Brauer groups

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Definition. An *algebra* is a vector space with an associative unital multiplication distributing over $+$.

Equivalently, (why?)

Definition. An algebra is a ring with a specifically chosen subfield in its center.

1 Tensor products

Definition. A tensor product of two algebras A and B is an algebra $A \otimes B$ with underlying vector space $A \otimes B$, and multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (a_1 a_2) \otimes (b_1 b_2).$$

Lemma. Let $M_i(K)$ be the K-algebra of $i \times i$ matrices with entries in K. Then, $\forall m, n > 0$,

$$M_n(K) \otimes_K M_m(K) \cong M_{nm}(K)$$

Proof. Note that $M_i(K) \cong \mathrm{End}_K(K^i)$. Given

$$\phi \in M_m(K), \psi \in M_n(K),$$

(ϕ, ψ) gives an element of

$$\mathrm{End}_K(K^m \otimes_K K^n) = \mathrm{End}_K(K^{mn}).$$

We get a map

$$h : M_m(K) \otimes M_n(K) \to M_{mn}(K)$$

which is injective by construction (check) and surjective by dimension reasons. □

1.1 Extension of scalars, a.k.a base change

Let A be a k-algebra, and F/k an extension of fields. Then $A \otimes_k F$ is an F-algebra (in the obvious way).

Example. $M_n(k) \otimes_k F \cong M_n(F)$.

1.2 Generalized quaternion algebras

Definition. Let k be a field (char $k \neq 2$). The *generalized quaternion algebra* $(a, b)_k$ is the 4-dimensional k-algebra with the basis $\{1, i, j, ij\}$, under the relations

$$i^2 = a, j^2 = b, ij = -ji.$$

Example.
1. $(1, 1)_{\mathbb{R}} \cong M_2(\mathbb{R})$.
2. $(-1, -1)_{\mathbb{R}} \cong \mathbb{H}$.

Theorem. $\mathbb{H}$ is not a matrix algebra.

Proof. $\dim \mathbb{H} > 1$ and $\mathbb{H}$ is a division ring. □

Proposition. If k is a field (char $k \neq 2$), either $(a, b)_k \cong M_2(k)$ or

$$(a, b)_k \otimes_k k(\sqrt{a}) \cong M_2(k(\sqrt{a})).$$

Proof. First, note that for $a, b, u \in k^\times$,

$$(a,b)_k \cong (u^2 a, b)_k \text{ and } (a,b)_k \cong (b,a)_k$$

First assume, wlog, that b is a square. Then $(a,b)_k \cong (1,a)_k \cong M_2(k)$ via the map

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, j \mapsto \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}, ij \mapsto \begin{bmatrix} 0 & b \\ -1 & 0 \end{bmatrix}.$$

If a, b are not square, the map $(a,b)_k \otimes_k k(\sqrt{a}) \to M_2(k(\sqrt{a}))$ is

$$1 \otimes 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i \otimes 1 \mapsto \begin{bmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{bmatrix}, j \otimes 1 \mapsto \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}, ij \otimes 1 \mapsto \begin{bmatrix} \sqrt{a} & 0 \\ 0 & b\sqrt{a} \end{bmatrix}$$

Since a and b are not squares, the image is linearly independent, so, by dimension-counting, this map is an iso. □

Definition. Let A, B be k-algebras. A is a *twisted form* of B if there is a finite extension F/k with

$$A \otimes_k F \cong B \otimes_k F.$$

Definition. An algebra A over a field K is *Brauer* if A is a twisted form of $M_n(k)$ for some n.

Lemma. Brauer algebras are finite-dimensional.

Proof. $A \otimes F \cong M_n(F)$. □

Definition. An algebra is *simple* if it has no nontrivial (2-sided) ideals.

Definition. An algerba is *central* if its center is just the base field.

Theorem (Wedderburn-Artin)**.** Let A be an f.d. central simple k-algebra. Then there exists an f.d. division ring $D \supset k$ and $n > 0$ such that $A \cong M_n(D)$, where n is unique and D is unique upto isomorphism.

Theorem. Finite-dimensional division rings over a field are Brauer algebras.

Lemma. If k is algerbaically closed, all division rings over k are k.

Definition. The *Brauer group* of a field is the group of Brauer algebras over that field, under the operation $\otimes_k$, up to the equivalence $A \sim M_n(A)$ for all algebras A. Equivalently, $A \sim B$ if, for some $m, n > 0$,

$$A \otimes_k M_n(k) \cong B \otimes_k M_m(k).$$

Theorem. If k is algebraically closed, $\mathrm{Br}(k) = 0$.

Theorem. $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2$, generated by $[(-1,-1)_\mathbb{R}]$.

Proof.

Lemma. Any Brauer group equivalence class has exactly one division algebra.

Proposition. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are all the division rings over $\mathbb{R}$.

□