Graph polynomials

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1 The chromatic polynomial

Definition 1. A *proper graph coloring* is an assignment of colors to vertices such that adjacent vertices are different colors.

Question 1. Given x colors, how many ways are there to color G?

Notation 1. $P_G(x)$ is defined to be the number of colorings of G using up to x colors.

For K_n, this is how it goes:

colors	1	2	3	$\cdots$	$n-1$	n	$n+1$	$\cdots$	x
# colorings	0	0	0	$\cdots$	0	$n!$	$\binom{n+1}{k}n!$	$\cdots$	$\binom{x}{n}n!$

For a tree T, we have:

colors	1	2	3	$\cdots$	x
colorings	0	1	$3 \cdot 2^{n-1}$	$\cdots$	$x(x-1)^{n-1}$

Let $f(k)$ denote the number of different k-partitions of G. Then

$$P_G(x) = \sum_{k=1}^{|V|} f(k) \cdot x(x-1)\cdots(x-n+1)$$

"It's a polynomial in x!"

2 Actually calculating that

How can we simplify a graph?

- Delete an edge to get $G - e$.
- Contract an edge to get G/e.

Proposition 1. The chromatic polynomial P_G satisfies

$$P_G = P_{G-e} - P_{G/e}.$$

3 The reliability polynomial

What is the probability that G remains connected after removing edges with probability $1-p$?
Call this $R_G(p)$. We have that

Theorem 1.

$$R_G(p) = pR_{G/e}(p) + (1-p)R_{G-e}(p)$$

In G/e, e does not fail. In $G-e$, e necessarily fails. This is essentially what the proof consists of.