DECISION MAKING BY POTENTIAL METHOD

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Abstract. A common procedure in Multi-Criteria Decision Analysis is pairwise comparison of alternatives with respect to some criteria. The result of this comparison, whether exact or subjective, is a weight associated to the pair of alternatives measured on some scales. Some authors organize their results in pairwise comparison matrix as a base for further calculation. Potential Method (PM), introduced here, uses a preference flow $\mathcal{F}$, defined on a directed graph, which generated a 'value function' called potential on the set of alternatives. Each criterion, or decision maker, has his own preference flow and MCDM based on PM preferes these flows (graphs) rather then potentials in aggregation process.

Preference flow can be complete (all pairs are evaluated) or incomplete (missing data). In both cases potential can be calculated as a solution of normal equation associated to the preference flow. If the preference graph is not connected potential is calculated for each connected component.

The notion of inconsistency measure ($i(\mathcal{F})$) is associated to the preference flow and has the property that $i(\mathcal{F}) = 0$ if and only if the flow is consistent. It is also shown that Geometric Mean Method (GMM) can be considered as a special case of PM which gives to GMM an inconsistency measure.

Comparison with PROMETHEE, eigenvalue method and Evidential Reasoning method is done, showing the priority of PM in both, computational complexity, and simplicity, specially if incompleteness in the data is present.

Contents

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1. Introduction

1.1. **Motivation.** Potential Method is the first step in analysis of `social network` in which each node has possibilities to communicate with its neighbors depending upon some criteria called state, colour or whatever. Generally speaking, the information can be split and sent to node's neighbors and the node has to decide about the proportion of information to be sent/received to/from each neighbor. Feedback in such networks is always present and it makes difficulties in finding the asymptotic state of such network, if such exists. There are some partial results of treating some simple feedback process by using PM based on the fix point theory in (Čaklović 2003). This article shows how PM can be aoolied on decision problems in hierarchical form, a 'directed version' of social network without feedback.

Potential Method is a ranking procedure that gives (as an output) a ranking function on the set of alternatives, whereas input is `preference flow` obtained in pairwise comparisons of alternatives. It can be used in Analytical Hierarchy Process (AHP) instead of eigenvalue method and it can be applied to other data structures such as decision tables as well.

In the case of missing data most of MCDM tools, like eigenvalue method and expected utility approach, cannot be applied directly without reconstructing initial data, while incomplete data set is equally well treated by PM as complete one. Another advantage of PM is clear geometric meaning of `inconsistency measure` of the preference flow $\mathcal{F}$, it is the angle between the flow vector $\mathcal{F}$ and the vector space of consistent flows. This vector space is the column space of the incidence matrix of the `preference graph`.

1.2. **Organization of the paper.** The paper is organized as follows. In section 1.3 some examples are given to show the power and flexibility of PM.

Section 2 describes PM in details, first for complete and then for incomplete graphs. Theorem 2 shows that normal integral of the given unimodular preference flow is an ordinal value function, extending thus the notion 'value function' on situations when preference flow does not reflect weak preference relation.

Aggregation procedure in MCDM PM is described in section 2.3. Two situations are described: hierarchical decision structure and decision table. Theorem 3 shows that expected utility approach is equivalent to PM, more precisely: expected utility equals normal integral of the consensus flow for decision table.

A brief description of WEB interface for PM, `http://decision.math.hr/`, is explained in section 2.4.

In section 3 we show that Geometric Mean Method can be considered as a special case of PM which gives to GMM an inconsistency measure, a feature of GMM not known in the literature, at least not to the author.

1.3. **Applications.** Before going to details let us show some applications of PM and its possibilities. We are giving three applications: an example of complex hierarchical data structure in motorcycle assessment, an example of group decision where group clustering is done based on `dissimilarity measure` of the flows, and a simple model of grading procedure where additive utility function is not a good model based on the notion of `coalition`.

1.3.1. *Incomplete data. Motorcycle assessment.* A complex data structures can be treated using PM. As an example the results of comparison are given with Yang's Evidential Reasoning method which is developed in (Yang 2001). A motorcycle assessment problem based on 29 attributes of a hierarchy is taken from (Yang 2001, Table 7, p. 52). Input data is given in Table 5 which present original data transformed to overall degrees of belief. Relative weights of the same group of attributes are shown in brackets. For criterion 'Fuel consumption', four consumption values are given: in/out urban area in winter/summer, with relative weights 1, 1, 1, 1.

Complexity of the problem arises from mixture of precise and imprecise data, and because of the missing data as well. Missing data is of two kinds in its nature: `non-existing data` and `incomplete belief`. For instance, the fuel consumption for Yamaha in urban area in winter is unknown and engine responsiveness of Yamaha has total belief degree of 0.9.

