Problem. Show that there are exactly three isomorphism classes of quotient rings of $\mathbb{F}_p[x]$ by (ideals generated by) quadratic polynomials.

Proof. Let the polynomial be $f = x^2 + mx + n$. We divide into cases based on whether or not f is reducible in $\mathbb{F}_p[x]$:

1. f is irreducible. Then (f) is maximal, and hence the quotient ring $\mathbb{F}_p[x]/(f)$ is a field, of order p^2 (since, by the division algorithm, every "coset" will correspond to a linear polynomial, and there are p^2 of those). Since there is a unique finite field for every prime power order, $\mathbb{F}_{p^2}$ forms one isomorphism class.

2. f is reducible and factors as $f = (x + a)(x + b)$. We divide further:

   (a) $a \neq b$. Then

   $$\frac{\mathbb{F}_p[x]}{\langle (x+a)(x+b) \rangle} \cong \frac{\mathbb{F}_p[x]}{(x+a)} \times \frac{\mathbb{F}_p[x]}{(x+b)} \cong \mathbb{F}_p \times \mathbb{F}_p$$

   which gives us another isomorphism class of quotient rings. The first isomorphism holds because both factors are prime, and the second isomorphism is obtained by applying the first isomorphism theorem to the map $\mathrm{ev}_\gamma : \mathbb{F}_p[x] \to \mathbb{F}_p$, which is surjective and has kernel $(x - \gamma)$, for $\gamma = -a$.

   (b) $a = b$. Then

   $$\frac{\mathbb{F}_p[x]}{\langle (x+a)^2 \rangle} \cong \mathbb{F}_p[x]/(x^2)$$

   with an explicit isomorphism given by $\overline{f(x)} \mapsto \overline{f(x-a)}$.

$\square$