ADS Coursework 2 - Task 3

s1140740

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We need to add some definitions that we will later use in our proofs. Let $S = \{y \in \{0,1\}^n : \sum_{i=1}^n w_i y_i \leq B\}$, the set of solutions for $count(n, B)$ and $S' = \{y \in \{0,1\}^n : \sum_{i=1}^n a_i y_i \leq n^2\}$ the set of solutions for $\widehat{count}(n, n^2)$. Without loss of generality, we can assume that the elements of the solution are sorted by the weights.

Step 1

Proof by contradiction. Suppose there is a solution $y \in S$ such that $y \notin S'$. Thus, we have:

$$\sum_{i=1}^n w_i y_i \leq B$$

$$n^2 \sum_{i=1}^n w_i y_i \leq Bn^2$$

$$\sum_{i=1}^n \frac{n^2 w_i}{B} y_i \leq n^2 \qquad \text{and since } \lfloor x \rfloor \leq x\text{, then}$$

$$\sum_{i=1}^n \left\lfloor \frac{n^2 w_i}{B} \right\rfloor y_i \leq n^2 \qquad \text{substituting } a_i = \left\lfloor \frac{n^2 w_i}{B} \right\rfloor\text{,we get}$$

$$\sum_{i=1}^n a_i y_i \leq n^2 \tag{0}$$

However, (0) is the condition elements must satisfy to belong to S'. This means we have arrived at a contradiction and therefore all elements that belong to S also belong to S'.

Step 2

We define k to be the index of the largest element w_k, such that:

$$w_k \leq \frac{B}{n}, \text{ which implies } \sum_{i=1}^k w_i \leq k\frac{B}{n} \leq B \qquad \text{since } k \leq n$$

and l to be the index of the largest element a_l of a solution y, $y \in S'$. Additionally, we define $T = S' \setminus S$.

Now, we define our function $f : S' \to S$ such that no element in S is the image of $f(y)$ for more than $n+1$ different y:

$$f(y) = y \qquad \text{if } y \in S \tag{1}$$

$$f(y) = y \setminus \{l\} \qquad \text{if } y \in T \tag{2}$$

It is clear that (1) is a mapping from S' to S. Now we need to prove the same for (2). We assume that we have a solution $y \in T$. There are two cases:

1. $l \leq k$
   Since w_l is the largest element of the solution and $\sum_{i=1}^{k} w_i \leq B$, hence $y \in S$ which contradicts the fact that $y \in T$ (remember: $T = S' \setminus S$). Thus, this is not possible.

2. $l > k$
   From definition of T, we know that:

$$\sum_{i \in T} \left\lfloor \frac{w_i n^2}{B} \right\rfloor y_i \leq n^2 \qquad \text{since } a_i = \left\lfloor \frac{n^2 w_i}{B} \right\rfloor$$

$$\sum_{i \in T} \left(\frac{w_i n^2}{B} - \epsilon_i \right) y_i \leq n^2 \qquad \text{where } 0 \leq \epsilon_i < 1$$

$$\frac{n^2}{B} \sum_{i \in T} (w_i - \sigma_i) \, y_i \leq n^2 \qquad \text{where } 0 \leq \sigma_i < \frac{B}{n^2} \text{ and } \sigma_i = \frac{B \epsilon_i}{n^2}$$

$$w_l y_l + \sum_{i \in f(T)} w_i y_i - \sum_{i \in T} \sigma_i y_i \leq B \qquad \text{since } f(T) = T \setminus \{l\}$$

$$\sum_{i \in f(T)} w_i y_i \leq B - \frac{B}{n} + \sum_{i \in T} \sigma_i y_i \qquad \text{since } w_l > \frac{B}{n} \text{ and } y_l = 1$$

$$\sum_{i \in f(T)} w_i y_i \leq B - \frac{B}{n} + \frac{B}{n^2} \sum_{i \in T} \epsilon_i \qquad \text{and since } \epsilon_i < 1$$

$$\sum_{i \in f(T)} w_i y_i \leq B \qquad \text{since } |T| \leq n \tag{2}$$

Notice, that (2) is the condition for y to be a member of S. Thus, we have found a function that maps elements from T to S. Furthermore, we prove that f maps at most $n+1$ elements of S' to S. This is because every solution in S is mapped to itself and for each solution in S there are n possible largest elements w_l which means at most n solutions from T are mapped to an element in S.

Step 3

Clearly lines 1 and 5 take $\Theta(1)$ time. The loop on lines 2 and 3 is executed exactly n times since k goes from 1 to n. The specification states that the call to countKnapsackDP on line 4 takes $\Theta(n \cdot n^2)$ time since the length of a is n. Combining this, we get a runtime of $\Theta(1) + n \cdot \Theta(1) + \Theta(n^3) = \Theta(n^3)$.