In ER approach the qualitative attributes are all assessed on the basis of the same five evaluation grades defined as poor, indifferent, average, good and excellent and abbreviated by P, I, A, G and E, respectively. The overall assessment of a motorcycle is also based on this set of grades. Overall degrees of belief in ER has to be set on the base of original input. This means that any inconsistence produced by subjective nature of the input data can be enlarged more than it is usual. While applying PM to motorcycle assessment problem the original data data were used as well, with the same result as with input data from Table 5.

PM is much more transparent. When dealing with exact data, a parameter called `flow-norm` (Fn) is introduced and its value here is 0.5 for criteria: range, displacement, speed ($\rightarrow$ max) and -0.5 for price ($\rightarrow$ min). All qualitative attributes P, I, A, G and E have $Fn = 4$ while all criteria for fuel consumption have $Fn = 0.5$. These values were chosen as a result of experimentation but do not have stronger influence on final ranking. It is only important to maintain the same Fn value for major criteria.

Flow-norm represents the subjective feeling for dispersion of ranks. It is defined as the maximum value of the flow components. If $Fn = 0$ then the flow equals zero and all alternatives have the same rank. To determine the Fn value one should experiment with it and find the right choice. More details about influence of flow-norm on ranking can be find in (Čaklović, Šego 2002). Table 1 gives comparative results of

Method		Kawasaki	Yamaha	Honda 3	BMW
ER	Average Utility	0.6232	0.666	0.7118	0.5847
	Rating	3	2	1	4
PM	Rank	0.251	0.253	0.260	0.235
	Rating	3	2	1	4

TABLE 1. Comparative results ER and PM for motorcycle assessment.

ER method and PM. While constructing consensus flow in PM we used formula (14), without attempt to make any kind of reconstruction of missing data. We repeated the calculation with data recovering using homogeneous distribution with minor differences in final numerical values of ranks.

1.3.2. *Multi-Criteria Group Decision and Group Clustering.* In this example 48 students were asked to give preference flows over the set of their lecturers. It was allowed for students to choose criteria and alternatives on their own choice and obtained consensus graph was not complete. First the students made pairwise comparisons of criteria (teaching qualities, professional competence, teachers' attitude towards students) on the scale 0-1-2-3-4, second they gave their preferences on the set of alternatives for each criteria. Aggregation was done according to formula (14) to calculate the `group flow`. The group ranks were calculated by formula (15).

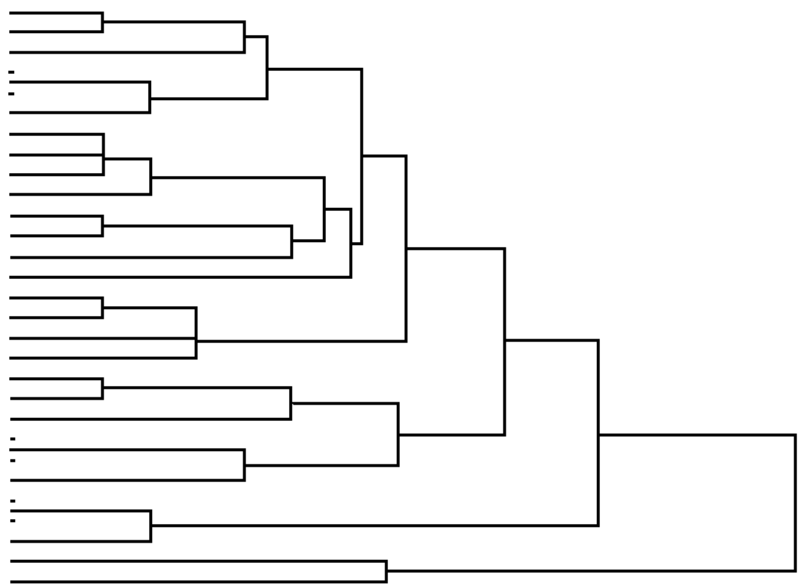

FIGURE 1. Dissimilarities for group decision. Weighted pair-group average.

It is easy to understand that inconsistency measure of consensus flow of the group is not a valuable information and some other techniques should be developed to analyze a group decision. In (Čaklović 2002) we developed a notion of graph distance in MCDM context and calculated the distances between individual preference flows, called dissimilarity matrix. From dendrogram of the group given in Figure 1 a cluster with two students, at the bottom, has been placed on a bit far from the other students. After careful examination of the data we discovered that those two students were strongly tendentious in their preferences and we decided to disregard their data in group consensus. A new group was created without those students and the group consensus was calculated again. It happened that one lecturer was lost from the list whereas the other one declined in rating by two steps.

