Problem. Show that there are exactly three isomorphism classes of quotient rings of $\mathbb{F}_p[x]$ by (ideals generated by) quadratic polynomials.

Proof. Let the polynomial be $f = x^2 + mx + n$. We divide into cases based on whether or not f is reducible in $\mathbb{F}_p[x]$:

1. f is irreducible. Then (f) is maximal, and hence the quotient ring $\mathbb{F}_p[x]/(f)$ is a field, of order p^2 (since, by the division algorithm, every "coset" will correspond to a linear polynomial, and there are p^2 of those). Since there is a unique finite field for every prime power order, $\mathbb{F}_{p^2}$ forms one isomorphism class.

2. f is reducible and factors as $f = (x+a)(x+b)$. We divide further:

   (a) $a \neq b$. Then

   $$\frac{\mathbb{F}_p[x]}{\langle (x+a)(x+b) \rangle} \cong \frac{\mathbb{F}_p[x]}{(x+a)} \times \frac{\mathbb{F}_p[x]}{(x+b)} \cong \mathbb{F}_p \times \mathbb{F}_p$$

   which gives us another isomorphism class of quotient rings. The first isomorphism holds because both factors are prime, and the second isomorphism is given by $(\overline{x+a}, \overline{x+b}) \mapsto (\overline{x}, \overline{x})$.

   (b) $a = b$. Then

   $$\frac{\mathbb{F}_p[x]}{\langle (x+a)^2 \rangle} \cong \mathbb{F}_p[x]/(x^2)$$

   with an explicit isomorphism given by $x + a \mapsto x$.

$\square$