Chapter 1

Algebraic integers

1.1 Integrality

Fix a domain A integrally closed in $K := K(A)$. Let $L|K$ be a finite extension, and B the integral closure of A in L. This is the $AKLB$ *diagram*:

$$\begin{tikzcd}
K \arrow[r, hook] & L \\
A \arrow[u] \arrow[r] & B \arrow[u]
\end{tikzcd}$$

Integrality is stable under the ring operations: one would like the following to hold, and they do:
This is a corollary

Theorem 1.1.1 (Module-theoretic characterization of integrality)**.** *A finite number of b_i are integral over A $\iff$ the ring $A[b_1, \dots, b_n]$ is finitely generated as an A-module.*

Proof. TODO. ■

Corollary 1.1.2. *If a and b are integral over A, so are $a + b$ and ab.*

Theorem 1.1.3 (Integrality is transitive)**.** *Consider ring extensions $A \subseteq B \subseteq C$. $A \subseteq B$ integral and $B \subseteq C$ integral $\iff$ $A \subseteq C$ integral.*

Proof. TODO. ■

Theorem 1.1.4. *Any element $l \in L$ is equal to b/a for $b \in B$ and $a \in A$.*

Proof. Consider an element $l \in L$. The minimal polynomial m_l of l over K gives rise to a polynomial over A

$$a_n l^n + a_{n-1} l^{n-1} + \cdots + a_0 = 0$$

by clearing denominators. Now observe that $\ell := a_n l$ is integral over A: multiplying by a_n^{n-1} gives an equation of the form

$$\ell^n + a'_{n-1}\ell^{n-1} + \cdots + a'_0 = 0.$$

This shows that taking $b/a = \ell/a_n$ works. ∎

Remark 1.1.5. Notice that $K(B) = L$. Indeed, $B \subset L$ so $K(B) \subset L$, and the result above shows that $L \subset K(B)$ (set-theoretically, $L \subset B \times A \subset B \times B$).

Theorem 1.1.6. $l \in L$ *is integral over* A *iff its minimal polynomial* μ_l *over* K *has coefficients in* A*.*

Proof. If $\mu := \mu_l \in A[x]$ then we have integrality of l over A by definition. Consider now the case of an integral l with minimal polynomial $\mu \in K[x]$. From integrality over A we know that l is a root of some $g \in A[x]$. Then $\mu | g$ in $K[x]$, so all zeros of μ are zeros of g and hence integral over A.

By Vietà, the coefficients a_i are given by elementary symmetric polynomials in the roots and are hence, by Corollary 1.1.2, integral over A themselves. The a_i are elements of K, so, in this case, integrality over A means that $a_i \in K$, and hence $\mu \in A[x]$. ∎

1.2 The trace and the norm

Given $x \in L$, multiplication by x determines an endomorphism

$$T_x : \alpha \mapsto x\alpha$$

of the K-vector space L. We define the trace and norm maps

$$\mathrm{Tr}_{L|K}(z) = \operatorname{tr} T_z$$
$$\mathrm{Nm}_{L|K}(z) = \det T_z$$

Theorem 1.2.7. *adsf in Corollary 1.1.2 k*

asdf