This example suggests that more complex analysis of public opinion should be done to find out possible hidden conflicts in society.

1.3.3. *An example of grading process.* This example is from Bouyssou and others (Bouysoou 1996). The four students enrolled in an undergraduate programme which consisted of three courses: Physics, Mathematics and Economics. For each course, a final grade between 0 and 20 is allocated. The results are given in Table 2. On the basis of these

	F	M	E
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

TABLE 2. Group profile

evaluations, it was suggested that student a should be ranked before the student b. Although a has low grade in Economics, he has reasonable good grades in both Mathematics and Physics which makes him a good candidate for an Engineering programme; b is weak in Mathematics and it seems difficult to recommend him for any programme with a strong formal component (Engineering or Economics). Using a similar type of reasoning, d appears to be a a fair candidate for a programme in Economics. Student c has two low grades and it seems difficult to recommend him for a programme in Engineering or in Economics. Therefore d is ranked before c, and ranking is:

$$a \succcurlyeq b \succcurlyeq d \succcurlyeq c.$$

Although these preferences appear reasonable, they are not compatible with the use of weighted average in order to aggregate the three grades, for details see (Bouysoou 1996). This example shows that that "criteria interact". Such interactions, though not unfrequent, cannot be dealt with using weighted averages. Such type of interaction can be treated using fuzzy decision making approach, see also Grabisch (Grabish 1996).

Interaction of criteria is called a `coalition`. It seems reasonable that coalition of two criteria become a new criterion. Three new coalitions are introduced: FM, ME and FE each consisting of two criteria. The `strength` of coalition is the sum of points of each criterion in coalition. We say that coalition is `weak` if its strength is less than 20 points, otherwise it is `strong`. This threshold of 20 points seem to be reasonable because the maximum grade, for each criteria, is 20. Taking only strong coalitions into account we obtain Table 3, where ($*$)

	F	M	E	FM	ME	FE
	5	5	5	3	3	3
a	18	12	6	30	$*$	24
b	18	7	11	25	$*$	29
c	5	17	8	22	25	$*$
d	5	12	13	$*$	25	$*$

TABLE 3. Coalition of criteria.

denotes weak coalitions (missing data). In the second row we specified the relative weight of each criterion including the coalition. PM gives the following ranking

	PM-rank
a	0.397
b	0.366
d	0.129
c	0.108

which is acceptable from decision maker's point of view. Details can be found in (Čaklović 2003).

2. A preference flow

2.1. **Potential of a complete graph.** Let us identify the set of alternatives with a set of n points (nodes) in the plane or in the space. For each pair of nodes decision maker gives a preference to one of them or express his indifference. Given preference can be drawn as an arc directed towards the more preferred node while indifference can be drawn as an edge without orientation or, which is equivalent, as a pair of opposite arcs. If this construction defines a weak preference relation (complete and transitive) $\succcurlyeq$ on the set of nodes then, an ordinal value function V on the set of nodes exists in the sense that

$$a \succcurlyeq b \Leftrightarrow V(a) \geq V(b). \tag{1}$$

The proof is straightforward if we define $V(a) = \#\{b \mid a \succcurlyeq b\}$. In Potential Method we associate to each arc a nonnegative real number (weight) with convention that the weight of an arc equals 0 if and only if it is indifference. Any real function on the set of arcs in directed graph is called a `flow`. Weighted preference relation is a flow that we call `preference flow`. Preference flow with binary values, i. e. in the set $\{0, 1\}$ we call `unimodular flow`. A weak preference relation can be considered as an example of unimodular flow.

In the following example zero weight is given to the preference $B \succcurlyeq A$ which means that also $A \succcurlyeq B$ or that decision maker expresses his indifference when comparing A and B. For a given preference flow

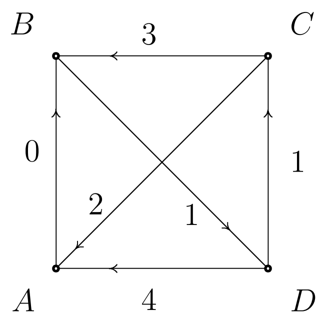

FIGURE 2. Preference flow.

decision maker wants to find a value function V that satisfies (1). This is possible, as we have already seen, for unimodular flow which arises from a weak preference relation. A construction of value function can be understood as the passage from preference flow to a function on the set of nodes. Such function is called `potential` analogous to an electrical network with electric flow and nodes having some electric potential.

