# PART I

## CHAPTER P
## Preparation for Calculus

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# Preparation for Calculus

## Section P.1 Graphs and Models

Solutions to Odd-Numbered Exercises

**1.** $y = -\frac{1}{2}x + 2$

$x$-intercept: $(4, 0)$

$y$-intercept: $(0, 2)$

Matches graph (b)

**3.** $y = 4 - x^2$

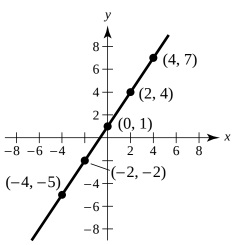
$x$-intercepts: $(2, 0), (-2, 0)$

$y$-intercept: $(0, 4)$

Matches graph (a)

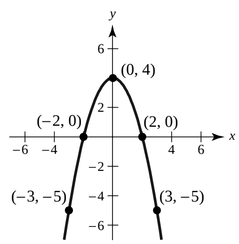
**5.** $y = \frac{3}{2}x + 1$

| $x$ | $-4$ | $-2$ | $0$ | $2$ | $4$ |
|---|---|---|---|---|---|
| $y$ | $-5$ | $-2$ | $1$ | $4$ | $7$ |



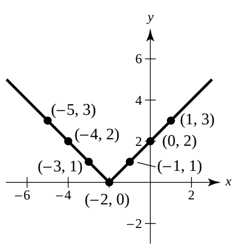

**7.** $y = 4 - x^2$

| $x$ | $-3$ | $-2$ | $0$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $y$ | $-5$ | $0$ | $4$ | $0$ | $-5$ |



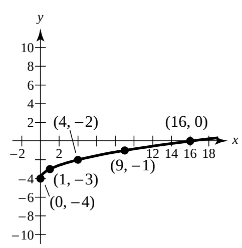

**9.** $y = |x + 2|$

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|---|---|---|---|
| $y$ | $3$ | $2$ | $1$ | $0$ | $1$ | $2$ | $3$ |




**11.** $y = \sqrt{x} - 4$

| $x$ | $0$ | $1$ | $4$ | $9$ | $16$ |
|---|---|---|---|---|---|
| $y$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |

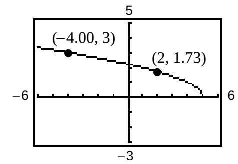


**13.**

Xmin = -3
Xmax = 5
Xscl = 1
Ymin = -3
Ymax = 5
Yscl = 1

Note that $y = 4$ when $x = 0$.

**15.**




(a) $(2, y) = (2, 1.73)$ $\quad \left(y = \sqrt{5 - 2} = \sqrt{3} \approx 1.73\right)$

(b) $(x, 3) = (-4, 3)$ $\quad \left(3 = \sqrt{5 - (-4)}\right)$

**17.** $y = x^2 + x - 2$

$y$-intercept: $y = 0^2 + 0 - 2$

$y = -2;\ (0, -2)$

$x$-intercepts: $0 = x^2 + x - 2$

$0 = (x + 2)(x - 1)$

$x = -2, 1;\ (-2, 0), (1, 0)$

**19.** $y = x^2\sqrt{25 - x^2}$

$y$-intercept: $y = 0^2\sqrt{25 - 0^2}$

$y = 0;\ (0, 0)$

$x$-intercepts: $0 = x^2\sqrt{25 - x^2}$

$0 = x^2\sqrt{(5 - x)(5 + x)}$

$x = 0, \pm 5;\ (0, 0);\ (\pm 5, 0)$

**21.** $y = \dfrac{3\left(2 - \sqrt{x}\right)}{x}$

$y$-intercept: None. $x$ cannot equal 0.

$x$-intercepts: $0 = \dfrac{3\left(2 - \sqrt{x}\right)}{x}$

$0 = 2 - \sqrt{x}$

$x = 4;\ (4, 0)$

**23.** $x^2y - x^2 + 4y = 0$

$y$-intercept:

$0^2(y) - 0^2 + 4y = 0$

$y = 0;\ (0, 0)$

$x$-intercept:

$x^2(0) - x^2 + 4(0) = 0$

$x = 0;\ (0, 0)$

**25.** Symmetric with respect to the $y$-axis since

$y = (-x)^2 - 2 = x^2 - 2.$

**27.** Symmetric with respect to the $x$-axis since

$(-y)^2 = y^2 = x^3 - 4x.$

**29.** Symmetric with respect to the origin since

$(-x)(-y) = xy = 4.$

**31.** $y = 4 - \sqrt{x + 3}$

No symmetry with respect to either axis or the origin.

**33.** Symmetric with respect to the origin since

$$-y = \frac{-x}{(-x)^2 + 1}$$

$$y = \frac{x}{x^2 + 1}.$$

**35.** $y = |x^3 + x|$ is symmetric with respect to the $y$-axis

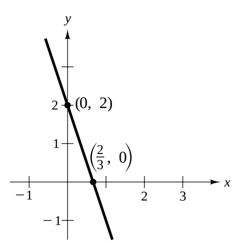
since $y = |(-x)^3 + (-x)| = |-(x^3 + x)| = |x^3 + x|$.

**37.** $y = -3x + 2$

Intercepts:

$\left(\frac{2}{3}, 0\right), (0, 2)$

Symmetry: none

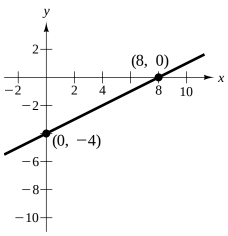


**39.** $y = \dfrac{x}{2} - 4$

Intercepts:

$(8, 0), (0, -4)$

Symmetry: none

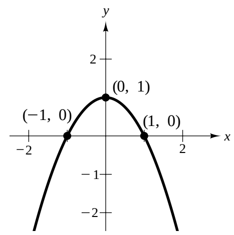



**41.** $y = 1 - x^2$

Intercepts:

$(1, 0), (-1, 0), (0, 1)$

Symmetry: $y$-axis

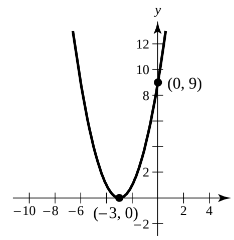



**43.** $y = (x + 3)^2$

Intercepts:

$(-3, 0), (0, 9)$

Symmetry: none

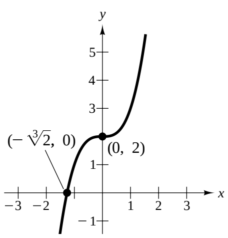



**45.** $y = x^3 + 2$

Intercepts:

$\left(-\sqrt[3]{2}, 0\right), (0, 2)$

Symmetry: none



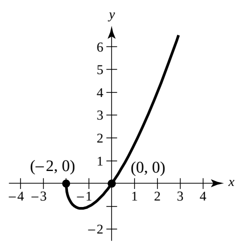

**47.** $y = x\sqrt{x + 2}$

Intercepts:

$(0, 0), (-2, 0)$

Symmetry: none

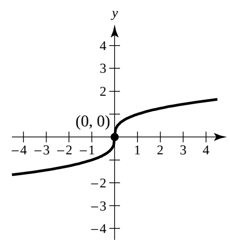
Domain: $x \geq -2$




**49.** $x = y^3$

Intercepts: $(0, 0)$

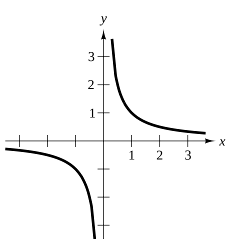
Symmetry: origin




**51.** $y = \dfrac{1}{x}$

Intercepts: none

Symmetry: origin

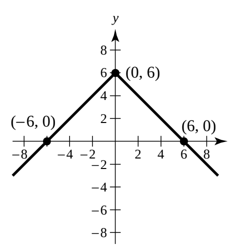


**53.** $y = 6 - |x|$

Intercepts:

$(0, 6), (-6, 0), (6, 0)$

Symmetry: $y$-axis




**55.** $y^2 - x = 9$

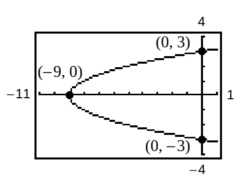
$y^2 = x + 9$

$y = \pm\sqrt{x + 9}$

Intercepts:

$(0, 3), (0, -3), (-9, 0)$

Symmetry: $x$-axis




**57.** $x + 3y^2 = 6$

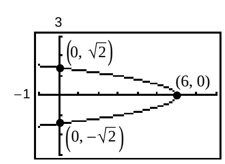
$3y^2 = 6 - x$

$y = \pm\sqrt{2 - \dfrac{x}{3}}$

Intercepts:

$(6, 0), \left(0, \sqrt{2}\right), \left(0, -\sqrt{2}\right)$

Symmetry: $x$-axis



**59.** $y = (x + 2)(x - 4)(x - 6)$ (other answers possible)

**61.** Some possible equations:

$$y = x$$
$$y = x^3$$
$$y = 3x^3 - x$$
$$y = \sqrt[3]{x}$$

**63.** $x + y = 2 \Rightarrow y = 2 - x$

$2x - y = 1 \Rightarrow y = 2x - 1$

$$2 - x = 2x - 1$$
$$3 = 3x$$
$$1 = x$$

The corresponding $y$-value is $y = 1$.

Point of intersection: $(1, 1)$

**65.** $x + y = 7 \Rightarrow y = 7 - x$

$3x - 2y = 11 \Rightarrow y = \dfrac{3x - 11}{2}$

$$7 - x = \frac{3x - 11}{2}$$
$$14 - 2x = 3x - 11$$
$$-5x = -25$$
$$x = 5$$

The corresponding $y$-value is $y = 2$.

Point of intersection: $(5, 2)$

**67.** $x^2 + y = 6 \Rightarrow y = 6 - x^2$

$x + y = 4 \Rightarrow y = 4 - x$

$$6 - x^2 = 4 - x$$
$$0 = x^2 - x - 2$$
$$0 = (x - 2)(x + 1)$$
$$x = 2, -1$$

The corresponding $y$-values are $y = 2$ (for $x = 2$) and $y = 5$ (for $x = -1$).

Points of intersection: $(2, 2), (-1, 5)$

**69.** $x^2 + y^2 = 5 \Rightarrow y^2 = 5 - x^2$

$x - y = 1 \Rightarrow y = x - 1$

$$5 - x^2 = (x - 1)^2$$
$$5 - x^2 = x^2 - 2x + 1$$
$$0 = 2x^2 - 2x - 4 = 2(x + 1)(x - 2)$$
$$x = -1 \text{ or } x = 2$$

The corresponding $y$-values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

**71.** $y = x^3$

$y = x$

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x + 1)(x - 1) = 0$$
$$x = 0, x = -1, \text{ or } x = 1$$

The corresponding $y$-values are $y = 0$, $y = -1$, and $y = 1$.

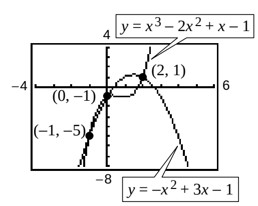
Points of intersection: $(0, 0), (-1, -1), (1, 1)$

**73.** $y = x^3 - 2x^2 + x - 1$

$y = -x^2 + 3x - 1$

$$x^3 - 2x^2 + x - 1 = -x^2 + 3x - 1$$
$$x^3 - x^2 - 2x = 0$$
$$x(x - 2)(x + 1) = 0$$
$$x = -1, 0, 2$$

$(-1, -5), (0, -1), (2, 1)$



**75.** $5.5\sqrt{x} + 10{,}000 = 3.29x$

$$\left(5.5\sqrt{x}\right)^2 = (3.29x - 10{,}000)^2$$
$$30.25x = 10.8241x^2 - 65{,}800x + 100{,}000{,}000$$
$$0 = 10.8241x^2 - 65{,}830.25x + 100{,}000{,}000 \qquad \text{Use the Quadratic Formula.}$$
$$x \approx 3133 \text{ units}$$

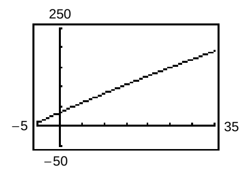
The other root, $x \approx 2949$, does not satisfy the equation $R = C$.

This problem can also be solved by using a graphing utility and finding the intersection of the graphs of $C$ and $R$.

**77.** (a) Using a graphing utility, you obtain

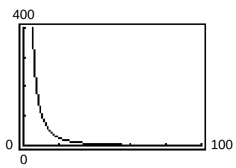
$$y = -0.0153t^2 + 4.9971t + 34.9405$$

(b)




(c) For the year 2004, $t = 34$ and $y \approx 187.2$ CPI.

**79.**




If the diameter is doubled, the resistance is changed by approximately a factor of $(1/4)$. For instance, $y(20) \approx 26.555$ and $y(40) \approx 6.36125$.

**81.** False; $x$-axis symmetry means that if $(1, -2)$ is on the graph, then $(1, 2)$ is also on the graph.

**83.** True; the $x$-intercepts are

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right).$$

**85.** Distance to the origin $= K \times$ Distance to $(2, 0)$

$$\sqrt{x^2 + y^2} = K\sqrt{(x-2)^2 + y^2}, K \neq 1$$
$$x^2 + y^2 = K^2(x^2 - 4x + 4 + y^2)$$

$(1 - K^2)x^2 + (1 - K^2)y^2 + 4K^2x - 4K^2 = 0$

**Note:** This is the equation of a circle!

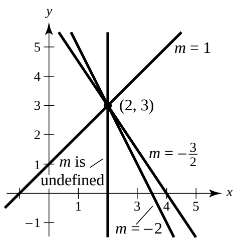
## Section P.2 Linear Models and Rates of Change

**1.** $m = 1$

**3.** $m = 0$

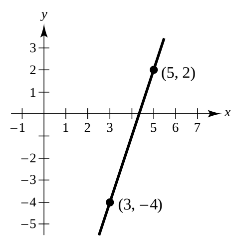
**5.** $m = -12$

**7.**




**9.** $m = \dfrac{2 - (-4)}{5 - 3}$

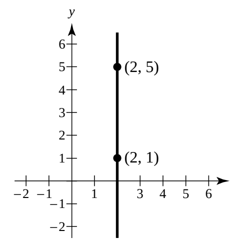
$= \dfrac{6}{2} = 3$




**11.** $m = \dfrac{5 - 1}{2 - 2}$

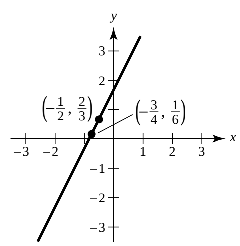
$= \dfrac{4}{0}$

undefined




**13.** $m = \dfrac{2/3 - 1/6}{-1/2 - (-3/4)}$

$= \dfrac{1/2}{1/4} = 2$




**15.** Since the slope is 0, the line is horizontal and its equation is $y = 1$. Therefore, three additional points are $(0, 1)$, $(1, 1)$, and $(3, 1)$.

**17.** The equation of this line is

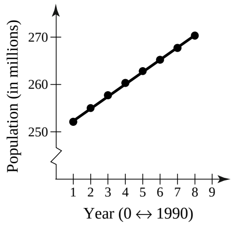
$$y - 7 = -3(x - 1)$$

$$y = -3x + 10.$$

Therefore, three additional points are $(0, 10)$, $(2, 4)$, and $(3, 1)$.

**19.** Given a line $L$, you can use any two distinct points to calculate its slope. Since a line is straight, the ratio of the change in $y$-values to the change in $x$-values will always be the same. See Section P.2 Exercise 93 for a proof.

**21.** (a)




(b) The slopes of the line segments are

$$\frac{255.0 - 252.1}{2 - 1} = 2.9$$
$$\frac{257.7 - 255.0}{3 - 2} = 2.7$$
$$\frac{260.3 - 257.7}{4 - 3} = 2.6$$
$$\frac{262.8 - 260.3}{5 - 4} = 2.5$$
$$\frac{265.2 - 262.8}{6 - 5} = 2.4$$
$$\frac{267.7 - 265.2}{7 - 6} = 2.5$$
$$\frac{270.3 - 267.7}{8 - 7} = 2.6$$

The population increased most rapidly from 1991 to 1992.

$(m = 2.9)$

**23.** $x + 5y = 20$

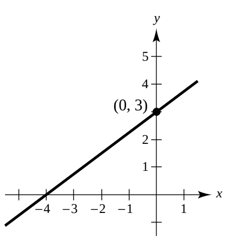
$$y = -\tfrac{1}{5}x + 4$$

Therefore, the slope is $m = -\frac{1}{5}$ and the $y$-intercept is $(0, 4)$.

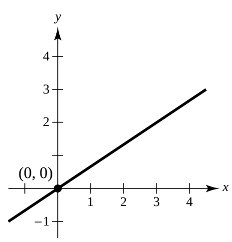
**25.** $x = 4$

The line is vertical. Therefore, the slope is undefined and there is no $y$-intercept.

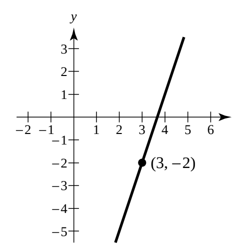
**27.**
$$y = \tfrac{3}{4}x + 3$$
$$4y = 3x + 12$$
$$0 = 3x - 4y + 12$$




**29.**
$$y = \tfrac{2}{3}x$$
$$3y = 2x$$
$$2x - 3y = 0$$

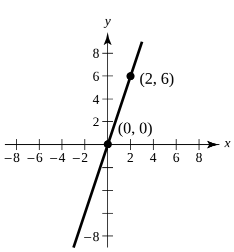



**31.**
$$y + 2 = 3(x - 3)$$
$$y + 2 = 3x - 9$$
$$y = 3x - 11$$
$$y - 3x + 11 = 0$$



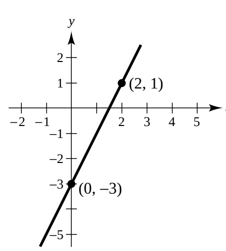

**33.** $m = \dfrac{6 - 0}{2 - 0} = 3$

$$y - 0 = 3(x - 0)$$
$$y = 3x$$



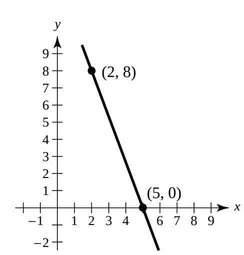

**35.** $m = \dfrac{1 - (-3)}{2 - 0} = 2$

$$y - 1 = 2(x - 2)$$
$$y - 1 = 2x - 4$$
$$0 = 2x - y - 3$$



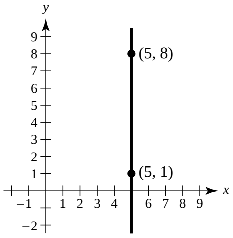

**37.** $m = \dfrac{8 - 0}{2 - 5} = -\dfrac{8}{3}$

$$y - 0 = -\frac{8}{3}(x - 5)$$
$$y = -\frac{8}{3}x + \frac{40}{3}$$
$$3y + 8x - 40 = 0$$



**39.** $m = \dfrac{8 - 1}{5 - 5}$ Undefined.

Vertical line $x = 5$

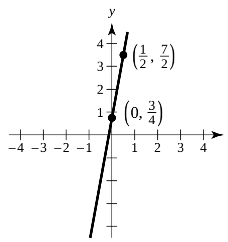



**41.** $m = \dfrac{7/2 - 3/4}{1/2 - 0} = \dfrac{11/4}{1/2} = \dfrac{11}{2}$

$$y - \frac{3}{4} = \frac{11}{2}(x - 0)$$

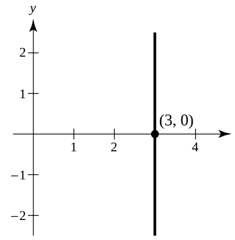
$$y = \frac{11}{2}x + \frac{3}{4}$$

$$22x - 4y + 3 = 0$$




**43.** $x = 3$

$$x - 3 = 0$$




**45.** $\dfrac{x}{2} + \dfrac{y}{3} = 1$

$$3x + 2y - 6 = 0$$

**47.** $\dfrac{x}{a} + \dfrac{y}{a} = 1$

$$\frac{1}{a} + \frac{2}{a} = 1$$

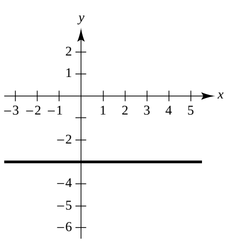
$$\frac{3}{a} = 1$$

$$a = 3 \Longrightarrow x + y = 3$$

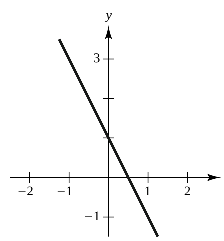
$$x + y - 3 = 0$$

**49.** $y = -3$

$$y + 3 = 0$$



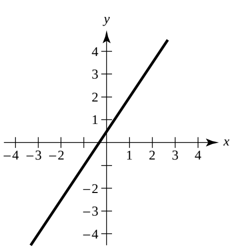
**51.** $y = -2x + 1$



**53.** $y - 2 = \frac{3}{2}(x - 1)$

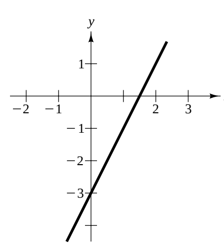
$$y = \tfrac{3}{2}x + \tfrac{1}{2}$$

$$2y - 3x - 1 = 0$$

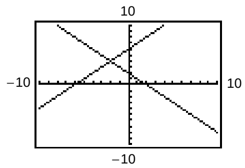


**55.** $2x - y - 3 = 0$

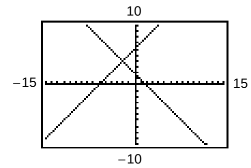
$$y = 2x - 3$$



**57.**




The lines do not appear perpendicular.




The lines appear perpendicular.

The lines are perpendicular because their slopes 1 and $-1$ are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.

**59.** $4x - 2y = 3$

$$y = 2x - \tfrac{3}{2}$$

$$m = 2$$

(a) $$y - 1 = 2(x - 2)$$
$$y - 1 = 2x - 4$$
$$2x - y - 3 = 0$$

(b) $$y - 1 = -\tfrac{1}{2}(x - 2)$$
$$2y - 2 = -x + 2$$
$$x + 2y - 4 = 0$$

**61.** $5x - 3y = 0$

$$y = \tfrac{5}{3}x$$

$$m = \tfrac{5}{3}$$

(a) $$y - \tfrac{7}{8} = \tfrac{5}{3}\left(x - \tfrac{3}{4}\right)$$
$$24y - 21 = 40x - 30$$
$$24y - 40x + 9 = 0$$

(b) $$y - \tfrac{7}{8} = -\tfrac{3}{5}\left(x - \tfrac{3}{4}\right)$$
$$40y - 35 = -24x + 18$$
$$40y + 24x - 53 = 0$$

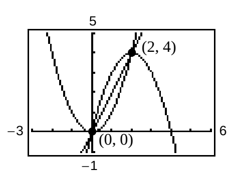
**63.** (a) $x = 2 \Longrightarrow x - 2 = 0$

(b) $y = 5 \Longrightarrow y - 5 = 0$

**65.** The slope is 125. Hence, $V = 125(t - 1) + 2540$
$$= 125t + 2415$$

**67.** The slope is $-2000$. Hence, $V = -2000(t - 1) + 20{,}400$
$$= -2000t + 22{,}400$$

**69.**




You can use the graphing utility to determine that the points of intersection are $(0, 0)$ and $(2, 4)$. Analytically,

$$x^2 = 4x - x^2$$
$$2x^2 - 4x = 0$$
$$2x(x - 2) = 0$$
$$x = 0 \Longrightarrow y = 0 \Longrightarrow (0, 0)$$
$$x = 2 \Longrightarrow y = 4 \Longrightarrow (2, 4).$$

The slope of the line joining $(0, 0)$ and $(2, 4)$ is $m = (4 - 0)/(2 - 0) = 2$. Hence, an equation of the line is

$$y - 0 = 2(x - 0)$$
$$y = 2x.$$

**71.** $m_1 = \dfrac{1 - 0}{-2 - (-1)} = -1$

$m_2 = \dfrac{-2 - 0}{2 - (-1)} = -\dfrac{2}{3}$

$m_1 \neq m_2$

The points are not collinear.

**73.** Equations of perpendicular bisectors:

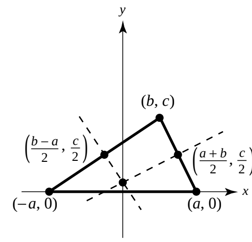
$$y - \frac{c}{2} = \frac{a - b}{c}\left(x - \frac{a + b}{2}\right)$$

$$y - \frac{c}{2} = \frac{a + b}{-c}\left(x - \frac{b - a}{2}\right)$$

Letting $x = 0$ in either equation gives the point of intersection:

$$\left(0, \frac{-a^2 + b^2 + c^2}{2c}\right).$$

This point lies on the third perpendicular bisector, $x = 0$.




**75.** Equations of altitudes:

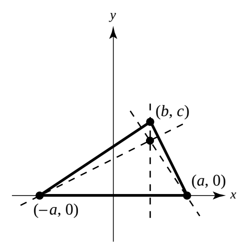
$$y = \frac{a - b}{c}(x + a)$$

$$x = b$$

$$y = -\frac{a + b}{c}(x - a)$$

Solving simultaneously, the point of intersection is

$$\left(b, \frac{a^2 - b^2}{c}\right).$$




**77.** Find the equation of the line through the points $(0, 32)$ and $(100, 212)$.

$$m = \tfrac{180}{100} = \tfrac{9}{5}$$

$$F - 32 = \tfrac{9}{5}(C - 0)$$

$$F = \tfrac{9}{5}C + 32$$

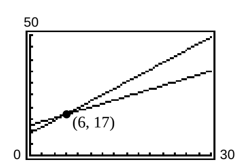
$$5F - 9C - 160 = 0$$

For $F = 72°$, $C \approx 22.2°$.

**79.** (a) $W_1 = 0.75x + 12.50$

$W_2 = 1.30x + 9.20$

(b)




Using a graphing utility, the point of intersection is approximately $(6, 17)$. Analytically,

$$0.75x + 12.50 = 1.30x + 9.20$$

$$3.3 = 0.55x \Longrightarrow x = 6$$

$$y = 0.75(6) + 12.50 = 17.$$

(c) Both jobs pay \$17 per hour if 6 units are produced. For someone who can produce more than 6 units per hour, the second offer would pay more. For a worker who produces less than 6 units per hour, the first offer pays more.

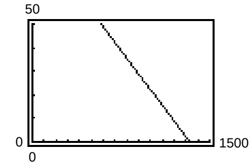
**81.** (a) Two points are $(50, 580)$ and $(47, 625)$. The slope is

$$m = \frac{625 - 580}{47 - 50} = -15.$$

$$p - 580 = -15(x - 50)$$

$$p = -15x + 750 + 580 = -15x + 1330$$

$$\text{or } x = \tfrac{1}{15}(1330 - p)$$

(b) 


If $p = 655$, $x = \frac{1}{15}(1330 - 655) = 45$ units.

(c) If $p = 595$, $x = \frac{1}{15}(1330 - 595) = 49$ units.

**83.** $4x + 3y - 10 = 0 \Longrightarrow d = \dfrac{|4(0) + 3(0) - 10|}{\sqrt{4^2 + 3^2}} = \dfrac{10}{5} = 2$

**85.** $x - y - 2 = 0 \Longrightarrow d = \dfrac{|1(-2) + (-1)(1) - 2|}{\sqrt{1^2 + 1^2}} = \dfrac{5}{\sqrt{2}} = \dfrac{5\sqrt{2}}{2}$

**87.** A point on the line $x + y = 1$ is $(0, 1)$. The distance from the point $(0, 1)$ to $x + y - 5 = 0$ is

$$d = \frac{|1(0) + 1(1) - 5|}{\sqrt{1^2 + 1^2}} = \frac{|1 - 5|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

**89.** If $A = 0$, then $By + C = 0$ is the horizontal line $y = -C/B$. The distance to $(x_1, y_1)$ is

$$d = \left|y_1 - \left(\frac{-C}{B}\right)\right| = \frac{|By_1 + C|}{|B|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

If $B = 0$, then $Ax + C = 0$ is the vertical line $x = -C/A$. The distance to $(x_1, y_1)$ is

$$d = \left|x_1 - \left(\frac{-C}{A}\right)\right| = \frac{|Ax_1 + C|}{|A|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

(Note that A and B cannot both be zero.)

The slope of the line $Ax + By + C = 0$ is $-A/B$. The equation of the line through $(x_1, y_1)$ perpendicular to $Ax + By + C = 0$ is:

$$y - y_1 = \frac{B}{A}(x - x_1)$$

$$Ay - Ay_1 = Bx - Bx_1$$

$$Bx_1 - Ay_1 = Bx - Ay$$

The point of intersection of these two lines is:

$$\begin{aligned}
Ax + By = -C \quad &\Longrightarrow \quad A^2x + ABy = -AC && (1)\\
Bx - Ay = Bx_1 - Ay_1 \quad &\Longrightarrow \quad \underline{B^2x - ABy} = \underline{B^2x_1 - ABy_1} && (2)\\
&(A^2 + B^2)x = -AC + B^2x_1 - ABy_1 && \text{(By adding equations (1) and (2))}\\
&x = \frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2}
\end{aligned}$$

$$\begin{aligned}
Ax + By = -C \quad &\Longrightarrow \quad ABx + B^2y = -BC && (3)\\
Bx - Ay = Bx_1 - Ay_1 \quad &\Longrightarrow \quad \underline{-AB_x + A^2y} = \underline{-ABx_1 + A^2y_1} && (4)\\
&(A^2 + B^2)y = -BC - ABx_1 + A^2y_1 && \text{(By adding equations (3) and (4))}\\
&y = \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}
\end{aligned}$$

**—CONTINUED—**

**89. —CONTINUED—**

$$\left(\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2}, \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}\right) \text{ point of intersection}$$

The distance between $(x_1, y_1)$ and this point gives us the distance between $(x_1, y_1)$ and the line $Ax + By + C = 0$.

$$\begin{aligned} d &= \sqrt{\left[\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2} - x_1\right]^2 + \left[\frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} - y_1\right]^2} \\ &= \sqrt{\left[\frac{-AC - ABy_1 - A^2x_1}{A^2 + B^2}\right]^2 + \left[\frac{-BC - ABx_1 - B^2y_1}{A^2 + B^2}\right]^2} \\ &= \sqrt{\left[\frac{-A(C + By_1 + Ax_1)}{A^2 + B^2}\right]^2 + \left[\frac{-B(C + Ax_1 + By_1)}{A^2 + B^2}\right]^2} \\ &= \sqrt{\frac{(A^2 + B^2)(C + Ax_1 + By_1)^2}{(A^2 + B^2)^2}} \\ &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \end{aligned}$$

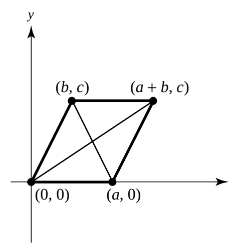
**91.** For simplicity, let the vertices of the rhombus be $(0, 0)$, $(a, 0)$, $(b, c)$, and $(a + b, c)$, as shown in the figure. The slopes of the diagonals are then

$$m_1 = \frac{c}{a + b} \text{ and } m_2 = \frac{c}{b - a}.$$

Since the sides of the Rhombus are equal, $a^2 = b^2 + c^2$, and we have

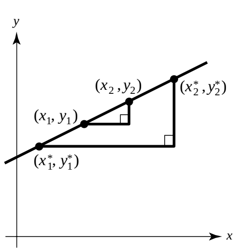
$$m_1 m_2 = \frac{c}{a + b} \cdot \frac{c}{b - a} = \frac{c^2}{b^2 - a^2} = \frac{c^2}{-c^2} = -1.$$

Therefore, the diagonals are perpendicular.




**93.** Consider the figure below in which the four points are collinear. Since the triangles are similar, the result immediately follows.

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$$




**95.** True.

$$ax + by = c_1 \Longrightarrow y = -\frac{a}{b}x + \frac{c_1}{b} \Longrightarrow m_1 = -\frac{a}{b}$$

$$bx - ay = c_2 \Longrightarrow y = \frac{b}{a}x - \frac{c_2}{a} \quad \Longrightarrow m_2 = \frac{b}{a}$$

$$m_2 = -\frac{1}{m_1}$$

## Section P.3 Functions and Their Graphs

**1.** (a) $f(0) = 2(0) - 3 = -3$

(b) $f(-3) = 2(-3) - 3 = -9$

(c) $f(b) = 2b - 3$

(d) $f(x - 1) = 2(x - 1) - 3 = 2x - 5$

**3.** (a) $g(0) = 3 - 0^2 = 3$

(b) $g(\sqrt{3}) = 3 - (\sqrt{3})^2 = 3 - 3 = 0$

(c) $g(-2) = 3 - (-2)^2 = 3 - 4 = -1$

(d) $g(t - 1) = 3 - (t - 1)^2 = -t^2 + 2t + 2$

**5.** (a) $f(0) = \cos(2(0)) = \cos 0 = 1$

(b) $f\left(-\frac{\pi}{4}\right) = \cos\left(2\left(-\frac{\pi}{4}\right)\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

(c) $f\left(\frac{\pi}{3}\right) = \cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$

**7.** $$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2, \Delta x \neq 0$$

**9.** $$\frac{f(x) - f(2)}{x - 2} = \frac{(1/\sqrt{x - 1} - 1)}{x - 2}$$

$$= \frac{1 - \sqrt{x - 1}}{(x - 2)\sqrt{x - 1}} \cdot \frac{1 + \sqrt{x - 1}}{1 + \sqrt{x - 1}} = \frac{2 - x}{(x - 2)\sqrt{x - 1}(1 + \sqrt{x - 1})} = \frac{-1}{\sqrt{x - 1}(1 + \sqrt{x - 1})}, x \neq 2$$

**11.** $h(x) = -\sqrt{x + 3}$

Domain: $x + 3 \geq 0 \Longrightarrow [-3, \infty)$

Range: $(-\infty, 0]$

**13.** $f(t) = \sec\frac{\pi t}{4}$

$\frac{\pi t}{4} \neq \frac{(2k + 1)\pi}{2} \Longrightarrow t \neq 4k + 2$

Domain: all $t \neq 4k + 2$, $k$ an integer

Range: $(-\infty, -1], [1, \infty)$

**15.** $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0), (0, \infty)$

Range: $(-\infty, 0), (0, \infty)$

**17.** $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1) = 2(-1) + 1 = -1$

(b) $f(0) = 2(0) + 2 = 2$

(c) $f(2) = 2(2) + 2 = 6$

(d) $f(t^2 + 1) = 2(t^2 + 1) = 2t^2 + 4$

(Note: $t^2 + 1 \geq 0$ for all $t$)

Domain: $(-\infty, \infty)$

Range: $(-\infty, 1), [2, \infty)$

**19.** $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$

(a) $f(-3) = |-3| + 1 = 4$

(b) $f(1) = -1 + 1 = 0$

(c) $f(3) = -3 + 1 = -2$

(d) $f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2$

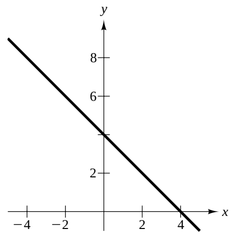
Domain: $(-\infty, \infty)$

Range: $(-\infty, 0] \cup [1, \infty)$

**21.** $f(x) = 4 - x$

Domain: $(-\infty, \infty)$

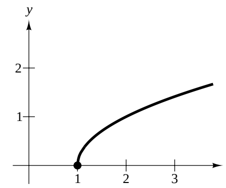
Range: $(-\infty, \infty)$



**23.** $h(x) = \sqrt{x - 1}$

Domain: $[1, \infty)$

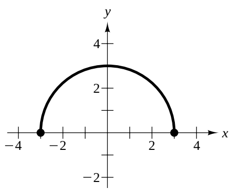
Range: $[0, \infty)$



**25.** $f(x) = \sqrt{9 - x^2}$

Domain: $[-3, 3]$

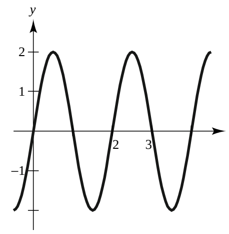
Range: $[0, 3]$



**27.** $g(t) = 2 \sin \pi t$

Domain: $(-\infty, \infty)$

Range: $[-2, 2]$



**29.** $x - y^2 = 0 \Rightarrow y = \pm\sqrt{x}$

$y$ is not a function of $x$. Some vertical lines intersect the graph twice.

**31.** $y$ is a function of $x$. Vertical lines intersect the graph at most once.

**33.** $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

$y$ is not a function of $x$ since there are two values of $y$ for some $x$.

**35.** $y^2 = x^2 - 1 \Rightarrow y = \pm\sqrt{x^2 - 1}$

$y$ is not a function of $x$ since there are two values of $y$ for some $x$.

**37.** $f(x) = |x| + |x - 2|$

If $x < 0$, then $f(x) = -x - (x - 2) = -2x + 2 = 2(1 - x)$.

If $0 \le x < 2$, then $f(x) = x - (x - 2) = 2$.

If $x \ge 2$, then $f(x) = x + (x - 2) = 2x - 2 = 2(x - 1)$.

Thus,

$$f(x) = \begin{cases} 2(1 - x), & x < 0 \\ 2, & 0 \le x < 2. \\ 2(x - 1), & x \ge 2. \end{cases}$$

**39.** The function is $g(x) = cx^2$. Since $(1, -2)$ satisfies the equation, $c = -2$. Thus, $g(x) = -2x^2$.

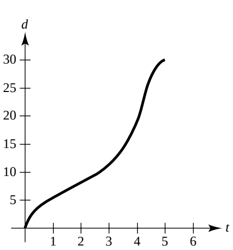
**41.** The function is $r(x) = c/x$, since it must be undefined at $x = 0$. Since $(1, 32)$ satisfies the equation, $c = 32$. Thus, $r(x) = 32/x$.

**43.** (a) For each time $t$, there corresponds a depth $d$.

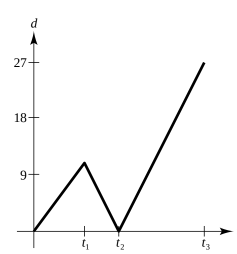
(b) Domain: $0 \le t \le 5$

Range: $0 \le d \le 30$

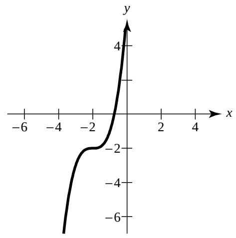
(c)



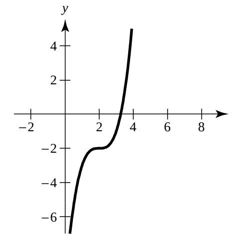
**45.**



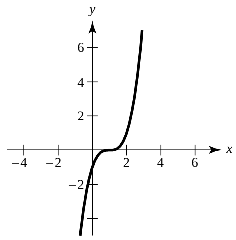
**47.** (a) The graph is shifted 3 units to the left.



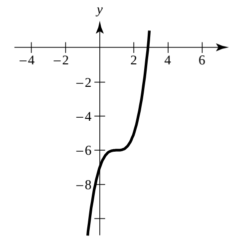
(b) The graph is shifted 1 unit to the right.



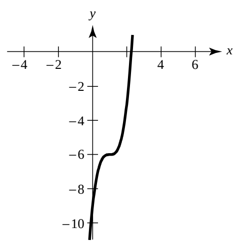
(c) The graph is shifted 2 units upward.



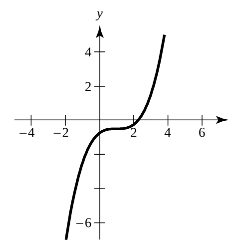
(d) The graph is shifted 4 units downward.



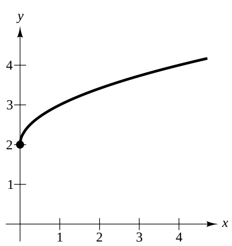
(e) The graph is stretched vertically by a factor of 3.



(f) The graph is stretched vertically by a factor of $\frac{1}{4}$.

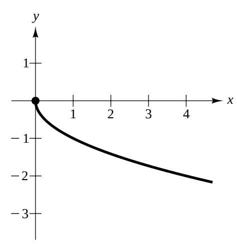


**49.** (a) $y = \sqrt{x} + 2$



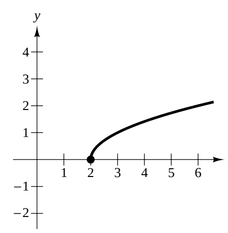
Vertical shift 2 units upward

(b) $y = -\sqrt{x}$



Reflection about the $x$-axis

(c) $y = \sqrt{x - 2}$



Horizontal shift 2 units to the right

**51.** (a) $T(4) = 16°$, $T(15) \approx 23°$

(b) If $H(t) = T(t - 1)$, then the program would turn on (and off) one hour later.

(c) If $H(t) = T(t) - 1$, then the overall temperature would be reduced 1 degree.

**53.** $f(x) = x^2$, $g(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x, \quad x \geq 0$

Domain: $[0, \infty)$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$$

Domain: $(-\infty, \infty)$

No. Their domains are different. $(f \circ g) = (g \circ f)$ for $x \geq 0$.

**55.** $f(x) = \dfrac{3}{x}$, $g(x) = x^2 - 1$

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \frac{3}{x^2 - 1}$$

Domain: all $x \neq \pm 1$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{3}{x}\right) = \left(\frac{3}{x}\right)^2 - 1 = \frac{9}{x^2} - 1 = \frac{9 - x^2}{x^2}$$

Domain: all $x \neq 0$

No, $f \circ g \neq g \circ f$.

**57.** $(A \circ r)(t) = A(r(t)) = A(0.6t) = \pi(0.6t)^2 = 0.36\pi t^2$

$(A \circ r)(t)$ represents the area of the circle at time $t$.

**59.** $f(-x) = (-x)^2(4 - (-x)^2) = x^2(4 - x^2) = f(x)$

Even

**61.** $f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$

Odd

**63.** (a) If $f$ is even, then $\left(\frac{3}{2}, 4\right)$ is on the graph.

(b) If $f$ is odd, then $\left(\frac{3}{2}, -4\right)$ is on the graph.

**65.**
$$\begin{aligned} f(-x) &= a_{2n+1}(-x)^{2n+1} + \cdots + a_3(-x)^3 + a_1(-x) \\ &= -[a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x] \\ &= -f(x) \end{aligned}$$

Odd

**67.** Let $F(x) = f(x)g(x)$ where $f$ and $g$ are even. Then

$$F(-x) = f(-x)g(-x) = f(x)g(x) = F(x).$$

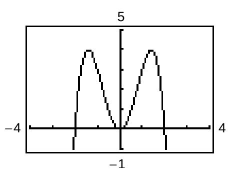
Thus, $F(x)$ is even. Let $F(x) = f(x)g(x)$ where $f$ and $g$ are odd. Then

$$F(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = F(x).$$

Thus, $F(x)$ is even.

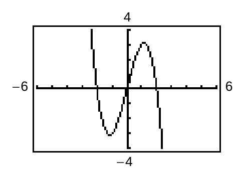
**69.** $f(x) = x^2 + 1$ and $g(x) = x^4$ are even.

$f(x)g(x) = (x^2 + 1)(x^4) = x^6 + x^4$ is even.




$f(x) = x^3 - x$ is odd and $g(x) = x^2$ is even.

$f(x)g(x) = (x^3 - x)(x^2) = x^5 - x^3$ is odd.

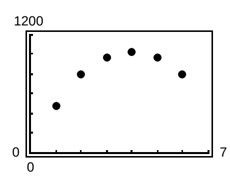



**71.** (a)

| $x$ | *length and width* | *volume V* |
|---|---|---|
| 1 | $24 - 2(1)$ | 484 |
| 2 | $24 - 2(2)$ | 800 |
| 3 | $24 - 2(3)$ | 972 |
| 4 | $24 - 2(4)$ | 1024 |
| 5 | $24 - 2(5)$ | 980 |
| 6 | $24 - 2(6)$ | 864 |

The maximum volume appears to be 1024 cm³.

(b)

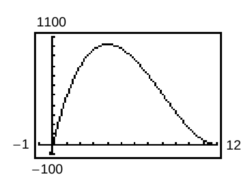



Yes, $V$ is a function of $x$.

(c) $V = x(24 - 2x)^2 = 4x(12 - x)^2$

Domain: $0 < x < 12$

(d)




Maximum volume is $V = 1024$ cm³ for box having dimensions $4 \times 16 \times 16$ cm.

**73.** False; let $f(x) = x^2$.

Then $f(-3) = f(3) = 9$, but $-3 \neq 3$.

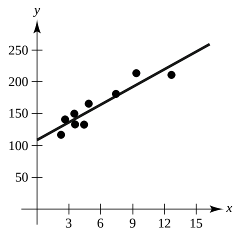
**75.** True, the function is even.

## Section P.4 Fitting Models to Data

**1.** Quadratic function

**3.** Linear function

**5.** (a), (b)

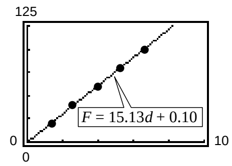



Yes. The cancer mortality increases linearly with increased exposure to the carcinogenic substance.

(c) If $x = 3$, then $y \approx 136$.

**7.** (a) $d = 0.066F$ or $F = 15.1d + 0.1$

(b)




The model fits well.

(c) If $F = 55$, then $d \approx 0.066(55) = 3.63$ cm.

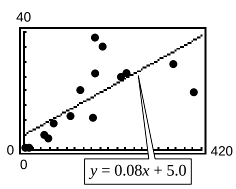
**9.** (a) Let $x =$ per capita energy usage (in millions of Btu)

$y =$ per capita gross national product (in thousands)

$y = 0.0764x + 4.9985 \approx 0.08x + 5.0$

$r = 0.7052$

(b)




(c) Denmark, Japan, and Canada

(d) Deleting the data for the three countries above,

$y = 0.0959x + 1.0539$

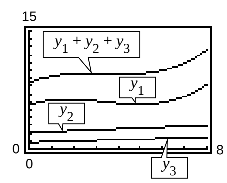
($r = 0.9202$ is much closer to 1.)

**11.** (a) $y_1 = 0.0343t^3 - 0.3451t^2 + 0.8837t + 5.6061$

$y_2 = 0.1095t + 2.0667$

$y_3 = 0.0917t + 0.7917$

(b)

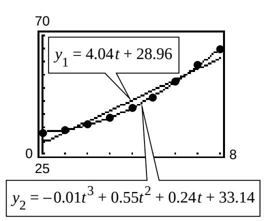



For 2002, $t = 12$ and $y_1 + y_2 + y_3 \approx 31.06$ cents/mile

**13.** (a) $y_1 = 4.0367t + 28.9644$

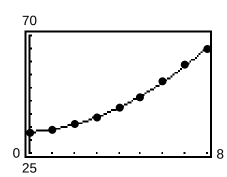
$y_2 = -0.0099t^3 + 0.5488t^2 + 0.2399t + 33.1414$

(b)




(c) The cubic model is better.

(d) $y_3 = 0.4297t^2 + 0.5994t + 32.9745$



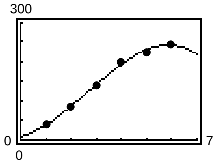

(e) The slope represents the average increase per year in the number of people (in millions) in HMOs.

(f) For 2000, $t = 10$, and $y_1 \approx 69.3$ million. (linear)

$y_2 \approx 80.5$ million (cubic)

**15.** (a) $y = -1.81x^3 + 14.58x^2 + 16.39x + 10$

(b)




(c) If $x = 4.5$, $y \approx 214$ horsepower.

**17.** (a) Yes, $y$ is a function of $t$. At each time $t$, there is one and only one displacement $y$.

(b) The amplitude is approximately

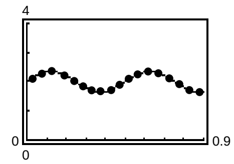
$$(2.35 - 1.65)/2 = 0.35.$$

The period is approximately

$$2(0.375 - 0.125) = 0.5.$$

(c) One model is $y = 0.35 \sin(4\pi t) + 2$.

(d)




**19.** Answers will vary.

## Review Exercises for Chapter P

**1.** $y = 2x - 3$

$x = 0 \Longrightarrow y = 2(0) - 3 = -3 \Longrightarrow (0, -3)$ $y$-intercept

$y = 0 \Longrightarrow 0 = 2x - 3 \Longrightarrow x = \frac{3}{2} \Longrightarrow \left(\frac{3}{2}, 0\right)$ $x$-intercept

**3.** $y = \dfrac{x-1}{x-2}$

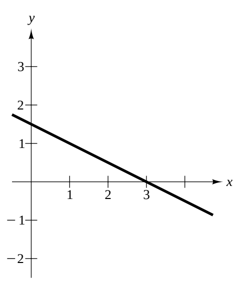
$x = 0 \Longrightarrow y = \dfrac{0-1}{0-2} = \dfrac{1}{2} \Longrightarrow \left(0, \dfrac{1}{2}\right)$ $y$-intercept

$y = 0 \Longrightarrow 0 = \dfrac{x-1}{x-2} \Longrightarrow x = 1 \Longrightarrow (1, 0)$ $x$-intercept

**5.** Symmetric with respect to $y$-axis since

$$(-x)^2y - (-x)^2 + 4y = 0$$
$$x^2y - x^2 + 4y = 0.$$

**7.** $y = -\frac{1}{2}x + \frac{3}{2}$



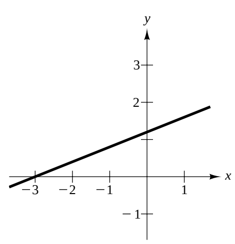

**9.** $-\frac{1}{3}x + \frac{5}{6}y = 1$

$-\frac{2}{5}x + y = \frac{6}{5}$

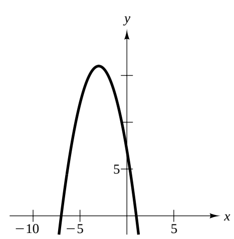
$y = \frac{2}{5}x + \frac{6}{5}$

Slope: $\frac{2}{5}$

$y$-intercept: $\frac{6}{5}$

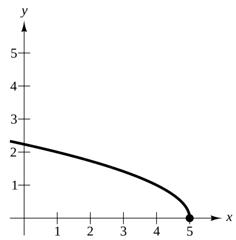



**11.** $y = 7 - 6x - x^2$



**13.** $y = \sqrt{5 - x}$

Domain: $(-\infty, 5]$



**15.** $y = 4x^2 - 25$

| |
|---|
| Xmin = -5 |
| Xmax = 5 |
| Xscl = 1 |
| Ymin = -30 |
| Ymax = 10 |
| Yscl = 5 |

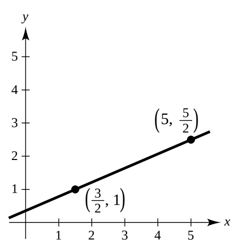
**17.**
$$\begin{aligned} 3x - 4y &= 8 \\ 4x + 4y &= 20 \\ \hline 7x \quad &= 28 \\ x &= 4 \\ y &= 1 \end{aligned}$$

Point: $(4, 1)$

**19.** You need factors $(x + 2)$ and $(x - 2)$. Multiply by $x$ to obtain origin symmetry

$$\begin{aligned} y &= x(x + 2)(x - 2). \\ &= x^3 - 4x. \end{aligned}$$

**21.**

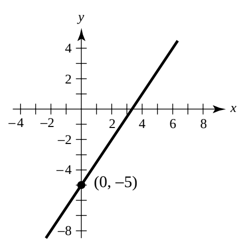



$$\text{Slope} = \frac{(5/2) - 1}{5 - (3/2)} = \frac{3/2}{7/2} = \frac{3}{7}$$

**23.**
$$\begin{aligned} \frac{1 - t}{1 - 0} &= \frac{1 - 5}{1 - (-2)} \\ 1 - t &= -\frac{4}{3} \\ t &= \frac{7}{3} \end{aligned}$$

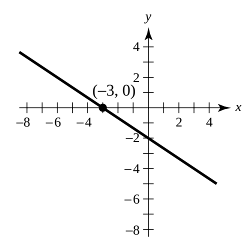
**25.**
$$\begin{aligned} y - (-5) &= \tfrac{3}{2}(x - 0) \\ y &= \tfrac{3}{2}x - 5 \end{aligned}$$

$$2y - 3x + 10 = 0$$




**27.**
$$\begin{aligned} y - 0 &= -\tfrac{2}{3}(x - (-3)) \\ y &= -\tfrac{2}{3}x - 2 \end{aligned}$$

$$3y + 2x + 6 = 0$$



**29.** (a) $y - 4 = \frac{7}{16}(x + 2)$

$16y - 64 = 7x + 14$

$0 = 7x - 16y + 78$

(c) $m = \frac{4 - 0}{-2 - 0} = -2$

$y = -2x$

$2x + y = 0$

(b) Slope of line is $\frac{5}{3}$.

$y - 4 = \frac{5}{3}(x + 2)$

$3y - 12 = 5x + 10$

$0 = 5x - 3y + 22$

(d) $x = -2$

$x + 2 = 0$

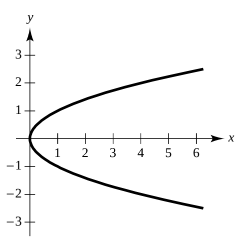
**31.** The slope is $-850$. $V = -850t + 12{,}500$.

$V(3) = -850(3) + 12{,}500 = \$9950$

**33.** $x - y^2 = 0$

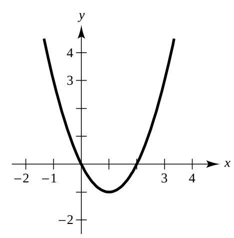
$y = \pm\sqrt{x}$

Not a function of $x$ since there are two values of $y$ for some $x$.




**35.** $y = x^2 - 2x$

Function of $x$ since there is one value of $y$ for each $x$.




**37.** $f(x) = \frac{1}{x}$

(a) $f(0)$ does not exist.

(b) $\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{\frac{1}{1 + \Delta x} - \frac{1}{1}}{\Delta x} = \frac{1 - 1 - \Delta x}{(1 + \Delta x)\Delta x}$

$= \frac{-1}{1 + \Delta x}, \Delta x \neq -1, 0$

**39.** (a) Domain: $36 - x^2 \geq 0 \Longrightarrow -6 \leq x \leq 6$ or $[-6, 6]$

Range: $[0, 6]$

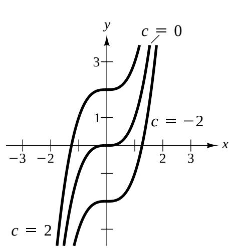
(b) Domain: all $x \neq 5$ or $(-\infty, 5), (5, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0), (0, \infty)$

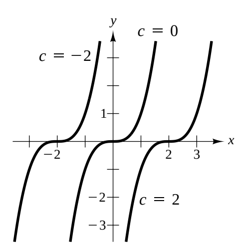
(c) Domain: all $x$ or $(-\infty, \infty)$

Range: all $y$ or $(-\infty, \infty)$

**41.** (a) $f(x) = x^3 + c, c = -2, 0, 2$



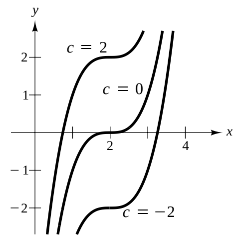

(b) $f(x) = (x - c)^3, c = -2, 0, 2$



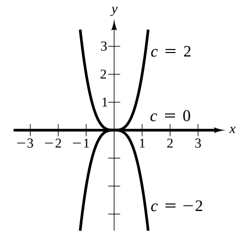

**—CONTINUED—**

**41.** **—CONTINUED—**

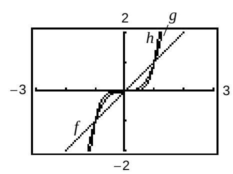
(c) $f(x) = (x - 2)^3 + c$, $c = -2, 0, 2$




(d) $f(x) = cx^3$, $c = -2, 0, 2$

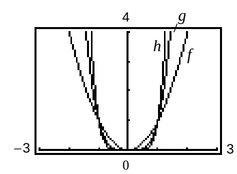



**43.** (a) Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$




The graphs of $f$, $g$, and $h$ all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$

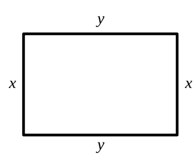



The graphs of $f$, $g$, and $h$ all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply.

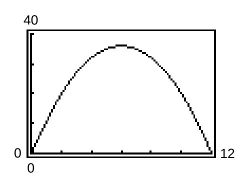
$y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

**45.** (a)




$$2x + 2y = 24$$
$$y = 12 - x$$
$$A = xy = x(12 - x) = 12x - x^2$$

(b) Domain: $0 < x < 12$




(c) Maximum area is $A = 36$. In general, the maximum area is attained when the rectangle is a square. In this case, $x = 6$.

**47.** (a) 3 (cubic), negative leading coefficient

(b) 4 (quartic), positive leading coefficient

(c) 2 (quadratic), negative leading coefficient

(d) 5, positive leading coefficient

**49.** (a) Yes, $y$ is a function of $t$. At each time $t$, there is one and only one displacement $y$.

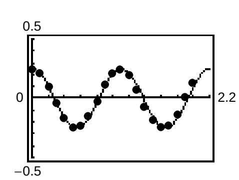
(b) The amplitude is approximately

$$(0.25 - (-0.25))/2 = 0.25.$$

The period is approximately 1.1.

(c) One model is $y = \frac{1}{4}\cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4}\cos(5.7t)$

(d)



# Problem Solving for Chapter P

**1.** (a)

$$x^2 - 6x + y^2 - 8y = 0$$
$$(x^2 - 6x + 9) + (y^2 - 8y + 16) = 9 + 16$$
$$(x - 3)^2 + (y - 4)^2 = 25$$

Center: $(3, 4)$ Radius: $5$

(b) Slope of line from $(0, 0)$ to $(3, 4)$ is $\frac{4}{3}$ Slope of tangent line is $-\frac{3}{4}$ Hence,

$$y - 0 = -\frac{3}{4}(x - 0) \implies y = -\frac{3}{4}x \quad \text{Tangent line}$$

(c) Slope of line from $(6, 0)$ to $(3, 4)$ is $\frac{4 - 0}{3 - 6} = -\frac{4}{3}$.

Slope of tangent line is $\frac{3}{4}$. Hence,

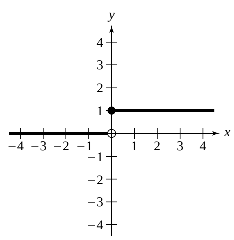
$$y - 0 = \frac{3}{4}(x - 6) \implies y = \frac{3}{4}x - \frac{9}{2} \quad \text{Tangent line}$$

(d)

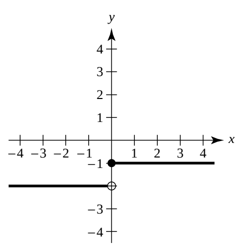
$$-\frac{3}{4}x = \frac{3}{4}x - \frac{9}{2}$$
$$\frac{3}{2}x = \frac{9}{2}$$
$$x = 3$$

Intersection: $\left(3, -\frac{9}{4}\right)$

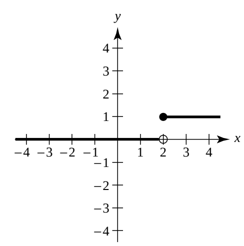
**3.** $H(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$



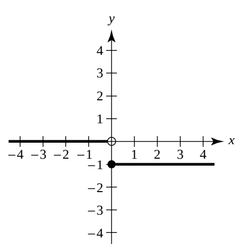
(a) $H(x) - 2$



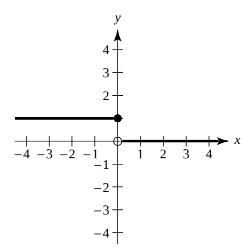
(b) $H(x - 2)$



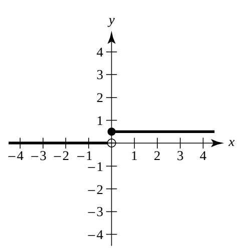
(c) $-H(x)$



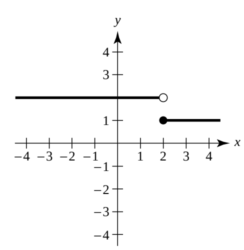
(d) $H(-x)$



(e) $\frac{1}{2}H(x)$



(f) $-H(x - 2) + 2$

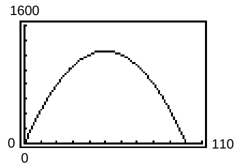


**5.** (a) $x + 2y = 100 \implies y = \dfrac{100 - x}{2}$

$$A(x) = xy = x\left(\frac{100 - x}{2}\right) = -\frac{x^2}{2} + 50x$$

Domain: $0 < x < 100$

(b)




Maximum of $1250\ \text{m}^2$ at $x = 50$ m, $y = 25$ m.

(c)
$$\begin{aligned} A(x) &= -\frac{1}{2}(x^2 - 100x) \\ &= -\frac{1}{2}(x^2 - 100x + 2500) + 1250 \\ &= -\frac{1}{2}(x - 50)^2 + 1250 \end{aligned}$$

$A(50) = 1250\ \text{m}^2$ is the maximum. $x = 50$ m, $y = 25$ m.

**7.** The length of the trip in the water is $\sqrt{2^2 + x^2}$, and the length of the trip over land is $\sqrt{1 + (3 - x)^2}$. Hence, the total time is

$$T = \frac{\sqrt{4 + x^2}}{2} + \frac{\sqrt{1 + (3 - x)^2}}{4} \text{ hours.}$$

**9.** (a) Slope $= \dfrac{9 - 4}{3 - 2} = 5$. Slope of tangent line is less than 5.

(b) Slope $= \dfrac{4 - 1}{2 - 1} = 3$. Slope of tangent line is greater than 3.

(c) Slope $= \dfrac{4.41 - 4}{2.1 - 2} = 4.1$. Slope of tangent line is less than 4.1.

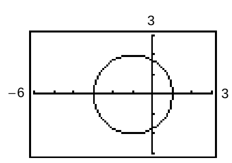
(d)
$$\begin{aligned} \text{Slope} &= \frac{f(2 + h) - f(2)}{(2 + h) - 2} \\ &= \frac{(2 + h)^2 - 4}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h,\ h \neq 0 \end{aligned}$$

(e) Letting $h$ get closer and closer to 0, the slope approaches 4. Hence, the slope at $(2, 4)$ is 4.

**11.** (a) At $x = 1$ and $x = -3$ the sounds are equal.

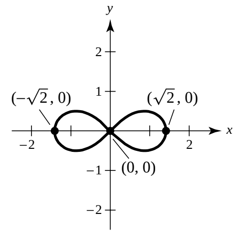
(b)
$$\begin{aligned} \frac{I}{\sqrt{x^2 + y^2}} &= \frac{2I}{\sqrt{(x - 3)^2 + y^2}} \\ (x - 3)^2 + y^2 &= 4(x^2 + y^2) \\ 3x^2 + 3y^2 + 6x &= 9 \\ x^2 + 2x + y^2 &= 3 \\ (x + 1)^2 + y^2 &= 4 \end{aligned}$$

Circle of radius 2 centered at $(-1, 0)$



**13.**

$$
\begin{aligned}
d_1 d_2 &= 1 \\
\left[(x+1)^2 + y^2\right]\left[(x-1)^2 + y^2\right] &= 1 \\
(x+1)^2(x-1)^2 + y^2\left[(x+1)^2 + (x-1)^2\right] + y^4 &= 1 \\
(x^2-1)^2 + y^2\left[2x^2 + 2\right] + y^4 &= 1 \\
x^4 - 2x^2 + 1 + 2x^2y^2 + 2y^2 + y^4 &= 1 \\
(x^4 + 2x^2y^2 + y^4) - 2x^2 + 2y^2 &= 0 \\
(x^2+y^2)^2 &= 2(x^2 - y^2)
\end{aligned}
$$




Let $y = 0$. Then $x^4 = 2x^2 \implies x = 0 \quad \text{or} \quad x^2 = 2$.

Thus, $(0, 0)$, $\left(\sqrt{2}, 0\right)$ and $\left(-\sqrt{2}, 0\right)$ are on the curve.

# CHAPTER 1
# Limits and Their Properties

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# Limits and Their Properties

## Section 1.1 A Preview of Calculus

Solutions to Odd-Numbered Exercises

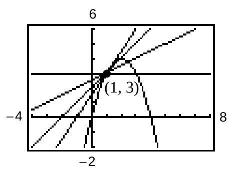
**1.** Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300$ feet

**3.** Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

**5.** Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$ sq. units

**7.** Precalculus: Volume $= (2)(4)(3) = 24$ cubic units

**9.** (a)




(b) The graphs of $y_2$ are approximations to the tangent line to $y_1$ at $x = 1$.

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:

$\{0.2, 0.1, 0.01, 0.001\}$

**11.** (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

## Section 1.2 Finding Limits Graphically and Numerically

**1.**

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.3448 | 0.3344 | 0.3334 | 0.3332 | 0.3322 | 0.3226 |

$\lim_{x\to 2} \frac{x-2}{x^2 - x - 2} \approx 0.3333$ (Actual limit is $\frac{1}{3}$.)

**3.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.2911 | 0.2889 | 0.2887 | 0.2887 | 0.2884 | 0.2863 |

$\lim_{x\to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \approx 0.2887$ (Actual limit is $1/(2\sqrt{3})$.)

**5.**

| $x$ | 2.9 | 2.99 | 2.999 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.0641$ | $-0.0627$ | $-0.0625$ | $-0.0625$ | $-0.0623$ | $-0.0610$ |

$$\lim_{x\to 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625 \quad \left(\text{Actual limit is } -\tfrac{1}{16}.\right)$$

**7.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.9983 | 0.99998 | 1.0000 | 1.0000 | 0.99998 | 0.9983 |

$\displaystyle\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

**9.** $\displaystyle\lim_{x\to 3} (4-x) = 1$

**11.** $\displaystyle\lim_{x\to 2} f(x) = \lim_{x\to 2} (4-x) = 2$

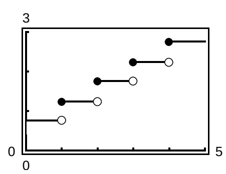
**13.** $\displaystyle\lim_{x\to 5} \frac{|x-5|}{x-5}$ does not exist. For values of $x$ to the left of 5, $|x-5|/(x-5)$ equals $-1$, whereas for values of $x$ to the right of 5, $|x-5|/(x-5)$ equals 1.

**15.** $\displaystyle\lim_{x\to \pi/2} \tan x$ does not exist since the function increases and decreases without bound as $x$ approaches $\pi/2$.

**17.** $\displaystyle\lim_{x\to 0} \cos(1/x)$ does not exist since the function oscillates between $-1$ and 1 as $x$ approaches 0.

**19.** $C(t) = 0.75 - 0.50[\![-(t-1)]\!]$

(a)




(b)

| $t$ | 3 | 3.3 | 3.4 | 3.5 | 3.6 | 3.7 | 4 |
|---|---|---|---|---|---|---|---|
| $C$ | 1.75 | 2.25 | 2.25 | 2.25 | 2.25 | 2.25 | 2.25 |

$\displaystyle\lim_{t\to 3.5} C(t) = 2.25$

(c)

| $t$ | 2 | 2.5 | 2.9 | 3 | 3.1 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $C$ | 1.25 | 1.75 | 1.75 | 1.75 | 2.25 | 2.25 | 2.25 |

$\displaystyle\lim_{t\to 3} C(t)$ does not exist. The values of $C$ jump from 1.75 to 2.25 at $t = 3$.

**21.** You need to find $\delta$ such that $0 < |x-1| < \delta$ implies $|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$\begin{aligned}
-0.1 &< \frac{1}{x} - 1 < 0.1 \\
1 - 0.1 &< \quad \frac{1}{x} \quad < 1 + 0.1 \\
\frac{9}{10} &< \quad \frac{1}{x} \quad < \frac{11}{10} \\
\frac{10}{9} &> \quad x \quad > \frac{10}{11} \\
\frac{10}{9} - 1 &> x - 1 > \frac{10}{11} - 1 \\
\frac{1}{9} &> x - 1 > -\frac{1}{11}.
\end{aligned}$$

So take $\delta = \dfrac{1}{11}$. Then $0 < |x-1| < \delta$ implies

$$\begin{aligned}
-\frac{1}{11} &< x - 1 < \frac{1}{11} \\
-\frac{1}{11} &< x - 1 < \frac{1}{9}.
\end{aligned}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < \epsilon < 0.1.$$

**23.** $\lim_{x\to 2}(3x+2) = 8 = L$

$$\begin{aligned}|(3x+2)-8| &< 0.01\\ |3x-6| &< 0.01\\ 3|x-2| &< 0.01\\ 0 < |x-2| &< \frac{0.01}{3} \approx 0.0033 = \delta\end{aligned}$$

Hence, if $0 < |x-2| < \delta = \dfrac{0.01}{3}$, you have

$$\begin{aligned}3|x-2| &< 0.01\\ |3x-6| &< 0.01\\ |(3x+2)-8| &< 0.01\\ |f(x)-L| &< 0.01\end{aligned}$$

**25.** $\lim_{x\to 2}(x^2-3) = 1 = L$

$$\begin{aligned}|(x^2-3)-1| &< 0.01\\ |x^2-4| &< 0.01\\ |(x+2)(x-2)| &< 0.01\\ |x+2|\,|x-2| &< 0.01\\ |x-2| &< \frac{0.01}{|x+2|}\end{aligned}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x-2| < \delta = 0.002$, you have

$$\begin{aligned}|x-2| < 0.002 = \frac{1}{5}(0.01) &< \frac{1}{|x+2|}(0.01)\\ |x+2||x-2| &< 0.01\\ |x^2-4| &< 0.01\\ |(x^2-3)-1| &< 0.01\\ |f(x)-L| &< 0.01\end{aligned}$$

**27.** $\lim_{x\to 2}(x+3) = 5$

Given $\epsilon > 0$:

$$\begin{aligned}|(x+3)-5| &< \epsilon\\ |x-2| &< \epsilon = \delta\end{aligned}$$

Hence, let $\delta = \epsilon$.

Hence, if $0 < |x-2| < \delta = \epsilon$, you have

$$\begin{aligned}|x-2| &< \epsilon\\ |(x+3)-5| &< \epsilon\\ |f(x)-L| &< \epsilon\end{aligned}$$

**29.** $\lim_{x\to -4}\left(\frac{1}{2}x-1\right) = \frac{1}{2}(-4)-1 = -3$

Given $\epsilon > 0$:

$$\begin{aligned}\left|\left(\tfrac{1}{2}x-1\right)-(-3)\right| &< \epsilon\\ \left|\tfrac{1}{2}x+2\right| &< \epsilon\\ \tfrac{1}{2}|x-(-4)| &< \epsilon\\ |x-(-4)| &< 2\epsilon\end{aligned}$$

Hence, let $\delta = 2\epsilon$.

Hence, if $0 < |x-(-4)| < \delta = 2\epsilon$, you have

$$\begin{aligned}|x-(-4)| &< 2\epsilon\\ \left|\tfrac{1}{2}x+2\right| &< \epsilon\\ \left|\left(\tfrac{1}{2}x-1\right)+3\right| &< \epsilon\\ |f(x)-L| &< \epsilon\end{aligned}$$

**31.** $\lim_{x\to 6} 3 = 3$

Given $\epsilon > 0$:

$$\begin{aligned}|3-3| &< \epsilon\\ 0 &< \epsilon\end{aligned}$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$\begin{aligned}|3-3| &< \epsilon\\ |f(x)-L| &< \epsilon\end{aligned}$$

**33.** $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\epsilon > 0$: $\left|\sqrt[3]{x}-0\right| < \epsilon$

$$\begin{aligned}\left|\sqrt[3]{x}\right| &< \epsilon\\ |x| &< \epsilon^3 = \delta\end{aligned}$$

Hence, let $\delta = \epsilon^3$.

Hence for $0 < |x-0| < \delta = \epsilon^3$, you have

$$\begin{aligned}|x| &< \epsilon^3\\ |\sqrt[3]{x}| &< \epsilon\\ |\sqrt[3]{x}-0| &< \epsilon\\ |f(x)-L| &< \epsilon\end{aligned}$$

**35.** $\lim_{x \to -2} |x - 2| = |(-2) - 2| = 4$

Given $\epsilon > 0$:

$$||x - 2| - 4| < \epsilon$$

$$|-(x - 2) - 4| < \epsilon \quad (x - 2 < 0)$$

$$|-x - 2| = |x + 2| = |x - (-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

Hence for $0 < |x - (-2)| < \delta = \epsilon$, you have

$$|x + 2| < \epsilon$$

$$|-(x + 2)| < \epsilon$$

$$|-(x - 2) - 4| < \epsilon$$

$$||x - 2| - 4| < \epsilon \quad \text{(because } x - 20\text{)}$$

$$|f(x) - L| < \epsilon$$

**37.** $\lim_{x \to 1} (x^2 + 1) = 2$

Given $\epsilon > 0$:

$$|(x^2 + 1) - 2| < \epsilon$$

$$|x^2 - 1| < \epsilon$$

$$|(x + 1)(x - 1)| < \epsilon$$

$$|x - 1| < \frac{\epsilon}{|x + 1|}$$

If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

Hence for $0 < |x - 1| < \delta = \dfrac{\epsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\epsilon < \frac{1}{|x + 1|}\epsilon$$

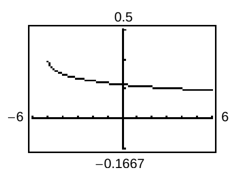
$$|x^2 - 1| < \epsilon$$

$$|(x^2 + 1) - 2| < \epsilon$$

$$|f(x) - 2| < \epsilon$$

**39.** $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

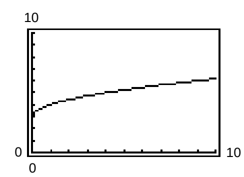
$\lim_{x \to 4} f(x) = \dfrac{1}{6}$




The domain is $[-5, 4) \cup (4, \infty)$.
The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

**41.** $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

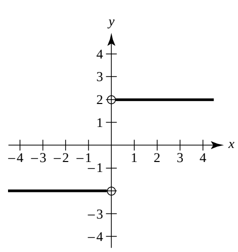
$\lim_{x \to 9} f(x) = 6$



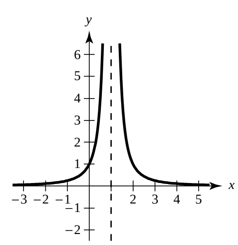

The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

**43.** $\lim_{x \to 8} f(x) = 25$ means that the values of $f$ approach 25 as $x$ gets closer and closer to 8.

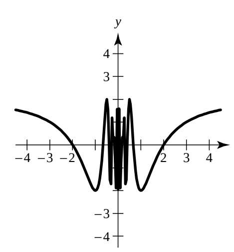
**45.** (i) The values of $f$ approach different numbers as $x$ approaches $c$ from different sides of $c$:



(ii) The values of $f$ increase without bound as $x$ approaches $c$:

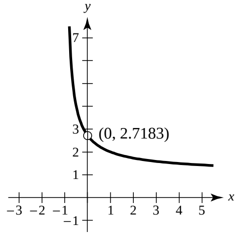


(iii) The values of $f$ oscillate between two fixed numbers as $x$ approaches $c$:



**47.** $f(x) = (1 + x)^{1/x}$

$\lim_{x \to 0} (1 + x)^{1/x} = e \approx 2.71828$




| $x$ | $f(x)$ |
|---|---|
| −0.1 | 2.867972 |
| −0.01 | 2.731999 |
| −0.001 | 2.719642 |
| −0.0001 | 2.718418 |
| −0.00001 | 2.718295 |
| −0.000001 | 2.718283 |

| $x$ | $f(x)$ |
|---|---|
| 0.1 | 2.593742 |
| 0.01 | 2.704814 |
| 0.001 | 2.716942 |
| 0.0001 | 2.718146 |
| 0.00001 | 2.718268 |
| 0.000001 | 2.718280 |

**49.** False; $f(x) = (\sin x)/x$ is undefined when $x = 0$. From Exercise 7, we have

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

**51.** False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}.$$

$f(4) = 10$

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} (x^2 - 4x) = 0 \neq 10$$

**53.** Answers will vary.

**55.** If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \implies |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \implies |f(x) - L_2| < \epsilon$. Let $\delta$ equal the smaller of $\delta_1$ and $\delta_2$. Then for $|x - c| < \delta$, we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

**57.** $\lim_{x\to c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \epsilon.$$

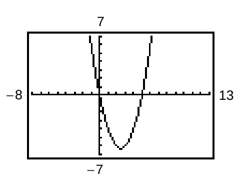
This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x\to c} f(x) = L$.

## Section 1.3 Evaluating Limits Analytically

**1.**

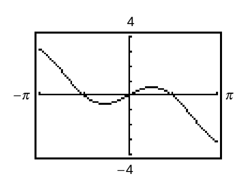



$h(x) = x^2 - 5x$

(a) $\lim_{x\to 5} h(x) = 0$

(b) $\lim_{x\to -1} h(x) = 6$

**3.**




$f(x) = x \cos x$

(a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to \pi/3} f(x) \approx 0.524$ $\left(= \dfrac{\pi}{6}\right)$

**5.** $\lim_{x\to 2} x^4 = 2^4 = 16$

**7.** $\lim_{x\to 0} (2x - 1) = 2(0) - 1 = -1$

**9.** $\lim_{x\to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

**11.** $\lim_{x\to -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

**13.** $\lim_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$

**15.** $\lim_{x\to 1} \dfrac{x - 3}{x^2 + 4} = \dfrac{1 - 3}{1^2 + 4} = \dfrac{-2}{5} = -\dfrac{2}{5}$

**17.** $\lim_{x\to 7} \dfrac{5x}{\sqrt{x + 2}} = \dfrac{5(7)}{\sqrt{7 + 2}} = \dfrac{35}{\sqrt{9}} = \dfrac{35}{3}$

**19.** $\lim_{x\to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

**21.** $\lim_{x\to -4} (x+3)^2 = (-4+3)^2 = 1$

**23.** (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

**25.** (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

**27.** $\lim_{x\to \pi/2} \sin x = \sin \frac{\pi}{2} = 1$

**29.** $\lim_{x\to 2} \cos \frac{\pi x}{3} = \cos \frac{\pi 2}{3} = -\frac{1}{2}$

**31.** $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

**33.** $\lim_{x\to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$

**35.** $\lim_{x\to 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$

**37.** (a) $\lim_{x\to c} [5g(x)] = 5 \lim_{x\to c} g(x) = 5(3) = 15$

(b) $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + 3 = 5$

(c) $\lim_{x\to c} [f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (2)(3) = 6$

(d) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{3}$

**39.** (a) $\lim_{x\to c} [f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c} [3f(x)] = 3 \lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c} [f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

**41.** $f(x) = -2x + 1$ and $g(x) = \frac{-2x^2 + x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = 1$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 3$

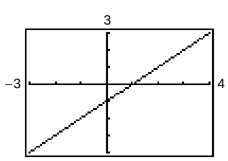
**43.** $f(x) = x(x+1)$ and $g(x) = \frac{x^3 - x}{x - 1}$ agree except at $x = 1$.

(a) $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 0$

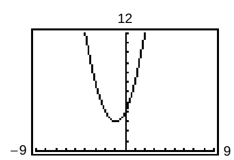
**45.** $f(x) = \frac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -2$




**47.** $f(x) = \frac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = 12$




**49.** $\lim_{x\to 5} \frac{x - 5}{x^2 - 25} = \lim_{x\to 5} \frac{x - 5}{(x+5)(x-5)}$

$= \lim_{x\to 5} \frac{1}{x + 5} = \frac{1}{10}$

**51.** $\lim_{x\to -3} \frac{x^2 + x - 6}{x^2 - 9} = \lim_{x\to -3} \frac{(x+3)(x-2)}{(x+3)(x-3)}$

$= \lim_{x\to -3} \frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6}$

**53.** $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}=\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$$=\lim_{x\to 0}\frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+5}+\sqrt{5}}=\frac{1}{2\sqrt{5}}=\frac{\sqrt{5}}{10}$$

**55.** $\displaystyle\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}=\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$$=\lim_{x\to 4}\frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)}=\lim_{x\to 4}\frac{1}{\sqrt{x+5}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

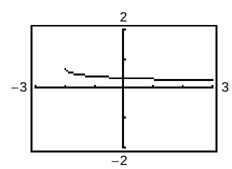
**57.** $\displaystyle\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}=\lim_{x\to 0}\frac{\frac{2-(2+x)}{2(2+x)}}{x}=\lim_{x\to 0}\frac{-1}{2(2+x)}=-\frac{1}{4}$

**59.** $\displaystyle\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x+2\Delta x-2x}{\Delta x}=\lim_{\Delta x\to 0}2=2$

**61.** $\displaystyle\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$$=\lim_{\Delta x\to 0}(2x+\Delta x-2)=2x-2$$

**63.** $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\approx 0.354$

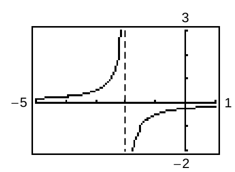


| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.358 | 0.354 | 0.345 | ? | 0.354 | 0.353 | 0.349 |

Analytically, $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}=\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$$=\lim_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}\approx 0.354$$

**65.** $\displaystyle\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}=-\frac{1}{4}$



| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-0.263$ | $-0.251$ | $-0.250$ | ? | $-0.250$ | $-0.249$ | $-0.238$ |

Analytically, $\displaystyle\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}=\lim_{x\to 0}\frac{2-(2+x)}{2(2+x)}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{-x}{2(2+x)}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{-1}{2(2+x)}=-\frac{1}{4}.$

**67.** $\lim_{x\to 0} \frac{\sin x}{5x} = \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

**69.** $\lim_{x\to 0} \frac{\sin x(1-\cos x)}{2x^2} = \lim_{x\to 0}\left[\frac{1}{2}\cdot\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$

$= \frac{1}{2}(1)(0) = 0$

**71.** $\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

**73.** $\lim_{h\to 0} \frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$

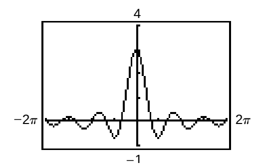
$= (0)(0) = 0$

**75.** $\lim_{x\to\pi/2} \frac{\cos x}{\cot x} = \lim_{x\to\pi/2} \sin x = 1$

**77.** $\lim_{t\to 0} \frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

**79.** $f(t) = \frac{\sin 3t}{t}$

| $t$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(t)$ | 2.96 | 2.9996 | 3 | ? | 3 | 2.9996 | 2.96 |

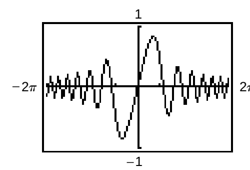



The limit appear to equal 3.

