1 Definitions

1. Prove that, for $n \neq 0$, $\cos(2\pi/n)$ is algebraic.
   Let $z = \cos(2\pi/n)$. Then, since $e^{2\pi i/n} = z + i\sqrt{1-z^2}$,

   $$1 = e^{2i\pi} = (z + i\sqrt{1-z^2})^n = p(z) + iq(z),$$

   where p and q are integer-coefficient polynomials. Squaring both sides, we have

   $$p(z)^2 - q(z)^2 - 1 = 2ip(z)q(z),$$

   and finally we deduce that

   $$(p(z)^2 - q(z)^2 - 1)^2 + 2p(z)q(z) = 0.$$

2. Let $\alpha = i/2$. Prove that $\mathbb{Z}[\alpha]$ is dense in $\mathbb{C}$.
   Taking powers of α, we can get all $z \in \mathbb{C}$ for which $Re(z)$ and $Im(z)$ are both dyadic rationals. Since the dyadic rationals are dense in $\mathbb{R}$, $\mathbb{Z}[\alpha]$ is dense in $\mathbb{C}$.

2 Zerodivisors etc.

1. If a and b are zerodivisors in a ring R, is $a + b$ one as well?
   No. Let $R = \mathbb{Z}/(6)$, and consider $[2]_6$ and $[3]_6$.

3 Nilpotents and stuff

1. Prove that, if x is nilpotent, $1 + x$ is a unit.
   Let $x^n = 0$. Then it is easy to see that

   $$(1 + x)(1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} = 1,$$

   as all terms but the 1 cancel out.

2. Prove that, in a ring R with prime characteristic p, if a is nilpotent, then $1 + a$ is unipotent.
   Let $a^k = 0$. Choose any multiple of p, say tp, greater than k. Then

   $$(1 + a)^{tp} = 1 + a^{tp} + \sum \binom{tp}{i} a^i.$$

   Now, $a^{tp} = a^k a^{tp-k} = 0$. As for the sum, well.

4 Polynomial rings

1. Prove that the multiplication in $R[x]$ is associative.

$$
\begin{aligned}
f(x)\cdot(g(x)\cdot h(x)) &= \left(\sum_{k\geq 0} f_i x^i\right)\left(\sum_{k\geq 0} x^k\left(\sum_{i+j=k} g_i h_j\right)\right)\\
&= \sum_{n\geq 0} x^n\left(\sum_{i+t=n} f_i\left(\sum_{j+k=t} g_j h_k\right)\right)\\
&= \sum_{n\geq 0} x^n\left(\sum_{i+j+k=n} f_i g_j h_k\right)\\
&= \sum_{n\geq 0} x^n\left(\sum_{i+j=t} f_i g_j\left(\sum_{k+t=n} h_k\right)\right)\\
&= \left(\sum_{k\geq 0} x^k\left(\sum_{i+j=k} f_i g_j\right)\right)\left(\sum_{k\geq 0} x^k h_k\right)\\
&= (f(x)\cdot g(x))\cdot h(x).
\end{aligned}
$$

2. What are the units in $F[[t]]$ for F a field?
Let $f = \sum f_i x^i$ be a formal power series with coefficients in F. For it to be a unit, there must exist some formal power series $g = \sum g_i x^i$ such that $fg = 1$. That is,

$$fg = \sum (x_k \sum_{i+j=k} f_i g_j) = 1.$$

For this, we must have $f_0 g_0 = 1$. This is possible iff f_0 is noninvertible, and the only such element in a field is 0. So polynomials with nonzero constant coefficients are units in $F[[t]]$. To prove the reverse inclusion, we note that

$$(\sum f_i x^i)(\sum g_i x^i) = (\sum h_i x^i) \text{ where } h_i = f_0 g_i + f_1 g_{i-1} + \cdots + f_i g_0$$

and choose g_js for which all $h_{\geq 1}$ are zero. To that end, set

$$g_j = -\frac{1}{f_0}(f_1 g_{i-1} + f_2 g_{i-2} + \cdots + f_i g_0).$$

Hence, the units in $F[[t]]$ are all the polynomials with nonzero constant coefficients.

5 Homomorphisms and ideals

1. Prove that every nonzero ideal in the ring of Gauss integers contains a nonzero integer.

Ideals are closed under multiplication. Choose some $z \in I$ and multiply it by $\bar{z}$. $z\bar{z}$ is an integer $\in I$.

2. Find generators for the kernels of the following maps:

   (a) $\phi : \mathbb{R}[x,y] \to \mathbb{R}$ defined by $f \mapsto f(0,0)$.
   For all $f \in \ker \phi$, $x|f$ and $y|f$, hence $\ker \phi = (x,y)$.

   (b) $\varphi : \mathbb{R}[x] \to \mathbb{C}$ defined by $f \mapsto f(2+i)$. Obviously, $K = \ker \varphi$ contains no polynomials of degree < 2. The minimal polynomial of $(2+i)$, $f = x^2 - 4x + 5$, is quadratic and belongs to K. Since it is monic as well, and $\mathbb{R}$ is a field, $K = (f)$.

   (c) $\eta : \mathbb{Z}[x] \to \mathbb{R}$ defined by $f \mapsto f(1+\sqrt{2})$. Consider the minimal polynomial of $1+\sqrt{2}$, $f = x^2 - 2x - 1$, and let g be any polynomial in the kernel.

   (d) $\sigma : \mathbb{Z}[x] \to \mathbb{C}$ defined by $x \mapsto \sqrt{2} + \sqrt{3}$.

3. Let R be a ring of prime characteristic p. Show that the Frobenius map $R \to R$ defined by $x \mapsto x^p$ is a ring homomorphism.
   Let $x, y \in R$. Then $(x+y)^p = x^p + y^p + \sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k}$. Since, for prime p, all $\binom{p}{t}$ are divisible by p, the sum is zero in R, and thus we have

   $$(x+y)^p = x^p + y^p \text{ in characteristic } p.$$

   This proves that $F(a+b) = F(a) + F(b)$. By commutativity, we also have

   $$(xy)^p = x^p y^p,$$

   so $F(ab) = F(a)F(b)$. Finally, $F(1) = 1$ is obvious.

4. Let I and J be ideals of a ring R. Show that the set $I + J$ of elements of the form $i + j$, where $i \in I$ and $j \in J$, is an ideal of R.
   $I + J$ is obviously the product of the subgroups I and J of R^+. It is also closed under R-multiplication, since if

   $$r(i+j) := ri + rj$$

   then $r(i+j) \in I + J$ since I and J are closed under R-multiplication. Hence $I + J$ is also an ideal.

5. For ideals I and J of a ring R, $I \cap J$ is also an ideal.
   For closure under addition, choose $k, k' \in I \cap J$. Since I and J are closed under addition, $k + k' \in I$ and $k + k' \in J$, so $k + k' \in I \cap J$.
   Similarly, if $k \in I \cap J$, closure under R-multiplication for $I \cap J$ follows from the corresponding property for I and J.