The situation in Figure 2, from the point of view of money flow, can be interpreted as follows. An arc $B \xrightarrow{1} D$ we interpret as one dollar exchange between B and D, $D \xrightarrow{4} A$ as 4 dollar exchange between D and A and so on. At the end, the ledger (potential) for each person is the following:

	X	$X/4$
A	6	1.5
B	2	0.5
C	-4	-1
D	-4	-1

For some reason, that we shall explain later, we divide above values of potential X by number of nodes, in this case this is 4. Let us note that C and D have the same potential value. For a preference flow in Figure 2 an ordinal value function does not exist because of the cycle $C \to B \to D \to C$ which violates transitivity.

To formalize the above construction of potential let us denote the set of nodes by $S = \{1, \ldots, n\}$ and suppose that graph is complete with $m = \binom{n}{2}$ arcs, the set of arcs being denoted by $\mathcal{A}$. The value of preference flow $\mathcal{F}$ on arc $\alpha = (j, i)$ we denote by $\mathcal{F}_\alpha$, i being more preferred than j. The value of potential X on i-th node is then

$$x_i = \frac{1}{n}\Bigg(\sum_j \mathcal{F}_{(j,i)} - \sum_j \mathcal{F}_{(i,j)}\Bigg), \tag{2}$$

which represents the difference of incoming and outgoing flow for i-th node. We can simplify above formula introducing a `flow matrix` F by

$$F_{ij} = \begin{cases} \mathcal{F}_{(j,i)} & \text{if } (j,i) \in \mathcal{A}, \\ -\mathcal{F}_{(i,j)} & \text{if } (i,j) \in \mathcal{A}, \end{cases}$$

with convention $F_{ii} = 0$. Matrix F is antisymmetric and potential X defined by formula (2) is arithmetic mean of the columns of F,

$$x_i = \frac{1}{n}\sum_{j=1}^{n} F_{ij}, \; i = 1, \ldots, n. \tag{3}$$

2.1.1. *Flow consistency.* To be consistent in linguistic sense means to be free of variations or contradictions. In terms of preference flow it means that for each cycle, as for this one in Figure 3, we have

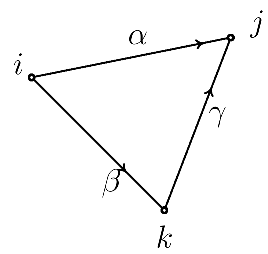

FIGURE 3. A consistent flow, $F_\beta + F_\gamma = F_\alpha$.

$$F_\beta + F_\gamma = F_\alpha. \tag{4}$$

Such a high precision is difficult to achieve especially if we are dealing with subjective preferences. In general, a flow $\mathcal{F}$ is consistent if for each cycle (not necessarily positive). $\sum_{i=1}^{k} y_i \alpha_i = 0,\ y_i = \pm 1$ in the graph we have

$$\sum_{i=1}^{k} y_i F_{\alpha_i} = 0. \tag{5}$$

Theorem 1. *A flow $\mathcal{F}$ is consistent if and only if $\mathcal{F}$ is a linear combination of the columns of the incidence matrix A of the graph, i. e. if and only if there exists a potential $X \in \mathbb{R}^n$ such that*

$$AX = \mathcal{F}. \tag{6}$$

Proof. The proof follows from orthogonal decomposition

$$R(A) \oplus N(A^\tau) = \mathbb{R}^m$$

and the fact that the space $N(A^\tau)$ of Kirchoff's flows is spanned by cycles. □

There is a notion of consistent matrix in Saaty's AHP method, (Saaty 1996), and a theorem which states that a positive reciprocal matrix $A = (a_{ij})$ is consistent if and only

$$a_{ij} a_{jk} = a_{ik}, \quad i, j, k = 1, \ldots, n. \tag{7}$$

Taking a logarithm of this relation one can recognize condition (4). In stochastic preference approach to group decision, (French 1986, p. 101), the author introduces a notion of stochastic preference p_{ab} as a probability of choosing a when offered the choice between a and b. Then, it is easy to show that if stochastic preference satisfies the consistency condition

$$\frac{p_{ab}}{p_{ba}} \cdot \frac{p_{bc}}{p_{cb}} = \frac{p_{ac}}{p_{ca}}$$

for all $a, b, c \in S$ then, it induces a weak preference order and generates an ordinal value function. If we define a flow by

$$\mathcal{F}_\alpha := \log \frac{p_{bc}}{p_{cb}}$$

then, stochastic preference is consistent if and only if the flow is consistent in the sense of (4). This shows that consistency of the flow has its counterparts in different approaches.

