Homotopy colimits

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1 Day 1

1.1 Preliminaries

Categories, functors, examples of categories, product, coproduct

1.2 Pushouts in Top

The pushout of the diagram

has the following universal property:

Example 1. In Top, pushout is "gluing together along C".

Let I be a (diagram) category (this is the only choice that makes sense), and let $A_\bullet \in \mathsf{Top}^{\mathsf{I}} = [\mathsf{I}, \mathsf{Top}]$ be a diagram in Top. We have a pair of functors

$$\operatorname*{colim}_{\mathsf{I}} : \mathsf{Top}^{\mathsf{I}} \leftrightarrows \mathsf{Top} : \mathsf{const}$$
$$A_\bullet \mapsto \operatorname*{colim}_{\mathsf{I}} A_\bullet$$
$$\mathsf{const}\, T \leftarrow T$$

Now consider that, in $\mathsf{Top}^{\mathsf{I}}$, $\mathsf{Hom}(A_\bullet, \mathsf{const}\, T)$ consists of maps from the diagram $A_\bullet$ to "trivial" diagrams, such that all the squares commute. But we defined const in such a way that all the maps in $\mathsf{const}\, T$ are identity maps, so we get the commutativity of squares for free. We can now collapse all the copies of T to a single T, and we see that an element of $\mathsf{Hom}(A_\bullet, \mathsf{const}\, T)$ is basically a way of mapping the diagram $A_\bullet$ into T.

But these "ways" all factor through the colimit of the diagram, so all we need to uniquely specify one of these is a map $\mathsf{colim}_{\mathsf{I}}\, A_\bullet \to T$.

Putting these together, we get the remarkable fact that

$$\mathsf{Hom}_{\mathsf{Top}}(\operatorname*{colim}_{\mathsf{I}} A_\bullet, T) \cong \mathsf{Hom}_{\mathsf{Top}^{\mathsf{I}}}(A_\bullet, \mathsf{const}\, T),$$

which, in the categorical language, finishes our proof of the fact that

$$\operatorname*{colim}_{\mathsf{I}} \dashv \mathsf{const}\,.$$

Question: Can you find all the pieces (intentional vagueness) of a category, a la singular homology, by mapping in from diagram categories and ... ?

2 Day 2

2.1 Topological tools

Definition 1. Two maps $f, g \in \mathsf{Hom}_{\mathsf{Top}}(X, Y)$ are *homotopic* if there is a continuous map

$$H : X \times I \to Y$$

etc

Definition 2. Two spaces X, Y are *homotopy equivalent* if there exist maps

$$f : X \leftrightarrows Y : g$$

with $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$.

Example 2. Let

$$L = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

What is a map in Top^L?

We want to find a homotopy invariant functor $\mathsf{hocolim} : \mathsf{Top}^{\mathsf{I}} \to \mathsf{Top}$ and look at adjunctions $\mathcal{F} : \mathsf{C} \leftrightarrows \mathsf{D} : \mathcal{G}$ and how they behave with respect to weak equivalences.

2.2 Model categories

A *model category* C has classes of

- weak equivalences $w\mathsf{C}$, "isos"
- fibrations $f\mathsf{C}$, "sur"
- cofibrations $co\mathsf{C}$, "inj"

such that

- C has all limits and colimits
- the classes satisfy 2-out-of-3
- cofibrations lift
- fibrations lift
- there are fibrant and cofibrant replacements
- other stuff

Examples:

- Top.
  - $w\mathsf{Top}$: homotopy equivalences
  - $co\mathsf{Top}$: inclusions
  - $f\mathsf{Top}$: ???
  - cofibrant replacements are mapping cylinders
- $\mathsf{Top}^{\mathsf{I}}$.
  - $w\mathsf{Top}$: pointwise homotopy equivalences
  - $co\mathsf{Top}$: ???
  - $f\mathsf{Top}$:
  - cofibrant replacements are mapping cylinders

Definition 3. An object A in a model category C is *cofibrant* if the map into it from the initial object is a cofibration.

Why do we care?

Theorem 1 (Quillen)**.** If $\mathcal{F} : \mathsf{C} \leftrightarrows \mathsf{D} : \mathcal{G}$ is an adjunction of model categories, such that $\mathcal{G}$ preserves $\twoheadrightarrow$ and $\xrightarrow{\sim}$,

- $\mathcal{F}$ preserves cofibrations and trivial cofibrations
- $\mathcal{F}$ preserves weak equivalences between cofibrant objects

$$
\begin{tikzcd}
& S^1 \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
S^1 \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & D^2 \arrow[ur] & \\
& * \arrow[rr] & & \color{red}{*} \\
D^2 \arrow[ur] \arrow[rr] & & \color{red}{S^2} \arrow[ur, red, "\not\simeq"] \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

3 Plan of attack

Given $A_\bullet \in \mathsf{Top}^\mathsf{I}$, our plan is now to:

- Find a cofibrant replacement ${A_\bullet}'$ of $A_\bullet$ (weakly equivalent)
- Define
$$\mathsf{hocolim}\, A_\bullet = \mathsf{colim}\, {A_\bullet}'$$

Definition 4. A small category I is a *directed Reedy category* if there exists a function

$$f : \mathsf{ob} I \to \mathbb{N}_0$$

such that $\exists b \to a$ implies

$$f(a) > f(b).$$

Let $i \in I$ be an object. Define a category $(I/i)'$

$$
\mathsf{hocolim} \left(\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd} \right)
$$

$$
\mathsf{hocolim} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & * \\ * & \end{tikzcd} \quad = \mathsf{colim} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}
$$