(0.0.1) Show that the localization of a ring R at a prime ideal $\mathfrak{p}$ is a local ring. What is the maximal ideal?

Solution. I claim that the unique maximal ideal is the ideal of nonunits of $R_{\mathfrak{p}}$, more usually defined as

$$\mathfrak{m} = \{\frac{a}{b} : \text{some element of } \mathfrak{p} \text{ divides } a\}$$

For any other ideal $\mathfrak{i}$, there are two options:

(a) It contains only nonunits. Then obviously $\mathfrak{i} \subset \mathfrak{m}$.

(b) It contains some element that is a unit. Since ideals are closed under multiplication, multiply by the inverse of the unit to get a 1 in the ideal, implying that $\mathfrak{i} = (1)$.

Hence all proper ideals are contained in $\mathfrak{m}$. $\square$

Chapter 1

Rings and ideals

1.1 Nilradical and Jacobson radical

Define the *nilradical* of a ring R as

$$N_R := \sqrt{(0)_R}$$

equivalently, $N(R)$ is the "ideal of nilpotents". The main use of the nilradical of a ring (as far as I know!) is to eliminate the problems associated with nilpotents by looking at R/N_R instead of R: it seems plausible (at least intuitively) that this ring has none.

(1.1.1) N_R is an ideal.

Proof. Closure under R-multiplication is clear. As for additive closure, consider $x, y \in N_R$, such that $x^m = y^n = 0$. We must show that $x + y$ is also a nilpotent.

Consider $(x + y)^{m+n-1}$. Every term of the binomial expansion contains either a power of $x^{r \geq m}$ or $y^{s \geq n}$ (else $r + s < m + n - 1$), so the entire thing is zero. $\square$

(1.1.2) R/N_R has no nonzero nilpotents.

Proof. Suppose $\bar{x}$ is a nilpotent in R/N_R, with $\bar{x}^n = 0$. Then x^n is a nilpotent in R and is mapped to $(0) \in R/N_R$, implying that $\bar{x} = 0$. $\square$