2.2. **Potential of incomplete graph.** Equation (6) determines a potential of the flow $\mathcal{F}$ if and only if it is consistent. If this is not the case the best we can do is to find the best approximation of $\mathcal{F}$ in the column space of incidence matrix A, which is consistent, and find its potential. This is the same as solving equation

$$A^\tau AX = A^\tau \mathcal{F}. \tag{8}$$

For the moment let us suppose that the graph is connected. In that case, the null space of incidence matrix is spanned by vector $\mathbb{1}$, with ones as its components, which implies that potential X given by (8) is unique up to a constant. If, however, we put the extra condition

$$\sum_{i=1}^{n} x_i = 0, \tag{9}$$

then, the solution of (8) and (9) is unique and we call it `normal integral` of $\mathcal{F}$.

`Laplace matrix` $A^\tau AX$ of the graph can be calculated using the formulas

$$(A^\tau A)_{jk} = \begin{cases} -1 & \text{if } j,k \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(A^\tau A)_{kk} = \text{the number of neighbors of } k.$$

Laplace matrix is symmetric and positive semi-definite with non-trivial kernel and dimension of that kernel equals the number of connected components of the graph. For complete graph, adding equation $\sum x_i = 0$ to each row of the system $A^\tau AX = A^\tau F$ we get

$$X = \frac{1}{n} \cdot A^\tau F,$$

which can be recognized as already given equation (2).

As an example let us solve the normal equation for the preference flow in Figure 2. Incidence matrix and the preference flow are

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 4 \\ 2 \\ 1 \end{bmatrix},$$

normal equation is

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} X = \begin{bmatrix} 6 \\ 2 \\ -4 \\ -4 \end{bmatrix},$$

with extra condition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} X = 0.$$

Adding this equation to each row of above system we obtain

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} X = \begin{bmatrix} 6 \\ 2 \\ -4 \\ -4 \end{bmatrix},$$

with the solution

$$X = \begin{bmatrix} 1.5 \\ 0.5 \\ -1 \\ -1 \end{bmatrix},$$

that is the same as the one we obtained earlier.

2.2.1. *Measure of inconsistency.* Measure of inconsistency $i(\mathcal{F})$ should satisfy:

$$i(\mathcal{F}) = 0 \text{ if and only if } \mathcal{F} \text{ is consistent.}$$

One such measure is given by

$$i(\mathcal{F}) = \frac{\|AX\|}{\|\mathcal{F}\|} \tag{10}$$

where X is normal integral of $\mathcal{F}$ and $\|\cdot\|$ is Euclidean norm. Measure $i(\mathcal{F})$ is the cosine of angle between the flow vector $\mathcal{F}$ and the column space of incidence matrix. It is invariant on positive affine transformations of the measurement scale in $\mathcal{F}$-space. Evidently, each new cycle in the preference graph adds a new component to $\mathcal{F}$ in $N(A^\tau)$ that is perpendicular to $R(A)$ and increases the inconsistency. Upper bound for admissible inconsistency measure is still an open question, due to the freshness of the method. Experiments suggest that the angle bellow 12 degrees is acceptable.

Another measure can be the angle itself or

$$\mu(\mathcal{F}) = \frac{1}{\sqrt{n-r}}\,\|\mathcal{F} - AX\|_2.$$

Some statistical considerations indicate that, if $\mathcal{F}$ is normally distributed, the square of $\mu(\mathcal{F})$ has $\chi^2(n-r)$ distribution, while invariant inconsistency $i(\mathcal{F})$ has Fisher's $F(n-r, n)$ distribution.

2.2.2. *Potential as ordinal value function.* A natural question arises whether X is an ordinal value function if its unimodular flow represents a weak preference relation. Generally, it is not the case and it is easy to find a counterexample. The following theorem is more precise.

Theorem 2. *If the flow $\mathcal{F}$ is unimodular and represents a weak preference relation on the set of nodes then, its normal integral is an ordinal value function and*

$$\mathcal{F}_{(b,a)} \geq 0 \Leftrightarrow X(a) \geq X(b).$$

The proof can be found in (Čaklović 2003) but we shall rewrite it here for the sake of completeness.

Proof. Let us denote by $\succcurlyeq$ weak preference relation

$$a \succcurlyeq b \Leftrightarrow \mathcal{F}_{(b,a)} \geq 0,$$

and by $\succ$ strict preference relation defined by

$$a \succ b \Leftrightarrow a \succcurlyeq b \text{ and } b \not\succcurlyeq a,$$

and by $\sim$ equivalence relation defined by

$$a \sim b \Leftrightarrow a \succcurlyeq b \text{ and } b \succcurlyeq a.$$

From the formula (2) it is easy to see that

$$n \cdot X(x) = \#\{y \in S \mid x \succ y\} - \#\{y \in S \mid y \succ x\} \tag{11}$$

and

$$n \cdot X(x) = 2V(x) - \#[x] - \#S$$

where $[x]$ denotes equivalence class of x i.e.