Analytically, $\lim_{t\to 0}\frac{\sin 3t}{t} = \lim_{t\to 0} 3\left(\frac{\sin 3t}{3t}\right) = 3(1) = 3.$

**81.** $f(x) = \frac{\sin x^2}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-0.099998$ | $-0.01$ | $-0.001$ | ? | 0.001 | 0.01 | 0.099998 |




Analytically, $\lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0} x\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0.$

**83.** $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)+3-(2x+3)}{h} = \lim_{h\to 0}\frac{2x+2h+3-2x-3}{h} = \lim_{h\to 0}\frac{2h}{h} = 2$

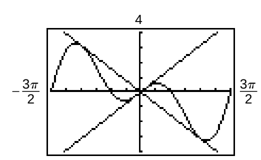
**85.** $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{4}{x+h}-\frac{4}{x}}{h} = \lim_{h\to 0}\frac{4x-4(x+h)}{(x+h)xh} = \lim_{h\to 0}\frac{-4}{(x+h)x} = \frac{-4}{x^2}$

**87.** $\lim_{x\to 0}(4-x^2) \le \lim_{x\to 0} f(x) \le \lim_{x\to 0}(4+x^2)$

$4 \le \lim_{x\to 0} f(x) \le 4$

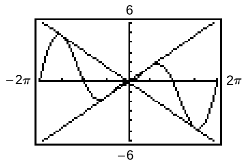
Therefore, $\lim_{x\to 0} f(x) = 4.$

**89.** $f(x) = x\cos x$



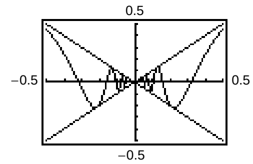

$\lim_{x\to 0}(x\cos x) = 0$

**91.** $f(x) = |x| \sin x$




$\lim_{x \to 0} |x| \sin x = 0$

**93.** $f(x) = x \sin \dfrac{1}{x}$




$\lim_{x \to 0} \left( x \sin \dfrac{1}{x} \right) = 0$

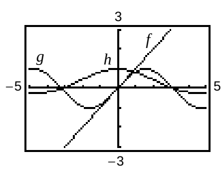
**95.** We say that two functions $f$ and $g$ agree at all but one point (on an open interval) if $f(x) = g(x)$ for all $x$ in the interval except for $x = c$, where $c$ is in the interval.

**97.** An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

**99.** $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$




When you are "close to" 0 the magnitude of $f$ is approximately equal to the magnitude of $g$. Thus, $|g|/|f| \approx 1$ when $x$ is "close to" 0.

**101.** $s(t) = -16t^2 + 1000$

$$\lim_{t \to 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \to 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \to 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \to 5} -16(t + 5) = -160 \text{ ft/sec.}$$

Speed $= 160$ ft/sec

**103.** $s(t) = -4.9t^2 + 150$

$$\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \to 3} \frac{-4.9(9 - t^2)}{3 - t}$$

$$= \lim_{x \to 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{x \to 3} -4.9(3 + t) = -29.4 \text{ m/sec}$$

**105.** Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist.

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \left[ \frac{1}{x} + \left( -\frac{1}{x} \right) \right] = \lim_{x \to 0} [0] = 0$$

**107.** Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

**109.** If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$, we have

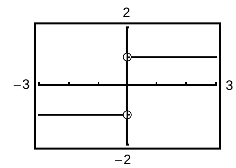
$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x \to c} [bf(x)] = bL$.

**111.**
$$-M|f(x)| \le f(x)g(x) \le M|f(x)|$$
$$\lim_{x\to c}(-M|f(x)|) \le \lim_{x\to c} f(x)g(x) \le \lim_{x\to c}(M|f(x)|)$$
$$-M(0) \le \lim_{x\to c} f(x)g(x) \le M(0)$$
$$0 \le \lim_{x\to c} f(x)g(x) \le 0$$

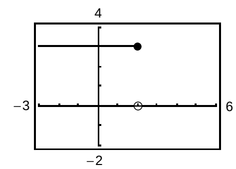
Therefore, $\lim_{x\to c} f(x)g(x) = 0$.

**113.** False. As $x$ approaches 0 from the left, $\dfrac{|x|}{x} = -1$.




**115.** True.

**117.** False. The limit does not exist.




**119.** Let

$$f(x) = \begin{cases} 4, & \text{if } x \ge 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4.$$

$\lim_{x\to 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \ge 0$, $f(x) = 4$.

**121.** $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 $x$ is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

$\lim_{x\to 0} g(x) = 0$.

When $x$ is "close to" 0, both parts of the function are "close to" 0.

**123.** (a)
$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}$$
$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$
$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$$
$$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

(b) Thus, $\dfrac{1-\cos x}{x^2} \approx \dfrac{1}{2} \implies 1 - \cos x \approx \dfrac{1}{2}x^2$

$$\implies \cos x \approx 1 - \frac{1}{2}x^2 \text{ for } x \approx 0.$$

(c) $\cos(0.1) \approx 1 - \dfrac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

## Section 1.4 Continuity and One-Sided Limits

**1.** (a) $\lim_{x\to 3^+} f(x) = 1$

(b) $\lim_{x\to 3^-} f(x) = 1$

(c) $\lim_{x\to 3} f(x) = 1$

The function is continuous at $x = 3$.

**3.** (a) $\lim_{x\to 3^+} f(x) = 0$

(b) $\lim_{x\to 3^-} f(x) = 0$

(c) $\lim_{x\to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

**5.** (a) $\lim_{x\to 4^+} f(x) = 2$

(b) $\lim_{x\to 4^-} f(x) = -2$

(c) $\lim_{x\to 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

**7.** $\lim_{x\to 5^+} \frac{x-5}{x^2-25} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$

**9.** $\lim_{x\to -3^-} \frac{x}{\sqrt{x^2-9}}$ does not exist because $\frac{x}{\sqrt{x^2-9}}$ grows without bound as $x \to -3^-$.

**11.** $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1.$

**13.**
$$\lim_{\Delta x\to 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)}$$
$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

**15.** $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+2}{2} = \frac{5}{2}$

**17.** $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3+1) = 2$

$\lim_{x\to 1} f(x) = 2$

**19.** $\lim_{x\to \pi} \cot x$ does not exist since

$\lim_{x\to \pi^+} \cot x$ and $\lim_{x\to \pi^-} \cot x$ do not exist.

**21.** $\lim_{x\to 4^-} (3[\![x]\!] - 5) = 3(3) - 5 = 4$
($[\![x]\!] = 3$ for $3 < x < 4$)

**23.** $\lim_{x\to 3} (2 - [\![-x]\!])$ does not exist because

$\lim_{x\to 3^-} (2 - [\![-x]\!]) = 2 - (-3) = 5$

and

$\lim_{x\to 3^+} (2 - [\![-x]\!]) = 2 - (-4) = 6.$

**25.** $f(x) = \frac{1}{x^2-4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

**27.** $f(x) = \frac{[\![x]\!]}{2} + x$

has discontinuities at each integer $k$ since $\lim_{x\to k^-} f(x) \neq \lim_{x\to k^+} f(x)$.

**29.** $g(x) = \sqrt{25-x^2}$ is continuous on $[-5, 5]$.

**31.** $\lim_{x\to 0^-} f(x) = 3 = \lim_{x\to 0^+} f(x)$.
$f$ is continuous on $[-1, 4]$.

**33.** $f(x) = x^2 - 2x + 1$ is continuous for all real $x$.

**35.** $f(x) = 3x - \cos x$ is continuous for all real $x$.

**37.** $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since

$\dfrac{x}{x^2 - x} = \dfrac{1}{x - 1}$ for $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

**39.** $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real $x$.

**41.** $f(x) = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

**43.** $f(x) = \dfrac{|x + 2|}{x + 2}$ has a nonremovable discontinuity at $x = -2$ since $\lim_{x \to -2} f(x)$ does not exist.

**45.** $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

**1.** $f(1) = 1$

**2.** $\left.\begin{array}{l} \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1 \\ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1 \end{array}\right\} \lim_{x \to 1} f(x) = 1$

**3.** $f(1) = \lim_{x \to 1} f(x)$

$f$ is continuous at $x = 1$, therefore, $f$ is continuous for all real $x$.

**47.** $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

**1.** $f(2) = \dfrac{2}{2} + 1 = 2$

**2.** $\left.\begin{array}{l} \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\dfrac{x}{2} + 1\right) = 2 \\ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3 - x) = 1 \end{array}\right\} \lim_{x \to 2} f(x)$ does not exist.

Therefore, $f$ has a nonremovable discontinuity at $x = 2$.

**49.** $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases} = \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$ has **possible** discontinuities at $x = -1$, $x = 1$.

**1.** $f(-1) = -1$ $\qquad f(1) = 1$

**2.** $\lim_{x \to -1} f(x) = -1$ $\qquad \lim_{x \to 1} f(x) = 1$

**3.** $f(-1) = \lim_{x \to -1} f(x)$ $\qquad f(1) = \lim_{x \to 1} f(x)$

$f$ is continuous at $x = \pm 1$, therefore, $f$ is continuous for all real $x$.

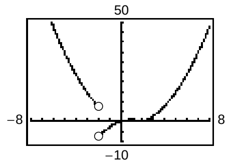
**51.** $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

**53.** $f(x) = [\![x - 1]\!]$ has nonremovable discontinuities at each integer $k$.

**55.** $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

$f$ is not continuous at $x = -2$.




**57.** $f(2) = 8$

Find $a$ so that $\lim_{x\to 2^+} ax^2 = 8 \implies a = \frac{8}{2^2} = 2$.

**59.** Find $a$ and $b$ such that $\lim_{x\to -1^+} (ax + b) = -a + b = 2$ and $\lim_{x\to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ \underline{(+)\ 3a + b} &\underline{= -2} \\ 4a \quad &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

**61.** $f(g(x)) = (x - 1)^2$

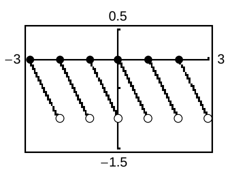
Continuous for all real $x$.

**63.** $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

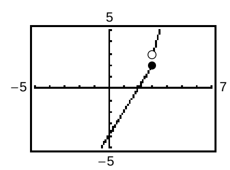
**65.** $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer




**67.** $f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



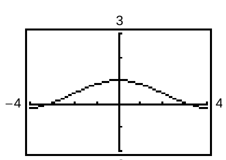

**69.** $f(x) = \dfrac{x}{x^2 + 1}$

Continuous on $(-\infty, \infty)$

**71.** $f(x) = \sec \dfrac{\pi x}{4}$

Continuous on:
$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

**73.** $f(x) = \dfrac{\sin x}{x}$




The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that $f$ has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

**75.** $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of $c$ between 1 and 2.

**77.** $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of $c$ between 0 and $\pi$.

**79.** $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

**81.** $g(t) = 2 \cos t - 3t$

$g$ is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value $c$ between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

**83.** $f(x) = x^2 + x - 1$

$f$ is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$
$$c = 3 \ (x = -4 \text{ is not in the interval.})$$

Thus, $f(3) = 11$.

**85.** $f(x) = x^3 - x^2 + x - 2$

$f$ is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

$(x^2 + x + 3$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

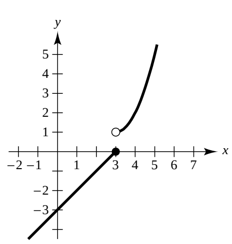
**87.** (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

**89.**




The function is not continuous at $x = 3$ because $\lim_{x\to3^+} f(x) = 1 \neq 0 = \lim_{x\to3^-} f(x)$.

**91.** The functions agree for integer values of $x$:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

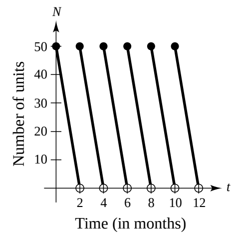
However, for non-integer values of $x$, the functions differ by 1.

$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!]$.

For example, $f\left(\frac{1}{2}\right) = 3 + 0 = 3$, $g\left(\frac{1}{2}\right) = 3 - (-1) = 4$.

**93.** $N(t) = 25\left(2\left[\!\left[\dfrac{t+2}{2}\right]\!\right] - t\right)$

| $t$ | 0 | 1 | 1.8 | 2 | 3 | 3.8 |
|---|---|---|---|---|---|---|
| $N(t)$ | 50 | 25 | 5 | 50 | 25 | 5 |




Discontinuous at every positive even integer.The company replenishes its inventory every two months.

**95.** Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius $r$.

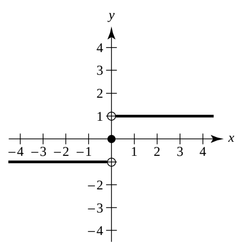
$V(1) = \frac{4}{3}\pi \approx 4.19$

$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value $r$ between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

**97.** Let $c$ be any real number. Then $\lim\limits_{x \to c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to $c$. Therefore, $f$ is not continuous at $c$.

**99.** $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$



(a) $\lim\limits_{x \to 0^-} \operatorname{sgn}(x) = -1$

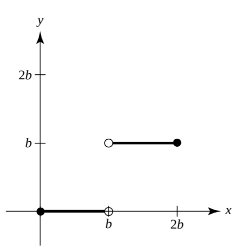
(b) $\lim\limits_{x \to 0^+} \operatorname{sgn}(x) = 1$

(c) $\lim\limits_{x \to 0} \operatorname{sgn}(x)$ does not exist.

**101.** True; if $f(x) = g(x)$, $x \neq c$, then $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$ and at least one of these limits (if they exist) does not equal the corresponding function at $x = c$.

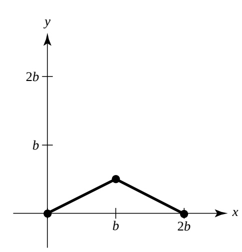
**103.** False; $f(1)$ is not defined and $\lim\limits_{x \to 1} f(x)$ does not exist.

**105.** (a) $f(x) = \begin{cases} 0 & 0 \le x < b \\ b & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2} & 0 \le x \le b \\ b - \dfrac{x}{2} & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

**107.** $f(x) = \dfrac{\sqrt{x + c^2} - c}{x},\ c > 0$

Domain: $x + c^2 \ge 0 \implies x \ge -c^2$ and $x \ne 0$, $[-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c}$$

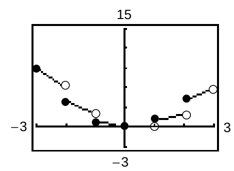
$$= \lim_{x\to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make $f$ continuous at $x = 0$.

**109.** $h(x) = x[\![x]\!]$

$h$ has nonremovable discontinuities at

$x = \pm 1, \pm 2, \pm 3, \ldots$ .




## Section 1.5 Infinite Limits

**1.** $\displaystyle\lim_{x\to -2^+} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

$\displaystyle\lim_{x\to -2^-} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

**3.** $\displaystyle\lim_{x\to -2^+} \tan\frac{\pi x}{4} = -\infty$

$\displaystyle\lim_{x\to -2^-} \tan\frac{\pi x}{4} = \infty$

**5.** $f(x) = \dfrac{1}{x^2 - 9}$

| $x$ | $-3.5$ | $-3.1$ | $-3.01$ | $-3.001$ | $-2.999$ | $-2.99$ | $-2.9$ | $-2.5$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.308 | 1.639 | 16.64 | 166.6 | $-166.7$ | $-16.69$ | $-1.695$ | $-0.364$ |

$\displaystyle\lim_{x\to -3^-} f(x) = \infty$

$\displaystyle\lim_{x\to -3^+} f(x) = -\infty$

**7.** $f(x) = \dfrac{x^2}{x^2 - 9}$

| $x$ | $-3.5$ | $-3.1$ | $-3.01$ | $-3.001$ | $-2.999$ | $-2.99$ | $-2.9$ | $-2.5$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 3.769 | 15.75 | 150.8 | 1501 | $-1499$ | $-149.3$ | $-14.25$ | $-2.273$ |

$\displaystyle\lim_{x\to -3^-} f(x) = \infty$

$\displaystyle\lim_{x\to -3^+} f(x) = -\infty$

**9.** $\lim_{x\to 0^+} \frac{1}{x^2} = \infty = \lim_{x\to 0^-} \frac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

**11.** $\lim_{x\to 2^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$

$\lim_{x\to 2^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

$\lim_{x\to -1^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$

$\lim_{x\to -1^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$

Therefore, $x = -1$ is a vertical asymptote.

**13.** $\lim_{x\to -2^-} \frac{x^2}{x^2 - 4} = \infty$ and $\lim_{x\to -2^+} \frac{x^2}{x^2 - 4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 2^-} \frac{x^2}{x^2 - 4} = -\infty$ and $\lim_{x\to 2^+} \frac{x^2}{x^2 - 4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

**15.** No vertical asymptote since the denominator is never zero.

**17.** $f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x}$ has vertical asymptotes at

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2},\ n \text{ any integer.}$$

**19.** $\lim_{t\to 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t\to 0^-} \left(1 - \frac{4}{t^2}\right)$

Therefore, $t = 0$ is a vertical asymptote.

**21.** $\lim_{x\to -2^+} \frac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to -2^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 1^+} \frac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to 1^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

**23.** $f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x+1)(x^2 - x + 1)}{x + 1}$

has no vertical asymptote since

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} (x^2 - x + 1) = 3$$

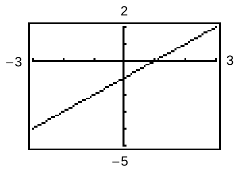
**25.** $f(x) = \frac{(x-5)(x+3)}{(x-5)(x^2+1)} = \frac{x+3}{x^2+1},\ x \neq 5$

No vertical asymptotes. The graph has a hole at $x = 5$.

**27.** $s(t) = \frac{t}{\sin t}$ has vertical asymptotes at $t = n\pi$, $n$ a nonzero integer. There is no vertical asymptote at $t = 0$ since

$$\lim_{t\to 0} \frac{t}{\sin t} = 1.$$

**29.** $\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} (x - 1) = -2$

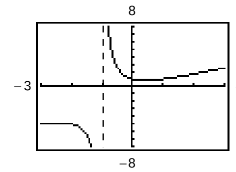



Removable discontinuity at $x = -1$

**31.** $\lim_{x \to -1^+} \frac{x^2 + 1}{x + 1} = \infty$

$\lim_{x \to -1^-} \frac{x^2 + 1}{x + 1} = -\infty$

Vertical asymptote at $x = -1$




**33.** $\lim_{x \to 2^+} \frac{x - 3}{x - 2} = -\infty$

**35.** $\lim_{x \to 3^+} \frac{x^2}{(x - 3)(x + 3)} = \infty$

**37.** $\lim_{x \to -3^-} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \to -3^-} \frac{x - 1}{x - 2} = \frac{4}{5}$

**39.** $\lim_{x \to 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x^2 + 1} = \frac{1}{2}$

**41.** $\lim_{x \to 0^-} \left(1 + \frac{1}{x}\right) = -\infty$

**43.** $\lim_{x \to 0^+} \frac{2}{\sin x} = \infty$

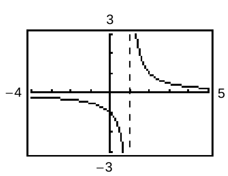
**45.** $\lim_{x \to \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \to \pi} \left(\sqrt{x} \sin x\right) = 0$

**47.** $\lim_{x \to (1/2)^-} x \sec(\pi x) = \infty$ and $\lim_{x \to (1/2)^+} x \sec(\pi x) = -\infty$.

Therefore, $\lim_{x \to (1/2)} x \sec(\pi x)$ does not exist.

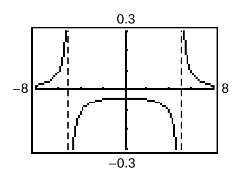
**49.** $f(x) = \frac{x^2 + x + 1}{x^3 - 1}$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x - 1} = \infty$




**51.** $f(x) = \frac{1}{x^2 - 25}$

$\lim_{x \to 5^-} f(x) = -\infty$



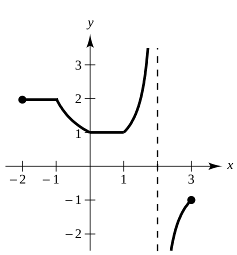

**53.** A limit in which $f(x)$ increases or decreases without bound as $x$ approaches $c$ is called an infinite limit. $\infty$ is not a number. Rather, the symbol

$$\lim_{x \to c} f(x) = \infty$$

says how the limit fails to exist.

**55.** One answer is $f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12}$.

**57.**




**59.** $S = \frac{k}{1 - r}$, $0 < |r| < 1$. Assume $k \neq 0$.

$\lim_{r \to 1^-} S = \lim_{r \to 1^-} \frac{k}{1 - r} = \infty \quad (\text{or } -\infty \text{ if } k < 0)$

**61.** $C = \dfrac{528x}{100 - x}, \; 0 \le x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\displaystyle\lim_{x \to 100^-} \frac{528}{100 - x} = \infty$ Thus, it is not possible.

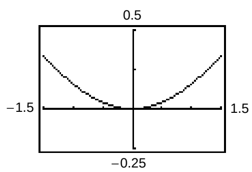
**63.** (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\displaystyle\lim_{x \to 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$

**65.** (a)

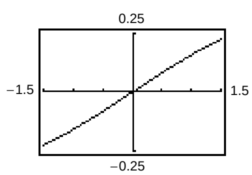
| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.0411 | 0.0067 | 0.0017 | $\approx 0$ | $\approx 0$ | $\approx 0$ |




$$\lim_{x \to 0^+} \frac{x - \sin x}{x} = 0$$

(b)

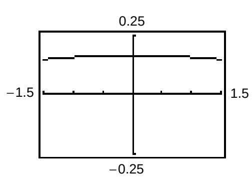
| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.0823 | 0.0333 | 0.0167 | 0.0017 | $\approx 0$ | $\approx 0$ |




$$\lim_{x \to 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

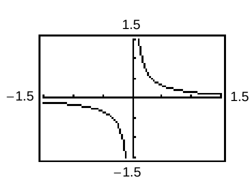
| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.1646 | 0.1663 | 0.1666 | 0.1667 | 0.1667 | 0.1667 |




$$\lim_{x \to 0^+} \frac{x - \sin x}{x^3} = 0.1167 \;\; (1/6)$$

(d)

| $x$ | 1 | 0.5 | 0.2 | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1585 | 0.3292 | 0.8317 | 1.6658 | 16.67 | 166.7 | 1667.0 |




$$\lim_{x \to 0^+} \frac{x - \sin x}{x^4} = \infty$$

For $n \ge 3$, $\displaystyle\lim_{x \to 0^+} \frac{x - \sin x}{x^n} = \infty$.

**67.** (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections. The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

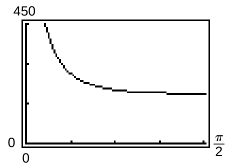
Total length $= 60 \cot \phi + 30(\pi + 2\phi)$

Domain: $\left(0, \frac{\pi}{2}\right)$

(d)

| $\phi$ | 0.3 | 0.6 | 0.9 | 1.2 | 1.5 |
|---|---|---|---|---|---|
| $L$ | 306.2 | 217.9 | 195.9 | 189.6 | 188.5 |

(e)




(f) $\lim_{\phi \to (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi \to 0^+} L = \infty$

**69.** False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

**71.** False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of $f$ has a vertical asymptote at $x = 0$, but $f(0) = 3$.

**73.** Given $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$:

(2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let $\delta$ be the smaller of $\delta_1$ and $\delta_2$. Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let $\delta$ be the smaller of $\delta_1$ and $\delta_2$. Then for $0 < |x - c| < \delta$, we have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

Therefore, $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$.

**75.** Given $\lim_{x \to c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \to c} f(x)$ exists and equals $L$. Then,

$$\lim_{x \to c} \frac{1}{f(x)} = \frac{\lim_{x \to c} 1}{\lim_{x \to c} f(x)} = \frac{1}{L} = 0.$$

This is not possible. Thus, $\lim_{x \to c} f(x)$ does not exist.

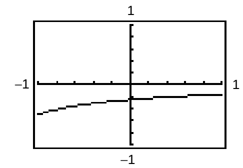
## Review Exercises for Chapter 1

**1.** Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, 8.25.

**3.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.26$ | $-0.25$ | $-0.250$ | $-0.2499$ | $-0.249$ | $-0.24$ |

$\lim_{x\to 0} f(x) \approx -0.25$




**5.** $h(x) = \dfrac{x^2 - 2x}{x}$

(a) $\lim_{x\to 0} h(x) = -2$

(b) $\lim_{x\to -1} h(x) = -3$

**7.** $\lim_{x\to 1} (3 - x) = 3 - 1 = 2$

Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for $0 < |x - 1| < \delta = \epsilon$, you have

$$\begin{aligned} |x - 1| &< \epsilon \\ |1 - x| &< \epsilon \\ |(3 - x) - 2| &< \epsilon \\ |f(x) - L| &< \epsilon \end{aligned}$$

**9.** $\lim_{x\to 2} (x^2 - 3) = 1$

Let $\epsilon > 0$ be given. We need $|x^2 - 3 - 1| < \epsilon \implies |x^2 - 4| = |(x - 2)(x + 2)| < \epsilon \implies |x - 2| < \dfrac{1}{|x + 2|}\epsilon$.

Assuming, $1 < x < 3$, you can choose $\delta = \epsilon/5$. Hence, for $0 < |x - 2| < \delta = \epsilon/5$ you have

$$\begin{aligned} |x - 2| &< \frac{\epsilon}{5} < \frac{1}{|x + 2|}\epsilon \\ |x - 2||x + 2| &< \epsilon \\ |x^2 - 4| &< \epsilon \\ |(x^2 - 3) - 1| &< \epsilon \\ |f(x) - L| &< \epsilon \end{aligned}$$

**11.** $\lim_{t\to 4} \sqrt{t + 2} = \sqrt{4 + 2} = \sqrt{6} \approx 2.45$

**13.** $\lim_{t\to -2} \dfrac{t + 2}{t^2 - 4} = \lim_{t\to -2} \dfrac{1}{t - 2} = -\dfrac{1}{4}$

**15.** $$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

**17.** $$\lim_{x\to 0} \frac{[1/(x + 1)] - 1}{x} = \lim_{x\to 0} \frac{1 - (x + 1)}{x(x + 1)} = \lim_{x\to 0} \frac{-1}{x + 1} = -1$$

**19.** $$\lim_{x\to -5} \frac{x^3 + 125}{x + 5} = \lim_{x\to -5} \frac{(x + 5)(x^2 - 5x + 25)}{x + 5} = \lim_{x\to -5} (x^2 - 5x + 25) = 75$$

**21.** $$\lim_{x\to 0} \frac{1 - \cos x}{\sin x} = \lim_{x\to 0} \left(\frac{x}{\sin x}\right)\left(\frac{1 - \cos x}{x}\right) = (1)(0) = 0$$

**23.** $$\lim_{\Delta x\to 0}\frac{\sin[(\pi/6)+\Delta x]-(1/2)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{1}{2}\cdot\frac{(\cos\Delta x-1)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x}$$
$$=0+\frac{\sqrt{3}}{2}(1)=\frac{\sqrt{3}}{2}$$

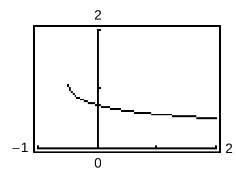
**25.** $\lim_{x\to c}[f(x)\cdot g(x)]=\left(-\frac{3}{4}\right)\left(\frac{2}{3}\right)=-\frac{1}{2}$

**27.** $f(x)=\dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}$

(a)

| $x$ | 1.1 | 1.01 | 1.001 | 1.0001 |
|---|---|---|---|---|
| $f(x)$ | 0.5680 | 0.5764 | 0.5773 | 0.5773 |

(b)




$$\lim_{x\to 1^+}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}\approx 0.577 \quad \left(\text{Actual limit is } \sqrt{3}/3.\right)$$

(c) $$\lim_{x\to 1^+}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}=\lim_{x\to 1^+}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}\cdot\frac{\sqrt{2x+1}+\sqrt{3}}{\sqrt{2x+1}+\sqrt{3}}$$
$$=\lim_{x\to 1^+}\frac{(2x+1)-3}{(x-1)\left(\sqrt{2x+1}+\sqrt{3}\right)}$$
$$=\lim_{x\to 1^+}\frac{2}{\sqrt{2x+1}+\sqrt{3}}$$
$$=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

**29.** $$\lim_{t\to a}\frac{s(a)-s(t)}{a-t}=\lim_{t\to 4}\frac{(-4.9(4)^2+200)-(-4.9t^2+200)}{4-t}$$
$$=\lim_{t\to 4}\frac{4.9(t-4)(t+4)}{4-t}$$
$$=\lim_{t\to 4}-4.9(t+4)=-39.2\text{ m/sec}$$

**31.** $\lim_{x\to 3^-}\dfrac{|x-3|}{x-3}=\lim_{x\to 3^-}\dfrac{-(x-3)}{x-3}=-1$

**33.** $\lim_{x\to 2}f(x)=0$

**35.** $\lim_{t\to 1}h(t)$ does not exist because $\lim_{t\to 1^-}h(t)=1+1=2$ and $\lim_{t\to 1^+}h(t)=\frac{1}{2}(1+1)=1$.

**37.** $f(x)=[\![x+3]\!]$

$\lim_{x\to k^+}[\![x+3]\!]=k+3$ where $k$ is an integer.

$\lim_{x\to k^-}[\![x+3]\!]=k+2$ where $k$ is an integer.

Nonremovable discontinuity at each integer $k$

Continuous on $(k, k+1)$ for all integers $k$

**39.** $f(x)=\dfrac{3x^2-x-2}{x-1}=\dfrac{(3x+2)(x-1)}{x-1}$

$\lim_{x\to 1}f(x)=\lim_{x\to 1}(3x+2)=5$

Removable discontinuity at $x=1$

Continuous on $(-\infty, 1)\cup(1,\infty)$

**41.** $f(x)=\dfrac{1}{(x-2)^2}$

$$\lim_{x\to 2}\frac{1}{(x-2)^2}=\infty$$

Nonremovable discontinuity at $x=2$

Continuous on $(-\infty, 2)\cup(2,\infty)$

**43.** $f(x)=\dfrac{3}{x+1}$

$\lim_{x\to 1^-}f(x)=-\infty$

$\lim_{x\to 1^+}f(x)=\infty$

Nonremovable discontinuity at $x=-1$

Continuous on $(-\infty, -1)\cup(-1,\infty)$

**45.** $f(x) = \csc \dfrac{\pi x}{2}$

Nonremovable discontinuities at each even integer. Continuous on

$(2k, 2k + 2)$

for all integers $k$.

**47.** $f(2) = 5$

Find $c$ so that $\lim_{x\to 2^+} (cx + 6) = 5$.

$$c(2) + 6 = 5$$
$$2c = -1$$
$$c = -\frac{1}{2}$$

**49.** $f$ is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value $c$ in $(1, 2)$ such that $2c^3 - 3 = 0$.

**51.** $f(x) = \dfrac{x^2 - 4}{|x - 2|} = (x + 2)\left[\dfrac{x - 2}{|x - 2|}\right]$

(a) $\lim_{x\to 2^-} f(x) = -4$

(b) $\lim_{x\to 2^+} f(x) = 4$

(c) $\lim_{x\to 2} f(x)$ does not exist.

**53.** $g(x) = 1 + \dfrac{2}{x}$

Vertical asymptote at $x = 0$

**55.** $f(x) = \dfrac{8}{(x - 10)^2}$

Vertical asymptote at $x = 10$

**57.** $\displaystyle\lim_{x\to -2^-} \frac{2x^2 + x + 1}{x + 2} = -\infty$

**59.** $\displaystyle\lim_{x\to -1^+} \frac{x + 1}{x^3 + 1} = \lim_{x\to -1^+} \frac{1}{x^2 - x + 1} = \frac{1}{3}$

**61.** $\displaystyle\lim_{x\to 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty$

**63.** $\displaystyle\lim_{x\to 0^+} \left(x - \frac{1}{x^3}\right) = -\infty$

**65.** $\displaystyle\lim_{x\to 0^+} \frac{\sin 4x}{5x} = \lim_{x\to 0^+} \left[\frac{4}{5}\left(\frac{\sin 4x}{4x}\right)\right] = \frac{4}{5}$

**67.** $\displaystyle\lim_{x\to 0^+} \frac{\csc 2x}{x} = \lim_{x\to 0^+} \frac{1}{x \sin 2x} = \infty$

**69.** $C = \dfrac{80{,}000p}{100 - p}$, $0 \le 0 < 100$

(a) $C(15) \approx \$14{,}117.65$  (b) $C(50) = \$80.000$

(c) $C(90) = \$720{,}000$  (d) $\displaystyle\lim_{p\to 100^-} \frac{80{,}000p}{100 - p} = \infty$

## Problem Solving for Chapter 1

**1.** (a) Perimeter $\triangle PAO = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + y^2} + 1$
$= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$

Perimeter $\triangle PBO = \sqrt{(x - 1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$
$= \sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$

(b) $r(x) = \dfrac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

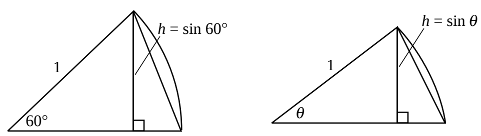
| $x$ | 4 | 2 | 1 | 0.1 | 0.01 |
|---|---|---|---|---|---|
| Perimeter $\triangle PAO$ | 33.02 | 9.08 | 3.41 | 2.10 | 2.01 |
| Perimeter $\triangle PBO$ | 33.77 | 9.60 | 3.41 | 2.00 | 2.00 |
| $r(x)$ | 0.98 | 0.95 | 1 | 1.05 | 1.005 |

(c) $\displaystyle\lim_{x\to 0^+} r(x) = \frac{1 + 0 + 1}{1 + 0 + 1} = \frac{2}{2} = 1$

**3.** (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. Hence,

$$\text{Area hexagon} = 6\left[\frac{1}{2}bh\right] = 6\left[\frac{1}{2}(1)\sin\frac{\pi}{3}\right]$$

$$= \frac{3\sqrt{3}}{2} \approx 2.598.$$




Error: $\pi - \dfrac{3\sqrt{3}}{2} \approx 0.5435.$

(b) There are $n$ triangles, each with central angle of $\theta = 2\pi/n$. Hence,

$$An = n\left[\frac{1}{2}bh\right] = n\left[\frac{1}{2}(1)\sin\frac{2\pi}{n}\right] = \frac{n\sin(2\pi/n)}{2}.$$

(c)

| $n$ | 6 | 12 | 24 | 48 | 96 |
|---|---|---|---|---|---|
| $An$ | 2.598 | 3 | 3.106 | 3.133 | 3.139 |

(d) As $n$ gets larger and larger, $2\pi/n$ approaches 0.

Letting $x = 2\pi/n$,

$$An = \frac{\sin(2\pi/n)}{2/n} = \frac{\sin(2\pi/n)}{(2\pi/n)}\pi = \frac{\sin x}{x}\pi$$

which approaches $(1)\pi = \pi$.

**5.** (a) Slope $= -\dfrac{12}{5}$

(b) Slope of tangent line is $\dfrac{5}{12}$.

$$y + 12 = \frac{5}{12}(x - 5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \quad \text{Tangent line}$$

(c) $Q = (x, y) = \left(x, \sqrt{169 - x^2}\right)$

$$m_x = \frac{-\sqrt{169 - x^2} + 12}{x - 5}$$

(d)
$$\lim_{x\to5} m_x = \lim_{x\to5} \frac{12 - \sqrt{169 - x^2}}{x - 5} \cdot \frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}}$$

$$= \lim_{x\to5} \frac{144 - (169 - x^2)}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$

$$= \lim_{x\to5} \frac{x^2 - 25}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$

$$= \lim_{x\to5} \frac{(x + 5)}{12 + \sqrt{169 - x^2}}$$

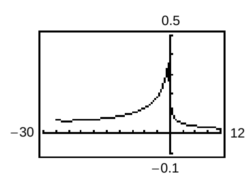
$$= \frac{10}{12 + 12} = \frac{5}{12}$$

This is the same slope as part (b).

**7.** (a)
$$3 + x^{1/3} \ge 0$$
$$x^{1/3} \ge -3$$
$$x \ge -27$$

Domain: $x \ge -27, x \ne 1$

(b)




(c)
$$\lim_{x\to-27^+} f(x) = \frac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1}$$

$$= \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$$

(d)
$$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2}$$

$$= \lim_{x\to1} \frac{3 + x^{1/3} - 4}{(x - 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \lim_{x\to1} \frac{x^{1/3} - 1}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \lim_{x\to1} \frac{1}{(x^{2/3} + x^{1/3} + 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

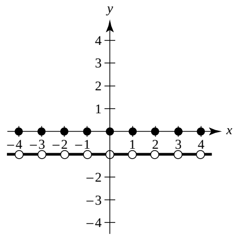
$$= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}$$

**9.** (a) $\displaystyle\lim_{x\to2} f(x) = 3$: $g_1, g_4$

(b) $f$ continuous at 2: $g_1$

(c) $\displaystyle\lim_{x\to2^-} f(x) = 3$: $g_1, g_3, g_4$

**11.**




(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\lim_{x \to 1^-} f(x) = -1$

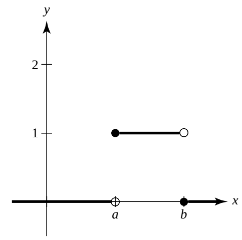
$\lim_{x \to 1^+} f(x) = -1$

$\lim_{x \to 1/2} f(x) = -1$

(c) $f$ is continuous for all real numbers except

$x = 0, \pm 1, \pm 2, \pm 3, \ldots$

**13.** (a)




(b) (i) $\lim_{x \to a^+} P_{a,\,b}(x) = 1$

(ii) $\lim_{x \to a^-} P_{a,\,b}(x) = 0$

(iii) $\lim_{x \to b^+} P_{a,\,b}(x) = 0$

(iv) $\lim_{x \to b^-} P_{a,\,b}(x) = 1$

(c) $P_{a,\,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of $u$, and above the $x$-axis, is 1.

# CHAPTER 2
# Differentiation

# CHAPTER 2
# Differentiation

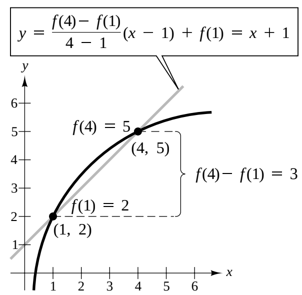
## Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Odd-Numbered Exercises

**1.** (a) $m = 0$

(b) $m = -3$

**3.** (a), (b)




(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

$= \dfrac{3}{3}(x - 1) + 2$

$= 1(x - 1) + 2$

$= x + 1$

**5.** $f(x) = 3 - 2x$ is a line. Slope $= -2$

**7.** Slope at $(1, -3) = \lim_{\Delta x \to 0} \dfrac{g(1 + \Delta x) - g(1)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [2 + 2(\Delta x)] = 2$$

**9.** Slope at $(0, 0) = \lim_{\Delta t \to 0} \dfrac{f(0 + \Delta t) - f(0)}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t}$$

$$= \lim_{\Delta t \to 0} (3 - \Delta t) = 3$$

**11.** $f(x) = 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3 - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0 = 0$$

**13.** $f(x) = -5x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -5 = -5$$

**15.** $h(s) = 3 + \dfrac{2}{3}s$

$$h'(s) = \lim_{\Delta s \to 0} \frac{h(s + \Delta s) - h(s)}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3}$$

**17.** $f(x) = 2x^2 + x - 1$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2\Delta x + 1) = 4x + 1
\end{aligned}$$

**19.** $f(x) = x^3 - 12x$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12
\end{aligned}$$

**21.** $f(x) = \dfrac{1}{x - 1}$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x - 1} - \dfrac{1}{x - 1}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} \\
&= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x - 1)(x - 1)} \\
&= -\frac{1}{(x - 1)^2}
\end{aligned}$$

**23.** $f(x) = \sqrt{x + 1}$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}\right) \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x\left[\sqrt{x + \Delta x + 1} + \sqrt{x + 1}\right]} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \\
&= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}
\end{aligned}$$

**25.** (a) $f(x) = x^2 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

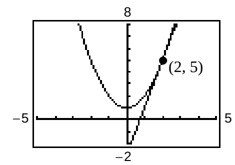
At $(2, 5)$, the slope of the tangent line is $m = 2(2) = 4$. The equation of the tangent line is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3.$$

(b)




**27.** (a) $f(x) = x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

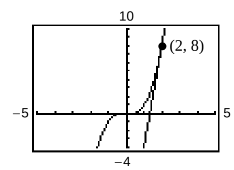
$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$. The equation of the tangent line is

$$y - 8 = 12(x - 2)$$

$$y = 12x - 16.$$

(b)




**29.** (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

At $(1, 1)$, the slope of the tangent line is

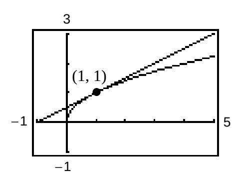
$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)



**31.** (a) $f(x) = \dfrac{4}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) + \dfrac{4}{x + \Delta x} - \left(x + \dfrac{4}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)}$$

$$= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

At $(4, 5)$, the slope of the tangent line is

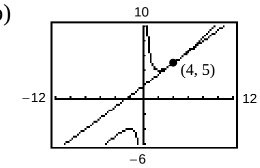
$$m = 1 - \frac{4}{16} = \frac{3}{4}$$

The equation of the tangent line is

$$y - 5 = \frac{3}{4}(x - 4)$$

$$y = \frac{3}{4}x + 2$$

(b)




**33.** From Exercise 27 we know that $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x = \pm 1.$$

Therefore, at the points $(1, 1)$ and $(-1, -1)$ the tangent lines are parallel to $3x - y + 1 = 0$. These lines have equations

$$y - 1 = 3(x - 1) \qquad \text{and} \qquad y + 1 = 3(x + 1)$$

$$y = 3x - 2 \qquad\qquad\qquad y = 3x + 2.$$

**35.** Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is $-\frac{1}{2}$, we have

$$-\frac{1}{2x\sqrt{x}} = -\frac{1}{2}$$

$$x = 1.$$

Therefore, at the point $(1, 1)$ the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{3}{2}.$$

**37.** $g(5) = 2$ because the tangent line passes through $(5, 2)$

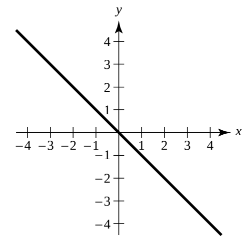
$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$

**39.** $f(x) = x \Rightarrow f'(x) = 1$ (b)

**41.** $f(x) = \sqrt{x} \Rightarrow f'(x)$ matches (a)

(decreasing slope as $x \to \infty$)

**43.**



Answers will vary.

Sample answer: $y = -x$

**45.** (a) If $f'(c) = 3$ and $f$ is odd, then $f'(-c) = f'(c) = 3$

(b) If $f'(c) = 3$ and $f$ is even, then $f'(-c) = -f(c) = -3$

**47.** Let $(x_0, y_0)$ be a point of tangency on the graph of $f$. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and $(x_0, y_0)$ equals the derivative of $f$ at $x_0$:

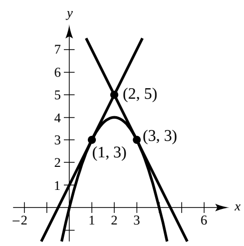
$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$

$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$

$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$

$$0 = x_0^2 - 4x_0 + 3$$

$$0 = (x_0 - 1)(x_0 - 3) \implies x_0 = 1, 3$$




Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and $-2$. The equations of the tangent lines are

$$y - 5 = 2(x - 2) \qquad y - 5 = -2(x - 2)$$

$$y = 2x + 1 \qquad y = -2x + 9$$

**49.** (a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, $g$ is decreasing (falling) at $x = 1$.

(d) Because $g'(-4) = \frac{7}{3}$, $g$ is increasing (rising) at $x = -4$.

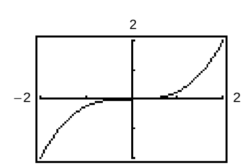
(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that $g$ is decreasing (falling) at $x = 2$.

**51.** $f(x) = \frac{1}{4}x^3$

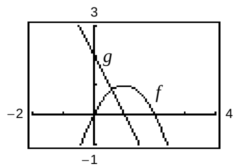
By the limit definition of the derivative we have $f'(x) = \frac{3}{4}x^2$.

| $x$ | $-2$ | $-1.5$ | $-1$ | $-0.5$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-2$ | $-\frac{27}{32}$ | $-\frac{1}{4}$ | $-\frac{1}{32}$ | 0 | $\frac{1}{32}$ | $\frac{1}{4}$ | $\frac{27}{32}$ | 2 |
| $f'(x)$ | 3 | $\frac{27}{16}$ | $\frac{3}{4}$ | $\frac{3}{16}$ | 0 | $\frac{3}{16}$ | $\frac{3}{4}$ | $\frac{27}{16}$ | 3 |



**53.** $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= (2(x + 0.01) - (x + 0.01)^2 - 2x + x^2) + 100$

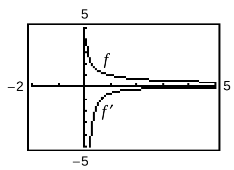



The graph of $g(x)$ is approximately the graph of $f'(x)$.

**55.** $f(2) = 2(4 - 2) = 4,\ f(2.1) = 2.1(4 - 2.1) = 3.99$

$f'(2) \approx \dfrac{3.99 - 4}{2.1 - 2} = -0.1$ [Exact: $f'(2) = 0$]

**57.** $f(x) = \dfrac{1}{\sqrt{x}}$ and $f'(x) = \dfrac{-1}{2x^{3/2}}$.




As $x \to \infty$, $f$ is nearly horizontal and thus $f' \approx 0$.

**59.** $f(x) = 4 - (x - 3)^2$

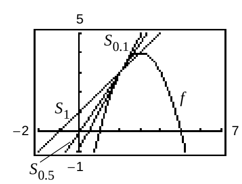
$S_{\Delta x}(x) = \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2)$

$= \dfrac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x}(x - 2) + 3 = \dfrac{1 - (\Delta x - 1)^2}{\Delta x}(x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3$

(a) $\Delta x = 1$: $S_{\Delta x} = (x - 2) + 3 = x + 1$

$\Delta x = 0.5$: $S_{\Delta x} = \left(\dfrac{3}{2}\right)(x - 2) + 3 = \dfrac{3}{2}x$

$\Delta x = 0.1$: $S_{\Delta x} = \left(\dfrac{19}{10}\right)(x - 2) + 3 = \dfrac{19}{10}x - \dfrac{4}{5}$




(b) As $\Delta x \to 0$, the line approaches the tangent line to $f$ at $(2, 3)$.

**61.** $f(x) = x^2 - 1, c = 2$

$f'(2) = \lim_{x\to2} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x\to2} \dfrac{(x^2 - 1) - 3}{x - 2} = \lim_{x\to2} \dfrac{(x - 2)(x + 2)}{x - 2} = \lim_{x\to2} (x + 2) = 4$

**63.** $f(x) = x^3 + 2x^2 + 1, c = -2$

$f'(-2) = \lim_{x\to-2} \dfrac{f(x) - f(-2)}{x + 2} = \lim_{x\to-2} \dfrac{x^2(x + 2)}{x + 2} = \lim_{x\to-2} \dfrac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x\to-2} x^2 = 4$

**65.** $g(x) = \sqrt{|x|}, c = 0$

$g'(0) = \lim_{x\to0} \dfrac{g(x) - g(0)}{x - 0} = \lim_{x\to0} \dfrac{\sqrt{|x|}}{x}$. Does not exist.

As $x \to 0^-$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{-1}{\sqrt{x}} \to -\infty$

As $x \to 0^+$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{1}{\sqrt{x}} \to \infty$

**67.** $f(x) = (x - 6)^{2/3}, c = 6$

$f'(6) = \lim_{x\to6} \dfrac{f(x) - f(6)}{x - 6}$

$= \lim_{x\to6} \dfrac{(x - 6)^{2/3} - 0}{x - 6}$

$= \lim_{x\to6} \dfrac{1}{(x - 6)^{1/3}}$

Does not exist.

**69.** $h(x) = |x + 5|, c = -5$

$$h'(-5) = \lim_{x\to-5} \frac{h(x) - h(-5)}{x - (-5)}$$
$$= \lim_{x\to-5} \frac{|x + 5| - 0}{x + 5}$$
$$= \lim_{x\to-5} \frac{|x + 5|}{x + 5}$$

Does not exist.

**71.** $f(x)$ is differentiable everywhere except at $x = -3$. (Sharp turn in the graph.)

**73.** $f(x)$ is differentiable everywhere except at $x = -1$. (Discontinuity)

**75.** $f(x)$ is differentiable everywhere except at $x = 3$. (Sharp turn in the graph)

**77.** $f(x)$ is differentiable on the interval $(1, \infty)$. (At $x = 1$ the tangent line is vertical)

**79.** $f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

**81.** $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x\to1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, $f$ is not differentiable at $x = 1$.

**83.** $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^-} \frac{(x - 1)^3 - 0}{x - 1}$$
$$= \lim_{x\to1^-} (x - 1)^2 = 0.$$

The derivative from the right is

$$\lim_{x\to1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^+} \frac{(x - 1)^2 - 0}{x - 1}$$
$$= \lim_{x\to1^+} (x - 1) = 0.$$

These one-sided limits are equal. Therefore, $f$ is differentiable at $x = 1$. $(f'(1) = 0)$

**85.** Note that $f$ is continuous at $x = 2$. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

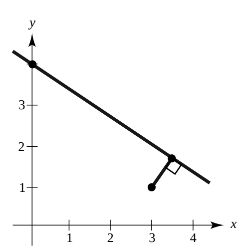
The derivative from the left is $\displaystyle\lim_{x\to2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x\to2^-} (x + 2) = 4.$

The derivative from the right is $\displaystyle\lim_{x\to2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x\to2^+} 4 = 4.$

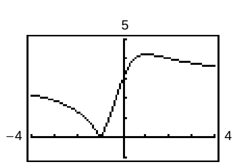
The one-sided limits are equal. Therefore, $f$ is differentiable at $x = 2$. $(f'(2) = 4)$

**87.** (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$
$$= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}.$$




(b)




The function $d$ is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.

**89.** False. the slope is $\lim_{\triangle x \to 0} \frac{f(2 + \triangle x) - f(2)}{\triangle x}$.

**91.** False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is $-1$ and the derivative from the right at $x = 0$ is 1. At $x = 0$, the derivative does not exist.

**93.** $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, we have $-|x| \leq x \sin(1/x) \leq |x|,\ x \neq 0$. Thus, $\lim_{x \to 0} x \sin(1/x) = 0 = f(0)$ and $f$ is continuous at $x = 0$. Using the alternative form of the derivative we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \left(\sin \frac{1}{x}\right).$$

Since this limit does not exist (it oscillates between $-1$ and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have $-x^2 \leq x^2 \sin(1/x) \leq x^2, x \neq 0$. Thus, $\lim_{x \to 0} x^2 \sin(1/x) = 0 = f(0)$ and $f$ is continuous at $x = 0$. Using the alternative form of the derivative again we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

Therefore, $g$ is differentiable at $x = 0$, $g'(0) = 0$.

## Section 2.2 Basic Differentiation Rules and Rates of Change

**1.** (a) $y = x^{1/2}$, $y' = \frac{1}{2}x^{-1/2}$, $y'(1) = \frac{1}{2}$

(b) $y = x^{3/2}$, $y' = \frac{3}{2}x^{1/2}$, $y'(1) = \frac{3}{2}$

(c) $y = x^2$, $y' = 2x$, $y'(1) = 2$

(d) $y = x^3$, $y' = 3x^2$, $y'(1) = 3$

**3.** $y = 8$

$y' = 0$

**5.** $y = x^6$

$y' = 6x^5$

**7.** $y = \frac{1}{x^7} = x^{-7}$

$y' = -7x^{-8} = \frac{-7}{x^8}$

**9.** $y = \sqrt[5]{x} = x^{1/5}$

$y' = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$

**11.** $f(x) = x + 1$

$f'(x) = 1$

**13.** $f(t) = -2t^2 + 3t - 6$

$f'(x) = -4t + 3$

**15.** $g(x) = x^2 + 4x^3$

$g'(x) = 2x + 12x^2$

**17.** $s(t) = t^3 - 2t + 4$

$s'(t) = 3t^2 - 2$

**19.** $y = \frac{\pi}{2} \sin \theta - \cos \theta$

$y' = \frac{\pi}{2} \cos \theta + \sin \theta$

**21.** $y = x^2 - \frac{1}{2} \cos x$

$y' = 2x + \frac{1}{2} \sin x$

**23.** $y = \frac{1}{x} - 3 \sin x$

$y' = -\frac{1}{x^2} - 3 \cos x$

| | Function | Rewrite | Derivative | Simplify |
|---|---|---|---|---|
| **25.** | $y = \dfrac{5}{2x^2}$ | $y = \dfrac{5}{2}x^{-2}$ | $y' = -5x^{-3}$ | $y' = \dfrac{-5}{x^3}$ |
| **27.** | $y = \dfrac{3}{(2x)^3}$ | $y = \dfrac{3}{8}x^{-3}$ | $y' = \dfrac{-9}{8}x^{-4}$ | $y' = \dfrac{-9}{8x^4}$ |
| **29.** | $y = \dfrac{\sqrt{x}}{x}$ | $y = x^{-1/2}$ | $y' = -\dfrac{1}{2}x^{-3/2}$ | $y' = -\dfrac{1}{2x^{3/2}}$ |

**31.** $f(x) = \dfrac{3}{x^2} = 3x^{-2}, (1, 3)$

$f'(x) = -6x^{-3} = \dfrac{-6}{x^3}$

$f'(1) = -6$

**33.** $f(x) = -\dfrac{1}{2} + \dfrac{7}{5}x^3, \left(0, -\dfrac{1}{2}\right)$

$f'(x) = \dfrac{21}{5}x^2$

$f'(0) = 0$

**35.** $y = (2x + 1)^2, \; (0, 1)$

$= 4x^2 + 4x + 1$

$y' = 8x + 4$

$y'(0) = 4$

**37.** $f(\theta) = 4 \sin \theta - \theta, \; (0, 0)$

$f'(\theta) = 4 \cos \theta - 1$

$f'(0) = 4(1) - 1 = 3$

**39.** $f(x) = x^2 + 5 - 3x^{-2}$

$f'(x) = 2x + 6x^{-3} = 2x + \dfrac{6}{x^3}$

**41.** $g(t) = t^2 - \dfrac{4}{t^3} = t^2 - 4t^{-3}$

$g'(t) = 2t + 12t^{-4} = 2t + \dfrac{12}{t^4}$

**43.** $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3 - 8}{x^3}$

**45.** $y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$

**47.** $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$f'(x) = \dfrac{1}{2}x^{-1/2} + 2x^{-2/3} = \dfrac{1}{2\sqrt{x}} + \dfrac{2}{x^{2/3}}$

**49.** $h(s) = s^{4/5} - s^{2/3}$

$h'(s) = \dfrac{4}{5}s^{-4/5} - \dfrac{2}{3}s^{-1/3} = \dfrac{4}{5s^{1/5}} - \dfrac{2}{3s^{1/3}}$

**51.** $f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x$

$f'(x) = 3x^{-1/2} - 5 \sin x = \dfrac{3}{\sqrt{x}} - 5 \sin x$

**53.** (a) $y = x^4 - 3x^2 + 2$

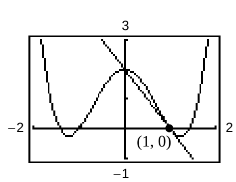
$y' = 4x^3 - 6x$

At $(1, 0)$: $y' = 4(1)^3 - 6(1) = -2$.

Tangent line: $y - 0 = -2(x - 1)$

$2x + y - 2 = 0$

(b)




**55.** (a) $f(x) = \dfrac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

$f'(x) = \dfrac{-3}{2}x^{-7/4} = \dfrac{-3}{2x^{7/4}}$

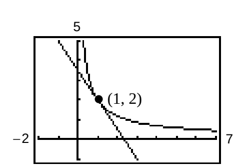
At $(1, 2)$, $f'(1) = \dfrac{-3}{2}$

Tangent line: $y - 2 = -\dfrac{3}{2}(x - 1)$

$y = -\dfrac{3}{2}x + \dfrac{7}{2}$

$3x + 2y - 7 = 0$

(b)



**57.** $y = x^4 - 8x^2 + 2$

$y' = 4x^3 - 16x$

$= 4x(x^2 - 4)$

$= 4x(x - 2)(x + 2)$

$y' = 0 \implies x = 0, \pm 2$

Horizontal tangents: $(0, 2), (2, -14), (-2, -14)$

**59.** $y = \dfrac{1}{x^2} = x^{-2}$

$y' = -2x^{-3} = \dfrac{-2}{x^3}$ cannot equal zero.

Therefore, there are no horizontal tangents.

**61.** $y = x + \sin x,\ 0 \le x < 2\pi$

$y' = 1 + \cos x = 0$

$\cos x = -1 \implies x = \pi$

At $x = \pi, y = \pi$.

Horizontal tangent: $(\pi, \pi)$

**63.** $x^2 - kx = 4x - 9$ Equate functions

$2x - k = 4$ Equate derivatives

Hence, $k = 2x - 4$ and

$x^2 - (2x - 4)x = 4x - 9 \Longrightarrow -x^2 = -9 \Longrightarrow x = \pm 3$.

For $x = 3, k = 2$ and for $x = -3, k = -10$.

**65.** $\dfrac{k}{x} = -\dfrac{3}{4}x + 3$ Equate functions

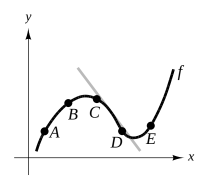
$-\dfrac{k}{x^2} = -\dfrac{3}{4}$ Equate derivatives

Hence, $k = \dfrac{3}{4}x^2$ and $\dfrac{\frac{3}{4}x^2}{x} = \dfrac{-3}{4}x + 3 \Longrightarrow \dfrac{3}{4}x = -\dfrac{3}{4}x + 3 \Longrightarrow \dfrac{3}{2}x = 3 \Longrightarrow x = 2 \Longrightarrow k = 3$.

**67.** (a) The slope appears to be steepest between $A$ and $B$.

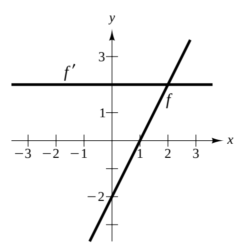
(b) The average rate of change between $A$ and $B$ is **greater** than the instantaneous rate of change at $B$.

(c)




**69.** $g(x) = f(x) + 6 \Longrightarrow g'(x) = f'(x)$

**71.**




If $f$ is linear then its derivative is a constant function.

$f(x) = ax + b$

$f'(x) = a$

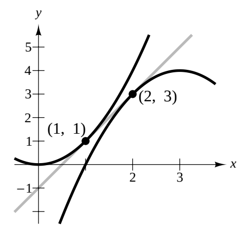
**73.** Let $(x_1, y_1)$ and $(x_2, y_2)$ be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively. The derivatives of these functions are

$$y' = 2x \implies m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \implies m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$,




$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (-x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

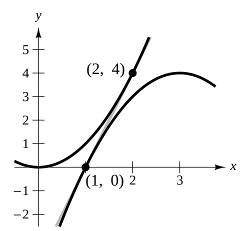
$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$




$x_2 = 1 \implies y_2 = 0$, $x_1 = 2$ and $y_1 = 4$

Thus, the tangent line through $(1, 0)$ and $(2, 4)$ is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \implies y = 4x - 4.$$

$x_2 = 2 \implies y_2 = 3$, $x_1 = 1$ and $y_1 = 1$

Thus, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \implies y = 2x - 1.$$

**75.** $f(x) = \sqrt{x}$, $(-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{x}y$$

$$4 + x = 2\sqrt{x}\sqrt{x}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

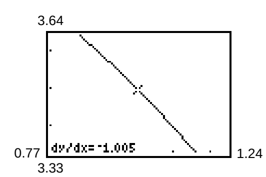
The point $(4, 2)$ is on the graph of $f$.

Tangent line: $\quad y - 2 = \dfrac{0 - 2}{-4 - 4}(x - 4)$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

**77.** $f'(1) = -1$

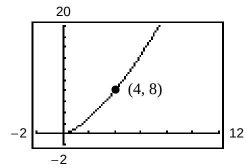


**79.** (a) One possible secant is between (3.9, 7.7019) and (4, 8):

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

$$y = S(x) = 2.981x - 3.924$$

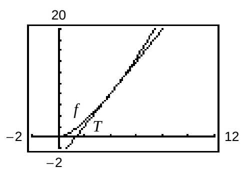



(b) $f'(x) = \frac{3}{2}x^{1/2} \implies f'(4) = \frac{3}{2}(2) = 3$

$T(x) = 3(x - 4) + 8 = 3x - 4$

$S(x)$ is an approximation of the tangent line $T(x)$.

(c) As you move further away from (4, 8), the accuracy of the approximation $T$ gets worse.




(d)

| $\Delta x$ | $-3$ | $-2$ | $-1$ | $-0.5$ | $-0.1$ | 0 | 0.1 | 0.5 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(4 + \Delta x)$ | 1 | 2.828 | 5.196 | 6.548 | 7.702 | 8 | 8.302 | 9.546 | 11.180 | 14.697 | 18.520 |
| $T(4 + \Delta x)$ | $-1$ | 2 | 5 | 6.5 | 7.7 | 8 | 8.3 | 9.5 | 11 | 14 | 17 |

**81.** False. Let $f(x) = x^2$ and $g(x) = x^2 + 4$. Then $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.

**83.** False. If $y = \pi^2$, then $dy/dx = 0$. ($\pi^2$ is a constant.)

**85.** True. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

**87.** $f(t) = 2t + 7, [1, 2]$

$f'(t) = 2$

Instantaneous rate of change is the constant 2.
Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because $f$ is a line of slope 2.)

**89.** $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$$(1, -1) \implies f'(1) = 1$$

$$\left(2, -\frac{1}{2}\right) \implies f'(2) = \frac{1}{4}$$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

**91.** (a) $s(t) = -16t^2 + 1362$

$v(t) = -32t$

(b) $\dfrac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48$ ft/sec

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32$ ft/sec.

When $t = 2$: $v(2) = -64$ ft/sec.

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \implies t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $v\left(\dfrac{\sqrt{1362}}{4}\right) = -32\left(\dfrac{\sqrt{1362}}{4}\right)$

$= -8\sqrt{1362} \approx -295.242$ ft/sec

**93.** $s(t) = -4.9t^2 + v_0 t + s_0$

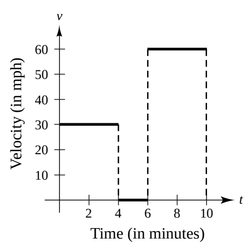
$= -4.9t^2 + 120t$

$v(t) = -9.8t + 120$

$v(5) = -9.8(5) + 120 = 71$ m/sec

$v(10) = -9.8(10) + 120 = 22$ m/sec

**95.**




(The velocity has been converted to miles per hour)

**97.** $v = 40 \text{ mph} = \frac{2}{3}$ mi/min

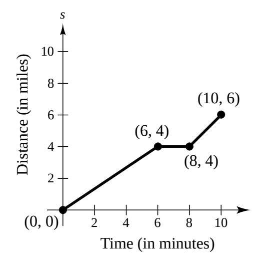
$\left(\frac{2}{3} \text{ mi/min}\right)(6 \text{ min}) = 4$ mi

$v = 0 \text{ mph} = 0$ mi/min

$(0 \text{ mi/min})(2 \text{ min}) = 0$ mi

$v = 60 \text{ mph} = 1$mi/min

$(1 \text{ mi/min})(2 \text{ min}) = 2$ mi




**99.** (a) Using a graphing utility, you obtain

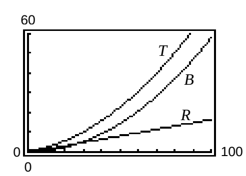
$R = 0.167v - 0.02.$

(b) Using a graphing utility, you obtain

$B = 0.00586v^2 - 0.0239v + 0.46.$

(c) $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(d)




(e) $\dfrac{dT}{dv} = 0.01172v + 0.1431$

For $v = 40$, $T'(40) \approx 0.612$.

For $v = 80$, $T'(80) \approx 1.081$.

For $v = 100$, $T'(100) \approx 1.315$.

(f) For increasing speeds, the total stopping distance increases.

**101.** $A = s^2, \dfrac{dA}{ds} = 2s$

When $s = 4$ m,

$\dfrac{dA}{ds} = 8$ square meters per meter change in $s$.

**103.** $$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

$$\frac{dC}{dQ} = -\frac{1{,}008{,}000}{Q^2} + 6.3$$

$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$

When $Q = 350$, $\dfrac{dC}{dQ} \approx -\$1.93$.

**105.** (a) $f'(1.47)$ is the rate of change of the amount of gasoline sold when the price is \$1.47 per gallon.

(b) $f'(1.47)$ is usually negative. As prices go up, sales go down.

**107.** $y = ax^2 + bx + c$

Since the parabola passes through $(0, 1)$ and $(1, 0)$, we have

$(0, 1)$: $1 = a(0)^2 + b(0) + c \implies c = 1$

$(1, 0)$: $0 = a(1)^2 + b(1) + 1 \implies b = -a - 1.$

Thus, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point $(1, 0)$.

$$\begin{aligned} y' &= 2ax + (-a - 1) \\ 1 &= 2a(1) + (-a - 1) \\ 1 &= a - 1 \\ a &= 2 \\ b &= -a - 1 = -3 \end{aligned}$$

Therefore, $y = 2x^2 - 3x + 1$.

**109.** $y = x^3 - 9x$

$y' = 3x^2 - 9$

Tangent lines through $(1, -9)$:

$$\begin{aligned} y + 9 &= (3x^2 - 9)(x - 1) \\ (x^3 - 9x) + 9 &= 3x^3 - 3x^2 - 9x + 9 \\ 0 &= 2x^3 - 3x^2 = x^2(2x - 3) \\ x &= 0 \text{ or } x = \tfrac{3}{2} \end{aligned}$$

The points of tangency are $(0, 0)$ and $\left(\frac{3}{2}, -\frac{81}{8}\right)$. At $(0, 0)$ the slope is $y'(0) = -9$. At $\left(\frac{3}{2}, -\frac{81}{8}\right)$ the slope is $y'\left(\frac{3}{2}\right) = -\frac{9}{4}$.

Tangent lines:

$$\begin{aligned} y - 0 &= -9(x - 0) \\ y &= -9x \\ 9x + y &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y + \tfrac{81}{8} &= -\tfrac{9}{4}\left(x - \tfrac{3}{2}\right) \\ y &= -\tfrac{9}{4}x - \tfrac{27}{4} \\ 9x + 4y + 27 &= 0 \end{aligned}$$

**111.** $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

$f$ must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} ax^3 = 8a \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x^2 + b) = 4 + b \end{aligned}\right\} \quad \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For $f$ to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$$\begin{aligned} 3a(2)^2 &= 2(2) \\ 12a &= 4 \\ a &= \tfrac{1}{3} \\ b &= 8a - 4 = -\tfrac{4}{3} \end{aligned}$$

**113.** Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \to 0} \sin x\left(\frac{\sin \Delta x}{\Delta x}\right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

## Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

**1.** $g(x) = (x^2 + 1)(x^2 - 2x)$

$$\begin{aligned} g'(x) &= (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x) \\ &= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2 \\ &= 4x^3 - 6x^2 + 2x - 2 \end{aligned}$$

**3.** $h(t) = \sqrt[3]{t}(t^2 + 4) = t^{1/3}(t^2 + 4)$

$$\begin{aligned} h'(t) &= t^{1/3}(2t) + (t^2 + 4)\frac{1}{3}t^{-2/3} \\ &= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}} \\ &= \frac{7t^2 + 4}{3t^{2/3}} \end{aligned}$$

**5.** $f(x) = x^3 \cos x$

$$\begin{aligned} f'(x) &= x^3(-\sin x) + \cos x(3x^2) \\ &= 3x^2 \cos x - x^3 \sin x \end{aligned}$$

**7.** $f(x) = \dfrac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

**9.** $h(x) = \dfrac{\sqrt[3]{x}}{x^3 + 1} = \dfrac{x^{1/3}}{x^3 + 1}$

$$\begin{aligned} h'(x) &= \frac{(x^3 + 1)\frac{1}{3}x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2} \\ &= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2} \\ &= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2} \end{aligned}$$

**11.** $g(x) = \dfrac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

**13.** $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$\begin{aligned} f'(x) &= (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15 \end{aligned}$$

$$f'(0) = -15$$

**15.** $f(x) = \dfrac{x^2 - 4}{x - 3}$

$$\begin{aligned} f'(x) &= \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2} \\ &= \frac{x^2 - 6x + 4}{(x - 3)^2} \end{aligned}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

**17.** $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}(4 - \pi)$$

| Function | Rewrite | Derivative | Simplify |
|---|---|---|---|
| **19.** $y = \dfrac{x^2 + 2x}{3}$ | $y = \dfrac{1}{3}x^2 + \dfrac{2}{3}x$ | $y' = \dfrac{2}{3}x + \dfrac{2}{3}$ | $y' = \dfrac{2x + 2}{3}$ |
| **21.** $y = \dfrac{7}{3x^3}$ | $y = \dfrac{7}{3}x^{-3}$ | $y' = -7x^{-4}$ | $y' = -\dfrac{7}{x^4}$ |
| **23.** $y = \dfrac{4x^{3/2}}{x}$ | $y = 4\sqrt{x}, x > 0$ | $y' = 2x^{-1/2}$ | $y' = \dfrac{2}{\sqrt{x}}$ |

**25.** $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$$
$$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$$
$$= \frac{2}{(x + 1)^2}, x \neq 1$$

**27.** $f(x) = x\left(1 - \dfrac{4}{x + 3}\right) = x - \dfrac{4x}{x + 3}$

$$f'(x) = 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} = \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2}$$
$$= \frac{x^2 + 6x - 3}{(x + 3)^2}$$

**29.** $f(x) = \dfrac{2x + 5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$

$$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$$
$$= \frac{2x - 5}{2x\sqrt{x}} = \frac{2x - 5}{2x^{3/2}}$$

**31.** $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$

$$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$$

**33.** $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3} = \dfrac{2x - 1}{x(x - 3)} = \dfrac{2x - 1}{x^2 - 3x}$

$$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$$
$$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$$

**35.** $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$

$$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$$
$$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$$
$$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$$
$$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$$

**37.** $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$

$$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$$
$$= \frac{-4xc^2}{(x^2 - c^2)^2}$$

**39.** $f(x) = t^2 \sin t$

$$f'(t) = t^2 \cos t + 2t \sin t$$
$$= t(t \cos t + 2 \sin t)$$

**41.** $f(t) = \dfrac{\cos t}{t}$

$$f'(t) = \frac{-t\sin t - \cos t}{t^2} = -\frac{t\sin t + \cos t}{t^2}$$

**43.** $f(x) = -x + \tan x$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

**45.** $g(t) = \sqrt[4]{t} + 8\sec t = t^{1/4} + 8\sec t$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8\sec t\tan t = \frac{1}{4t^{3/4}} + 8\sec t\tan t$$

**47.** $y = \dfrac{3(1 - \sin x)}{2\cos x} = \dfrac{3}{2}(\sec x - \tan x)$

$$y' = \frac{3}{2}(\sec x\tan x - \sec^2 x) = \frac{3}{2}\sec x(\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x\tan x - \tan^2 x - 1)$$

**49.** $y = -\csc x - \sin x$

$$y' = \csc x\cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x\cot^2 x$$

**51.** $f(x) = x^2\tan x$

$$f'(x) = x^2\sec^2 x + 2x\tan x$$

$$= x(x\sec^2 x + 2\tan x)$$

**53.** $y = 2x\sin x + x^2\cos x$

$$y' = 2x\cos x + 2\sin x + x^2(-\sin x) + 2x\cos x$$

$$= 4x\cos x + 2\sin x - x^2\sin x$$

**55.** $g(x) = \left(\dfrac{x+1}{x+2}\right)(2x - 5)$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2}$$ (form of answer may vary)

**57.** $g(\theta) = \dfrac{\theta}{1 - \sin\theta}$

$$g'(\theta) = \frac{1 - \sin\theta + \theta\cos\theta}{(\sin\theta - 1)^2}$$ (form of answer may vary)

**59.** $y = \dfrac{1 + \csc x}{1 - \csc x}$

$$y' = \frac{(1 - \csc x)(-\csc x\cot x) - (1 + \csc x)(\csc x\cot x)}{(1 - \csc x)^2} = \frac{-2\csc x\cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)\left(\sqrt{3}\right)}{(1-2)^2} = -4\sqrt{3}$$

**61.** $h(t) = \dfrac{\sec t}{t}$

$$h'(t) = \frac{t(\sec t\tan t) - (\sec t)(1)}{t^2}$$

$$= \frac{\sec t(t\tan t - 1)}{t^2}$$

$$h'(\pi) = \frac{\sec\pi(\pi\tan\pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

**63.** (a) $f(x) = (x^3 - 3x + 1)(x + 2), \quad (1, -3)$

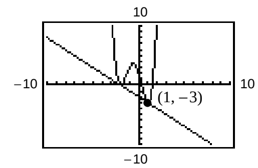
$f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$

$= 4x^3 + 6x^2 - 6x - 5$

$f'(1) = -1 =$ slope at $(1, -3)$.

Tangent line: $y + 3 = -1(x - 1) \Longrightarrow y = -x - 2$

(b)



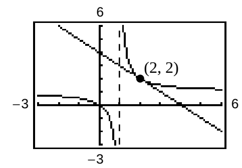

**65.** (a) $f(x) = \dfrac{x}{x - 1}$, $(2, 2)$

$f'(x) = \dfrac{(x - 1)(1) - x(1)}{(x - 1)^2} = \dfrac{-1}{(x - 1)^2}$

$f'(2) = \dfrac{-1}{(2 - 1)^2} = -1 =$ slope at $(2, 2)$.

Tangent line: $y - 2 = -1(x - 2) \Longrightarrow y = -x + 4$

(b)




**67.** (a) $f(x) = \tan x, \quad \left(\dfrac{\pi}{4}, 1\right)$

$f'(x) = \sec^2 x$

$f'\left(\dfrac{\pi}{4}\right) = 2 =$ slope at $\left(\dfrac{\pi}{4}, 1\right)$.

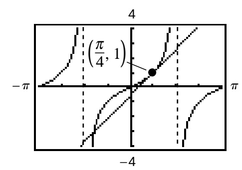
Tangent line:

$y - 1 = 2\left(x - \dfrac{\pi}{4}\right)$

$y - 1 = 2x - \dfrac{\pi}{2}$

$4x - 2y - \pi + 2 = 0$

(b)




**69.** $f(x) = \dfrac{x^2}{x - 1}$

$f'(x) = \dfrac{(x - 1)(2x) - x^2(1)}{(x - 1)^2}$

$= \dfrac{x^2 - 2x}{(x - 1)^2} = \dfrac{x(x - 2)}{(x - 1)^2}$

$f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

**71.** $f'(x) = \dfrac{(x + 2)3 - 3x(1)}{(x + 2)^2} = \dfrac{6}{(x + 2)^2}$

$g'(x) = \dfrac{(x + 2)5 - (5x + 4)(1)}{(x + 2)^2} = \dfrac{6}{(x + 2)^2}$

$g(x) = \dfrac{5x + 4}{(x + 2)} = \dfrac{3x}{(x + 2)} + \dfrac{2x + 4}{(x + 2)} = f(x) + 2$

$f$ and $g$ differ by a constant.

**73.** $f(x) = x^n \sin x$

$f'(x) = x^n \cos x + nx^{n-1} \sin x$

$= x^{n-1}(x \cos x + n \sin x)$

When $n = 1$: $f'(x) = x \cos x + \sin x$.

When $n = 2$: $f'(x) = x(x \cos x + 2 \sin x)$.

When $n = 3$: $f'(x) = x^2(x \cos x + 3 \sin x)$.

When $n = 4$: $f'(x) = x^3(x \cos x + 4 \sin x)$.

For general $n$, $f'(x) = x^{n-1}(x \cos x + n \sin x)$.

**75.** Area $= A(t) = (2t + 1)\sqrt{t} = 2t^{3/2} + t^{1/2}$

$A'(t) = 2\left(\dfrac{3}{2}t^{1/2}\right) + \dfrac{1}{2}t^{-1/2}$

$= 3t^{1/2} + \dfrac{1}{2}t^{-1/2}$

$= \dfrac{6t + 1}{2\sqrt{t}}$ cm$^2$/sec

**77.** $C = 100\left(\dfrac{200}{x^2} + \dfrac{x}{x+30}\right), \ 1 \le x$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

(a) When $x = 10$: $\dfrac{dC}{dx} = -\$38.13$.

(b) When $x = 15$: $\dfrac{dC}{dx} = -\$10.37$.

(c) When $x = 20$: $\dfrac{dC}{dx} = -\$3.80$.

As the order size increases, the cost per item decreases.

**79.** $P(t) = 500\left[1 + \dfrac{4t}{50 + t^2}\right]$

$$P'(t) = 500\left[\frac{(50+t^2)(4) - (4t)(2t)}{(50+t^2)^2}\right]$$

$$= 500\left[\frac{200 - 4t^2}{(50+t^2)^2}\right]$$

$$= 2000\left[\frac{50 = t^2}{(50+t^2)^2}\right]$$

$P'(2) \approx 31.55$ bacteria per hour

**81.** (a) $\sec x = \dfrac{1}{\cos x}$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(b) $\csc x = \dfrac{1}{\sin x}$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

(c) $\cot x = \dfrac{\cos x}{\sin x}$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

**83.** $f(x) = 4x^{3/2}$

$f'(x) = 6x^{1/2}$

$f''(x) = 3x^{-1/2} = \dfrac{3}{\sqrt{x}}$

**85.** $f(x) = \dfrac{x}{x-1}$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

**87.** $f(x) = 3\sin x$

$f'(x) = 3\cos x$

$f''(x) = -3\sin x$

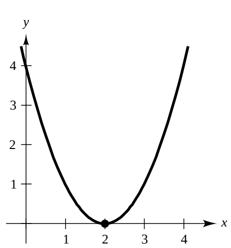
**89.** $f'(x) = x^2$

$f''(x) = 2x$

**91.** $f'''(x) = 2\sqrt{x}$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$

**93.**




$f(2) = 0$

One such function is $f(x) = (x-2)^2$.

**95.** $f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + 4$

$= 0$

**97.** $f(x) = \dfrac{g(x)}{h(x)}$

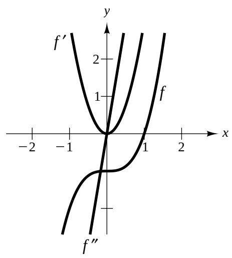
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

**99.**




It appears that $f$ is cubic; so $f'$ would be quadratic and $f''$ would be linear.

**101.** $v(t) = 36 - t^2,\ 0 \le t \le 6$

$a(t) = -2t$

$v(3) = 27$ m/sec

$a(3) = -6$ m/sec

The speed of the object is decreasing.

**103.** $v(t) = \dfrac{100t}{2t + 15}$

$$a(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2}$$

$$= \frac{1500}{(2t + 15)^2}$$

(a) $a(5) = \dfrac{1500}{[2(5) + 15]^2} = 2.4$ ft/sec$^2$

(b) $a(10) = \dfrac{1500}{[2(10) + 15]^2} \approx 1.2$ ft/sec$^2$

(c) $a(20) = \dfrac{1500}{[2(20) + 15]^2} \approx 0.5$ ft/sec$^2$

**105.** $f(x) = g(x)h(x)$

(a) $f'(x) = g(x)h'(x) + h(x)g'(x)$

$$f''(x) = g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)$$
$$= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)$$
$$f'''(x) = g(x)h'''(x) + g'(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x)$$
$$= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$$
$$f^{(4)}(x) = g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g'(x)h'''(x) + 3g''(x)h''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x)$$
$$+ g'''(x)h'(x) + g^{(4)}(x)h(x)$$
$$= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$$

(b) $$f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{1[(n-1)(n-2)\cdots(2)(1)]}g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{(2)(1)[(n-2)(n-3)\cdots(2)(1)]}g''(x)h^{(n-2)}(x)$$
$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{(3)(2)(1)[(n-3)(n-4)\cdots(2)(1)]}g'''(x)h^{(n-3)}(x) + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{[(n-1)(n-2)\cdots(2)(1)](1)}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$
$$= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots$$
$$+ \frac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

**Note:** $n! = n(n-1)\ldots 3 \cdot 2 \cdot 1$ (read "$n$ factorial.")

**107.** $f(x) = \cos x$ $\quad f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

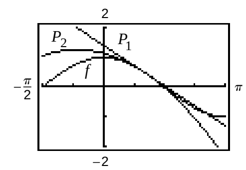
$f'(x) = -\sin x$ $\quad f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$ $\quad f''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

(a) $P_1(x) = f'(a)(x - a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

$$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a)$$
$$= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

(b)




(c) $P_2$ is a better approximation.

(d) The accuracy worsens as you move farther away from $x = a = (\pi/3)$.

**109.** False. If $y = f(x)g(x)$, then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

**111.** True

$$h'(c) = f(c)g'(c) + g(c)f'(c)$$
$$= f(c)(0) + g(c)(0)$$
$$= 0$$

**113.** True

**115.** $f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$$f'(x) = \left.\begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases}\right\} = 2|x|$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

$f''(0)$ does not exist since the left and right derivatives are not equal.

## Section 2.4 The Chain Rule

| | $y = f(g(x))$ | $u = g(x)$ | $y = f(u)$ |
|---|---|---|---|
| **1.** | $y = (6x - 5)^4$ | $u = 6x - 5$ | $y = u^4$ |
| **3.** | $y = \sqrt{x^2 - 1}$ | $u = x^2 - 1$ | $y = \sqrt{u}$ |
| **5.** | $y = \csc^3 x$ | $u = \csc x$ | $y = u^3$ |

**7.** $y = (2x - 7)^3$

$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$

**9.** $g(x) = 3(4 - 9x)^4$

$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$

**11.** $f(x) = (9 - x^2)^{2/3}$

$$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\frac{4x}{3(9 - x^2)^{1/3}}$$

**13.** $f(t) = (1 - t)^{1/2}$

$$f'(t) = \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}}$$

**15.** $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

**17.** $y = 2(4 - x^2)^{1/4}$

$$y' = 2\left(\frac{1}{4}\right)(4 - x^2)^{-3/4}(-2x)$$

$$= \frac{-x}{\sqrt[4]{(4 - x^2)^3}}$$

**19.** $y = (x - 2)^{-1}$

$$y' = -1(2 - x)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

**21.** $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

**23.** $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

**25.** $f(x) = x^2(x - 2)^4$

$$f'(x) = x^2[4(x - 2)^3(1)] + (x - 2)^4(2x)$$

$$= 2x(x - 2)^3[2x + (x - 2)]$$

$$= 2x(x - 2)^3(3x - 2)$$

**27.** $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$y' = x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1)$$

$$= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2}$$

$$= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)]$$

$$= \frac{1 - 2x^2}{\sqrt{1 - x^2}}$$

**29.** $y = \dfrac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$y' = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}(1)$$

$$= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2}$$

$$= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)]$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

**31.** $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

$$g'(x) = 2\left(\frac{x + 5}{x^2 + 2}\right)\left(\frac{(x^2 + 2) - (x + 5)(2x)}{(x^2 + 2)^2}\right)$$

$$= \frac{2(x + 5)(2 - 10x - x^2)}{(x^2 + 2)^3}$$

**33.** $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

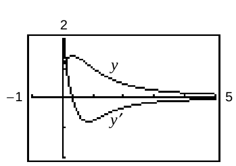
$$f'(v) = 3\left(\frac{1 - 2v}{1 + v}\right)^2\left(\frac{(1 + v)(-2) - (1 - 2v)}{(1 + v)^2}\right)$$

$$= \frac{-9(1 - 2v)^2}{(1 + v)^4}$$

**35.** $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

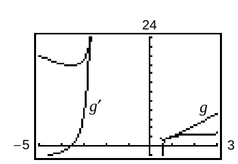
The zero of $y'$ corresponds to the point on the graph of $y$ where the tangent line is horizontal.




