

Homotopy colimits

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1 Day 1

1.1 Preliminaries

Categories, functors, examples of categories, product, coproduct

1.2 Pushouts in Top

The pushout of the diagram

has the following universal property:

Example 1. In \mathbf{Top} , pushout is “gluing together along C ”.

Let I be a (diagram) category (this is the only choice that makes sense), and let $A_\bullet \in \mathbf{Top}^I = [I, \mathbf{Top}]$ be a diagram in \mathbf{Top} . We have a pair of functors

$$\begin{array}{ccc}
\mathrm{colim} : \mathbf{Top}^I & \rightleftarrows & \mathbf{Top} : \mathrm{const} \\
& & \downarrow \\
& & A_\bullet \mapsto \mathrm{colim} A_\bullet \\
& & \downarrow \\
& & \mathrm{const} T \leftarrow T
\end{array}$$

Now consider that, in \mathbf{Top}^I , $\mathrm{Hom}(A_\bullet, \mathrm{const} T)$ consists of maps from the diagram A_\bullet to “trivial” diagrams, such that all the squares commute. But we defined const in such a way that all the maps in $\mathrm{const} T$ are identity maps, so we get the commutativity of squares for free. We can now collapse all the copies of T to a single T , and we see that an element of $\mathrm{Hom}(A_\bullet, \mathrm{const} T)$ is basically a way of mapping the diagram A_\bullet into T .

But these “ways” all factor through the colimit of the diagram, so all we need to uniquely specify one of these is a map $\mathrm{colim}_I A_\bullet \rightarrow T$.

Putting these together, we get the remarkable fact that

$$\mathrm{Hom}_{\mathbf{Top}}(\mathrm{colim}_I A_\bullet, T) \cong \mathrm{Hom}_{\mathbf{Top}^I}(A_\bullet, \mathrm{const} T),$$

which, in the categorical language, finishes our proof of the fact that

$$\mathrm{colim} \dashv \mathrm{const}.$$

Question: Can you find all the pieces (intentional vagueness) of a category, a la singular homology, by mapping in from diagram categories and ... ?

2 Day 2

2.1 Topological tools

Definition 1. Two maps $f, g \in \mathrm{Hom}_{\mathbf{Top}}(X, Y)$ are *homotopic* if there is a continuous map

$$H : X \times I \rightarrow Y$$

etc

Definition 2. Two spaces X, Y are *homotopy equivalent* if there exist maps

$$f : X \rightrightarrows Y : g$$

with $fg \sim \text{id}_Y$ and $gf \sim \text{id}_X$.

Example 2. Let

$$L = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

What is a map in Top^L ?

We want to find a homotopy invariant functor $\text{hocolim} : \text{Top}^L \rightarrow \text{Top}$ and look at adjunctions $\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$ and how they behave with respect to weak equivalences.

2.2 Model categories

A *model category* \mathcal{C} has classes of

- weak equivalences $w\mathcal{C}$, “isos”
- fibrations $f\mathcal{C}$, “sur”
- cofibrations $co\mathcal{C}$, “inj”

such that

- \mathcal{C} has all limits and colimits
- the classes satisfy 2-out-of-3
- cofibrations lift
- fibrations lift
- there are fibrant and cofibrant replacements
- other stuff

Examples:

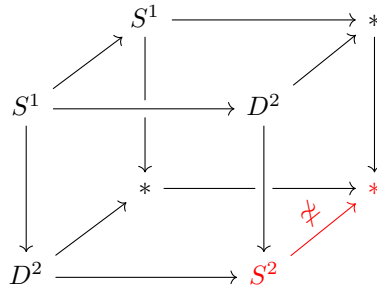
- Top .
 - $w\text{Top}$: homotopy equivalences
 - $co\text{Top}$: inclusions
 - $f\text{Top}$: ???
 - cofibrant replacements are mapping cylinders
- Top^L .
 - $w\text{Top}$: pointwise homotopy equivalences
 - $co\text{Top}$: ???
 - $f\text{Top}$:
 - cofibrant replacements are mapping cylinders

Definition 3. An object A in a model category \mathcal{C} is *cofibrant* if the map into it from the initial object is a cofibration.

Why do we care?

Theorem 1 (Quillen). If $\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$ is an adjunction of model categories, such that \mathcal{G} preserves \twoheadrightarrow and $\xrightarrow{\sim}$,

- \mathcal{F} preserves cofibrations and trivial cofibrations
- \mathcal{F} preserves weak equivalences between cofibrant objects



3 Plan of attack

Given $A_\bullet \in \mathbf{Top}^I$, our plan is now to:

- Find a cofibrant replacement A_\bullet' of A_\bullet (weakly equivalent)
- Define

$$\mathrm{hocolim} A_\bullet = \mathrm{colim} A_\bullet'$$

Definition 4. A small category I is a *directed Reedy category* if there exists a function

$$f : \mathbf{ob} I \rightarrow \mathbb{N}_0$$

such that $\exists b \rightarrow a$ implies

$$f(a) > f(b).$$

Let $i \in I$ be an object. Define a category $(I/i)'$

$$\mathrm{hocolim} \left(\begin{array}{ccc} & A & \longrightarrow B \\ & \downarrow & \\ & C & \end{array} \right)$$

$$\mathrm{hocolim} \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} = \mathrm{colim} \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$