbers, but here i reported the passages.
The request was to find the expression of the EMF (E and H equation) given γ, the direction z and supposing no losses ($\alpha = 0$). The field of E is on his peak E_0 wen $t = 0$ and $z = 50$
First of all the peak of the field E_0 is obtained when $cos(\omega t - \beta z + \phi_0) = 1$, so when $(\omega t - \beta z + \phi_0) = 0$.
We can simplify saying that $\omega t = 0$ (we can assume the initial time $t_0 = 0$ because of the given data).
From eq. (3.16) we can obtain ω from γ ($\omega^2 = c^2 - \omega^2$).
From eq. (2.18) we also know that $\beta = \frac{\omega}{c}$.
Then we obtain φ_0 from $-\frac{\omega}{c} + \phi_0 = 0$.
We have all we need to write down the equation for E in time

$$E(z,t) = E_0\,cos(-\beta z + \phi_0)\,\hat{\imath}$$

What about H?

$$\nabla \times \overline{E} = \frac{\partial E_x}{\partial z}\hat{\jmath} = -\mu\frac{\partial \overline{H}}{\partial t}\hat{\jmath} \tag{3.28}$$

Doing some strange calculation we can obtain the H equation, where the argument of cos are the same, but H_0 changes:

$$H(z,t) = H_0\,cos(-\beta z + \phi_0)\,\hat{\jmath}$$

Frequency domain

In frequency it is much more simple:

$$\overset{\circ\!\rightarrow}{E}(z) = \overline{E_0}\,\cancelto{0}{e^{-\alpha z}}\;e^{-j\beta z} = |E_0|\,e^{-j\beta z}\,e^{j\varphi_0}$$

We already have E_0, β and φ_0 can be simply calculated as before.
Now $\overset{\circ\!\!\rightarrow}{H}(z)$??

$$
\begin{aligned}
&\frac{\partial \overset{\circ\!\!\rightarrow}{E_x}}{\partial z} = -j\omega\mu \overset{\circ\!\!\rightarrow}{H} \\
&\overset{\circ\!\!\rightarrow}{H} = \frac{1}{-j\omega\mu}\frac{\partial \overset{\circ\!\!\rightarrow}{E_x}}{\partial z} = \\
&= \frac{j}{\omega\beta}\overline{E_0}e^{-j\beta z} = \\
&= \frac{\beta}{\mu\omega}|E_0|\, e^{j\varphi_0}e^{-j\beta z}
\end{aligned}
\tag{3.29}
$$

We know that $\beta = \frac{\omega}{c}$, then:

$$
\frac{\omega}{c}\frac{1}{\omega\mu} = \frac{\sqrt{\mu\varepsilon}}{\mu} = \sqrt{\frac{\epsilon}{\mu}} = \eta
$$

So we obtain a very useful equation:

$$
\overset{\circ\!\!\rightarrow}{H} = \frac{1}{\eta}\,\overline{E_0}e^{-j\beta z} \tag{3.30}
$$

And we can also say that if the field propagates along z:

$$
\begin{aligned}
&\overset{\circ\!\!\rightarrow}{H} = \frac{1}{\eta}\,\hat{k} \times \overset{\circ\!\!\rightarrow}{E} \\
&\overset{\circ\!\!\rightarrow}{E} = \eta\,\overset{\circ\!\!\rightarrow}{H} \times \hat{k}
\end{aligned}
\tag{3.31}
$$

Note that the second equation in eq. (3.31) is very similar to the first hom law because we have:

$$
\left[\frac{V}{m}\right] = \eta\,\left[\frac{A}{m}\right]
$$

Just like $[V] = \Omega\ [A]$

One more thing: Poynting vector

The Poynting vector $\overline{S}$ describe the density of power of our radiation, and it is defined by:

$$
\overline{S} = \overline{E} \times \overline{H} \rightarrow \left[\frac{w}{m}\right] \tag{3.32}
$$

To obtain $\overline{S}$ is not very simple, but in phasor domain it is *na crema* (italian way to say "very beautiful"):

$$
\overline{S} = \frac{\overset{\circ\!\!\rightarrow}{E} \times \overset{\circ\!\!\rightarrow}{H}^*}{2} = \frac{1}{2}\,\overline{E}_x \cdot \overline{H}_y^*\,\hat{k} \tag{3.33}
$$