**37.** $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

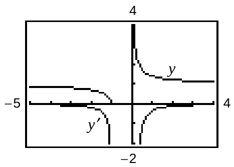
The zeros of $g'$ correspond to the points on the graph of $g$ where the tangent lines are horizontal.



**39.** $y = \sqrt{\dfrac{x+1}{x}}$

$y' = -\dfrac{\sqrt{(x+1)/x}}{2x(x+1)}$

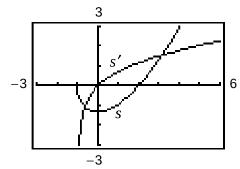
$y'$ has no zeros.




**41.** $s(t) = \dfrac{-2(2-t)\sqrt{1+t}}{3}$

$s'(t) = \dfrac{t}{\sqrt{1+t}}$

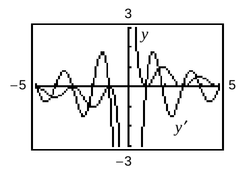
The zero of $s'(t)$ corresponds to the point on the graph of $s(t)$ where the tangent line is horizontal.




**43.** $y = \dfrac{\cos \pi x + 1}{x}$

$$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$$

$$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$$




The zeros of $y'$ correspond to the points on the graph of $y$ where the tangent lines are horizontal.

**45.** (a) $y = \sin x$

$y' = \cos x$

$y'(0) = 1$

1 cycle in $[0, 2\pi]$

(b) $y = \sin 2x$

$y' = 2\cos 2x$

$y'(0) = 2$

2 cycles in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is $a$.

**47.** $y = \cos 3x$

$\dfrac{dy}{dx} = -3\sin 3x$

**49.** $g(x) = 3\tan 4x$

$g'(x) = 12\sec^2 4x$

**51.** $y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$

$y' = \cos(\pi x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$

**53.** $h(x) = \sin 2x \cos 2x$

$h'(x) = \sin 2x(-2\sin 2x) + \cos 2x(2\cos 2x)$

$= 2\cos^2 2x - 2\sin^2 2x$

$= 2\cos 4x.$

Alternate solution: $h(x) = \dfrac{1}{2}\sin 4x$

$h'(x) = \dfrac{1}{2}\cos 4x(4) = 2\cos 4x$

**55.** $f(x) = \dfrac{\cot x}{\sin x} = \dfrac{\cos x}{\sin^2 x}$

$$f'(x) = \frac{\sin^2 x(-\sin x) - \cos x(2\sin x \cos x)}{\sin^4 x}$$

$$= \frac{-\sin^2 x - 2\cos^2 x}{\sin^3 x} = \frac{-1 - \cos^2 x}{\sin^3 x}$$

**57.** $y = 4\sec^2 x$

$y' = 8\sec x \cdot \sec x \tan x = 8\sec^2 x \tan x$

**59.** $f(\theta) = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$f'(\theta) = 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2)$

$= \sin 2\theta \cos 2\theta = \frac{1}{2}\sin 4\theta$

**61.** $f(x) = 3\sec^2(\pi t - 1)$

$f'(t) = 6\sec(\pi t - 1)\sec(\pi t - 1)\tan(\pi t - 1)(\pi)$

$= 6\pi\sec^2(\pi t - 1)\tan(\pi t - 1) = \dfrac{6\pi\sin(\pi t - 1)}{\cos^3(\pi t - 1)}$

**63.** $y = \sqrt{x} + \dfrac{1}{4}\sin(2x)^2$

$= \sqrt{x} + \dfrac{1}{4}\sin(4x^2)$

$\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} + \dfrac{1}{4}\cos(4x^2)(8x)$

$= \dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

**65.** $y = \sin(\cos x)$

$\dfrac{dy}{dx} = \cos(\cos x) \cdot (-\sin x)$

$= -\sin x \cos(\cos x)$

**67.** $s(t) = (t^2 + 2t + 8)^{1/2}, \quad (2, 4)$

$s'(t) = \dfrac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$

$= \dfrac{t + 1}{\sqrt{t^2 + 2t + 8}}$

$s'(2) = \dfrac{3}{4}$

**69.** $f(x) = \dfrac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \quad \left(-1, -\dfrac{3}{5}\right)$

$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\dfrac{9x^2}{(x^3 - 4)^2}$

$f'(-1) = -\dfrac{9}{25}$

**71.** $f(t) = \dfrac{3t + 2}{t - 1}, \quad (0, -2)$

$f'(t) = \dfrac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \dfrac{-5}{(t - 1)^2}$

$f'(0) = -5$

**73.** $y = 37 - \sec^3(2x), \quad (0, 36)$

$y' = -3\sec^2(2x)[2\sec(2x)\tan(2x)]$

$= -6\sec^3(2x)\tan(2x)$

$y'(0) = 0$

**75.** (a) $f(x) = \sqrt{3x^2 - 2}, \quad (3, 5)$

$f'(x) = \dfrac{1}{2}(3x^2 - 2)^{-1/2}(6x)$

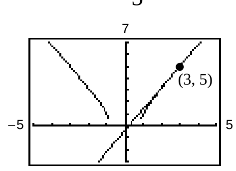
$= \dfrac{3x}{\sqrt{3x^2 - 2}}$

$f'(3) = \dfrac{9}{5}$

Tangent line:

$y - 5 = \dfrac{9}{5}(x - 3) \Longrightarrow 9x - 5y - 2 = 0$

(b)




**77.** (a) $f(x) = \sin 2x, \quad (\pi, 0)$

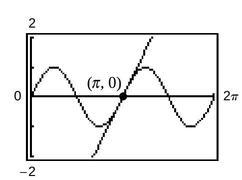
$f'(x) = 2\cos 2x$

$f'(\pi) = 2$

Tangent line:

$y = 2(x - \pi) \Longrightarrow 2x - y - 2\pi = 0$

(b)



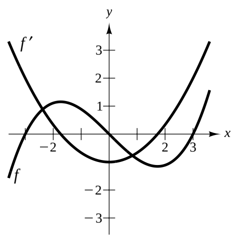
**79.** $f(x) = 2(x^2 - 1)^3$

$$f'(x) = 6(x^2 - 1)^2(2x)$$
$$= 12x(x^4 - 2x^2 + 1)$$
$$= 12x^5 - 24x^3 + 12x$$
$$f''(x) = 60x^4 - 72x^2 + 12$$
$$= 12(5x^2 - 1)(x^2 - 1)$$

**81.** $f(x) = \sin x^2$

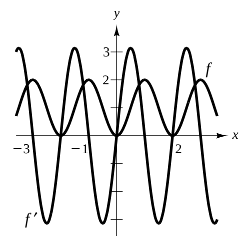
$$f'(x) = 2x \cos x^2$$
$$f''(x) = 2x[2x(-\sin x^2)] + 2 \cos x^2$$
$$= 2[\cos x^2 - 2x^2 \sin x^2]$$

**83.**




The zeros of $f'$ correspond to the points where the graph of $f$ has horizontal tangents.

**85.**




The zeros of $f'$ correspond to the points where the graph of $f$ has horizontal tangents.

**87.** $g(x) = f(3x)$

$$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$$

**89.** (a) $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$
$$f'(5) = (-3)(-2) + (6)(3) = 24$$

(b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x))h'(x)$$
$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need $g'(3)$ to find $f'(5)$.

(c) $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$
$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

(d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2 g'(x)$$
$$f'(5) = 3(-3)^2(6) = 162$$

**91.** (a) $f = 132{,}400(331 - v)^{-1}$

$$f' = (-1)(132{,}400)(331 - v)^{-2}(-1)$$
$$= \frac{132{,}400}{(331 - v)^2}$$

When $v = 30$, $f' \approx 1.461$.

(b) $f = 132{,}400(331 + v)^{-1}$

$$f' = (-1)(132{,}400)(331 + v)^{-2}(1)$$
$$= \frac{-132{,}400}{(331 + v)^2}$$

When $v = 30$, $f' \approx -1.016$.

**93.** $\theta = 0.2 \cos 8t$

The maximum angular displacement is $\theta = 0.2$ (since $-1 \le \cos 8t \le 1$).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

When $t = 3$, $d\theta/dt = -1.6 \sin 24 \approx 1.4489$ radians per second.

**95.** $S = C(R^2 - r^2)$

$$\frac{dS}{dt} = C\left(2R\frac{dR}{dt} - 2r\frac{dr}{dt}\right)$$

Since $r$ is constant, we have $dr/dt = 0$ and

$$\frac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$$
$$= 4.224 \times 10^{-2} = 0.04224.$$

**97.** (a) $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161$

(b) $C = 60x + 1350$

$$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161) + 1350$$

$$\frac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$$

$$= -294.696t^2 + 2317.44t - 30.492$$

The function $\frac{dC}{dt}$ is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

**99.** $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$$f''(x) = -\beta^2 \sin \beta x$$

$$f'''(x) = -\beta^3 \cos \beta x$$

$$f^{(4)} = \beta^4 \sin \beta x$$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$$

**101.** (a) $r'(x) = f'(g(x))g'(x)$

$$r'(1) = f'(g(1))g'(1)$$

Note that $g(1) = 4$ and $f'(4) = \frac{5-0}{6-2} = \frac{5}{4}$

Also, $g'(1) = 0$. Thus, $r'(1) = 0$

(b) $s'(x) = g'(f(x))f'(x)$

$$s'(4) = g'(f(4))f'(4)$$

Note that $f(4) = \frac{5}{2}$, $g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2}$ and

$f'(4) = \frac{5}{4}$.

Thus, $s'(4) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$.

**103.** $g = \sqrt{x(x+n)}$

$$= \sqrt{x^2 + nx}$$

$$\frac{dg}{dx} = \frac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$$

$$= \frac{2x + n}{2\sqrt{x^2 + nx}}$$

$$= \frac{(2x + n)/2}{\sqrt{x(x+n)}}$$

$$= \frac{[x + (x + n)]/2}{\sqrt{x(x+n)}}$$

$$= \frac{a}{g}$$

**105.** $g(x) = |2x - 3|$

$$g'(x) = 2\left(\frac{2x - 3}{|2x - 3|}\right), \quad x \neq \frac{3}{2}$$

**107.** $h(x) = |x|\cos x$

$$h'(x) = -|x| \sin x + \frac{x}{|x|} \cos x, \quad x \neq 0$$

**109.** (a) $f(x) = \tan \dfrac{\pi x}{4}$ $\qquad f(1) = 1$

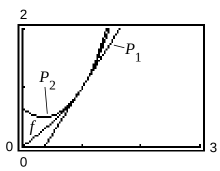
$f'(x) = \dfrac{\pi}{4}\sec^2 \dfrac{\pi x}{4}$ $\qquad f'(1) = \dfrac{\pi}{4}(2) = \dfrac{\pi}{2}$

$f''(x) = \dfrac{\pi}{2}\sec^2 \dfrac{\pi x}{4} \cdot \tan \dfrac{\pi x}{4}\left(\dfrac{\pi}{4}\right)$ $\qquad f''(1) = \dfrac{\pi}{8}(2)(1) = \dfrac{\pi}{4}$

$P_1(x) = f'(1)(x - 1) + f(1) = \dfrac{\pi}{2}(x - 1) + 1.$

$P_2(x) = \dfrac{1}{2}\left(\dfrac{\pi}{4}\right)(x - 1)^2 + f'(1)(x - 1) + f(1) = \dfrac{\pi}{8}(x - 1)^2 + \dfrac{\pi}{2}(x - 1) + 1$

(b)




(c) $P_2$ is a better approximation than $P_1$

(d) The accuracy worsens as you move away from $x = c = 1$.

**111.** False. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$.

**113.** True

# Section 2.5 Implicit Differentiation

**1.** $x^2 + y^2 = 36$

$2x + 2yy' = 0$

$y' = \dfrac{-x}{y}$

**3.** $x^{1/2} + y^{1/2} = 9$

$\dfrac{1}{2}x^{-1/2} + \dfrac{1}{2}y^{-1/2}y' = 0$

$y' = -\dfrac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\dfrac{y}{x}}$

**5.** $x^3 - xy + y^2 = 4$

$3x^2 - xy' - y + 2yy' = 0$

$(2y - x)y' = y - 3x^2$

$y' = \dfrac{y - 3x^2}{2y - x}$

**7.** $x^3y^3 - y - x = 0$

$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$y' = \dfrac{1 - 3x^2y^3}{3x^3y^2 - 1}$

**9.** $x^3 - 3x^2 + 2xy^2 = 12$

$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$

$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$

$y^1 = \dfrac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$

**11.** $\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$y' = \dfrac{\cos x}{4 \sin 2y}$

**13.** $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

**15.** $y = \sin(xy)$

$$y' = [xy' + y]\cos(xy)$$

$$y' - x\cos(xy)y' = y\cos(xy)$$

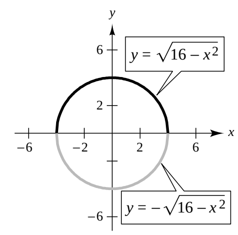
$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

**17.** (a) $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

(b)




(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}(16 - x^2)^{-1/2}(-2x)$$

$$= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm\sqrt{16 - x^2}} = \frac{-x}{y}$$

(d) Implicitly:

$$2x + 2yy' = 0$$

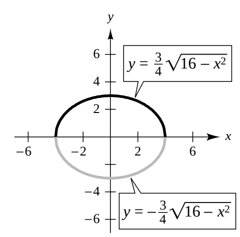
$$y' = -\frac{x}{y}$$

**19.** (a) $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm\frac{3}{4}\sqrt{16 - x^2}$$

(b)




(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{3}{8}(16 - x^2)^{-1/2}(-2x)$$

$$= \mp\frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y}$$

(d) Implicitly:

$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

**21.** $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

At $(-4, -1)$: $y' = -\frac{1}{4}$

**23.** $y^2 = \frac{x^2 - 4}{x^2 + 4}$

$$2yy' = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$2yy' = \frac{16x}{(x^2 + 4)^2}$$

$$y' = \frac{8x}{y(x^2 + 4)^2}$$

At $(2, 0)$, $y'$ is undefined.

**25.** $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At $(8, 1)$: $y' = -\frac{1}{2}$.

**27.** $\tan(x + y) = x$

$$(1 + y')\sec^2(x + y) = 1$$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} = -\sin^2(x + y)$$

$$= -\frac{x^2}{x^2 + 1}$$

At $(0, 0)$: $y' = 0$.

**29.** $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At $(2, 1)$: $y' = \frac{-32}{64} = -\frac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \frac{8}{x^2 + 4}\right)$

**31.** $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At $(1, 1)$: $y' = 0$.

**33.** $\tan y = x$

$$y'\sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

**35.** $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

**37.** $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

**39.** $y^2 = x^3$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

**41.** $\sqrt{x} + \sqrt{y} = 4$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

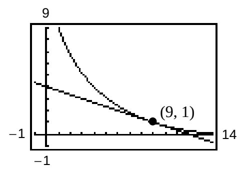
$$y' = \frac{-\sqrt{y}}{\sqrt{x}}$$

At $(9, 1)$, $y' = -\dfrac{1}{3}$

Tangent line: $y - 1 = -\dfrac{1}{3}(x - 9)$

$$y = -\frac{1}{3}x + 4$$

$$x + 3y - 12 = 0$$



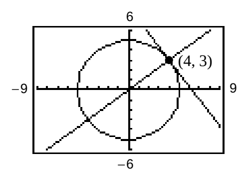

**43.** $x^2 + y^2 = 25$

$$y' = \frac{-x}{y}$$

At $(4, 3)$:

Tangent line: $y - 3 = \dfrac{-4}{3}(x - 4) \Longrightarrow 4x + 3y - 25 = 0$

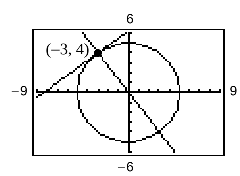
Normal line: $y - 3 = \dfrac{3}{4}(x - 4) \Longrightarrow 3x - 4y = 0.$




At $(-3, 4)$:

Tangent line: $y - 4 = \dfrac{3}{4}(x + 3) \Longrightarrow 3x - 4y + 25 = 0$

Normal line: $y - 4 = \dfrac{-4}{3}(x + 3) \Longrightarrow 4x + 3y = 0.$




**45.** $x^2 + y^2 = r^2$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let $(x_0, y_0)$ be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

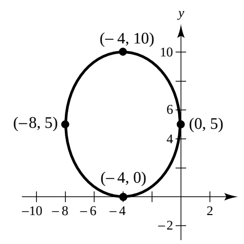
$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

**47.** $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$




Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$y(y - 10) = 0 \Longrightarrow y = 0, 10$$

Horizontal tangents: $(-4, 0)$, $(-4, 10)$.

Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

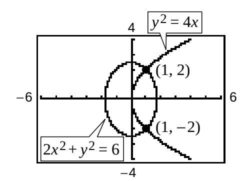
$$25x(x + 8) = 0 \Longrightarrow x = 0, -8$$

Vertical tangents: $(0, 5)$, $(-8, 5)$.

**49.** Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.




| *Ellipse:* | *Parabola:* |
| --- | --- |
| $4x + 2yy' = 0$ | $2yy' = 4$ |
| $y' = -\dfrac{2x}{y}$ | $y' = \dfrac{2}{y}$ |

At $(1, 2)$, the slopes are:

| | |
| --- | --- |
| $y' = -1$ | $y' = 1.$ |

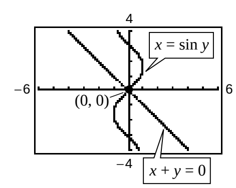
At $(1, -2)$, the slopes are:

| | |
| --- | --- |
| $y' = 1$ | $y' = -1.$ |

Tangents are perpendicular.

**51.** $y = -x$ and $x = \sin y$

Point of intersection: $(0, 0)$




| $\underline{y = -x:}$ | $\underline{x = \sin y:}$ |
| --- | --- |
| $y' = -1$ | $1 = y' \cos y$ |
| | $y' = \sec y$ |

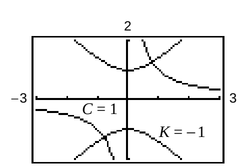
At $(0, 0)$, the slopes are:

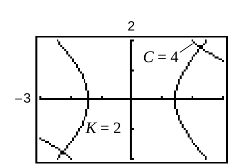
| | |
| --- | --- |
| $y' = -1$ | $y' = 1.$ |

Tangents are perpendicular.

**53.**

| | |
| --- | --- |
| $xy = C$ | $x^2 - y^2 = K$ |
| $xy' + y = 0$ | $2x - 2yy' = 0$ |
| $y' = -\dfrac{y}{x}$ | $y' = \dfrac{x}{y}$ |




At any point of intersection $(x, y)$ the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

**55.** $2y^2 - 3x^4 = 0$

(a) $4yy' - 12x^3 = 0$

$$4yy' = 12x^3$$

$$y' = \frac{12x^3}{4y} = \frac{3x^3}{y}$$

(b) $4y\frac{dy}{dt} - 12x^3\frac{dx}{dt} = 0$

$$y\frac{dy}{dt} = 3x^3\frac{dx}{dt}$$

**57.** $\cos \pi y - 3 \sin \pi x = 1$

(a) $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

$$y' = \frac{-3 \cos \pi x}{\sin \pi y}$$

(b) $-\pi \sin(\pi y)\frac{dy}{dt} - 3\pi \cos(\pi x)\frac{dx}{dt} = 0$

$$-\sin(\pi y)\frac{dy}{dt} = 3 \cos(\pi x)\frac{dx}{dt}$$

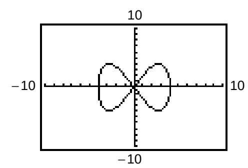
**59.** A function is in explicit form if $y$ is written as a function of $x$: $y = f(x)$. For example, $y = x^3$. An implicit equation is not in the form $y = f(x)$. For example, $x^2 + y^2 = 5$.

**61.** (a) $x^4 = 4(4x^2 - y^2)$

$$4y^2 = 16x^2 - x^4$$

$$y^2 = 4x^2 - \frac{1}{4}x^4$$

$$y = \pm\sqrt{4x^2 - \frac{1}{4}x^4}$$




(b) $y = 3 \Longrightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$$36 = 16x^2 - x^4$$

$$x^4 - 16x^2 + 36 = 0$$

$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

**Note** that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = \left(1 \pm \sqrt{7}\right)^2$.

Hence, there are four values of $x$:

$$-1 - \sqrt{7},\ 1 - \sqrt{7},\ -1 + \sqrt{7},\ 1 + \sqrt{7}$$

To find the slope, $2yy' = 8x - x^3 \Longrightarrow y' = \frac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$$y_1 = \frac{1}{3}\left(\sqrt{7} + 7\right)\left(x + 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + 8\sqrt{7} + 23\right].$$

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

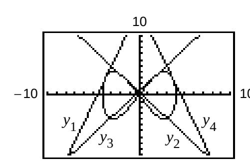
$$y_2 = \frac{1}{3}\left(\sqrt{7} - 7\right)\left(x - 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} - 7\right)x + 23 - 8\sqrt{7}\right].$$

For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$$y_3 = -\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right].$$

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$$y_4 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right].$$




**—CONTINUED—**

**61. —CONTINUED—**

(c) Equating $y_3$ and $y_4$,

$$-\frac{1}{3}(\sqrt{7}-7)(x+1-\sqrt{7})+3=-\frac{1}{3}(\sqrt{7}+7)(x-1-\sqrt{7})+3$$
$$(\sqrt{7}-7)(x+1-\sqrt{7})=(\sqrt{7}+7)(x-1-\sqrt{7})$$
$$\sqrt{7}x+\sqrt{7}-7-7x-7+7\sqrt{7}=\sqrt{7}x-\sqrt{7}-7+7x-7-7\sqrt{7}$$
$$16\sqrt{7}=14x$$
$$x=\frac{8\sqrt{7}}{7}$$

If $x=\dfrac{8\sqrt{7}}{7}$, then $y=5$ and the lines intersect at $\left(\dfrac{8\sqrt{7}}{7}, 5\right)$.

**63.** Let $f(x)=x^n=x^{p/q}$, where $p$ and $q$ are nonzero integers and $q>0$. First consider the case where $p=1$. The derivative of $f(x)=x^{1/q}$ is given by

$$\frac{d}{dx}[x^{1/q}]=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

where $t=x+\Delta x$. Observe that

$$\frac{f(t)-f(x)}{t-x}=\frac{t^{1/q}-x^{1/q}}{t-x}=\frac{t^{1/q}-x^{1/q}}{(t^{1/q})^q-(x^{1/q})^q}$$
$$=\frac{t^{1/q}-x^{1/q}}{(t^{1/q}-x^{1/q})(t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdots+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)})}$$
$$=\frac{1}{t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdots+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)}}.$$

As $t\to x$, the denominator approaches $qx^{1-(1/q)}$. That is,

$$\frac{d}{dx}[x^{1/q}]=\frac{1}{qx^{1-(1/q)}}=\frac{1}{q}x^{(1/q)-1}.$$

Now consider $f(x)=x^{p/q}=(x^p)^{1/q}$. From the Chain Rule,

$$f'(x)=\frac{1}{q}(x^p)^{(1/q)-1}\frac{d}{dx}[x^p]=\frac{1}{q}(x^p)^{(1/q)-1}px^{p-1}=\frac{p}{q}x^{[(p/q)-p]+(p-1)}=\frac{p}{q}x^{(p/q)-1}=nx^{n-1}\left(n=\frac{p}{q}\right).$$

## Section 2.6 Related Rates

**1.** $y=\sqrt{x}$

$$\frac{dy}{dt}=\left(\frac{1}{2\sqrt{x}}\right)\frac{dx}{dt}$$
$$\frac{dx}{dt}=2\sqrt{x}\frac{dy}{dt}$$

(a) When $x=4$ and $dx/dt=3$,

$$\frac{dy}{dt}=\frac{1}{2\sqrt{4}}(3)=\frac{3}{4}.$$

(b) When $x=25$ and $dy/dt=2$,

$$\frac{dx}{dt}=2\sqrt{25}(2)=20.$$

**3.** $xy=4$

$$x\frac{dy}{dt}+y\frac{dx}{dt}=0$$
$$\frac{dy}{dt}=\left(-\frac{y}{x}\right)\frac{dx}{dt}$$
$$\frac{dx}{dt}=\left(-\frac{x}{y}\right)\frac{dy}{dt}$$

(a) When $x=8$, $y=1/2$, and $dx/dt=10$,

$$\frac{dy}{dt}=-\frac{1/2}{8}(10)=-\frac{5}{8}.$$

(b) When $x=1$, $y=4$, and $dy/dt=-6$,

$$\frac{dx}{dt}=-\frac{1}{4}(-6)=\frac{3}{2}.$$

**5.** $y = x^2 + 1$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt} = 2x\,\frac{dx}{dt}$

(a) When $x = -1$,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When $x = 1$,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

**7.** $y = \tan x$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt} = \sec^2 x\,\frac{dx}{dt}$

(a) When $x = -\pi/3$,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When $x = -\pi/4$,

$$\frac{dy}{dt} = \left(\sqrt{2}\right)^2(2) = 4 \text{ cm/sec.}$$

(c) When $x = 0$,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

**9.** (a) $\frac{dx}{dt}$ negative $\Rightarrow \frac{dy}{dt}$ positive

(b) $\frac{dy}{dt}$ positive $\Rightarrow \frac{dx}{dt}$ negative

**11.** Yes, $y$ changes at a constant rate: $\frac{dy}{dt} = a \cdot \frac{dx}{dt}$.

No, the rate $\frac{dy}{dt}$ is a multiple of $\frac{dx}{dt}$.

**13.** $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$\frac{dx}{dt} = 2$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x)\frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}}\frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

**15.** $A = \pi r^2$

$\frac{dr}{dt} = 3$

$\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}$

(a) When $r = 6$,

$$\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2\text{/min.}$$

(b) When $r = 24$,

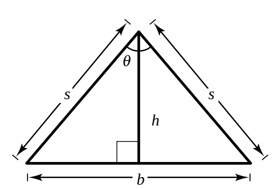
$$\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2\text{/min.}$$

**17.** (a) $\sin\frac{\theta}{2} = \frac{(1/2)b}{s} \Rightarrow b = 2s\sin\frac{\theta}{2}$

$\cos\frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s\cos\frac{\theta}{2}$

$$A = \frac{1}{2}bh = \frac{1}{2}\left(2s\sin\frac{\theta}{2}\right)\left(s\cos\frac{\theta}{2}\right)$$

$$= \frac{s^2}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{s^2}{2}\sin\theta$$




(b) $\frac{dA}{dt} = \frac{s^2}{2}\cos\theta\frac{d\theta}{dt}$ where $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min.

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$

When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{s^2}{8}$

(c) If $d\theta/dt$ is constant, $dA/dt$ is proportional to $\cos\theta$.

**19.** $V = \frac{4}{3}\pi r^3, \ \frac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2 \ \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right) = \frac{1}{4\pi r^2}(800)$$

(a) When $r = 30$, $\frac{dr}{dt} = \frac{1}{4\pi(30)^2}(800) = \frac{2}{9\pi}$ cm/min.

(b) When $r = 60$, $\frac{dr}{dt} = \frac{1}{4\pi(60)^2}(800) = \frac{1}{18\pi}$ cm/min.

**21.** $s = 6x^2$

$$\frac{dx}{dt} = 3$$

$$\frac{ds}{dt} = 12x \ \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec.}$$

(b) When $x = 10$,

$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec.}$$

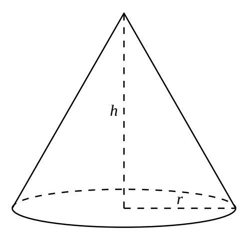
**23.** $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{4}h^2\right)h \quad [\text{since } 2r = 3h]$

$$= \frac{3\pi}{4}h^3$$

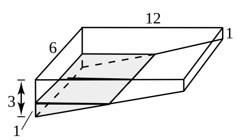
$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \ \frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

When $h = 15$, $\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.



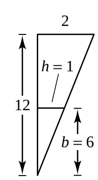

**25.**




(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1m. of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$

(see similar triangle diagram)




% pool filled $= \frac{18}{144}(100\%) = 12.5\%$

(b) Since for $0 \le h \le 2$, $b = 6h$, you have

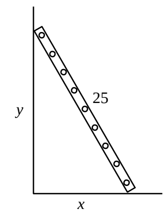
$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h\frac{dh}{dt} = \frac{1}{4} \Longrightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min.}$$

**27.** $x^2 + y^2 = 25^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$




(a) When $x = 7$, $y = \sqrt{576} = 24$, $\dfrac{dy}{dt} = \dfrac{-2(7)}{24} = \dfrac{-7}{12}$ ft/sec.

When $x = 15$, $y = \sqrt{400} = 20$, $\dfrac{dy}{dt} = \dfrac{-2(15)}{20} = \dfrac{-3}{2}$ ft/sec.

When $x = 24$, $y = 7$, $\dfrac{dy}{dt} = \dfrac{-2(24)}{7} = \dfrac{-48}{7}$ ft/sec.

(b) $A = \dfrac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

From part (a) we have $x = 7$, $y = 24$, $\dfrac{dx}{dt} = 2$, and $\dfrac{dy}{dt} = -\dfrac{7}{12}$.

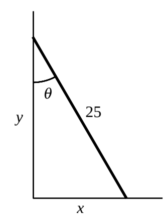
Thus, $\dfrac{dA}{dt} = \dfrac{1}{2}\left[7\left(-\dfrac{7}{12}\right) + 24(2)\right]$

$$= \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec.}$$

(c) $\tan\theta = \dfrac{x}{y}$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left[\frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}\right]$$

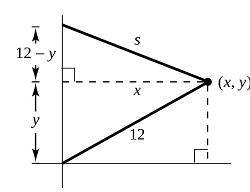



Using $x = 7$, $y = 24$, $\dfrac{dx}{dt} = 2$, $\dfrac{dy}{dt} = -\dfrac{7}{12}$ and $\cos\theta = \dfrac{24}{25}$, we have $\dfrac{d\theta}{dt} = \left(\dfrac{24}{25}\right)^2\left[\dfrac{1}{24}(2) - \dfrac{7}{(24)^2}\left(-\dfrac{7}{12}\right)\right] = \dfrac{1}{12}$ rad/sec.

**29.** When $y = 6$, $x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and

$$s = \sqrt{x^2 + (12 - y)^2}$$

$$= \sqrt{108 + 36} = 12.$$




$$x^2 + (12 - y)^2 = s^2$$

$$2x\frac{dx}{dt} + 2(12 - y)(-1)\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} + (y - 12)\frac{dy}{dt} = s\frac{ds}{dt}$$

Also, $x^2 + y^2 = 12^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longrightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $x\dfrac{dx}{dt} + (y - 12)\left(\dfrac{-x}{y}\dfrac{dx}{dt}\right) = s\dfrac{ds}{dt}$

$$\frac{dx}{dt}\left[x - x + \frac{12x}{y}\right] = s\frac{ds}{dt} \Longrightarrow \frac{dx}{dt} = \frac{sy}{12x}\cdot\frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})}(-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt} = \frac{-6\sqrt{3}}{6}\cdot\frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical).}$$

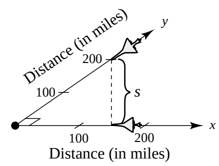
**31.** (a) $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$




When $x = 150$ and $y = 200$, $s = 250$ and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

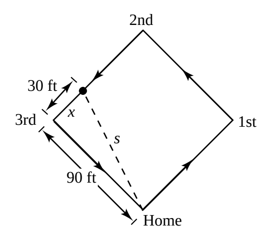
(b) $t = \dfrac{250}{750} = \dfrac{1}{3}$ hr $= 20$ min

**33.** $s^2 = 90^2 + x^2$

$$x = 30$$

$$\frac{dx}{dt} = -28$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Longrightarrow \frac{ds}{dt} = \frac{x}{s}\cdot\frac{dx}{dt}$$




When $x = 30$,

$$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$$

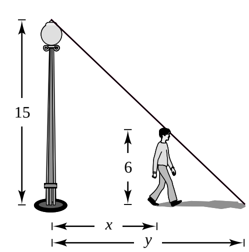
$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$

**35.** (a) $\dfrac{15}{6} = \dfrac{y}{y - x} \Longrightarrow 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3}\cdot\frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$




(b) $\dfrac{d(y - x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt} = \dfrac{25}{3} - 5 = \dfrac{10}{3}$ ft/sec

**37.** $x(t) = \frac{1}{2}\sin\frac{\pi t}{6}, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}, y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{\pi}{6}\right)\cos\frac{\pi t}{6} = \frac{\pi}{12}\cos\frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longrightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus,

$$\frac{dy}{dt} = -\frac{1/4}{\sqrt{15}/4}\cdot\frac{\pi}{12}\cos\left(\frac{\pi}{6}\right)$$

$$= \frac{-\pi}{\sqrt{15}}\left(\frac{1}{12}\right)\frac{\sqrt{3}}{2} = \frac{-\pi}{24}\frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120}.$$

Speed $= \left|\frac{-\sqrt{5}\pi}{120}\right| = \frac{\sqrt{5}\pi}{120}$ m/sec

**39.** Since the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, since $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Therefore,

$$k(4\pi r^2) = 4\pi r^2\frac{dr}{dt} \Longrightarrow k = \frac{dr}{dt}.$$

**41.**

$$pV^{1.3} = k$$

$$1.3\,pV^{0.3}\frac{dV}{dt} + V^{1.3}\frac{dp}{dt} = 0$$

$$V^{0.3}\left(1.3p\frac{dV}{dt} + V\frac{dp}{dt}\right) = 0$$

$$1.3p\frac{dV}{dt} = -V\frac{dp}{dt}$$

**43.**

$$\tan\theta = \frac{y}{30}$$

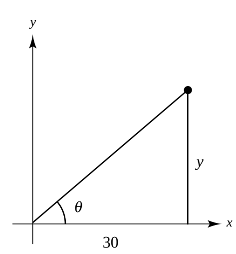
$$\frac{dy}{dt} = 3 \text{ m/sec.}$$

$$\sec^2\theta\cdot\frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$$

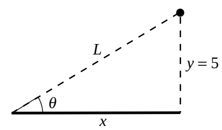
$$\frac{d\theta}{dt} = \frac{1}{30}\cos^2\theta\cdot\frac{dy}{dt}$$

When $y = 30$, $\theta = \pi/4$ and $\cos\theta = \sqrt{2}/2$. Thus,

$$\frac{d\theta}{dt} = \frac{1}{30}\left(\frac{1}{2}\right)(3) = \frac{1}{20} \text{ rad/sec.}$$



**45.** $\tan\theta = \dfrac{y}{x}, y = 5$




$$\frac{dx}{dt} = -600 \text{ mi/hr}$$

$$(\sec^2\theta)\frac{d\theta}{dt} = -\frac{5}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left(-\frac{5}{x^2}\right)\frac{dx}{dt} = \frac{x^2}{L^2}\left(-\frac{5}{x^2}\right)\frac{dx}{dt}$$

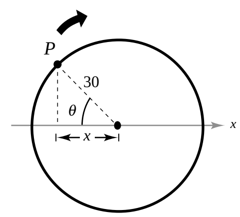
$$= \left(-\frac{5^2}{L^2}\right)\left(\frac{1}{5}\right)\frac{dx}{dt} = (-\sin^2\theta)\left(\frac{1}{5}\right)(-600) = 120\sin^2\theta$$

(a) When $\theta = 30°$, $\dfrac{d\theta}{dt} = \dfrac{120}{4} = 30$ rad/hr $= \dfrac{1}{2}$ rad/min.

(b) When $\theta = 60°$, $\dfrac{d\theta}{dt} = 120\left(\dfrac{3}{4}\right) = 90$ rad/hr $= \dfrac{3}{2}$ rad/min.

(c) When $\theta = 75°$, $\dfrac{d\theta}{dt} = 120\sin^2 75° \approx 111.96$ rad/hr $\approx 1.87$ rad/min.

**47.** $\dfrac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi$ rad/sec

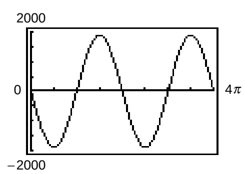



(a) $\cos\theta = \dfrac{x}{30}$

$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$$\frac{dx}{dt} = -30\sin\theta\frac{d\theta}{dt} = -30\sin\theta(20\pi) = -600\pi\sin\theta$$

(b)




(c) $|dx/dt| = |-600\pi\sin\theta|$ is greatest when $\sin\theta = 1 \implies \theta = (\pi/2) + n\pi$ (or $90° + n\cdot 180°$)

$|dx/dt|$ is least when $\theta = n\pi$ (or $n\cdot 180°$).

(d) For $\theta = 30°$, $\dfrac{dx}{dt} = -600\pi\sin(30°) = -600\pi\dfrac{1}{2} = -300\pi$ cm/sec.

For $\theta = 60°$, $\dfrac{dx}{dt} = -600\pi\sin(60°) = -600\pi\dfrac{\sqrt{3}}{2} = -300\sqrt{3}\,\pi$ cm/sec

**49.** $\tan\theta = \dfrac{x}{50} \implies x = 50\tan\theta$

$$\frac{dx}{dt} = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$2 = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25}\cos^2\theta,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

**51.** $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \dfrac{d^2y}{dt^2}$

First derivative: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Second derivative: $x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt} + y\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt}\cdot\dfrac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right)\left[-x\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right]$$

When $x = 7$, $y = 24$, $\dfrac{dy}{dt} = -\dfrac{7}{12}$, and $\dfrac{dx}{dt} = 2$ (see Exercise 27). Since $\dfrac{dx}{dt}$ is constant, $\dfrac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24}\left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2\right] = \frac{1}{24}\left[-4 - \frac{49}{144}\right] = \frac{1}{24}\left[-\frac{625}{144}\right] \approx -0.1808 \text{ ft/sec}^2$$

**53.** (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\dfrac{dm}{dt} = \dfrac{dm}{ds}\dfrac{ds}{dt} = (-1.762s + 29.10)\dfrac{ds}{dt}$

(c) If $t = s$ (1995), then $s = 15.5$ and $\dfrac{ds}{dt} = 1.2$.

Thus, $\dfrac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

## Review Exercises for Chapter 2

**1.** $f(x) = x^2 - 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{[(x+\Delta x)^2 - 2(x+\Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x - 2) = 2x - 2 \end{aligned}$$

**3.** $f(x) = \sqrt{x} + 1$

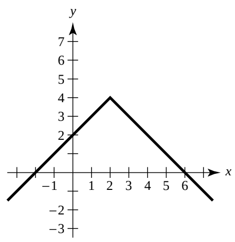
$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{(\sqrt{x+\Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x\to 0}\frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

**5.** $f$ is differentiable for all $x \neq -1$.

**7.** $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.



**9.** Using the limit definition, you obtain $g'(x) = \frac{4}{3}x - \frac{1}{6}$.

At $x = -1$, $g'(-1) = -\frac{4}{3} - \frac{1}{6} = \frac{-3}{2}$

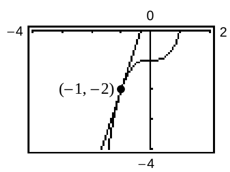
**11.** (a) Using the limit defintion, $f'(x) = 3x^2$.

At $x = -1$, $f'(-1) = 3$. The tangent line is

$$y - (-2) = 3(x - (-1))$$

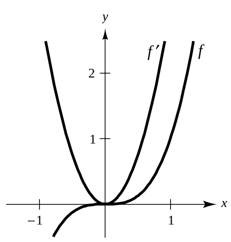
$$y = 3x + 1$$

(b)




**13.**
$$\begin{aligned} g'(2) &= \lim_{x\to 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{x^2(x-1) - 4}{x - 2} \\ &= \lim_{x\to 2} \frac{x^3 - x^2 - 4}{x - 2} \\ &= \lim_{x\to 2} \frac{(x-2)(x^2 + x + 2)}{x - 2} \\ &= \lim_{x\to 2} (x^2 + x + 2) = 8 \end{aligned}$$

**15.**




**17.** $y = 25$

$y' = 0$

**19.** $f(x) = x^8$

$f'(x) = 8x^7$

**21.** $h(t) = 3t^4$

$h'(t) = 12t^3$

**23.** $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

**25.** $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$

**27.** $g(t) = \frac{2}{3}t^{-2}$

$g'(x) = \frac{-4}{3}t^{-3} = \frac{-4}{3t^3}$

**29.** $f(\theta) = 2\theta - 3\sin\theta$

$f'(\theta) = 2 - 3\cos\theta$

**31.** $f(\theta) = 3\cos\theta - \frac{\sin\theta}{4}$

$f'(\theta) = -3\sin\theta - \frac{\cos\theta}{4}$

**33.** $F = 200\sqrt{T}$

$F'(t) = \dfrac{100}{\sqrt{T}}$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

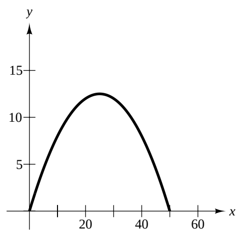
**35.** $s(t) = -16t^2 + s_0$

$s(9.2) = -16(9.2)^2 + s_0 = 0$

$s_0 = 1354.24$

The building is approximately 1354 feet high (or 415 m).

**37.** (a)




Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$0 = x\left(1 - \dfrac{x}{50}\right)$ implies $x = 50$.

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$y' = 1 - 0.04x$

$y'(0) = 1$

$y'(10) = 0.6$

$y'(25) = 0$

$y'(30) = -0.2$

$y'(50) = -1$

(e) $y'(25) = 0$

**39.** $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$a(t) = v'(t) = 2$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$

(d) $x(t) = 0$ for $t = 1, 2$.

$|v(1)| = |2(1) - 3| = 1$

$|v(2)| = |2(2) - 3| = 1$

The speed is 1 when the position is 0.

**41.** $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$f'(x) = (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x)$

$= 2(6x^3 - 9x^2 + 16x - 7)$

**43.** $h(x) = \sqrt{x}\sin x = x^{1/2}\sin x$

$h'(x) = \dfrac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x$

**45.** $f(x) = 2x - x^{-2}$

$f'(x) = 2 + 2x^{-3} = 2\left(1 + \dfrac{1}{x^3}\right)$

$= \dfrac{2(x^3 + 1)}{x^3}$

**47.** $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

$f'(x) = \dfrac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2}$

$= \dfrac{-(x^2 + 1)}{(x^2 - 1)^2}$

**49.** $f(x) = (4 - 3x^2)^{-1}$

$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \dfrac{6x}{(4 - 3x^2)^2}$

**51.** $y = \dfrac{x^2}{\cos x}$

$y' = \dfrac{\cos x\,(2x) - x^2(-\sin x)}{\cos^2 x} = \dfrac{2x\cos x + x^2\sin x}{\cos^2 x}$

**53.** $y = 3x^2\sec x$

$y' = 3x^2\sec x\tan x + 6x\sec x$

**55.** $y = -x\tan x$

$y' = -x\sec^2 x - \tan x$

**57.** $y = x\cos x - \sin x$

$y' = -x\sin x + \cos x - \cos x = -x\sin x$

**59.** $g(t) = t^3 - 3t + 2$

$g'(t) = 3t^2 - 3$

$g''(t) = 6t$

**61.** $f(\theta) = 3\tan\theta$

$f'(\theta) = 3\sec^2\theta$

$f''(\theta) = 6\sec\theta(\sec\theta\tan\theta) = 6\sec^2\theta\tan\theta$

**63.** $y = 2\sin x + 3\cos x$

$y' = 2\cos x - 3\sin x$

$y'' = -2\sin x - 3\cos x$

$$\begin{aligned} y'' + y &= -(2\sin x + 3\cos x) + (2\sin x + 3\cos x) \\ &= 0 \end{aligned}$$

**65.** $f(x) = (1 - x^3)^{1/2}$

$$\begin{aligned} f'(x) &= \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2) \\ &= -\frac{3x^2}{2\sqrt{1 - x^3}} \end{aligned}$$

**67.** $h(x) = \left(\dfrac{x - 3}{x^2 + 1}\right)^2$

$$\begin{aligned} h'(x) &= 2\left(\frac{x - 3}{x^2 + 1}\right)\left(\frac{(x^2 + 1)(1) - (x - 3)(2x)}{(x^2 + 1)^2}\right) \\ &= \frac{2(x - 3)(-x^2 + 6x + 1)}{(x^2 + 1)^3} \end{aligned}$$

**69.** $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$\begin{aligned} f'(s) &= (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\tfrac{5}{2}\right)(s^2 - 1)^{3/2}(2s) \\ &= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)] \\ &= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25) \end{aligned}$$

**71.** $y = 3\cos(3x + 1)$

$y' = -9\sin(3x + 1)$

**73.** $y = \dfrac{1}{2}\csc 2x$

$$\begin{aligned} y' &= \frac{1}{2}(-\csc 2x\cot 2x)(2) \\ &= -\csc 2x\cot 2x \end{aligned}$$

**75.** $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

$$\begin{aligned} y' &= \frac{1}{2} - \frac{1}{4}\cos 2x(2) \\ &= \frac{1}{2}(1 - \cos 2x) = \sin^2 x \end{aligned}$$

**77.** $y = \dfrac{2}{3}\sin^{3/2}x - \dfrac{2}{7}\sin^{7/2}x$

$$\begin{aligned} y' &= \sin^{1/2}x\cos x - \sin^{5/2}x\cos x \\ &= (\cos x)\sqrt{\sin x}(1 - \sin^2 x) \\ &= (\cos^3 x)\sqrt{\sin x} \end{aligned}$$

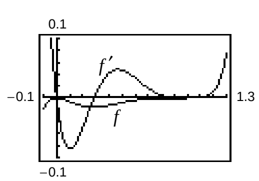
**79.** $y = \dfrac{\sin\pi x}{x + 2}$

$$y' = \frac{(x + 2)\pi\cos\pi x - \sin\pi x}{(x + 2)^2}$$

**81.** $f(t) = t^2(t - 1)^5$

$f'(t) = t(t - 1)^4(7t - 2)$

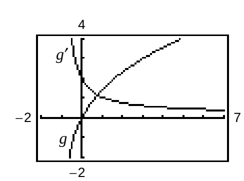
The zeros of $f'$ correspond to the points on the graph of $f$ where the tangent line is horizontal.




**83.** $g(x) = 2x(x + 1)^{-1/2}$

$$g'(x) = \frac{x + 2}{(x + 1)^{3/2}}$$

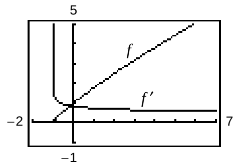
$g'$ does not equal zero for any value of $x$ in the domain. The graph of $g$ has no horizontal tangent lines.



**85.** $f(t) = (t+1)^{1/2}(t+1)^{1/3} = (t+1)^{5/6}$

$$f'(t) = \frac{5}{6(t+1)^{1/6}}$$

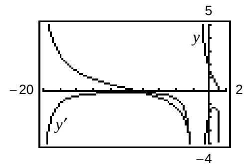
$f'$ does not equal zero for any $x$ in the domain. The graph of $f$ has no horizontal tangent lines.




**87.** $y = \tan\sqrt{1-x}$

$$y' = -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$$

$y'$ does not equal zero for any $x$ in the domain. The graph has no horizontal tangent lines.




**89.** $y = 2x^2 + \sin 2x$

$y' = 4x + 2\cos 2x$

$y'' = 4 - 4\sin 2x$

**91.** $f(x) = \cot x$

$f'(x) = -\csc^2 x$

$$f'' = -2\csc x(-\csc x \cdot \cot x) = 2\csc^2 x \cot x$$

**93.** $f(t) = \dfrac{t}{(1-t)^2}$

$$f'(t) = \frac{t+1}{(1-t)^3}$$

$$f''(t) = \frac{2(t+2)}{(1-t)^4}$$

**95.** $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$g'(\theta) = 3\sec^2 3\theta - \cos(\theta - 1)$

$g''(\theta) = 18\sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$

**97.** $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t+2)}{(t^2+4t+10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1+2)}{(1+4+10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3+2)}{(9+12+10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5+2)}{(25+30+10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10+2)}{(100+40+10)^2} \approx -0.747 \text{ deg/hr.}$$

**99.**

$$x^2 + 3xy + y^3 = 10$$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x+y^2)y' = -(2x+3y)$$

$$y' = \frac{-(2x+3y)}{3(x+y^2)}$$

**101.**

$$y\sqrt{x} - x\sqrt{y} = 16$$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

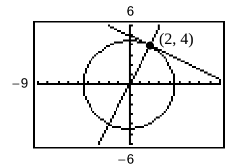
$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

**103.**
$$x \sin y = y \cos x$$
$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$
$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$
$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

**105.** $x^2 + y^2 = 20$




$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$
At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$
$$x + 2y - 10 = 0$$
Normal line: $y - 4 = 2(x - 2)$
$$2x - y = 0$$

**107.** $y = \sqrt{x}$

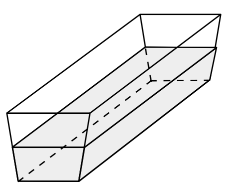
$$\frac{dy}{dt} = 2 \text{ units/sec}$$
$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt} \Longrightarrow \frac{dx}{dt} = 2\sqrt{x}\frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

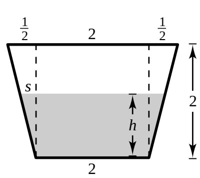
(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

**109.** $\frac{s}{h} = \frac{1/2}{2}$



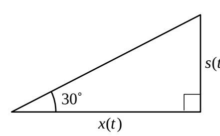
$$s = \frac{1}{4}h$$
$$\frac{dV}{dt} = 1$$

Width of water at depth $h$:
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$
$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$
$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$




When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.

**111.** $s(t) = 60 - 4.9t^2$

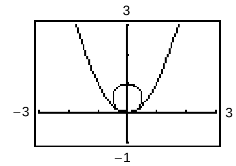



$$s'(t) = -9.8t$$
$$s = 35 = 60 - 4.9t^2$$
$$4.9t^2 = 25$$
$$t = \frac{5}{\sqrt{4.9}}$$
$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$
$$x(t) = \sqrt{3}\,s(t)$$
$$\frac{dx}{dt} = \sqrt{3}\frac{ds}{dt} = \sqrt{3}(-9.8)\frac{5}{\sqrt{4.9}}$$
$$\approx -38.34 \text{ m/sec}$$

## Problem Solving for Chapter 2

**1.** (a) $x^2 + (y - r)^2 = r^2$ Circle

$x^2 = y$ Parabola




Substituting,

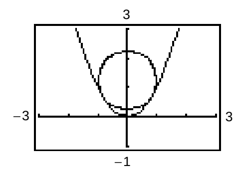
$$(y - r)^2 = r^2 - y$$
$$y^2 - 2ry + r^2 = r^2 - y$$
$$y^2 - 2ry + y = 0$$
$$y(y - 2r + 1) = 0$$

Since you want only one solution, let $1 - 2r = 0 \Rightarrow r = \frac{1}{2}$

Graph $y = x^2$ and $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$

(b) Let $(x, y)$ be a point of tangency: $x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \dfrac{x}{b - y}$ (circle).

$y = x^2 \Rightarrow y' = 2x$ (parabola). Equating,




$$2x = \frac{x}{b - y}$$
$$2(b - y) = 1$$
$$b - y = \frac{1}{2} \Rightarrow b = y + \frac{1}{2}$$

Also, $x^2 + (y - b)^2 = 1$ and $y = x^2$ imply

$$y + (y - b)^2 = 1 \Rightarrow y + \left[y - \left(y + \frac{1}{2}\right)\right] = 1 \Rightarrow y - \frac{1}{2} = 1 \Rightarrow y = \frac{3}{4} \text{ and } b = \frac{5}{4}.$$

Center: $\left(0, \dfrac{5}{4}\right)$

Graph $y = x^2$ and $x^2 + \left(y - \dfrac{5}{4}\right)^2 = 1$

**3.** (a) $f(x) = \cos x$ $\quad P_1(x) = a_0 + a_1x$

$f(0) = 1$ $\quad P_1(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $\quad P'_1(0) = a_1 \Rightarrow a_1 = 0$

$P_1(x) = 1$

(b) $f(x) = \cos x$ $\quad P_2(x) = a_0 + a_1x + a_2x^2$

$f(0) = 1$ $\quad P_2(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $\quad P'_2(0) = a_1 \Rightarrow a_1 = 0$

$f''(0) = -1$ $\quad P''_2(0) = 2a_2 \Rightarrow a_2 = -\frac{1}{2}$

$P_2(x) = 1 - \frac{1}{2}x^2$

(c)

| $x$ | $-1.0$ | $-0.1$ | $-0.001$ | 0 | 0.001 | 0.1 | 1.0 |
|---|---|---|---|---|---|---|---|
| $\cos x$ | 0.5403 | 0.9950 | $\approx 1$ | 1 | $\approx 1$ | 0.9950 | 0.5403 |
| $P_2(x)$ | 0.5 | 0.9950 | $\approx 1$ | 1 | $\approx 1$ | 0.9950 | 0.5 |

$P_2(x)$ is a good approximation of $f(x) = \cos x$ when $x$ is near 0.

(d) $f(x) = \sin x$ $\quad P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$f(0) = 0$ $\quad P_3(0) = a_0 \Rightarrow a_0 = 0$

$f'(0) = 1$ $\quad P'_3(0) = a_1 \Rightarrow a_1 = 1$

$f''(0) = 0$ $\quad P''_3(0) = 2a_2 \Rightarrow a_2 = 0$

$f'''(0) = -1$ $\quad P'''_3(0) = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$

$P_3(x) = x - \frac{1}{6}x^3$

**5.** Let $p(x) = Ax^3 + Bx^2 + Cx + D$

$p'(x) = 3Ax^2 + 2Bx + C$

At $(1, 1)$: $A + B + C + D = 1$ Equation 1

$3A + 2B + C = 14$ Equation 2

At $(-1, -3)$: $-A + B - C + D = -3$ Equation 3

$3A - 2B + C = -2$ Equation 4

Adding Equations 1 and 3: $2B + 2D = -2$

Subtracting Equations 1 and 3: $2A + 2C = 4$

Adding Equations 2 and 4: $6A + 2C = 12$

Subtracting Equations 2 and 4: $4B = 16$

Hence, $B = 4$ and $D = \frac{1}{2}(-2 - 2B) = -5$

Subtracting $2A + 2C = 4$ and $6A + 2C = 12$, you obtain $4A = 8 \Rightarrow A = 2$. Finally, $C = \frac{1}{2}(4 - 2A) = 0$

Thus, $p(x) = 2x^3 + 4x^2 - 5$.

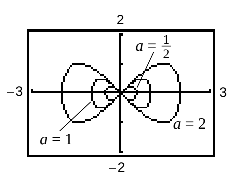
**7.** (a) $x^4 = a^2x^2 - a^2y^2$

$a^2y^2 = a^2x^2 - x^4$

$$y = \frac{\pm\sqrt{a^2x^2 - x^4}}{a}$$

Graph: $y_1 = \dfrac{\sqrt{a^2x^2 - x^4}}{a}$ and $y_2 = -\dfrac{\sqrt{a^2x^2 - x^4}}{a}$

(b)




$(\pm a, 0)$ are the $x$-intercepts, along with $(0, 0)$.

(c) Differentiating implicitly,

$4x^3 = 2a^2x - 2a^2yy'$

$$y' = \frac{2a^2x - 4x^3}{2a^2y} = \frac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{\pm a}{\sqrt{2}}.$$

$$\left(\frac{a^2}{2}\right)^2 = a^2\left(\frac{a^2}{2}\right) - a^2y^2$$

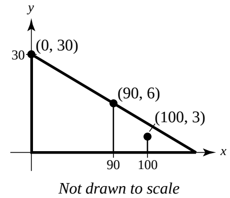
$$\frac{a^4}{4} = \frac{a^4}{2} - a^2y^2$$

$$a^2y^2 = \frac{a^4}{4}$$

$$y^2 = \frac{a^2}{4}$$

$$y = \pm\frac{a}{2}$$

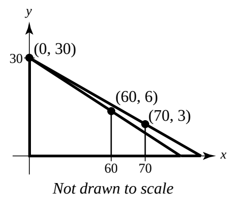
Four points: $\left(\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(\frac{a}{\sqrt{2}}, -\frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(\frac{-a}{\sqrt{2}}, -\frac{a}{2}\right)$

**9.** (a) 


Line determined by $(0, 30)$ and $(90, 6)$:

$$y - 30 = \frac{30 - 6}{0 - 90}(x - 0) = -\frac{24}{90}x = -\frac{4}{15}x \Longrightarrow y = -\frac{4}{15}x + 30$$

When $x = 100$, $y = \frac{-4}{15}(100) + 30 = \frac{10}{3} > 3 \Longrightarrow$ Shadow determined by man.

(b) 


Line determined by $(0, 30)$ and $(60, 6)$:

$$y - 30 = \frac{30 - 6}{0 - 60}(x - 0) = -\frac{2}{5}x \Longrightarrow y = -\frac{2}{5}x + 30$$

When $x = 70$, $y = \frac{-2}{5}(70) + 30 = 2 < 3 \Longrightarrow$ Shadow determined by child.

(c) Need $(0, 30)$, $(d, 6)$, $(d + 10, 3)$ collinear.

$$\frac{30 - 6}{0 - d} = \frac{6 - 3}{d - (d + 10)} \Longrightarrow \frac{24}{d} = \frac{3}{10} \Longrightarrow d = 80 \text{ feet}$$

(d) Let $y$ be the length of the street light to the tip of the shadow. We know that $\frac{dx}{dt} = -5$.

For $x > 80$, the shadow is determined by the man.

$$\frac{y}{30} = \frac{y - x}{6} \Longrightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4}\frac{dx}{dt} = \frac{-25}{4}.$$

For $x < 80$, the shadow is determined by the child.

$$\frac{y}{30} = \frac{y - x - 10}{3} \Longrightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9}\frac{dx}{dt} = \frac{-50}{9}.$$

Therefore,

$$\frac{dy}{dt} = \begin{cases} \dfrac{-25}{4} & x > 80 \\[2ex] \dfrac{-50}{9} & 0 < x < 80 \end{cases}$$

$\frac{dy}{dt}$ is not continuous at $x = 80$.

**11.** $L'(x) = \lim_{\Delta x \to 0} \frac{L(x + \Delta x) - L(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{L(x) + L(\Delta x) - L(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{L(\Delta x)}{\Delta x}$$

Also, $L'(0) = \lim_{\Delta x \to 0} \frac{L(\Delta x) - L(0)}{\Delta x}$

But, $L(0) = 0$ because $L(0) = L(0 + 0) = L(0) + L(0) \Longrightarrow L(0) = 0$.

Thus, $L'(x) = L'(0)$, for all $x$.

The graph of $L$ is a line through the origin of slope $L'(0)$.

**13.** (a)

| $z$ (degrees) | 0.1 | 0.01 | 0.0001 |
|---|---|---|---|
| $\dfrac{\sin z}{z}$ | 0.0174524 | 0.0174533 | 0.0174533 |

(b) $\lim\limits_{z\to 0} \dfrac{\sin z}{z} \approx 0.0174533$

In fact, $\lim\limits_{z\to 0} \dfrac{\sin z}{z} = \dfrac{\pi}{180}$

(c) $$\frac{d}{dz}(\sin z) = \lim_{\Delta z \to 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\sin z\left(\frac{\cos \Delta z - 1}{\Delta z}\right)\right] + \lim_{\Delta z \to 0} \left[\cos z\left(\frac{\sin \Delta z}{\Delta z}\right)\right]$$

$$= \sin z(0) + \cos z\left(\frac{\pi}{180}\right) = \frac{\pi}{180}\cos z$$

(d) $S(90) = \sin\left(\dfrac{\pi}{180}\, 90\right) = \sin\dfrac{\pi}{2} = 1;\ C(180) = \cos\left(\dfrac{\pi}{180}\, 180\right) = -1$

$$\frac{d}{dz}S(z) = \frac{d}{dz}\sin(cz) = c \cdot \cos(cz) = \frac{\pi}{180}C(z)$$

(e) The formulas for the derivatives are more complicated in degrees.

**15.** $j(t) = a'(t)$

(a) $j(t)$ is the rate of change of the acceleration.

(b) From Exercise 102 in Section 2.3,

$s(t) = -8.25t^2 + 66t$

$v(t) = -16.5t + 66$

$a(t) = -16.5$

$a'(t) = j(t) = 0$

# CHAPTER 3
# Applications of Differentiation

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# Applications of Differentiation

## Section 3.1 Extrema on an Interval

Solutions to Odd-Numbered Exercises

**1.** $f(x) = \dfrac{x^2}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \dfrac{8x}{(x^2 + 4)^2}$

$f'(0) = 0$

**3.** $f(x) = x + \dfrac{27}{2x^2} = x + \dfrac{27}{x}x^{-2}$

$f'(x) = 1 - 27x^{-3} = 1 - \dfrac{27}{x^3}$

$f'(3) = 1 - \dfrac{27}{3^3} = 1 - 1 = 0$

**5.** $f(x) = (x + 2)^{2/3}$

$f'(x) = \dfrac{2}{3}(x + 2)^{-1/3}$

$f'(-2)$ is undefined.

**7.** Critical numbers: $x = 2$

$x = 2$: absolute maximum

**9.** Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maximum

$x = 2$: absolute minimum

**11.** $f(x) = x^2(x - 3) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Critical numbers: $x = 0, x = 2$

**13.** $g(t) = t\sqrt{4 - t},\ t < 3$

$g'(t) = t\left[\dfrac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$

$= \dfrac{1}{2}(4 - t)^{-1/2}[-t + 2(4 - t)]$

$= \dfrac{8 - 3t}{2\sqrt{4 - t}}$

Critical number is $t = \dfrac{8}{3}$.

**15.** $h(x) = \sin^2 x + \cos x, 0 < x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

On $(0, 2\pi)$, critical numbers: $x = \dfrac{\pi}{3}, x = \pi, x = \dfrac{5\pi}{3}$

**17.** $f(x) = 2(3 - x), [-1, 2]$

$f'(x) = -2 \Longrightarrow$ No critical numbers

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

**19.** $f(x) = -x^2 + 3x, [0, 3]$

$f'(x) = -2x + 3$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3}{2}, \frac{9}{4}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

**21.** $f(x) = x^3 - \frac{3}{2}x^2,\ [-1, 2]$

$f'(x) = 3x^2 - 3x = 3x(x - 1)$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

**23.** $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$f'(x) = 2x^{-1/3} - 2 = \dfrac{2\left(1 - \sqrt[3]{x}\right)}{\sqrt[3]{x}}$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

**25.** $g(t) = \dfrac{t^2}{t^2 + 3}, [-1, 1]$

$g'(t) = \dfrac{6t}{(t^2 + 3)^2}$

Left endpoint: $\left(-1, \frac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \frac{1}{4}\right)$ Maximum

**27.** $h(s) = \dfrac{1}{s - 2}, [0, 1]$

$h'(s) = \dfrac{-1}{(s - 2)^2}$

Left endpoint: $\left(0, -\frac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

**29.** $f(x) = \cos \pi x, \left[0, \frac{1}{6}\right]$

$f'(x) = -\pi \sin \pi x$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\frac{1}{6}, \frac{\sqrt{3}}{2}\right)$ Minimum

**31.** $y = \dfrac{4}{x} + \tan\dfrac{\pi x}{8},\ [1, 2]$

$$y' = \frac{-4}{x^2} + \frac{\pi}{8}\sec^2\frac{\pi x}{8} = 0$$

$$\frac{\pi}{8}\sec^2\frac{\pi x}{8} = \frac{4}{x^2}$$

On the interval $[1, 2]$, this equation has no solutions.
Thus, there are no critical numbers.

Left endpoint: $\left(1, \sqrt{2} + 3\right) \approx (1, 4.4142)$ Maximum

Right endpoint: $(2, 3)$ Minimum

**33.** (a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

**35.** $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

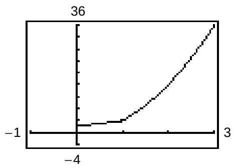
(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

**37.** $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$

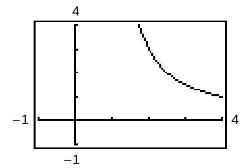
Left endpoint: $(0, 2)$ Minimum

Right endpoint: $(3, 36)$ Maximum

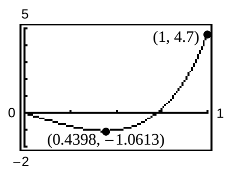



**39.** $f(x) = \dfrac{3}{x - 1}, (1, 4]$

Right endpoint: $(4, 1)$ Minimum




**41.** (a)




Maximum: $(1, 4.7)$ (endpoint)

Minimum: $(0.4398, -1.0613)$

(b)

$$f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

$$f(0) = 0$$

$$f(1) = 4.7 \text{ Maximum (endpoint)}$$

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: $(0.4398, -1.0613)$

**43.** $f(x) = (1 + x^3)^{1/2}, [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, we have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$$\left| f''\left(\sqrt[3]{-10 + \sqrt{108}}\right) \right| \approx 1.47 \text{ is the maximum value.}$$

**45.** $f(x) = (x + 1)^{2/3}, [0, 2]$

$$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$$

$$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$$

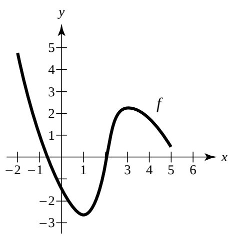
$$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$$

$$|f^{(4)}(0)| = \frac{56}{81} \text{ is the maximum value.}$$

**47.** $f(x) = \tan x$

$f$ is continuous on $[0, \pi/4]$ but not on $[0, \pi]$. $\lim_{x \to \pi/2^-} \tan x = \infty$.

**49.**




**51.** (a) Yes

(b) No

**53.** (a) No

(b) Yes

**55.** $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. $P$ is decreasing for $I > 12$.

**57.**

$$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \frac{\pi}{6} \le \theta \le \frac{\pi}{2}$$

$$\frac{dS}{d\theta} = \frac{3s^2}{2}\left(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta\right)$$

$$= \frac{3s^2}{2}\csc\theta\left(-\sqrt{3}\cot\theta + \csc\theta\right) = 0$$

$$\csc\theta = \sqrt{3}\cot\theta$$

$$\sec\theta = \sqrt{3}$$

$$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553 \text{ radians}$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

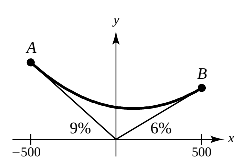
$$S\left(\frac{\pi}{2}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\operatorname{arcsec}\sqrt{3}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{2}\right)$$

$S$ is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radians.

**59.** (a) $y = ax^2 + bx + c$

$y' = 2ax + b$




The coordinates of $B$ are $(500, 30)$, and those of $A$ are $(-500, 45)$. From the slopes at $A$ and $B$,

$$-1000a + b = -0.09$$

$$1000a + b = 0.06.$$

Solving these two equations, you obtain $a = 3/40000$ and $b = -3/200$. From the points $(500, 30)$ and $(-500, 45)$, you obtain

$$30 = \frac{3}{40000}500^2 + 500\left(\frac{-3}{200}\right) + c$$

$$45 = \frac{3}{40000}500^2 - 500\left(\frac{-3}{200}\right) + c.$$

In both cases, $c = 18.75 = \dfrac{75}{4}$. Thus,

$$y = \frac{3}{40000}x^2 - \frac{3}{200}x + \frac{75}{4}.$$

**—CONTINUED—**

**59. —CONTINUED—**

(b)

| $x$ | $-500$ | $-400$ | $-300$ | $-200$ | $-100$ | 0 | 100 | 200 | 300 | 400 | 500 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | 0 | .75 | 3 | 6.75 | 12 | 18.75 | 12 | 6.75 | 3 | .75 | 0 |

For $-500 \le x \le 0$, $d = (ax^2 + bx + c) - (-0.09x)$.

For $0 \le x \le 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(c) The lowest point on the highway is $(100, 18)$, which is not directly over the point where the two hillsides come together.

**61.** True. See Exercise 25.

**63.** True.

## Section 3.2 Rolle's Theorem and the Mean Value Theorem

**1.** Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ since $f$ is not differentiable at $x = 1$.

**3.** $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

$x$-intercepts: $(-1, 0), (2, 0)$

$f'(x) = 2x - 1 = 0$ at $x = \frac{1}{2}$.

**5.** $f(x) = x\sqrt{x + 4}$

$x$-intercepts: $(-4, 0), (0, 0)$

$$f'(x) = x\frac{1}{2}(x + 4)^{-1/2} + (x + 4)^{1/2}$$

$$= (x + 4)^{-1/2}\left(\frac{x}{2} + (x + 4)\right)$$

$$f'(x) = \left(\frac{3}{2}x + 4\right)(x + 4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

**7.** $f(x) = x^2 - 2x, [0, 2]$

$f(0) = f(2) = 0$

$f$ is continuous on $[0, 2]$. $f$ is differentiable on $(0, 2)$. Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Longrightarrow x = 1$$

$c$ value: 1

**9.** $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = f(3) = 0$

$f$ is continuous on $[1, 3]$. $f$ is differentiable on $(1, 3)$. Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Longrightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}$$

**11.** $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = f(8) = 3$

$f$ is continuous on $[-8, 8]$. $f$ is not differentiable on $(-8, 8)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.

**13.** $f(x) = \frac{x^2 - 2x - 3}{x + 2}, \ [-1, 3]$

$f(-1) = f(3) = 0$

$f$ is continuous on $[-1, 3]$. (**Note:** The discontinuity, $x = -2$, is not in the interval.) $f$ is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

$c$ value: $-2+\sqrt{5}$

**15.** $f(x) = \sin x, \ [0, 2\pi]$

$f(0) = f(2\pi) = 0$

$f$ is continuous on $[0, 2\pi]$. $f$ is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = \cos x$$

$c$ values: $\frac{\pi}{2}, \frac{3\pi}{2}$

**17.** $f(x) = \frac{6x}{\pi} - 4\sin^2 x, \ \left[0, \frac{\pi}{6}\right]$

$f(0) = f\left(\frac{\pi}{6}\right) = 0$

$f$ is continuous on $[0, \pi/6]$. $f$ is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8\sin x\cos x = 0$$

$$\frac{6}{\pi} = 8\sin x\cos x$$

$$\frac{3}{4\pi} = \frac{1}{2}\sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2}\arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

$c$ value: $0.2489$

**19.** $f(x) = \tan x, \ [0, \pi]$

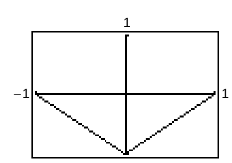
$f(0) = f(\pi) = 0$

$f$ is not continuous on $[0, \pi]$ since $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

**21.** $f(x) = |x| - 1, \ [-1, 1]$

$f(-1) = f(1) = 0$

$f$ is continuous on $[-1, 1]$. $f$ is not differentiable on $(-1, 1)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.



**23.** $f(x) = 4x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$

$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$

$f$ is continuous on $[-1/4, 1/4]$. $f$ is differentiable on $(-1/4, 1/4)$. Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

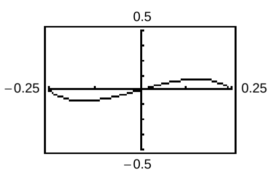
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm\frac{2}{\sqrt{\pi}}$$

$$x = \pm\frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm\frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

$c$ values: $\pm 0.1533$ radian




**25.** $f(t) = -16t^2 + 48t + 32$

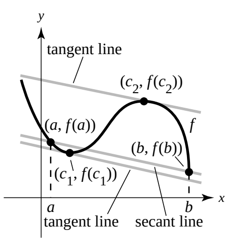
(a) $f(1) = f(2) = 64$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

**27.**




**29.** $f(x) = \frac{1}{x - 3}, [0, 6]$

$f$ has a discontinuity at $x = 3$.

**31.** $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$ when $x = -\frac{1}{2}$. Therefore,

$$c = -\frac{1}{2}.$$

**33.** $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

**35.** $f(x) = \sqrt{2 - x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2 - x}} = -\frac{1}{3}$$

$$2\sqrt{2 - x} = 3$$

$$\sqrt{2 - x} = \frac{3}{2}$$

$$2 - x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

**37.** $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

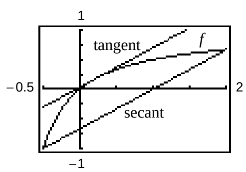
$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$c = \frac{\pi}{2}$$

**39.** $f(x) = \dfrac{x}{x + 1}$ on $\left[-\dfrac{1}{2}, 2\right]$.

(a)




(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$3y - 2 = 2x - 4$$

$$3y - 2x + 2 = 0$$

(c) $f'(x) = \dfrac{1}{(x + 1)^2} = \dfrac{2}{3}$

$$(x + 1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$, $c = -1 + \left(\sqrt{6}/2\right)$.

$$f(c) = \frac{-1 + \left(\sqrt{6}/2\right)}{\left[-1 + \left(\sqrt{6}/2\right)\right] + 1} = \frac{-2 + \sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

Tangent line: $y - 1 + \dfrac{2}{\sqrt{6}} = \dfrac{2}{3}\left(x - \dfrac{\sqrt{6}}{2} + 1\right)$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

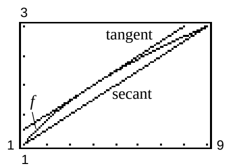
$$3y - 2x - 5 + 2\sqrt{6} = 0$$

**41.** $f(x) = \sqrt{x},\ [1, 9]$

$(1, 1), (9, 3)$

$m = \dfrac{3-1}{9-1} = \dfrac{1}{4}$

(a)




(b) Secant line: $y - 1 = \dfrac{1}{4}(x-1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c) $f'(x) = \dfrac{1}{2\sqrt{x}}$

$$\frac{f(9)-f(1)}{9-1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line: $y - 2 = \dfrac{1}{4}(x-4)$

$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

**43.** $s(t) = -4.9t^2 + 500$

(a) $V_{\text{avg}} = \dfrac{s(3)-s(0)}{3-0} = \dfrac{455.9-500}{3} = -14.7$ m/sec

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

**45.** No. Let $f(x) = x^2$ on $[-1, 2]$.

$$f'(x) = 2x$$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

**47.** Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time $t_0$, $0 < t_0 < 5.5$, such that

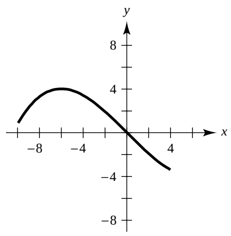
$$S'(t_0) = v(t_0) = \frac{2500-0}{5.5-0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

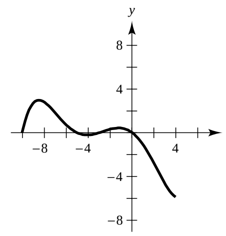
**49.** (a) $f$ is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0,\ f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of $x$ in $[-10, 4]$ satisfying $f(x) = 0$.

(b) There exist real numbers $a$ and $b$ such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number $c$ in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c)

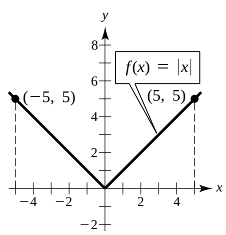



(d)




(e) No, $f'$ did not have to be continuous on $[-10, 4]$.

**51.** $f$ is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem.
$\Longrightarrow f$ is not differentiable on $(-5, 5)$.
Example: $f(x) = |x|$




**53.** False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

**55.** True. A polynomial is continuous and differentiable everywhere.

**57.** Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots $x_1$ and $x_2$. Then by Rolle's Theorem, since $p(x_1) = p(x_2) = 0$, there exists $c$ in $(x_1, x_2)$ such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, since $n > 0$, $a > 0$. Therefore, $p(x)$ cannot have two real roots.

**59.** If $p(x) = Ax^2 + Bx + C$, then

$$
\begin{aligned}
p'(x) = 2Ax + B = \frac{f(b) - f(a)}{b - a} &= \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\
&= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\
&= \frac{(b - a)[A(b + a) + B]}{b - a} \\
&= A(b + a) + B.
\end{aligned}
$$

Thus, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

**61.** $f(x) = \frac{1}{2}\cos x$ differentiable on $(-\infty, \infty)$.

$f'(x) = -\frac{1}{2}\sin x$

$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Longrightarrow f'(x) < 1$ for all real numbers.

Thus, from Exercise 60, $f$ has, at most, one fixed point. $(x \approx 0.4502)$

## Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

**1.** $f(x) = x^2 - 6x + 8$

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

**3.** $y = \dfrac{x^3}{4} - 3x$

Increasing on: $(-\infty, -2), (2, \infty)$

Decreasing on: $(-2, 2)$

**5.** $f(x) = \dfrac{1}{x^2} = x^{-2}$

$f'(x) = \dfrac{-2}{x^3}$

Discontinuity: $x = 0$

| Test intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

**7.** $g(x) = x^2 - 2x - 8$

$g'(x) = 2x - 2$

Critical number: $x = 1$

| Test intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $g'(x)$: | $g' < 0$ | $g' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

**9.** $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}\left(x - 2\sqrt{2}\right)\left(x + 2\sqrt{2}\right)$$

Critical numbers: $x = \pm 2\sqrt{2}$

| Test intervals: | $-4 < x < -2\sqrt{2}$ | $-2\sqrt{2} < x < 2\sqrt{2}$ | $2\sqrt{2} < x < 4$ |
|---|---|---|---|
| Sign of $y'$: | $y' < 0$ | $y' > 0$ | $y' < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

Increasing on $\left(-2\sqrt{2}, 2\sqrt{2}\right)$

Decreasing on $\left(-4, -2\sqrt{2}\right), \left(2\sqrt{2}, 4\right)$

**11.** $f(x) = x^2 - 6x$

$f'(x) = 2x - 6 = 0$

Critical number: $x = 3$

| Test intervals: | $-\infty < x < 3$ | $3 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

Relative minimum: $(3, -9)$

**13.** $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4 = 0$

Critical number: $x = 1$

| Test intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

Relative maximum: $(1, 5)$

**15.** $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$

Critical numbers: $x = -2, 1$

| Test intervals: | $-\infty < x < -2$ | $-2 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, -2), (1, \infty)$

Decreasing on: $(-2, 1)$

Relative maximum: $(-2, 20)$

Relative minimum: $(1, -7)$

**17.** $f(x) = x^2(3 - x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2 = 3x(2 - x)$

Critical numbers: $x = 0, 2$

| Test intervals: | $-\infty < x < 0$ | $0 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ | $f' < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

Increasing on: $(0, 2)$

Decreasing on: $(-\infty, 0), (2, \infty)$

Relative maximum: $(2, 4)$

Relative minimum: $(0, 0)$

**19.** $f(x) = \dfrac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers: $x = -1, 1$

| Test intervals: | $-\infty < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

Relative maximum: $\left(-1, \frac{4}{5}\right)$

Relative minimum: $\left(1, -\frac{4}{5}\right)$

**21.** $f(x) = x^{1/3} + 1$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

Critical number: $x = 0$

| Test intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' > 0$ |
| Conclusion: | Increasing | Increasing |

Increasing on: $(-\infty, \infty)$

No relative extrema

**23.** $f(x) = (x - 1)^{2/3}$

$f'(x) = \frac{2}{3(x - 1)^{1/3}}$

Critical number: $x = 1$

| Test intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

Relative minimum: $(1, 0)$

**25.** $f(x) = 5 - |x - 5|$

$f'(x) = -\frac{x - 5}{|x - 5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$

Critical number: $x = 5$

| Test intervals: | $-\infty < x < 5$ | $5 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $(-\infty, 5)$

Decreasing on: $(5, \infty)$

Relative maximum: $(5, 5)$

**27.** $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

Critical numbers: $x = -1, 1$

Discontinuity: $x = 0$

| Test intervals: | $-\infty < x < -1$ | $-1 < x < 0$ | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 0), (0, 1)$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

**29.** $f(x) = \dfrac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

| Test intervals: | $-\infty < x < -3$ | $-3 < x < 0$ | $0 < x < 3$ | $3 < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' > 0$ | $f' < 0$ | $f' < 0$ |
| Conclusion: | Increasing | Increasing | Decreasing | Decreasing |

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

Relative maximum: $(0, 0)$

**31.** $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

| Test intervals: | $-\infty < x < -3$ | $-3 < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

**33.** $f(x) = \dfrac{x}{2} + \cos x, 0 < x < 2\pi$

$$f'(x) = \frac{1}{2} - \sin x = 0$$

Critical numbers: $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

| Test intervals: | $0 < x < \dfrac{\pi}{6}$ | $\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$ | $\dfrac{5\pi}{6} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$

Relative maximum: $\left(\dfrac{\pi}{6}, \dfrac{\pi + 6\sqrt{3}}{12}\right)$

Relative minimum: $\left(\dfrac{5\pi}{6}, \dfrac{5\pi - 6\sqrt{3}}{12}\right)$

**35.** $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

| Test intervals: | $0 < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{7\pi}{6}$ | $\frac{7\pi}{6} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < \frac{11\pi}{6}$ | $\frac{11\pi}{6} < x < 2\pi$ |
|---|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{7\pi}{6}, \dfrac{3\pi}{2}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3\pi}{2}, \dfrac{11\pi}{6}\right)$

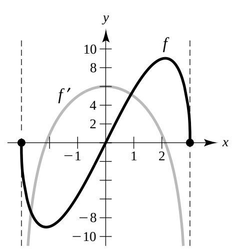
Relative minima: $\left(\dfrac{7\pi}{6}, -\dfrac{1}{4}\right), \left(\dfrac{11\pi}{6}, -\dfrac{1}{4}\right)$

Relative maxima: $\left(\dfrac{\pi}{2}, 2\right), \left(\dfrac{3\pi}{2}, 0\right)$

**37.** $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

(a) $f'(x) = \dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)




(c) $\dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}} = 0$

Critical numbers: $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

(d) Intervals:

| $\left(-3, -\frac{3\sqrt{2}}{2}\right)$ | $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ | $\left(\frac{3\sqrt{2}}{2}, 3\right)$ |
|---|---|---|
| $f'(x) < 0$ | $f'(x) > 0$ | $f'(x) < 0$ |
| Decreasing | Increasing | Decreasing |

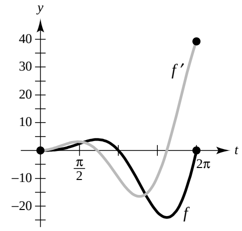
$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**39.** $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t$

$= t(t \cos t + 2 \sin t)$

(b)




(c) $t(t \cos t + 2 \sin t) = 0$

$t = 0$ or $t = -2 \tan t$

$t \cot t = -2$

$t \approx 2.2889, 5.0870$ (graphing utility)

Critical numbers: $t = 2.2889, t = 5.0870$

(d) Intervals:

| $(0, 2.2889)$ | $(2.2889, 5.0870)$ | $(5.0870, 2\pi)$ |
|---|---|---|
| $f'(t) > 0$ | $f'(t) < 0$ | $f'(t) > 0$ |
| Increasing | Decreasing | Increasing |

$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**41.** $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1} = \dfrac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x, \; x \neq \pm 1$

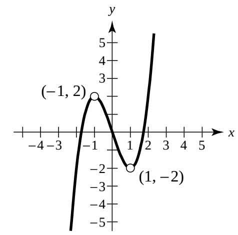
$f(x) = g(x) = x^3 - 3x$ for all $x \neq \pm 1$.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \quad f'(x) \neq 0$

$f$ symmetric about origin

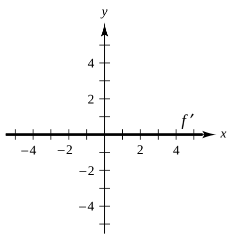
zeros of $f$: $(0, 0), \left(\pm\sqrt{3}, 0\right)$

No relative extrema

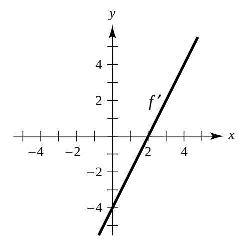



Holes at $(-1, 2)$ and $(1, -2)$

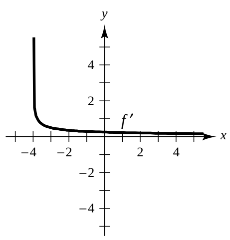
**43.** $f(x) = c$ is constant $\Longrightarrow f'(x) = 0$




**45.** $f$ is quadratic $\Longrightarrow f'$ is a line.




**47.** $f$ has positive, but decreasing slope




**In Exercises 49–53, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.**

**49.** $g(x) = f(x) + 5$

$g'(x) = f'(x)$

$g'(0) = f'(0) < 0$

**51.** $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(-6) = -f'(-6) < 0$

**53.** $g(x) = f(x - 10)$

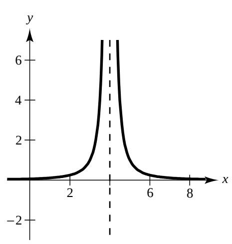
$g'(x) = f'(x - 10)$

$g'(0) = f'(-10) > 0$

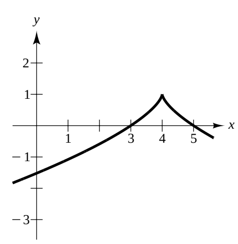
**55.** $f'(x) = \begin{cases} > 0, & x < 4 \Longrightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Longrightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$

Two possibilities for $f(x)$ are given below.