$$[x] := \{y \in S \mid y \sim x\},$$

and V is an ordinal value function given by (1). If $y \sim x$ then $[x] = [y]$ and $V(x) = V(y)$ which implies $X(x) = X(y)$. Let us suppose now that $x \succ y$, i.e. $x \succcurlyeq y$ i $x \not\prec y$. Then

$$\begin{aligned} n \cdot \big(X(x) - X(y)\big) &= V(x) - V(y) + V(x) - \#[x] - (V(y) - \#[y]) \\ &> V(x) - \#[x] - (V(y) - \#[y]) \\ &= \#\{z \in S \mid x \succ z\} - \#\{z \in S \mid y \succ z\}, \end{aligned}$$

using transitivity of $\succ$ and the fact that $x \succ y$ we obtain

$$X(x) - X(y) \geq 0.$$

This proves

$$x \succcurlyeq y \Rightarrow X(x) \geq X(y). \tag{12}$$

To complete this we have to prove the implication $X(x) \geq X(y) \Rightarrow x \succcurlyeq y$. Let us suppose that $X(x) \geq X(y)$ and $y \succ x$ for some $x, y \in S$. Because of (12) we conclude that $X(y) \geq X(x)$, and $y \not\prec x$ implies $X(y) > X(x)$ which contradicts the supposition and proves the implication. □

2.3. **Aggregation of flows in MCDM.** We are going now to explain aggregation procedure of individual flows to obtain `consensus flow` for several criteria or group of decision makers. If one criterion is present then, for a given preference flow $\mathcal{F}$, and incidence matrix A of the preference graph, normal integral X is given as a solution of equations

$$A^\tau AX = A^\tau \mathcal{F}, \quad \sum_{i=1}^{m} X_i = 0. \tag{13}$$

If the graph is not connected, normal integral is unique on each connected component of the graph.

If more than one criterion is present, the procedure of making a `consensus graph` $(V, \mathcal{A})$ and consensus flow $\mathcal{F}$ is the following. Each criterion C_i generates its own preference graph $(V, \mathcal{A}_i)$ and its own preference flow $\mathcal{F}_i$. Let us denote the weight of i-th criterion by w_i. First, for a given pair $\alpha = (u, v)$ of alternatives we calculate

$$F_\alpha := \sum_{\substack{i=1 \\ \pm\alpha\in\mathcal{A}_i}}^{k} w_i \mathcal{F}_i(\alpha) \tag{14}$$

where the summand $w_i\mathcal{F}_i(\alpha)$ is taken in account if and only if $\alpha \in \mathcal{A}_i$ or $-\alpha \in \mathcal{A}_i$. If this sum is non-negative, then we include α in the set $\mathcal{A}$ of arcs of consensus graph, and we put $\mathcal{F}(\alpha) := F_\alpha$. Otherwise, we define $-\alpha = (v, u)$ as an arc in $\mathcal{A}$ and $\mathcal{F}(-\alpha) := -F_\alpha$. The flow $\mathcal{F}$ becomes a non-negative flow.

2.3.1. *Hierarchical decision structure.* In hierarchical decision structure decision is made through the hierarchy from upper to lower levels until the bottom level of alternatives is reached. The highest level has one node (aim) or several nodes in group decision (group members). The sum of weights of nodes in the first and the last level should be one, which reflects the conservation of information.

Each node in the hierarchical structure, except the nodes in last level, is a parent node for its children (leaves) in some other level. Parent node can be considered as criterion for evaluation of its children. In PM the only restriction is that children of a parent should be in the same level set. The parents of a node can be from different levels sets. Restriction made by conservation law is that sum of the weights of nodes in some level set should be the sum of weights of their parents.

PM calculates the weights of nodes in some level in the following way. First, the aggregation of flows is done over the set of all parents for particular level that is not ranked yet. Consensus graph need not be connected. For each connected component $\mathcal{C}$ the normal integral $X_{\mathcal{C}}$ is a solution of the normal equation (8). The weight function on the connected component is then

$$w_{\mathcal{C}} = \alpha_{\mathcal{C}} \cdot \frac{a^{X_{\mathcal{C}}}}{\|a^{X_{\mathcal{C}}}\|_1} \tag{15}$$

where $\|\cdot\|_1$ represents l_1 norm and $\alpha_{\mathcal{C}}$ is the sum of the weights of the parents for that particular component $\mathcal{C}$. Exponential function $X \mapsto a^X$ is defined by components and a is a positive constant that can depend, roughly speaking, upon the set of parent nodes. In implementation, the value of a is controlled through a subjective parameter called `flow-norm` associated to each parent, see (Čaklović, Šego 2002) for details. In fact we fix the value $a = 2$ and after that we change Fn if necessary. We repeat the process until the bottom level is ranked.

