LP/CFT

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August 2016

1 Preliminaries from Galois theory

We will let K be a number field. Denote by Fld_k the category of field extensions of k.

Theorem 1 (Fundamental theorem of Galois theory)**.** There is a functor

$$\mathrm{Gal}(-/k)\colon \mathsf{Fld}_k^{\mathsf{op}} \to \mathsf{Grp},$$

the *Galois group functor*.

In particular, this means that given a k-automorphism $K \to L$, we get a morphism of Galois groups

$$\mathrm{Gal}(L/k) \to \mathrm{Gal}(K/k)$$

since any automorphism of L fixes K.

Recall that the field of *cyclotomic numbers*, $\mathbb{Q}(\zeta_n)$, has Galois group

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

where $[n] \in \mathbb{Z}/n\mathbb{Z}$ acts as the n-th power map.

2 Class field theory

Theorem 2 (Kronecker-Weber)**.** The maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$ satisfies

$$\mathbb{Q}^{\mathrm{ab}} = \bigcup_n \mathbb{Q}(\zeta_n)$$

where, for $m|n$, we identify $\mathbb{Q}(\zeta_m)$ with the canonically given subfield of $\mathbb{Q}(\zeta_n)$.

In particular, we may now apply $\mathrm{Gal}(-/\mathbb{Q})$ to get the following:

$$\Gamma^{\mathrm{ab}} := \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^\times$$

Here the limit is taken with respect to the system of surjections

$$\pi_n^m\colon (\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times$$

that sends, for instance, $[5] \in \mathbb{Z}/6\mathbb{Z}$ to $[1] \in \mathbb{Z}/3\mathbb{Z}$.

What does an element of Γ^{ab} look like? By the definition of the inverse limit of a filtered set (TODO check this), an element of Γ^{ab} is a collection of elements

$$\alpha_n \in \mathbb{Z}/n\mathbb{Z}$$

compatible with the π_m^n, where by *compatibility* we mean that

$$m|n \implies \pi_n^m(\alpha_m) = \alpha_n.$$

2.1 Describing Γ^{ab} with p-adics

(fill in defns of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ later)

We have the following classical result:

Theorem 3 (Chinese remainder theorem)**.** There exists an isomorphism

$$\mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}/p^{\nu_p(n)}\mathbb{Z}.$$

Definition 1. We denote by

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

the *profinite completion* of $\mathbb{Z}$, where the limit is taken with respect to the natural system of surjections considered in the previous section.

Now note that

$$\begin{aligned}\hat{\mathbb{Z}} &= \varprojlim_n \mathbb{Z}/n\mathbb{Z} \\ &\cong \varprojlim_n \prod_p \mathbb{Z}/p^{\nu_p(n)}\mathbb{Z} \\ &\cong \prod_p \varprojlim_r \mathbb{Z}/p^r\mathbb{Z}\end{aligned}$$

which finally gives us

$$\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p.$$

Now observe that the Kronecker-Weber theorem can be understood as saying that $\Gamma^{\mathrm{ab}} \cong \hat{\mathbb{Z}}^\times$. Using the product expression for $\hat{\mathbb{Z}}$, we find that

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \prod_p \mathbb{Z}_p^\times.$$

3 Class field theory

The obvious next step is, given a number field $F/\mathbb{Q}$, to try to "upgrade" the Kronecker-Weber theorem and describe its maximal abelian extension F^{ab}.

No such analog is known. However, we do have a description of $\text{Gal}(F^{\text{ab}}/F)$, the abelianized Galois group of F, via class field theory.

3.1 Adeles and ideles

Define the ring of *integral adeles*

$$\mathbb{A}_{\mathbb{Z}} = \mathbb{R} \times \hat{\mathbb{Z}}$$

and the ring of *adeles* as

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$