(a)



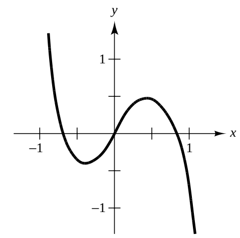
(b)



**57.** The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ since the sign of $f'$ changes in these intervals. $f$ is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

Relative minimum when $x \approx -0.40$.

Relative maximum when $x \approx 0.48$.



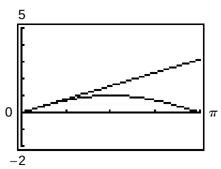

**59.** $f(x) = x, g(x) = \sin x, 0 < x < \pi$

(a)

| $x$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| $g(x)$ | 0.479 | 0.841 | 0.997 | 0.909 | 0.598 | 0.141 |

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.

(b)




$x > \sin x$ on $(0, \pi)$

(c) Let $h(x) = f(x) - g(x) = x - \sin x$

$$h'(x) = 1 - \cos x > 0 \text{ on } (0, \pi).$$

Therefore, $h(x)$ is increasing on $(0, \pi)$. Since $h(0) = 0$, $h(x) > 0$ on $(0, \pi)$. Thus,

$$x - \sin x > 0$$
$$x > \sin x$$
$$f(x) > g(x) \text{ on } (0, \pi).$$

**61.** $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$v' = k(2Rr - 3r^2)$

$= kr(2R - 3r) = 0$

$r = 0$ or $\frac{2}{3}R$

Maximum when $r = \frac{2}{3}R$.

**63.** $P = \dfrac{vR_1R_2}{(R_1 + R_2)^2}$, $v$ and $R_1$ are constant

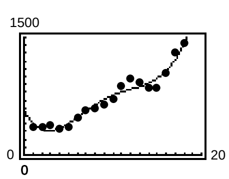
$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Longrightarrow R_2 = R_1$$

Maximum when $R_1 = R_2$.

**65.** (a) $B = 0.1198t^4 - 4.4879t^3 + 56.9909t^2 - 223.0222t + 579.9541$

(b)




(c) $B' = 0$ for $t \approx 2.78$, or 1983, (311.1 thousand bankruptcies)
Actual minimum: 1984 (344.3 thousand bankruptcies)

**67.** (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

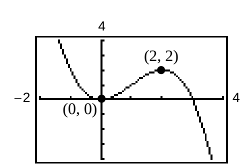
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$.

| | | |
|---|---|---|
| $(0, 0)$: | $0 = a_0$ | $(f(0) = 0)$ |
| | $0 = a_1$ | $(f'(0) = 0)$ |
| $(2, 2)$: | $2 = 8a_3 + 4a_2$ | $(f(2) = 2)$ |
| | $0 = 12a_3 + 4a_2$ | $(f'(2) = 0)$ |

(c) The solution is $a_0 = a_1 = 0, a_2 = \dfrac{3}{2}, a_3 = -\dfrac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$

(d)



**69.** (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

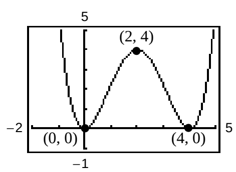
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

| | | |
|---|---|---|
| $(0, 0)$: | $0 = a_0$ | $(f(0) = 0)$ |
| | $0 = a_1$ | $(f'(0) = 0)$ |
| $(4, 0)$: | $0 = 256a_4 + 64a_3 + 16a_2$ | $(f(4) = 0)$ |
| | $0 = 256a_4 + 48a_3 + 8a_2$ | $(f'(4) = 0)$ |
| $(2, 4)$: | $4 = 16a_4 + 8a_3 + 4a_2$ | $(f(2) = 4)$ |
| | $0 = 32a_4 + 12a_3 + 4a_2$ | $(f'(2) = 0)$ |

(c) The solution is $a_0 = a_1 = 0$, $a_2 = 4$, $a_3 = -2$, $a_4 = \frac{1}{4}$.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)




**71.** True

Let $h(x) = f(x) + g(x)$ where $f$ and $g$ are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ since $f'(x) > 0$ and $g'(x) > 0$.

**73.** False

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and $f$ only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

**75.** False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

**77.** Assume that $f'(x) < 0$ for all $x$ in the interval $(a, b)$ and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number $c$ such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, $f$ is decreasing on the interval.

**79.** Let $f(x) = (1 + x)^n - nx - 1$. Then

$$\begin{aligned} f'(x) &= n(1 + x)^{n-1} - n \\ &= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1. \end{aligned}$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Longrightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Longrightarrow (1 + x)^n > 1 + nx.$$

## Section 3.4 Concavity and the Second Derivative Test

**1.** $y = x^2 - x - 2, y'' = 2$

Concave upward: $(-\infty, \infty)$

**3.** $f(x) = \dfrac{24}{x^2 + 12}, y'' = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

**5.** $f(x) = \dfrac{x^2 + 1}{x^2 - 1}, \; y'' = \dfrac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

**7.** $f(x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2$

$f''(x) = 6 - 6x$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

**9.** $y = 2x - \tan x, \; \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2 \sec^2 x \tan x$

Concave upward: $\left(-\dfrac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \dfrac{\pi}{2}\right)$

**11.** $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

The concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

**13.** $f(x) = \dfrac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm\dfrac{2}{\sqrt{3}}$.

| Test interval: | $-\infty < x < -\dfrac{2}{\sqrt{3}}$ | $-\dfrac{2}{\sqrt{3}} < x < \dfrac{2}{\sqrt{3}}$ | $\dfrac{2}{\sqrt{3}} < x < \infty$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f''(x) > 0$ | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Points of inflection: $\left(\pm\dfrac{2}{\sqrt{3}}, -\dfrac{20}{9}\right)$

**15.** $f(x) = x(x-4)^3$

$$f'(x) = x[3(x-4)^2] + (x-4)^3$$
$$= (x-4)^2(4x-4)$$
$$f''(x) = 4(x-1)[2(x-4)] + 4(x-4)^2$$
$$= 4(x-4)[2(x-1) + (x-4)]$$
$$= 4(x-4)(3x-6) = 12(x-4)(x-2)$$

$f''(x) = 12(x-4)(x-2) = 0$ when $x = 2, 4$.

| Test interval: | $-\infty < x < 2$ | $2 < x < 4$ | $4 < x < \infty$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f''(x) > 0$ | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion: | Concave upward | Concave downward | Concave upward |

Points of inflection: $(2, -16), (4, 0)$

**17.** $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

$f''(x) > 0$ on the entire domain of $f$ (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection. Concave upward on $(-3, \infty)$

**19.** $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}$$

| Test intervals: | $-\infty < x < -\sqrt{3}$ | $-\sqrt{3} < x < 0$ | $0 < x < \sqrt{3}$ | $\sqrt{3} < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $f'' < 0$ | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward | Concave upward |

Points of inflection: $\left(-\sqrt{3}, -\dfrac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \dfrac{\sqrt{3}}{4}\right)$

**21.** $f(x) = \sin\left(\dfrac{x}{2}\right), 0 \le x \le 4\pi$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$f''(x) = 0$ when $x = 0, 2\pi, 4\pi$.

Point of inflection: $(2\pi, 0)$

| Test interval: | $0 < x < 2\pi$ | $2\pi < x < 4\pi$ |
|---|---|---|
| Sign of $f''(x)$: | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave downward | Concave upward |

**23.** $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$f'(x) = \sec\left(x - \frac{\pi}{2}\right)\tan\left(x - \frac{\pi}{2}\right)$

$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right)\tan^2\left(x - \frac{\pi}{2}\right) \neq 0$ for any $x$ in the domain of $f$.

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No points of inflection

**25.** $f(x) = 2\sin x + \sin 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x + 2\cos 2x$

$f''(x) = -2\sin x - 4\sin 2x = -2\sin x(1 + 4\cos x)$

$f''(x) = 0$ when $x = 0, 1.823, \pi, 4.460$.

| Test interval: | $0 < x < 1.823$ | $1.823 < x < \pi$ | $\pi < x < 4.460$ | $4.460 < x < 2\pi$ |
|---|---|---|---|---|
| Sign of $f''(x)$: | $f'' < 0$ | $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward | Concave upward |

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

**27.** $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

**29.** $f(x) = (x - 5)^2$

$f'(x) = 2(x - 5)$

$f''(x) = 2$

Critical number: $x = 5$

$f''(5) > 0$

Therefore, $(5, 0)$ is a relative minimum.

**31.** $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

**33.** $g(x) = x^2(6 - x)^3$

$g'(x) = x(x - 6)^2(12 - 5x)$

$g''(x) = 4(6 - x)(5x^2 - 24x + 18)$

Critical numbers: $x = 0, \frac{12}{5}, 6$

$g''(0) = 432 > 0$

Therefore, $(0, 0)$ is a relative minimum.

$g''\left(\frac{12}{5}\right) = -155.52 < 0$

Therefore, $\left(\frac{12}{5}, 268.7\right)$ is a relative minimum.

$g''(6) = 0$

Test fails by the First Derivative Test, $(6, 0)$ is not an extremum.

**35.** $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

**37.** $f(x) = x + \dfrac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

Critical numbers: $x = \pm 2$

$f''(-2) < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) > 0$

Therefore, $(2, 4)$ is a relative minimum.

**39.** $f(x) = \cos x - x,\ 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, $f$ is non-increasing and there are no relative extrema.

**41.** $f(x) = 0.2x^2(x - 3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x - 3)(10x^2 - 24x + 9)$

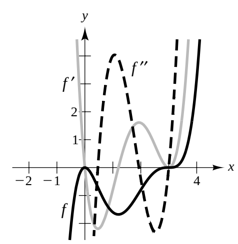
(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$

(c)




$f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

**43.** $f(x) = \sin x - \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x, [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{\pi}{2}, x = \dfrac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$

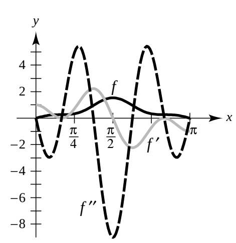
(b) $f''\left(\dfrac{\pi}{2}\right) < 0 \Rightarrow \left(\dfrac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\dfrac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$(1.9685, 0.9637), \left(\dfrac{5\pi}{6}, 0.2667\right)$

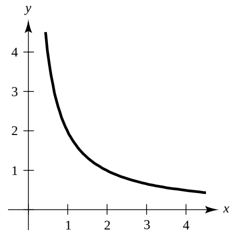
**Note:** $(0, 0)$ and $(\pi, 0)$ are not points of inflection since they are endpoints.

(c)




The graph of $f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

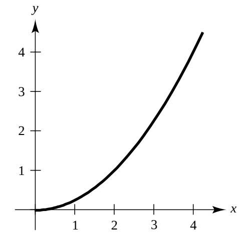
**45.** (a)




$f' < 0$ means $f$ decreasing

$f'$ increasing means concave upward

(b)



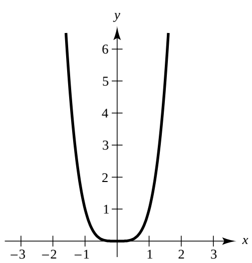

$f' > 0$ means $f$ increasing

$f'$ increasing means concave upward

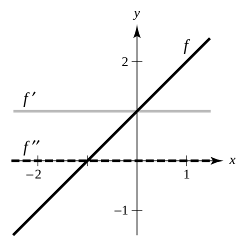
**47.** Let $f(x) = x^4$.

$$f''(x) = 12x^2$$

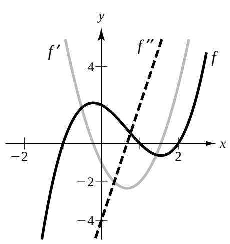
$f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



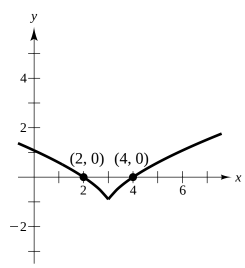

**49.**



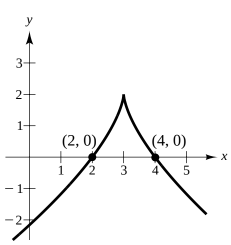

**51.**



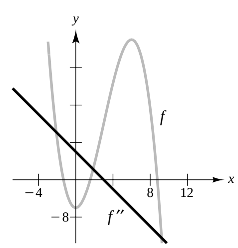

**53.**




**55.**




**57.**




$f''$ is linear.

$f'$ is quadratic.

$f$ is cubic.

$f$ concave upwards on $(-\infty, 3)$, downward on $(3, \infty)$.

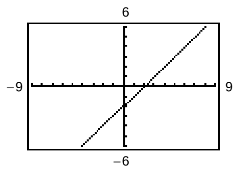
**59.** (a) $n = 1$:

$f(x) = x - 2$

$f'(x) = 1$

$f''(x) = 0$

No inflection points




$n = 2$:

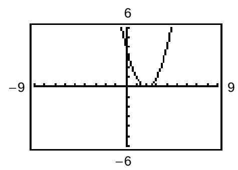
$f(x) = (x - 2)^2$

$f'(x) = 2(x - 2)$

$f''(x) = 2$

No inflection points

Relative minimum: $(2, 0)$



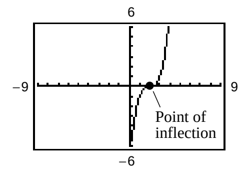

$n = 3$:

$f(x) = (x - 2)^3$

$f'(x) = 3(x - 2)^2$

$f''(x) = 6(x - 2)$

Inflection point: $(2, 0)$




$n = 4$:

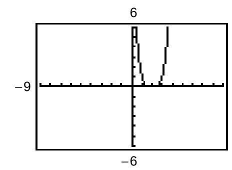
$f(x) = (x - 2)^4$

$f'(x) = 4(x - 2)^3$

$f''(x) = 12(x - 2)^2$

No inflection points:

Relative minimum: $(2, 0)$




**Conclusion:** If $n \geq 3$ and $n$ is odd, then $(2, 0)$ is an inflection point. If $n \geq 2$ and $n$ is even, then $(2, 0)$ is a relative minimum.

(b) Let $f(x) = (x - 2)^n$, $f'(x) = n(x - 2)^{n-1}$, $f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

Thus, $x = 2$ is an inflection point if and only if $n \geq 3$ is odd.

**61.** $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(3, 3)$

Relative minimum: $(5, 1)$

Point of inflection: $(4, 2)$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned}\right\} 98a + 16b + 2c = -2 \Longrightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0,\ f''(4) = 24a + 2b = 0$

$$\begin{array}{rcl} 49a + 8b + c = -1 & \qquad & 24a + 2b = \phantom{-}0 \\ \underline{27a + 6b + c = \phantom{-}0} & & \underline{22a + 2b = -1} \\ 22a + 2b \phantom{{}+c} = -1 & & 2a \phantom{{}+2b} = \phantom{-}1 \end{array}$$

$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

**63.** $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \quad f''(x) = 6ax + 2b$

$$f(0) = 0 \Rightarrow d = 0$$

$$\begin{aligned} f(-4) = 1 &\Rightarrow -64a + 16b - 4c = 1 \\ f'(-4) = 0 &\Rightarrow \quad 48a - 8b + c = 0 \\ f'(0) = 0 &\Rightarrow \qquad\qquad\qquad c = 0 \end{aligned}$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$$f(x) = \tfrac{1}{32}x^3 + \tfrac{3}{16}x^2$$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$$

Two miles from touchdown.

**65.** $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

**67.** $C = 0.5x^2 + 15x + 5000$

$$\overline{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\overline{C}$ = average cost per unit

$$\frac{d\overline{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

**69.** $S = \dfrac{5000t^2}{8 + t^2}$

$$S'(t) = \frac{80{,}000t}{(8 + t^2)^2}$$

$$S''(t) = \frac{80{,}000(8 - 3t^2)}{(8 + t^2)^3}$$

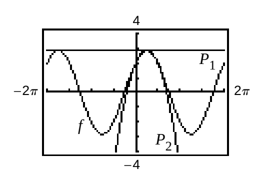
$S''(t) = 0$ for $t = \sqrt{8/3} \approx 1.633$.

Sales are increasing at the greatest rate at $t = 1.633$ years.

**71.** $f(x) = 2(\sin x + \cos x), \qquad f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$$f'(x) = 2(\cos x - \sin x), \qquad f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = 2(-\sin x - \cos x), \qquad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$$




$$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$$

$$P_1'(x) = 0$$

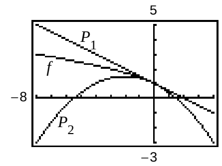
$$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}\left(-2\sqrt{2}\right)\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$P_2''(x) = -2\sqrt{2}$$

The values of $f$, $P_1$, $P_2$, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

**73.** $f(x) = \sqrt{1-x}, \qquad f(0) = 1$




$f'(x) = -\dfrac{1}{2\sqrt{1-x}}, \qquad f'(0) = -\dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4(1-x)^{3/2}}, \qquad f''(0) = -\dfrac{1}{4}$

$P_1(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) = 1 - \dfrac{x}{2}$

$P_1'(x) = -\dfrac{1}{2}$

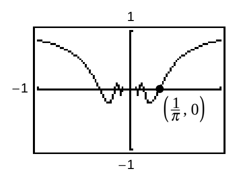
$P_2(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) + \dfrac{1}{2}\left(-\dfrac{1}{4}\right)(x-0)^2 = 1 - \dfrac{x}{2} - \dfrac{x^2}{8}$

$P_2'(x) = -\dfrac{1}{2} - \dfrac{x}{4}$

$P_2''(x) = -\dfrac{1}{4}$

The values of $f$, $P_1$, $P_2$, and their first derivatives are equal at $x = 0$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

**75.** $f(x) = x\sin\left(\dfrac{1}{x}\right)$




$f'(x) = x\left[-\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)\right] + \sin\left(\dfrac{1}{x}\right) = -\dfrac{1}{x}\cos\left(\dfrac{1}{x}\right) + \sin\left(\dfrac{1}{x}\right)$

$f''(x) = -\dfrac{1}{x}\left[\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)\right] + \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) - \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^3}\sin\left(\dfrac{1}{x}\right) = 0$

$x = \dfrac{1}{\pi}$

Point of inflection: $\left(\dfrac{1}{\pi}, 0\right)$

When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.

**77.** Assume the zeros of $f$ are all real. Then express the function as $f(x) = a(x - r_1)(x - r_2)(x - r_3)$ where $r_1$, $r_2$, and $r_3$ are the distinct zeros of $f$. From the Product Rule for a function involving three factors, we have

$$\begin{aligned} f'(x) &= a[(x-r_1)(x-r_2) + (x-r_1)(x-r_3) + (x-r_2)(x-r_3)] \\ f''(x) &= a[(x-r_1) + (x-r_2) + (x-r_1) + (x-r_3) + (x-r_2) + (x-r_3)] \\ &= a[6x - 2(r_1 + r_2 + r_3)]. \end{aligned}$$

Consequently, $f''(x) = 0$ if

$$x = \frac{2(r_1 + r_2 + r_3)}{6} = \frac{r_1 + r_2 + r_3}{3} = \text{(Average of } r_1, r_2, \text{ and } r_3\text{)}.$$

**79.** True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

**81.** False.

$$f(x) = 3\sin x + 2\cos x$$
$$f'(x) = 3\cos x - 2\sin x$$
$$3\cos x - 2\sin x = 0$$
$$3\cos x = 2\sin x$$
$$\tfrac{3}{2} = \tan x$$

Critical number: $x = \tan^{-1}\left(\tfrac{3}{2}\right)$

$f\left(\tan^{-1}\tfrac{3}{2}\right) \approx 3.60555$ is the maximum value of $y$.

**83.** False. Concavity is determined by $f''$.

## Section 3.5 Limits at Infinity

**1.** $f(x) = \dfrac{3x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 3$

Matches (f)

**3.** $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

Matches (d)

**5.** $f(x) = \dfrac{4\sin x}{x^2 + 1}$

No vertical asymptotes

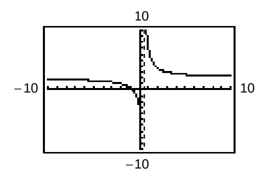
Horizontal asymptotes: $y = 0$

Matches (b)

**7.** $f(x) = \dfrac{4x + 3}{2x - 1}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 7 | 2.26 | 2.025 | 2.0025 | 2.0003 | 2 | 2 |

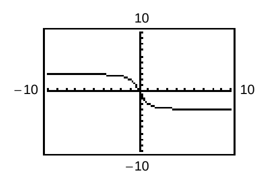
$\lim_{x\to\infty} f(x) = 2$




**9.** $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-2$ | $-2.98$ | $-2.9998$ | $-3$ | $-3$ | $-3$ | $-3$ |

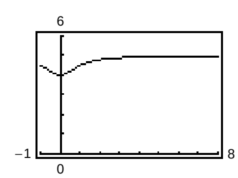
$\lim_{x\to\infty} f(x) = -3$




**11.** $f(x) = 5 - \dfrac{1}{x^2 + 1}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 4.5 | 4.99 | 4.9999 | 4.999999 | 5 | 5 | 5 |

$\lim_{x\to\infty} f(x) = 5$



**13.** (a) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \dfrac{10}{x^2}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^3 - 3x^2 + 10}{x^3} = 5 - \dfrac{3}{x} + \dfrac{10}{x^3}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^4} = \dfrac{5x^3 - 3x^2 + 10}{x^4} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{10}{x^4}$

$\lim_{x\to\infty} h(x) = 0$

**15.** (a) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1} = 1$

(c) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x - 1} = \infty$ (Limit does not exist)

**17.** (a) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^2 - 4} = 0$

(b) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\dfrac{2}{3}$

(c) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x - 4} = -\infty$ (Limit does not exist)

**19.** $\lim_{x\to\infty} \dfrac{2x - 1}{3x + 2} = \lim_{x\to\infty} \dfrac{2 - (1/x)}{3 + (2/x)} = \dfrac{2 - 0}{3 + 0} = \dfrac{2}{3}$

**21.** $\lim_{x\to\infty} \dfrac{x}{x^2 - 1} = \lim_{x\to\infty} \dfrac{1/x}{1 - (1/x^2)} = \dfrac{0}{1} = 0$

**23.** $\lim_{x\to-\infty} \dfrac{5x^2}{x + 3} = \lim_{x\to-\infty} \dfrac{5x}{1 + (3/x)} = -\infty$

Limit does not exist.

**25.** $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2 - x}} = \lim_{x\to-\infty} \dfrac{1}{\dfrac{\sqrt{x^2 - x}}{-\sqrt{x^2}}}$, $\left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

$= \lim_{x\to-\infty} \dfrac{-1}{\sqrt{1 - (1/x)}} = -1$

**27.** $\lim_{x\to-\infty} \dfrac{2x + 1}{\sqrt{x^2 - x}} = \lim_{x\to-\infty} \dfrac{2 + \dfrac{1}{x}}{\left(\dfrac{\sqrt{x^2 - x}}{-\sqrt{x^2}}\right)}$ $\left(\text{for } x < 0,\ x = -\sqrt{x^2}\right)$

$= \lim_{x\to\infty} \dfrac{-2 - (1/x)}{\sqrt{x + (1/x)}} = -2$

**29.** Since $(-1/x) \le (\sin(2x))/x \le (1/x)$ for all $x \ne 0$, we have by the Squeeze Theorem,

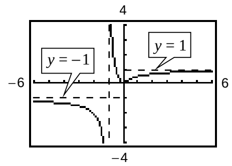
$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin(2x)}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin(2x)}{x} \le 0.$$

Therefore, $\lim_{x\to\infty} \dfrac{\sin(2x)}{x} = 0.$

**31.** $\lim_{x\to\infty} \dfrac{1}{2x + \sin x} = 0$

**33.** (a) $f(x) = \dfrac{|x|}{x+1}$




$$\lim_{x\to\infty} \frac{|x|}{x+1} = 1$$

$$\lim_{x\to-\infty} \frac{|x|}{x+1} = -1$$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.

**35.** $\displaystyle\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1$

(Let $x = 1/t$.)

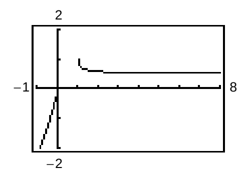
**37.** $\displaystyle\lim_{x\to-\infty} \left(x + \sqrt{x^2+3}\right) = \lim_{x\to-\infty} \left[\left(x + \sqrt{x^2+3}\right) \cdot \frac{x - \sqrt{x^2+3}}{x - \sqrt{x^2+3}}\right] = \lim_{x\to-\infty} \frac{-3}{x - \sqrt{x^2+3}} = 0$

**39.** $\displaystyle\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right) = \lim_{x\to\infty} \left[\left(x - \sqrt{x^2+x}\right) \cdot \frac{x + \sqrt{x^2+x}}{x + \sqrt{x^2+x}}\right]$

$$= \lim_{x\to\infty} \frac{-x}{x + \sqrt{x^2+x}} = \lim_{x\to\infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$$

**41.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 0.513 | 0.501 | 0.500 | 0.500 | 0.500 | 0.500 |




$$\lim_{x\to\infty} \left(x - \sqrt{x(x-1)}\right) = \lim_{x\to\infty} \frac{x - \sqrt{x^2-x}}{1} \cdot \frac{x + \sqrt{x^2-x}}{x + \sqrt{x^2-x}}$$

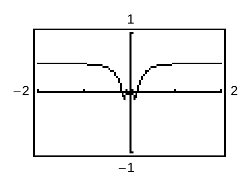
$$= \lim_{x\to\infty} \frac{x}{x + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{1}{1 + \sqrt{1 - (1/x)}}$$

$$= \frac{1}{2}$$

**43.**

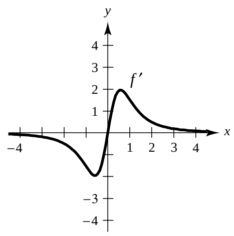
| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.479 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |




Let $x = 1/t$.

$$\lim_{x\to\infty} x \sin\left(\frac{1}{2x}\right) = \lim_{t\to 0^+} \frac{\sin(t/2)}{t} = \lim_{t\to 0^+} \frac{1}{2} \frac{\sin(t/2)}{t/2} = \frac{1}{2}$$

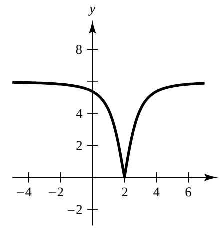
**45.** (a)




(b) $\lim_{x\to\infty} f(x) = 3 \qquad \lim_{x\to\infty} f'(x) = 0$

(c) Since $\lim_{x\to\infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x\to\infty} f'(x) = 0$.

**47.** Yes. For example, let $f(x) = \dfrac{6|x-2|}{\sqrt{(x-2)^2+1}}$.




**49.** $y = \dfrac{2+x}{1-x}$

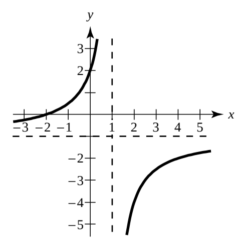
Intercepts: $(-2, 0), (0, 2)$

Symmetry: none

Horizontal asymptote: $y = -1$ since

$$\lim_{x\to-\infty} \frac{2+x}{1-x} = -1 = \lim_{x\to\infty} \frac{2+x}{1-x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



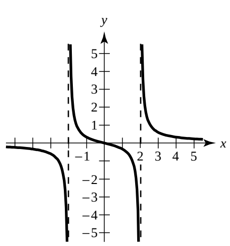

**51.** $y = \dfrac{x}{x^2-4}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$




**53.** $y = \dfrac{x^2}{x^2+9}$

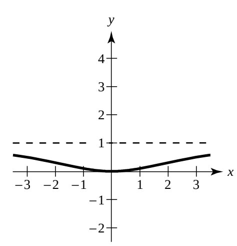
Intercept: $(0, 0)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x\to-\infty} \frac{x^2}{x^2+9} = 1 = \lim_{x\to\infty} \frac{x^2}{x^2+9}.$$

Relative minimum: $(0, 0)$



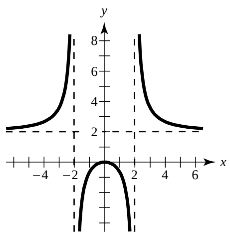
**55.** $y = \dfrac{2x^2}{x^2 - 4}$

Intercept: $(0, 0)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = \pm 2$




**57.** $xy^2 = 4$

Domain: $x > 0$

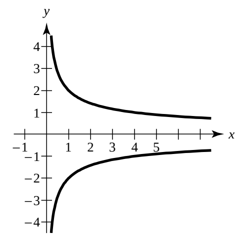
Intercepts: none

Symmetry: $x$-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \to \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity: $x = 0$ (Vertical asymptote)




**59.** $y = \dfrac{2x}{1 - x}$

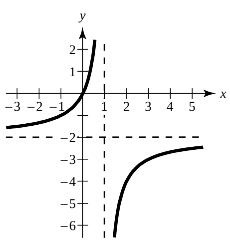
Intercept: $(0, 0)$

Symmetry: none

Horizontal asymptote: $y = -2$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \to \infty} \frac{2x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)




**61.** $y = 2 - \dfrac{3}{x^2}$

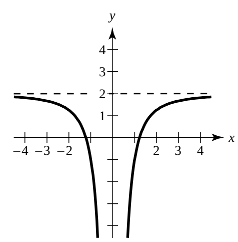
Intercepts: $\left(\pm\sqrt{3/2}, 0\right)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 2$ since

$$\lim_{x \to -\infty} \left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x \to \infty} \left(2 - \frac{3}{x^2}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



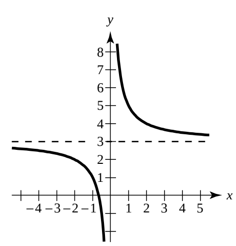

**63.** $y = 3 + \dfrac{2}{x}$

Intercept: $y = 0 = 3 + \dfrac{2}{x} \Rightarrow \dfrac{2}{x} = -3 \Rightarrow x = -\dfrac{2}{3}\left(-\dfrac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$



**65.** $y = \dfrac{x^3}{\sqrt{x^2 - 4}}$

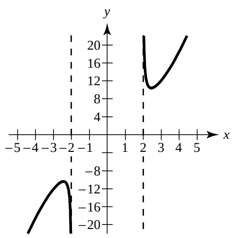
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)



**67.** $f(x) = 5 - \dfrac{1}{x^2} = \dfrac{5x^2 - 1}{x^2}$

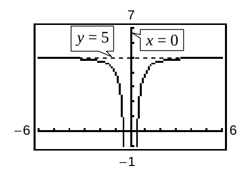
Domain: $(-\infty, 0), (0, \infty)$

$f'(x) = \dfrac{2}{x^3} \Longrightarrow$ No relative extrema

$f''(x) = -\dfrac{6}{x^4} \Longrightarrow$ No points of inflection

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 5$




**69.** $f(x) = \dfrac{x}{x^2 - 4}$

$$f'(x) = \frac{(x^2 - 4) - x(2x)}{(x^2 - 4)^2}$$

$$= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \neq 0 \text{ for any } x \text{ in the domain of } f.$$

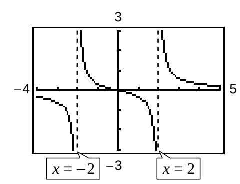
$$f''(x) = \frac{(x^2 - 4)^2(-2x) + (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^2}$$

$$= \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0 \text{ when } x = 0.$$

Since $f''(x) > 0$ on $(-2, 0)$ and $f''(x) < 0$ on $(0, 2)$, then $(0, 0)$ is a point of inflection.

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 0$




**71.** $f(x) = \dfrac{x - 2}{x^2 - 4x + 3} = \dfrac{x - 2}{(x - 1)(x - 3)}$

$$f'(x) = \frac{(x^2 - 4x + 3) - (x - 2)(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$$

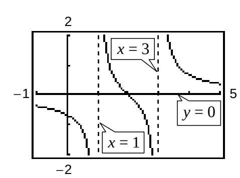
$$f''(x) = \frac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4}$$

$$= \frac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = 0 \text{ when } x = 2.$$

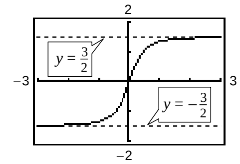
Since $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptote: $x = 1, x = 3$

Horizontal asymptote: $y = 0$



**73.** $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$




$f'(x) = \dfrac{3}{(4x^2 + 1)^{3/2}} \Longrightarrow$ No relative extrema

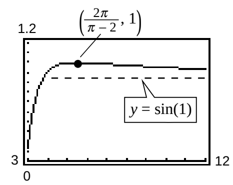
$f''(x) = \dfrac{-36x}{(4x^2 + 1)^{5/2}} = 0$ when $x = 0$.

Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm\dfrac{3}{2}$

No vertical asymptotes

**75.** $g(x) = \sin\left(\dfrac{x}{x - 2}\right),\ 3 < x < \infty$




$g'(x) = \dfrac{-2\cos\left(\dfrac{x}{x - 2}\right)}{(x - 2)^2}$

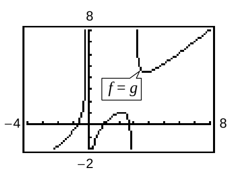
Horizontal asymptote: $y = 1$

Relative maximum: $\dfrac{x}{x - 2} = \dfrac{\pi}{2} \Longrightarrow x = \dfrac{2\pi}{\pi - 2} \approx 5.5039$

No vertical asymptotes

**77.** $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)},\ g(x) = x + \dfrac{2}{x(x - 3)}$

(a)

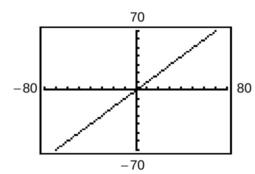



(b) $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$$= \frac{x^2(x - 3)}{x(x - 3)} + \frac{2}{x(x - 3)}$$

$$= x + \frac{2}{x(x - 3)} = g(x)$$

(c)




The graph appears as the slant asymptote $y = x$.

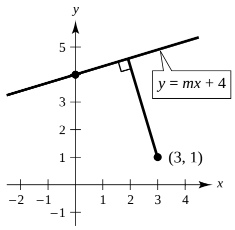
**79.** $C = 0.5x + 500$

$\overline{C} = \dfrac{C}{x}$

$\overline{C} = 0.5 + \dfrac{500}{x}$

$\displaystyle\lim_{x \to \infty}\left(0.5 + \frac{500}{x}\right) = 0.5$

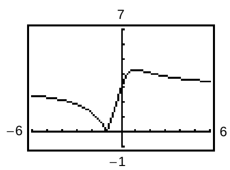
**81.** line: $mx - y + 4 = 0$




(a) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}}$

$= \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

(b)

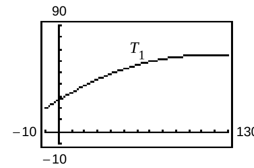



(c) $\lim_{m \to \infty} d(m) = 3 = \lim_{m \to -\infty} d(m)$

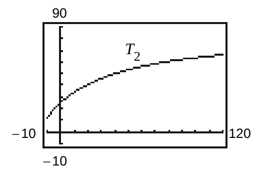
The line approaches the vertical line $x = 0$. Hence, the distance approaches 3.

**83.** (a) $T_1(t) = -0.003t^2 + 0.677t + 26.564$

(b)




(c)




$T_2 = \dfrac{1451 + 86t}{58 + t}$

(d) $T_1(0) \approx 26.6$

$T_2(0) \approx 25.0$

(e) $\lim_{t \to \infty} T_2 = \dfrac{86}{1} = 86$

(f) The limiting temperature is 86.
$T_1$ has no horizontal asymptote.

**85.** Answers will vary. See page 195.

**87.** False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$. (See Exercise 2.)

## Section 3.6 A Summary of Curve Sketching

**1.** $f$ has constant negative slope. Matches (D)

**3.** The slope is periodic, and zero at $x = 0$. Matches (A)

**5.** (a) $f'(x) = 0$ for $x = -2$ and $x = 2$

$f'$ is negative for $-2 < x < 2$ (decreasing function).

$f'$ is positive for $x > 2$ and $x < -2$
(increasing function).

(b) $f''(x) = 0$ at $x = 0$ (Inflection point).

$f''$ is positive for $x > 0$ (Concave upwards).

$f''$ is negative for $x < 0$ (Concave downward).

(c) $f'$ is increasing on $(0, \infty)$. $(f'' > 0)$

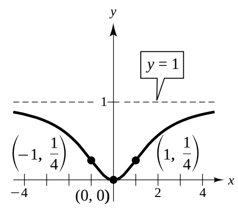
(d) $f'(x)$ is minimum at $x = 0$. The rate of change of $f$ at $x = 0$ is less than the rate of change of $f$ for all other values of $x$.

**7.** $y = \dfrac{x^2}{x^2 + 3}$

$y' = \dfrac{6x}{(x^2 + 3)^2} = 0$ when $x = 0$.

$y'' = \dfrac{18(1 - x^2)}{(x^2 + 3)^3} = 0$ when $x = \pm 1$.

Horizontal asymptote: $y = 1$




| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $-$ | Decreasing, concave down |
| $x = -1$ | $\frac{1}{4}$ | $-$ | $0$ | Point of inflection |
| $-1 < x < 0$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 0$ | $0$ | $0$ | $+$ | Relative minimum |
| $0 < x < 1$ | | $+$ | $+$ | Increasing, concave up |
| $x = 1$ | $\frac{1}{4}$ | $+$ | $0$ | Point of inflection |
| $1 < x < \infty$ | | $+$ | $-$ | Increasing, concave down |

**9.** $y = \dfrac{1}{x - 2} - 3$

$y' = -\dfrac{1}{(x - 2)^2} < 0$ when $x \neq 2$.

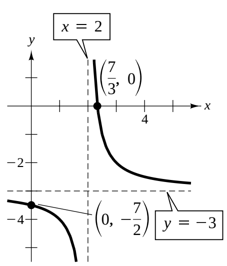
$y'' = \dfrac{2}{(x - 2)^3}$

No relative extrema, no points of inflection

Intercepts: $\left(\dfrac{7}{3}, 0\right), \left(0, -\dfrac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$




**11.** $y = \dfrac{2x}{x^2 - 1}$

$y' = \dfrac{-2(x^2 + 1)}{(x^2 - 1)^2} < 0$ if $x \neq \pm 1$.

$y'' = \dfrac{4x(x^2 + 3)}{(x^2 - 1)^3} = 0$ if $x = 0$.

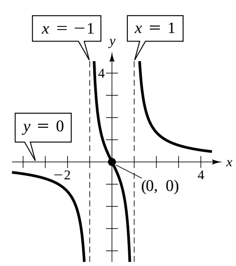
Inflection point: $(0, 0)$

Intercept: $(0, 0)$

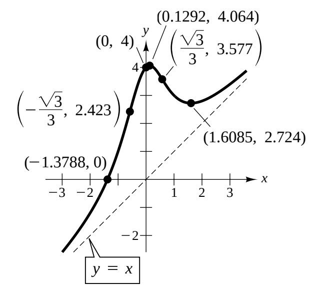
Vertical asymptote: $x = \pm 1$

Horizontal asymptote: $y = 0$

Symmetry with respect to the origin



**13.** $g(x) = x + \dfrac{4}{x^2 + 1}$




$$g'(x) = 1 - \frac{8x}{(x^2 + 1)^2} = \frac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0 \text{ when } x \approx 0.1292, 1.6085$$

$$g''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0 \text{ when } x = \pm\frac{\sqrt{3}}{3}$$

$g''(0.1292) < 0$, therefore, $(0.1292, 4.064)$ is relative maximum.

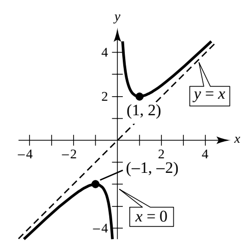
$g''(1.6085) > 0$, therefore, $(1.6085, 2.724)$ is a relative minimum.

Points of inflection: $\left(-\dfrac{\sqrt{3}}{3}, 2.423\right), \left(\dfrac{\sqrt{3}}{3}, 3.577\right)$

Intercepts: $(0, 4), (-1.3788, 0)$

Slant asymptote: $y = x$

**15.** $f(x) = \dfrac{x^2 + 1}{x} = x + \dfrac{1}{x}$




$f'(x) = 1 - \dfrac{1}{x^2} = 0$ when $x = \pm 1$.

$f''(x) = \dfrac{2}{x^3} \neq 0$

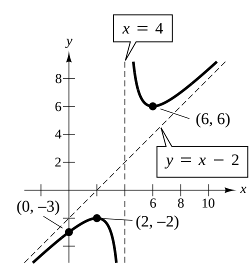
Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$

**17.** $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$




$$y' = 1 - \frac{4}{(x - 4)^2}$$

$$= \frac{(x - 2)(x - 6)}{(x - 4)^2} = 0 \text{ when } x = 2, 6.$$

$$y'' = \frac{8}{(x - 4)^3}$$

$y'' < 0$ when $x = 2$.

Therefore, $(2, -2)$ is a relative maximum.

$y'' > 0$ when $x = 6$.

Therefore, $(6, 6)$ is a relative minimum.

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

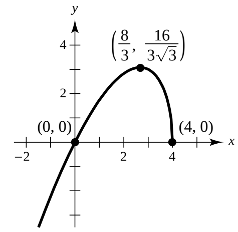
**19.** $y = x\sqrt{x-4}$,

Domain: $(-\infty, 4]$

$$y' = \frac{8-3x}{2\sqrt{4-x}} = 0 \text{ when } x = \frac{8}{3} \text{ and undefined when } x = 4.$$

$$y'' = \frac{3x-16}{4(4-x)^{3/2}} = 0 \text{ when } x = \frac{16}{3} \text{ and undefined when } x = 4.$$

**Note:** $x = \frac{16}{3}$ is not in the domain.

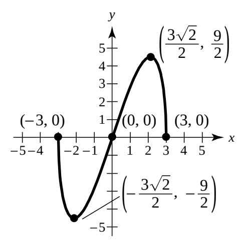



| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < \frac{8}{3}$ | | $+$ | $-$ | Increasing, concave down |
| $x = \frac{8}{3}$ | $\frac{16}{3\sqrt{3}}$ | 0 | $-$ | Relative maximum |
| $\frac{8}{3} < x < 4$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 4$ | 0 | Undefined | Undefined | Endpoint |

**21.** $h(x) = x\sqrt{9-x^2}$ Domain: $-3 \le x \le 3$

$$h'(x) = \frac{9-2x^2}{\sqrt{9-x^2}} = 0 \text{ when } x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$$

$$h''(x) = \frac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0 \text{ when } x = 0$$




Relative maximum: $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$

Relative minimum: $\left(-\frac{3\sqrt{2}}{2}, -\frac{9}{2}\right)$

Intercepts: $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

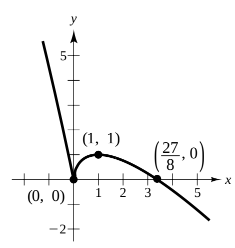
Point of inflection: $(0, 0)$

**23.** $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1-x^{1/3})}{x^{1/3}}$$

$$= 0 \text{ when } x = 1 \text{ and undefined when } x = 0.$$

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \ne 0.$$

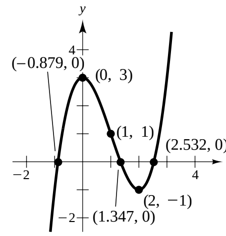



| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 0$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 0$ | 0 | Undefined | Undefined | Relative minimum |
| $0 < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | 1 | 0 | $-$ | Relative maximum |
| $1 < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

**25.** $y = x^3 - 3x^2 + 3$

$y' = 3x^2 - 6x = 3x(x - 2) = 0$ when $x = 0, x = 2$

$y'' = 6x - 6 = 6(x - 1) = 0$ when $x = 1$



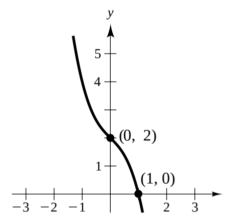

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 0$ | | $+$ | $-$ | Increasing, concave down |
| $x = 0$ | 3 | 0 | $-$ | Relative maximum |
| $0 < x < 1$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 1$ | 1 | $-$ | 0 | Point of inflection |
| $1 < x < 2$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 2$ | $-1$ | 0 | $+$ | Relative minimum |
| $2 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**27.** $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

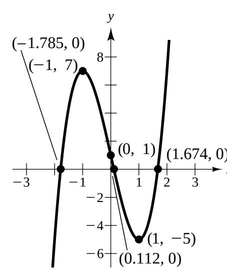



| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 0$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 0$ | 2 | $-$ | 0 | Point of inflection |
| $0 < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

**29.** $f(x) = 3x^3 - 9x + 1$

$f'(x) = 9x^2 - 9 = 9(x^2 - 1) = 0$ when $x = \pm 1$

$f''(x) = 18x = 0$ when $x = 0$

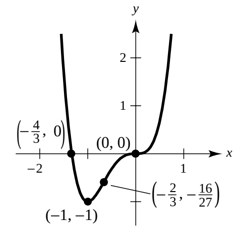



| | $f(x)$ | $f'(x)$ | $f''(x)$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $+$ | $-$ | Increasing, concave down |
| $x = -1$ | 7 | 0 | $-$ | Relative maximum |
| $-1 < x < 0$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 0$ | 1 | $-$ | 0 | Point of inflection |
| $0 < x < 1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 1$ | $-5$ | 0 | $+$ | Relative minimum |
| $1 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**31.** $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

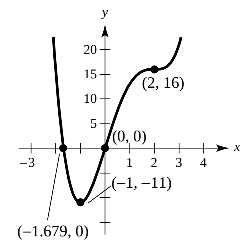



| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-1$ | $0$ | $+$ | Relative minimum |
| $-1 < x < -\frac{2}{3}$ | | $+$ | $+$ | Increasing, concave up |
| $x = -\frac{2}{3}$ | $-\frac{16}{27}$ | $+$ | $0$ | Point of inflection |
| $-\frac{2}{3} < x < 0$ | | $+$ | $-$ | Increasing, concave down |
| $x = 0$ | $0$ | $0$ | $0$ | Point of inflection |
| $0 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**33.** $f(x) = x^4 - 4x^3 + 16x$

$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2 = 0$ when $x = -1, x = 2$.

$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0$ when $x = 0, x = 2$.

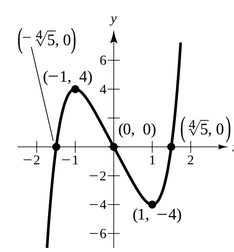



| | $f(x)$ | $f'(x)$ | $f''(x)$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-11$ | $0$ | $+$ | Relative minimum |
| $-1 < x < 0$ | | $+$ | $+$ | Increasing, concave up |
| $x = 0$ | $0$ | $+$ | $0$ | Point of inflection |
| $0 < x < 2$ | | $+$ | $-$ | Increasing, concave down |
| $x = 2$ | $16$ | $0$ | $0$ | Point of inflection |
| $2 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**35.** $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 0$ when $x = \pm 1$.

$y'' = 20x^3 = 0$ when $x = 0$.



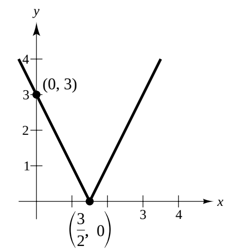

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $+$ | $-$ | Increasing, concave down |
| $x = -1$ | $4$ | $0$ | $-$ | Relative maximum |
| $-1 < x < 0$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 0$ | $0$ | $-$ | $0$ | Point of inflection |
| $0 < x < 1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 1$ | $-4$ | $0$ | $+$ | Relative minimum |
| $1 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**37.** $y = |2x - 3|$

$y' = \dfrac{2(2x-3)}{|2x-3|}$ undefined at $x = \dfrac{3}{2}$.

$y'' = 0$

| | $y$ | $y'$ | Conclusion |
|---|---|---|---|
| $-\infty < x < \frac{3}{2}$ | | $-$ | Decreasing |
| $x = \frac{3}{2}$ | 0 | Undefined | Relative minimum |
| $\frac{3}{2} < x < \infty$ | | $+$ | Increasing |




**39.** $y = \sin x - \dfrac{1}{18}\sin 3x,\ 0 \le x \le 2\pi$

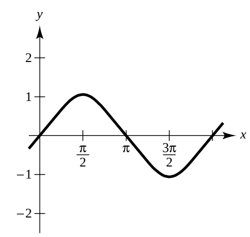
$y' = \cos x - \dfrac{1}{6}\cos 3x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

$y'' = -\sin x + \dfrac{1}{2}\sin 3x = 0$ when $x = 0, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \pi, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

Relative maximum: $\left(\dfrac{\pi}{2}, \dfrac{19}{18}\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -\dfrac{19}{18}\right)$

Inflection points: $\left(\dfrac{\pi}{6}, \dfrac{4}{9}\right), \left(\dfrac{5\pi}{6}, \dfrac{4}{9}\right), (\pi, 0), \left(\dfrac{7\pi}{6}, -\dfrac{4}{9}\right), \left(\dfrac{11\pi}{6}, -\dfrac{4}{9}\right)$




**41.** $y = 2x - \tan x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$y' = 2 - \sec^2 x = 0$ when $x = \pm\dfrac{\pi}{4}$.

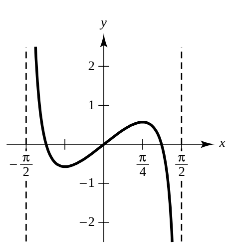
$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

Relative maximum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\dfrac{\pi}{4}, 1 - \dfrac{\pi}{2}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm\dfrac{\pi}{2}$

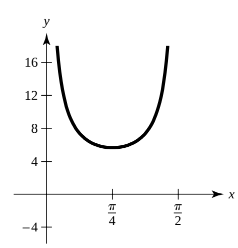



**43.** $y = 2(\csc x + \sec x),\ 0 < x < \dfrac{\pi}{2}$

$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \implies x = \pi/4$

Relative minimum: $\left(\dfrac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, x = \dfrac{\pi}{2}$



**45.** $g(x) = x \tan x, \ -\dfrac{3\pi}{2} < x < \dfrac{3\pi}{2}$

$g'(x) = \dfrac{x + \sin x \cos x}{\cos^2 x} = 0$ when $x = 0$

$g''(x) = \dfrac{2(\cos x + x \sin x)}{\cos^3 x}$

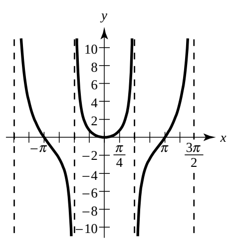
Vertical asymptotes: $x = -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Intercepts: $(-\pi, 0), (0, 0), (\pi, 0)$

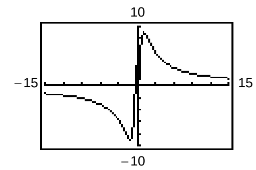
Symmetric with respect to $y$-axis.

Increasing on $\left(0, \dfrac{\pi}{2}\right)$ and $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Points of inflection: $(\pm 2.80, 0)$



**47.** $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2 + 1)}$



$x = 0$ vertical asymptote

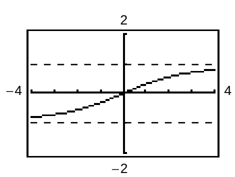
$y = 0$ horizontal asymptote

Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$

**49.** $y = \dfrac{x}{\sqrt{x^2 + 7}}$



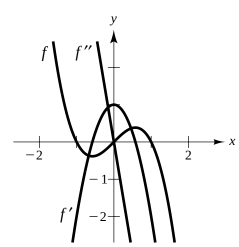
$(0, 0)$ point of inflection

$y = \pm 1$ horizontal asymptotes

**51.** $f$ is cubic.

$f'$ is quadratic.

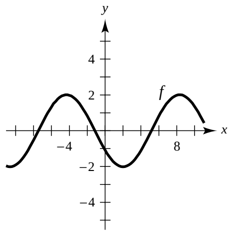
$f''$ is linear.

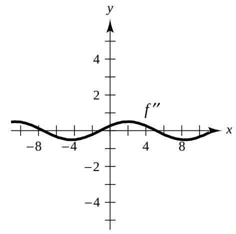


**53.**

(any vertical translate of $f$ will do)

**55.**



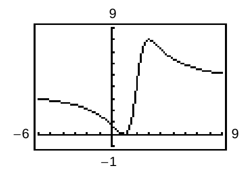

(any vertical translate of $f$ will do)

**57.** Since the slope is negative, the function is decreasing on $(2, 8)$, and hence $f(3) > f(5)$.

**59.** $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

Horizontal asymptote: $y = 4$

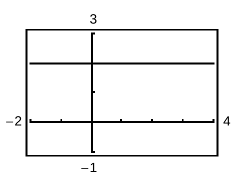



The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

**61.** $h(x) = \dfrac{6 - 2x}{3 - x}$

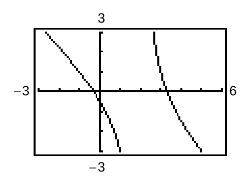
$$= \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.




hole at $(3, 2)$

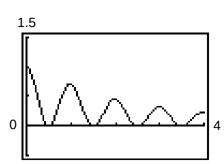
**63.** $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2} = -x + 1 + \dfrac{3}{x - 2}$




The graph appears to approach the slant asymptote $y = -x + 1$.

**65.** $f(x) = \dfrac{\cos^2 \pi x}{\sqrt{x^2 + 1}}$, $(0, 4)$

(a)




On $(0, 4)$ there seem to be 7 critical numbers:

0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

(b) $f'(x) = \dfrac{-\cos \pi x(x \cos \pi x + 2\pi(x^2 + 1)\sin \pi x)}{(x^2 + 1)^{3/2}} = 0$

Critical numbers $\approx \frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}$.

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using $f'$ shows that they are not integers.

**67.** Vertical asymptote: $x = 5$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x-5}$$

**69.** Vertical asymptote: $x = 5$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x-5} = \frac{3x^2 - 13x - 9}{x-5}$$

**71.** $f(x) = \dfrac{ax}{(x-b)^2}$

(a) The graph has a vertical asymptote at $x = b$. If $a > 0$, the graph approaches $\infty$ as $x \to b$. If $a < 0$, the graph approaches $-\infty$ as $x \to b$. The graph approaches its vertical asymptote faster as $|a| \to 0$.

(b) As $b$ varies, the position of the vertical asymptote changes: $x = b$. Also, the coordinates of the minimum $(a > 0)$ or maximum $(a < 0)$ are changed.

**73.** $f(x) = \dfrac{3x^n}{x^4 + 1}$

(a) For $n$ even, $f$ is symmetric about the $y$-axis. For $n$ odd, $f$ is symmetric about the origin.

(b) The $x$-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 3$ as the horizontal asymptote.

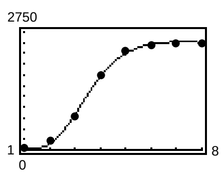
(d) There is a slant asymptote $y = 3x$ if $n = 5$:

$$\frac{3x^5}{x^4+1} = 3x - \frac{3x}{x^4+1}.$$

(e)

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $M$ | 1 | 2 | 3 | 2 | 1 | 0 |
| $N$ | 2 | 3 | 4 | 5 | 2 | 3 |

**75.** (a)




(b) When $t = 10$, $N(10) \approx 2434$ bacteria.

(c) $N$ is a maximum when $t \approx 7.2$ (seventh day).

(d) $N''(t) = 0$ for $t \approx 3.2$

(e) $\displaystyle\lim_{t\to\infty} N(t) = \frac{13{,}250}{7} \approx 1892.86$

## Section 3.7 Optimization Problems

**1.** (a)

| First Number, $x$ | Second Number | Product, $P$ |
|---|---|---|
| 10 | $110 - 10$ | $10(110 - 10) = 1000$ |
| 20 | $110 - 20$ | $20(110 - 20) = 1800$ |
| 30 | $110 - 30$ | $30(110 - 30) = 2400$ |
| 40 | $110 - 40$ | $40(110 - 40) = 2800$ |
| 50 | $110 - 50$ | $50(110 - 50) = 3000$ |
| 60 | $110 - 60$ | $60(110 - 60) = 3000$ |

**—CONTINUED—**

**1. —CONTINUED—**

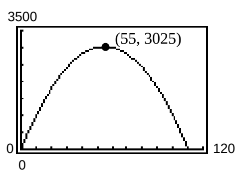
(b)

| First Number, $x$ | Second Number | Product, $P$ |
|---|---|---|
| 10 | $110 - 10$ | $10(110 - 10) = 1000$ |
| 20 | $110 - 20$ | $20(110 - 20) = 1800$ |
| 30 | $110 - 30$ | $30(110 - 30) = 2400$ |
| 40 | $110 - 40$ | $40(110 - 40) = 2800$ |
| 50 | $110 - 50$ | $50(110 - 50) = 3000$ |
| 60 | $110 - 60$ | $60(110 - 60) = 3000$ |
| 70 | $110 - 70$ | $70(110 - 70) = 2800$ |
| 80 | $110 - 80$ | $80(110 - 80) = 2400$ |
| 90 | $110 - 90$ | $90(110 - 90) = 1800$ |
| 100 | $110 - 100$ | $100(110 - 100) = 1000$ |

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x) = 110x - x^2$

(d)




The solution appears to be $x = 55$.

(e) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$\dfrac{d^2P}{dx^2} = -2 < 0$

$P$ is a maximum when $x = 110 - x = 55$.

The two numbers are 55 and 55.

**3.** Let $x$ and $y$ be two positive numbers such that $xy = 192$.

$$S = x + y = x + \frac{192}{x}$$

$\dfrac{dS}{dx} = 1 - \dfrac{192}{x^2} = 0$ when $x = \sqrt{192}$.

$\dfrac{d^2S}{dx^2} = \dfrac{384}{x^3} > 0$ when $x = \sqrt{192}$.

$S$ is a minimum when $x = y = \sqrt{192}$.

**5.** Let $x$ be a positive number.

$$S = x + \frac{1}{x}$$

$\dfrac{dS}{dx} = 1 - \dfrac{1}{x^2} = 0$ when $x = 1$.

$\dfrac{d^2S}{dx^2} = \dfrac{2}{x^3} > 0$ when $x = 1$.

The sum is a minimum when $x = 1$ and $1/x = 1$.

**7.** Let $x$ be the length and $y$ the width of the rectangle.

$$2x + 2y = 100$$

$$y = 50 - x$$

$$A = xy = x(50 - x)$$

$\dfrac{dA}{dx} = 50 - 2x = 0$ when $x = 25$.

$\dfrac{d^2A}{dx^2} = -2 < 0$ when $x = 25$.

$A$ is maximum when $x = y = 25$ meters.

**9.** Let $x$ be the length and $y$ the width of the rectangle.

$$xy = 64$$

$$y = \frac{64}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$\dfrac{dP}{dx} = 2 - \dfrac{128}{x^2} = 0$ when $x = 8$.

$\dfrac{d^2P}{dx^2} = \dfrac{256}{x^3} > 0$ when $x = 8$.

$P$ is minimum when $x = y = 8$ feet.

**11.** $d = \sqrt{(x-4)^2 + \left(\sqrt{x} - 0\right)^2}$

$= \sqrt{x^2 - 7x + 16}$

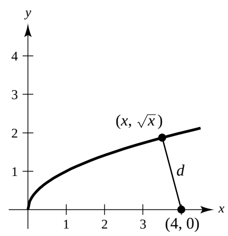
Since $d$ is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$

$$f'(x) = 2x - 7 = 0$$

$$x = \frac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $\left(7/2, \sqrt{7/2}\right)$.




**13.** $d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$

$= \sqrt{x^4 - 4x + (17/4)}$

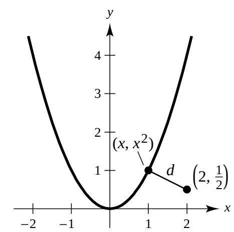
Since $d$ is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \tfrac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.




**15.** $\dfrac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$

$$\frac{d^2Q}{dx^2} = kQ_0 - 2kx$$

$$= k(Q_0 - 2x) = 0 \text{ when } x = \frac{Q_0}{2}.$$

$$\frac{d^3Q}{dx^3} = -2k < 0 \text{ when } x = \frac{Q_0}{2}.$$

$dQ/dx$ is maximum when $x = Q_0/2$.

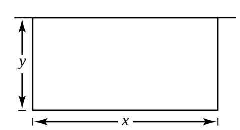
**17.** $xy = 180{,}000$ (see figure)

$S = x + 2y = \left(x + \dfrac{360{,}000}{x}\right)$ where $S$ is the length of fence needed.

$$\frac{dS}{dx} = 1 - \frac{360{,}000}{x^2} = 0 \text{ when } x = 600.$$

$$\frac{d^2S}{dx^2} = \frac{720{,}000}{x^3} > 0 \text{ when } x = 600.$$

$S$ is a minimum when $x = 600$ meters and $y = 300$ meters.




**19.** (a) $A = 4(\text{area of side}) + 2(\text{area of Top})$

(a) $A = 4(3)(11) + 2(3)(3) = 150$ square inches

(b) $A = 4(5)(5) + 2(5)(5) = 150$ square inches

(c) $A = 4(3.25)(6) + 2(6)(6) = 150$ square inches

(b) $V = (\text{length})(\text{width})(\text{height})$

(a) $V = (3)(3)(11) = 99$ cubic inches

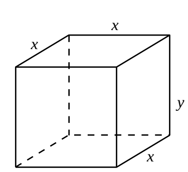
(b) $V = (5)(5)(5) = 125$ cubic inches

(c) $V = (6)(6)(3.25) = 117$ cubic inches

(c) $S = 4xy + 2x^2 = 150 \Longrightarrow y = \dfrac{150 - 2x^2}{4x}$

$$V = x^2y = x^2\left(\frac{150 - 2x^2}{4x}\right) = \frac{75}{2}x - \frac{1}{2}x^3$$

$$V' = \frac{75}{2} - \frac{3}{2}x^2 = 0 \Longrightarrow x = \pm 5$$




By the First Derivative Test, $x = 5$ yields the maximum volume. Dimensions: $5 \times 5 \times 5$. (A cube!)

**21.** (a) $V = x(s - 2x)^2,\ 0 < x < \dfrac{s}{2}$

$$\frac{dV}{dx} = 2x(s - 2x)(-2) + (s - 2x)^2$$

$$= (s - 2x)(s - 6x) = 0 \text{ when } x = \frac{s}{2}, \frac{s}{6}\ (s/2 \text{ is not in the domain}).$$

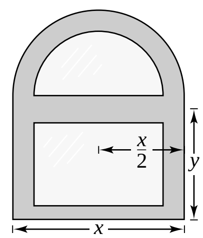
$$\frac{d^2V}{dx^2} = 24x - 8s$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = \frac{s}{6}.$$

$$V = \frac{2s^3}{27} \text{ is maximum when } x = \frac{5}{6}.$$

(b) If the length is doubled, $V = \frac{2}{27}(2s)^3 = 8\left(\frac{2}{27}s^3\right)$. Volume is increased by a factor of 8.

**23.** $16 = 2y + x + \pi\left(\dfrac{x}{2}\right)$




$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}$$

$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \dfrac{16}{4 + \pi}$ feet and $x = \dfrac{32}{4 + \pi}$ feet.

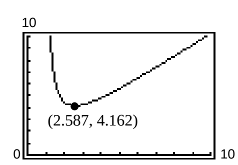
**25.** (a) $\dfrac{y - 2}{0 - 1} = \dfrac{0 - 2}{x - 1}$

$$y = 2 + \frac{2}{x - 1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}},\quad x > 1$$

(b)




$L$ is minimum when $x \approx 2.587$ and $L \approx 4.162$.

**—CONTINUED—**

**25. —CONTINUED—**

(c) Area $= A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x-1}\right) = x + \frac{x}{x-1}$

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

$$(x-1)^2 = 1$$

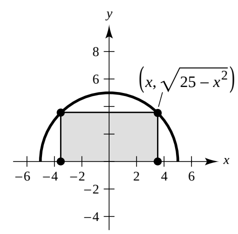
$$x - 1 = \pm 1$$

$$x = 0, 2 \quad (\text{select } x = 2)$$

Then $y = 4$ and $A = 4$.

Vertices: $(0, 0), (2, 0), (0, 4)$

**27.** $A = 2xy = 2x\sqrt{25 - x^2}$ (see figure)




$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25 - x^2}}\right) + 2\sqrt{25 - x^2}$$

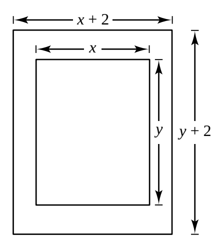
$$= 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$

**29.** $xy = 30 \Longrightarrow y = \frac{30}{x}$




$$A = (x + 2)\left(\frac{30}{x} + 2\right) \quad \text{(see figure)}$$

$$\frac{dA}{dx} = (x + 2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2 - 30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$

$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$

By the First Derivative Test, the dimensions $(x + 2)$ by $(y + 2)$ are $\left(2 + \sqrt{30}\right)$ by $\left(2 + \sqrt{30}\right)$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.

**31.** $V = \pi r^2 h = 22$ cubic inches or $h = \frac{22}{\pi r^2}$

(a)

| Radius, $r$ | Height | Surface Area |
|---|---|---|
| 0.2 | $\frac{22}{\pi(0.2)^2}$ | $2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$ |
| 0.4 | $\frac{22}{\pi(0.4)^2}$ | $2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$ |
| 0.6 | $\frac{22}{\pi(0.6)^2}$ | $2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$ |
| 0.8 | $\frac{22}{\pi(0.8)^2}$ | $2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$ |

**—CONTINUED—**

**31. —CONTINUED—**

(b)

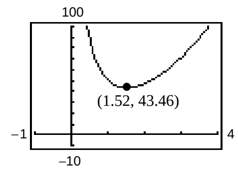
| Radius, $r$ | Height | Surface Area |
|---|---|---|
| 0.2 | $\dfrac{22}{\pi(0.2)^2}$ | $2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$ |
| 0.4 | $\dfrac{22}{\pi(0.4)^2}$ | $2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$ |
| 0.6 | $\dfrac{22}{\pi(0.6)^2}$ | $2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$ |
| 0.8 | $\dfrac{22}{\pi(0.8)^2}$ | $2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$ |
| 1.0 | $\dfrac{22}{\pi(1.0)^2}$ | $2\pi(1.0)\left[1.0 + \dfrac{22}{\pi(1.0)^2}\right] \approx 50.3$ |
| 1.2 | $\dfrac{22}{\pi(1.2)^2}$ | $2\pi(1.2)\left[1.2 + \dfrac{22}{\pi(1.2)^2}\right] \approx 45.7$ |
| 1.4 | $\dfrac{22}{\pi(1.4)^2}$ | $2\pi(1.4)\left[1.4 + \dfrac{22}{\pi(1.4)^2}\right] \approx 43.7$ |
| 1.6 | $\dfrac{22}{\pi(1.6)^2}$ | $2\pi(1.6)\left[1.6 + \dfrac{22}{\pi(1.6)^2}\right] \approx 43.6$ |
| 1.8 | $\dfrac{22}{\pi(1.8)^2}$ | $2\pi(1.8)\left[1.8 + \dfrac{22}{\pi(1.8)^2}\right] \approx 44.8$ |
| 2.0 | $\dfrac{22}{\pi(2.0)^2}$ | $2\pi(2.0)\left[2.0 + \dfrac{22}{\pi(2.0)^2}\right] \approx 47.1$ |

The minimum seems to be about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)




The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\dfrac{dS}{dr} = 4\pi r - \dfrac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$h = \dfrac{22}{\pi r^2} \approx 3.04$ in.

**Note:** Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

**33.** Let $x$ be the sides of the square ends and $y$ the length of the package.

$P = 4x + y = 108 \implies y = 108 - 4x$

$V = x^2 y = x^2(108 - 4x) = 108x^2 - 4x^3$

$\dfrac{dV}{dx} = 216x - 12x^2$

$= 12x(18 - x) = 0$ when $x = 18$.

$\dfrac{d^2V}{dx^2} = 216 - 24x = -216 < 0$ when $x = 18$.

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

**35.** $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right)\right] = \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0$$

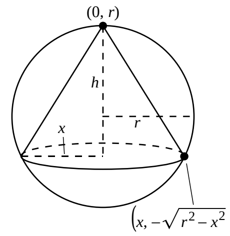
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$




By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81} \text{cubic units.}$$

**37.** No, there is no minimum area. If the sides are $x$ and $y$, then $2x + 2y = 20 \implies y = 10 - x$.
The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

**39.** $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

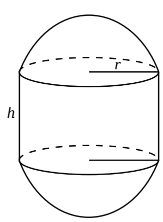
$$S = 4\pi r^2 + 2\pi rh = 4\pi r^2 + 2\pi r\left(\frac{12}{\pi r^2} - \frac{4}{3}r\right)$$

$$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0 \text{ when } r = \sqrt[3]{9/\pi} \approx 1.42 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0 \text{ when } r = \sqrt[3]{9/\pi} \text{ cm.}$$

The surface area is minimum when $r = \sqrt[3]{9/\pi}$ cm and $h = 0$. The resulting solid is a sphere of radius $r \approx 1.42$ cm.




**41.** Let $x$ be the length of a side of the square and $y$ the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

$A$ is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

**43.** Let $S$ be the strength and $k$ the constant of proportionality. Given $h^2 + w^2 = 24^2$, $h^2 = 24^2 - w^2$,

$$S = kwh^2$$

$$S = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^3) = 0 \text{ when } w = 8\sqrt{3},\ h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

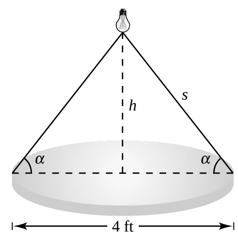
**45.** $R = \dfrac{v_0^2}{g}\sin 2\theta$

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g}\cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g}\sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test, $R$ is maximum when $\theta = \pi/4$.

**47.** $\sin\alpha = \dfrac{h}{s} \Longrightarrow s = \dfrac{h}{\sin\alpha},\ 0 < \alpha < \dfrac{\pi}{2}$




$$\tan\alpha = \frac{h}{2} \Longrightarrow h = 2\tan\alpha \Longrightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$$

$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4}[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)]$$

$$= \frac{k}{4}\cos\alpha[\cos^2\alpha - 2\sin^2\alpha]$$

$$= \frac{k}{4}\cos\alpha[1 - 3\sin^2\alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin\alpha = \pm\frac{1}{\sqrt{3}}.$$

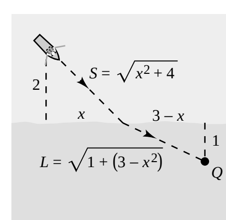
Since $\alpha$ is acute, we have

$$\sin\alpha = \frac{1}{\sqrt{3}} \Longrightarrow h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \text{ feet.}$$

Since $(d^2I)/(d\alpha^2) = (k/4)\sin\alpha(9\sin^2\alpha - 7) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.

**49.**

$$S = \sqrt{x^2 + 4},\ L = \sqrt{1 + (3 - x)^2}$$




$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

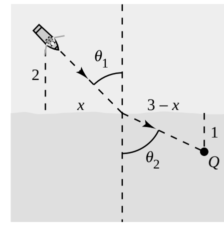
$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphics calculator, you can determine that this equation has only one root in this interval ($x = 1$). Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

**51.** $T = \dfrac{\sqrt{x^2 + 4}}{v_1} + \dfrac{\sqrt{x^2 - 6x + 10}}{v_2}$




$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + 4}} = \sin\theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin\theta_2$$

we have

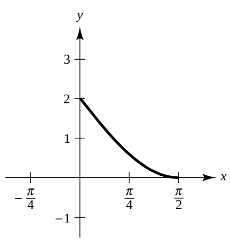
$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Longrightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

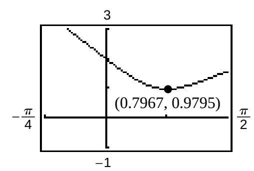
this condition yields a minimum time.

**53.** $f(x) = 2 - 2\sin x$



(a) Distance from origin to $y$-intercept is 2.
Distance from origin to $x$-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2\sin x)^2}$



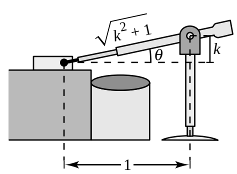

Minimum distance $= 0.9795$ at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

**55.** $F\cos\theta = k(W - F\sin\theta)$



$$F = \frac{kW}{\cos\theta + k\sin\theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k\cos\theta - \sin\theta)}{(\cos\theta + k\sin\theta)^2} = 0$$

$k\cos\theta = \sin\theta \Longrightarrow k = \tan\theta \Longrightarrow \theta = \arctan k$

Since

$$\cos\theta + k\sin\theta = \frac{1}{\sqrt{k^2 + 1}} + \frac{k^2}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1},$$

the minimum force is

$$F = \frac{kW}{\cos\theta + k\sin\theta} = \frac{kW}{\sqrt{k^2 + 1}}.$$

**57.** (a)

| Base 1 | Base 2 | Altitude | Area |
|---|---|---|---|
| 8 | $8 + 16 \cos 10°$ | $8 \sin 10°$ | $\approx 22.1$ |
| 8 | $8 + 16 \cos 20°$ | $8 \sin 20°$ | $\approx 42.5$ |
| 8 | $8 + 16 \cos 30°$ | $8 \sin 30°$ | $\approx 59.7$ |
| 8 | $8 + 16 \cos 40°$ | $8 \sin 40°$ | $\approx 72.7$ |
| 8 | $8 + 16 \cos 50°$ | $8 \sin 50°$ | $\approx 80.5$ |
| 8 | $8 + 16 \cos 60°$ | $8 \sin 60°$ | $\approx 83.1$ |

(b)

| Base 1 | Base 2 | Altitude | Area |
|---|---|---|---|
| 8 | $8 + 16 \cos 10°$ | $8 \sin 10°$ | $\approx 22.1$ |
| 8 | $8 + 16 \cos 20°$ | $8 \sin 20°$ | $\approx 42.5$ |
| 8 | $8 + 16 \cos 30°$ | $8 \sin 30°$ | $\approx 59.7$ |
| 8 | $8 + 16 \cos 40°$ | $8 \sin 40°$ | $\approx 72.7$ |
| 8 | $8 + 16 \cos 50°$ | $8 \sin 50°$ | $\approx 80.5$ |
| 8 | $8 + 16 \cos 60°$ | $8 \sin 60°$ | $\approx 83.1$ |
| 8 | $8 + 16 \cos 70°$ | $8 \sin 70°$ | $\approx 80.7$ |
| 8 | $8 + 16 \cos 80°$ | $8 \sin 80°$ | $\approx 74.0$ |
| 8 | $8 + 16 \cos 90°$ | $8 \sin 90°$ | $\approx 64.0$ |

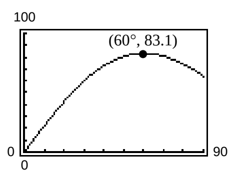
The maximum cross-sectional area is approximately 83.1 square feet.

(c) $A = (a + b)\dfrac{h}{2}$

$$= [8 + (8 + 16 \cos \theta)]\frac{8 \sin \theta}{2}$$

$$= 64(1 + \cos \theta)\sin \theta,\ 0° < \theta < 90°$$

(e)




(d) $\dfrac{dA}{d\theta} = 64(1 + \cos \theta)\cos \theta + (-64 \sin \theta)\sin \theta$

$$= 64(\cos \theta + \cos^2 \theta - \sin^2 \theta)$$

$$= 64(2 \cos^2 \theta + \cos \theta - 1)$$

$$= 64(2 \cos \theta - 1)(\cos \theta + 1)$$

$$= 0 \text{ when } \theta = 60°, 180°, 300°.$$

The maximum occurs when $\theta = 60°$.

**59.** $C = 100\left(\dfrac{200}{x^2} + \dfrac{x}{x + 30}\right),\ 1 \le x$

$$C' = 100\left(-\frac{400}{x^3} + \frac{30}{(x + 30)^2}\right)$$

Approximation: $x \approx 40.45$ units, or 4045 units

**61.** $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}.$$

Line: $y = \dfrac{64}{141}x$

$$S = \left|4\left(\frac{64}{141}\right) - 1\right| + \left|5\left(\frac{64}{141}\right) - 6\right| + \left|10\left(\frac{64}{141}\right) - 3\right|$$

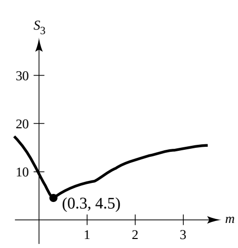
$$= \left|\frac{256}{141} - 1\right| + \left|\frac{320}{141} - 6\right| + \left|\frac{640}{141} - 3\right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

**63.** $S_3 = \dfrac{|4m - 1|}{\sqrt{m^2 + 1}} + \dfrac{|5m - 6|}{\sqrt{m^2 + 1}} + \dfrac{|10m - 3|}{\sqrt{m^2 + 1}}$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$



## Section 3.8 Newton's Method

**1.** $f(x) = x^2 - 3$

$f'(x) = 2x$

$x_1 = 1.7$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.7000 | $-0.1100$ | 3.4000 | $-0.0324$ | 1.7324 |
| 2 | 1.7324 | 0.0012 | 3.4648 | 0.0003 | 1.7321 |

**3.** $f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 3$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 3.0000 | 0.1411 | $-0.9900$ | $-0.1425$ | 3.1425 |
| 2 | 3.1425 | $-0.0009$ | $-1.0000$ | 0.0009 | 3.1416 |

**5.** $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

Approximation of the zero of $f$ is 0.682.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.5000 | $-0.3750$ | 1.7500 | $-0.2143$ | 0.7143 |
| 2 | 0.7143 | 0.0788 | 2.5307 | 0.0311 | 0.6832 |
| 3 | 0.6832 | 0.0021 | 2.4003 | 0.0009 | 0.6823 |

**7.** $f(x) = 3\sqrt{x - 1} - x$

$$f'(x) = \frac{3}{2\sqrt{x - 1}} - 1$$

Approximation of the zero of $f$ is 1.146.

Similarly, the other zero is approximately 7.854.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.2000 | 0.1416 | 2.3541 | 0.0602 | 1.1398 |
| 2 | 1.1398 | $-0.0181$ | 3.0118 | $-0.0060$ | 1.1458 |
| 3 | 1.1458 | $-0.0003$ | 2.9284 | $-0.0001$ | 1.1459 |

**9.** $f(x) = x^3 + 3$

$f'(x) = 3x^2$

Approximation of the zero of $f$ is $-1.442$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $-1.5000$ | $-0.3750$ | 6.7500 | $-0.0556$ | $-1.4444$ |
| 2 | $-1.4444$ | $-0.0134$ | 6.2589 | $-0.0021$ | $-1.4423$ |
| 3 | $-1.4423$ | $-0.0003$ | 6.2407 | $-0.0001$ | $-1.4422$ |

**11.** $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.5000 | $-0.3360$ | 1.6400 | $-0.2049$ | 0.7049 |
| 2 | 0.7049 | $-0.0921$ | 0.7824 | $-0.1177$ | 0.8226 |
| 3 | 0.8226 | $-0.0231$ | 0.4037 | $-0.0573$ | 0.8799 |
| 4 | 0.8799 | $-0.0045$ | 0.2495 | $-0.0181$ | 0.8980 |
| 5 | 0.8980 | $-0.0004$ | 0.2048 | $-0.0020$ | 0.9000 |
| 6 | 0.9000 | 0.0000 | 0.2000 | 0.0000 | 0.9000 |

Approximation of the zero of $f$ is 0.900.

**—CONTINUED—**

**11. —CONTINUED—**

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.1 | 0.0000 | $-0.1600$ | $-0.0000$ | 1.1000 |

Approximation of the zero of $f$ is 1.100.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.9 | 0.0000 | 0.8000 | 0.0000 | 1.9000 |

Approximation of the zero of $f$ is 1.900.

**13.** $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of $f$ is $-0.489$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $-0.5000$ | $-0.0206$ | 1.8776 | $-0.0110$ | $-0.4890$ |
| 2 | $-0.4890$ | 0.0000 | 1.8723 | 0.0000 | $-0.4890$ |

**15.** $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$$h'(x) = 2 - \frac{1}{2\sqrt{x + 4}}$$

Point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 0.569$.

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\dfrac{h(x_n)}{h'(x_n)}$ | $x_n - \dfrac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.6000 | 0.0552 | 1.7669 | 0.0313 | 0.5687 |
| 2 | 0.5687 | $-0.0001$ | 1.7661 | 0.0000 | 0.5687 |

**17.** $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 4.493$.

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\dfrac{h(x_n)}{h'(x_n)}$ | $x_n - \dfrac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 4.5000 | $-0.1373$ | $-21.5048$ | 0.0064 | 4.4936 |
| 2 | 4.4936 | $-0.0039$ | $-20.2271$ | 0.0002 | 4.4934 |

**19.** $f(x) = x^2 - a = 0$

$f'(x) = 2x$

$$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i}$$

$$= \frac{2x_i^2 - x_i^2 + a}{2x_i} = \frac{x_i^2 + a}{2x_i} = \frac{x_i}{2} + \frac{a}{2x_i}$$

**21.** $x_{i+1} = \dfrac{x_i^2 + 7}{2x_i}$

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $x_i$ | 2.0000 | 2.7500 | 2.6477 | 2.6458 | 2.6458 |

$\sqrt{7} \approx 2.646$

**23.** $x_{i+1} = \dfrac{3x_i^4 + 6}{4x_i^3}$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 1.5000 | 1.5694 | 1.5651 | 1.5651 |

$\sqrt[4]{6} \approx 1.565$

**25.** $f(x) = 1 + \cos x$

$f'(x) = -\sin x$

Approximation of the zero: 3.141

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 3.0000 | 0.0100 | −0.1411 | −0.0709 | 3.0709 |
| 2 | 3.0709 | 0.0025 | −0.0706 | −0.0354 | 3.1063 |
| 3 | 3.1063 | 0.0006 | −0.0353 | −0.0176 | 3.1239 |
| 4 | 3.1239 | 0.0002 | −0.0177 | −0.0088 | 3.1327 |
| 5 | 3.1327 | 0.0000 | −0.0089 | −0.0044 | 3.1371 |
| 6 | 3.1371 | 0.0000 | −0.0045 | −0.0022 | 3.1393 |
| 7 | 3.1393 | 0.0000 | −0.0023 | −0.0011 | 3.1404 |
| 8 | 3.1404 | 0.0000 | −0.0012 | −0.0006 | 3.1410 |

**27.** $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |

**29.** $y = -x^3 + 6x^2 - 10x + 6 = f(x)$

$y' = -3x^2 + 12x - 10 = f'(x)$

$x_1 = 2$

$x_2 = 1$

$x_3 = 2$

$x_4 = 1$ and so on.