2.3.2. *Consensus flow for decision table.* For decision table, Table 4, the consensus flow, defined on the graph with actions $a_1, \dots a_m$ as vertices and with states $\theta_1, \dots \theta_n$ as criteria, according to formula (14), is defined by

$$F_{kj} = \sum_i P(\theta_i)(v_{ki} - v_{ji}), \quad k, j = 1, \dots, m. \tag{16}$$

Here F_{kj} denotes the flow component on arc (j,k) and represents the kj component of the `flow matrix`. Note that such flow is complete and that the flow matrix exists only for complete flows. The following theorem justifies above definition by proving the equivalence of PM and expected utility

$$U_i = U(a_i) := \textstyle\sum_j P(\theta_j) v_{ij}. \tag{17}$$

		States of nature			
		θ_1	θ_2	$\cdots$	θ_n
Actions	a_1	v_{11}	v_{12}	$\cdots$	v_{1n}
	a_2	v_{21}	v_{21}	$\cdots$	v_{21}
	.	.	.	$\cdots$	.
	a_m	v_{m1}	v_{m2}	$\cdots$	v_{mn}

TABLE 4. Decision table.

Theorem 3. *Ranking over the set of alternatives given by Expected Utility is the same as ranking given by Potential Method. More precisely, if X is normal integral of the flow* (16) then $X = U$ and

$$X_k \geq X_l \iff U_k \geq U_l.$$

The proof can be found in (Čaklović, Šego 2002). If some data are missing the flow is not complete, normal integral and weights can be calculated, and utilities U_k cannot which proves that PM extends in some way the utility approach.

2.4. **Implementation and software.** Online decision making site based on PM has URL `http://decision.math.hr/english.htm`. Web interface for creating complex hierarchies has the following steps:

(1) defining project name and levels,
(2) specifying nodes for each level,
(3) specifying leaves-level for the nodes,
(4) specifying the leaves for each parent,
(5) pairwise comparison,
(6) and the last step — calculation. An useful option is 'input' which gives back the input data in ASCII form. After saving the input in the file, the preferences can be changed and the file can be uploaded for reprocessing.

Group decision making is not automatized yet, but it can be done on demand for educational purpose or research.

3. Potential and geometric mean

Some decision making tools use ratio scale in pairwise comparisons. In this case a positive reciprocal matrix A is obtained. Logarithm of A, taken by components, is an antisymmetric matrix F representing the logarithmic flow

$$\mathcal{F}_\alpha = F_{ij} = \log_a a_{ij},$$

where $\alpha = (j, i)$. Using formula (3), the normal integral X of $\mathcal{F}$ can be expressed in terms of matrix A

$$x_i = \frac{1}{n}\sum_j F_{ij} = \frac{1}{n}\sum_j \log_a a_{ij} = \log_a \Big(\prod_j a_{ij}\Big)^{\frac{1}{n}},$$

and un-normalized weight w_i, using (15), can be written as

$$w_i = \Big(\prod_j a_{ij}\Big)^{\frac{1}{n}}, \; i = 1, \ldots, n.$$

Another measure of inconsistency of $\mathcal{F}$ is evidently given by

$$\begin{aligned}\mu(F) &= \sum_{\alpha\in\mathcal{A}} \left|F_\alpha - \big(AX\big)_\alpha\right| \\ &= \sum_{\alpha\in\mathcal{A}} \left|\log_a a_{ij} - \log_a w_i + \log_a w_j\right|,\end{aligned}$$

recalling that $a_{ij} = a_{ji}^{-1}$,

$$= \sum_{\substack{\alpha\in\mathcal{A}\\ a_{ij}\frac{w_j}{w_i}\geq 1}} \log_a\left(a_{ij}\frac{w_j}{w_i}\right) - \sum_{\substack{\alpha\in\mathcal{A}\\ a_{ij}\frac{w_j}{w_i}<1}} \log_a\left(a_{ij}\frac{w_j}{w_i}\right)$$

$$= \log_a \frac{\displaystyle\prod_{\substack{a_{ij}\geq 1\\ a_{ij}\frac{w_j}{w_i}\geq 1}} a_{ij}\frac{w_j}{w_i}}{\displaystyle\prod_{\substack{a_{ij}\geq 1\\ a_{ij}\frac{w_j}{w_i}<1}} a_{ij}\frac{w_j}{w_i}}.$$

