**(0.0.1)** Show that the localization of a ring $R$ at a prime ideal $\mathfrak{p}$ is a local ring. What is the maximal ideal?

*Solution.* I claim that the unique maximal ideal is the ideal of nonunits of $R_{\mathfrak{p}}$, more usually defined as

$$\mathfrak{m} = \{\frac{a}{b} : \text{some element of } \mathfrak{p} \text{ divides } a\}$$

For any other ideal $\mathfrak{i}$, there are two options:

(a) It contains only nonunits. Then obviously $\mathfrak{i} \subset \mathfrak{m}$.

(b) It contains some element that is a unit. Since ideals are closed under multiplication, multiply by the inverse of the unit to get a 1 in the ideal, implying that $\mathfrak{i} = (1)$.

Hence all proper ideals are contained in $\mathfrak{m}$. $\square$

# Chapter 1

# Rings and ideals

## 1.1 Nilradical and Jacobson radical

Define the *nilradical* of a ring $R$ as

$$N_R := \sqrt{(0)_R}$$

equivalently, $N(R)$ is the "ideal of nilpotents". The main use of the nilradical of a ring (as far as I know!) is to eliminate the problems associated with nilpotents by looking at $R/N_R$ instead of $R$: it seems plausible (at least intuitively) that this ring has none.

**(1.1.1)** $N_R$ is an ideal.

*Proof.* Closure under $R$-multiplication is clear. As for additive closure, consider $x, y \in N_R$, such that $x^m = y^n = 0$. We must show that $x+y$ is also a nilpotent.

Consider $(x+y)^{m+n-1}$. Every term of the binomial expansion contains either a power of $x^{r \geq m}$ or $y^{s \geq n}$ (else $r+s < m+n-1$), so the entire thing is zero. □

**(1.1.2)** $R/N_R$ has no nonzero nilpotents.

*Proof.* Suppose $\bar{x}$ is a nilpotent in $R/N_R$, with $\bar{x}^n = 0$. Then $x^n$ is a nilpotent in $R$ and is mapped to $(0) \in R/N_R$, implying that $\bar{x} = 0$. □

**(1.1.3)** The nilradical of a ring $R$ is the intersection of all its prime ideals.

*Proof.* Let $\mathfrak{P}$ denote the intersection of all the prime ideals. If $x \in R$ is nilpotent and $\mathfrak{p}$ is a prime ideal, then (for some $n$) $x^n = 0 \in \mathfrak{p}$, hence $x \in \sqrt{\mathfrak{p}} = \mathfrak{p}$. Hence $N_R \subset \mathfrak{p}$.

Now we prove that $x$ not nilpotent $\implies x \notin \mathfrak{P}$. Let $\Sigma$ be the set of ideals whose radicals do not contain $x$. Since $(0) \in \Sigma$, it is nonempty and can be ordered by inclusion. Applying Zorn's lemma, we get a maximal element $\mathfrak{p}$ of $\Sigma$. We will show that $\mathfrak{p}$ is prime.

Choose $s, t \notin \mathfrak{p}$. Then $\mathfrak{p} + (s), \mathfrak{p} + (t)$ strictly contain $\mathfrak{p}$, so they are not in $\Sigma$. By the definition of $\Sigma$, this means that some power of $x$ is in both those sum ideals:

$$x^m \in \mathfrak{p} + (s), x^n \in \mathfrak{p} + (t)$$

for some $m$ and $n$. Then

$$x^{m+n} \in \mathfrak{p} + (xy)$$

so $\mathfrak{p} + (xy)$ is not in $\Sigma$ either, so $xy \notin \mathfrak{p}$. Hence we have a prime ideal $\mathfrak{p}$ not containing $x$, so $x \notin \mathfrak{P}$ either. □

*Balarka's proof.* Assume $x$ in $A$ is not nilpotent. Then $A_x$, the localization at powers of $x$, is nonzero. Pick a maximal ideal $\mathfrak{m}$ (¡– here lies disguised Zorn). Contract to get a prime ideal in $A$ not containing any power of $x$. Hence, $x$ is not contained in a prime ideal of $A$ and thus not in intersection of all prime ideals in $A$. □