Fails to converge

**31.** Answers will vary. See page 222.

Newton's Method uses tangent lines to approximate $c$ such that $f(c) = 0$.

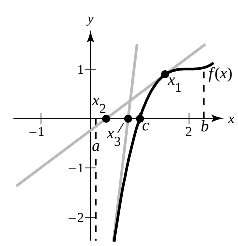
First, estimate an initial $x_1$ close to $c$ (see graph).

Then determine $x_2$ by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

Calculate a third estimate by $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$.

Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

Let $x_{n+1}$ be the final approximation of $c$.




**33.** Let $g(x) = f(x) - x = \cos x - x$

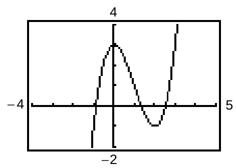
$g'(x) = -\sin x - 1$.

The fixed point is approximately 0.74.

| $n$ | $x_n$ | $g(x_n)$ | $g'(x_n)$ | $\dfrac{g(x_n)}{g'(x_n)}$ | $x_n - \dfrac{g(x_n)}{g'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | −0.4597 | −1.8415 | 0.2496 | 0.7504 |
| 2 | 0.7504 | −0.0190 | −1.6819 | 0.0113 | 0.7391 |
| 3 | 0.7391 | 0.0000 | −1.6736 | 0.0000 | 0.7391 |

**35.** $f(x) = x^3 - 3x^2 + 3,\ f'(x) = 3x^2 - 6x$

(a)




(b) $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

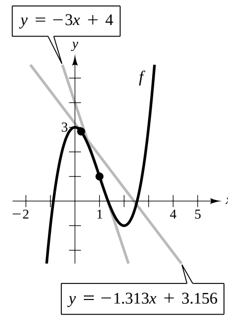
Continuing, the zero is 1.347.

(c) $x_1 = \dfrac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

Continuing, the zero is 2.532.

(d)




The $x$-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess $x_1$ is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

**37.** $f(x) = \dfrac{1}{x} - a = 0$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

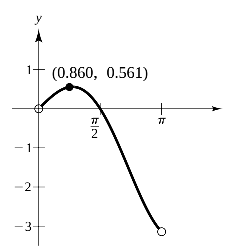
**39.** $f(x) = x\cos x,\ (0, \pi)$

$f'(x) = -x\sin x + \cos x = 0$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$[F'(x) = -2\sin x + x\cos x]$

| $n$ | $x_n$ | $F(x_n)$ | $F'(x_n)$ | $\dfrac{F(x_n)}{F'(x_n)}$ | $x_n - \dfrac{F(x_n)}{F'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.9000 | −0.0834 | −2.1261 | 0.0392 | 0.8608 |
| 2 | 0.8608 | −0.0010 | −2.0778 | 0.0005 | 0.8603 |



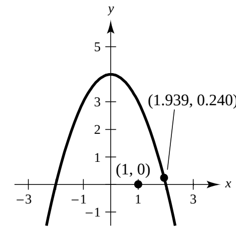

Approximation to the critical number: 0.860

**41.** $y = f(x) = 4 - x^2, (1, 0)$

$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$$

$d$ is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

$$g(x) = D' = 4x^3 - 14x - 2$$
$$g'(x) = 12x^2 - 14$$




| $n$ | $x_n$ | $g(x_n)$ | $g'(x_n)$ | $\dfrac{g(x_n)}{g'(x_n)}$ | $x_n - \dfrac{g(x_n)}{g'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.0000 | 2.0000 | 34.0000 | 0.0588 | 1.9412 |
| 2 | 1.9412 | 0.0830 | 31.2191 | 0.0027 | 1.9385 |
| 3 | 1.9385 | −0.0012 | 31.0934 | 0.0000 | 1.9385 |

$x \approx 1.939$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.

**43.**

$$\text{Minimize: } T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$$

$$T = \frac{\sqrt{x^2+4}}{3} + \frac{\sqrt{x^2-6x+10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}} = 0$$

$$4x\sqrt{x^2-6x+10} = -3(x-3)\sqrt{x^2+4}$$

$$16x^2(x^2-6x+10) = 9(x-3)^2(x^2+4)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Since $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.7000 | 19.5887 | 135.6240 | 0.1444 | 1.5556 |
| 2 | 1.5556 | −1.0480 | 150.2780 | −0.0070 | 1.5626 |
| 3 | 1.5626 | 0.0014 | 49.5591 | 0.0000 | 1.5626 |

Approximation: $x \approx 1.563$ miles

**45.**

$$2{,}500{,}000 = -76x^3 + 4830x^2 - 320{,}000$$

$$76x^3 - 4830x^2 + 2{,}820{,}000 = 0$$

Let $f(x) = 76x^3 - 4830x^2 + 2{,}820{,}000$

$$f'(x) = 228x^2 - 9660x.$$

From the graph, choose $x_1 = 40$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 40.0000 | −44000.0000 | −21600.0000 | 2.0370 | 37.9630 |
| 2 | 37.9630 | 17157.6209 | −38131.4039 | −0.4500 | 38.4130 |
| 3 | 38.4130 | 780.0914 | −34642.2263 | −0.0225 | 38.4355 |
| 4 | 38.4355 | 2.6308 | −34465.3435 | −0.0001 | 38.4356 |

The zero occurs when $x \approx 38.4356$ which corresponds to \$384,356.

**47.** False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ is given in **reduced** form.

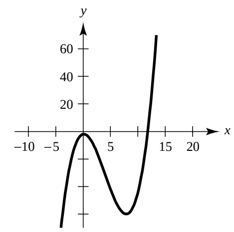
**49.** True

**51.** $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

$f'(x) = \frac{3}{4}x^2 - 6x + \frac{3}{4}$

Let $x_1 = 12$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 12.0000 | 7.0000 | 36.7500 | 0.1905 | 11.8095 |
| 2 | 11.8095 | 0.2151 | 34.4912 | 0.0062 | 11.8033 |
| 3 | 11.8033 | 0.0015 | 34.4186 | 0.0000 | 11.8033 |




Approximation: $x \approx 11.803$

## Section 3.9 Differentials

**1.** $f(x) = x^2$

$f'(x) = 2x$

Tangent line at $(2, 4)$: $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = x^2$ | 3.6100 | 3.9601 | 4 | 4.0401 | 4.4100 |
| $T(x) = 4x - 4$ | 3.6000 | 3.9600 | 4 | 4.0400 | 4.4000 |

**3.** $f(x) = x^5$

$f'(x) = 5x^4$

Tangent line at $(2, 32)$: $y - f(2) = f'(2)(x - 2)$

$$y - 32 = 80(x - 2)$$

$$y = 80x - 128$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = x^5$ | 24.7610 | 31.2080 | 32 | 32.8080 | 40.8410 |
| $T(x) = 80x - 128$ | 24.0000 | 31.2000 | 32 | 32.8000 | 40.0000 |

**5.** $f(x) = \sin x$

$f'(x) = \cos x$

Tangent line at $(2, \sin 2)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sin 2 = (\cos 2)(x - 2)$$

$$y = (\cos 2)(x - 2) + \sin 2$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \sin x$ | 0.9463 | 0.9134 | 0.9093 | 0.9051 | 0.8632 |
| $T(x) = (\cos 2)(x - 2) + \sin 2$ | 0.9509 | 0.9135 | 0.9093 | 0.9051 | 0.8677 |

**7.** $y = f(x) = \frac{1}{2}x^3, f'(x) = \frac{3}{2}x^2, x = 2, \Delta x = dx = 0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$
$$= f(2.1) - f(2)$$
$$= 0.6305$$

$$dy = f'(x)dx$$
$$= f'(2)(0.1)$$
$$= 6(0.1) = 0.6$$

**9.** $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(-0.99) - f(-1)\\ &= [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx\\ &= f'(-1)(0.01)\\ &= (-4)(0.01) = -0.04\end{aligned}$$

**11.** $y = 3x^2 - 4$

$dy = 6x\,dx$

**13.** $y = \dfrac{x + 1}{2x - 1}$

$dy = \dfrac{-3}{(2x - 1)^2}\,dx$

**15.** $y = x\sqrt{1 - x^2}$

$dy = \left(x\dfrac{-x}{\sqrt{1 - x^2}} + \sqrt{1 - x^2}\right)dx = \dfrac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx$

**17.** $y = 2x - \cot^2 x$

$$\begin{aligned}dy &= (2 + 2\cot x \csc^2 x)dx\\ &= (2 + 2\cot x + 2\cot^3 x)dx\end{aligned}$$

**19.** $y = \dfrac{1}{3}\cos\left(\dfrac{6\pi x - 1}{2}\right)$

$dy = -\pi\sin\left(\dfrac{6\pi x - 1}{2}\right)dx$

**21.** (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$
$\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$
$\approx 1 + (1)(0.04) = 1.04$

**23.** (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$
$\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$
$\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$

**25.** (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$
$\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$
$\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$

**27.** (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$
$\approx 8 + 0(-0.07) = 8$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$
$\approx 8 + 0(0.1) = 8$

**29.** $A = x^2$

$x = 12$

$\Delta x = dx = \pm\frac{1}{64}$

$dA = 2x\,dx$

$\Delta A \approx dA = 2(12)\left(\pm\frac{1}{64}\right)$

$= \pm\frac{3}{8}$ square inches

**31.** $A = \pi r^2$

$r = 14$

$\Delta r = dr = \pm\frac{1}{4}$

$\Delta A \approx dA = 2\pi r\,dr = \pi(28)\left(\pm\frac{1}{4}\right)$

$= \pm 7\pi$ square inches

**33.** (a) $x = 15$ centimeter

$\Delta x = dx = \pm 0.05$ centimeters

$A = x^2$

$dA = 2x\,dx = 2(15)(\pm 0.05)$

$= \pm 1.5$ square centimeters

Percentage error:

$$\frac{dA}{A} = \frac{\pm 1.5}{(15)^2} = 0.00666\ldots = \frac{2}{3}\%$$

(b) $\dfrac{dA}{A} = \dfrac{2x\,dx}{x^2} = \dfrac{2\,dx}{x} \le 0.025$

$$\frac{dx}{x} \le \frac{0.025}{2} = 0.0125 = 1.25\%$$

**35.** $r = 6$ inches

$\Delta r = dr = \pm 0.02$ inches

(a) $V = \dfrac{4}{3}\pi r^3$

$dV = 4\pi r^2\,dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi$ cubic inches

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi$ square inches

(c) Relative error: $\dfrac{dV}{V} = \dfrac{4\pi r^2\,dr}{(4/3)\pi r^3} = \dfrac{3dr}{r}$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

Relative error: $\dfrac{dS}{S} = \dfrac{8\pi r\,dr}{4\pi r^2} = \dfrac{2dr}{r}$

$$= \frac{2(0.02)}{6} = 0.000666\ldots = \frac{2}{3}\%$$

**37.** $V = \pi r^2 h = 40\pi r^2$, $r = 5$ cm, $h = 40$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 80\pi r\,dr = 80\pi(5)(0.2) = 80\pi \text{ cm}^3$

**39.** (a) $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}}\,dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi\,dL)/\left(g\sqrt{L/g}\right)}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2}(\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

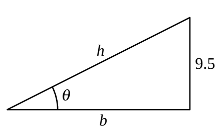
Percentage error: $\dfrac{dT}{T}(100) = 0.25\% = \dfrac{1}{4}\%$

(b) $(0.0025)(3600)(24) = 216$ seconds

$= 3.6$ minutes

**41.** $\theta = 26°45' = 26.75°$

$d\theta = \pm 15' = \pm 0.25°$




(a) $h = 9.5 \csc\theta$

$dh = -9.5\csc\theta\cot\theta\,d\theta$

$$\frac{dh}{h} = -\cot\theta\,d\theta$$

$$\left|\frac{dh}{h}\right| = (\cot 26.75°)(0.25°)$$

Converting to radians, $(\cot 0.4669)(0.0044)$
$\approx 0.0087 = 0.87\%$ (in radians).

(b) $\left|\dfrac{dh}{h}\right| = \cot\theta\,d\theta \le 0.02$

$$\frac{d\theta}{\theta} \le \frac{0.02}{\theta(\cot\theta)} = \frac{0.02\tan\theta}{\theta}$$

$$\frac{d\theta}{\theta} \le \frac{0.02\tan 26.75°}{26.75°} \approx \frac{0.02\tan 0.4669}{0.4669}$$

$$\approx 0.0216 = 2.16\% \text{ (in radians)}$$

**43.** $r = \dfrac{v_0^{\,2}}{32}(\sin 2\theta)$

$v_0 = 2200$ ft/sec

$\theta$ changes from $10°$ to $11°$

$$dr = \frac{(2200)^2}{16}(\cos 2\theta)\,d\theta$$

$$\theta = 10\left(\frac{\pi}{180}\right)$$

$$d\theta = (11 - 10)\frac{\pi}{180}$$

$$\Delta r \approx dr$$

$$= \frac{(2200)^2}{16}\cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right) \approx 4961 \text{ feet}$$

$$\approx 4961 \text{ feet}$$

**45.** Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$$

$$f(x + \Delta x) = \sqrt{99.4}$$

$$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

**47.** Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}\,dx$$

$$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4\left(\sqrt[4]{625}\right)^3}(-1)$$

$$= 5 - \frac{1}{500} = 4.998$$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

**49.** Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/\left(2\sqrt{x}\right)$.

Then

$$f(4.02) \approx f(4) + f'(4)\,dx$$

$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$$

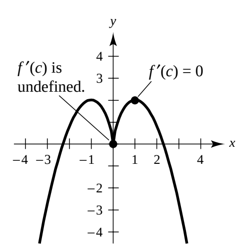
**51.** In general, when $\Delta x \to 0$, $dy$ approaches $\Delta y$.

**53.** True

**55.** True

## Review Exercises for Chapter 3

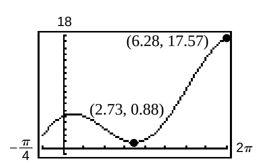
**1.** A number $c$ in the domain of $f$ is a critical number if $f'(c) = 0$ or $f'$ is undefined at $c$.




**3.** $g(x) = 2x + 5\cos x$, $[0, 2\pi]$

$g'(x) = 2 - 5\sin x$

$\quad = 0$ when $\sin x = \frac{2}{5}$.




Critical numbers: $x \approx 0.41$, $x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum

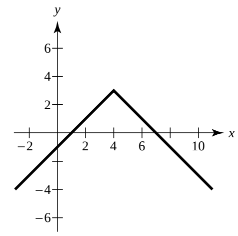
**5.** Yes. $f(-3) = f(2) = 0$. $f$ is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$f'(x) = (x + 3)(3x - 1) = 0$ for $x = \frac{1}{3}$.

$c = \frac{1}{3}$ satisfies $f'(c) - 0$.

**7.** $f(x) = 3 - |x - 4|$

(a)




$f(1) = f(7) = 0$

(b) $f$ is not differentiable at $x = 4$.

**9.**
$$f(x) = x^{2/3},\ 1 \le x \le 8$$
$$f'(x) = \frac{2}{3}x^{-1/3}$$
$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$
$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$
$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

**11.**
$$f(x) = x - \cos x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$f'(x) = 1 + \sin x$$
$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$
$$f'(c) = 1 + \sin c = 1$$
$$c = 0$$

**13.**
$$f(x) = Ax^2 + Bx + C$$
$$f'(x) = 2Ax + B$$
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2{}^2 - x_1{}^2) + B(x_2 - x_1)}{x_2 - x_1}$$
$$= A(x_1 + x_2) + B$$
$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$
$$2Ac = A(x_1 + x_2)$$
$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

**15.** $f(x) = (x - 1)^2(x - 3)$

$f'(x) = (x - 1)^2(1) + (x - 3)(2)(x - 1)$

$= (x - 1)(3x - 7)$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

| Interval: | $-\infty < x < 1$ | $1 < x < \frac{7}{3}$ | $\frac{7}{3} < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f'(x) > 0$ | $f'(x) < 0$ | $f'(x) > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

**17.** $h(x) = \sqrt{x}(x - 3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$

$= \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$

Critical number: $x = 1$

| Interval: | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $h'(x)$: | $h'(x) < 0$ | $h'(x) > 0$ |
| Conclusion: | Decreasing | Increasing |

**19.** $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

| Test Interval: | $-\infty < t < 2$ | $2 < t < \infty$ |
|---|---|---|
| Sign of $h'(t)$: | $h'(t) < 0$ | $h'(t) > 0$ |
| Conclusion: | Decreasing | Increasing |

**21.** $y = \dfrac{1}{3}\cos(12t) - \dfrac{1}{4}\sin(12t)$

$v = y' = -4\sin(12t) - 3\cos(12t)$

(a) When $t = \dfrac{\pi}{8}$, $y = \dfrac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4\sin(12t) - 3\cos(12t) = 0$ when $\dfrac{\sin(12t)}{\cos(12t)} = -\dfrac{3}{4} \Rightarrow \tan(12t) = -\dfrac{3}{4}$.

Therefore, $\sin(12t) = -\dfrac{3}{5}$ and $\cos(12t) = \dfrac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\dfrac{2\pi}{12} = \dfrac{\pi}{6}$

Frequency: $\dfrac{1}{\pi/6} = \dfrac{6}{\pi}$

**23.** $f(x) = x + \cos x, \ 0 \le x \le 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

| Test Interval: | $0 < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f''(x) < 0$ | $f''(x) > 0$ | $f''(x) < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

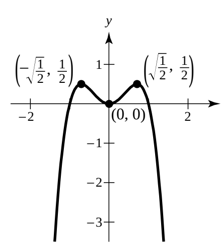
**25.** $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm\dfrac{1}{\sqrt{2}}$

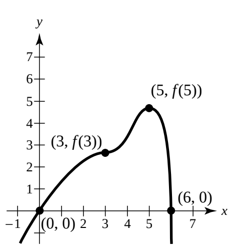
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm\dfrac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$



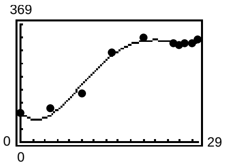

**27.**




**29.** The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

**31.** (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$

(b)




(c) Maximum at $(21.9, 319.5)$ $(\approx 1992)$

Minimum at $(2.6, 69.6)$ $(\approx 1972)$

(d) Outlays increasing at greatest rate at the point of inflection $(9.8, 173.7)$ $(\approx 1979)$

**33.** $\lim_{x\to\infty} \dfrac{2x^2}{3x^2+5} = \lim_{x\to\infty} \dfrac{2}{3+5/x^2} = \dfrac{2}{3}$

**35.** $\lim_{x\to\infty} \dfrac{5\cos x}{x} = 0$, since $|5\cos x| \le 5$.

**37.** $h(x) = \dfrac{2x+3}{x-4}$

Discontinuity: $x = 4$

$$\lim_{x\to\infty} \frac{2x+3}{x-4} = \lim_{x\to\infty} \frac{2+(3/x)}{1-(4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

**39.** $f(x) = \dfrac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x\to\infty} \left(\frac{3}{x} - 2\right) = -2$$

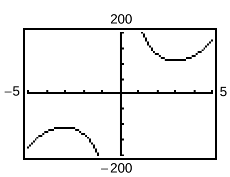
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

**41.** $f(x) = x^3 + \dfrac{243}{x}$

Relative minimum: $(3, 108)$

Relative maximum: $(-3, -108)$

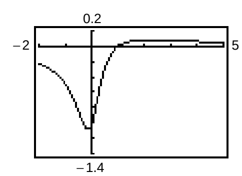



Vertical asymptote: $x = 0$

**43.** $f(x) = \dfrac{x-1}{1+3x^2}$

Relative minimum: $(-0.155, -1.077)$

Relative maximum: $(2.155, 0.077)$




Horizontal asymptote: $y = 0$

**45.** $f(x) = 4x - x^2 = x(4-x)$

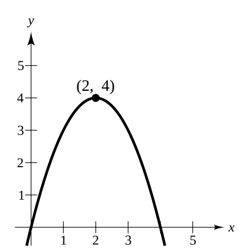
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$f'(x) = 4 - 2x = 0$ when $x = 2$.

$f''(x) = -2$

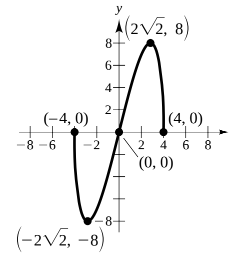
Therefore, $(2, 4)$ is a relative maximum.

Intercepts: $(0, 0)$, $(4, 0)$



**47.** $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$

Domain: $[-4, 4]$; Range: $[-8, 8]$




$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$$f'\left(-2\sqrt{2}\right) > 0$$

Therefore, $\left(-2\sqrt{2}, -8\right)$ is a relative minimum.

$$f''\left(2\sqrt{2}\right) < 0$$

Therefore, $\left(2\sqrt{2}, 8\right)$ is a relative maximum.

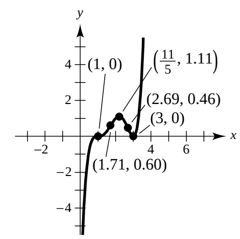
Point of inflection: $(0, 0)$

Intercepts: $(-4, 0), (0, 0), (4, 0)$

Symmetry with respect to origin

**49.** $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$




$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$$f''(3) > 0$$

Therefore, $(3, 0)$ is a relative minimum.

$$f''\left(\frac{11}{5}\right) < 0$$

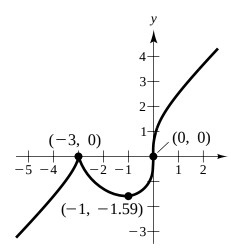
Therefore, $\left(\frac{11}{5}, \frac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0), \left(\frac{11 - \sqrt{6}}{5}, 0.60\right), \left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9), (1, 0), (3, 0)$

**51.** $f(x) = x^{1/3}(x + 3)^{2/3}$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$



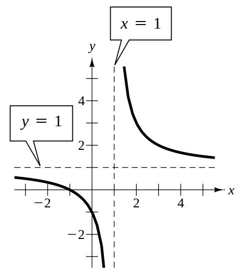

$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $\left(-1, -\sqrt[3]{4}\right)$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0), (0, 0)$

**53.** $f(x) = \dfrac{x+1}{x-1}$




Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

$f'(x) = \dfrac{-2}{(x-1)^2} < 0$ if $x \neq 1$.

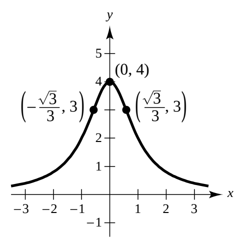
$f''(x) = \dfrac{4}{(x-1)^3}$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$

**55.** $f(x) = \dfrac{4}{1+x^2}$




Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$f'(x) = \dfrac{-8x}{(1+x^2)^2} = 0$ when $x = 0$.

$f''(x) = \dfrac{-8(1-3x^2)}{(1+x^2)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 4)$ is a relative maximum.

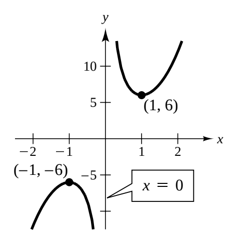
Points of inflection: $\left(\pm\sqrt{3}/3, 3\right)$

Intercept: $(0, 4)$

Symmetric to the $y$-axis

Horizontal asymptote: $y = 0$

**57.** $f(x) = x^3 + x + \dfrac{4}{x}$




Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$f'(x) = 3x^2 + 1 - \dfrac{4}{x^2} = \dfrac{3x^4 + x^2 - 4}{x^2} = 0$ when $x = \pm 1$.

$f''(x) = 6x + \dfrac{8}{x^3} = \dfrac{6x^4 + 8}{x^3} \neq 0$

$f''(-1) < 0$

Therefore, $(-1, -6)$ is a relative maximum.

$f''(1) > 0$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin

**59.** $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = \dfrac{2x(x^2 - 9)}{|x^2 - 9|} = 0$ when $x = 0$ and is undefined when $x = \pm 3$.

$f''(x) = \dfrac{2(x^2 - 9)}{|x^2 - 9|}$ is undefined at $x = \pm 3$.

$f''(0) < 0$

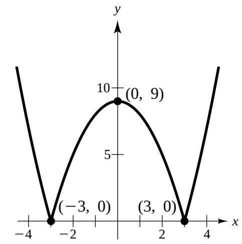
Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the $y$-axis




**61.** $f(x) = x + \cos x$

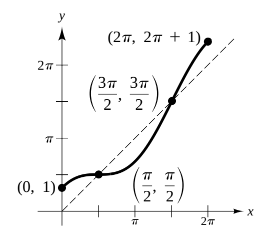
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$f'(x) = 1 - \sin x \geq 0$, $f$ is increasing.

$f''(x) = -\cos x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

Intercept: $(0, 1)$




**63.** $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

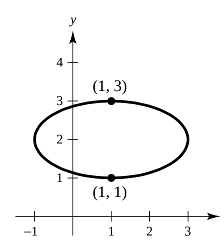
$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$




(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

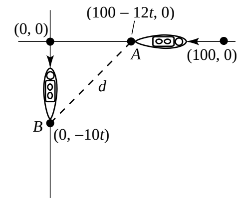
$$2x + 8y\frac{dy}{dx} - 2 - 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

**65.** Let $t = 0$ at noon.




$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10{,}000 - 2400t + 244t^2$$

$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

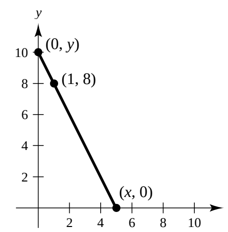
Ship A at $(40.98, 0)$; Ship B at $(0, -49.18)$

$$d^2 = 10{,}000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4{:}55 \text{ P.M.}.$$

$$d \approx 64 \text{ km}$$

**67.** We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,




$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

Let $f(x) = L^2 = x^2 + \left(\frac{8x}{x - 1}\right)^2$.

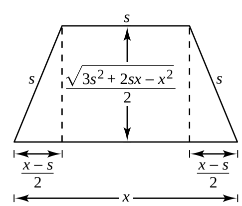
$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x[(x - 1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum).}$$

Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$

**69.** $A = $ (Average of bases)(Height)




$$= \left(\frac{x + s}{2}\right)\frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4}\left[\frac{(s - x)(s + x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2}\right]$$

$$= \frac{2(2s - x)(s + x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

$A$ is a maximum when $x = 2s$.

**71.** You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$. Assume that the hypotenuse of length $L$ passes through $(4, 6)$.

$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

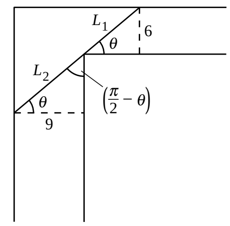
Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x - 4}\right)^2$.

$$f'(x) = 2x + 72\left(\frac{x}{x - 4}\right)\left[\frac{-4}{(x - 4)^2}\right] = 0$$

$$x[(x - 4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

**73.** $\csc\theta = \dfrac{L_1}{6}$ or $L_1 = 6\csc\theta$ (see figure)




$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9\csc\left(\frac{\pi}{2} - \theta\right)$$

$$L = L_1 + L_2 = 6\csc\theta + 9\csc\left(\frac{\pi}{2} - \theta\right) = 6\csc\theta + 9\sec\theta$$

$$\frac{dL}{d\theta} = -6\csc\theta\cot\theta + 9\sec\theta\tan\theta = 0$$

$$\tan^3\theta = \frac{2}{3} \Longrightarrow \tan\theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6\,\frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9\,\frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft}$$ (Compare to Exercise 72 using $a = 9$ and $b = 6$.)

**75.** Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33{,}000}{60v^2}$$

$$= 0 \text{ when } v = \sqrt{3000} = 10\sqrt{30} \approx 54.8 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30}$$ so this value yields a minimum.

**77.** $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$f'(x) = 3x^2 - 3$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | −1.5000 | 0.1250 | 3.7500 | 0.0333 | −1.5333 |
| 2 | −1.5333 | −0.0049 | 4.0530 | −0.0012 | −1.5321 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | −0.5000 | 0.3750 | −2.2500 | −0.1667 | −0.3333 |
| 2 | −0.3333 | −0.0371 | −2.6667 | 0.0139 | −0.3472 |
| 3 | −0.3472 | −0.0003 | −2.6384 | 0.0001 | −0.3473 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | −1.9000 | 0.1590 | 7.8300 | 0.0203 | 1.8797 |
| 2 | 1.8797 | 0.0024 | 7.5998 | 0.0003 | 1.8794 |

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

**79.** Find the zeros of $f(x) = x^4 - x - 3$.

$f'(x) = 4x^3 - 1$

From the graph you can see that $f(x)$ has two real zeros.

$f$ changes sign in $[-2, -1]$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $-1.2000$ | 0.2736 | $-7.9120$ | $-0.0346$ | $-1.1654$ |
| 2 | $-1.1654$ | 0.0100 | $-7.3312$ | $-0.0014$ | $-1.1640$ |

On the interval $[-2, -1]$: $x \approx -1.164$.

$f$ changes sign in $[1, 2]$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.5000 | 0.5625 | 12.5000 | 0.0450 | 1.4550 |
| 2 | 1.4550 | 0.0268 | 11.3211 | 0.0024 | 1.4526 |
| 3 | 1.4526 | $-0.0003$ | 11.2602 | 0.0000 | 1.4526 |

On the interval $[1, 2]$; $x \approx 1.453$.

**81.** $y = x(1 - \cos x) = x - x\cos x$

$$\frac{dy}{dx} = 1 + x\sin x - \cos x$$

$$dy = (1 + x\sin x - \cos x)\,dx$$

**83.**

$$S = 4\pi r^2.\ dr = \Delta r = \pm 0.025$$

$$dS = 8\pi r\,dr = 8\pi(9)(\pm 0.025)$$

$$= \pm 1.8\pi \text{ square cm}$$

$$\frac{dS}{S}(100) = \frac{8\pi r\,dr}{4\pi r^2}(100) = \frac{2\,dr}{r}(100)$$

$$= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\%$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\,dr = 4\pi(9)^2(\pm 0.025)$$

$$= \pm 8.1\pi \text{ cubic cm}$$

$$\frac{dV}{V}(100) = \frac{4\pi r^2\,dr}{(4/3)\pi r^3}(100) = \frac{3\,dr}{r}(100)$$

$$= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\%$$

## Problem Solving for Chapter 3

**1.** Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. $g$ is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point $c$ cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$

$$g'(b) = f'(b) - d = y_2 - d > 0$$

Hence, $a < c < b$ and $g'(c) = 0 \implies f'(c) = d$.

**3.** (a) For $a = -3, -2, -1, 0$, $p$ has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, $p$ has a relative maximum at $(0, 0)$ and 2 relative minima.

(b) $p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \implies x = 0, \pm\sqrt{\dfrac{3}{a}}$

$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$

For $x = 0$, $p''(0) = -12 < 0 \implies p$ has a relative maximum at $(0, 0)$.

(c) If $a > 0$, $x = \pm\sqrt{\dfrac{3}{a}}$ are the remaining critical numbers.

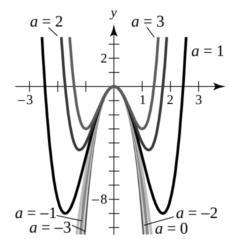
$p''\left(\pm\sqrt{\dfrac{3}{a}}\right) = 12\left(\dfrac{3}{a}\right) - 12 = 24 > 0 \implies p$ has relative minima for $a > 0$.

(d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\dfrac{3}{a}}$. Then

$$p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}.$$

Thus, $y = -\dfrac{9}{a} = -3\left(\pm\sqrt{\dfrac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of $p$.




**5.** $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^4 + 2ax = 2x(2x^2 + a)$

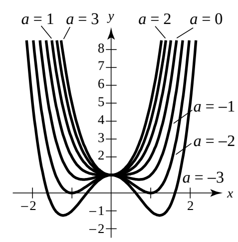
$p''(x) = 16x^2 + 2a$

For $a \geq 0$, there is one relative minimum at $(0, 0)$.

(b) For $a < 0$, there is a relative maximum at $(0, 1)$.

(c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\dfrac{a}{2}}$.

(d) There are either 1 or 3 critical points. The above analysis shows that there cannot be exactly two relative extrema.




**7.** $f(x) = \dfrac{c}{x} + x^2$

$f'(x) = -\dfrac{c}{x^2} + 2x = 0 \implies \dfrac{c}{x^2} = 2x \implies x^3 = \dfrac{c}{2} \implies x = \sqrt[3]{\dfrac{c}{x}}$

$f''(x) = \dfrac{2c}{x^3} + 2$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\dfrac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\dfrac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\dfrac{c}{2}}$ is a relative minimum too.

Answer: all $c$.

**9.** Set $\dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k$.

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2$.

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

$F$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

There exists $c_1$, $a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0, F'(c_1) = 0$.

There exists $c_2$, $a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$$k = \frac{f''(c_2)}{2}.$$

Thus, $k = \dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \dfrac{f''(c_2)}{2} \Longrightarrow f(b) = f(a) + f'(a)(b - a) + \dfrac{1}{2}f''(c_2)(b - a)^2$.

**11.** $E(\phi) = \dfrac{\tan\phi(1 - 0.1\tan\phi)}{0.1 + \tan\phi} = \dfrac{10\tan\phi - \tan^2\phi}{1 + 10\tan\phi}$

$$E'(\phi) = \frac{(1 + 10\tan\phi)(10\sec^2\phi - 2\tan\phi\sec^2\phi) - (10\tan\phi - \tan^2\phi)10\sec^2\phi}{(1 + 10\tan\phi)} = 0$$

$\Longrightarrow (1 + 10\tan\phi)(10\sec^2\phi - 2\tan\phi\sec^2\phi) = (10\tan\phi - \tan^2\phi)10\sec^2\phi$

$\Longrightarrow 10\sec^2\phi - 2\tan\phi\sec^2\phi + 100\tan\phi\sec^2\phi - 20\tan^2\phi\sec^2\phi$

$= 100\tan\phi\sec^2\phi - 10\tan^2\phi\sec^2\phi$

$\Longrightarrow 10 - 2\tan\phi = 10\tan^2\phi$

$\Longrightarrow 10\tan^2\phi + 2\tan\phi - 10 = 0$

$\tan\phi = \dfrac{-2 \pm \sqrt{4 + 400}}{20} \approx 0.90499, -1.10499$

Using the positive value, $\phi \approx 0.7356$, or $42.14^\circ$.

**13.** $v = -2400\pi\sin\theta$

$v' = -2400\pi\cos\theta = 0$

$\theta = \dfrac{\pi}{2} + 2n\pi, \dfrac{3\pi}{2} + 2n\pi$, $n$ an integer

**15.** The line has equation $\dfrac{x}{3} + \dfrac{y}{4} = 1$ or $y = -\dfrac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \implies \frac{8}{3}x = 4 \implies x = \frac{3}{2}$$

Dimensions: $\dfrac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center $(r, r)$ to the line $\dfrac{x}{3} + \dfrac{y}{4} - 1 = 0$ must be $r$:

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12}{5}\left|\frac{7r - 12}{12}\right| = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \implies r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

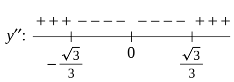
Semicircle: The center lies on the line $\dfrac{x}{3} + \dfrac{y}{4} = 1$ and satisfies $x = y = r$.

Thus $\dfrac{r}{3} + \dfrac{r}{4} = 1 \implies \dfrac{7}{12}r = 1 \implies r = \dfrac{12}{7}$. No calculus necessary.

**17.** $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \implies x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$




The tangent line has greatest slope at $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$ and least slope at $\left(\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$.

**19.** (a)

| $x$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 1.0 |
|---|---|---|---|---|---|---|
| $\sin x$ | 0.09983 | 0.19867 | 0.29552 | 0.38942 | 0.47943 | 0.84147 |

$\sin x \le x$

(b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and on $[0, x]$ you have by the Mean Value Theorem,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad 0 < c < x$$

$$\cos(c) = \frac{\sin x}{x}$$

Hence, $\left|\dfrac{\sin x}{x}\right| = |\cos(c)| \le 1$

$$\implies |\sin x| \le |x|$$

$$\implies \sin x \le x$$

# CHAPTER 4
# Integration

# CHAPTER 4
# Integration

## Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Odd-Numbered Exercises

**1.** $\dfrac{d}{dx}\left(\dfrac{3}{x^3} + C\right) = \dfrac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \dfrac{-9}{x^4}$

**3.** $\dfrac{d}{dx}\left(\dfrac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$

**5.** $\dfrac{dy}{dt} = 3t^2$

$y = t^3 + C$

**Check:** $\dfrac{d}{dt}[t^3 + C] = 3t^2$

**7.** $\dfrac{dy}{dx} = x^{3/2}$

$y = \dfrac{2}{5}x^{5/2} + C$

**Check:** $\dfrac{d}{dx}\left[\dfrac{2}{5}x^{5/2} + C\right] = x^{3/2}$

| | Given | Rewrite | Integrate | Simplify |
|---|---|---|---|---|
| **9.** | $\int \sqrt[3]{x}\, dx$ | $\int x^{1/3}\, dx$ | $\dfrac{x^{4/3}}{4/3} + C$ | $\dfrac{3}{4}x^{4/3} + C$ |
| **11.** | $\int \dfrac{1}{x\sqrt{x}}\, dx$ | $\int x^{-3/2}\, dx$ | $\dfrac{x^{-1/2}}{-1/2} + C$ | $-\dfrac{2}{\sqrt{x}} + C$ |
| **13.** | $\int \dfrac{1}{2x^3}\, dx$ | $\dfrac{1}{2}\int x^{-3}\, dx$ | $\dfrac{1}{2}\left(\dfrac{x^{-2}}{-2}\right) + C$ | $-\dfrac{1}{4x^2} + C$ |

**15.** $\int (x + 3)dx = \dfrac{x^2}{2} + 3x + C$

**Check:** $\dfrac{d}{dx}\left[\dfrac{x^2}{2} + 3x + C\right] = x + 3$

**17.** $\int (2x - 3x^2)dx = x^2 - x^3 + C$

**Check:** $\dfrac{d}{dx}[x^2 - x^3 + C] = 2x - 3x^2$

**19.** $\int (x^3 + 2)\, dx = \dfrac{1}{4}x^4 + 2x + C$

**Check:** $\dfrac{d}{dx}\left(\dfrac{1}{4}x^4 + 2x + C\right) = x^3 + 2$

**21.** $\int (x^{3/2} + 2x + 1)\, dx = \dfrac{2}{5}x^{5/2} + x^2 + x + C$

**Check:** $\dfrac{d}{dx}\left(\dfrac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$

**23.** $\int \sqrt[3]{x^2}\, dx = \int x^{2/3}\, dx = \dfrac{x^{5/3}}{5/3} + C = \dfrac{3}{5}x^{5/3} + C$

**Check:** $\dfrac{d}{dx}\left(\dfrac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$

**25.** $\int \dfrac{1}{x^3}\, dx = \int x^{-3}\, dx = \dfrac{x^{-2}}{-2} + C = -\dfrac{1}{2x^2} + C$

**Check:** $\dfrac{d}{dx}\left(-\dfrac{1}{2x^2} + C\right) = \dfrac{1}{x^3}$

**27.** $\displaystyle\int \frac{x^2+x+1}{\sqrt{x}}\,dx = \int (x^{3/2}+x^{1/2}+x^{-1/2})\,dx = \frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+2x^{1/2}+C = \frac{2}{15}x^{1/2}(3x^2+5x+15)+C$

**Check:** $\displaystyle\frac{d}{dx}\left(\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+2x^{1/2}+C\right) = x^{3/2}+x^{1/2}+x^{-1/2} = \frac{x^2+x+1}{\sqrt{x}}$

**29.** $\displaystyle\int (x+1)(3x-2)\,dx = \int (3x^2+x-2)\,dx$

$\displaystyle = x^3+\frac{1}{2}x^2-2x+C$

**Check:** $\displaystyle\frac{d}{dx}\left(x^3+\frac{1}{2}x^2-2x+C\right) = 3x^2+x-2$

$= (x+1)(3x-2)$

**31.** $\displaystyle\int y^2\sqrt{y}\,dy = \int y^{5/2}\,dy = \frac{2}{7}y^{7/2}+C$

**Check:** $\displaystyle\frac{d}{dy}\left(\frac{2}{7}y^{7/2}+C\right) = y^{5/2} = y^2\sqrt{y}$

**33.** $\displaystyle\int dx = \int 1\,dx = x+C$

**Check:** $\displaystyle\frac{d}{dx}(x+C) = 1$

**35.** $\displaystyle\int (2\sin x+3\cos x)\,dx = -2\cos x+3\sin x+C$

**Check:** $\displaystyle\frac{d}{dx}(-2\cos x+3\sin x+C) = 2\sin x+3\cos x$

**37.** $\displaystyle\int (1-\csc t\cot t)\,dt = t+\csc t+C$

**Check:** $\displaystyle\frac{d}{dt}(t+\csc t+C) = 1-\csc t\cot t$

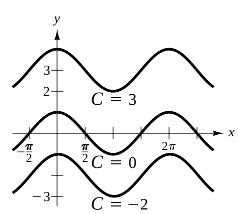
**39.** $\displaystyle\int (\sec^2\theta-\sin\theta)\,d\theta = \tan\theta+\cos\theta+C$

**Check:** $\displaystyle\frac{d}{d\theta}(\tan\theta+\cos\theta+C) = \sec^2\theta-\sin\theta$

**41.** $\displaystyle\int (\tan^2 y+1)\,dy = \int \sec^2 y\,dy = \tan y+C$

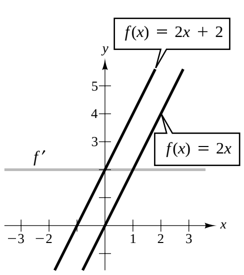
**Check:** $\displaystyle\frac{d}{dy}(\tan y+C) = \sec^2 y = \tan^2 y+1$

**43.** $f(x) = \cos x$




**45.** $f'(x) = 2$

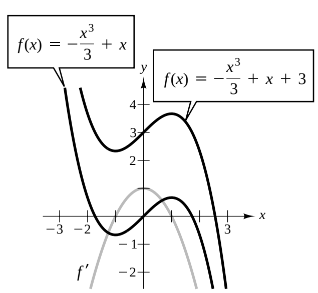
$f(x) = 2x + C$




Answers will vary.

**47.** $f'(x) = 1 - x^2$

$\displaystyle f(x) = x - \frac{x^3}{3} + C$




Answers will vary.

**49.** $\displaystyle\frac{dy}{dx} = 2x - 1,\ (1, 1)$

$\displaystyle y = \int (2x-1)\,dx = x^2 - x + C$

$1 = (1)^2 - (1) + C \Rightarrow C = 1$

$y = x^2 - x + 1$

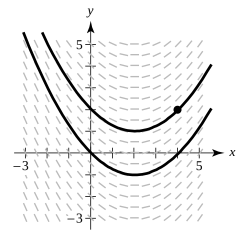
**51.** $\dfrac{dy}{dx} = \cos x,\ (0, 4)$

$$y = \int \cos x\, dx = \sin x + C$$

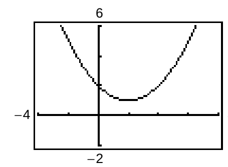
$$4 = \sin 0 + C \implies C = 4$$

$$y = \sin x + 4$$

**53.** (a) Answers will vary.




(b) $\dfrac{dy}{dx} = \dfrac{1}{2}x - 1,\ (4, 2)$




$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$

**55.** $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x\, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \implies C = 6$$

$$f(x) = 2x^2 + 6$$

**57.** $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5)dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \implies C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

**59.** $f''(x) = 2$

$f'(2) = 5$

$f(2) = 10$

$$f'(x) = \int 2\, dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \implies C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1)\, dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \implies C_2 = 4$$

$$f(x) = x^2 + x + 4$$

**61.** $f''(x) = x^{-3/2}$

$f'(4) = 2$

$f(0) = 0$

$$f'(x) = \int x^{-3/2}\, dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$$

$$f'(4) = -\frac{2}{2} + C_1 = 2 \implies C_1 = 3$$

$$f'(x) = -\frac{2}{\sqrt{x}} + 3$$

$$f(x) = \int (-2x^{-1/2} + 3)\, dx = -4x^{1/2} + 3x + C_2$$

$$f(0) = 0 + 0 + C_2 = 0 \implies C_2 = 0$$

$$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$$

**63.** (a) $h(t) = \displaystyle\int (1.5t + 5)\, dt = 0.75t^2 + 5t + C$

$$h(0) = 0 + 0 + C = 12 \implies C = 12$$

$$h(t) = 0.75t^2 + 5t + 12$$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

**65.** $f(0) = -4$. Graph of $f'$ is given.

(a) $f'(4) \approx -1.0$

(b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, $f$ must increase more than 4 units on $[0, 4]$.

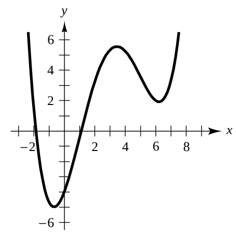
(c) No, $f(5) < f(4)$ because $f$ is decreasing on $[4, 5]$.

(d) $f$ is an maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the first derivative test.

(e) $f$ is concave upward when $f'$ is increasing on $(-\infty, 1)$ and $(5, \infty)$. $f$ is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) $f''$ is a minimum at $x = 3$.

(g)




**67.** $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32\,dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int (-32t + 60)dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$s(t) = -16t^2 + 60t + 6$ Position function

The ball reaches its maximim height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \frac{15}{8} \text{ seconds}$$

$$s\left(\frac{15}{18}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 \approx 62.26 \text{ feet}$$

**69.** From Exercise 68, we have:

$$s(t) = -16t^2 + v_0 t$$

$s'(t) = -32t + v_0 = 0$ when $t = \dfrac{v_0}{32}$ = time to reach maximum height.

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35{,}200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

**71.** $a(t) = -9.8$

$$v(t) = \int -9.8\,dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \implies v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0)\,dt = -4.9t^2 + v_0 t + C_2$$

$$f(0) = s_0 = C_2 \implies f(t) = -4.9t^2 + v_0 t + s_0$$

**73.** From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$v(t) = -9.8t + 10 = 0$ (Maximum height when $v = 0$.)

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

**75.** $a = -1.6$

$v(t) = \int -1.6\,dt = -1.6t + v_0 = -1.6t$, since the stone was dropped, $v_0 = 0$.

$$s(t) = \int (-1.6t)\,dt = -0.8t^2 + s_0$$

$$s(20) = 0 \implies -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

**77.** $x(t) = t^3 - 6t^2 + 9t - 2 \qquad 0 \le t \le 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3) = 3(t-1)(t-3)$

$a(t) = v'(t) = 6t - 12 = 6(t-2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t-2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

**79.** $v(t) = \dfrac{1}{\sqrt{t}} = t^{-1/2} \qquad t > 0$

$x(t) = \int v(t)\,dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \implies C = 2$

$x(t) = 2t^{1/2} + 2$ position function

$a(t) = v'(t) = -\dfrac{1}{2}t^{-3/2} = \dfrac{-1}{2t^{3/2}}$ acceleration

**81.** (a) $v(0) = 25 \text{ km/hr} = 25 \cdot \dfrac{1000}{3600} = \dfrac{250}{36} \text{ m/sec}$

$v(13) = 80 \text{ km/hr} = 80 \cdot \dfrac{1000}{3600} = \dfrac{800}{36} \text{ m/sec}$

$a(t) = a$ (constant acceleration)

$v(t) = at + C$

$v(0) = \dfrac{250}{36} \implies v(t) = at + \dfrac{250}{36}$

$v(13) = \dfrac{800}{36} = 13a + \dfrac{250}{36}$

$\dfrac{550}{36} = 13a$

$a = \dfrac{550}{468} = \dfrac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a\dfrac{t^2}{2} + \dfrac{250}{36}t \qquad (s(0) = 0)$

$s(13) = \dfrac{275}{234}\dfrac{(13)^2}{2} + \dfrac{250}{36}(13) \approx 189.58 \text{ m}$

**83.** Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t - 10)$ when $t = 10$ sec.

(a) $s(10) = 3(10)^2 = 300$ ft

(b) $v(10) = 6(10) = 60 \text{ ft/sec} \approx 41$ mph

**85.** $\dfrac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \dfrac{22}{15} \text{ ft/sec}$

(a)

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| $V_1$(ft/sec) | 0 | 3.67 | 10.27 | 23.47 | 42.53 | 66 | 95.33 |
| $V_2$(ft/sec) | 0 | 30.8 | 55.73 | 74.8 | 88 | 93.87 | 95.33 |

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

(c) $S_1(t) = \int V_1(t)\,dt = \dfrac{0.1068}{3}t^3 - \dfrac{0.0416}{2}t^2 + 0.3679t$

$S_2(t) = \int V_2(t)\,dt = -\dfrac{0.1208t^3}{3} + \dfrac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$]

$S_1(30) \approx 953.5$ feet

$S_2(30) \approx 1970.3$ feet

The second car was going faster than the first until the end.

**87.** $a(t) = k$

$v(t) = kt$

$s(t) = \dfrac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \implies k = \frac{160^2}{1.4}$$

$$\approx 18{,}285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

**89.** True

**91.** True

**93.** False. For example, $\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx$ because $\dfrac{x^3}{3} + C \neq \left(\dfrac{x^2}{2} + C_1\right)\left(\dfrac{x^2}{2} + C_2\right)$

**95.** $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$

$f(x) = \begin{cases} x + C_1, & 0 \le x < 2 \\ \dfrac{3x^2}{2} + C_2, & 2 \le x \le 5 \end{cases}$

$f(1) = 3 \implies 1 + C_1 = 3 \implies C_1 = 2$

$f$ is continuous: Values must agree at $x = 2$:

$4 = 6 + C_2 \implies C_2 = -2$

$f(x) = \begin{cases} x + 2, & 0 \le x < 2 \\ \dfrac{3x^2}{2} - 2, & 2 \le x \le 5 \end{cases}$

The left and right hand derivatives at $x = 2$ do not agree. Hence $f$ is not differentiable at $x = 2$.

## Section 4.2 Area

**1.** $\sum_{i=1}^{5}(2i + 1) = 2\sum_{i=1}^{5} i + \sum_{i=1}^{5} 1 = 2(1 + 2 + 3 + 4 + 5) + 5 = 35$

**3.** $\sum_{k=0}^{4} \dfrac{1}{k^2 + 1} = 1 + \dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{10} + \dfrac{1}{17} = \dfrac{158}{85}$

**5.** $\sum_{k=1}^{4} c = c + c + c + c = 4c$

**7.** $\sum_{i=1}^{9} \dfrac{1}{3i}$

**9.** $\sum_{j=1}^{8}\left[5\left(\dfrac{j}{8}\right) + 3\right]$

**11.** $\dfrac{2}{n}\sum_{i=1}^{n}\left[\left(\dfrac{2i}{n}\right)^3 - \left(\dfrac{2i}{n}\right)\right]$

**13.** $\dfrac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \dfrac{3i}{n}\right)^2\right]$

**15.** $\sum_{i=1}^{20} 2i = 2\sum_{i=1}^{20} i = 2\left[\dfrac{20(21)}{2}\right] = 420$

**17.** $\sum_{i=1}^{20}(i - 1)^2 = \sum_{i=1}^{19} i^2$

$$= \left[\frac{19(20)(39)}{6}\right] = 2470$$

**19.** $\displaystyle\sum_{i=1}^{15} i(i-1)^2 = \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i$

$$= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2}$$

$$= 14{,}400 - 2{,}480 + 120$$

$$= 12{,}040$$

**21.** sum seq($x$ $\boxed{\wedge}$ 2 + 3, $x$, 1, 20, 1) = 2930 (*TI-82*)

$$\sum_{i=1}^{20}(i^2+3) = \frac{20(20+1)(2(20)+1)}{6} + 3(20)$$

$$= \frac{(20)(21)(41)}{6} + 60 = 2930$$

**23.** $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

**25.** $S = [3 + 3 + 5](1) = 11$

$s = [2 + 2 + 3](1) = 7$

**27.** $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \dfrac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \dfrac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$

**29.** $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

**31.** $\displaystyle\lim_{n\to\infty}\left[\left(\frac{81}{n^4}\right)\frac{n^2(n+1)^2}{4}\right] = \frac{81}{4}\lim_{n\to\infty}\left[\frac{n^4+2n^3+n^2}{n^4}\right] = \frac{81}{4}(1) = \frac{81}{4}$

**33.** $\displaystyle\lim_{n\to\infty}\left[\left(\frac{18}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{18}{2}\lim_{n\to\infty}\left[\frac{n^2+n}{n^2}\right] = \frac{18}{2}(1) = 9$

**35.** $\displaystyle\sum_{i=1}^{n}\frac{2i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i+1) = \frac{1}{n^2}\left[2\frac{n(n+1)}{2} + n\right] = \frac{n+2}{n} = S(n)$

$S(10) = \frac{12}{10} = 1.2$

$S(100) = 1.02$

$S(1000) = 1.002$

$S(10{,}000) = 1.0002$

**37.** $\displaystyle\sum_{k=1}^{n}\frac{6k(k-1)}{n^3} = \frac{6}{n^3}\sum_{k=1}^{n}(k^2-k) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$

$$= \frac{6}{n^2}\left[\frac{2n^2+3n+1-3n-3}{6}\right] = \frac{1}{n^2}[2n^2-2] = S(n)$$

$S(10) = 1.98$

$S(100) = 1.9998$

$S(1000) = 1.999998$

$S(10{,}000) = 1.99999998$

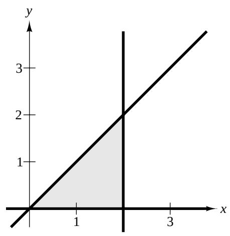
**39.** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i}{n^2}\right) = \lim_{n\to\infty}\frac{16}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[8\left(\frac{n^2+n}{n^2}\right)\right] = 8\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 8$

**41.** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$$= \lim_{n\to\infty}\frac{1}{6}\left[\frac{2n^3-3n^2+n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2-(3/n)+(1/n^2)}{1}\right)\right] = \frac{1}{3}$$

**43.** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}1+\frac{1}{n}\sum_{i=1}^{n}i\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1+\frac{n^2+n}{2n^2}\right] = 2\left(1+\frac{1}{2}\right) = 3$

**45.** (a)




(b) $\displaystyle\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

| $x$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| $s(n)$ | 1.6 | 1.8 | 1.96 | 1.98 |
| $S(n)$ | 2.4 | 2.2 | 2.04 | 2.02 |

(f) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}(i-1)$

$$= \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2}-n\right]$$

$$= \lim_{n\to\infty}\left[\frac{2(n+1)}{n}-\frac{4}{n}\right] = 2$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2}$$

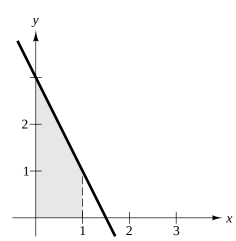
$$= \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

**47.** $y = -2x+3$ on $[0, 1]$. $\displaystyle\left(\textbf{Note: } \Delta x = \frac{1-0}{n} = \frac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-2\left(\frac{i}{n}\right)+3\right]\left(\frac{1}{n}\right)$$

$$= 3-\frac{2}{n^2}\sum_{i=1}^{n} i = 3-\frac{2(n+1)n}{2n^2} = 2-\frac{1}{n}$$

$\text{Area} = \displaystyle\lim_{n\to\infty} s(n) = 2$

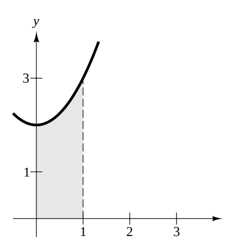



**49.** $y = x^2+2$ on $[0, 1]$. $\displaystyle\left(\textbf{Note: } \Delta x = \frac{1}{n}\right)$

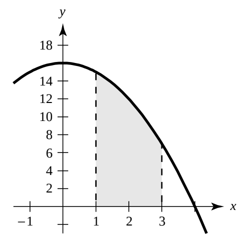
$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2+2\right]\left(\frac{1}{n}\right)$$

$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right]+2 = \frac{n(n+1)(2n+1)}{6n^3}+2 = \frac{1}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+2$$

$\text{Area} = \displaystyle\lim_{n\to\infty} S(n) = \frac{7}{3}$



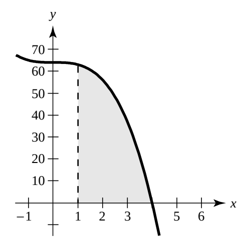
**51.** $y = 16 - x^2$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{2}{n}\right)$




$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= \frac{2}{n}\sum_{i=1}^{n}\left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right] \\ &= \frac{2}{n}\left[15n - \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{4}{n}\frac{n(n+1)}{2}\right] \\ &= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1) \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$

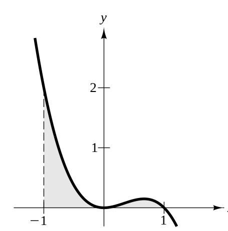
**53.** $y = 64 - x^3$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}\right)$




$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right) \\ &= \frac{3}{n}\sum_{i=1}^{n}\left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right] \\ &= \frac{3}{n}\left[63n - \frac{27}{n^3}\frac{n^2(n+1)^2}{4} - \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{9}{n}\frac{n(n+1)}{2}\right] \\ &= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2}\frac{n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

**55.** $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}\right)$

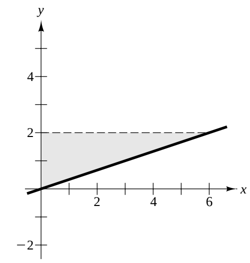



Again, $T(n)$ is neither an upper nor a lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n}1 - \frac{20}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2 - \frac{16}{n^4}\sum_{i=1}^{n}i^3 \\ &= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} \\ &= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

**57.** $f(y) = 3y,\ 0 \le y \le 2$ $\quad \left(\textbf{Note: } \Delta y = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

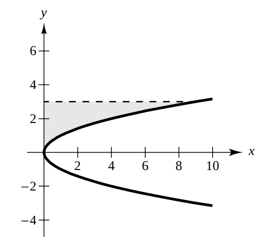



$$S(n) = \sum_{i=1}^{n} f(m_i)\,\Delta y = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2}\sum_{i=1}^{n} i = \left(\frac{12}{n^2}\right)\cdot\frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(6 + \frac{6}{n}\right) = 6$$

**59.** $f(y) = y^2,\ 0 \le y \le 3$ $\quad \left(\textbf{Note: } \Delta y = \dfrac{3-0}{n} = \dfrac{3}{n}\right)$

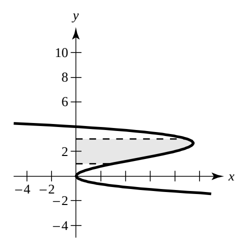



$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right) = \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2}\left(\frac{2n^2+3n+1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$

**61.** $g(y) = 4y^2 - y^3,\ 1 \le y \le 3.$ $\quad \left(\textbf{Note: } \Delta y = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$




$$S(n) = \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right] = \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$

**63.** $f(x) = x^2 + 3,\ 0 \le x \le 2,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{1}{2},\ c_1 = \frac{1}{4},\ c_2 = \frac{3}{4},\ c_3 = \frac{5}{4},\ c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 3\right]\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right]$$

$$= \frac{69}{8}$$

**65.** $f(x) = \tan x,\ 0 \le x \le \dfrac{\pi}{4},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{16},\ c_1 = \frac{\pi}{32},\ c_2 = \frac{3\pi}{32},\ c_3 = \frac{5\pi}{32},\ c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

**67.** $f(x) = \sqrt{x}$ on $[0, 4]$.

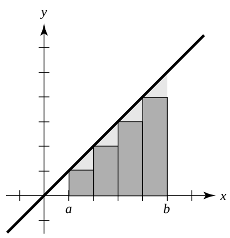
| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| Approximate area | 5.3838 | 5.3523 | 5.3439 | 5.3403 | 5.3384 |

(Exact value is 16/3)

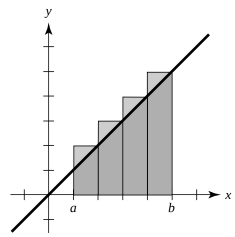
**69.** $f(x) = \tan\left(\dfrac{\pi x}{8}\right)$ on $[1, 3]$.

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| Approximate area | 2.2223 | 2.2387 | 2.2418 | 2.2430 | 2.2435 |

**71.** We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



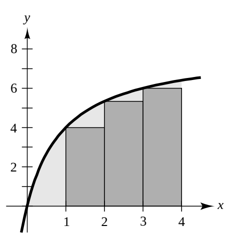

The sum of the areas of these circumscribed rectangles is the upper sum.




We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

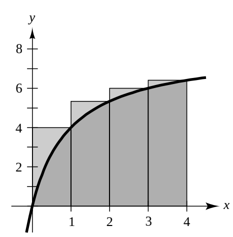
**73.** (a)




Lower sum:

$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

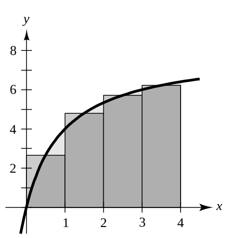
(b)




Upper sum:

$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

(c)




Midpoint Rule:

$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

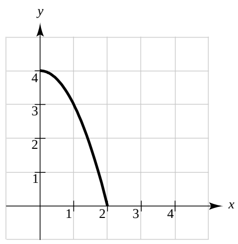
(d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

| $n$ | 4 | 8 | 20 | 100 | 200 |
|---|---|---|---|---|---|
| $s(n)$ | 15.333 | 17.368 | 18.459 | 18.995 | 19.06 |
| $S(n)$ | 21.733 | 20.568 | 19.739 | 19.251 | 19.188 |
| $M(n)$ | 19.403 | 19.201 | 19.137 | 19.125 | 19.125 |

(f) $s(n)$ increases because the lower sum approaches the exact value as $n$ increases. $S(n)$ decreases because the upper sum approaches the exact value as $n$ increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

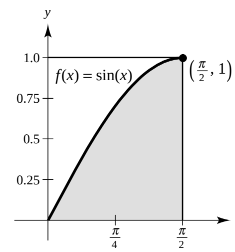
**75.**



b. $A \approx 6$ square units

**77.** True. (Theorem 4.2 (2))

**79.** $f(x) = \sin x, \left[0, \dfrac{\pi}{2}\right]$




Let $A_1 =$ area bounded by $f(x) = \sin x$, the $x$-axis, $x = 0$ and $x = \pi/2$. Let $A_2 =$ area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$.
In this program, the computer is generating $N_2$ pairs of random points in the rectangle whose area is represented by $A_2$. It is keeping track of how many of these points, $N_1$, lie in the region whose area is represented by $A_1$. Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Longrightarrow A_1 \approx \frac{N_1}{N_2} A_2.$$

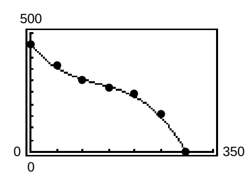
The larger $N_2$ is the better the approximation to $A_1$.

**81.** Suppose there are $n$ rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

$$2[1 + 2 + \cdots + n] = n(n + 1)$$
$$1 + 2 + \cdots + n = \tfrac{1}{2}(n)(n + 1).$$

**83.** (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)

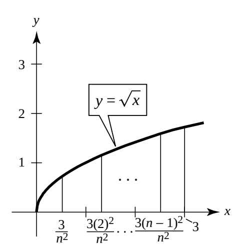



(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76{,}897.5 \text{ ft}^2.$$

## Section 4.3 Riemann Sums and Definite Integrals

**1.** $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \dfrac{3i^2}{n^2}$




$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i - 1)^2}{n^2} = \frac{3}{n^2}(2i - 1)$$

$$\begin{aligned}
\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x_i &= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{3i^2}{n^2}}\, \frac{3}{n^2}(2i - 1) \\
&= \lim_{n \to \infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^{n} (2i^2 - i) \\
&= \lim_{n \to \infty} \frac{3\sqrt{3}}{n^3}\left[2\frac{n(n + 1)(2n + 1)}{6} - \frac{n(n + 1)}{2}\right] \\
&= \lim_{n \to \infty} 3\sqrt{3}\left[\frac{(n + 1)(2n + 1)}{3n^2} - \frac{n + 1}{2n^2}\right] \\
&= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464
\end{aligned}$$

**3.** $y = 6$ on $[4, 10]$. $\left(\textbf{Note}: \Delta x = \dfrac{10-4}{n} = \dfrac{6}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(4 + \frac{6i}{n}\right)\left(\frac{6}{n}\right) = \sum_{i=1}^{n} 6\left(\frac{6}{n}\right) = \sum_{i=1}^{n} \frac{36}{n} = 36$$

$$\int_4^{10} 6\,dx = \lim_{n\to\infty} 36 = 36$$

**5.** $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note:}\ \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty} \frac{2}{n} = 0$$

**7.** $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note:}\ \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1)\,dx = \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

**9.** $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i = \int_{-1}^{5} (3x + 10)\,dx$

on the interval $[-1, 5]$.

**11.** $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i = \int_0^3 \sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

**13.** $\displaystyle\int_0^5 3\,dx$

**15.** $\displaystyle\int_{-4}^{4} (4 - |x|)\,dx$

**17.** $\displaystyle\int_{-2}^{2} (4 - x^2)\,dx$

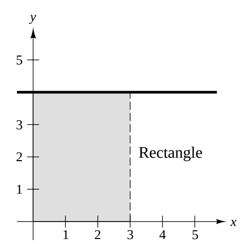
**19.** $\displaystyle\int_0^{\pi} \sin x\,dx$

**21.** $\displaystyle\int_0^2 y^3\,dy$

**23.** Rectangle

$A = bh = 3(4)$

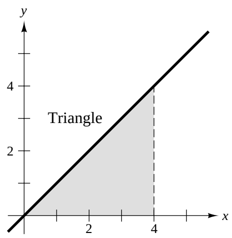
$A = \displaystyle\int_0^3 4\,dx = 12$



**25.** Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$$

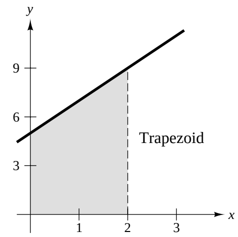
$$A = \int_0^4 x\,dx = 8$$




**27.** Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{5+9}{2}\right)2$$

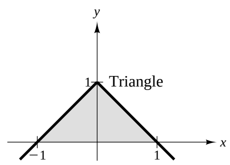
$$A = \int_0^2 (2x + 5)\,dx = 14$$




**29.** Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

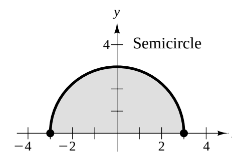
$$A = \int_{-1}^1 (1 - |x|)\,dx = 1$$




**31.** Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

$$A = \int_{-3}^3 \sqrt{9 - x^2}\,dx = \frac{9\pi}{2}$$




**In Exercises 33–39, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$**

**33.** $\int_4^2 x\,dx = -\int_2^4 x\,dx = -6$

**35.** $\int_2^4 4x\,dx = 4\int_2^4 x\,dx = 4(6) = 24$

**37.** $\int_2^4 (x - 8)\,dx = \int_2^4 x\,dx - 8\int_2^4 dx = 6 - 8(2) = -10$

**39.** $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx = \frac{1}{2}\int_2^4 x^3\,dx - 3\int_2^4 x\,dx + 2\int_2^4 dx$

$$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$$

**41.** (a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\int_5^5 f(x)\,dx = 0$

(d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

**43.** (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$

$$= 10 + (-2) = 8$$

(b) $\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$

$$= -2 - 10 = -12$$

(c) $\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

**45.** (a) Quarter circle below $x$-axis: $-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below $x$-axis: $-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c): $4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answer to (d) plus $2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

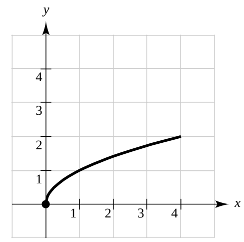
**47.** The left endpoint approximation will be greater than the actual area: $>$

**49.** Because the curve is concave upward, the midpoint approximation will be less than the actual area: $<$

**51.** $f(x) = \dfrac{1}{x-4}$

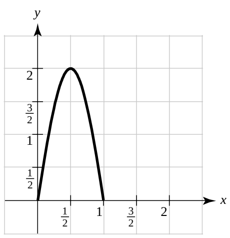
is not integrable on the interval $[3, 5]$ and $f$ has a discontinuity at $x = 4$.

**53.**



a. $A \approx 5$ square units

**55.**



d. $\displaystyle\int_0^1 2 \sin \pi x \, dx \approx \frac{1}{2}(1)(2) \approx 1$

**57.** $\displaystyle\int_0^3 x\sqrt{3-x}\, dx$

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $L(n)$ | 3.6830 | 3.9956 | 4.0707 | 4.1016 | 4.1177 |
| $M(n)$ | 4.3082 | 4.2076 | 4.1838 | 4.1740 | 4.1690 |
| $R(n)$ | 3.6830 | 3.9956 | 4.0707 | 4.1016 | 4.1177 |

**59.** $\displaystyle\int_0^{\pi/2} \sin^2 x \, dx$

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $L(n)$ | 0.5890 | 0.6872 | 0.7199 | 0.7363 | 0.7461 |
| $M(n)$ | 0.7854 | 0.7854 | 0.7854 | 0.7854 | 0.7854 |
| $R(n)$ | 0.9817 | 0.8836 | 0.8508 | 0.8345 | 0.8247 |

**61.** True

**63.** True

**65.** False

$\displaystyle\int_0^2 (-x)\, dx = -2$

**67.** $f(x) = x^2 + 3x, [0, 8]$

$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$

$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$

$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x = f(1)\,\Delta x_1 + f(2)\,\Delta x_2 + f(5)\,\Delta x_3 + f(8)\,\Delta x_4$$

$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

**69.** $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

**71.** Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

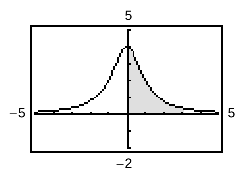
$$\lim_{n\to\infty}\frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2] = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(2n+1)(n+1)}{6}$$

$$= \lim_{n\to\infty}\frac{2n^2 + 3n + 1}{6n^2} = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

## Section 4.4 The Fundamental Theorem of Calculus

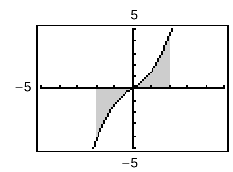
**1.** $f(x) = \dfrac{4}{x^2 + 1}$

$\displaystyle\int_0^{\pi}\frac{4}{x^2+1}\,dx$ is positive.




**3.** $f(x) = x\sqrt{x^2 + 1}$

$$\int_{-2}^{2} x\sqrt{x^2+1}\,dx = 0$$




**5.** $\displaystyle\int_0^1 2x\,dx = \Big[x^2\Big]_0^1 = 1 - 0 = 1$

**7.** $\displaystyle\int_{-1}^{0}(x-2)\,dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^{0} = 0 - \left(\frac{1}{2} + 2\right) = -\frac{5}{2}$

**9.** $\displaystyle\int_{-1}^{1}(t^2 - 2)\,dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^{1} = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$

**11.** $\displaystyle\int_0^1 (2t-1)^2\,dt = \int_0^1 (4t^2 - 4t + 1)\,dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

**13.** $\displaystyle\int_1^2\left(\frac{3}{x^2} - 1\right)dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3 - 1) = \frac{1}{2}$

**15.** $\displaystyle\int_1^4 \frac{u-2}{\sqrt{u}}\,du = \int_1^4 (u^{1/2} - 2u^{-1/2})\,du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}\left(\sqrt{4}\right)^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$

**17.** $\displaystyle\int_{-1}^{1}\left(\sqrt[3]{t} - 2\right)dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^{1} = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$

**19.** $\displaystyle\int_0^1 \frac{x - \sqrt{x}}{3}\,dx = \frac{1}{3}\int_0^1 (x - x^{1/2})\,dx = \frac{1}{3}\left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$

**21.** $\displaystyle\int_{-1}^{0}(t^{1/3} - t^{2/3})\,dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^{0} = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$

**23.** $\displaystyle\int_0^3 |2x - 3|\,dx = \int_0^{3/2}(3 - 2x)\,dx + \int_{3/2}^{3}(2x-3)\,dx$ $\left(\text{split up the integral at the zero } x = \dfrac{3}{2}\right)$

$$= \Big[3x - x^2\Big]_0^{3/2} + \Big[x^2 - 3x\Big]_{3/2}^{3} = \left(\frac{9}{2} - \frac{9}{4}\right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2}\right) = 2\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9}{2}$$

**25.** $\displaystyle\int_0^3 |x^2 - 4|\,dx = \int_0^2 (4 - x^2)\,dx + \int_2^3 (x^2 - 4)\,dx$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$

$$= \left(8 - \frac{8}{3}\right) + (9 - 12) - \left(\frac{8}{3} - 8\right)$$

$$= \frac{23}{3}$$

**27.** $\displaystyle\int_0^{\pi} (1 + \sin x)\,dx = \Big[x - \cos x\Big]_0^{\pi} = (\pi + 1) - (0 - 1) = 2 + \pi$

**29.** $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx = \Big[\tan x\Big]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}$

**31.** $\displaystyle\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta = \Big[4\sec\theta\Big]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$

**33.** $\displaystyle\int_0^3 10{,}000(t - 6)\,dt = 10{,}000\left[\frac{t^2}{2} - 6t\right]_0^3 = -\$135{,}000$

**35.** $\displaystyle A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

**37.** $\displaystyle A = \int_0^3 (3 - x)\sqrt{x}\,dx = \int_0^3 (3x^{1/2} - x^{3/2})\,dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x)\right]_0^3 = \frac{12\sqrt{3}}{5}$

**39.** $\displaystyle A = \int_0^{\pi/2} \cos x\,dx = \Big[\sin x\Big]_0^{\pi/2} = 1$

**41.** Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1)\,dx = \Big[x^3 + x\Big]_0^2 = 8 + 2 = 10.$$

**43.** Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x)\,dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2 = 4 + 2 = 6.$$

**45.** $\displaystyle\int_0^2 \left(x - 2\sqrt{x}\right)dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3}\right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$\left(\sqrt{c} - 1\right)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm\sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}\right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

**47.** $\displaystyle\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \Big[2\tan x\Big]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$

$$f(c)\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 4$$

$$2\sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm\frac{2}{\sqrt{\pi}}$$

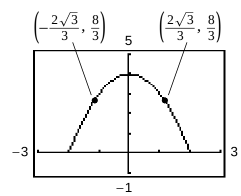
$$c = \pm\operatorname{arcsec}\left(\frac{2}{\sqrt{\pi}}\right)$$

$$= \pm\arccos\frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

**49.** $\displaystyle\frac{1}{2-(-2)}\int_{-2}^{2}(4-x^2)\,dx = \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{1}{4}\left[\left(8-\frac{8}{3}\right) - \left(-8+\frac{8}{3}\right)\right] = \frac{8}{3}$

Average value $= \dfrac{8}{3}$

$4 - x^2 = \dfrac{8}{3}$ when $x^2 = 4 - \dfrac{8}{3}$ or $x = \pm\dfrac{2\sqrt{3}}{3} \approx \pm 1.155.$

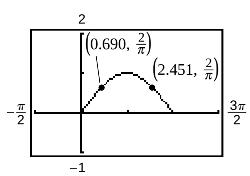



**51.** $\displaystyle\frac{1}{\pi-0}\int_0^{\pi}\sin x\,dx = \left[-\frac{1}{\pi}\cos x\right]_0^{\pi} = \frac{2}{\pi}$

Average value $= \dfrac{2}{\pi}$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690, 2.451$$




**53.** If $f$ is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

**55.** $\displaystyle\int_0^2 f(x)\,dx = -(\text{area of region } A) = -1.5$

**57.** $\displaystyle\int_0^6 |f(x)|\,dx = -\int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = 1.5 + 5.0 = 6.5$

**59.** $\displaystyle\int_0^6 [2 + f(x)]\,dx = \int_0^6 2\,dx + \int_0^6 f(x)\,dx$

$$= 12 + 3.5 = 15.5$$

**61.** (a) $F(x) = k\sec^2 x$

$F(0) = k = 500$

$F(x) = 500\sec^2 x$

(b) $\displaystyle\frac{1}{\pi/3 - 0}\int_0^{\pi/3} 500\sec^2 x\,dx = \frac{1500}{\pi}\Big[\tan x\Big]_0^{\pi/3}$

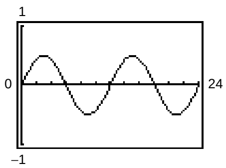
$$= \frac{1500}{\pi}\left(\sqrt{3} - 0\right)$$

$\approx 826.99$ newtons

$\approx 827$ newtons

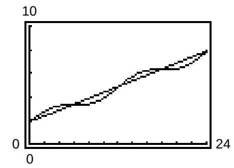
**63.** $\displaystyle\frac{1}{5-0}\int_0^5 (0.1729t + 0.1552t^2 - 0.0374t^3)\,dt \approx \frac{1}{5}\Big[0.08645t^2 + 0.05073t^3 - 0.00935t^4\Big]_0^5 \approx 0.5318$ liter

**65.** (a)




The area above the $x$-axis equals the area below the $x$-axis. Thus, the average value is zero.

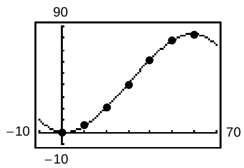
(b)




The average value of $S$ appears to be $g$.

**67.** (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$

(b)




(c) $\displaystyle\int_0^{60} v(t)\,dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t\right]_0^{60} \approx 2476 \text{ meters}$

**69.** $F(x) = \displaystyle\int_0^x (t - 5)\,dt = \left[\frac{t^2}{2} - 5t\right]_0^x = \frac{x^2}{2} - 5x$

$F(2) = \dfrac{4}{2} - 5(2) = -8$

$F(5) = \dfrac{25}{2} - 5(5) = -\dfrac{25}{2}$

$F(8) = \dfrac{64}{2} - 5(8) = -8$

**71.** $F(x) = \displaystyle\int_1^x \frac{10}{v^2}\,dv = \int_1^x 10v^{-2}\,dv = \frac{-10}{v}\Bigg]_1^x$

$= -\dfrac{10}{x} + 10 = 10\left(1 - \dfrac{1}{x}\right)$

$F(2) = 10\left(\dfrac{1}{2}\right) = 5$

$F(5) = 10\left(\dfrac{4}{5}\right) = 8$

$F(8) = 10\left(\dfrac{7}{8}\right) = \dfrac{35}{4}$

**73.** $F(x) = \displaystyle\int_1^x \cos\theta\,d\theta = \sin\theta\Big]_1^x = \sin x - \sin 1$

$F(2) = \sin 2 - \sin 1 = 0.0678$

$F(5) = \sin 5 - \sin 1 \approx -1.8004$

$F(8) = \sin 8 - \sin 1 \approx 0.1479$

**75.** (a) $\displaystyle\int_0^x (t + 2)\,dt = \left[\frac{t^2}{2} + 2t\right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\dfrac{d}{dx}\left[\dfrac{1}{2}x^2 + 2x\right] = x + 2$

**77.** (a) $\displaystyle\int_8^x \sqrt[3]{t}\,dt = \left[\frac{3}{4}t^{4/3}\right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\dfrac{d}{dx}\left[\dfrac{3}{4}x^{4/3} - 12\right] = x^{1/3} = \sqrt[3]{x}$

**79.** (a) $\displaystyle\int_{\pi/4}^x \sec^2 t\,dt = \Big[\tan t\Big]_{\pi/4}^x = \tan x - 1$

(b) $\dfrac{d}{dx}[\tan x - 1] = \sec^2 x$

**81.** $F(x) = \displaystyle\int_{-2}^x (t^2 - 2t)\,dt$

$F'(x) = x^2 - 2x$

**83.** $F(x) = \displaystyle\int_{-1}^x \sqrt{t^4 + 1}\,dt$

$F'(x) = \sqrt{x^4 + 1}$

**85.** $F(x) = \displaystyle\int_0^x t\cos t\,dt$

$F'(x) = x\cos x$

**87.** $F(x) = \int_x^{x+2} (4t + 1)\, dt$

$= \left[2t^2 + t\right]_x^{x+2}$

$= [2(x + 2)^2 + (x + 2)] - [2x^2 + x]$

$= 8x + 10$

$F'(x) = 8$

**Alternate solution:**

$F(x) = \int_x^{x+2} (4t + 1)\, dt$

$= \int_x^0 (4t + 1)\, dt + \int_0^{x+2} (4t + 1)\, dt$

$= -\int_0^x (4t + 1)\, dt + \int_0^{x+2} (4t + 1)\, dt$

$F'(x) = -(4x + 1) + 4(x + 2) + 1 = 8$

**89.** $F(x) = \int_0^{\sin x} \sqrt{t}\, dt = \left[\frac{2}{3}t^{3/2}\right]_0^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$

$F'(x) = (\sin x)^{1/2} \cos x = \cos x\sqrt{\sin x}$

**Alternate solution**

$F(x) = \int_0^{\sin x} \sqrt{t}\, dt$

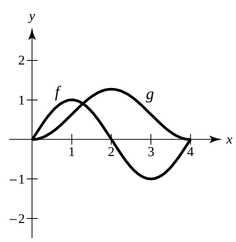
$F'(x) = \sqrt{\sin x}\,\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$

**91.** $F(x) = \int_0^{x^3} \sin t^2\, dt$

$F'(x) = \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6$

**93.** $g(x) = \int_0^x f(t)\, dt$

$g(0) = 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0$




$g$ has a relative maximum at $x = 2$.

**95.** (a) $C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\, dt\right)$

$= 5000\left(25 + 3\left[\frac{4}{5}t^{5/4}\right]_0^x\right)$

$= 5000\left(25 + \frac{12}{5}x^{5/4}\right) = 1000(125 + 12x^{5/4})$

(b) $C(1) = 1000(125 + 12(1)) = \$137{,}000$

$C(5) = 1000(125 + 12(5)^{5/4}) \approx \$214{,}721$

$C(10) = 1000(125 + 12(10)^{5/4}) \approx \$338{,}394$

**97.** True

**99.** False; $\int_{-1}^1 x^{-2}\, dx = \int_{-1}^0 x^{-2}\, dx + \int_0^1 x^{-2}\, dx$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

**101.** $f(x) = \int_0^{1/x} \frac{1}{t^2 + 1}\, dt + \int_0^x \frac{1}{t^2 + 1}\, dt$

By the Second Fundamental Theorem of Calculus, we have

$$f'(x) = \frac{1}{(1/x)^2 + 1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2 + 1}$$

$$= -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0.$$

Since $f'(x) = 0$, $f(x)$ must be constant.