Above expression suggests to define `geometric-mean inconsistency` gmi(A) of matrix A by

$$\mathrm{gmi}(A) = \frac{\displaystyle\prod_{\substack{a_{ij}\geq 1\\ a_{ij}\frac{w_j}{w_i}\geq 1}} a_{ij}\frac{w_j}{w_i}}{\displaystyle\prod_{\substack{a_{ij}\geq 1\\ a_{ij}\frac{w_j}{w_i}<1}} a_{ij}\frac{w_j}{w_i}} = \frac{\displaystyle\prod_{\substack{a_{ij}\geq 1\\ a_{ij}\frac{w_j}{w_i}\geq 1}} a_{ij}\left(\prod_k \frac{a_{jk}}{a_{ik}}\right)^{\frac{1}{n}}}{\displaystyle\prod_{\substack{a_{ij}\geq 1\\ a_{ij}\frac{w_j}{w_i}<1}} a_{ij}\left(\prod_k \frac{a_{jk}}{a_{ik}}\right)^{\frac{1}{n}}} \tag{18}$$

Formula for gmi(A) looks complicated, it is easier to write a computer program to calculate gmi(A) than to write it.

Evidently, $\mathrm{gmi}(A) \geq 1$ and $\mathrm{gmi}(A) = 1$ if and only if A is consistent reciprocal matrix in the sense of (7). Moreover, changing the unit measure we do not influence the gmi(A).

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Attribute	**Kawasaki**	**Yamaha**	**Honda**	**BMW**
Price, pounds (9)	6499	5199	6199	8220
Displacement, cc (5)	1052	1188	998	987
Range, miles (7)	175	160	170	200
Top speed, mph (7)	160	155	160	145
Engine (14)				
Responsiveness, (0.2)	(E, 0.8)	(G, 0.3) (E, 0.6)	(G, 1.0)	(I, 1.0)
Fuel consumption, mpg (0.4)	(32, 0.25) (36, 0.25) (41, 0.25) (43, 0.25)	(28, 0.25) (34, 0.25) (38, 0.25)	(31, 0.25) (35, 0.25) (39, 0.25) (43, 0.25)	(35, 0.25) (39, 0.25) (46, 0.25) (48, 0.25)
Quietness (0.1)	(I, 0.5) (A, 0.5)	(A, 1.0)	(G, 0.5) (E, 0.3)	
Vibration (0.1)	(G, 1.0)	(I, 1.0)	(G, 0.5) (E, 0.5)	(P, 1.0)
Starting (0.2)	(G, 1.0)	(A, 0.6) (G, 0.3)	(G, 1.0)	(A, 1.0)
Operation (7)				
Handling (0.5)				
Steering (0.3)	(E, 0.9)	(G, 1.0)	(A, 1.0)	(A, 0.6)
Bumpy bends (0.1)	(A, 0.5) (G, 0.5)	(G, 1.0)	(G, 0.8) (E, 0.1)	(P, 0.5) (I, 0.5)
Manoeuvrability (0.4)	(A, 1.0)	(E, 0.9)		(P, 1.0)
Top speed stability (0.3)	(E, 1.0)	(G, 1.0)	(G, 1.0)	(G, 0.6) (E, 0.4)
Transmission (0.167)				
Clutch operation (0.5)	(A, 0.8)	(G, 1.0)	(E, 0.85)	(I, 0.2) (A, 0.8)
Gearbox operation (0.5)	(A, 0.5) (G, 0.5)	(I, 0.5) (A, 0.5)	(E, 1.0)	(P, 1.0)
Brakes (0.333)				
Stopping power (0.4)	(G, 1.0)	(A, 0.3) (G, 0.6)	(G, 1.0)	
Braking stability (0.3)	(G, 0.5) (E, 0.5)	(G, 1.0)	(A, 0.5) (G, 0.5)	(E, 1.0)
Feel at control (0.3)	(P, 1.0)	(G, 0.5) (E, 0.5)	(G, 1.0)	(G, 0.5) (E, 0.5)
General (14)				
Quality of finish (0.4)	(P, 0.5) (I, 0.5)	(G, 1.0)	(E, 1.0)	(G, 0.5) (E, 0.5)
Seat comfort (0.3)	(G, 1.0)	(G, 0.5) (E, 0.5)	(G, 1.0)	(E, 1.0)
Headlight (0.1)	(G, 1.0)	(A, 1.0)	(E, 1.0)	(G, 0.5) (E, 0.5)
Mirrors (0.1)	(A, 0.5) (G, 0.5)	(G, 0.5) (E, 0.5)	(E, 1.0)	(G, 1.0)
Horn (0.1)		(G, 1.0)	(G, 0.5) (E, 0.5)	(E, 1.0)

TABLE 5. Data for motorcycle selection problem.