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1 Tensor products of characters

Theorem. The product of any two characters of the same finite group G is again a character of G

Proof. The representation is $\rho = \rho_1 \otimes \rho_2$. Taking its trace, we get a block-diagonal matrix whose character is

$$\chi = \chi_1 \chi_2.$$

□

Let's compute the character table of S_4. The conjugacy classes are:

There is a natural character that sends a conjugacy class to the number of axes in $\mathbb{C}^4$ which it preserves:

χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
χ_4					
χ_5					

How about squaring χ_3? We get a character that contains a copy of χ_1

$$9, 1, 0, 1, 1$$

and subtracting that off, and then doing the same with a copy of χ_3 gives

$$5, -1, -1, 1, 1$$

But $\langle \chi, \chi \rangle = \frac{1}{24}(25, 6, 8, 6, 3) = 2$, so it's not irreducible. So it must be equal to $\chi_4 + \chi_5$. Note that, wlog, $\chi_4 = \chi_2 \chi_3$. We finally get

χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1
χ_5	2	0	-1	0	2

2 Induced representations

For any subgroup H of a group G, and any rep $\rho : H \to GL(V)$, we can define a representation of G, called the induced representation.

1. Find a set of representations for the left cosets of H in G., i.e., decompose G as

$$G = eH \cup a_2 H \cup \cdots a_q H,\ q = |G|/|H|$$

2. The space for our new rep has dimension dq. Our new representation σ sends g to a $q \times q$ block matrix, where each block is $d \times d$. We put $\rho(a_i^{-1} g a_j)$ in the i, j block if $a_i^{-1} g a_j \in H$ and 0 otherwise. Note that this is actually a "block permutation" matrix.

2.1 Example: $S_4 \subset S_5$

Let's look at S_4 as a subset of S_5, and see what the natural representation induces. Decompose S_5 as

I originally thought of a 5-cycle in class, but Mark's better sense prevailed and we used (15).

$$S_5 = S_4 \cup (15)S_4 \cup (25)S_4 \cup (35)S_4 \cup (45)S_4.$$