**103.** $x(t) = t^3 - 6t^2 + 9t - 2$

$$\begin{aligned} x'(t) &= 3t^2 - 12t + 9 \\ &= 3(t^2 - 4t + 3) \\ &= 3(t - 3)(t - 1) \end{aligned}$$

$$\begin{aligned} \text{Total distance} &= \int_0^5 |x'(t)|\,dt \\ &= \int_0^5 3|(t-3)(t-1)|\,dt \\ &= 3\int_0^1 (t^2 - 4t + 3)\,dt - 3\int_1^3 (t^2 - 4t + 3)\,dt + 3\int_3^5 (t^2 - 4t + 3)\,dt \\ &= 4 + 4 + 20 \\ &= 28 \text{ units} \end{aligned}$$

**105.**
$$\begin{aligned} \text{Total distance} &= \int_1^4 |x'(t)|\,dt \\ &= \int_1^4 |v(t)|\,dt \\ &= \int_1^4 \frac{1}{\sqrt{t}}\,dt \\ &= 2t^{1/2}\Big]_1^4 \\ &= 2(2 - 1) = 2 \text{ units} \end{aligned}$$

## Section 4.5 Integration by Substitution

| $\int f(g(x))g'(x)\,dx$ | $u = g(x)$ | $du = g'(x)\,dx$ |
| --- | --- | --- |
| **1.** $\int (5x^2 + 1)^2(10x)\,dx$ | $5x^2 + 1$ | $10x\,dx$ |
| **3.** $\int \frac{x}{\sqrt{x^2 + 1}}\,dx$ | $x^2 + 1$ | $2x\,dx$ |
| **5.** $\int \tan^2 x \sec^2 x\,dx$ | $\tan x$ | $\sec^2 x\,dx$ |

**7.** $\int (1 + 2x)^4\, 2\,dx = \frac{(1 + 2x)^5}{5} + C$

**Check:** $\frac{d}{dx}\left[\frac{(1 + 2x)^5}{5} + C\right] = 2(1 + 2x)^4$

**9.** $\int (9 - x^2)^{1/2}(-2x)\,dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$

**Check:** $\frac{d}{dx}\left[\frac{2}{3}(9 - x^2)^{3/2} + C\right] = \frac{2}{3} \cdot \frac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$

**11.** $\int x^3(x^4+3)^2\,dx = \frac{1}{4}\int (x^4+3)^2(4x^3)\,dx = \frac{1}{4}\frac{(x^4+3)^3}{3} + C = \frac{(x^4+3)^3}{12} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^4+3)^3}{12} + C\right] = \frac{3(x^4+3)^2}{12}(4x^3) = (x^4+3)^2(x^3)$

**13.** $\int x^2(x^3-1)^4\,dx = \frac{1}{3}\int (x^3-1)^4(3x^2)\,dx = \frac{1}{3}\left[\frac{(x^3-1)^5}{5}\right] + C = \frac{(x^3-1)^5}{15} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^3-1)^5}{15} + C\right] = \frac{5(x^3-1)^4(3x^2)}{15} = x^2(x^3-1)^4$

**15.** $\int t\sqrt{t^2+2}\,dt = \frac{1}{2}\int (t^2+2)^{1/2}(2t)\,dt = \frac{1}{2}\frac{(t^2+2)^{3/2}}{3/2} + C = \frac{(t^2+2)^{3/2}}{3} + C$

**Check:** $\frac{d}{dt}\left[\frac{(t^2+2)^{3/2}}{3} + C\right] = \frac{3/2(t^2+2)^{1/2}(2t)}{3} = (t^2+2)^{1/2}t$

**17.** $\int 5x(1-x^2)^{1/3}\,dx = -\frac{5}{2}\int (1-x^2)^{1/3}(-2x)\,dx = -\frac{5}{2}\cdot\frac{(1-x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1-x^2)^{4/3} + C$

**Check:** $\frac{d}{dx}\left[-\frac{15}{8}(1-x^2)^{4/3} + C\right] = -\frac{15}{8}\cdot\frac{4}{3}(1-x^2)^{1/3}(-2x) = 5x(1-x^2)^{1/3} = 5x\sqrt[3]{1-x^2}$

**19.** $\int \frac{x}{(1-x^2)^3}\,dx = -\frac{1}{2}\int (1-x^2)^{-3}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{-2}}{-2} + C = \frac{1}{4(1-x^2)^2} + C$

**Check:** $\frac{d}{dx}\left[\frac{1}{4(1-x^2)^2} + C\right] = \frac{1}{4}(-2)(1-x^2)^{-3}(-2x) = \frac{x}{(1-x^2)^3}$

**21.** $\int \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3}\int (1+x^3)^{-2}(3x^2)\,dx = \frac{1}{3}\left[\frac{(1+x^3)^{-1}}{-1}\right] + C = -\frac{1}{3(1+x^3)} + C$

**Check:** $\frac{d}{dx}\left[-\frac{1}{3(1+x^3)} + C\right] = -\frac{1}{3}(-1)(1+x^3)^{-2}(3x^2) = \frac{x^2}{(1+x^3)^2}$

**23.** $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$

**Check:** $\frac{d}{dx}[-(1-x^2)^{1/2} + C] = -\frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$

**25.** $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = -\int\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right)dt = -\frac{[1+(1/t)]^4}{4} + C$

**Check:** $\frac{d}{dt}\left[-\frac{[1+(1/t)]^4}{4} + C\right] = -\frac{1}{4}(4)\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right) = \frac{1}{t^2}\left(1+\frac{1}{t}\right)^3$

**27.** $\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{2}\int (2x)^{-1/2}\,2\,dx = \frac{1}{2}\left[\frac{(2x)^{1/2}}{1/2}\right] + C = \sqrt{2x} + C$

**Check:** $\frac{d}{dx}\left[\sqrt{2x} + C\right] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$

**29.** $\displaystyle\int \frac{x^2 + 3x + 7}{\sqrt{x}}\,dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2 + 5x + 35) + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C\right] = \frac{x^2 + 3x + 7}{\sqrt{x}}$

**31.** $\displaystyle\int t^2\left(t - \frac{2}{t}\right)dt = \int (t^3 - 2t)\,dt = \frac{1}{4}t^4 - t^2 + C$

**Check:** $\displaystyle\frac{d}{dt}\left[\frac{1}{4}t^4 - t^2 + C\right] = t^3 - 2t = t^2\left(t - \frac{2}{t}\right)$

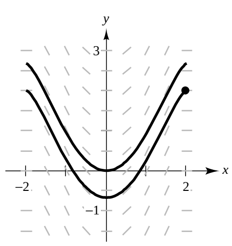
**33.** $\displaystyle\int (9 - y)\sqrt{y}\,dy = \int (9y^{1/2} - y^{3/2})\,dy = 9\left(\frac{2}{3}y^{3/2}\right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$

**Check:** $\displaystyle\frac{d}{dy}\left[\frac{2}{5}y^{3/2}(15 - y) + C\right] = \frac{d}{dy}\left[6y^{3/2} - \frac{2}{5}y^{5/2} + C\right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$

**35.**
$$\begin{aligned} y &= \int\left[4x + \frac{4x}{\sqrt{16 - x^2}}\right]dx \\ &= 4\int x\,dx - 2\int (16 - x^2)^{-1/2}(-2x)\,dx \\ &= 4\left(\frac{x^2}{2}\right) - 2\left[\frac{(16 - x^2)^{1/2}}{1/2}\right] + C \\ &= 2x^2 - 4\sqrt{16 - x^2} + C \end{aligned}$$

**37.**
$$\begin{aligned} y &= \int \frac{x + 1}{(x^2 + 2x - 3)^2}\,dx \\ &= \frac{1}{2}\int (x^2 + 2x - 3)^{-2}(2x + 2)\,dx \\ &= \frac{1}{2}\left[\frac{(x^2 + 2x - 3)^{-1}}{-1}\right] + C \\ &= -\frac{1}{2(x^2 + 2x - 3)} + C \end{aligned}$$

**39.** (a)

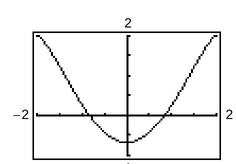



(b) $\displaystyle\frac{dy}{dx} = x\sqrt{4 - x^2}$, $(2, 2)$

$$\begin{aligned} y &= \int x\sqrt{4 - x^2}\,dx = -\frac{1}{2}\int (4 - x^2)^{1/2}(-2x\,dx) \\ &= -\frac{1}{2}\cdot\frac{2}{3}(4 - x^2)^{3/2} + C = -\frac{1}{3}(4 - x^2)^{3/2} + C \end{aligned}$$

$(2, 2)$: $\displaystyle 2 = -\frac{1}{3}(4 - 2^2)^{3/2} + C \implies C = 2$

$$y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$$




**41.** $\displaystyle\int \pi \sin \pi x\,dx = -\cos \pi x + C$

**43.** $\displaystyle\int \sin 2x\,dx = \frac{1}{2}\int (\sin 2x)(2x)\,dx = -\frac{1}{2}\cos 2x + C$

**45.** $\displaystyle\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta = -\int \cos\frac{1}{\theta}\left(-\frac{1}{\theta^2}\right)d\theta = -\sin\frac{1}{\theta} + C$

**47.** $\displaystyle\int \sin 2x \cos 2x\, dx = \frac{1}{2}\int (\sin 2x)(2\cos 2x)\, dx = \frac{1}{2}\frac{(\sin 2x)^2}{2} + C = \frac{1}{4}\sin^2 2x + C$ OR

$\displaystyle\int \sin 2x \cos 2x\, dx = -\frac{1}{2}\int (\cos 2x)(-2\sin 2x)\, dx = -\frac{1}{2}\frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4}\cos^2 2x + C_1$ OR

$\displaystyle\int \sin 2x \cos 2x\, dx = \frac{1}{2}\int 2\sin 2x \cos 2x\, dx = \frac{1}{2}\int \sin 4x\, dx - \frac{1}{8}\cos 4x + C_2$

**49.** $\displaystyle\int \tan^4 x \sec^2 x\, dx = \frac{\tan^5 x}{5} + C = \frac{1}{5}\tan^5 x + C$

**51.** $\displaystyle\int \frac{\csc^2 x}{\cot^3 x}\, dx = -\int (\cot x)^{-3}(-\csc^2 x)\, dx$

$$= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2\cot^2 x} + C = \frac{1}{2}\tan^2 x + C = \frac{1}{2}(\sec^2 x - 1) + C = \frac{1}{2}\sec^2 x + C_1$$

**53.** $\displaystyle\int \cot^2 x\, dx = \int (\csc^2 x - 1)\, dx = -\cot x - x + C$

**55.** $\displaystyle f(x) = \int \cos\frac{x}{2}\, dx = 2\sin\frac{x}{2} + C$

Since $f(0) = 3 = 2\sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2\sin\frac{x}{2} + 3.$$

**57.** $u = x + 2,\ x = u - 2,\ dx = du$

$$\begin{aligned}
\int x\sqrt{x+2}\, dx &= \int (u-2)\sqrt{u}\, du \\
&= \int (u^{3/2} - 2u^{1/2})\, du \\
&= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\
&= \frac{2u^{3/2}}{15}(3u - 10) + C \\
&= \frac{2}{15}(x+2)^{3/2}[3(x+2) - 10] + C \\
&= \frac{2}{15}(x+2)^{3/2}(3x - 4) + C
\end{aligned}$$

**59.** $u = 1 - x,\ x = 1 - u,\ dx = -du$

$$\begin{aligned}
\int x^2\sqrt{1-x}\, dx &= -\int (1-u)^2\sqrt{u}\, du \\
&= -\int (u^{1/2} - 2u^{3/2} + u^{5/2})\, du \\
&= -\left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right) + C \\
&= -\frac{2u^{3/2}}{105}(35 - 42u + 15u^2) + C \\
&= -\frac{2}{105}(1-x)^{3/2}[35 - 42(1-x) + 15(1-x)^2] + C \\
&= -\frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C
\end{aligned}$$

**61.** $u = 2x - 1,\ x = \frac{1}{2}(u + 1),\ dx = \frac{1}{2}\,du$

$$\begin{aligned}\int \frac{x^2 - 1}{\sqrt{2x - 1}}\,dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}}\,\frac{1}{2}\,du\\ &= \frac{1}{8}\int u^{-1/2}[(u^2 + 2u + 1) - 4]\,du\\ &= \frac{1}{8}\int (u^{3/2} + 2u^{1/2} - 3u^{-1/2})\,du\\ &= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} - 6u^{1/2}\right) + C\\ &= \frac{u^{1/2}}{60}(3u^2 + 10u - 45) + C\\ &= \frac{\sqrt{2x - 1}}{60}[3(2x - 1)^2 + 10(2x - 1) - 45] + C\\ &= \frac{1}{60}\sqrt{2x - 1}(12x^2 + 8x - 52) + C\\ &= \frac{1}{15}\sqrt{2x - 1}(3x^2 + 2x - 13) + C\end{aligned}$$

**63.** $u = x + 1,\ x = u - 1,\ dx = du$

$$\begin{aligned}\int \frac{-x}{(x + 1) - \sqrt{x + 1}}\,dx &= \int \frac{-(u - 1)}{u - \sqrt{u}}\,du\\ &= -\int \frac{\left(\sqrt{u} + 1\right)\left(\sqrt{u} - 1\right)}{\sqrt{u}\left(\sqrt{u} - 1\right)}\,du\\ &= -\int (1 + u^{-1/2})\,du\\ &= -(u + 2u^{1/2}) + C\\ &= -u - 2\sqrt{u} + C\\ &= -(x + 1) - 2\sqrt{x + 1} + C\\ &= -x - 2\sqrt{x + 1} - 1 + C\\ &= -\left(x + 2\sqrt{x + 1}\right) + C_1\end{aligned}$$

where $C_1 = -1 + C$.

**65.** Let $u = x^2 + 1,\ du = 2x\,dx$.

$$\int_{-1}^{1} x(x^2 + 1)^3\,dx = \frac{1}{2}\int_{-1}^{1}(x^2 + 1)^3(2x)\,dx = \left[\frac{1}{8}(x^2 + 1)^4\right]_{-1}^{1} = 0$$

**67.** Let $u = x^3 + 1,\ du = 3x^2\,dx$

$$\begin{aligned}\int_1^2 2x^2\sqrt{x^3 + 1}\,dx &= 2\cdot\frac{1}{3}\int_1^2 (x^3 + 1)^{1/2}(3x^2)\,dx\\ &= \left[\frac{2}{3}\,\frac{(x^3 + 1)^{3/2}}{3/2}\right]_1^2\\ &= \frac{4}{9}\left[(x^3 + 1)^{3/2}\right]_1^2\\ &= \frac{4}{9}\left[27 - 2\sqrt{2}\right] = 12 - \frac{8}{9}\sqrt{2}\end{aligned}$$

**69.** Let $u = 2x + 1$, $du = 2\,dx$.

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int_0^4 (2x+1)^{-1/2}(2)\,dx = \left[\sqrt{2x+1}\right]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

**71.** Let $u = 1 + \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int_1^9 \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx = 2\int_1^9 \left(1+\sqrt{x}\right)^{-2}\left(\frac{1}{2\sqrt{x}}\right)dx = \left[-\frac{2}{1+\sqrt{x}}\right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

**73.** $u = 2 - x$, $x = 2 - u$, $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_1^2 (x-1)\sqrt{2-x}\,dx = \int_1^0 -[(2-u)-1]\sqrt{u}\,du = \int_1^0 (u^{3/2} - u^{1/2})\,du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^0 = -\left[\frac{2}{5} - \frac{2}{3}\right] = \frac{4}{15}$$

**75.** $\displaystyle\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right)dx = \left[\frac{3}{2}\sin\left(\frac{2}{3}x\right)\right]_0^{\pi/2} = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}$

**77.** $u = x + 1$, $x = u - 1$, $dx = du$

When $x = 0$, $u = 1$. When $x = 7$, $u = 8$.

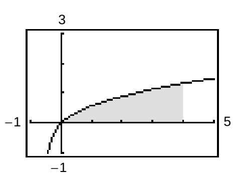
$$\text{Area} = \int_0^7 x\sqrt[3]{x+1}\,dx = \int_1^8 (u-1)\sqrt[3]{u}\,du$$

$$= \int_1^8 (u^{4/3} - u^{1/3})\,du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right]_1^8 = \left(\frac{384}{7} - 12\right) - \left(\frac{3}{7} - \frac{3}{4}\right) = \frac{1209}{28}$$

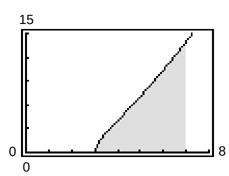
**79.** $\displaystyle A = \int_0^{\pi} (2\sin x + \sin 2x)\,dx = -\left[2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi} = 4$

**81.** $\displaystyle \text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx = 2\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \left[2\tan\left(\frac{x}{2}\right)\right]_{\pi/2}^{2\pi/3} = 2\left(\sqrt{3} - 1\right)$

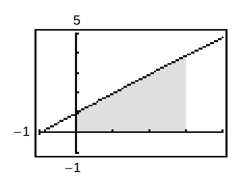
**83.** $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx \approx 3.333 = \frac{10}{3}$




**85.** $\displaystyle\int_3^7 x\sqrt{x-3}\,dx \approx 28.8 = \frac{144}{5}$




**87.** $\displaystyle\int_0^3 \left(\theta + \cos\frac{\theta}{6}\right)d\theta \approx 7.377$




**89.** $\displaystyle\int (2x-1)^2\,dx = \frac{1}{2}\int (2x-1)^2\,2\,dx = \frac{1}{6}(2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$\displaystyle\int (2x-1)^2\,dx = \int (4x^2 - 4x + 1)\,dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$

They differ by a constant: $C_2 = C_1 - \dfrac{1}{6}$.

**91.** $f(x) = x^2(x^2 + 1)$ is even.

$$\int_{-2}^{2} x^2(x^2+1)\,dx = 2\int_0^2 (x^4 + x^2)\,dx = 2\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_0^2$$

$$= 2\left[\frac{32}{5} + \frac{8}{3}\right] = \frac{272}{15}$$

**93.** $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^{2} x(x^2+1)^3\,dx = 0$$

**95.** $\displaystyle\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$; the function $x^2$ is an even function.

(a) $\displaystyle\int_{-2}^{0} x^2\,dx = \int_0^2 x^2\,dx = \frac{8}{3}$

(b) $\displaystyle\int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx = \frac{16}{3}$

(c) $\displaystyle\int_0^2 (-x^2)\,dx = -\int_0^2 x^2\,dx = -\frac{8}{3}$

(d) $\displaystyle\int_{-2}^{0} 3x^2\,dx = 3\int_0^2 x^2\,dx = 8$

**97.** $\displaystyle\int_{-4}^{4} (x^3 + 6x^2 - 2x - 3)\,dx = \int_{-4}^{4} (x^3 - 2x)\,dx + \int_{-4}^{4} (6x^2 - 3)\,dx = 0 + 2\int_0^4 (6x^2 - 3)\,dx = 2\Big[2x^3 - 3x\Big]_0^4 = 232$

**99.** Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

**101.** $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\displaystyle\int_{-2}^{2} x(x^2+1)^2\,dx = 0.$

**103.** $\displaystyle\frac{dV}{dt} = \frac{k}{(t+1)^2}$

$$V(t) = \int \frac{k}{(t+1)^2}\,dt = -\frac{k}{t+1} + C$$

$$V(0) = -k + C = 500{,}000$$

$$V(1) = -\frac{1}{2}k + C = 400{,}000$$

Solving this system yields $k = -200{,}000$ and $C = 300{,}000$. Thus,

$$V(t) = \frac{200{,}000}{t+1} + 300{,}000.$$

When $t = 4$, $V(4) = \$340{,}000$.

**105.** $\displaystyle\frac{1}{b-a}\int_a^b \left[74.50 + 43.75 \sin\frac{\pi t}{6}\right] dt = \frac{1}{b-a}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_a^b$

(a) $\displaystyle\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^3 = \frac{1}{3}\left(223.5 + \frac{262.5}{\pi}\right) \approx 102.352$ thousand units

(b) $\displaystyle\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_3^6 = \frac{1}{3}\left(447 + \frac{262.5}{\pi} - 223.5\right) \approx 102.352$ thousand units

(c) $\displaystyle\frac{1}{12}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^{12} = \frac{1}{12}\left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi}\right) = 74.5$ thousand units

**107.** $\dfrac{1}{b-a}\displaystyle\int_a^b [2\sin(60\pi t) + \cos(120\pi t)]\,dt = \dfrac{1}{b-a}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\,\pi}\sin(120\pi t)\right]_a^b$

(a) $\dfrac{1}{(1/60)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/60} = 60\left[\left(\dfrac{1}{30\pi}+0\right) - \left(-\dfrac{1}{30\pi}\right)\right] = \dfrac{4}{\pi} \approx 1.273$ amps

(b) $\dfrac{1}{(1/240)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/240} = 240\left[\left(-\dfrac{1}{30\sqrt{2}\pi} + \dfrac{1}{120\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right]$

$= \dfrac{2}{\pi}\left(5 - 2\sqrt{2}\right) \approx 1.382$ amps

(c) $\dfrac{1}{(1/30)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/30} = 30\left[\left(\dfrac{1}{30\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right] = 0$ amps

**109.** False

$$\int (2x+1)^2\,dx = \frac{1}{2}\int (2x+1)^2\, 2\,dx = \frac{1}{6}(2x+1)^3 + C$$

**111.** True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d)\,dx = \underbrace{\int_{-10}^{10} (ax^3 + cx)\,dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10} (bx^2 + d)\,dx}_{\text{Even}} = 0 + 2\int_0^{10} (bx^2 + d)\,dx$$

**113.** True

$$4\int \sin x \cos x\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

**115.** Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(u)\,du = \int_{a+h}^{b+h} f(x)\,dx.$$

## Section 4.6 Numerical Integration

**1.** Exact: $\displaystyle\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

**3.** Exact: $\displaystyle\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.000$

Trapezoidal: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

**5.** Exact: $\displaystyle\int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^2 = 4.0000$

Trapezoidal: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{8}\left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8\right] = 4.0625$

Simpson's: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{12}\left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8\right] = 4.0000$

**7.** Exact: $\displaystyle\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\displaystyle\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right]$

$\approx 12.6640$

Simpson's: $\displaystyle\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

**9.** Exact: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{x+1}\right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$\displaystyle = \frac{1}{8}\left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9}\right) \approx 0.1676$

Simpson's: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$\displaystyle = \frac{1}{12}\left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9}\right) \approx 0.1667$

**11.** Trapezoidal: $\displaystyle\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}[1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\displaystyle\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}[1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

**13.** $\displaystyle\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\displaystyle\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\displaystyle\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

**15.** Trapezoidal: $\displaystyle\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{8}\left[\cos 0 + 2\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

$\approx 0.957$

Simpson's: $\displaystyle\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{12}\left[\cos 0 + 4\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

$\approx 0.978$

Graphing utility: 0.977

**17.** Trapezoidal: $\displaystyle\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{80}[\sin(1) + 2\sin(1.025)^2 + 2\sin(1.05)^2 + 2\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Simpson's: $\displaystyle\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{120}[\sin(1) + 4\sin(1.025)^2 + 2\sin(1.05)^2 + 4\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

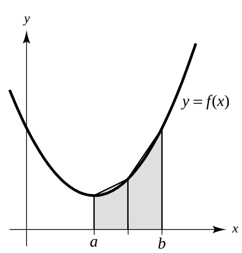
Graphing utility: 0.089

**19.** Trapezoidal: $\displaystyle\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\displaystyle\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

Graphing utility: 0.186

**21.** (a)




The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

**23.** $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\le \dfrac{(2-0)^3}{12(4^2)}(12) = 0.5$ since

$f''(x)$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\le \dfrac{(2-0)^5}{180(4^4)}(0) = 0$ since

$f^{(4)}(x) = 0$.

**25.** $f''(x) = \dfrac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\le \dfrac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133{,}333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \dfrac{24}{x^5}$ in $[1, 3]$

(b) $|f^{(4)(}x)|$ is maximum when $x = 1$ and when $|f^{(4)}(1)| = 24$.

Simpson's: Error $\le \dfrac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426{,}666.67$, $n > 25.56$; let $n = 26$.

**27.** $f(x) = \sqrt{1 + x}$

(a) $f''(x) = -\dfrac{1}{4(1 + x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{1}{4}$.

Trapezoidal: Error $\leq \dfrac{8}{12n^2}\left(\dfrac{1}{4}\right) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \dfrac{-15}{16(1 + x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{15}{16}$.

Simpson's: Error $\leq \dfrac{32}{180n^4}\left(\dfrac{15}{16}\right) < 0.00001$, $n^4 > 16{,}666.67$, $n > 11.36$; let $n = 12$.

**29.** $f(x) = \tan(x^2)$

(a) $f''(x) = 2\sec^2(x^2)[1 + 4x^2\tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\leq \dfrac{(1 - 0)^3}{12n^2}(49.5305) < 0.00001$, $n^2 > 412{,}754.17$, $n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8\sec^2(x^2)[12x^2 + (3 + 32x^4)\tan(x^2) + 36x^2\tan^2(x^2) + 48x^4\tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\leq \dfrac{(1 - 0)^5}{180n^4}(9184.4734) < 0.00001$, $n^4 > 5{,}102{,}485.22$, $n > 47.53$; let $n = 48$.

**31.** Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\leq \dfrac{(b - a)^5}{180n^4}(0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\displaystyle\int_0^1 x^3\,dx = \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 1\right] = \frac{1}{4}$

This is the exact value of the integral.

**33.** $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

| $n$ | $L(n)$ | $M(n)$ | $R(n)$ | $T(n)$ | $S(n)$ |
|---|---|---|---|---|---|
| 4 | 12.7771 | 15.3965 | 18.4340 | 15.6055 | 15.4845 |
| 8 | 14.0868 | 15.4480 | 16.9152 | 15.5010 | 15.4662 |
| 10 | 14.3569 | 15.4544 | 16.6197 | 15.4883 | 15.4658 |
| 12 | 14.5386 | 15.4578 | 16.4242 | 15.4814 | 15.4657 |
| 16 | 14.7674 | 15.4613 | 16.1816 | 15.4745 | 15.4657 |
| 20 | 14.9056 | 15.4628 | 16.0370 | 15.4713 | 15.4657 |

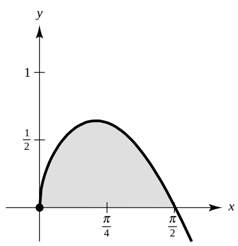
**35.** $f(x) = \sin\sqrt{x}$ on $[0, 4]$.

| $n$ | $L(n)$ | $M(n)$ | $R(n)$ | $T(n)$ | $S(n)$ |
|---|---|---|---|---|---|
| 4 | 2.8163 | 3.5456 | 3.7256 | 3.2709 | 3.3996 |
| 8 | 3.1809 | 3.5053 | 3.6356 | 3.4083 | 3.4541 |
| 10 | 3.2478 | 3.4990 | 3.6115 | 3.4296 | 3.4624 |
| 12 | 3.2909 | 3.4952 | 3.5940 | 3.4425 | 3.4674 |
| 16 | 3.3431 | 3.4910 | 3.5704 | 3.4568 | 3.4730 |
| 20 | 3.3734 | 3.4888 | 3.5552 | 3.4643 | 3.4759 |

**37.** $A = \int_0^{\pi/2} \sqrt{x}\cos x\,dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{84}\left[\sqrt{0}\cos 0 + 4\sqrt{\frac{\pi}{28}}\cos\frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}}\cos\frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}}\cos\frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}}\cos\frac{\pi}{2}\right]$$
$$\approx 0.701$$




**39.** $W = \int_0^5 100x\sqrt{125 - x^3}\,dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x\sqrt{125 - x^3}\,dx \approx \frac{5}{3(12)}\left[0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3}\right.$$
$$\left. + 400\left(\frac{15}{12}\right)\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0\right] \approx 10{,}233.58 \text{ ft} \cdot \text{lb}$$

**41.** $\int_0^{1/2} \frac{6}{\sqrt{1 - x^2}}\,dx$ Simpson's Rule, $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)}[6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282]$$
$$\approx \frac{1}{36}[113.098] \approx 3.1416$$

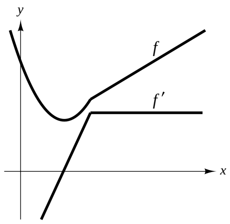
**43.** Area $\approx \frac{1000}{2(10)}[125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89{,}250$ sq m

**45.** $\int_0^t \sin\sqrt{x}\,dx = 2$, $n = 10$

By trial and error, we obtain $t \approx 2.477$.

## Review Exercises for Chapter 4

**1.**




**3.** $\displaystyle\int (2x^2 + x - 1)\,dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

**5.** $\displaystyle\int \frac{x^3+1}{x^2}\,dx = \int \left(x + \frac{1}{x^2}\right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$

**7.** $\displaystyle\int (4x - 3\sin x)\,dx = 2x^2 + 3\cos x + C$

**9.** $f'(x) = -2x,\ (-1, 1)$

$$f(x) = \int -2x\,dx = -x^2 + C$$

When $x = -1$:

$$y = -1 + C = 1$$
$$C = 2$$
$$y = 2 - x^2$$

**11.** $a(t) = a$

$$v(t) = \int a\,dt = at + C_1$$

$v(0) = 0 + C_1 = 0$ when $C_1 = 0$.

$$v(t) = at$$

$$s(t) = \int at\,dt = \frac{a}{2}t^2 + C_2$$

$s(0) = 0 + C_2 = 0$ when $C_2 = 0$.

$$s(t) = \frac{a}{2}t^2$$

$s(30) = \dfrac{a}{2}(30)^2 = 3600$ or

$a = \dfrac{2(3600)}{(30)^2} = 8\text{ ft/sec}^2$.

$v(30) = 8(30) = 240$ ft/sec

**13.** $a(t) = -32$

$v(t) = -32t + 96$

$s(t) = -16t^2 + 96t$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

(b) $s(3) = -144 + 288 = 144$ ft

(c) $v(t) = -32t + 96 = \dfrac{96}{2}$ when $t = \dfrac{3}{2}$ sec.

(d) $s\left(\dfrac{3}{2}\right) = -16\left(\dfrac{9}{4}\right) + 96\left(\dfrac{3}{2}\right) = 108$ ft

**15.** (a) $\displaystyle\sum_{i=1}^{10} (2i - 1)$

(b) $\displaystyle\sum_{i=1}^{n} i^3$

(c) $\displaystyle\sum_{i=1}^{10} (4i + 2)$

**17.** $y = \dfrac{10}{x^2+1}, \Delta x = \dfrac{1}{2}, n = 4$

$$S(n) = S(4) = \frac{1}{2}\left[\frac{10}{1} + \frac{10}{(1/2)^2+1} + \frac{10}{(1)^2+1} + \frac{10}{(3/2)^2+1}\right]$$

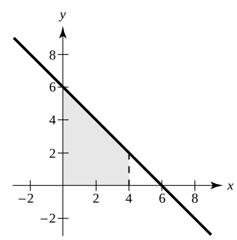
$$\approx 13.0385$$

$$s(n) = s(4) = \frac{1}{2}\left[\frac{10}{(1/2)^2+1} + \frac{10}{1+1} + \frac{10}{(3/2)^2+1} + \frac{10}{2^2+1}\right]$$

$$\approx 9.0385$$

$9.0385 < \text{Area of Region} < 13.0385$

**19.** $y = 6 - x, \Delta x = \dfrac{4}{n}$, right endpoints



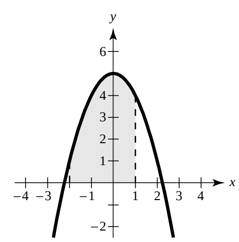

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(6 - \frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n}\left[6n - \frac{4}{n}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[24 - 8\,\frac{n+1}{n}\right] = 24 - 8 = 16$$

**21.** $y = 5 - x^2, \Delta x = \dfrac{3}{n}$




$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[5 - \left(-2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

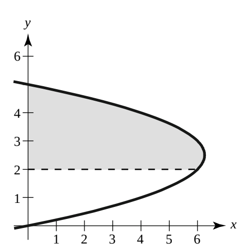
$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[1 + \frac{12i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[n + \frac{12}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[3 + 18\,\frac{n+1}{n} - \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= 3 + 18 - 9 = 12$$

**23.** $x = 5y - y^2, 2 \le y \le 5, \Delta y = \dfrac{3}{n}$




$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[5\left(2 + \frac{3i}{n}\right) - \left(2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[10 + \frac{15i}{n} - 4 - 12\frac{i}{n} - \frac{9i^2}{n^2}\right]$$

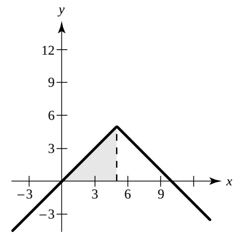
$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[6 + \frac{3i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[6n + \frac{3}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \left[18 + \frac{9}{2} - 9\right] = \frac{27}{2}$$

**25.** $\lim_{\|\Delta\|\to\infty} \sum_{i=1}^{n} (2ci - 3)\,\Delta xi = \int_4^6 (2x - 3)\,dx$

**27.**




$$\int_0^5 \Bigl(5 - |x - 5|\,dx = \int_0^5 (5 - (5 - x))\,dx = \int_0^5 x\,dx = \frac{25}{2}$$

(triangle)

**29.** (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx = 10 + 3 = 13$

(b) $\int_2^6 [f(x) - g(x)]\,dx = \int_2^6 f(x)\,dx - \int_2^6 g(x)\,dx = 10 - 3 = 7$

(c) $\int_2^6 [2f(x) - 3g(x)]\,dx = 2\int_2^6 f(x)\,dx - 3\int_2^6 g(x)\,dx = 2(10) - 3(3) = 11$

(d) $\int_2^6 5f(x)\,dx = 5\int_2^6 f(x)\,dx = 5(10) = 50$

**31.** $\int_1^8 \left(\sqrt[3]{x} + 1\right) dx = \left[\frac{3}{4}x^{4/3} + x\right]_1^8 = \left[\frac{3}{4}(16) + 8\right] - \left[\frac{3}{4} + 1\right] = \frac{73}{4}$ (c)

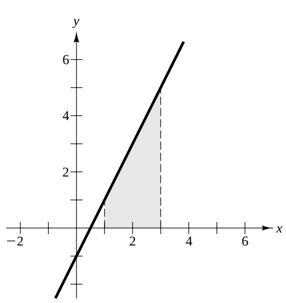
**33.** $\int_0^4 (2 + x)\,dx = \left[2x + \frac{x^2}{2}\right]_0^4 = 8 + \frac{16}{2} = 16$

**35.** $\int_{-1}^{1} (4t^3 - 2t)\,dt = \left[t^4 - t^2\right]_{-1}^{1} = 0$

**37.** $\int_4^9 x\sqrt{x}\,dx = \int_4^9 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_4^9 = \frac{2}{5}\left[\left(\sqrt{9}\right)^5 - \left(\sqrt{4}\right)^5\right] = \frac{2}{5}(243 - 32) = \frac{422}{5}$

**39.** $\int_0^{3\pi/4} \sin\theta\,d\theta = \Bigl[-\cos\theta\Bigr]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2}\right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$

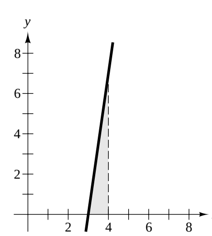
**41.** $\int_1^3 (2x - 1)\,dx = \left[x^2 - x\right]_1^3 = 6$



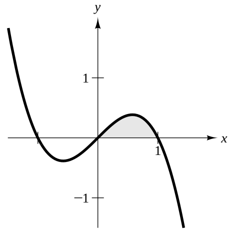

**43.** $\int_3^4 (x^2 - 9)\,dx = \left[\frac{x^3}{3} - 9x\right]_3^4$

$$= \left(\frac{64}{3} - 36\right) - (9 - 27)$$

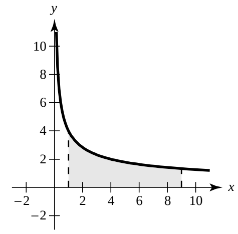
$$= \frac{64}{3} - \frac{54}{3} = \frac{10}{3}$$



**45.** $\displaystyle\int_0^1 (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$




**47.** Area $\displaystyle= \int_1^9 \frac{4}{\sqrt{x}}\,dx = \left[\frac{4x^{1/2}}{(1/2)}\right]_1^9 = 8(3-1) = 16$

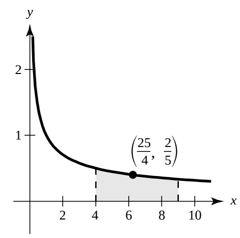



**49.** $\displaystyle\frac{1}{9-4}\int_4^9 \frac{1}{\sqrt{x}}\,dx = \left[\frac{1}{5}2\sqrt{x}\right]_4^9 = \frac{2}{5}(3-2) = \frac{2}{5}$ Average value

$$\frac{2}{5} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x} = \frac{5}{2}$$

$$x = \frac{25}{4}$$




**51.** $F'(x) = x^2\sqrt{1+x^3}$

**53.** $F'(x) = x^2 + 3x + 2$

**55.** $\displaystyle\int (x^2+1)^3\,dx = \int (x^6 + 3x^4 + 3x^2 + 1)\,dx = \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$

**57.** $u = x^3 + 3,\ du = 3x^2\,dx$

$$\int \frac{x^2}{\sqrt{x^3+3}}\,dx = \int (x^3+3)^{-1/2}\,x^2\,dx = \frac{1}{3}\int (x^3+3)^{-1/2}\,3x^2\,dx = \frac{2}{3}(x^3+3)^{1/2} + C$$

**59.** $u = 1 - 3x^2,\ du = -6x\,dx$

$$\int x(1-3x^2)^4\,dx = -\frac{1}{6}\int (1-3x^2)^4(-6x\,dx) = -\frac{1}{30}(1-3x^2)^5 + C = \frac{1}{30}(3x^2-1)^5 + C$$

**61.** $\displaystyle\int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

**63.** $\displaystyle\int \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,dx = \int (1-\cos\theta)^{-1/2}\sin\theta\,d\theta = 2(1-\cos\theta)^{1/2} + C = 2\sqrt{1-\cos\theta} + C$

**65.** $\displaystyle\int \tan^n x\sec^2 x\,dx = \frac{\tan^{n+1}x}{n+1} + C,\ n \neq -1$

**67.** $\displaystyle\int (1+\sec\pi x)^2\sec\pi x\tan\pi x\,dx = \frac{1}{\pi}\int (1+\sec\pi x)^2(\pi\sec\pi x\tan\pi x)\,dx = \frac{1}{3\pi}(1+\sec\pi x)^3 + C$

**69.** $\displaystyle\int_{-1}^2 x(x^2-4)\,dx = \frac{1}{2}\int_{-1}^2 (x^2-4)(2x)\,dx = \frac{1}{2}\left.\frac{(x^2-4)^2}{2}\right]_{-1}^2 = \frac{1}{4}[0-9] = -\frac{9}{4}$

**71.** $\int_0^3 \frac{1}{\sqrt{1+x}}\,dx = \int_0^3 (1+x)^{-1/2}\,dx = \Big[2(1+x)^{1/2}\Big]_0^3 = 4 - 2 = 2$

**73.** $u = 1 - y, y = 1 - u, dy = -du$

When $y = 0$, $u = 1$. When $y = 1$, $u = 0$.

$$2\pi\int_0^1 (y+1)\sqrt{1-y}\,dy = 2\pi\int_1^0 -[(1-u)+1]\sqrt{u}\,du$$
$$= 2\pi\int_1^0 (u^{3/2} - 2u^{1/2})\,du = 2\pi\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right]_1^0 = \frac{28\pi}{15}$$

**75.** $\int_0^\pi \cos\left(\frac{x}{2}\right)dx = 2\int_0^\pi \cos\left(\frac{x}{2}\right)\frac{1}{2}\,dx = \left[2\sin\left(\frac{x}{2}\right)\right]_0^\pi = 2$

**77.** $u = 1 - x, x = 1 - u, dx = -du$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx = \frac{15}{4}\int_{1-a}^{1-b} -(1-u)\sqrt{u}\,du$$
$$= \frac{15}{4}\int_{1-a}^{1-b}(u^{3/2} - u^{1/2})\,du = \frac{15}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1-a}^{1-b} = \frac{15}{4}\left[\frac{2u^{3/2}}{15}(3u-5)\right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_a^b$$

(a) $P_{0.50,\,0.75} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_{0.50}^{0.75} = 0.353 = 35.3\%$

(b) $P_{0,\,b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_0^b = -\frac{(1-b)^{3/2}}{2}(3b+2) + 1 = 0.5$

$(1-b)^{3/2}(3b+2) = 1$

$b \approx 0.586 = 58.6\%$

**79.** $p = 1.20 + 0.04t$

$$C = \frac{15{,}000}{M}\int_t^{t+1} p\,ds$$

(a) 2000 corresponds to $t = 10$.

$$C = \frac{15{,}000}{M}\int_{10}^{11}[1.20 + 0.04t]\,dt$$
$$= \frac{15{,}000}{M}\Big[1.20t + 0.02t^2\Big]_{10}^{11} = \frac{24{,}300}{M}$$

(b) 2005 corresponds to $t = 15$.

$$C = \frac{15{,}000}{M}\Big[1.20t + 0.02t^2\Big]_{15}^{16} = \frac{27{,}300}{M}$$

**81.** Trapezoidal Rule $(n = 4)$: $\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{8}\left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.257$

Simpson's Rule $(n = 4)$: $\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{12}\left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.254$

Graphing utility: 0.254

**83.** Trapezoidal Rule $(n = 4)$: $\int_0^{\pi/2}\sqrt{x}\cos x\,dx \approx 0.637$

Simpson's Rule $(n = 4)$: 0.685

Graphing Utility: 0.704

**85.** (a) $R < I < T < L$

(b) $S(4) = \frac{4-0}{3(4)}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$

$$\approx \frac{1}{3}\left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4}\right] \approx 5.417$$

## Problem Solving for Chapter 4

**1.** (a) $L(1) = \int_1^1 \frac{1}{t}\,dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$L'(1) = 1$

(c) $L(x) = 1 = \int_1^x \frac{1}{t}\,dt$ for $x \approx 2.718$

$\int_1^{2.718} \frac{1}{t}\,dt = 0.999896$ (**Note:** The exact value of $x$ is $e$, the base of the natural logarithm function.)

(d) We first show that $\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^{1} \frac{1}{t}\,dt$.

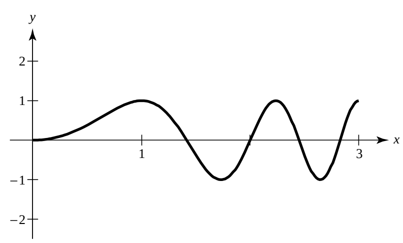
To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1}\,dt$.

Then $\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^{1} \frac{1}{ux_1}(x_1\,du) = \int_{1/x_1}^{1} \frac{1}{u}\,du = \int_{1/x_1}^{1} \frac{1}{t}\,dt$.

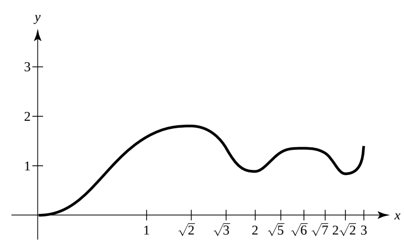
Now,
$$\begin{aligned} L(x_1x_2) &= \int_1^{x_1x_2} \frac{1}{t}\,dt = \int_{1/x_1}^{x_2} \frac{1}{u}\,du \left(\text{using } u = \frac{t}{x_1}\right) \\ &= \int_{1/x_1}^{1} \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du \\ &= \int_1^{x_1} \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du \\ &= L(x_1) + L(x_2). \end{aligned}$$

**3.** $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$

(a)




(b)




The zeros of $y = \sin\frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin\frac{\pi x^2}{2} = 0 \implies \frac{\pi x^2}{2} = n\pi \implies x^2 = 2n \implies x = \sqrt{2n}$, $n$ integer.

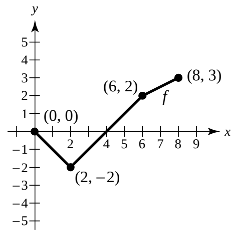
Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \implies \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \implies x^2 = 1 + 2n \implies x = \sqrt{1 + 2n}$, $n$ integer

Points of inflection at $x = 1$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$.

**5.** (a)




(b)

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $F(x)$ | 0 | $-\frac{1}{2}$ | $-2$ | $-\frac{7}{2}$ | $-4$ | $-\frac{7}{2}$ | $-2$ | $\frac{1}{4}$ | 3 |

(c) $f(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x < 6 \\ \frac{1}{2}x - 1, & 6 \le x \le 8 \end{cases}$

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} (-x^2/2), & 0 \le x < 2 \\ (x^2/2) - 4x + 4, & 2 \le x < 6 \\ (1/4)x^2 - x - 5, & 6 \le x \le 8 \end{cases}$$

$F'(x) = f(x)$. $F$ is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is $-4$ at $x = 4$, and the maximum is 3 at $x = 8$.

(d) $F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$

$x = 2$ is a point of inflection, whereas $x = 6$ is not. ($f$ is not continuous at $x = 6$.)

**7.** (a) $\displaystyle\int_{-1}^{1} \cos x\,dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$$\int_{-1}^{1} \cos x\,dx = \sin x\Big]_{-1}^{1} = 2\sin(1) \approx 1.6829$$

Error. $|1.6829 - 1.6758| = 0.0071$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{1 + x^2}\,dx \approx \frac{1}{1 + (1/3)} + \frac{1}{1 + (1/3)} = \frac{3}{2}$

(Note: exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$$\int_{-1}^{1} p(x)\,dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{2b}{3} + 2d$$

$$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$$

**9.** Consider $F(x) = [f(x)]^2 \implies F'(x) = 2f(x)\,f'(x)$. Thus,

$$\begin{aligned} \int_a^b f(x)\,f'(x)\,dx &= \int_a^b \frac{1}{2}F'(x)\,dx \\ &= \left[\frac{1}{2}F(x)\right]_a^b \\ &= \frac{1}{2}[F(b) - F(a)] \\ &= \frac{1}{2}[f(b)^2 - f(a)^2] \end{aligned}$$

**11.** Consider $\displaystyle\int_0^1 x^5\,dx = \frac{x^6}{6}\Big]_0^1 = \frac{1}{6}$.

The corresponding Riemann Sum using right endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5\right] \\ &= \frac{1}{n^6}[1^5 + 2^5 + \cdots + n^5] \end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}$.

**13.** By Theorem 4.8, $0 < f(x) \le M \implies \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b - a)$.

Similarly, $m \le f(x) \implies m(b - a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx$.

Thus, $m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)$. On the interval $[0, 1]$, $1 \le \sqrt{1 + x^4} \le \sqrt{2}$ and $b - a = 1$.

Thus, $1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}$. $\left(\textbf{Note: } \int_0^1 \sqrt{1 + x^4}\,dx \approx 1.0894\right)$

**15.** Since $-|f(x)| \le f(x) \le |f(x)|$,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \implies \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

**17.** $\dfrac{1}{365}\displaystyle\int_0^{365} 100{,}000\left[1 + \sin\frac{2\pi(t - 60)}{365}\right]dt = \frac{100{,}000}{365}\left[t - \frac{365}{2\pi}\cos\frac{2\pi(t - 60)}{365}\right]_0^{365} = 100{,}000$ lbs.

# CHAPTER 5
# Logarithmic, Exponential, and Other Transcendental Functions

# CHAPTER 5
# Logarithmic, Exponential, and Other Transcendental Functions

## Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Odd-Numbered Exercises

**1.** Simpson's Rule: $n = 10$

| $x$ | 0.5 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $\int_1^x \frac{1}{t}\,dt$ | $-0.6932$ | 0.4055 | 0.6932 | 0.9163 | 1.0987 | 1.2529 | 1.3865 |

**Note:** $\int_1^{0.5} \frac{1}{t}\,dt = -\int_{0.5}^{1} \frac{1}{t}\,dt$

**3.** (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

**5.** (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

**7.** $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

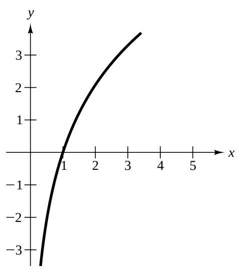
**9.** $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

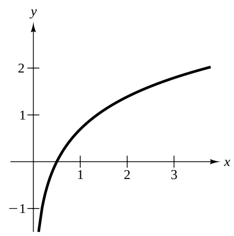
**11.** $f(x) = 3 \ln x$

Domain: $x > 0$



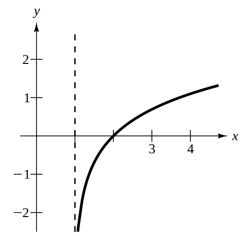
**13.** $f(x) = \ln 2x$

Domain: $x > 0$



**15.** $f(x) = \ln(x - 1)$

Domain: $x > 1$



**17.** (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

**19.** $\ln \frac{2}{3} = \ln 2 - \ln 3$

**21.** $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

**23.** $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

**25.** $\ln\left(\dfrac{x^2-1}{x^3}\right)^3 = 3[\ln(x^2-1) - \ln x^3]$

$= 3[\ln(x+1) + \ln(x-1) - 3\ln x]$

**27.** $\ln z(z-1)^2 = \ln z + \ln(z-1)^2$

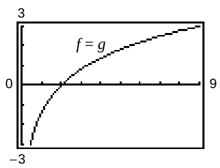
$= \ln z + 2\ln(z-1)$

**29.** $\ln(x-2) - \ln(x+2) = \ln\dfrac{x-2}{x+2}$

**31.** $\dfrac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)] = \dfrac{1}{3}\ln\dfrac{x(x+3)^2}{x^2-1} = \ln\sqrt[3]{\dfrac{x(x+3)^2}{x^2-1}}$

**33 .** $2\ln 3 - \dfrac{1}{2}\ln(x^2+1) = \ln 9 - \ln\sqrt{x^2+1} = \ln\dfrac{9}{\sqrt{x^2+1}}$

**35.**




**37.** $\lim_{x\to 3^+} \ln(x-3) = -\infty$

**39.** $\lim_{x\to 2^-} \ln[x^2(3-x)] = \ln 4 \approx 1.3863$

**41.** $y = \ln x^3 = 3\ln x$

$y' = \dfrac{3}{x}$

At $(1, 0)$, $y' = 3$.

**43.** $y = \ln x^2 = 2\ln x$

$y' = \dfrac{2}{x}$

At $(1, 0)$, $y' = 2$.

**45.** $g(x) = \ln x^2 = 2\ln x$

$g'(x) = \dfrac{2}{x}$

**47.** $y = (\ln x)^4$

$\dfrac{dy}{dx} = 4(\ln x)^3\left(\dfrac{1}{x}\right) = \dfrac{4(\ln x)^3}{x}$

**49.** $y = \ln x\sqrt{x^2-1} = \ln x + \dfrac{1}{2}\ln(x^2-1)$

$\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{2}\left(\dfrac{2x}{x^2-1}\right) = \dfrac{2x^2-1}{x(x^2-1)}$

**51.** $f(x) = \ln\dfrac{x}{x^2+1} = \ln x - \ln(x^2+1)$

$f'(x) = \dfrac{1}{x} - \dfrac{2x}{x^2+1} = \dfrac{1-x^2}{x(x^2+1)}$

**53.** $g(t) = \dfrac{\ln t}{t^2}$

$g'(t) = \dfrac{t^2(1/t) - 2t\ln t}{t^4} = \dfrac{1-2\ln t}{t^3}$

**55.** $y = \ln(\ln x^2)$

$\dfrac{dy}{dx} = \dfrac{1}{\ln x^2}\dfrac{d}{dx}(\ln x^2) = \dfrac{(2x/x^2)}{\ln x^2} = \dfrac{2}{x\ln x^2} = \dfrac{1}{x\ln x}$

**57.** $y = \ln\sqrt{\dfrac{x+1}{x-1}} = \dfrac{1}{2}[\ln(x+1) - \ln(x-1)]$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left[\dfrac{1}{x+1} - \dfrac{1}{x-1}\right] = \dfrac{1}{1-x^2}$

**59.** $f(x) = \ln\dfrac{\sqrt{4+x^2}}{x} = \dfrac{1}{2}\ln(4+x^2) - \ln x$

$f'(x) = \dfrac{x}{4+x^2} - \dfrac{1}{x} = \dfrac{-4}{x(x^2+4)}$

**61.** $y = \dfrac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = \frac{-x\left(x/\sqrt{x^2+1}\right) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x + \sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2}$$

**63.** $y = \ln|\sin x|$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

**65.** $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

**67.** $y = \ln\left|\dfrac{-1+\sin x}{2+\sin x}\right|$

$$= \ln|-1+\sin x| - \ln|2+\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1+\sin x} - \frac{\cos x}{2+\sin x}$$

$$= \frac{3\cos x}{(\sin x - 1)(\sin x + 2)}$$

**69.** $f(x) = \sin 2x \ln x^2 = 2 \sin 2x \ln x$

$$f'(x) = (2\sin 2x)\left(\frac{1}{x}\right) + 4\cos 2x \ln x$$

$$= \frac{2}{x}(\sin 2x + 2x\cos 2x \ln x)$$

$$= \frac{2}{x}(\sin 2x + x\cos 2x \ln x^2)$$

**71.** (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

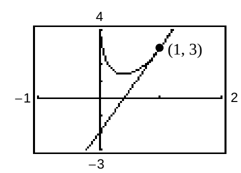
When $x = 1$, $\dfrac{dy}{dx} = 5$.

Tangent line: $y - 3 = 5(x - 1)$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)




**73.** $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

**75.** $y = 2(\ln x) + 3$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

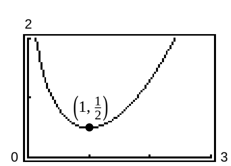
**77.** $y = \dfrac{x^2}{2} - \ln x$

Domain: $x > 0$

$$y' = x - \frac{1}{x} = \frac{(x+1)(x-1)}{x} = 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

Relative minimum: $\left(1, \frac{1}{2}\right)$



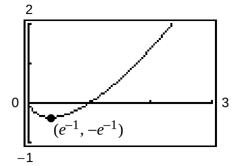
**79.** $y = x \ln x$

Domain: $x > 0$

$y' = x\left(\dfrac{1}{x}\right) + \ln x = 1 + \ln x = 0$ when $x = e^{-1}$.

$y'' = \dfrac{1}{x} > 0$

Relative minimum: $(e^{-1}, -e^{-1})$




**81.** $y = \dfrac{x}{\ln x}$

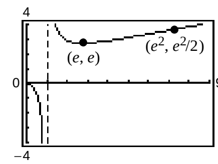
Domain: $0 < x < 1, x > 1$

$y' = \dfrac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} = 0$ when $x = e$.

$y'' = \dfrac{(\ln x)^2(1/x) - (\ln x - 1)(2/x)\ln x}{(\ln x)^4} = \dfrac{2 - \ln x}{x(\ln x)^3} = 0$ when $x = e^2$.

Relative minimum: $(e, e)$

Point of inflection: $(e^2, e^2/2)$




**83.** $f(x) = \ln x, \quad f(1) = 0$

$f'(x) = \dfrac{1}{x}, \quad f'(1) = 1$

$f''(x) = -\dfrac{1}{x^2}, \quad f''(1) = -1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \dfrac{1}{2} f''(1)(x - 1)^2$

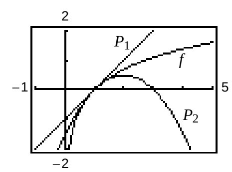
$= (x - 1) - \dfrac{1}{2}(x - 1)^2, \quad P_2(1) = 0$

$P_1'(x) = 1, \quad P_1'(1) = 1$

$P_2'(x) = 1 - (x - 1) = 2 - x, \quad P_2'(1) = 1$

$P_2''(x) = -1, \quad P_2''(1) = -1$

The values of $f$, $P_1$, $P_2$, and their first derivatives agree at $x = 1$. The values of the second derivatives of $f$ and $P_2$ agree at $x = 1$.




**85.** Find $x$ such that $\ln x = -x$.

$f(x) = (\ln x) + x = 0$

$f'(x) = \dfrac{1}{x} + 1$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n\left[\dfrac{1 - \ln x_n}{1 + x_n}\right]$

| $n$ | 1 | 2 | 3 |
|---|---|---|---|
| $x_n$ | 0.5 | 0.5644 | 0.5671 |
| $f(x_n)$ | −0.1931 | −0.0076 | −0.0001 |

Approximate root: $x = 0.567$

**87.** $y = x\sqrt{x^2 - 1}$

$\ln y = \ln x + \dfrac{1}{2}\ln(x^2 - 1)$

$\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{x} + \dfrac{x}{x^2 - 1}$

$\dfrac{dy}{dx} = y\left[\dfrac{2x^2 - 1}{x(x^2 - 1)}\right] = \dfrac{2x^2 - 1}{\sqrt{x^2 - 1}}$

**89.** $$y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$$

$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x-1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$\frac{dy}{dx} = y\left[\frac{3x^2-15x+8}{2x(3x-2)(x-1)}\right]$$

$$= \frac{3x^3-15x^2+8x}{2(x-1)^3\sqrt{3x-2}}$$

**91.** $$y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]$$

$$= \frac{y}{2}\left[\frac{4x^2+4x-2}{x(x^2-1)}\right] = \frac{(2x^2+2x-1)\sqrt{x-1}}{(x+1)^{3/2}}$$

**93.** Answers will vary. See Theorem 5.1 and 5.2.

**95.** $\ln e^x = x$ because $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions.

**97.** (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \dfrac{2}{x} = 0$ for $x = 2$.

**99.** $$\beta = 10\log_{10}\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}\ln\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}[\ln I + 16\ln 10] = 160 + 10\log_{10} I$$

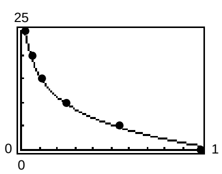
$$\beta(10^{-10}) = \frac{10}{\ln 10}[\ln 10^{-10} + 16\ln 10] = \frac{10}{\ln 10}[-10\ln 10 + 16\ln 10] = \frac{10}{\ln 10}[6\ln 10] = 60 \text{ decibels}$$

**101.** (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$$h = 0.8627 - 6.4474\ln p.$$

(c)




(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $$h = 0.8627 - 6.4474\ln p$$

$$1 = -6.4474\frac{1}{p}\frac{dp}{dh} \quad \text{(implicit differentiation)}$$

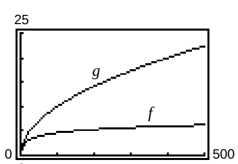
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.55$ and $dp/dh = -0.0853$ atmos/km.

For $h = 20$, $p = 0.06$ and $dp/dh = -0.00931$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

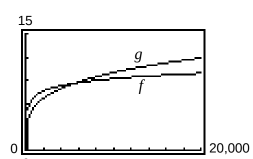
**103.** (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$




$$f'(x) = \frac{1}{x},\ g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. $g$ is increasing at a faster rate than $f$ for "large" values of $x$.

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$




$$f'(x) = \frac{1}{x},\ g'(x) = \frac{1}{4\sqrt[4]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. $g$ is increasing at a faster rate than $f$ for "large" values of $x$. $f(x) = \ln x$ increases very slowly for "large" values of $x$.

**105.** False

$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$

## Section 5.2 The Natural Logarithmic Function: Integration

**1.** $\displaystyle\int \frac{5}{x}dx = 5\int \frac{1}{x}dx = 5 \ln |x| + C$

**3.** $u = x + 1,\ du = dx$

$$\int \frac{1}{x+1}\,dx = \ln|x + 1| + C$$

**5.** $u = 3 - 2x,\ du = -2\,dx$

$$\int \frac{1}{3-2x}\,dx = -\frac{1}{2}\int \frac{1}{3-2x}(-2)\,dx = -\frac{1}{2}\ln|3 - 2x| + C$$

**7.** $u = x^2 + 1,\ du = 2x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{x^2+1}(2x)\,dx = \frac{1}{2}\ln(x^2 + 1) + C = \ln\sqrt{x^2+1} + C$$

**9.** $\displaystyle\int \frac{x^2-4}{x}\,dx = \int\left(x - \frac{4}{x}\right)dx = \frac{x^2}{2} - 4\ln|x| + C$

**11.** $u = x^3 + 3x^2 + 9x,\ du = 3(x^2 + 2x + 3)\,dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3(x^2+2x+3)}{x^3+3x^2+9x}\,dx = \frac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$$

**13.** $\displaystyle\int \frac{x^2-3x+2}{x+1}dx = \int\left(x - 4 + \frac{6}{x+1}\right)dx = \frac{x^2}{2} - 4x + 6\ln|x + 1| + C$

**15.** $\displaystyle\int \frac{x^3-3x^2+5}{x-3}\,dx = \int\left(x^2 + \frac{5}{x-3}\right)dx = \frac{x^3}{3} + 5\ln|x - 3| + C$

**17.** $\displaystyle\int \frac{x^4+x-4}{x^2+2}dx = \int\left(x^2 - 2 + \frac{x}{x^2+2}\right)dx = \frac{x^3}{3} - 2x + \frac{1}{2}\ln(x^2 + 2) + C$

**19.** $u = \ln x,\ du = \dfrac{1}{x}\,dx$

$$\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3 + C$$

**21.** $u = x + 1,\ du = dx$

$$\int \frac{1}{\sqrt{x+1}}\,dx = \int (x+1)^{-1/2}\,dx = 2(x+1)^{1/2} + C = 2\sqrt{x+1} + C$$

**23.** $\displaystyle\int \frac{2x}{(x-1)^2}\,dx = \int \frac{2x-2+2}{(x-1)^2}\,dx$

$$= \int \frac{2(x-1)}{(x-1)^2}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$

$$= 2\int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$

$$= 2\ln|x - 1| - \frac{2}{(x-1)} + C$$

**25.** $u = 1 + \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx \implies (u - 1)\,du = dx$

$$\begin{aligned}\int \frac{1}{1 + \sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du &= \int \left(u - \frac{1}{u}\right)du\\ &= u - \ln|u| + C_1\\ &= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1\\ &= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C\end{aligned}$$

where $C = C_1 + 1$.

**27.** $u = \sqrt{x} - 3,\ du = \dfrac{1}{2\sqrt{x}}\,dx \implies 2(u + 3)\,du = dx$

$$\begin{aligned}\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx = 2\int \frac{(u+3)^2}{u}\,du = 2\int \frac{u^2 + 6u + 9}{u}\,du &= 2\int \left(u + 6 + \frac{9}{u}\right)du\\ &= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1 = u^2 + 12u + 18\ln|u| + C_1\\ &= \left(\sqrt{x} - 3\right)^2 + 12\left(\sqrt{x} - 3\right) + 18\ln\left|\sqrt{x} - 3\right| + C_1\\ &= x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C \text{ where } C = C_1 - 27.\end{aligned}$$

**29.** $\displaystyle\int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|\sin\theta| + C$

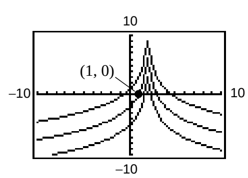
$(u = \sin\theta,\ du = \cos\theta\,d\theta)$

**31.** $$\begin{aligned}\int \csc 2x\,dx &= \frac{1}{2}\int (\csc 2x)(2)\,dx\\ &= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C\end{aligned}$$

**33.** $\displaystyle\int \frac{\cos t}{1 + \sin t}\,dt = \ln|1 + \sin t| + C$

**35.** $\displaystyle\int \frac{\sec x \tan x}{\sec x - 1}\,dx = \ln|\sec x - 1| + C$

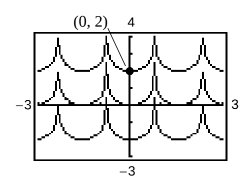
**37.** $$\begin{aligned}y &= \int \frac{3}{2 - x}\,dx\\ &= -3\int \frac{1}{x - 2}\,dx\\ &= -3\ln|x - 2| + C\end{aligned}$$




$(1, 0)$: $0 = -3\ln|1 - 2| + C \implies C = 0$

$y = -3\ln|x - 2|$

**39.** $$\begin{aligned}s &= \int \tan(2\theta)\,d\theta\\ &= \frac{1}{2}\int \tan(2\theta)(2\,d\theta)\\ &= -\frac{1}{2}\ln|\cos 2\theta| + C\end{aligned}$$

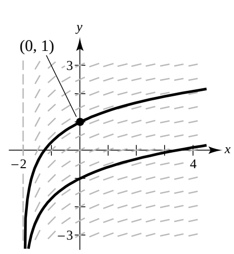



$(0, 2)$: $2 = -\dfrac{1}{2}\ln|\cos(0)| + C \implies C = 2$

$s = -\dfrac{1}{2}\ln|\cos 2\theta| + 2$

**41.** $\dfrac{dy}{dx} = \dfrac{1}{x + 2},\ (0, 1)$

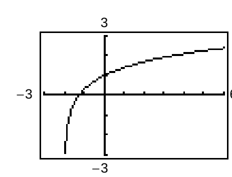
(a)




(b) $\displaystyle y = \int \frac{1}{x + 2}\,dx = \ln|x + 2| + C$

$y(0) = 1 \implies 1 = \ln 2 + C \implies C = 1 - \ln 2$

Hence, $y = \ln|x + 2| + 1 - \ln 2 = \ln\left|\dfrac{x + 2}{2}\right| + 1$.



**43.** $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx = \left[\frac{5}{3}\ln|3x+1|\right]_0^4$

$\displaystyle= \frac{5}{3}\ln 13 \approx 4.275$

**45.** $u = 1 + \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx = \left[\frac{1}{3}(1+\ln x)^3\right]_1^e = \frac{7}{3}$

**47.** $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx = \int_0^2 \left(x - 1 - \frac{1}{x+1}\right)dx$

$\displaystyle= \left[\frac{1}{2}x^2 - x - \ln|x+1|\right]_0^2 = -\ln 3$

**49.** $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta - \sin\theta}\,d\theta = \Big[\ln|\theta - \sin\theta|\Big]_1^2$

$\displaystyle= \ln\left|\frac{2-\sin 2}{1-\sin 1}\right| \approx 1.929$

**51.** $\displaystyle -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + \mathrm{C} = \ln|\sec x| + C$

**53.** $\displaystyle \ln|\sec x + \tan x| + C = \ln\left|\frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C = \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$

$\displaystyle= \ln\left|\frac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C$

**55.** $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx = 2\big(1+\sqrt{x}\big) - 2\ln\big(1+\sqrt{x}\big) + C_1$

$= 2\left[\sqrt{x} - \ln\big(1+\sqrt{x}\big)\right] + C$ where $C = C_1 + 2$.

**57.** $\displaystyle\int \cos(1-x)\,dx = -\sin(1-x) + C$

**59.** $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x - \sin x)\,dx = \Big[-\ln|\csc x + \cot x| + \cos x\Big]_{\pi/4}^{\pi/2} = \ln\big(\sqrt{2}+1\big) - \frac{\sqrt{2}}{2} \approx 0.174$

**Note: In Exercises 61 and 63, you can use the Second Fundamental Theorem of Calculus or integrate the function.**

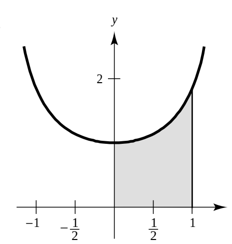
**61.** $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

$\displaystyle F'(x) = \frac{1}{x}$

**63.** $\displaystyle F(x) = \int_x^{3x} \frac{1}{t}\,dt = \int_1^{3x} \frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt$

$\displaystyle F'(x) = \frac{3}{3x} - \frac{1}{x} = 0$

**65.**

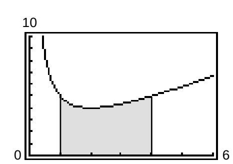


$A \approx 1.25$
Matches (d)

**67.** $\displaystyle A = \int_1^4 \frac{x^2+4}{x}\,dx = \int_1^4 \left(x + \frac{4}{x}\right)dx$

$\displaystyle= \left[\frac{x^2}{2} + 4\ln x\right]_1^4 = (8 + 4\ln 4) - \frac{1}{2}$

$\displaystyle= \frac{15}{2} + 8\ln 2 \approx 13.045$ square units

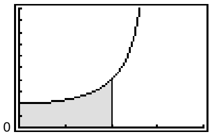


**69.** $\displaystyle\int_0^2 2\sec\frac{\pi x}{6}\,dx = \frac{12}{\pi}\int_0^2 \sec\left(\frac{\pi x}{6}\right)\frac{\pi}{6}\,dx$

$$= \left[\frac{12}{\pi}\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$$

$$= \frac{12}{\pi}\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \frac{12}{\pi}\ln|1 + 0|$$

$$= \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.03041$$




**71.** Power Rule

**73.** Substitution: $(u = x^2 + 4)$ and Log Rule

**75.** Divide the polynomials:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

**77.** Average value $\displaystyle = \frac{1}{4-2}\int_2^4 \frac{8}{x^2}dx = 4\int_2^4 x^{-2}dx$

$$= \left[-4\frac{1}{x}\right]_2^4$$

$$= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1$$

**79.** Average value $\displaystyle = \frac{1}{e-1}\int_1^e \frac{\ln x}{x}dx = \frac{1}{e-1}\left[\frac{(\ln x)^2}{2}\right]_1^e$

$$= \frac{1}{e-1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2e-2} \approx 0.291$$

**81.** $\displaystyle P(t) = \int \frac{3000}{1 + 0.25t}\,dt = (3000)(4)\int \frac{0.25}{1 + 0.25t}\,dt = 12{,}000\ln|1 + 0.25t| + C$

$P(0) = 12{,}000\ln|1 + 0.25(0)| + C = 1000$

$C = 1000$

$P(t) = 12{,}000\ln|1 + 0.25t| + 1000 = 1000[12\ln|1 + 0.25t| + 1]$

$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$

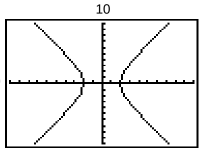
**83.** $\displaystyle\frac{1}{50 - 40}\int_{40}^{50} \frac{90{,}000}{400 + 3x}\,dx = \Big[3000\ln|400 + 3x|\Big]_{40}^{50} \approx \$168.27$

**85.** (a) $2x^2 - y^2 = 8$

$y^2 = 2x^2 - 8$

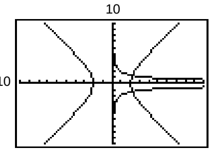
$y_1 = \sqrt{2x^2 - 8}$

$y_2 = -\sqrt{2x^2 - 8}$




(b) $y^2 = e^{-\int(1/x)dx} = e^{-\ln x + C} = e^{\ln(1/x)}(e^C) = \dfrac{1}{x}k$

Let $k = 4$ and graph $y^2 = \dfrac{4}{x}$ $\quad\begin{pmatrix} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x}\end{pmatrix}$




(c) In part (a), $2x^2 - y^2 = 8$

$4x - 2yy' = 0$

$$y' = \frac{2x}{y}.$$

In part (b), $y^2 = \dfrac{4}{x} = 4x^{-1}$

$$2yy' = \frac{-4}{x^2}$$

$$y' = \frac{-2}{yx^2} = \frac{-2y}{y^2x^2} = \frac{-2y}{4x} = \frac{-y}{2x}.$$

Using a graphing utility the graphs intersect at $(2.214, 1.344)$. The slopes are $3.295$ and $-0.304 = (-1)/3.295$, respectively.

**87.** False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$$

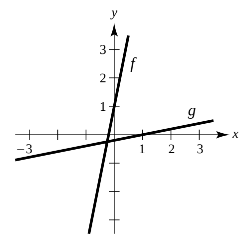
**89.** True

$$\int \frac{1}{x}\,dx = \ln|x| + C_1$$
$$= \ln|x| + \ln|C| = \ln|Cx|,\ C \neq 0$$

## Section 5.3 Inverse Functions

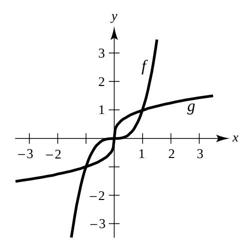
**1.** (a)
$$f(x) = 5x + 1$$
$$g(x) = \frac{x-1}{5}$$
$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$
$$g((f(x)) = g(5x+1) = \frac{(5x+1)-1}{5} = x$$

(b)



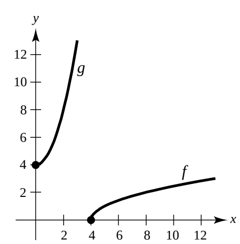

**3.** (a)
$$f(x) = x^3$$
$$g(x) = \sqrt[3]{x}$$
$$f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$$
$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



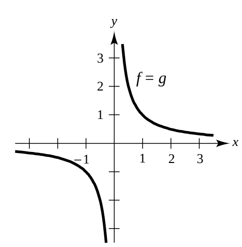

**5.** (a)
$$f(x) = \sqrt{x-4}$$
$$g(x) = x^2 + 4,\ x \geq 0$$
$$f(g(x)) = f(x^2+4)$$
$$= \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$
$$g(f(x)) = g\left(\sqrt{x-4}\right)$$
$$= \left(\sqrt{x-4}\right)^2 + 4 = x - 4 + 4 = x$$

(b)




**7.** (a)
$$f(x) = \frac{1}{x}$$
$$g(x) = \frac{1}{x}$$
$$f(g(x)) = \frac{1}{1/x} = x$$
$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

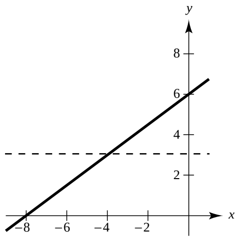



**9.** Matches (c)

**11.** Matches (a)

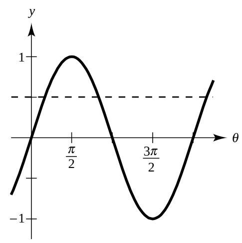
**13.** $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



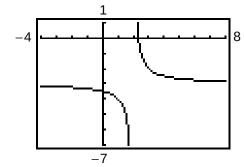

**15.** $f(\theta) = \sin \theta$

Not one-to-one; does not have an inverse



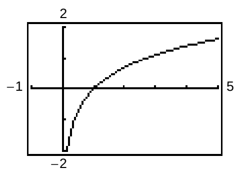

**17.** $h(s) = \dfrac{1}{s - 2} - 3$

One-to-one; has an inverse



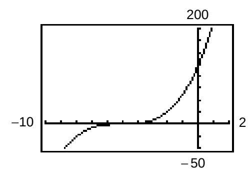

**19.** $f(x) = \ln x$

One-to-one; has an inverse




**21.** $g(x) = (x + 5)^3$

One-to-one; has an inverse




**23.** $f(x) = (x + a)^3 + b$

$f'(x) = 3(x + a)^2 \geq 0$ for all $x$.

$f$ is increasing on $(-\infty, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**25.** $f(x) = \dfrac{x^4}{4} - 2x^2$

$f'(x) = x^3 - 4x = 0$ when $x = 0, 2, -2$.

$f$ is not strictly monotonic on $(-\infty, \infty)$. Therefore, $f$ does not have an inverse.

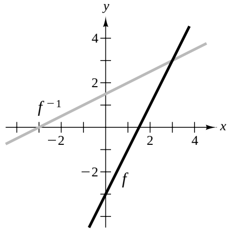
**27.** $f(x) = 2 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$ for all $x$.

$f$ is decreasing on $(-\infty, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

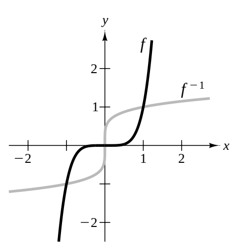
**29.**

$$f(x) = 2x - 3 = y$$
$$x = \frac{y + 3}{2}$$
$$y = \frac{x + 3}{2}$$
$$f^{-1}(x) = \frac{x + 3}{2}$$



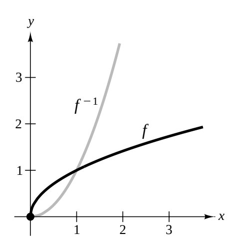

**31.**

$$f(x) = x^5 = y$$
$$x = \sqrt[5]{y}$$
$$y = \sqrt[5]{x}$$
$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$




**33.**

$$f(x) = \sqrt{x} = y$$
$$x = y^2$$
$$y = x^2$$
$$f^{-1}(x) = x^2, \quad x \geq 0$$

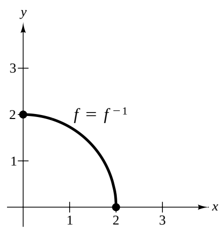


**35.** $f(x) = \sqrt{4 - x^2} = y, \quad 0 \le x \le 2$

$x = \sqrt{4 - y^2}$

$y = \sqrt{4 - x^2}$

$f^{-1}(x) = \sqrt{4 - x^2}, \ 0 \le x \le 2$

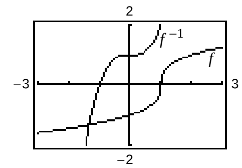



**37.** $f(x) = \sqrt[3]{x - 1} = y$

$x = y^3 + 1$

$y = x^3 + 1$

$f^{-1}(x) = x^3 + 1$



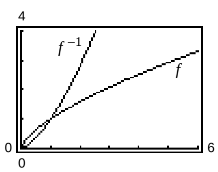

The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

**39.** $f(x) = x^{2/3} = y, \ x \ge 0$

$x = y^{3/2}$

$y = x^{3/2}$

$f^{-1}(x) = x^{3/2}, \ x \ge 0$



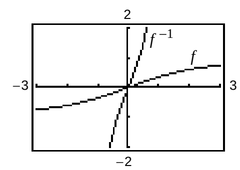

The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

**41.** $f(x) = \dfrac{x}{\sqrt{x^2 + 7}} = y$

$x = \dfrac{\sqrt{7}y}{\sqrt{1 - y^2}}$

$y = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}$

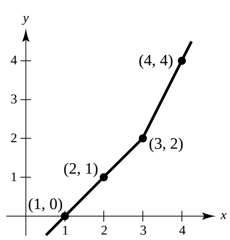
$f^{-1}(x) = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}, \ -1 < x < 1$




The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

**43.**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f^{-1}(x)$ | 0 | 1 | 2 | 4 |




**45.** (a) Let $x$ be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is $50 - x$. The total cost is

$$y = 1.25x + 1.60(50 - x)$$
$$= -0.35x + 80 \qquad 0 \le x \le 50.$$

(b) We find the inverse of the original function:

$$y = -0.35x + 80$$
$$0.35x = 80 - y$$
$$x = \tfrac{100}{35}(80 - y)$$

Inverse: $y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x)$.

$x$ represents cost and $y$ represents pounds.

(c) Domain of inverse is $62.5 \le x \le 80$.

(d) If $x = 73$ in the inverse function, $y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20$ pounds.

**47.** $f(x) = (x - 4)^2$ on $[4, \infty)$

$f'(x) = 2(x - 4) > 0$ on $(4, \infty)$

$f$ is increasing on $[4, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**49.** $f(x) = \dfrac{4}{x^2}$ on $(0, \infty)$

$f'(x) = -\dfrac{8}{x^3} < 0$ on $(0, \infty)$

$f$ is decreasing on $(0, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**51.** $f(x) = \cos x$ on $[0, \pi]$

$f'(x) = -\sin x < 0$ on $(0, \pi)$

$f$ is decreasing on $[0, \pi]$. Therefore, $f$ is strictly monotonic and has an inverse.

**53.** $f(x) = \dfrac{x}{x^2 - 4} = y$ on $(-2, 2)$

$x^2y - 4y = x$

$x^2y - x - 4y = 0$

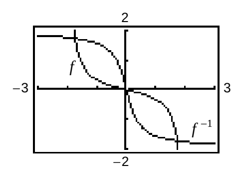
$a = y, b = -1, c = -4y$

$x = \dfrac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \dfrac{1 \pm \sqrt{1 + 16y^2}}{2y}$

$y = f^{-1}(x) = \begin{cases} \left(1 - \sqrt{1 + 16x^2}\right)/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

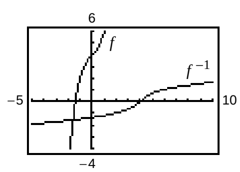
Domain: all $x$

Range: $-2 < y < 2$



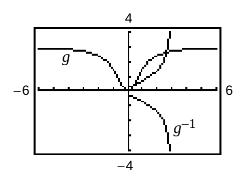

The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

**55.** (a), (b)




(c) Yes, $f$ is one-to-one and has an inverse. The inverse relation is an inverse function.

**57.** (a), (b)




(c) $g$ is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

**59.** $f(x) = \sqrt{x - 2}$, Domain: $x \geq 2$

$f'(x) = \dfrac{1}{2\sqrt{x - 2}} > 0$ for $x > 2$.

$f$ is one-to-one; has an inverse

$\sqrt{x - 2} = y$

$x - 2 = y^2$

$x = y^2 + 2$

$y = x^2 + 2$

$f^{-1}(x) = x^2 + 2, x \geq 0$

**61.** $f(x) = |x - 2|, x \leq 2$

$= -(x - 2)$

$= 2 - x$

$f$ is one-to-one; has an inverse

$2 - x = y$

$2 - y = x$

$f^{-1}(x) = 2 - x, \; x \geq 0$

**63.** $f(x) = (x - 3)^2$ is one-to-one for $x \geq 3$.

$(x - 3)^2 = y$

$x - 3 = \sqrt{y}$

$x = \sqrt{y} + 3$

$y = \sqrt{x} + 3$

$f^{-1}(x) = \sqrt{x} + 3, \; x \geq 0$

(Answer is not unique)

**65.** $f(x) = |x + 3|$ is one-to-one for $x \geq -3$.

$x + 3 = y$

$x = y - 3$

$y = x - 3$

$f^{-1}(x) = x - 3, \; x \geq 0$

(Answer is not unique)

**67.** Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time $t$ corresponding to the volume $V$.

**69.** No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

**71.**
$$f(x) = x^3 + 2x - 1,\ f(1) = 2 = a$$
$$f'(x) = 3x^2 + 2$$
$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

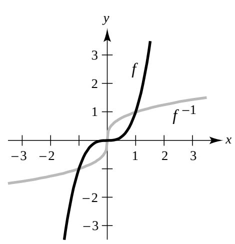
**73.**
$$f(x) = \sin x,\ f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$$
$$f'(x) = \cos x$$
$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)} = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

**75.**
$$f(x) = x^3 - \frac{4}{x},\ f(2) = 6 = a$$
$$f'(x) = 3x^2 + \frac{4}{x^2}$$
$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13}$$

**77.** (a) Domain $f$ = Domain $f^{-1} = (-\infty, \infty)$

(b) Range $f$ = Range $f^{-1} = (-\infty, \infty)$

(c)

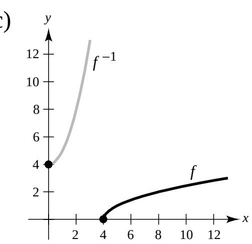


(d)
$$f(x) = x^3,\ \left(\frac{1}{2}, \frac{1}{8}\right)$$
$$f'(x) = 3x^2$$
$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$
$$f^{-1}(x) = \sqrt[3]{x},\ \left(\frac{1}{8}, \frac{1}{2}\right)$$
$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$
$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

**79.** (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)



(d)
$$f(x) = \sqrt{x - 4},\ (5, 1)$$
$$f'(x) = \frac{1}{2\sqrt{x - 4}}$$
$$f'(5) = \frac{1}{2}$$
$$f^{-1}(x) = x^2 + 4,\ (1, 5)$$
$$(f^{-1})'(x) = 2x$$
$$(f^{-1})'(1) = 2$$

**81.** $x = y^3 - 7y^2 + 2$

$$1 = 3y^2\frac{dy}{dx} - 14y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}. \text{ At } (-4, 1), \frac{dy}{dx} = \frac{1}{3 - 14} = \frac{-1}{11}.$$

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.

Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

Hence, $\frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}$.

**In Exercises 83 and 85, use the following.**

$f(x) = \frac{1}{8}x - 3$ **and** $g(x) = x^3$

$f^{-1}(x) = 8(x + 3)$ **and** $g^{-1}(x) = \sqrt[3]{x}$

**83.** $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

**85.** $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

**In Exercises 87 and 89, use the following.**

$f(x) = x + 4$ **and** $g(x) = 2x - 5$

$f^{-1}(x) = x - 4$ **and** $g^{-1}(x) = \frac{x + 5}{2}$

**87.**
$$\begin{aligned}(g^{-1} \circ f^{-1})(x) &= g^{-1}(f^{-1}(x))\\ &= g^{-1}(x - 4)\\ &= \frac{(x - 4) + 5}{2}\\ &= \frac{x + 1}{2}\end{aligned}$$

**89.**
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(2x - 5)\\ &= (2x - 5) + 4\\ &= 2x - 1\end{aligned}$$

Hence, $(f \circ g)^{-1}(x) = \frac{x + 1}{2}$

(Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$)

**91.** Answers will vary. See page 335 and Example 3.

**93.** $y = x^2$ on $(-\infty, \infty)$ does not have an inverse.

**95.** $f$ is not one-to-one because many different $x$-values yield the same $y$-value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\frac{(2n - 1)\pi}{2}$, where $n$ is an integer

**97.** Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$$\begin{aligned}(f \circ g)(x) &= y\\ f(g(x)) &= y\\ g(x) &= f^{-1}(y)\\ x &= g^{-1}(f^{-1}(y))\\ &= (g^{-1} \circ f^{-1})(y)\end{aligned}$$

Since $f$ and $g$ are one-to-one functions, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

**99.** Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point $(a, b)$ if and only if the graphs of $g(x)$ and $h(x)$ contain the point $(b, a)$. Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

**101.** False

Let $f(x) = x^2$.

**103.** True

**105.** Not true

Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - x, & 1 < x \le 2 \end{cases}$.

$f$ is one-to-one, but not strictly monotonic.

**107.**
$$f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}},\ f(2) = 0$$
$$f'(x) = \frac{1}{\sqrt{1+x^4}}$$
$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

## Section 5.4 Exponential Functions: Differentiation and Integration

**1.** $e^0 = 1$

$\ln 1 = 0$

**3.** $\ln 2 = 0.6931$

$e^{0.6931\ldots} = 2$

**5.** $e^{\ln x} = 4$

$x = 4$

**7.** $e^x = 12$

$x = \ln 12 \approx 2.485$

**9.** $9 - 2e^x = 7$

$2e^x = 7$

$e^x = 1$

$x = 0$

**11.** $50e^{-x} = 30$

$$e^{-x} = \frac{3}{5}$$
$$-x = \ln\left(\frac{3}{5}\right)$$
$$x = \ln\left(\frac{5}{3}\right) \approx 0.511$$

**13.** $\ln x = 2$

$x = e^2 \approx 7.3891$

**15.** $\ln(x-3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

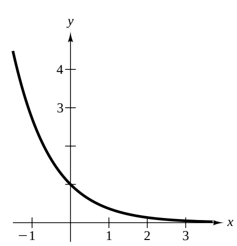
**17.** $\ln\sqrt{x+2} = 1$

$\sqrt{x+2} = e^1 = e$

$x + 2 = e^2$

$x = e^2 - 2 \approx 5.389$

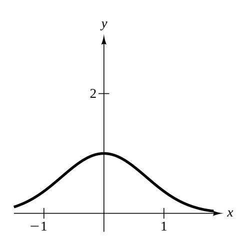
**19.** $y = e^{-x}$



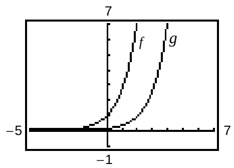

**21.** $y = e^{-x^2}$

Symmetric with respect to the $y$-axis

Horizontal asymptote: $y = 0$

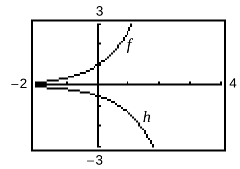


**23.** (a)



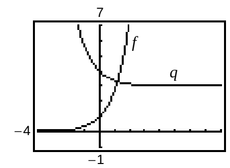

Horizontal shift 2 units to the right

(b)




A reflection in the $x$-axis and a vertical shrink

(c)




Vertical shift 3 units upward and a reflection in the $y$-axis

**25.** $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

**27.** $y = C(1 - e^{-ax})$

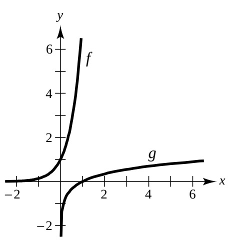
Vertical shift $C$ units

Reflection in both the $x$- and $y$-axes

Matches (a)

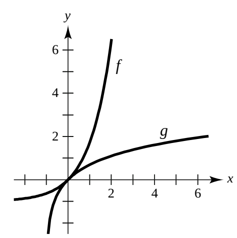
**29.** $f(x) = e^{2x}$

$g(x) = \ln\sqrt{x} = \dfrac{1}{2}\ln x$

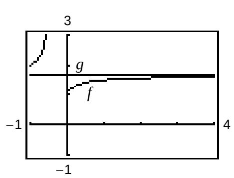



**31.** $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$




**33.**




As $x \to \infty$, the graph of $f$ approaches the graph of $g$.

$$\lim_{x\to\infty}\left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

**35.** $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000} \approx 2.718280469$

$$e \approx 2.718281828$$

$$e > \left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$$

**37.** (a) $y = e^{3x}$

$y' = 3e^{3x}$

At $(0, 1)$, $y' = 3$.

(b) $y = e^{-3x}$

$y' = -3e^{-3x}$

At $(0, 1)$, $y' = -3$.

**39.** $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

**41.** $f(x) = e^{-2x+x^2}$

$\dfrac{dy}{dx} = 2(x - 1)e^{-2x+x^2}$

**43.** $y = e^{\sqrt{x}}$

$\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

**45.** $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$

**47.** $y = \ln e^{x^2} = x^2$

$\dfrac{dy}{dx} = 2x$

**49.** $y = \ln(1 + e^{2x})$

$\dfrac{dy}{dx} = \dfrac{2e^{2x}}{1 + e^{2x}}$

**51.** $y = \dfrac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$

$\dfrac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x})$

$= \dfrac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$

**53.** $y = x^2e^x - 2xe^x + 2e^x = e^x(x^2 - 2x + 2)$

$\dfrac{dy}{dx} = e^x(2x - 2) + e^x(x^2 - 2x + 2) = x^2e^x$

**55.** $f(x) = e^{-x}\ln x$

$f'(x) = e^{-x}\left(\dfrac{1}{x}\right) - e^{-x}\ln x = e^{-x}\left(\dfrac{1}{x} - \ln x\right)$

**57.** $y = e^x(\sin x + \cos x)$

$\dfrac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)(e^x)$

$= e^x(2\cos x) = 2e^x\cos x$

**59.** $xe^y - 10x + 3y = 0$

$xe^y\dfrac{dy}{dx} + e^y - 10 + 3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(xe^y + 3) = 10 - e^y$

$\dfrac{dy}{dx} = \dfrac{10 - e^y}{xe^y + 3}$

**61.** $f(x) = (3 + 2x)e^{-3x}$

$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$

$= (-7 - 6x)e^{-3x}$

$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$

$= 3(6x + 5)e^{-3x}$

**63.** $y = e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$

$y' = e^x\left(-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x\right) + e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$

$= e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$

$y'' = e^x\left[-\left(\sqrt{2} + 2\right)\sin\sqrt{2}x + \left(\sqrt{2} - 2\right)\cos\sqrt{2}x\right] + e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$

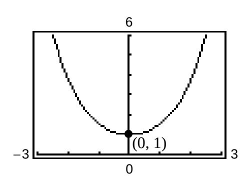
$= e^x\left[\left(-1 - 2\sqrt{2}\right)\sin\sqrt{2}x + \left(-1 + 2\sqrt{2}\right)\cos\sqrt{2}x\right]$

$-2y' + 3y = -2e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right] + 3e^x\left[\cos\sqrt{2}x + \sin\sqrt{2}x\right]$

$= e^x\left[\left(1 - 2\sqrt{2}\right)\cos\sqrt{2}x + \left(1 + 2\sqrt{2}\right)\sin\sqrt{2}x\right] = -y''$

Therefore, $-2y' + 3y = -y'' \implies y'' - 2y' + 3y = 0.$

**65.** $f(x) = \dfrac{e^x + e^{-x}}{2}$

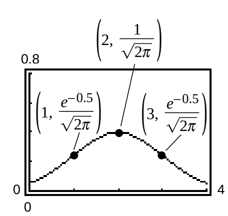



$f'(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

$f''(x) = \dfrac{e^x + e^{-x}}{2} > 0$

Relative minimum: $(0, 1)$

**67.** $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$



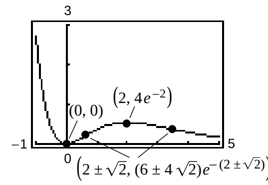

$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x - 2)e^{-(x-2)^2/2}$

$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x - 1)(x - 3)e^{-(x-2)^2/2}$

Relative maximum: $\left(2, \dfrac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(3, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$

**69.** $f(x) = x^2e^{-x}$




$$f'(x) = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2 - x) = 0 \text{ when } x = 0, 2.$$

$$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$$

$$= e^{-x}(x^2 - 4x + 2) = 0 \text{ when } x = 2 \pm \sqrt{2}.$$

Relative minimum: $(0, 0)$

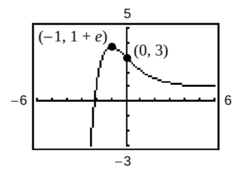
Relative maximum: $(2, 4e^{-2})$

$$x = 2 \pm \sqrt{2}$$

$$y = (2 \pm \sqrt{2})^2e^{-(2 \pm \sqrt{2})}$$

Points of inflection: $(3.414, 0.384), (0.586, 0.191)$

**71.** $g(t) = 1 + (2 + t)e^{-t}$



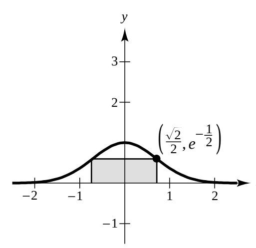

$$g'(t) = (1 + t)e^{-t}$$

$$g''(t) = te^{-t}$$

Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$

Point of inflection: $(0, 3)$

**73.** $A = (\text{base})(\text{height}) = 2xe^{-x^2}$




$$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$$A = \sqrt{2}e^{-1/2}$$

**75.** $y = \dfrac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2(1 + ae^{-x/b})\left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

$$= \frac{Lae^{-x/b}[ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Longrightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Longrightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the $y$-coordinate of the inflection point is $L/2$.

**77.** $e^{-x} = x \implies f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$x_1 = 1$

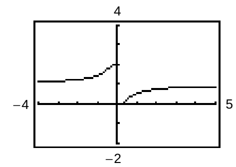
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

We approximate the root of $f$ to be $x = 0.567$.

**79.** (a)

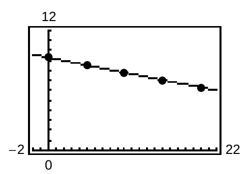



(b) When $x$ increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As $x$ approaches zero from the right, $1/x$ approaches $\infty$, $e^{1/x}$ approaches $\infty$ and $f(x)$ approaches zero. As $x$ approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

**81.**

| $h$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $P$ | 10,332 | 5,583 | 2,376 | 1,240 | 517 |
| $\ln P$ | 9.243 | 8.627 | 7.773 | 7.123 | 6.248 |

(a)




$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.

(b) $\ln P = ah + b$

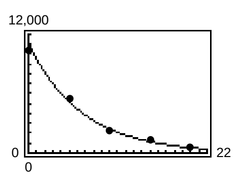
$P = e^{ah+b} = e^b e^{ah}$

$P = Ce^{ah}$, $C = e^b$

For our data, $a = -0.1499$ and $C = e^{9.3018} = 10{,}957.7$

$P = 10{,}957.7e^{-0.1499h}$

(c)




(d) $\dfrac{dP}{dh} = (10{,}957.71)(-0.1499)e^{-0.1499h}$

$= -1642.56e^{-0.1499h}$

For $h = 5$, $\dfrac{dP}{dh} = -776.3$. For $h = 18$, $\dfrac{dP}{dh} \approx -110.6$.

**83.** $f(x) = e^{x/2}$, $f(0) = 1$

$f'(x) = \frac{1}{2}e^{x/2}$, $f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{4}e^{x/2}$, $f''(0) = \frac{1}{4}$

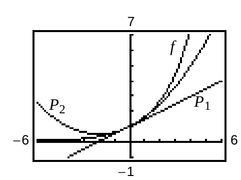
$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1$, $P_1(0) = 1$

$P_1'(x) = \frac{1}{2}$, $P_1'(0) = \frac{1}{2}$

$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 = \frac{x^2}{8} + \frac{x}{2} + 1$, $P_2(0) = 1$

$P_2'(x) = \frac{1}{4}x + \frac{1}{2}$, $P_2'(0) = \frac{1}{2}$

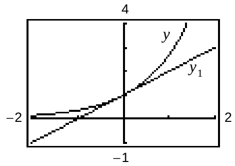
$P_2''(x) = \frac{1}{4}$, $P_2''(0) = \frac{1}{4}$




The values of $f$, $P_1$, $P_2$ and their first derivatives agree at $x = 0$. The values of the second derivatives of $f$ and $P_2$ agree at $x = 0$.

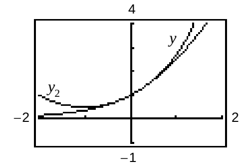
**85.** (a) $y = e^x$

$y_1 = 1 + x$



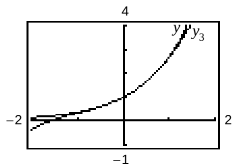

(b) $y = e^x$

$y_2 = 1 + x + \left(\dfrac{x^2}{2}\right)$




(c) $y = e^x$

$y_3 = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$




**87.** Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}\,5dx = e^{5x} + C$$

**89.** Let $u = -2x$, $du = -2\,dx$.

$$\int_0^1 e^{-2x}dx = -\frac{1}{2}\int_0^1 e^{-2x}(-2)\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= \frac{1}{2}(1 - e^{-2}) = \frac{e^2 - 1}{2e^2}$$

**91.** $\displaystyle\int xe^{-x^2}dx = -\frac{1}{2}\int e^{-x^2}(-2x)dx = -\frac{1}{2}e^{-x^2} + C$

**93.** $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2e^{\sqrt{x}} + C$

**95.** Let $u = 1 + e^{-x}$, $du = -e^{-x}\,dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}}\,dx = -\int \frac{-e^{-x}}{1 + e^{-x}}\,dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

**97.** Let $u = \dfrac{3}{x}$, $du = -\dfrac{3}{x^2}\,dx$.

$$\int_1^3 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_1^3 e^{3/x}\left(-\frac{3}{x^2}\right)dx$$
$$= \left[-\frac{1}{3}e^{3/x}\right]_1^3 = \frac{e}{3}(e^2 - 1)$$

**99.** Let $u = 1 - e^x$, $du = -e^x\,dx$.

$$\int e^x\sqrt{1 - e^x}\,dx = -\int (1 - e^x)^{1/2}(-e^x)\,dx$$
$$= -\frac{2}{3}(1 - e^x)^{3/2} + C$$

**101.** Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x})\,dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln|e^x - e^{-x}| + C$$

**103.** $\displaystyle\int \frac{5 - e^x}{e^{2x}}\,dx = \int 5e^{-2x}\,dx - \int e^{-x}\,dx$

$$= -\frac{5}{2}e^{-2x} + e^{-x} + C$$

**105.** $\displaystyle\int e^{\sin \pi x}\cos \pi x\,dx = \frac{1}{\pi}\int e^{\sin \pi x}(\pi \cos \pi x)\,dx$

$$= \frac{1}{\pi}e^{\sin \pi x} + C$$

**107.** $\displaystyle\int e^{-x}\tan(e^{-x})\,dx = -\int [\tan(e^{-x})](-e^{-x})\,dx$

$$= \ln|\cos(e^{-x})| + C$$

**109.** Let $u = ax^2$, $du = 2ax\,dx$. (Assume $a \neq 0$)

$$y = \int xe^{ax^2}\,dx$$

$$= \frac{1}{2a}\int e^{ax^2}(2ax)\,dx = \frac{1}{2a}e^{ax^2} + C$$

**111.** $f'(x) = \int \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}(e^x - e^{-x}) + C_1$

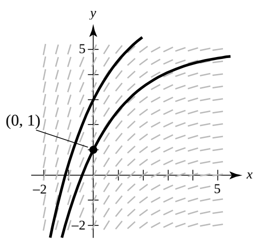
$f'(0) = C_1 = 0$

$f(x) = \int \frac{1}{2}(e^x - e^{-x})\,dx = \frac{1}{2}(e^x + e^{-x}) + C_2$

$f(0) = 1 + C_2 = 1 \implies C_2 = 0$

$f(x) = \frac{1}{2}(e^x + e^{-x})$

**113.** (a)



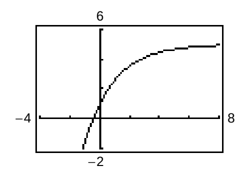

(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2}\,dx = -4\int e^{-x/2}\left(-\frac{1}{2}\,dx\right)$$

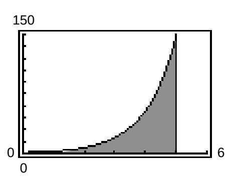
$$= -4e^{-x/2} + C$$

$(0, 1)$: $1 = -4e^0 + C = -4 + C \implies C = 5$

$y = -4e^{-x/2} + 5$

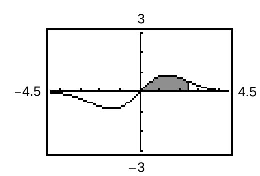



**115.** $\int_0^5 e^x\,dx = \Big[e^x\Big]_0^5 = e^5 - 1 \approx 147.413$




**117.** $\int_0^{\sqrt{6}} xe^{-x^2/4}dx = \Big[-2e^{-x^2/4}\Big]_0^{\sqrt{6}}$

$$= -2e^{-3/2} + 2 \approx 1.554$$




**119.** (a) $f(u - v) = e^{u-v} = (e^u)(e^{-v}) = \frac{e^u}{e^v} = \frac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$.

**121.** $0.0665\int_{48}^{60} e^{-0.0139(t-48)^2}\,dt$

Graphing Utility: $0.4772 = 47.72\%$

**123.** $\int_0^x e^t\,dt \geq \int_0^x 1\,dt$

$\Big[e^t\Big]_0^x \geq \Big[t\Big]_0^x$

$e^x - 1 \geq x \implies e^x \geq 1 + x$ for $x \geq 0$

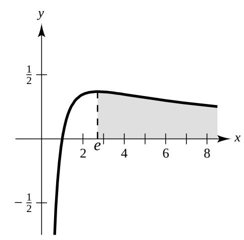
**125.** $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. $f$ is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

**127.** Yes. $f(x) = Ce^x$, $C$ a constant.

**129.** $e^{-x} > 0 \implies \int_0^2 e^{-x}dx > 0$.

**131.** $f(x) = \dfrac{\ln x}{x}$



(a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e)$, $f'(x) > 0 \implies f$ is increasing.

On $(e, \infty)$, $f'(x) < 0 \implies f$ is decreasing.

(b) For $e \le A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$
$$B \ln A > A \ln B$$
$$\ln A^B > \ln B^A$$
$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.

## Section 5.5 Bases Other than $e$ and Applications

**1.** $y = \left(\dfrac{1}{2}\right)^{t/3}$

At $t_0 = 6$, $y = \left(\dfrac{1}{2}\right)^{6/3} = \dfrac{1}{4}$

**3.** $y = \left(\dfrac{1}{2}\right)^{t/7}$

At $t_0 = 10$, $y = \left(\dfrac{1}{2}\right)^{10/7} \approx 0.3715$

**5.** $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

**7.** $\log_7 1 = 0$

**9.** (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \dfrac{1}{3}$

$\log_3 \dfrac{1}{3} = -1$

**11.** (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

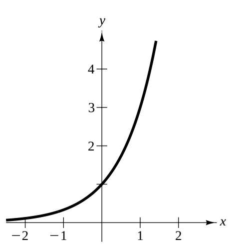
(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

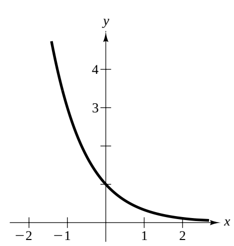
**13.** $y = 3^x$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 |



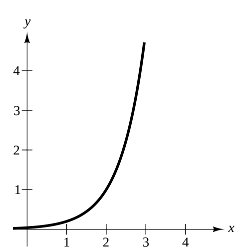
**15.** $y = \left(\dfrac{1}{3}\right)^x = 3^{-x}$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 9 | 3 | 1 | $\frac{1}{3}$ | $\frac{1}{9}$ |



**17.** $h(x) = 5^{x-2}$

| $x$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | $\frac{1}{125}$ | $\frac{1}{25}$ | $\frac{1}{5}$ | 1 | 5 |



**19.** (a) $\log_{10} 1000 = x$

$$10^x = 1000$$
$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$
$$x = -1$$

**21.** (a) $\log_3 x = -1$

$$3^{-1} = x$$
$$x = \tfrac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$
$$x = \tfrac{1}{16}$$

**23.** (a)

$$x^2 - x = \log_5 25$$
$$x^2 - x = \log_5 5^2 = 2$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1 \text{ OR } x = 2$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$
$$3x = 1$$
$$x = \tfrac{1}{3}$$

**25.** $3^{2x} = 75$

$$2x \ln 3 = \ln 75$$
$$x = \frac{1}{2}\frac{\ln 75}{\ln 3} \approx 1.965$$

**27.** $2^{3-x} = 625$

$$(3 - x)\ln 2 = \ln 625$$
$$3 - x = \frac{\ln 625}{\ln 2}$$
$$x = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

**29.** $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$
$$t = \frac{1}{12}\frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

**31.** $\log_2(x - 1) = 5$

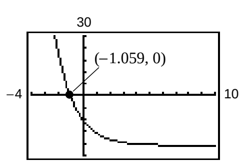
$$x - 1 = 2^5 = 32$$
$$x = 33$$

**33.** $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$
$$x = \pm\sqrt{3^{4.5}} \approx \pm 11.845$$

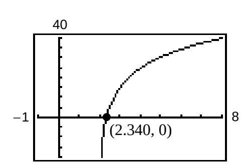
**35.** $g(x) = 6(2^{1-x}) - 25$

Zero: $x \approx -1.059$




**37.** $h(s) = 32 \log_{10}(s - 2) + 15$

Zero: $s \approx 2.340$

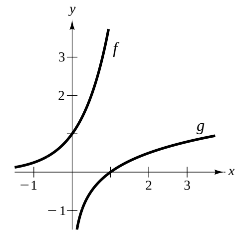


**39.** $f(x) = 4^x$

$g(x) = \log_4 x$

| $x$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{16}$ | $\frac{1}{4}$ | $1$ | $2$ | $4$ |

| $x$ | $\frac{1}{16}$ | $\frac{1}{4}$ | $1$ | $2$ | $4$ |
|---|---|---|---|---|---|
| $g(x)$ | $-2$ | $-1$ | $0$ | $\frac{1}{2}$ | $1$ |




**41.** $f(x) = 4^x$

$f'(x) = (\ln 4)\, 4^x$

**43.** $y = 5^{x-2}$

$$\frac{dy}{dx} = (\ln 5)\, 5^{x-2}$$

**45.** $g(t) = t^2 2^t$

$$g'(t) = t^2 (\ln 2)\, 2^t + (2t)\, 2^t$$
$$= t\,2^t (t \ln 2 + 2)$$
$$= 2^t t(2 + t \ln 2)$$

**47.** $h(\theta) = 2^{-\theta} \cos \pi\theta$

$$h'(\theta) = 2^{-\theta}(-\pi \sin \pi\theta) - (\ln 2)2^{-\theta} \cos \pi\theta$$
$$= -2^{-\theta}[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta]$$

**49.** $y = \log_3 x$

$$\frac{dy}{dx} = \frac{1}{x \ln 3}$$

**51.** $f(x) = \log_2 \dfrac{x^2}{x-1}$

$$= 2 \log_2 x - \log_2 (x - 1)$$
$$f'(x) = \frac{2}{x \ln 2} - \frac{1}{(x-1) \ln 2}$$
$$= \frac{x-2}{(\ln 2)x(x-1)}$$

**53.** $y = \log_5 \sqrt{x^2 - 1} = \dfrac{1}{2} \log_5 (x^2 - 1)$

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{(x^2-1)\ln 5} = \frac{x}{(x^2-1)\ln 5}$$

**55.** $g(t) = \dfrac{10 \log_4 t}{t} = \dfrac{10}{\ln 4}\left(\dfrac{\ln t}{t}\right)$

$$g'(t) = \frac{10}{\ln 4}\left[\frac{t(1/t) - \ln t}{t^2}\right]$$
$$= \frac{10}{t^2 \ln 4}[1 - \ln t] = \frac{5}{t^2 \ln 2}(1 - \ln t)$$

**57.** $y = x^{2/x}$

$$\ln y = \frac{2}{x} \ln x$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) = \frac{2}{x^2}(1 - \ln x)$$
$$\frac{dy}{dx} = \frac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$$

**59.** $y = (x-2)^{x+1}$

$$\ln y = (x+1) \ln(x-2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x+1)\left(\frac{1}{x-2}\right) + \ln(x-2)$$
$$\frac{dy}{dx} = y\left[\frac{x+1}{x-2} + \ln(x-2)\right]$$
$$= (x-2)^{x+1}\left[\frac{x+1}{x-2} + \ln(x-2)\right]$$

**61.** $\displaystyle\int 3^x\, dx = \frac{3^x}{\ln 3} + C$

**63.** $\displaystyle\int_{-1}^{2} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2}$

$\displaystyle= \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right]$

$\displaystyle= \frac{7}{2\ln 2} = \frac{7}{\ln 4}$

**65.** $\displaystyle\int x\,5^{-x^2}\,dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\,dx$

$\displaystyle= -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$

$\displaystyle= \frac{-1}{2\ln 5}\left(5^{-x^2}\right) + C$

**67.** $\displaystyle\int \frac{3^{2x}}{1+3^{2x}}\,dx,\ u = 1 + 3^{2x},\ du = 2(\ln 3)3^{2x}\,dx$

$\displaystyle\frac{1}{2\ln 3}\int \frac{(2\ln 3)3^{2x}}{1+3^{2x}}\,dx = \frac{1}{2\ln 3}\ln(1+3^{2x}) + C$

**69.** $\displaystyle\frac{dy}{dx} = 0.4^{x/3},\ \left(0, \frac{1}{2}\right)$

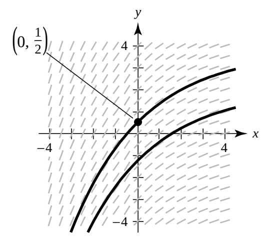
$\displaystyle y = \int 0.4^{x/3}\,dx = 3\int 0.4^{x/3}\left(\frac{1}{3}\,dx\right)$

$\displaystyle= \frac{3}{\ln 0.4}0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C$

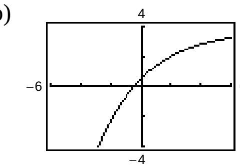
$\displaystyle y = 3\ln 2.5(0.4)^{x/3} + \frac{1}{2} - 3\ln 2.5$

$\displaystyle= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}$

(a)

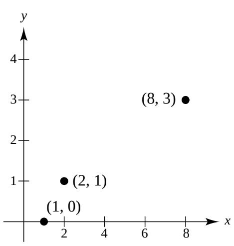



(b)




**71.** Answers will vary. Example: Growth and decay problems.

**73.**




| $x$ | 1 | 2 | 8 |
|---|---|---|---|
| $y$ | 0 | 1 | 3 |

(a) $y$ is an exponential function of $x$: False

(b) $y$ is a logarithmic function of $x$: True; $y = \log_2 x$

(c) $x$ is an exponential function of $y$: True, $2^y = x$

(d) $y$ is a linear function of $x$: False

**75.** $\displaystyle f(x) = \log_2 x \implies f'(x) = \frac{1}{x\ln 2}$

$g(x) = x^x \implies g'(x) = x^x(1 + \ln x)$

[**Note:** Let $y = g(x)$. Then: $\ln y = \ln x^x = x\ln x$

$\displaystyle\frac{1}{y}y' = x\cdot\frac{1}{x} + \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x) = g'(x)$.]

$h(x) = x^2 \implies h'(x) = 2x$

$k(x) = 2^x \implies k'(x) = (\ln 2)2^x$

From greatest to smallest rate of growth:
$g(x), k(x), h(x), f(x)$

**77.** $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$

$\approx \$40.64$

(b) $\displaystyle\frac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$: $\displaystyle\frac{dC}{dt} \approx 0.051P$

When $t = 8$: $\displaystyle\frac{dC}{dt} \approx 0.072P$

(c) $\displaystyle\frac{dC}{dt} = (\ln 1.05)[P(1.05)^t]$

$= (\ln 1.05)C(t)$

The constant of proportionality is $\ln 1.05$.

**79.** $P = \$1000,\ r = 3\frac{1}{2}\% = 0.035,\ t = 10$

$A = 1000\left(1 + \dfrac{0.035}{n}\right)^{10n}$

$A = 1000e^{(0.035)(10)} = 1419.07$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | 1410.60 | 1414.78 | 1416.91 | 1418.34 | 1419.04 | 1419.07 |

**81.** $P = \$1000,\ r = 5\% = 0.05,\ t = 30$

$A = 1000\left(1 + \dfrac{0.05}{n}\right)^{30n}$

$A = 1000e^{(0.05)30} = 4481.69$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | 4321.94 | 4399.79 | 4440.21 | 4467.74 | 4481.23 | 4481.69 |

**83.** $100{,}000 = Pe^{0.05t} \implies P = 100{,}000e^{-0.05t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | 95,122.94 | 60,653.07 | 36,787.94 | 22,313.02 | 13,583.53 | 8208.50 |

**85.** $100{,}000 = P\left(1 + \dfrac{0.05}{12}\right)^{12t} \implies P = 100{,}000\left(1 + \dfrac{0.05}{12}\right)^{-12t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | 95,132.82 | 60,716.10 | 36,864.45 | 22,382.66 | 13,589.88 | 8251.24 |

**87.** (a) $A = 20{,}000\left(1 + \dfrac{0.06}{365}\right)^{(365)(8)} \approx \$32{,}320.21$

(b) $A = \$30{,}000$

(c) $A = 8000\left(1 + \dfrac{0.06}{365}\right)^{(365)(8)} + 20{,}000\left(1 + \dfrac{0.06}{365}\right)^{(365)(4)}$

$\approx \$12{,}928.09 + 25{,}424.48 = \$38{,}352.57$

(d) $A = 9000\left[\left(1 + \dfrac{0.06}{365}\right)^{(365)(8)} + \left(1 + \dfrac{0.06}{365}\right)^{(365)(4)} + 1\right]$

$\approx \$34{,}985.11$

Take option (c).

**89.** (a) $\lim_{t\to\infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft$^3$

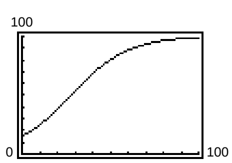
(b) $V' = \dfrac{322.27}{t^2}\, e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft$^3$/yr

$V'(60) \approx 0.040$ million ft$^3$/yr

**91.** $y = \dfrac{300}{3 + 17e^{-0.0625x}}$

(a)




(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

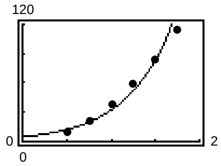
(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$y' = \dfrac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$

$y'' = \dfrac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$

$17e^{-0.0625x} - 3 = 0 \implies x \approx 27.8$ or 27,800 egg masses.

**93.** (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$

(b) 


(c) $B'(d) = 9.0952e^{1.9132d}$

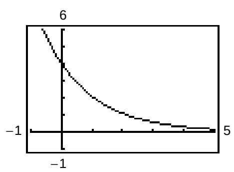
$B'(0.8) \approx 42.03$ tons/inch

$B'(1.5) \approx 160.38$ tons/inch

**95.** (a) $\displaystyle\int_0^4 f(t)\,dt \approx 5.67$

$\displaystyle\int_0^4 g(t)\,dt \approx 5.67$

$\displaystyle\int_0^4 h(t)\,dt \approx 5.67$

(b) 


(c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t = 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t = 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

**97.** $\displaystyle P = \int_0^{10} 2000e^{-0.06t}\,dt$

$$= \left[\frac{2000}{-0.06}e^{-0.06t}\right]_0^{10}$$

$\approx \$15,039.61$

**99.**

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 1200 | 720 | 432 | 259.20 | 155.52 |

$y = C(k^t)$

When $t = 0, y = 1200 \implies C = 1200.$

$y = 1200(k^t)$

$$\frac{720}{1200} = 0.6,\ \frac{432}{720} = 0.6,\ \frac{259.20}{432} = 0.6,\ \frac{155.52}{259.20} = 0.6$$

Let $k = 0.6.$

$y = 1200(0.6)^t$

**101.** False. $e$ is an irrational number.

**103.** True.

$$\begin{aligned} f(g(x)) &= 2 + e^{\ln(x-2)} \\ &= 2 + x - 2 = x \\ g(f(x)) &= \ln(2 + e^x - 2) \\ &= \ln e^x = x \end{aligned}$$

**105.** True.

$\dfrac{d}{dx}[e^x] = e^x$ and $\dfrac{d}{dx}[e^{-x}] = -e^{-x}$

$e^x = e^{-x}$ when $x = 0.$

$(e^0)(-e^{-0}) = -1$

**107.** $$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}\,dt \implies \frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int \frac{8}{25}\,dt \implies$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1 e^{(2/5)t}$$

$$y(0) = 1 \implies C_1 = 4 \implies 4e^{(2/5)t} = \frac{y}{(5/4) - y}$$

$$\implies 4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \implies 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$\implies y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

## Section 5.6 Differential Equations: Growth and Decay

**1.** $$\frac{dy}{dx} = x + 2$$

$$y = \int (x + 2)dx = \frac{x^2}{2} + 2x + C$$

**3.** $$\frac{dy}{dx} = y + 2$$

$$\frac{dy}{y + 2} = dx$$

$$\int \frac{1}{y + 2}dy = \int dx$$

$$\ln|y + 2| = x + C_1$$

$$y + 2 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 2$$

**5.** $$y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy'\,dx = \int 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

**7.** $$y' = \sqrt{x}\,y$$

$$\frac{y'}{y} = \sqrt{x}$$

$$\int \frac{y'}{y}\,dx = \int \sqrt{x}\,dx$$

$$\int \frac{dy}{y} = \int \sqrt{x}\,dx$$

$$\ln y = \frac{2}{3}x^{3/2} + C_1$$

$$y = e^{(2/3)x^{3/2} + C_1}$$

$$= e^{C_1}\,e^{(2/3)x^{3/2}}$$

$$= Ce^{(2/3)x^{3/2}}$$

**9.** $(1 + x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{1 + x^2}$$
$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$
$$\int \frac{y'}{y}\,dx = \int \frac{2x}{1 + x^2}\,dx$$
$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2}\,dx$$
$$\ln y = \ln(1 + x^2) + C_1$$
$$\ln y = \ln(1 + x^2) + \ln C$$
$$\ln y = \ln C(1 + x^2)$$
$$y = C(1 + x^2)$$

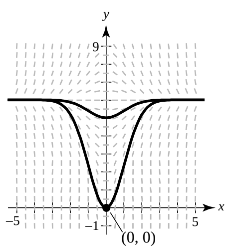
**11.** $\dfrac{dQ}{dt} = \dfrac{k}{t^2}$

$$\int \frac{dQ}{dt}\,dt = \int \frac{k}{t^2}\,dt$$
$$\int dQ = -\frac{k}{t} + C$$
$$Q = -\frac{k}{t} + C$$

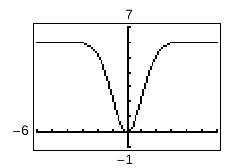
**13.** $\dfrac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds}\,ds = \int k(250 - s)\,ds$$
$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$
$$N = -\frac{k}{2}(250 - s)^2 + C$$

**15.** (a)

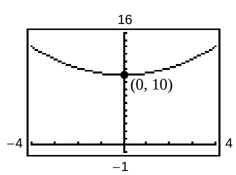



(b) $\dfrac{dy}{dx} = x(6 - y), \quad (0, 0)$



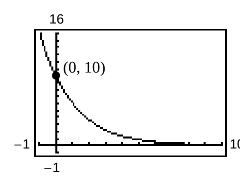

$$\frac{dy}{y - 6} = -x$$
$$\ln|y - 6| = \frac{-x^2}{2} + C$$
$$y - 6 = e^{-x^2/2 + C} = C_1e^{-x^2/2}$$
$$y = 6 + C_1e^{-x^2/2}$$
$$(0, 0)\text{:}\ \ 0 = 6 + C_1 \implies C_1 = -6 \implies y = 6 - 6e^{-x^2/2}$$

**17.** $\dfrac{dy}{dt} = \dfrac{1}{2}t, \ (0, 10)$




$$\int dy = \int \frac{1}{2}t\,dt$$
$$y = \frac{1}{4}t^2 + C$$
$$10 = \frac{1}{4}(0)^2 + C \implies C = 10$$
$$y = \frac{1}{4}t^2 + 10$$

**19.** $\dfrac{dy}{dt} = -\dfrac{1}{2}y, \ (0, 10)$




$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dt$$
$$\ln y = -\frac{1}{2}t + C_1$$
$$y = e^{-(t/2) + C_1} = e^{C_1}\,e^{-t/2} = Ce^{-t/2}$$
$$10 = Ce^0 \implies C = 10$$
$$y = 10e^{-t/2}$$

**21.** $\dfrac{dy}{dx} = ky$

$y = Ce^{kx}$ (Theorem 5.16)

$(0, 4)$: $4 = Ce^0 = C$

$(3, 10)$: $10 = 4e^{3k} \implies k = \dfrac{1}{3}\ln\left(\dfrac{5}{2}\right)$

When $x = 6$, $y = 4e^{1/3\ln(5/2)(6)} = 4e^{\ln(5/2)^2}$

$$= 4\left(\frac{5}{2}\right)^2 = 25$$

**23.** $\dfrac{dV}{dt} = kV$

$V = Ce^{kt}$ (Theorem 5.16)

$(0, 20{,}000)$: $C = 20{,}000$

$(4, 12{,}500)$: $12{,}500 = 20{,}000e^{4k} \implies k = \dfrac{1}{4}\ln\left(\dfrac{5}{8}\right)$

When $t = 6$, $V = 20{,}000e^{1/4\ln(5/8)(6)} = 20{,}000e^{\ln(5/8)^{3/2}}$

$$= 20{,}000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118$$

**25.** $y = Ce^{kt},\ \left(0, \frac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5} \approx 0.4605$$

$$y = \frac{1}{2}e^{0.4605t}$$

**27.** $y = Ce^{kt},\ (1, 1),\ (5, 5)$

$$\begin{aligned} 1 &= Ce^{k} \\ 5 &= Ce^{5k} \\ 5Ce^{k} &= Ce^{5k} \\ 5e^{k} &= e^{5k} \\ 5 &= e^{4k} \\ k &= \frac{\ln 5}{4} \approx 0.4024 \\ y &= Ce^{0.4024t} \\ 1 &= Ce^{0.4024} \\ C &\approx 0.6687 \\ y &= 0.6687e^{0.4024t} \end{aligned}$$

**29.** A differential equation in $x$ and $y$ is an equation that involves $x$, $y$ and derivatives of $y$.

**31.** $\dfrac{dy}{dx} = \dfrac{1}{2}xy$

$\dfrac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

**33.** Since the initial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10{,}000$, $y = 10e^{[\ln(1/2)/1620](10{,}000)} \approx 0.14$ gram.

**35.** Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $0.5 = Ce^{[\ln(1/2)/1620](10{,}000)} \implies C \approx 36.07$.

Initial quantity: 36.07 grams.

When $t = 1000$, we have $y = Ce^{[\ln(1/2)/1620](1000)} \approx 23.51$ grams.

**37.** Since the initial quantity is 5 grams, we have $y = 5.0e^{[\ln(1/2)/5730]t}$.

When $t = 1000$, $y \approx 4.43$ g.

When $t = 10{,}000$, $y \approx 1.49$ g.

**39.** Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24{,}360](1000)} \implies C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10{,}000$, $y = 2.16e^{[\ln(1/2)/24{,}360](10{,}000)} \approx 1.63$ grams.

**41.** Since $\dfrac{dy}{dx} = ky,\ y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{1620k}$$

$$k = \frac{-\ln 2}{1620}$$

$$y = y_0e^{-(\ln 2)t/1620}.$$

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$. Therefore, 95.81% of the present amount still exists.

**43.** Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$\begin{aligned} 2 &= e^{0.06t} \\ \ln 2 &= 0.06t \\ t &= \frac{\ln 2}{0.06} \approx 11.55 \text{ years.} \end{aligned}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \$1822.12$

**45.** Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

**47.** Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$

$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$

$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

**49.** $500{,}000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$

$$\approx \$112{,}087.09$$

**51.** $500{,}000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 500{,}000\left(1 + \frac{0.08}{12}\right)^{-420}$$

$$= \$30{,}688.87$$

**53.** (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$

$$\ln 2 = t \ln 1.07$$

$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$

$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$

$$\ln 2 = 0.07t$$

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

**55.** (a) $2000 = 1000(1 + 0.085)^t$

$$2 = 1.085^t$$

$$\ln 2 = t \ln 1.085$$

$$t = \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.085}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.085}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.085}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{12}\right)} \approx 8.18 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.085}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.085}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.085}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{365}\right)} \approx 8.16 \text{ years}$$

(d) $2000 = 1000e^{0.085t}$

$$2 = e^{0.085t}$$

$$\ln 2 = 0.085t$$

$$t = \frac{\ln 2}{0.085} \approx 8.15 \text{ years}$$

**57.** $P = Ce^{kt} = Ce^{-0.009t}$

$P(-1) = 8.2 = Ce^{-0.009(-1)} \implies C = 8.1265$

$P = 8.1265e^{-0.009t}$

$P(10) \approx 7.43$ or 7,430,000 people in 2010

**59.** $P = Ce^{kt} = Ce^{0.036t}$

$P(-1) = 4.6 = Ce^{0.036(-1)} \implies C = 4.7686$

$P = 4.7686e^{0.036t}$

$P(10) \approx 6.83$ or 6,830,000 people in 2010

**61.** If $k < 0$, the population decreases.

If $k > 0$, the population increases.

**63.**
$$P = Ce^{kx},\ (0, 760),\ (1000, 672.71)$$
$$C = 760$$
$$672.71 = 760e^{1000x}$$
$$x = \frac{\ln(672.71/760)}{1000} \approx -0.000122$$
$$P \approx 760e^{-0.000122x}$$

When $x = 3000$, $P \approx 527.06$ mm Hg.

**65.** (a)
$$19 = 30(1 - e^{20k})$$
$$30e^{20k} = 11$$
$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$
$$N \approx 30(1 - e^{-0.0502t})$$

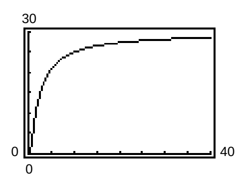
(b)
$$25 = 30(1 - e^{-0.0502t})$$
$$e^{-0.0502t} = \frac{1}{6}$$
$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

**67.** $S = Ce^{k/t}$

(a)
$$S = 5 \text{ when } t = 1$$
$$5 = Ce^{k}$$
$$\lim_{t\to\infty} Ce^{k/t} = C = 30$$
$$5 = 30e^{k}$$
$$k = \ln\tfrac{1}{6} \approx -1.7918$$
$$S \approx 30e^{-1.7918/t}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)




**69.** $A(t) = V(t)e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}}\,e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}-0.10t}$

$$\frac{dA}{dt} = 100{,}000\left(\frac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t}-0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, $(1998 + 16)$. **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \le x \le 30$ and $0 \le y \le 600{,}000$.

**71.** $\beta(I) = 10\log_{10}\dfrac{I}{I_0},\ I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10\log_{10}\dfrac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10\log_{10}\dfrac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10\log_{10}\dfrac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10\log_{10}\dfrac{10^{-4}}{10^{-16}} = 120$ decibels

**73.** $R = \dfrac{\ln I - 0}{\ln 10},\ I = e^{R\ln 10} = 10^{R}$

(a) $8.3 = \dfrac{\ln I - 0}{\ln 10}$

$I = 10^{8.3} \approx 199{,}526{,}231.5$

(b) $2R = \dfrac{\ln I - 0}{\ln 10}$

$I = e^{2R\ln 10} = e^{2R\ln 10} = (e^{R\ln 10})^2 = (10^{R})^2$

Increases by a factor of $e^{2R\ln 10}$ or $10^{R}$.

(c) $\dfrac{dR}{dI} = \dfrac{1}{I\ln 10}$

**75.** False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

**77.** True

## Section 5.7 Differential Equations: Separation of Variables

**1.** Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

**3.** Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check:
$$\begin{aligned} y' &= -C_1 \sin x + C_2 \cos x \\ y'' &= -C_1 \cos x - C_2 \sin x \\ y'' + y &= -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0 \end{aligned}$$

**5.** $y = -\cos x \ln|\sec x + \tan x|$

$$\begin{aligned} y' &= (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x| \\ &= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x| \\ &= -1 + \sin x \ln|\sec x + \tan x| \\ y'' &= (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x| \\ &= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| \end{aligned}$$

Substituting,

$$\begin{aligned} y'' + y &= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x| \\ &= \tan x. \end{aligned}$$

**In Exercises 7–11, the differential equation is $y^{(4)} - 16y = 0$.**

**7.**
$$\begin{aligned} y &= 3 \cos x \\ y^{(4)} &= 3 \cos x \\ y^{(4)} - 16y &= -45 \cos x \neq 0, \end{aligned}$$

No.

**9.**
$$\begin{aligned} y &= e^{-2x} \\ y^{(4)} &= 16e^{-2x} \\ y^{(4)} - 16y &= 16e^{-2x} - 16e^{-2x} = 0, \end{aligned}$$

Yes.

**11.**
$$\begin{aligned} y &= C_1e^{2x} + C_2e^{-2x} + C_3 \sin 2x + C_4 \cos 2x \\ y^{(4)} &= 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3 \sin 2x + 16C_4 \cos 2x \\ y^{(4)} - 16y &= 0, \end{aligned}$$

Yes.

**In 13–17, the differential equation is $xy' - 2y = x^3e^x$.**

**13.** $y = x^2,\ y' = 2x$

$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$

No.

**15.** $y = x^2(2 + e^x),\quad y' = x^2(e^x) + 2x(2 + e^x)$

$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x,$

Yes.

**17.** $y = \ln x, y' = \dfrac{1}{x}$

$xy' - 2y = x\left(\dfrac{1}{x}\right) - 2\ln x \neq x^3e^x,\qquad$ No.

**19.** $y = Ce^{kx}$

$\dfrac{dy}{dx} = Cke^{kx}$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

**21.** $y^2 = Cx^3$ passes through $(4, 4)$

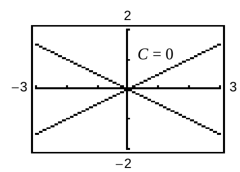
$16 = C(64) \implies C = \frac{1}{4}$

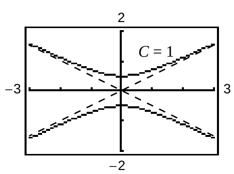
Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

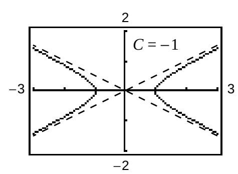
**23.** Differential equation: $4yy' - x = 0$

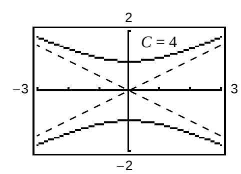
General solution: $4y^2 - x^2 = C$

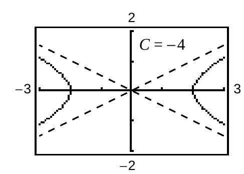
Particular solutions: $C = 0$, Two intersecting lines
$C = \pm 1,\ C = \pm 4$, Hyperbolas




**25.** Differential equation: $y' + 2y = 0$

General Solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

**27.** Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$

$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$

$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) +$
$9(C_1 \sin 3x + C_2 \cos 3x) = 0$

Initial conditions: $y\left(\dfrac{\pi}{6}\right) = 2,\ y'\left(\dfrac{\pi}{6}\right) = 1$

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \implies C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right)$$

$$= -3C_2 \implies C_2 = -\frac{1}{3}$$

Particular solution: $y = 2\sin 3x - \dfrac{1}{3}\cos 3x$

**29.** Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2, y'' = 6C_2x$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3) = 0$$

Initial conditions: $y(2) = 0,\ y'(2) = 4$

$$0 = 2C_1 + 8C_2$$

$$y' = C_1 + 3C_2x^2$$

$$4 = C_1 + 12C_2$$

$$\left.\begin{aligned} C_1 + 4C_2 &= 0 \\ C_1 + 12C_2 &= 4 \end{aligned}\right\} \quad C_2 = \frac{1}{2},\quad C_1 = -2$$

Particular solution: $y = -2x + \dfrac{1}{2}x^3$

**31.** $\dfrac{dy}{dx} = 3x^2$

$$y = \int 3x^2\,dx = x^3 + C$$

**33.** $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

$$y = \int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln(1 + x^2) + C$$

$(u = 1 + x^2, du = 2x\,dx)$

**35.** $\dfrac{dy}{dx} = \dfrac{x - 2}{x} = 1 - \dfrac{2}{x}$

$$y = \int \left[1 - \frac{2}{x}\right] dx$$

$$= x - 2\ln|x| + C = x - \ln x^2 + C$$

**37.** $\dfrac{dy}{dx} = \sin 2x$

$$y = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$$

$(u = 2x, du = 2dx)$

**39.** $\dfrac{dy}{dx} = x\sqrt{x - 3}$ Let $u = \sqrt{x - 3}$, then $x = u^2 + 3$ and $dx = 2u\,du$.

$$y = \int x\sqrt{x - 3}\,dx = \int (u^2 + 3)(u)(2u)du$$

$$= 2\int (u^4 + 3u^2)\,du = 2\left(\frac{u^5}{5} + u^3\right) + C = \frac{2}{5}(x - 3)^{5/2} + 2(x - 3)^{3/2} + C$$

**41.** $\dfrac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

$(u = x^2,\ du = 2x\,dx)$

**43.** $\dfrac{dy}{dx} = \dfrac{x}{y}$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = C$$

**45.** $\dfrac{dr}{ds} = 0.05r$

$$\int \frac{dr}{r} = \int 0.05\,ds$$

$$\ln|r| = 0.05s + C_1$$

$$r = e^{0.05s + C_1} = Ce^{0.05s}$$

**47.** $(2 + x)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$$

$$\ln y = 3\ln(2 + x) + \ln C = \ln C(2 + x)^3$$

$$y = C(x + 2)^3$$

**49.** $yy' = \sin x$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2\cos x + C$$

**51.** $\sqrt{1 - 4x^2}\,\dfrac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1 - 4x^2}}\,dx$$

$$\int dy = \int \frac{x}{\sqrt{1 - 4x^2}}\,dx$$

$$= -\frac{1}{8}\int (1 - 4x^2)^{-1/2}(-8x\,dx)$$

$$y = -\frac{1}{4}(1 - 4x^2)^{1/2} + C$$

**53.** $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\, du = \frac{dx}{x}\right)$$

$$\ln y = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

**55.** $yy' - e^x = 0$

$$\int y\,dy = \int e^x\,dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4,\ 16 = 2 + C,\ C = 14$

Particular solution: $y^2 = 2e^x + 14$

**57.** $y(x + 1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x + 1)\,dx$$

$$\ln y = -\frac{(x + 1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1,\ 1 = Ce^{-1/2},\ C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

**59.** $y(1 + x^2)\,\dfrac{dy}{dx} = x(1 + y^2)$

$$\frac{y}{1 + y^2}\,dy = \frac{x}{1 + x^2}\,dx$$

$$\frac{1}{2}\ln(1 + y^2) = \frac{1}{2}\ln(1 + x^2) + C_1$$

$$\ln(1 + y^2) = \ln(1 + x^2) + \ln C = \ln[C(1 + x^2)]$$

$$1 + y^2 = C(1 + x^2)$$

$$y(0) = \sqrt{3}:\ 1 + 3 = C \implies C = 4$$

$$1 + y^2 = 4(1 + x^2)$$

$$y^2 = 3 + 4x^2$$

**61.** $\dfrac{du}{dv} = uv \sin v^2$

$$\int \frac{du}{u} = \int v \sin v^2\,dv$$

$$\ln u = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1,\ C = \dfrac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

**63.** $dP - kP\,dt = 0$

$$\int \frac{dP}{P} = k\int dt$$

$$\ln P = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0,\ P_0 = Ce^0 = C$

Particular solution: $P = P_0 e^{kt}$

**65.** $\dfrac{dy}{dx} = \dfrac{-9x}{16y}$

$$\int 16y\,dy = -\int 9x\,dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition: $y(1) = 1,\ 8 = -\dfrac{9}{2} + C,\ C = \dfrac{25}{2}$

Particular solution: $8y^2 = \dfrac{-9}{2}x^2 + \dfrac{25}{2},$

$$16y^2 + 9x^2 = 25$$

**67.** $m = \dfrac{dy}{dx} = \dfrac{0 - y}{(x + 2) - x} = -\dfrac{y}{2}$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$

$$\ln y = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

**69.** $f(x, y) = x^3 - 4xy^2 + y^3$

$$f(tx, ty) = t^3x^3 - 4t\,xt^2y^2 + t^3y^3$$

$$= t^3(x^3 - 4xy^2 + y^3)$$

Homogeneous of degree 3

**71.** $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{t^4x^2y^2}{\sqrt{t^2x^2 + t^2y^2}} = t^3\frac{x^2y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

**73.** $f(x, y) = 2\ln xy$

$$f(tx, ty) = 2\ln tx\,ty$$

$$= 2\ln t^2xy = 2(\ln t^2 + \ln xy)$$

Not homogeneous

**75.** $f(x, y) = 2\ln\dfrac{x}{y}$

$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous degree 0

**77.** $y' = \dfrac{x + y}{2x},\ y = vx$

$$v + x\frac{dv}{dx} = \frac{x + vx}{2x}$$

$$x\frac{dv}{dx} = \frac{1 + v}{2} - v$$

$$2\int \frac{dv}{1 - v} = \int \frac{dx}{x}$$

$$-\ln(1 - v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1 - v^2)} = |Cx|$$

$$\frac{1}{[1 - (y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x - y)^2} = |Cx|$$

$$|x| = C(x - y)^2$$

**79.** $y' = \dfrac{x - y}{x + y},\ y = vx$

$$v + x\frac{dv}{dx} = \frac{x - xv}{x + xv}$$

$$v\,dx + x\,dv = \frac{1 - v}{1 + v}dx$$

$$\int \frac{v + 1}{v^2 + 2v - 1}\,dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln|v^2 + 2v - 1| = -\ln|x| + \ln C_1 = \ln\left|\frac{C_1}{x}\right|$$

$$|v^2 + 2v - 1| = \frac{C}{x^2}$$

$$\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = \frac{C}{x^2}$$

$$|y^2 + 2xy - x^2| = C$$

**81.**
$$y' = \frac{xy}{x^2 - y^2}, y = vx$$
$$v + x\frac{dv}{dx} = \frac{x^2 v}{x^2 - x^2 v^2}$$
$$v\,dx + x\,dv = \frac{v}{1 - v^2}\,dx$$
$$\int \frac{1 - v^2}{v^3}\,dv = \int \frac{dx}{x}$$
$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1 x|$$
$$\frac{-1}{2v^2} = \ln|C_1 x v|$$
$$\frac{-x^2}{2y^2} = \ln|C_1 y|$$
$$y = Ce^{-x^2/2y^2}$$

**83.**
$$x\,dy - (2xe^{-y/x} + y)\,dx = 0, y = vx$$
$$x(v\,dx + x\,dv) - (2xe^{-v} + vx)\,dx = 0$$
$$\int e^v\,dv = \int \frac{2}{x}\,dx$$
$$e^v = \ln C_1 x^2$$
$$e^{y/x} = \ln C_1 + \ln x^2$$
$$e^{y/x} = C + \ln x^2$$

Initial condition: $y(1) = 0, 1 = C$

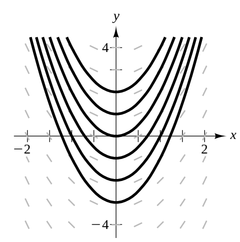
Particular solution: $e^{y/x} = 1 + \ln x^2$

**85.**
$$\left(x \sec \frac{y}{x} + y\right)dx - x\,dy = 0, y = vx$$
$$(x \sec v + xv)dx - x(v\,dx + x\,dv) = 0$$
$$(\sec v + v)\,dx = v\,dx + x\,dv$$
$$\int \cos v\,dv = \int \frac{dx}{x}$$
$$\sin v = \ln x + \ln C_1$$
$$x = Ce^{\sin v}$$
$$= Ce^{\sin(y/x)}$$

Initial condition: $y(1) = 0, 1 = Ce^0 = C$

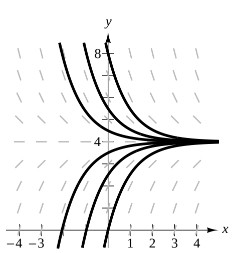
Particular solution: $x = e^{\sin(y/x)}$

**87.** $\dfrac{dy}{dx} = x$



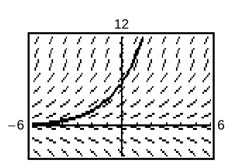

$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

**89.** $\dfrac{dy}{dx} = 4 - y$

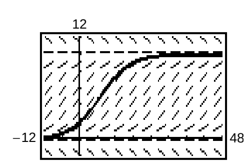



$$\int \frac{dy}{4 - y} = \int dx$$
$$\ln|4 - y| = -x + C_1$$
$$4 - y = e^{-x + C_1}$$
$$y = 4 + Ce^{-x}$$

**91.** $\dfrac{dy}{dx} = 0.5y, y(0) = 6$




**93.** $\dfrac{dy}{dx} = 0.02y(10 - y), y(0) = 2$



**95.** $\dfrac{dy}{dt} = ky,\ y = Ce^{kt}$

Initial conditions: $y(0) = y_0$

$$y(1620) = \frac{y_0}{2}$$

$$C = y_0$$

$$\frac{y_0}{2} = y_0 e^{1620k}$$

$$k = \frac{\ln(1/2)}{1620}$$

Particular solution: $y = y_0 e^{-t(\ln 2)/1620}$

When $t = 25$, $y \approx 0.989 y_0$, $y = 98.9\%$ of $y_0$.

**97.** $\dfrac{dy}{dx} = k(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

**99.** $\dfrac{dy}{dx} = ky(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

**101.**
$$\frac{dw}{dt} = k(1200 - w)$$

$$\int \frac{dw}{1200 - w} = \int k\,dt$$

$$\ln(1200 - w) = -kt + C_1$$

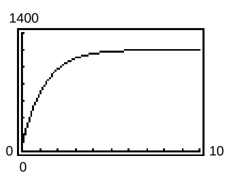
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

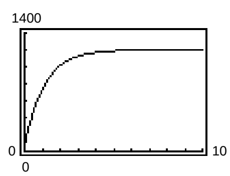
$$w = 1200 - Ce^{-kt}$$

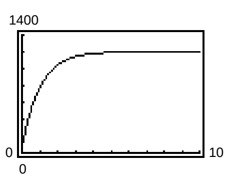
$$w(0) = 60 = 1200 - C \implies C = 1200 - 60 = 1140$$

$$w = 1200 - 1140e^{-kt}$$

(a)




(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$$\lim_{t \to 0} w = 1200$$

**103.** (a)
$$\frac{dv}{dt} = k(W - v)$$

$$\int \frac{dv}{W - v} = \int k\,dt$$

$$-\ln(W - v) = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20$, $v = 0$ when $t = 0$, and

$v = 5$ when $t = 1$.

$C = 20$, $k = -\ln(3/4)$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) \approx 20(1 - e^{-0.2877t})$$

(b) $$s = \int 20(1 - e^{-0.2877t})\,dt$$

$$\approx 20\left[t + 3.4761e^{-0.2877t}\right] + C$$

Since $S(0) = 0$, $C \approx -69.5$ and we have $s \approx 20t + 69.5(e^{-0.2877t} - 1)$.

**105.** Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

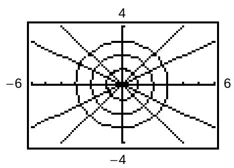
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$y = Kx$$




**107.** Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$

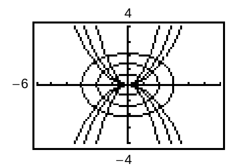
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\dfrac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$




**109.** Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

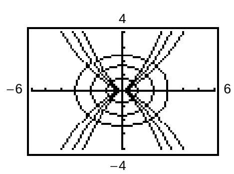
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\dfrac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$




**111.** A general solution of order $n$ has $n$ arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

**113.** $M(x, y)dx + N(x, y)dy = 0$, where $M$ and $N$ are homogeneous functions of the same degree.

**115.** False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

**117.** False

$$f(tx, ty) = t^2x^2 + t^2xy + 2$$
$$\neq t^2 f(x, y)$$

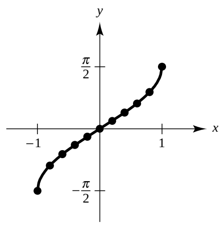
# Section 5.8 Inverse Trigonometric Functions: Differentiation

**1.** $y = \arcsin x$

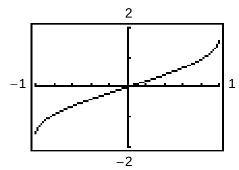
(a)

| $x$ | $-1$ | $-0.8$ | $-0.6$ | $-0.4$ | $-0.2$ | $0$ | $0.2$ | $0.4$ | $0.6$ | $0.8$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $-1.571$ | $-0.927$ | $-0.644$ | $-0.412$ | $-0.201$ | $0$ | $0.201$ | $0.412$ | $0.644$ | $0.927$ | $1.571$ |

(b)




(c)




(d) Symmetric about origin:
$\arcsin(-x) = -\arcsin x$

Intercept: $(0, 0)$

**3.** False.

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

**5.** $\arcsin \dfrac{1}{2} = \dfrac{\pi}{6}$

**7.** $\arccos \dfrac{1}{2} = \dfrac{\pi}{3}$

**9.** $\arctan \dfrac{\sqrt{3}}{3} = \dfrac{\pi}{6}$

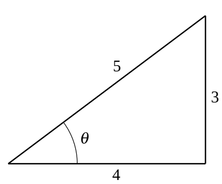
**11.** $\operatorname{arccsc}\left(-\sqrt{2}\right) = -\dfrac{\pi}{4}$

**13.** $\arccos(-0.8) \approx 2.50$

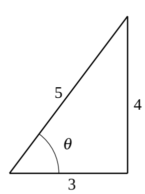
**15.** $\operatorname{arcsec}(1.269) = \arccos\left(\dfrac{1}{1.269}\right)$

$\approx 0.66$

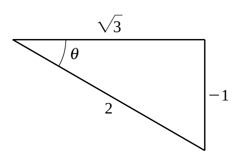
**17.** (a) $\sin\left(\arctan \frac{3}{4}\right) = \frac{3}{5}$



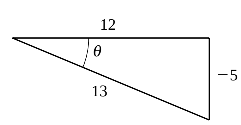

(b) $\sec\left(\arcsin \frac{4}{5}\right) = \frac{5}{3}$




**19.** (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right] = \cot\left(-\dfrac{\pi}{6}\right) = -\sqrt{3}$



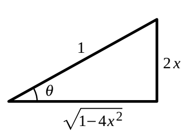

(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right] = -\dfrac{13}{5}$




**21.** $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

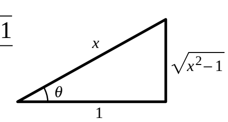
$y = \cos\theta = \sqrt{1 - 4x^2}$




**23.** $y = \sin(\operatorname{arcsec} x)$

$\theta = \operatorname{arcsec} x,\ 0 \le \theta \le \pi,\ \theta \ne \dfrac{\pi}{2}$

$y = \sin\theta = \dfrac{\sqrt{x^2 - 1}}{|x|}$

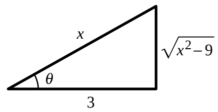



The absolute value bars on $x$ are necessary because of the restriction $0 \le \theta \le \pi$, $\theta \ne \pi/2$, and $\sin\theta$ for this domain must always be nonnegative.

**25.** $y = \tan\left(\operatorname{arcsec}\dfrac{x}{3}\right)$

$\theta = \operatorname{arcsec}\dfrac{x}{3}$

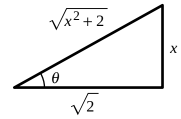
$y = \tan\theta = \dfrac{\sqrt{x^2 - 9}}{3}$



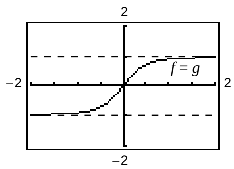

**27.** $y = \csc\left(\arctan\dfrac{x}{\sqrt{2}}\right)$

$\theta = \arctan\dfrac{x}{\sqrt{2}}$

$y = \csc\theta = \dfrac{\sqrt{x^2 + 2}}{x}$




**29.** $\sin(\arctan 2x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

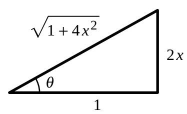



Asymptotes: $y = \pm 1$

$\arctan 2x = \theta$

$\tan\theta = 2x$

$\sin\theta = \dfrac{2x}{\sqrt{1 + 4x^2}}$




**31.** $\arcsin(3x - \pi) = \frac{1}{2}$

$3x - \pi = \sin\left(\frac{1}{2}\right)$

$x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right) + \pi\right] \approx 1.207$

**33.** $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

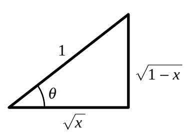
$\sqrt{2x} = \sin\left(\arccos\sqrt{x}\right)$

$\sqrt{2x} = \sqrt{1 - x},\ 0 \le x \le 1$

$2x = 1 - x$

$3x = 1$

$x = \dfrac{1}{3}$




**35.** (a) $\operatorname{arccsc} x = \arcsin\dfrac{1}{x},\ |x| \ge 1$

Let $y = \operatorname{arccsc} x$. Then for

$-\dfrac{\pi}{2} \le y < 0$ and $0 < y \le \dfrac{\pi}{2}$,

$\csc y = x \implies \sin y = 1/x$. Thus, $y = \arcsin(1/x)$. Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2},\ x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x)\tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

**37.** $f(x) = \arcsin(x - 1)$

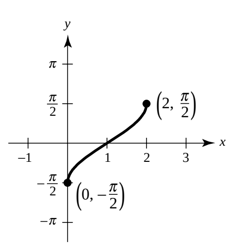
$x - 1 = \sin y$

$x = 1 + \sin y$

Domain: $[0, 2]$

Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.



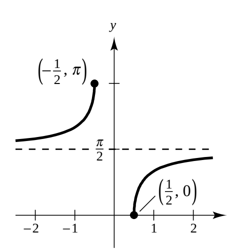

**39.** $f(x) = \operatorname{arcsec} 2x$

$2x = \sec y$

$x = \dfrac{1}{2}\sec y$

Domain: $\left(-\infty, -\dfrac{1}{2}\right], \left[\dfrac{1}{2}, \infty\right)$

Range: $\left[0, \dfrac{\pi}{2}\right), \left(\dfrac{\pi}{2}, \pi\right]$



**41.** $f(x) = 2\arcsin(x-1)$

$$f'(x) = \frac{2}{\sqrt{1-(x-1)^2}} = \frac{2}{\sqrt{2x-x^2}}$$

**43.** $g(x) = 3\arccos\dfrac{x}{2}$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1-(x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$$

**45.** $f(x) = \arctan\dfrac{x}{a}$

$$f'(x) = \frac{1/a}{1+(x^2/a^2)} = \frac{a}{a^2+x^2}$$

**47.** $g(x) = \dfrac{\arcsin 3x}{x}$

$$g'(x) = \frac{x\left(3/\sqrt{1-9x^2}\right) - \arcsin 3x}{x^2}$$

$$= \frac{3x - \sqrt{1-9x^2}\arcsin 3x}{x^2\sqrt{1-9x^2}}$$

**49.** $h(t) = \sin(\arccos t) = \sqrt{1-t^2}$

$$h'(t) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1-t^2}}$$

**51.** $y = x\arccos x - \sqrt{1-x^2}$

$$y' = \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \arccos x$$

**53.** $y = \dfrac{1}{2}\left(\dfrac{1}{2}\ln\dfrac{x+1}{x-1} + \arctan x\right)$

$$= \frac{1}{4}[\ln(x+1) - \ln(x-1)] + \frac{1}{2}\arctan x$$

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

**55.** $y = x\arcsin x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1-x^2}}\right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

**57.** $y = 8\arcsin\dfrac{x}{4} - \dfrac{x\sqrt{16-x^2}}{2}$

$$y' = 2\frac{1}{\sqrt{1-(x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}(16-x^2)^{-1/2}(-2x)$$

$$= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}}$$

$$= \frac{16-(16-x^2)+x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}}$$

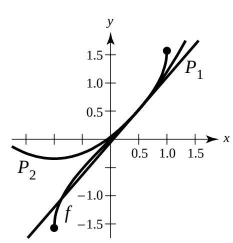
**59.** $y = \arctan x + \dfrac{x}{1+x^2}$

$$y' = \frac{1}{1+x^2} + \frac{(1+x^2) - x(2x)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)+(1-x^2)}{(1+x^2)^2}$$

$$= \frac{2}{(1+x^2)^2}$$

**61.** $f(x) = \arcsin x,\ a = \dfrac{1}{2}$




$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x-\frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x-\frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x-\frac{1}{2}\right)^2$$

**63.** $f(x) = \text{arcsec } x - x$

$$f'(x) = \frac{1}{|x|\sqrt{x^2 - 1}} - 1$$

$$= 0 \text{ when } |x|\sqrt{x^2 - 1} = 1.$$

$$x^2(x^2 - 1) = 1$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or}$$

$$x = \pm\sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

**65.** $f(x) = \arctan x - \arctan(x - 4)$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0$$

$$1 + x^2 = 1 + (x - 4)^2$$

$$0 = -8x + 16$$

$$x = 2$$

By the First Derivative Test, $(2, 2.214)$ is a relative maximum.

**67.** The trigonometric functions are not one-to-one on $(-\infty, \infty)$, sot their domains must be restricted to intervals on which they are one-to-one.

**69.** $y = \text{arccot } x,\ 0 < y < \pi$

$x = \cot y$

$$\tan y = \frac{1}{x}$$

So, graph the function

$$y = \arctan\left(\frac{1}{x}\right) \text{ for } x > 0 \text{ and } y = \arctan\left(\frac{1}{x}\right) + \pi \text{ for } x < 0.$$

**71.** (a) $\cot \theta = \dfrac{x}{5}$

$$\theta = \text{arccot}\left(\frac{x}{5}\right)$$

(b) $\dfrac{d\theta}{dt} = \dfrac{-\frac{1}{5}}{1 + \left(\frac{x}{5}\right)^2}\dfrac{dx}{dt} = \dfrac{-5}{x^2 + 25}\dfrac{dx}{dt}$

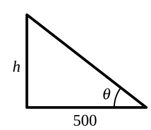
If $\dfrac{dx}{dt} = -400$ and $x = 10$, $\dfrac{d\theta}{dt} = 16$ rad/hr.

If $\dfrac{dx}{dt} = -400$ and $x = 3$, $\dfrac{d\theta}{dt} \approx 58.824$ rad/hr.

**73.** (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec.

(b) $\tan \theta = \dfrac{h}{500} = \dfrac{-16t^2 + 256}{500}$




$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15{,}625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

**75.** $\tan(\arctan x + \arctan y) = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \dfrac{x + y}{1 - xy},\ xy \neq 1$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right),\ xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\frac{(1/2) + (1/3)}{1 - [(1/2)\cdot(1/3)]} = \arctan\frac{5/6}{1 - (1/6)} = \arctan\frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

**77.** $f(x) = kx + \sin x$

$f'(x) = k + \cos x \geq 0$ for $k \geq 1$

$f'(x) = k + \cos x \leq 0$ for $k \leq -1$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

**79.** True

$\dfrac{d}{dx}[\arctan x] = \dfrac{1}{1 + x^2} > 0$ for all $x$.

**81.** True

$\dfrac{d}{dx}[\arctan(\tan x)] = \dfrac{\sec^2 x}{1 + \tan^2 x} = \dfrac{\sec^2 x}{\sec^2 x} = 1$

## Section 5.9 Inverse Trigonometric Functions: Integration

**1.** $\displaystyle\int \frac{5}{\sqrt{9 - x^2}}\,dx = 5\arcsin\left(\frac{x}{3}\right) + C$

**3.** Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{1/6} \frac{1}{\sqrt{1 - 9x^2}}\,dx = \frac{1}{3}\int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}}(3)\,dx = \left[\frac{1}{3}\arcsin(3x)\right]_0^{1/6} = \frac{\pi}{18}$$

**5.** $\displaystyle\int \frac{7}{16 + x^2}\,dx = \frac{7}{4}\arctan\left(\frac{x}{4}\right) + C$

**7.** Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{3}/2} \frac{1}{1 + 4x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1 + (2x)^2}\,dx = \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

**9.** $\displaystyle\int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx = \operatorname{arcsec}|2x| + C$

**11.** $\displaystyle\int \frac{x^3}{x^2 + 1}\,dx = \int\left[x - \frac{x}{x^2 + 1}\right]dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$ (Use long division.)

**13.** $\displaystyle\int \frac{1}{\sqrt{1 - (x + 1)^2}}\,dx = \arcsin(x + 1) + C$

**15.** Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1 - (t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin(t^2) + C$$

**17.** Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin}{\sqrt{1 - x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

**19.** Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_{-1/2}^{0} \frac{x}{\sqrt{1 - x^2}}\,dx = -\frac{1}{2}\int_{-1/2}^{0} (1 - x^2)^{-1/2}(-2x)\,dx$$

$$= \left[-\sqrt{1 - x^2}\right]_{-1/2}^{0} = \frac{\sqrt{3} - 2}{2}$$

$$\approx -0.134$$

**21.** Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{4 + e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4 + (e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$$

**23.** Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1 + \cos^2 x}\,dx = -\int_{\pi/2}^{\pi} \frac{-\sin x}{1 + \cos^2 x}\,dx$$

$$= \left[-\arctan(\cos x)\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

**25.** $\displaystyle\int \frac{1}{\sqrt{x}\,\sqrt{1 - x}}\,dx$. $u = \sqrt{x}$, $x - u^2$, $dx = 2u\,du$

$$\int \frac{1}{u\sqrt{1 - u^2}}(2u\,du) = 2\int \frac{du}{\sqrt{1 - u^2}} = 2\arcsin u + C$$

$$= 2\arcsin\sqrt{x} + C$$

**27.** $$\int \frac{x - 3}{x^2 + 1}\,dx = \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx - 3\int \frac{1}{x^2 + 1}\,dx$$

$$= \frac{1}{2}\ln(x^2 + 1) - 3\arctan x + C$$

**29.** $$\int \frac{x + 5}{\sqrt{9 - (x - 3)^2}}\,dx = \int \frac{(x - 3)}{\sqrt{9 - (x - 3)^2}}\,dx + \int \frac{8}{\sqrt{9 - (x - 3)^2}}\,dx$$

$$= -\sqrt{9 - (x - 3)^2} - 8\arcsin\left(\frac{x - 3}{3}\right) + C$$

$$= -\sqrt{6x - x^2} + 8\arcsin\left(\frac{x}{3} - 1\right) + C$$

**31.** $$\int_0^2 \frac{1}{x^2 - 2x + 2}\,dx = \int_0^2 \frac{1}{1 + (x - 1)^2}\,dx = \left[\arctan(x - 1)\right]_0^2 = \frac{\pi}{2}$$

**33.** $$\int \frac{2x}{x^2 + 6x + 13}\,dx = \int \frac{2x + 6}{x^2 + 6x + 13}dx - 6\int \frac{1}{x^2 + 6x + 13}\,dx = \int \frac{2x + 6}{x^2 + 6x + 13}\,dx - 6\int \frac{1}{4 + (x + 3)^2}\,dx$$

$$= \ln|x^2 + 6x + 13| - 3\arctan\left(\frac{x + 3}{2}\right) + C$$

**35.** $$\int \frac{1}{\sqrt{-x^2 - 4x}}\,dx = \int \frac{1}{\sqrt{4 - (x + 2)^2}}\,dx = \arcsin\left(\frac{x + 2}{2}\right) + C$$

**37.** Let $u = -x^2 - 4x$, $du = (-2x - 4)\,dx$.

$$\int \frac{x + 2}{\sqrt{-x^2 - 4x}}\,dx = -\frac{1}{2}\int (-x^2 - 4x)^{-1/2}(-2x - 4)\,dx = -\sqrt{-x^2 - 4x} + C$$

**39.** $$\int_2^3 \frac{2x - 3}{\sqrt{4x - x^2}}\,dx = \int_2^3 \frac{2x - 4}{\sqrt{4x - x^2}}\,dx + \int_2^3 \frac{1}{\sqrt{4x - x^2}}\,dx = -\int_2^3 (4x - x^2)^{-1/2}(4 - 2x)\,dx + \int_2^3 \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx$$

$$= \left[-2\sqrt{4x - x^2} + \arcsin\left(\frac{x - 2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

**41.** Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^2 + 1}\,dx = \frac{1}{2}\arctan(x^2 + 1) + C$$

**43.** Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2+3} = dt$.

$$\int \sqrt{e^t - 3}\,dt = \int \frac{2u^2}{u^2+3}\,du = \int 2\,du - \int 6\,\frac{1}{u^2+3}\,du$$

$$= 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t - 3}{3}} + C$$

**45.** A perfect square trinomial is an expression in $x$ with three terms that factor as a perfect square.

Example: $x^2 + 6x + 9 = (x+3)^2$

**47.** (a) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C,\ u = x$

(b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + C,\ u = 1 - x^2$

(c) $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\,dx$ cannot be evaluated using the basic integration rules.

**49.** (a) $\displaystyle\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2} + C,\ u = x - 1$

(b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int x\sqrt{x-1}\,dx = \int (u^2+1)(u)(2u)\,du = 2\int (u^4 + u^2)\,du = 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C$$

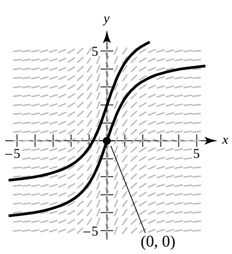
$$= \frac{2}{15}u^3(3u^2 + 5) + C = \frac{2}{15}(x-1)^{3/2}[3(x-1) + 5] + C = \frac{2}{15}(x-1)^{3/2}(3x+2) + C$$

(c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u^2+1}{u}(2u)\,du = 2\int (u^2+1)\,du = 2\left(\frac{u^3}{3} + u\right) + C = \frac{2}{3}u(u^2+3) + C = \frac{2}{3}\sqrt{x-1}(x+2) + C$$

**Note:** In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

**51.** (a)

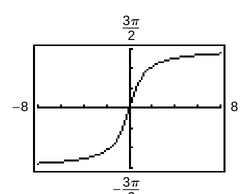



(b) $\dfrac{dy}{dx} = \dfrac{3}{1+x^2}$, $(0, 0)$

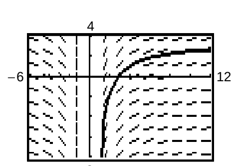
$$y = 3\int \frac{dx}{1+x^2} = 3\arctan x + C$$

$(0, 0)$: $0 = 3\arctan(0) + C \implies C = 0$

$$y = 3\arctan x$$




**53.** $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2-1}}$, $y(3) = 0$

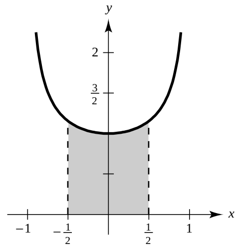



**55.** $\displaystyle A = \int_1^3 \frac{1}{x^2 - 2x + 1 + 4}\,dx = \int_1^3 \frac{1}{(x-1)^2 + 2^2}\,dx$

$$= \left[\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\right]_1^3 = \frac{1}{2}\arctan(1) = \frac{\pi}{8} \approx 0.3927$$

**57.** Area $\approx (1)(1) = 1$

Matches (c)



**59.** (a) $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \Big[4\arctan x\Big]_0^1 = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4\int_0^1 \frac{4}{1+x^2}\,dx \approx 4\left(\frac{1}{36}\right)\left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2}\right] \approx 3.1415918$$

(c) 3.1415927

**61.** (a) $\displaystyle\frac{d}{dx}\left[\arcsin\left(\frac{u}{a}\right) + C\right] = \frac{1}{\sqrt{1-(u^2/a^2)}}\left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$

Thus, $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$

(b) $\displaystyle\frac{d}{dx}\left[\frac{1}{a}\arctan\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{1+(u/a)^2}\right] = \frac{1}{a^2}\left[\frac{u'}{(a^2+u^2)/a^2}\right] = \frac{u'}{a^2+u^2}$

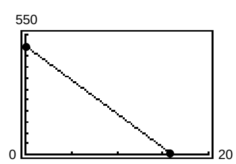
Thus, $\displaystyle\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C.$

(c) Assume $u > 0$.

$\displaystyle\frac{d}{dx}\left[\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}}\right] = \frac{1}{a}\left[\frac{u'}{u\sqrt{u^2-a^2)/a^2}}\right] = \frac{u'}{u\sqrt{u^2-a^2}}$. The case $u < 0$ is handled in a similar manner.

Thus, $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}}\,dx = \frac{1}{a}\operatorname{arsec}\frac{|u|}{a} + C.$

**63.** (a) $v(t) = -32t + 500$




(b) $\displaystyle s(t) = \int v(t)\,dt = \int(-32t + 500)\,dt$

$= -16t^2 + 500t + C$

$s(0) = -16(0) + 500(0) + C = 0 \implies C = 0$

$s(t) = -16t^2 + 500t$

When the object reaches its maximum height, $v(t) = 0$.

$v(t) = -32t + 500 = 0$

$-32t = -500$

$t = 15.625$

$s(15.625) = -16(15.625)^2 + 500(15.625)$

$= 3906.25$ ft (Maximum height)

**—CONTINUED—**

**63. —CONTINUED—**

(c) $$\int \frac{1}{32 + kv^2}\,dv = -\int dt$$

$$\frac{1}{\sqrt{32k}}\arctan\left(\sqrt{\frac{k}{32}}v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}}v\right) = -\sqrt{32k}\,t + C$$

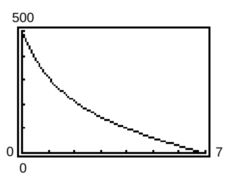
$$\sqrt{\frac{k}{32}}v = \tan\left(C - \sqrt{32k}\,t\right)$$

$$v = \sqrt{\frac{32}{k}}\tan\left(C - \sqrt{32k}\,t\right)$$

When $t = 0$, $v = 500$, $C = \arctan\left(500\sqrt{k/32}\right)$, and we have

$$v(t) = \sqrt{\frac{32}{k}}\tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}\,t\right].$$

(d) When $k = 0.001$, $v(t) = \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]$.




$v(t) = 0$ when $t_0 \approx 6.86$ sec.

(e) $$h = \int_0^{6.86} \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right] dt$$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

## Section 5.10 Hyperbolic Functions

**1.** (a) $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.018$

(b) $\tanh(-2) = \dfrac{\sinh(-2)}{\cosh(-2)} = \dfrac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$

**3.** (a) $\operatorname{csch}(\ln 2) = \dfrac{2}{e^{\ln 2} - e^{-\ln 2}} = \dfrac{2}{2 - (1/2)} = \dfrac{4}{3}$

(b) $$\coth(\ln 5) = \frac{\cosh(\ln 5)}{\sinh(\ln 5)} = \frac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}} = \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}$$

**5.** (a) $\cosh^{-1}(2) = \ln\left(2 + \sqrt{3}\right) \approx 1.317$

(b) $\operatorname{sech}^{-1}\left(\dfrac{2}{3}\right) = \ln\left(\dfrac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$

**7.** $$\tanh^2 x + \operatorname{sech}^2 x = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

**9.** $$\sinh x \cosh y + \cosh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$
$$= \frac{1}{4}[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}]$$
$$= \frac{1}{4}[2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x + y)$$

**11.** $$3 \sinh x + 4 \sinh^3 x = \sinh x(3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2}\right)\left[3 + 4\left(\frac{e^x - e^{-x}}{2}\right)^2\right]$$
$$= \left(\frac{e^x - e^{-x}}{2}\right)[3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2}(e^x - e^{-x})(e^{2x} + e^{-2x} + 1)$$
$$= \frac{1}{2}[e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)$$

**13.** $$\sinh x = \frac{3}{2}$$
$$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \implies \cosh^2 x = \frac{13}{4} \implies \cosh x = \frac{\sqrt{13}}{2}$$
$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$
$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$
$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$
$$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

**15.** $y = \sinh(1 - x^2)$

$y' = -2x \cosh(1 - x^2)$

**17.** $f(x) = \ln(\sinh x)$

$$f'(x) = \frac{1}{\sinh}(\cosh x) = \coth x$$

**19.** $$y = \ln\left(\tanh \frac{x}{2}\right)$$
$$y' = \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2 \sinh(x/2) \cosh(x/2)}$$
$$= \frac{1}{\sinh x} = \operatorname{csch} x$$

**21.** $$h(x) = \frac{1}{4}\sinh(2x) - \frac{x}{2}$$
$$h'(x) = \frac{1}{2}\cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

**23.** $f(t) = \arctan(\sinh t)$

$$f'(t) = \frac{1}{1 + \sinh^2 t}(\cosh t)$$
$$= \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$$

**25.** Let $y = g(x)$.

$$y = x^{\cosh x}$$
$$\ln y = \cosh x \ln x$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{\cosh x}{x} + \sinh x \ln x$$
$$\frac{dy}{dx} = \frac{y}{x}[\cosh x + x(\sinh x) \ln x]$$
$$= \frac{x^{\cosh x}}{x}[\cosh x + x(\sinh x) \ln x]$$

**27.** $y = (\cosh x - \sinh x)^2$

$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$

$= -2(\cosh x - \sinh x)^2 = -2e^{-2x}$

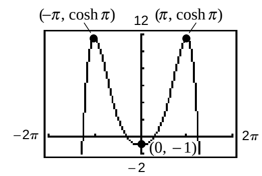
**29.** $f(x) = \sin x \sinh x - \cos x \cosh x, \; -4 \le x \le 4$

$f'(x) = \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x$

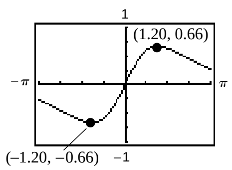
$= 2 \sin x \cosh x = 0$ when $x = 0, \pm\pi$.

Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$




**31.** $g(x) = x \operatorname{sech} x = \dfrac{x}{\cosh x}$




Relative maximum: $(1.20, 0.66)$

Relative minimum: $(-1.20, -0.66)$

**33.** $y = a \sinh x$

$y' = a \cosh x$

$y'' = a \sinh x$

$y''' = a \cosh x$

Therefore, $y''' - y' = 0$.

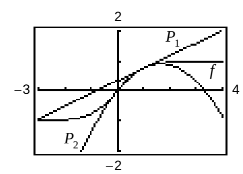
**35.** $f(x) = \tanh x \qquad f(1) = \tanh(1) \approx 0.7616$

$f'(x) = \operatorname{sech}^2 x \qquad f'(1) = \dfrac{1}{\cosh^2(1)} \approx 0.4200$

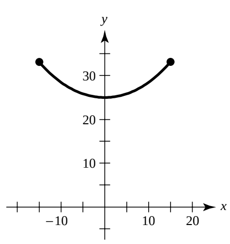
$f''(x) = -2 \operatorname{sech}^2 x \cdot \tanh x \qquad f''(1) \approx -0.6397$

$P_1(x) = f(1) + f'(1)(x - 1) = 0.7616 + 0.42(x - 1)$

$P_2(x) = 0.7616 + 0.42(x - 1) - \dfrac{0.6397}{2}(x - 1)^2$




**37.** (a) $y = 10 + 15 \cosh \dfrac{x}{15}, \; -15 \le x \le 15$




(b) At $x = \pm 15$, $y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0$, $y = 10 + 15 \cosh(1) = 25$.

(c) $y' = \sinh \dfrac{x}{15}$. At $x = 15$, $y' = \sinh(1) \approx 1.175$

**39.** Let $u = 1 - 2x$, $du = -2\,dx$.

$$\int \sinh(1 - 2x)\,dx = -\frac{1}{2}\int \sinh(1 - 2x)(-2)\,dx$$

$$= -\frac{1}{2}\cosh(1 - 2x) + C$$

**41.** Let $u = \cosh(x - 1)$, $du = \sinh(x - 1)\,dx$.

$$\int \cosh^2(x - 1)\sinh(x - 1)\,dx = \frac{1}{3}\cosh^3(x - 1) + C$$

**43.** Let $u = \sinh x$, $du = \cosh x\, dx$.

$$\int \frac{\cosh x}{\sinh x}\, dx = \ln|\sinh x| + C$$

**45.** Let $u = \dfrac{x^2}{2}$, $du = x\, dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2}\, dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2}\right) x\, dx = -\coth \frac{x^2}{2} + C$$

**47.** Let $u = \dfrac{1}{x}$, $du = -\dfrac{1}{x^2}\, dx$.

$$\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2}\, dx = -\int \operatorname{csch} \frac{1}{x} \coth \frac{1}{x}\left(-\frac{1}{x^2}\right) dx = \operatorname{csch} \frac{1}{x} + C$$

**49.** $\displaystyle\int_0^4 \frac{1}{25 - x^2}\, dx = \left[\frac{1}{10} \ln\left|\frac{5 + x}{5 - x}\right|\right]_0^4 = \frac{1}{10} \ln 9 = \frac{1}{5} \ln 3$

**51.** Let $u = 2x$, $du = 2\, dx$.

$$\int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}}(2)\, dx = \Big[\arcsin(2x)\Big]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

**53.** Let $u = x^2$, $du = 2x\, dx$.

$$\int \frac{x}{x^4 + 1}\, dx = \frac{1}{2} \int \frac{2x}{(x^2)^2 + 1}\, dx = \frac{1}{2} \arctan(x^2) + C$$

**55.** $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

**57.** $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}}(\sec^2 x) = |\sec x|$$

**59.** $y = \coth^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x}(2 \cos 2x) = 2 \sec 2x$$

**61.** $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x\left(\frac{2}{\sqrt{1 + 4x^2}}\right) + 2 \sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} = 2 \sinh^{-1}(2x)$$

**63.** See page 395.

**65.** $y = a \operatorname{sech}^{-1}\left(\dfrac{x}{a}\right) - \sqrt{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

**67.** $\displaystyle\int \frac{1}{\sqrt{1 + e^{2x}}}\, dx = \int \frac{e^x}{e^x\sqrt{1 + (e^x)^2}}\, dx = -\operatorname{csch}^{-1}(e^x) + C = -\ln\left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x}\right) + C$

**69.** Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\, dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\, dx = 2\int \frac{1}{\sqrt{1 + \left(\sqrt{x}\right)^2}}\left(\frac{1}{2\sqrt{x}}\right) dx = 2 \sinh^{-1}\sqrt{x} + C = 2 \ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$$

**71.** $\displaystyle\int \frac{-1}{4x - x^2}\, dx = \int \frac{1}{(x - 2)^2 - 4}\, dx = \frac{1}{4} \ln\left|\frac{(x - 2) - 2}{(x - 2) + 2}\right| = \frac{1}{4} \ln\left|\frac{x - 4}{x}\right| + C$

**73.** $\displaystyle\int \frac{1}{1 - 4x - 2x^2}\,dx = \int \frac{1}{3 - 2(x+1)^2}\,dx = \frac{-1}{\sqrt{2}}\int \frac{\sqrt{2}}{\left[\sqrt{2}(x+1)\right]^2 - \left(\sqrt{3}\right)^2}\,dx$

$$= \frac{-1}{2\sqrt{6}} \ln\left|\frac{\sqrt{2}(x+1) - \sqrt{3}}{\sqrt{2}(x+1) + \sqrt{3}}\right| + C = \frac{1}{2\sqrt{6}} \ln\left|\frac{\sqrt{2}(x+1) + \sqrt{3}}{\sqrt{2}(x+1) - \sqrt{3}}\right| + C$$

**75.** Let $u = 4x - 1$, $du = 4\,dx$.

$$y = \int \frac{1}{\sqrt{80 + 8x - 16x^2}}\,dx = \frac{1}{4}\int \frac{4}{\sqrt{81 - (4x-1)^2}}\,dx = \frac{1}{4}\arcsin\left(\frac{4x-1}{9}\right) + C$$

**77.** $\displaystyle y = \int \frac{x^3 - 21x}{5 + 4x - x^2}\,dx = \int\left(-x - 4 + \frac{20}{5 + 4x - x^2}\right)dx = \int(-x - 4)\,dx + 20\int \frac{1}{3^2 - (x-2)^2}\,dx$

$$= -\frac{x^2}{2} - 4x + \frac{20}{6}\ln\left|\frac{(x-2)+3}{(x-2)-3}\right| + C = -\frac{x^2}{2} - 4x + \frac{10}{3}\ln\left|\frac{x+1}{x-5}\right| + C = \frac{-x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{x-5}{x+1}\right| + C$$

**79.**
$$\begin{aligned} A &= 2\int_0^4 \operatorname{sech}\frac{x}{2}\,dx \\ &= 2\int_0^4 \frac{2}{e^{x/2} + e^{-x/2}}\,dx \\ &= 4\int_0^4 \frac{e^{x/2}}{(e^{x/2})^2 + 1}\,dx \\ &= \Big[8\arctan(e^{x/2})\Big]_0^4 \\ &= 8\arctan(e^2) - 2\pi \approx 5.207 \end{aligned}$$

**81.**
$$\begin{aligned} A &= \int_0^2 \frac{5x}{\sqrt{x^4 + 1}}\,dx \\ &= \frac{5}{2}\int_0^2 \frac{2x}{\sqrt{(x^2)^2 + 1}}\,dx \\ &= \left[\frac{5}{2}\ln\left(x^2 + \sqrt{x^4 + 1}\right)\right]_0^2 \\ &= \frac{5}{2}\ln\left(4 + \sqrt{17}\right) \approx 5.237 \end{aligned}$$

**83.** $\displaystyle\int \frac{3k}{16}\,dt = \int \frac{1}{x^2 - 12x + 32}\,dx$

$$\frac{3kt}{16} = \int \frac{1}{(x-6)^2 - 4}\,dx = \frac{1}{2(2)}\ln\left|\frac{(x-6)-2}{(x-6)+2}\right| + C = \frac{1}{4}\ln\left|\frac{x-8}{x-4}\right| + C$$

When $x = 0$: $\quad t = 0$

$$C = -\frac{1}{4}\ln(2)$$

When $x = 1$: $\quad t = 10$

$$\frac{30k}{16} = \frac{1}{4}\ln\left|\frac{-7}{-3}\right| - \frac{1}{4}\ln(2) = \frac{1}{4}\ln\left(\frac{7}{6}\right)$$

$$k = \frac{2}{15}\ln\left(\frac{7}{6}\right)$$

When $t = 20$: $\quad \displaystyle\left(\frac{3}{16}\right)\left(\frac{2}{15}\right)\ln\left(\frac{7}{6}\right)(20) = \frac{1}{4}\ln\frac{x-8}{2x-8}$

$$\ln\left(\frac{7}{6}\right)^2 = \ln\frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

**85.** As $k$ increases, the time required for the object to reach the ground increases.

**87.** $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$y' = \dfrac{e^x - e^{-x}}{2} = \sinh x$

**89.** $y = \cosh^{-1} x$

$\cosh y = x$

$(\sinh y)(y') = 1$

$y' = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$

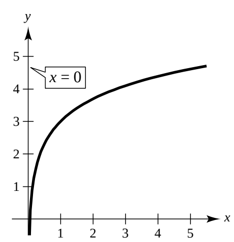
**91.** $y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\dfrac{-2}{e^x + e^{-x}}\right)\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$

## Review Exercises for Chapter 5

**1.** $f(x) = \ln x + 3$

Vertical shift 3 units upward
Vertical asymptote: $x = 0$




**3.** $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}} = \dfrac{1}{5} \ln \dfrac{(2x - 1)(2x + 1)}{4x^2 + 1} = \dfrac{1}{5}[\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$

**5.** $\ln 3 + \dfrac{1}{3}\ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln\left(\dfrac{3\sqrt[3]{4 - x^2}}{x}\right)$

**7.** $\ln\sqrt{x + 1} = 2$

$\sqrt{x + 1} = e^2$

$x + 1 = e^4$

$x = e^4 - 1 \approx 53.598$

**9.** $g(x) = \ln\sqrt{x} = \dfrac{1}{2}\ln x$

$g'(x) = \dfrac{1}{2x}$

**11.** $f(x) = x\sqrt{\ln x}$

$f'(x) = \left(\dfrac{x}{2}\right)(\ln x)^{-1/2}\left(\dfrac{1}{x}\right) + \sqrt{\ln x}$

$= \dfrac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \dfrac{1 + 2\ln x}{2\sqrt{\ln x}}$

**13.** $y = \dfrac{1}{b^2}\left[\ln(a + bx) + \dfrac{a}{a + bx}\right]$

$\dfrac{dy}{dx} = \dfrac{1}{b^2}\left[\dfrac{b}{a + bx} - \dfrac{ab}{(a + bx)^2}\right] = \dfrac{x}{(a + bx)^2}$

**15.** $y = -\dfrac{1}{a}\ln\left(\dfrac{a + bx}{x}\right) = -\dfrac{1}{a}[\ln(a + bx) - \ln x]$

$\dfrac{dy}{dx} = -\dfrac{1}{a}\left(\dfrac{b}{a + bx} - \dfrac{1}{x}\right) = \dfrac{1}{x(a + bx)}$

**17.** $u = 7x - 2,\ du = 7\,dx$

$\displaystyle\int \frac{1}{7x - 2}\,dx = \frac{1}{7}\int \frac{1}{7x - 2}(7)\,dx = \frac{1}{7}\ln|7x - 2| + C$

**19.** $\displaystyle\int \frac{\sin x}{1+\cos x}\,dx = -\int \frac{-\sin x}{1+\cos x}\,dx$

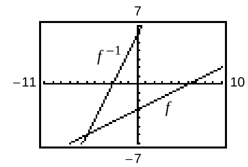
$= -\ln|1+\cos x| + C$

**21.** $\displaystyle\int_1^4 \frac{x+1}{x}\,dx = \int_1^4 \left(1+\frac{1}{x}\right)dx = \Big[x + \ln|x|\Big]_1^4 = 3 + \ln 4$

**23.** $\displaystyle\int_0^{\pi/3} \sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = \ln\left(2+\sqrt{3}\right)$

**25.** (a)
$$f(x) = \tfrac{1}{2}x - 3$$
$$y = \tfrac{1}{2}x - 3$$
$$2(y+3) = x$$
$$2(x+3) = y$$
$$f^{-1}(x) = 2x + 6$$

(b)

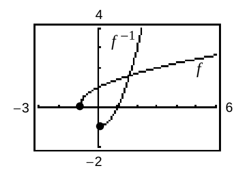



(c) $f^{-1}(f(x)) = f^{-1}\left(\tfrac{1}{2}x - 3\right) = 2\left(\tfrac{1}{2}x - 3\right) + 6 = x$

$f(f^{-1}(x)) = f(2x+6) = \tfrac{1}{2}(2x+6) - 3 = x$

**27.** (a)
$$f(x) = \sqrt{x+1}$$
$$y = \sqrt{x+1}$$
$$y^2 - 1 = x$$
$$x^2 - 1 = y$$
$$f^{-1}(x) = x^2 - 1,\ x \ge 0$$

(b)



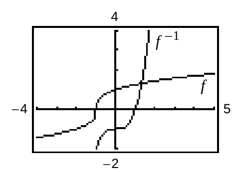

(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x+1}\right) = \sqrt{(x^2-1)^2} - 1 = x$

$f(f^{-1}(x)) = f(x^2 - 1) = \sqrt{(x^2-1)+1}$

$= \sqrt{x^2} = x$ for $x \ge 0$.

**29.** (a)
$$f(x) = \sqrt[3]{x+1}$$
$$y = \sqrt[3]{x+1}$$
$$y^3 - 1 = x$$
$$x^3 - 1 = y$$
$$f^{-1}(x) = x^3 - 1$$

(b)




(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = x$

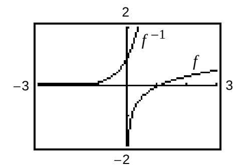
$f(f^{-1}(x)) = f(x^3 - 1) = \sqrt[3]{(x^3-1)+1} = x$

**31.**
$$f(x) = x^3 + 2$$
$$f^{-1}(x) = (x-2)^{1/3}$$
$$(f^{-1})'(x) = \frac{1}{3}(x-2)^{-2/3}$$
$$(f^{-1})'(-1) = \frac{1}{3}(-1-2)^{-2/3} = \frac{1}{3(-3)^{2/3}}$$
$$= \frac{1}{3^{5/3}} \approx 0.160$$

**33.**
$$f(x) = \tan x$$
$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$
$$f'(x) = \sec^2 x$$
$$f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$$
$$(f^{-1})'\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{f'(\pi/6)} = \frac{3}{4}$$

**35.** (a)
$$\begin{aligned} f(x) &= \ln \sqrt{x} \\ y &= \ln \sqrt{x} \\ e^y &= \sqrt{x} \\ e^{2y} &= x \\ e^{2x} &= y \\ f^{-1}(x) &= e^{2x} \end{aligned}$$

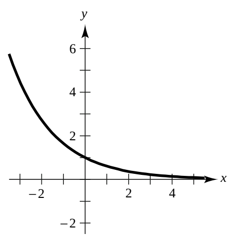
(b)




(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$

**37.** $y = e^{-x/2}$




**39.** $f(x) = \ln(e^{-x^2}) = -x^2$

$f'(x) = -2x$

**41.** $g(t) = t^2 e^t$

$g'(x) = t^2 e^t + 2te^t = te^t(t + 2)$

**43.** $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

**45.** $g(x) = \dfrac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

**47.**
$$\begin{aligned} y(\ln x) + y^2 &= 0 \\ y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) &= 0 \\ (2y + \ln x)\frac{dy}{dx} &= \frac{-y}{x} \\ \frac{dy}{dx} &= \frac{-y}{x(2y + \ln x)} \end{aligned}$$

**49.** Let $u = -3x^2$, $du = -6x\,dx$.

$$\int xe^{-3x^2}\,dx = -\frac{1}{6}\int e^{-3x^2}(-6x)\,dx = -\frac{1}{6}e^{-3x^2} + C$$

**51.**
$$\begin{aligned} \int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx &= \int (e^{3x} - e^x + e^{-x})\,dx \\ &= \frac{1}{3}e^{3x} - e^x - e^{-x} + C \\ &= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C \end{aligned}$$

**53.**
$$\begin{aligned} \int xe^{1-x^2}\,dx &= -\frac{1}{2}\int e^{1-x^2}(-2x)\,dx \\ &= -\frac{1}{2}e^{1-x^2} + C \end{aligned}$$

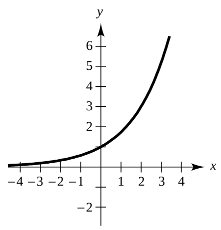
**55.** Let $u = e^x - 1$, $du = e^x\,dx$.

$$\int \frac{e^x}{e^x - 1}\,dx = \ln|e^x - 1| + C$$

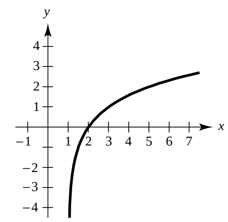
**57.**
$$\begin{aligned} y &= e^x(a \cos 3x + b \sin 3x) \\ y' &= e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x) \\ &= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x] \\ y'' &= e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x] \\ &= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x] \\ y'' - 2y' + 10y &= e^x\{[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x\} = 0 \end{aligned}$$

**59.** Area $= \int_0^4 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$

**61.** $y = 3^{3/2}$




**63.** $y = \log_2(x - 1)$




**65.** $f(x) = 3^{x-1}$

$f'(x) = 3^{x-1}\ln 3$

**67.**
$$y = x^{2x+1}$$
$$\ln y = (2x + 1)\ln x$$
$$\frac{y'}{y} = \frac{2x + 1}{x} + 2\ln x$$
$$y' = y\left(\frac{2x + 1}{x} + 2\ln x\right) = x^{2x+1}\left(\frac{2x + 1}{x} + 2\ln x\right)$$

**69.** $g(x) = \log_3\sqrt{1 - x} = \frac{1}{2}\log_3(1 - x)$

$$g'(x) = \frac{1}{2}\frac{-1}{(1 - x)\ln 3} = \frac{1}{2(x - 1)\ln 3}$$

**71.** $\int (x + 1)5^{(x+1)^2}\,dx = \frac{1}{2}\frac{1}{\ln 5}5^{(x+1)^2} + C$

**73.** (a) $y = x^a$

$y' = ax^{a-1}$

(b) $y = a^x$

$y' = (\ln a)a^x$

(c)
$$y = x^x$$
$$\ln y = x\ln x$$
$$\frac{1}{y}y' = x \cdot \frac{1}{x} + (1)\ln x$$
$$y' = y(1 + \ln x)$$
$$y' = x^x(1 + \ln x)$$

(d) $y = a^a$

$y' = 0$

**75.** $10{,}000 = Pe^{(0.07)(15)}$

$$P = \frac{10{,}000}{e^{1.05}} \approx \$3499.38$$

**77.**
$$P(h) = 30e^{kh}$$
$$P(18{,}000) = 30e^{18{,}000k} = 15$$
$$k = \frac{\ln(1/2)}{18{,}000} = \frac{-\ln 2}{18{,}000}$$
$$P(h) = 30e^{-(h\ln 2)/18{,}000}$$
$$P(35{,}000) = 30e^{-(35{,}000\ln 2)/18{,}000} \approx 7.79 \text{ inches}$$

**79.**
$$P = Ce^{0.015t}$$
$$2C = Ce^{0.015t}$$
$$2 = e^{0.015t}$$
$$\ln 2 = 0.015t$$
$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

**81.**
$$\frac{dy}{dx} = \frac{x^2 + 3}{x}$$
$$\int dy = \int\left(x + \frac{3}{x}\right)dx$$
$$y = \frac{x^2}{2} + 3\ln|x| + C$$

**83.** $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y}\,dy = \int 2x\,dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2 + C_1} = y$$

$$y = Ce^{x^2}$$

**85.** $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2)\,dx - 2xy\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$(x^2 + v^2x^2)\,dx - 2x(vx)(x\,dv + v\,dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2)\,dx - 2x^3v\,dv = 0$$

$$(x^2 - x^2v^2)\,dx = 2x^3v\,dv$$

$$(1 - v^2)\,dx = 2x\,\,dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2}\,dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

**87.** $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2$

$y'' = 6C_2x$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x = C_2x^3)$$

$$= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0$$

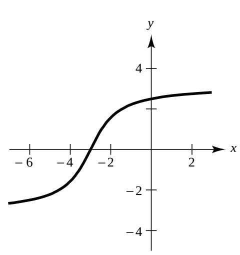
$x = 2, y = 0$: $0 = 2C_1 + 8C_2 \implies C_1 = -4C_2$

$x = 2, y' = 4$: $4 = C_1 + 12C_2$

$$4 = (-4C_2) + 12C_2 = 8C_2 \implies C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

**89.** $f(x) = 2\arctan(x + 3)$

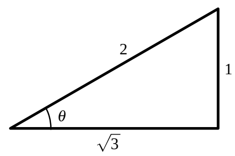


**91.** (a)

$$\text{Let } \theta = \arcsin\frac{1}{2}$$

$$\sin\theta = \frac{1}{2}$$

$$\sin\left(\arcsin\frac{1}{2}\right) = \sin\theta = \frac{1}{2}.$$




(b)

$$\text{Let } \theta = \arcsin\frac{1}{2}$$

$$\sin\theta = \frac{1}{2}$$

$$\cos\left(\arcsin\frac{1}{2}\right) = \cos\theta = \frac{\sqrt{3}}{2}.$$

**93.** $y = \tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

**95.** $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

**97.** $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

$$y' = \frac{2x\arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}}\arcsin x = (\arcsin x)^2$$

**99.** Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{1}{e^{2x}+e^{-2x}}\,dx = \int \frac{e^{2x}}{1+e^{4x}}\,dx = \frac{1}{2}\int \frac{1}{1+(e^{2x})^2}(2e^{2x})\,dx = \frac{1}{2}\arctan(e^{2x}) + C$$

**101.** Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-(x^2)^2}}(2x)\,dx = \frac{1}{2}\arcsin x^2 + C$$

**103.** Let $u = 16 + x^2$, $du = 2x\,dx$.

$$\int \frac{x}{16+x^2}\,dx = \frac{1}{2}\int \frac{1}{16+x^2}(2x)\,dx = \frac{1}{2}\ln(16+x^2) + C$$

**105.** Let $u = \arctan\left(\dfrac{x}{2}\right)$, $du = \dfrac{2}{4+x^2}\,dx$.

$$\int \frac{\arctan(x/2)}{4+x^2}\,dx = \frac{1}{2}\int\left(\arctan\frac{x}{2}\right)\left(\frac{2}{4+x^2}\right)dx = \frac{1}{4}\left(\arctan\frac{x}{2}\right)^2 + C$$

**107.** $\displaystyle\int \frac{dy}{\sqrt{A^2-y^2}} = \int\sqrt{\frac{k}{m}}\,dt$

$$\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}}\,t + C$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}}\,t\right) = \frac{y}{A}$$

$$y = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

**109.** $y = 2x - \cosh\sqrt{x}$

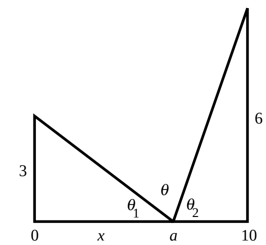
$$y' = 2 - \frac{1}{2\sqrt{x}}\left(\sinh\sqrt{x}\right) = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

**111.** Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{x^4-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2-1}}(2x)\,dx = \frac{1}{2}\ln\left(x^2 + \sqrt{x^4-1}\right) + C$$

## Problem Solving for Chapter 5

**1.** $\tan\theta_1 = \dfrac{3}{x}$




$\tan\theta_2 = \dfrac{6}{10-x}$

Minimize $\theta_1 + \theta_2$:

$f(x) = \theta_1 + \theta_2 = \arctan\left(\dfrac{3}{x}\right) + \arctan\left(\dfrac{6}{10-x}\right)$

$f'(x) = \dfrac{1}{1+\dfrac{9}{x^2}}\left(\dfrac{-3}{x^2}\right) + \dfrac{1}{1+\dfrac{36}{(10-x)^2}}\left(\dfrac{6}{(10-x)^2}\right) = 0$

$$\begin{aligned}
\frac{3}{x^2+9} &= \frac{6}{(10-x)^2+36}\\
(10-x)^2 + 36 &= 2(x^2+9)\\
100 - 20x + x^2 + 36 &= 2x^2 + 18\\
x^2 + 20x - 118 &= 0\\
x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} &= -10 \pm \sqrt{218}
\end{aligned}$$

$a = -10 + \sqrt{218} \approx 4.7648 \qquad f(a) \approx 1.4153$

$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263$ or $98.9°$

Endpoints: $a = 0$: $\theta \approx 1.0304$

$a = 10$: $\theta \approx 1.2793$

Maximum is 1.7263 at $a = -10 + \sqrt{218} \approx 4.7648$.

**3.** $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \implies \ln x = \dfrac{\pi}{2} + 2k\pi$.

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \implies \ln x = \dfrac{3\pi}{2} + 2k\pi$.

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of $f$ is $[-1, 1]$.

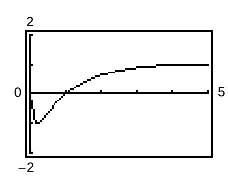
(e) $f'(x) = \dfrac{1}{x}\cos(\ln x)$

$f'(x) = 0 \implies \cos(\ln x) = 0 \implies \ln x = \dfrac{\pi}{2} + k\pi \implies$

$x = e^{\pi/2}$ on $[1, 10]$

$$\left.\begin{aligned} f(e^{\pi/2}) &= 1\\ f(1) &= 0\\ f(10) &\approx 0.7440 \end{aligned}\right\} \text{Maximum is 1 at } x = e^{\pi/2} \approx 4.8105$$

(f)




$\lim\limits_{x\to 0^+} f(x)$ seems to be $-\dfrac{1}{2}$. (This in incorrect.)

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \ldots$

we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots$

we have $f(x) = -1$.

That is, as $x \to 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus $\lim\limits_{x\to 0+} \sin(\ln x)$ does not exist.

You can verifiy this by graphing $f(x)$ on small intervals close to the origin.

**5.** (a) $\dfrac{\text{Area sector}}{\text{Area circle}} = \dfrac{t}{2\pi} \Rightarrow \text{Area sector} = \dfrac{t}{2\pi}(\pi) = \dfrac{t}{2}$

(b) $\text{Area } AOP = \dfrac{1}{2}(\text{base})(\text{height}) - \displaystyle\int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$

$$A(t) = \frac{1}{2}\cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

$$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1}\sinh t$$

$$= \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sinh^2 t$$

$$= \frac{1}{2}[\cosh^2 t - \sinh^2 t] = \frac{1}{2}$$

$A(t) = \dfrac{1}{2}t + C$. But, $A(0) = C = 0 \Rightarrow C = 0$

Thus, $A(t) = \dfrac{1}{2}t$ or $t = 2A(t)$.

**7.**
$$y = \ln x$$
$$y' = \frac{1}{x}$$
$$y - b = \frac{1}{a}(x - a)$$
$$y = \frac{1}{a}x + b - 1 \quad \text{Tangent line}$$

If $x = 0$, $c = b - 1$. Thus, $b - c = b - (b - 1) = 1$.

**9.** Let $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$, $x = u^2 - 2u + 1$, $dx = (2u - 2)du$.

$$\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x}\, dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)}\, du$$
$$= \int_2^3 \frac{2(u - 1)}{u^2 - u}\, du$$
$$= \int_2^3 \frac{2}{u}\, du$$
$$= \Big[2 \ln u\Big]_2^3$$
$$= 2\ln 3 - 2\ln 2 = 2\ln\left(\frac{3}{2}\right)$$
$$\approx 0.8109$$

**11.** (a)
$$\frac{dy}{dt} = y^{1.01}$$
$$\int y^{-1.01}\, dy = \int dt$$
$$\frac{y^{-0.01}}{-0.01} = t + C_1$$
$$\frac{1}{y^{0.01}} = -0.01t + C$$
$$y^{0.01} = \frac{1}{C - 0.01t}$$
$$y = \frac{1}{(C - 0.01t)^{100}}$$

$y(0) = 1$: $1 = \dfrac{1}{C^{100}} \Rightarrow C = 1$

Hence, $y = \dfrac{1}{(1 - 0.01t)^{100}}$.

For $T = 100$, $\displaystyle\lim_{t\to T^-} y = \infty$.

(b)
$$\int y^{-(1+\varepsilon)}\, dy = \int k\, dt$$
$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$
$$y^{-\varepsilon} = -\varepsilon kt + C$$
$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$y(0) = y_0 = \dfrac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \dfrac{1}{y_0} \Rightarrow C = \left(\dfrac{1}{y_0}\right)^{\varepsilon}$

Hence, $y = \dfrac{1}{\left(\dfrac{1}{y_0^{\varepsilon}} - \varepsilon kt\right)^{1/\varepsilon}}$.

For $t \to \dfrac{1}{y_0^{\varepsilon}\varepsilon k}$, $y \to \infty$.

**13.** Since $\dfrac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y-20}\,dy = \int k\,dt$$

$$\ln(y - 20) = kt + C$$

$$y = Ce^{kt} + 20.$$

When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = (28/52) = (7/13)$, and $k = \ln(7/13)$.
Thus, $y = 52e^{[\ln(7/13)]t} + 20$.

When $t = 5$, $y = 52e^{5\ln(7/13)} + 20 \approx 22.35°$.

**15.** (a) $\dfrac{dS}{dt} = k_1 S(L - S)$

$S = \dfrac{L}{1 + Ce^{-kt}}$ is a solution because

$$\frac{dS}{dt} = -L(1 + Ce^{-kt})^{-2}(-C\,ke^{-kt})$$

$$= \frac{LC\,ke^{-kt}}{(1 + Ce^{-kt})^2}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \frac{C\,Le^{-kt}}{1 + Ce^{-kt}}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$

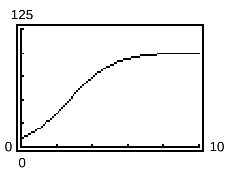
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \implies C = 9$. And,
$S = 20$ when $t = 1 \implies k = -\ln(4/9)$.

Particular Solution. $S = \dfrac{100}{1 + 9e^{\ln(4/9)t}}$

$$= \frac{100}{1 + 9e^{-0.8109t}}$$

(c)




(b) $\dfrac{dS}{dt} = \ln\left(\dfrac{4}{9}\right) S(100 - S)$

$$\frac{d^2S}{dt^2} = \ln\left(\frac{4}{9}\right)\left[S\left(-\frac{dS}{dt}\right) + (100 - S)\frac{dS}{dt}\right]$$

$$= \ln\left(\frac{4}{9}\right)(100 - 2S)\frac{dS}{dt}$$

$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$

Choosing $S = 50$, we have:

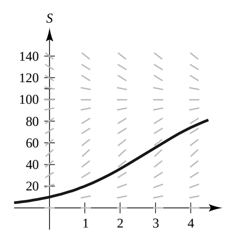
$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$2 = 1 + 9e^{\ln(4/9)t}$$

$$\frac{\ln(1/9)}{\ln(4/9)} = t$$

$$t \approx 2.7 \text{ months}$$

(d)




(e) Sales will decrease toward the line $S = L$.

# PART I

# CHAPTER 6
## Applications of Integration

# CHAPTER 6
# Applications of Integration

## Section 6.1 Area of a Region Between Two Curves

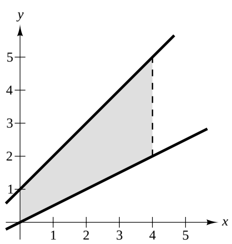
Solutions to Odd-Numbered Exercises

**1.** $A = \int_0^6 [0 - (x^2 - 6x)]\,dx = -\int_0^6 (x^2 - 6x)\,dx$

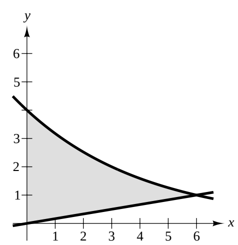
**3.** $A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)]\,dx = \int_0^3 (-2x^2 + 6x)\,dx$

**5.** $A = 2\int_{-1}^0 3(x^3 - x)\,dx = 6\int_{-1}^0 (x^3 - x)\,dx$ or $-6\int_0^1 (x^3 - x)\,dx$

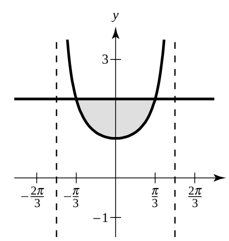
**7.** $\int_0^4 \left[(x + 1) - \dfrac{x}{2}\right] dx$



**9.** $\int_0^6 \left[4(2^{-x/3}) - \dfrac{x}{6}\right] dx$



**11.** $\int_{-\pi/3}^{\pi/3} [2 - \sec x]\,dx$

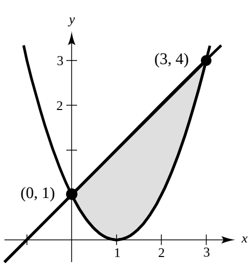


**13.** $f(x) = x + 1$

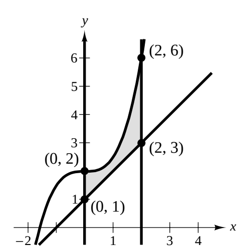
$g(x) = (x - 1)^2$

$A \approx 4$

Matches (d)




**15.** $$A = \int_0^2 \left[\left(\frac{1}{2}x^3 + 2\right) - (x + 1)\right] dx$$
$$= \int_0^2 \left(\frac{1}{2}x^3 - x + 1\right) dx$$
$$= \left[\frac{x^4}{8} - \frac{x^2}{2} + x\right]_0^2$$
$$= \left(\frac{16}{8} - \frac{4}{2} + 2\right) - 0 = 2$$



**17.** The points of intersection are given by:

$$x^2 - 4x = 0$$

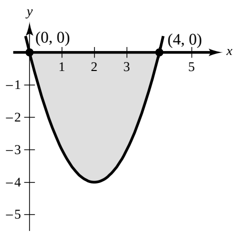
$$x(x-4) = 0 \quad \text{when} \quad x = 0, 4$$

$$A = \int_0^4 [g(x) - f(x)]\,dx$$

$$= -\int_0^4 (x^2 - 4x)\,dx$$

$$= -\left[\frac{x^3}{3} - 2x^2\right]_0^4$$

$$= \frac{32}{3}$$




**19.** The points of intersection are given by:

$$x^2 + 2x + 1 = 3x + 3$$

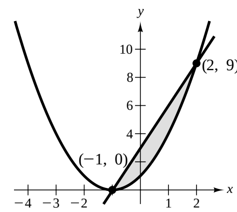
$$(x-2)(x+1) = 0 \quad \text{when} \quad x = -1, 2$$

$$A = \int_{-1}^2 [g(x) - f(x)]\,dx$$

$$= \int_{-1}^2 [(3x+3) - (x^2 + 2x + 1)]\,dx$$

$$= \int_{-1}^2 (2 + x - x^2)\,dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}$$



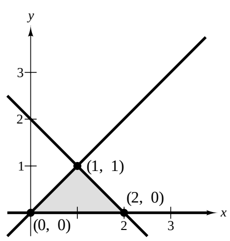

**21.** The points of intersection are given by:

$$x = 2 - x \quad \text{and} \quad x = 0 \quad \text{and} \quad 2 - x = 0$$

$$x = 1 \qquad\qquad x = 0 \qquad\qquad x = 2$$

$$A = \int_0^1 [(2-y) - (y)]\,dy = \left[2y - y^2\right]_0^1 = 1$$

Note that if we integrate with respect to x, we need two integrals. Also, note that the region is a triangle.




**23.** The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

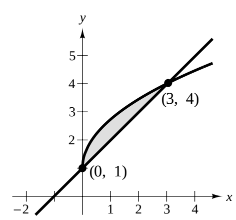
$$\sqrt{3x} = x \quad \text{when} \quad x = 0, 3$$

$$A = \int_0^3 [f(x) - g(x)]\,dx$$

$$= \int_0^3 \left[\left(\sqrt{3x} + 1\right) - (x+1)\right]\,dx$$

$$= \int_0^3 \left[(3x)^{1/2} - x\right]\,dx$$

$$= \left[\frac{2}{9}(3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2}$$




**25.** The points of intersection are given by:

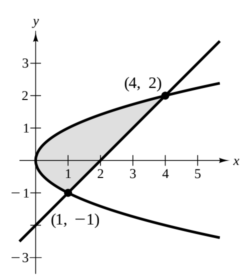
$$y^2 = y + 2$$

$$(y-2)(y+1) = 0 \quad \text{when} \quad y = -1, 2$$

$$A = \int_{-1}^2 [g(y) - f(y)]\,dy$$

$$= \int_{-1}^2 [(y+2) - y^2]\,dy$$

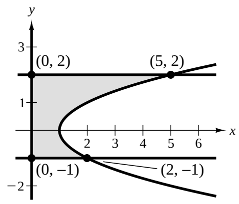
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2}$$



**27.** $A = \int_{-1}^{2} [f(y) - g(y)]\, dy$

$= \int_{-1}^{2} [(y^2 + 1) - 0]\, dy$

$= \left[\frac{y^3}{3} + y\right]_{-1}^{2} = 6$



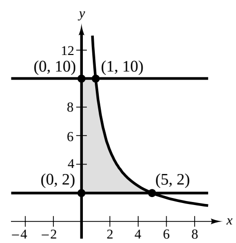

**29.** $y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$

$A = \int_{2}^{10} \frac{10}{y}\, dy$

$= \left[10 \ln y\right]_{2}^{10}$

$= 10(\ln 10 - \ln 2)$

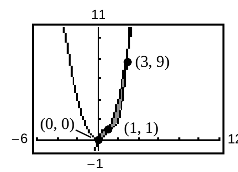
$= 10 \ln 5 \approx 16.0944$




**31.** The points of intersection are given by:

$x^3 - 3x^2 + 3x = x^2$

$x(x - 1)(x - 3) = 0 \quad \text{when} \quad x = 0, 1, 3$




$A = \int_{0}^{1} [f(x) - g(x)]\, dx + \int_{1}^{3} [g(x) - f(x)]\, dx$

$= \int_{0}^{1} [(x^3 - 3x^2 + 3x) - x^2]\, dx + \int_{1}^{3} [x^2 - (x^3 - 3x^2 + 3x)]\, dx$

$= \int_{0}^{1} (x^3 - 4x^2 + 3x)\, dx + \int_{1}^{3} (-x^3 + 4x^2 - 3x)\, dx$

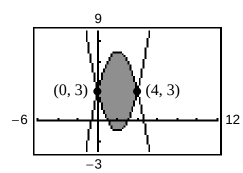
$= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_{0}^{1} + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_{1}^{3} = \frac{37}{12}$

Numerical Approximation: $0.417 + 2.667 \approx 3.083$

**33.** The points of intersection are given by:

$x^2 - 4x + 3 = 3 + 4x - x^2$

$2x(x - 4) = 0 \quad \text{when} \quad x = 0, 4$




$A = \int_{0}^{4} [(3 + 4x - x^2) - (x^2 - 4x + 3)]\, dx$

$= \int_{0}^{4} (-2x^2 + 8x)\, dx$

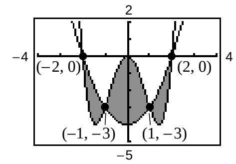
$= \left[-\frac{2x^3}{3} + 4x^2\right]_{0}^{4} = \frac{64}{3}$

Numerical Approximation: 21.333

**35.** $f(x) = x^4 - 4x^2, \quad g(x) = x^2 - 4$

The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$
$$x^4 - 5x^2 + 4 = 0$$
$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when} \quad x = \pm 2, \pm 1$$



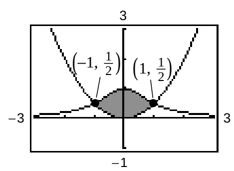

By symmetry,

$$\begin{aligned} A &= 2\int_0^1 [(x^4 - 4x^2) - (x^2 - 4)]\,dx + 2\int_1^2 [(x^2 - 4) - (x^4 - 4x^2)]\,dx \\ &= 2\int_0^1 (x^4 - 5x^2 + 4)\,dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\,dx \\ &= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_0^1 + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2 \\ &= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8. \end{aligned}$$

Numerical Approximation: $5.067 + 2.933 = 8.0$

**37.** The points of intersection are given by:

$$\begin{aligned} \frac{1}{1 + x^2} &= \frac{x^2}{2} \\ x^4 + x^2 - 2 &= 0 \\ (x^2 + 2)(x^2 - 1) &= 0 \\ x &= \pm 1 \end{aligned}$$



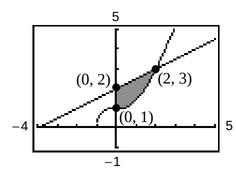

$$\begin{aligned} A &= 2\int_0^1 [f(x) - g(x)]\,dx \\ &= 2\int_0^1 \left[\frac{1}{1 + x^2} - \frac{x^2}{2}\right]dx \\ &= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1 \\ &= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237 \end{aligned}$$

Numerical Approximation: 1.237

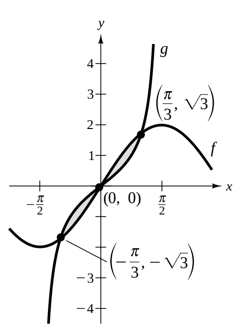
**39.** $\sqrt{1 + x^3} \le \frac{1}{2}x + 2$ on $[0, 2]$

Numerical approximation: 1.759

$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1 + x^3}\right]dx \approx 1.759$$



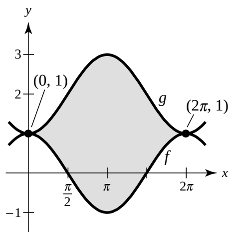

**41.** 
$$\begin{aligned} A &= 2\int_0^{\pi/3} [f(x) - g(x)]\,dx \\ &= 2\int_0^{\pi/3} (2\sin x - \tan x)\,dx \\ &= 2\Big[-2\cos x + \ln|\cos x|\Big]_0^{\pi/3} \\ &= 2(1 - \ln 2) \approx 0.614 \end{aligned}$$



**43.** $A = \int_0^{2\pi} [(2 - \cos x) - \cos x]\,dx$

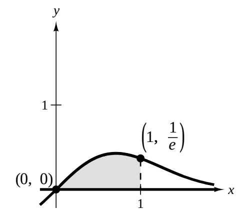
$= 2\int_0^{2\pi} (1 - \cos x)\,dx$

$= 2\Big[x - \sin x\Big]_0^{2\pi} = 4\pi \approx 12.566$



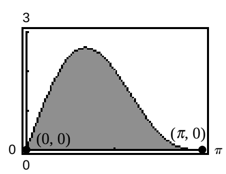

**45.** $A = \int_0^{1} [xe^{-x^2} - 0]\,dx$

$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$



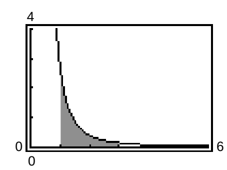

**47.** $A = \int_0^{\pi} [(2\sin x + \sin 2x) - 0]\,dx$

$= \left[-2\cos x - \frac{1}{2}\cos 2x\right]_0^{\pi} = 4.0$




**49.** $A = \int_1^{3} \left[\frac{1}{x^2}e^{1/x} - 0\right]dx$

$= \Big[-e^{1/x}\Big]_1^3 = e - e^{1/3} \approx 1.323$

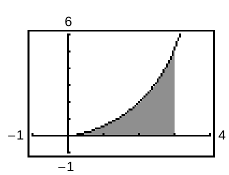



**51.** (a) $y = \sqrt{\dfrac{x^3}{4 - x}}, \quad y = 0, \quad x = 3$

(b) $A = \int_0^3 \sqrt{\dfrac{x^3}{4 - x}}\,dx,$

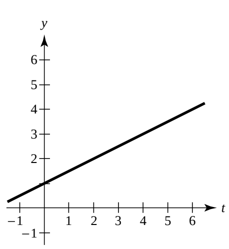
No, it cannot be evaluated by hand.

(c) 4.7721

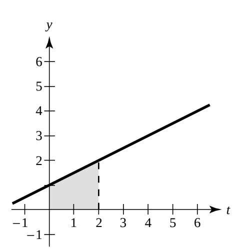



**53.** $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right)dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

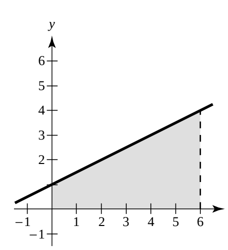
(a) $F(0) = 0$




(b) $F(2) = \dfrac{2^2}{4} + 2 = 3$

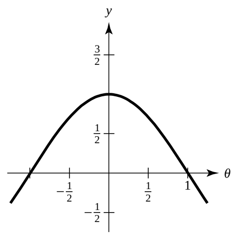



(c) $F(6) = \dfrac{6^2}{4} + 6 = 15$

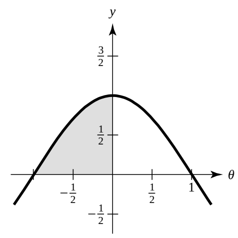


**55.** $F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2}\, d\theta = \left[\frac{2}{\pi} \sin \frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi} \sin \frac{\pi\alpha}{2} + \frac{2}{\pi}$

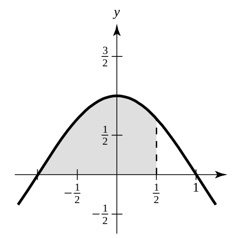
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$



(c) $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

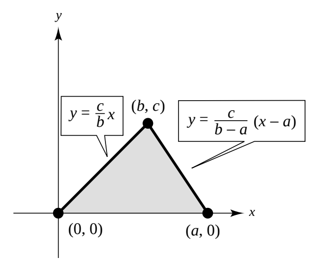


**57.** $A = \int_0^c \left[\left(\frac{b-a}{c} y + a\right) - \frac{b}{c} y\right] dy$

$= \int_0^c \left(-\frac{a}{c} y + a\right) dy$

$= \left[-\frac{a}{2c} y^2 + ay\right]_0^c$

$= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left(= \frac{1}{2}\text{(base)(height)}\right)$




**59.** $f(x) = x^3$

$f'(x) = 3x^2$

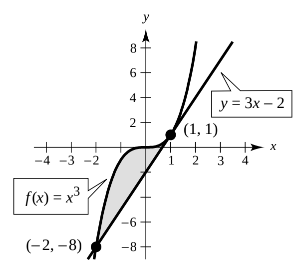
At $(1, 1)$, $f'(1) = 3$.

Tangent line:

$y - 1 = 3(x - 1)$ or $y = 3x - 2$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^{1} [x^3 - (3x - 2)]\, dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1} = \frac{27}{4}$$



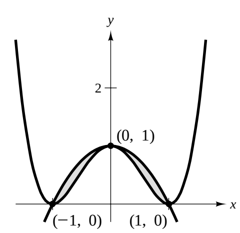

**61.** The variable is $y$.

**63.** $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$A = \int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\, dx$

$= \int_{-1}^{1} (x^2 - x^4)\, dx$

$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$




You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.

**65.** Offer 2 is better because the accumulated salary (area under the curve) is larger.

**67.** $$A = \int_{-3}^{3} (9 - x^2)\, dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b]\, dx = 18$$

$$\int_{0}^{\sqrt{9-b}} [(9 - b) - x^2]\, dx = 9$$

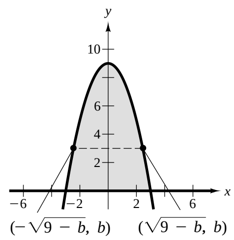
$$\left[(9 - b)x - \frac{x^3}{3}\right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3}(9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

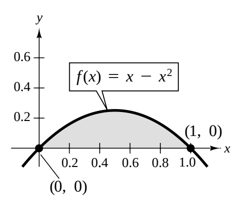
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$




**69.** $\displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\, \Delta x$

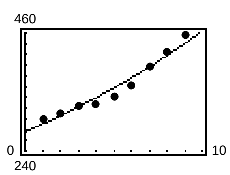
where $x_i = \dfrac{i}{n}$ and $\Delta x = \dfrac{1}{n}$ is the same as

$$\int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

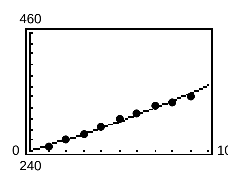



**71.** $$\int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)]\, dt = \int_0^5 0.13t\, dt = \left[\frac{0.13t^2}{2}\right]_0^5 = \$1.625 \text{ billion}$$

**73.** (a) $y_1 = (275.0675)(1.0537)^t = (275.0675)e^{0.0523t}$]




(b) $y_2 = (239.9407)(1.0417)^t = (239.9407)e^{0.0408t}$

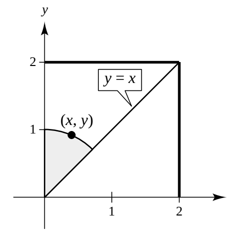



(c) $\displaystyle\int_{10}^{15} (y_1 - y_2)\, dt \approx 649.5$ billion dollars

(d) No, model $y_1 > y_2$ forever because $1.0537 > 1.0417$.

No, these models are not accurate. According to news reports, $E > R$ eventually.

**75.** The total area is 8 times the area of the shaded region to the right. A point $(x, y)$ is on the upper boundary of the region if




$$\sqrt{x^2 + y^2} = 2 - y$$
$$x^2 + y^2 = 4 - 4y + y^2$$
$$x^2 = 4 - 4y$$
$$4y = 4 - x^2$$
$$y = 1 - \frac{x^2}{4}.$$

We now determine where this curve intersects the line $y = x$.

$$x = 1 - \frac{x^2}{4}$$
$$x^2 + 4x - 4 = 0$$
$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \implies x = -2 + 2\sqrt{2}$$

$$\text{Total area} = 8\int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx$$

$$= 8\left[x - \frac{x^3}{12} - \frac{x^2}{2}\right]_0^{-2+2\sqrt{2}} = \frac{16}{3}\left(4\sqrt{2} - 5\right) \approx 8(0.4379) = 3.503$$

**77.** (a) $A = 2\left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5 - x}\right) dx + \int_5^{5.5} (1 - 0)\, dx\right]$

$$= 2\left(\left[x + \frac{2}{9}(5 - x)^{3/2}\right]_0^5 + \left[x\right]_5^{5.5}\right) = 2\left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5\right) \approx 6.031 \text{ m}^2$$

(b) $V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$

(c) $5000\,V \approx 5000(12.062) = 60{,}310$ pounds

**79.** True

**81.** False. Let $f(x) = x$ and $g(x) = 2x - x^2$. $f$ and $g$ intersect at $(1, 1)$, the midpoint of $[0, 2]$. But

$$\int_a^b [f(x) - g(x)]\, dx = \int_0^2 [x - (2x - x^2)]\, dx = \frac{2}{3} \neq 0.$$

## Section 6.2 Volume: The Disk Method

**1.** $V = \pi\int_0^1 (-x + 1)^2\, dx = \pi\int_0^1 (x^2 - 2x + 1)\, dx = \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{\pi}{3}$

**3.** $V = \pi\int_1^4 \left(\sqrt{x}\right)^2 dx = \pi\int_1^4 x\, dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \frac{15\pi}{2}$

**5.** $V = \pi\int_0^1 [(x^2)^2 - (x^3)^2]\, dx = \pi\int_0^1 (x^4 - x^6)\, dx = \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{2\pi}{35}$

**7.** $y = x^2 \implies x = \sqrt{y}$

$$V = \pi\int_0^4 \left(\sqrt{y}\right)^2 dy = \pi\int_0^4 y\, dy$$
$$= \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi$$

**9.** $y = x^{2/3} \implies x = y^{3/2}$

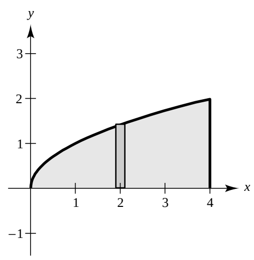
$$V = \pi\int_0^1 (y^{3/2})^2\, dy = \pi\int_0^1 y^3\, dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}$$

**11.** $y = \sqrt{x},\ y = 0,\ x = 4$

(a) $R(x) = \sqrt{x},\ r(x) = 0$

$$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx$$

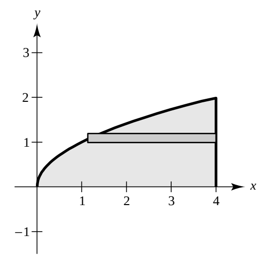
$$= \pi \int_0^4 x\,dx = \left[\frac{\pi}{2}x^2\right]_0^4 = 8\pi$$




(b) $R(y) = 4,\ r(y) = y^2$

$$V = \pi \int_0^2 \left(16 - y^4\right) dy$$

$$= \pi\left[16y - \frac{1}{5}y^5\right]_0^2 = \frac{128\pi}{5}$$

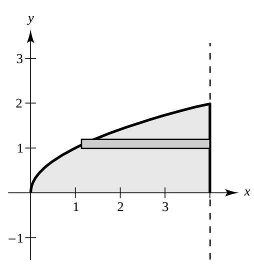



(c) $R(y) = 4 - y^2,\ r(y) = 0$

$$V = \pi\int_0^2 \left(4 - y^2\right)^2 dy$$

$$= \pi\int_0^2 \left(16 - 8y^2 + y^4\right) dy$$

$$= \pi\left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{256\pi}{15}$$

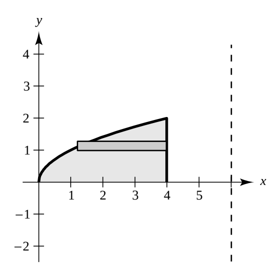



(d) $R(y) = 6 - y^2,\ r(y) = 2$

$$V = \pi\int_0^2 \left[\left(6 - y^2\right)^2 - 4\right] dy$$

$$= \pi\int_0^2 \left(32 - 12y^2 + y^4\right) dy$$

$$= \pi\left[32y - 4y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{192\pi}{5}$$



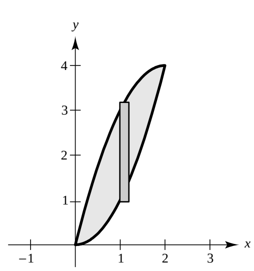

**13.** $y = x^2,\ y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2\ \ r(x) = x^2$

$$V = \pi\int_0^2 \left[\left(4x - x^2\right)^2 - x^4\right] dx$$

$$= \pi\int_0^2 \left(16x^2 - 8x^3\right) dx$$

$$= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3}$$

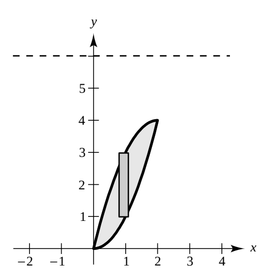



(b) $R(x) = 6 - x^2,\ r(x) = 6 - \left(4x - x^2\right)$

$$V = \pi\int_0^2 \left[\left(6 - x^2\right)^2 - \left(6 - 4x + x^2\right)^2\right] dx$$

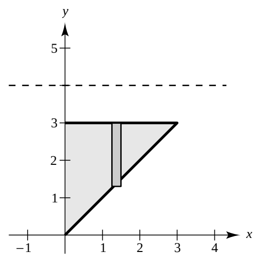
$$= 8\pi\int_0^2 \left(x^3 - 5x^2 + 6x\right) dx$$

$$= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3}$$



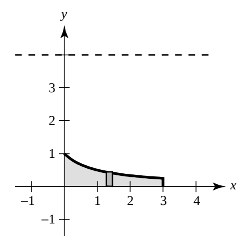
**15.** $R(x) = 4 - x, \; r(x) = 1$

$$V = \pi \int_0^3 \left[(4 - x)^2 - (1)^2\right] dx$$
$$= \pi \int_0^3 (x^2 - 8x + 15)\, dx$$
$$= \pi \left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3 = 18\pi$$



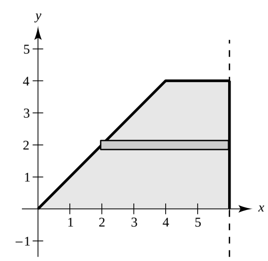

**17.** $R(x) = 4, \; r(x) = 4 - \dfrac{1}{1 + x}$

$$V = \pi \int_0^3 \left[4^2 - \left(4 - \frac{1}{1 + x}\right)^2\right] dx$$
$$= \pi \int_0^3 \left[\frac{8}{1 + x} - \frac{1}{(1 + x)^2}\right] dx$$
$$= \pi \left[8 \ln(1 + x) + \frac{1}{1 + x}\right]_0^3$$
$$= \pi \left[8 \ln 4 + \frac{1}{4} - 1\right]$$
$$= \left(8 \ln 4 - \frac{3}{4}\right)\pi \approx 32.485$$




**19.** $R(y) = 6 - y, \; r(y) = 0$

$$V = \pi \int_0^4 (6 - y)^2\, dy$$
$$= \pi \int_0^4 (y^2 - 12y + 36)\, dy$$
$$= \pi \left[\frac{y^3}{3} - 6y^2 + 36y\right]_0^4$$
$$= \frac{208\pi}{3}$$

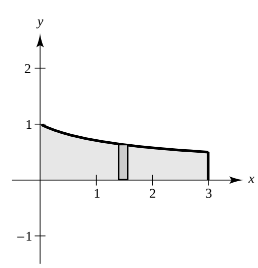



**21.** $R(y) = 6 - y^2, \; r(y) = 2$

$$V = \pi \int_{-2}^2 \left[(6 - y^2)^2 - (2)^2\right] dy$$
$$= 2\pi \int_0^2 (y^4 - 12y^2 + 32)\, dy$$
$$= 2\pi \left[\frac{y^5}{5} - 4y^3 + 32y\right]_0^2$$
$$= \frac{384\pi}{5}$$

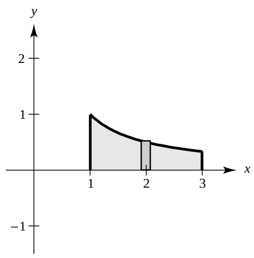
**23.** $R(x) = \dfrac{1}{\sqrt{x + 1}}, \; r(x) = 0$

$$V = \pi \int_0^3 \left(\frac{1}{\sqrt{x + 1}}\right)^2 dx$$
$$= \pi \int_0^3 \frac{1}{x + 1}\, dx$$
$$= \left[\pi \ln|x + 1|\right]_0^3 = \pi \ln 4$$



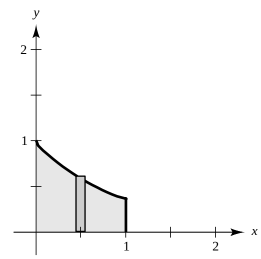
**25.** $R(x) = \dfrac{1}{x},\ r(x) = 0$

$$V = \pi\int_1^4 \left(\frac{1}{x}\right)^2 dx$$
$$= \pi\left[-\frac{1}{x}\right]_1^4$$
$$= \frac{3\pi}{4}$$



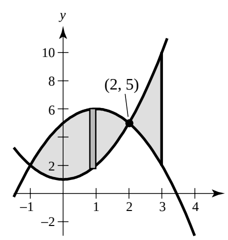

**27.** $R(x) = e^{-x},\ r(x) = 0$

$$V = \pi\int_0^1 (e^{-x})^2\, dx$$
$$= \pi\int_0^1 e^{-2x}\, dx$$
$$= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$
$$= \frac{\pi}{2}(1 - e^{-2}) \approx 1.358$$



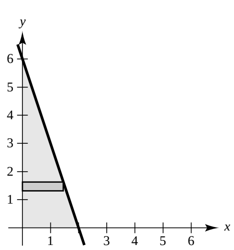

**29.** $V = \pi\displaystyle\int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2)]\, dx + \pi\int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2]\, dx$

$$= \pi\int_0^2 (-4x^3 - 8x^2 + 20x + 24)\, dx + \pi\int_2^3 (4x^3 + 8x^2 - 20x - 24)\, dx$$
$$= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^3 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3$$
$$= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3}$$

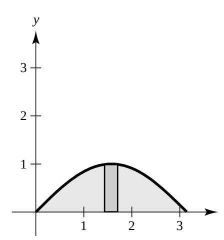



**31.** $y = 6 - 3x \implies x = \dfrac{1}{3}(6 - y)$

$$V = \pi\int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy$$
$$= \frac{\pi}{9}\int_0^6 [36 - 12y + y^2]\, dy$$
$$= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6$$
$$= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right]$$
$$= 8\pi$$




**33.** $V = \pi\displaystyle\int_0^{\pi} [\sin x]^2\, dx \approx 4.9348$

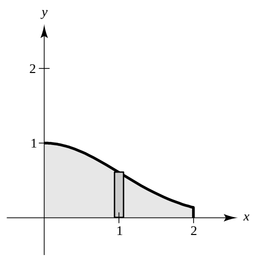


**35.** $V = \pi \int_0^2 [e^{-x^2}]^2\, dx \approx 1.9686$

**37.** $V = \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2\, dx \approx 49.0218$

**39.** $A \approx 3$

Matches (a)




**41.** Disk Method:

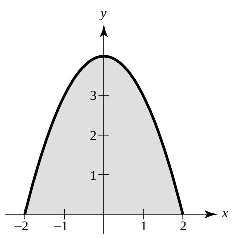
$$V = \pi\int_a^b [R(x)]^2\, dx \quad \text{or} \quad V = \pi\int_c^d [R(y)]^2\, dy$$

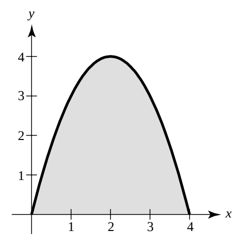
Washer Method:

$$V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)\, dx \quad \text{or}$$

$$V = \pi\int_c^d ([R(y)]^2 - [r(y)]^2)\, dy$$

**43.**




The volumes are the same because the solid has been translated horizontally.

**45.** $R(x) = \dfrac{1}{2}x,\ r(x) = 0$

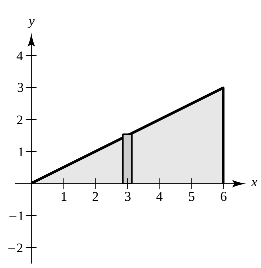
$$V = \pi\int_0^6 \frac{1}{4}x^2\, dx$$

$$= \left[\frac{\pi}{12}x^3\right]_0^6 = 18\pi$$

**Note:** $V = \dfrac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$



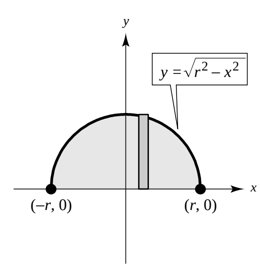

**47.** $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$V = \pi\int_{-r}^r (r^2 - x^2)\, dx$$

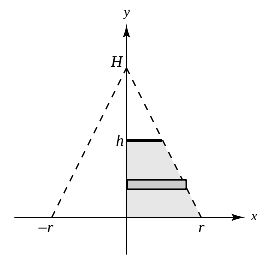
$$= 2\pi\int_0^r (r^2 - x^2)\, dx$$

$$= 2\pi\left[r^2x - \frac{1}{3}x^3\right]_0^r$$

$$= 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3$$



**49.** $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right),\ R(y) = r\left(1 - \frac{y}{H}\right),\ r(y) = 0$

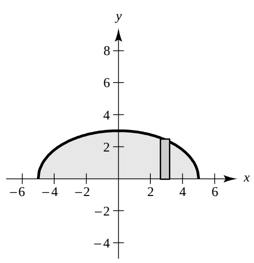



$$\begin{aligned}V &= \pi\int_0^h \left[r\left(1 - \frac{y}{H}\right)\right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right) dy\\ &= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h\\ &= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right)\\ &= \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)\end{aligned}$$

**51.** $V = \pi\int_0^2 \left(\frac{1}{8}x^2\sqrt{2 - x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2 - x)\, dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}$

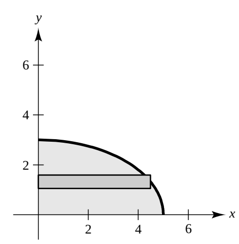
**53.** (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2},\ r(x) = 0$

$$\begin{aligned}V &= \frac{9\pi}{25}\int_{-5}^5 (25 - x^2)\, dx\\ &= \frac{18\pi}{25}\int_0^5 (25 - x^2)\, dx\\ &= \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5 = 60\pi\end{aligned}$$

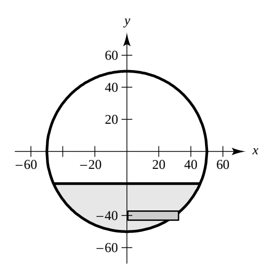



(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2},\ r(y) = 0,\ x \geq 0$

$$\begin{aligned}V &= \frac{25\pi}{9}\int_0^3 (9 - y^2)\, dy\\ &= \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3 = 50\pi\end{aligned}$$




**55.** Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500{,}000}{3}\ \text{ft}^3$




Volume of water in the tank:

$$\begin{aligned}\pi\int_{-50}^{y_0} \left(\sqrt{2500 - y^2}\right)^2 dy &= \pi\int_{-50}^{y_0} (2500 - y^2)\, dy\\ &= \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0}\\ &= \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)\end{aligned}$$

When the tank is one-fourth of its capacity:

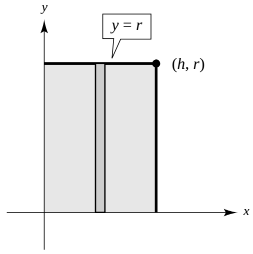
$$\begin{aligned}\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) &= \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}00}{3}\right)\\ 125{,}000 &= 7500y_0 - y_0^3 + 250{,}000\\ y_0^3 - 7500y_0 - 125{,}000 &= 0\\ y_0 &\approx -17.36\end{aligned}$$

Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

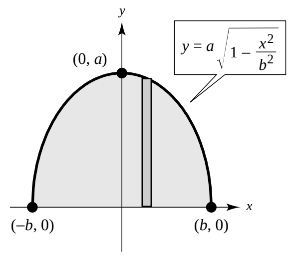
**57.** (a) $\pi\int_0^h r^2\,dx$ (ii)

is the volume of a right circular cylinder with radius $r$ and height $h$.



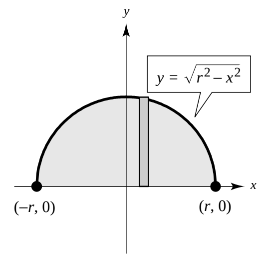

(b) $\pi\int_{-b}^{b}\left(a\sqrt{1-\frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



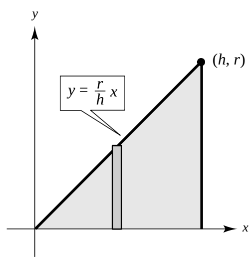

(c) $\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius $r$.



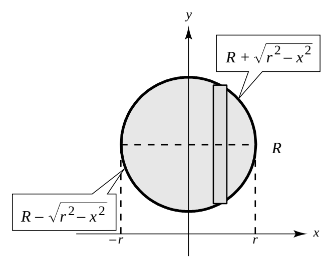

(d) $\pi\int_0^h\left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as $r$ and height $h$.

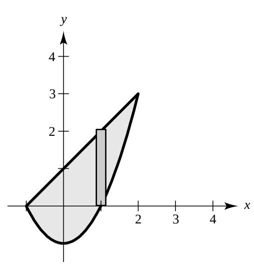



(e) $\pi\int_{-r}^{r}\left[\left(R+\sqrt{r^2-x^2}\right)^2-\left(R-\sqrt{r^2-x^2}\right)^2\right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as $r$ and the distance from the axis of the torus to the center of its cross section as $R$.




**59.**



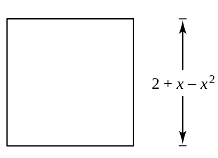

Base of Cross Section $= (x+1)-(x^2-1) = 2+x-x^2$

(a) $A(x) = b^2 = (2+x-x^2)^2$

$= 4+4x-3x^2-2x^3+x^4$

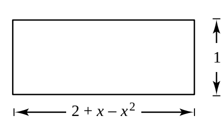
$$V = \int_{-1}^{2}(4+4x-3x^2-2x^3+x^4)\,dx$$

$$= \left[4x+2x^3-x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$

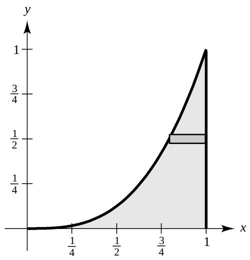



(b) $A(x) = bh = (2+x-x^2)1$

$$V = \int_{-1}^{2}(2+x-x^2)\,dx = \left[2x+\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



**61.**



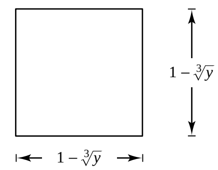

Base of Cross Section $= 1 - \sqrt[3]{y}$

(a) $A(y) = b^2 = \left(1 - \sqrt[3]{y}\right)^2$

$$V = \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy$$

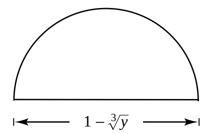
$$= \int_0^1 (1 - 2y^{1/3} + y^{2/3})\, dy$$

$$= \left[y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3}\right]_0^1 = \frac{1}{10}$$




(b) $A(y) = \dfrac{1}{2}\pi r^2 = \dfrac{1}{2}\pi\left(\dfrac{1 - \sqrt[3]{y}}{2}\right)^2 = \dfrac{1}{8}\pi\left(1 - \sqrt[3]{y}\right)^2$

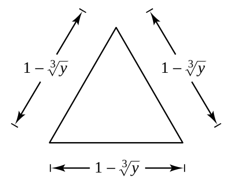
$$V = \frac{1}{8}\pi \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{8}\left(\frac{1}{10}\right) = \frac{\pi}{80}$$




(c) $A(y) = \dfrac{1}{2}bh = \dfrac{1}{2}\left(1 - \sqrt[3]{y}\right)\left(\dfrac{\sqrt{3}}{2}\right)\left(1 - \sqrt[3]{y}\right)$

$$= \frac{\sqrt{3}}{4}\left(1 - \sqrt[3]{y}\right)^2$$

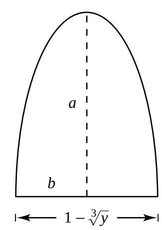
$$V = \frac{\sqrt{3}}{4}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\sqrt{3}}{4}\left(\frac{1}{10}\right) = \frac{\sqrt{3}}{40}$$




(d) $A(y) = \dfrac{1}{2}\pi ab = \dfrac{\pi}{2}(2)\left(1 - \sqrt[3]{y}\right)\dfrac{1 - \sqrt[3]{y}}{2}$

$$= \frac{\pi}{2}\left(1 - \sqrt[3]{y}\right)^2$$

$$V = \frac{\pi}{2}\int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = \frac{\pi}{2}\left(\frac{1}{10}\right) = \frac{\pi}{20}$$




**63.** Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$. Since $A_1(x) = A_2(x)$, we have

$$V_1 = \int_a^b A_1(x)\, dx = \int_a^b A_2(x)\, dx = V_2$$

Thus, the volumes are the same.

**65.** $\dfrac{4}{3}\pi(25 - r^2)^{3/2} = \dfrac{1}{2}\left(\dfrac{4}{3}\right)\pi(125)$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2}\right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

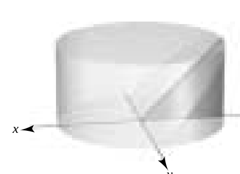
$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

**67.** (a) Since the cross sections are isosceles right triangles:

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\sqrt{r^2 - y^2}\right)\left(\sqrt{r^2 - y^2}\right) = \frac{1}{2}(r^2 - y^2)$$

$$V = \frac{1}{2}\int_{-r}^{r} (r^2 - y^2)\, dy = \int_0^r (r^2 - y^2)\, dy = \left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3$$




(b) $A(x) = \dfrac{1}{2}bh = \dfrac{1}{2}\sqrt{r^2 - y^2}\left(\sqrt{r^2 - y^2}\tan\theta\right) = \dfrac{\tan\theta}{2}(r^2 - y^2)$

$$V = \frac{\tan\theta}{2}\int_{-r}^{r} (r^2 - y^2)\, dy = \tan\theta\int_0^r (r^2 - y^2)\, dy = \tan\theta\left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{2}{3}r^3\tan\theta$$

As $\theta \to 90°$, $V \to \infty$.

# Section 6.3 Volume: The Shell Method

**1.** $p(x) = x$

$h(x) = x$

$V = 2\pi\int_0^2 x(x)\,dx = \left[\dfrac{2\pi x^3}{3}\right]_0^2 = \dfrac{16\pi}{3}$

$= \left[\dfrac{2\pi x^3}{3}\right]_0^2 = \dfrac{16\pi}{3}$

**3.** $p(x) = x$

$h(x) = \sqrt{x}$

$V = 2\pi\int_0^4 x\sqrt{x}\,dx$

$= 2\pi\int_0^4 x^{3/2}\,dx$

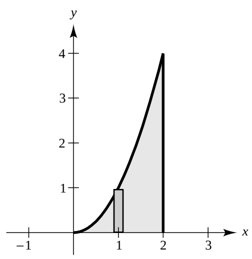
$= \left[\dfrac{4\pi}{5}x^{5/2}\right]_0^4 = \dfrac{128\pi}{5}$

**5.** $p(x) = x$

$h(x) = x^2$

$V = 2\pi\int_0^2 x^3\,dx$

$= \left[\dfrac{\pi}{2}x^4\right]_0^2 = 8\pi$



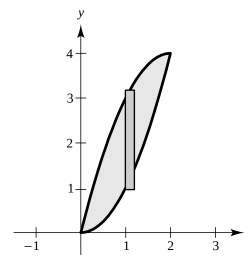

**7.** $p(x) = x$

$h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$V = 2\pi\int_0^2 x(4x - 2x^2)\,dx$

$= 4\pi\int_0^2 (2x^2 - x^3)\,dx$

$= 4\pi\left[\dfrac{2}{3}x^3 - \dfrac{1}{4}x^4\right]_0^2 = \dfrac{16\pi}{3}$

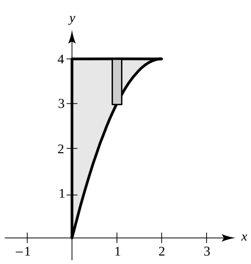



**9.** $p(x) = x$

$h(x) = 4 - (4x - x^2) = x^2 - 4x + 4$

$V = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx$

$= 2\pi\left[\dfrac{x^4}{4} - \dfrac{4}{3}x^3 + 2x^2\right]_0^2 = \dfrac{8\pi}{3}$



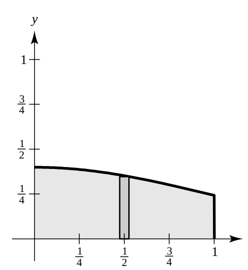

**11.** $p(x) = x$

$h(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$V = 2\pi\int_0^1 x\left(\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx$

$= \sqrt{2\pi}\int_0^1 e^{-x^2/2}x\,dx$

$= \left[-\sqrt{2\pi}\,e^{-x^2/2}\right]_0^1 = \sqrt{2\pi}\left(1 - \dfrac{1}{\sqrt{e}}\right) \approx 0.986$




**13.** $p(y) = y$

$h(y) = 2 - y$

$V = 2\pi\int_0^2 y(2 - y)\,dy$

$= 2\pi\int_0^2 (2y - y^2)\,dy$

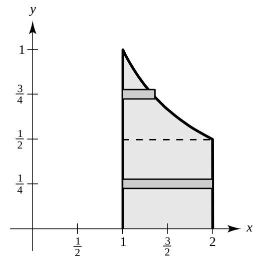
$= 2\pi\left[y^2 - \dfrac{y^3}{3}\right]_0^2 = \dfrac{8\pi}{3}$

**15.** $p(y) = y$ and $h(y) = 1$ if $0 \le y < \frac{1}{2}$.

$p(y) = y$ and $h(y) = \frac{1}{y} - 1$ if $\frac{1}{2} \le y \le 1$.

$$V = 2\pi \int_0^{1/2} y\,dy + 2\pi \int_{1/2}^{1} (1 - y)\,dy$$

$$= 2\pi\left[\frac{y^2}{2}\right]_0^{1/2} + 2\pi\left[y - \frac{y^2}{2}\right]_{1/2}^{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$



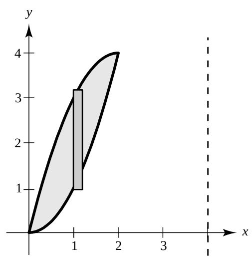
**17.** $p(x) = 4 - x$

$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 (4 - x)(4x - 2x^2)\,dx$$

$$= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x)\,dx$$

$$= 4\pi\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 = 16\pi$$



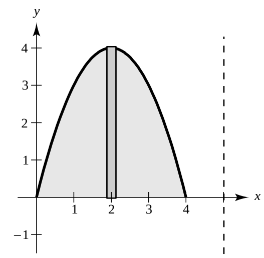
**19.** $p(x) = 5 - x$

$h(x) = 4x - x^2$

$$V = 2\pi \int_0^4 (5 - x)(4x - x^2)\,dx$$

$$= 2\pi \int_0^4 (x^3 - 9x^2 + 20x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = 64\pi$$

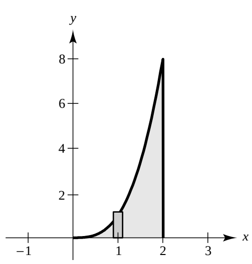


**21.** (a) **Disk**

$R(x) = x^3$

$r(x) = 0$

$$V = \pi \int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$

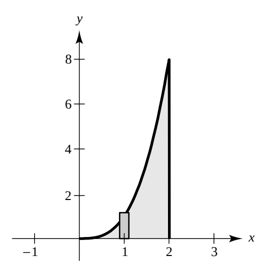


(b) **Shell**

$p(x) = x$

$h(x) = x^3$

$$V = 2\pi \int_0^2 x^4\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64\pi}{5}$$



(c) **Shell**

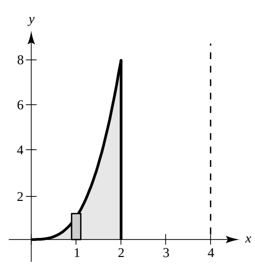
$p(x) = 4 - x$

$h(x) = x^3$

$$V = 2\pi \int_0^2 (4 - x)x^3\,dx$$

$$= 2\pi \int_0^2 (4x^3 - x^4)\,dx$$

$$= 2\pi\left[x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{96\pi}{5}$$



**23.** (a) **Shell**

$$p(y) = y$$

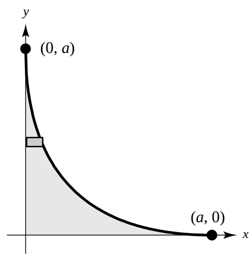
$$h(y) = (a^{1/2} - y^{1/2})^2$$

$$V = 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y)\,dy$$

$$= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2)\,dy$$

$$= 2\pi\left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3}\right]_0^a$$

$$= 2\pi\left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3}\right] = \frac{\pi a^3}{15}$$




(b) Same as part (a) by symmetry

(c) **Shell**

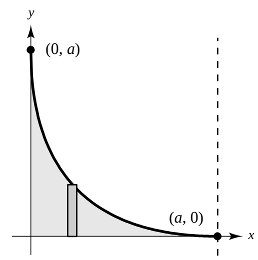
$$p(x) = a - x$$

$$h(x) = (a^{1/2} - x^{1/2})^2$$

$$V = 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2\,dx$$

$$= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2)\,dx$$

$$= 2\pi\left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3\right]_0^a = \frac{4\pi a^3}{15}$$



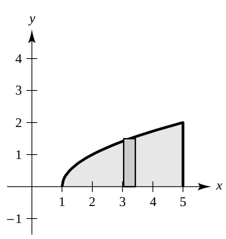

**25.** $V = 2\pi\int_x^d p(y)h(y)\,dy$ or $V = 2\pi\int_a^b p(x)h(x)\,dx$

**27.** $\pi \int_1^5 (x - 1)\,dx = \pi \int_1^5 \left(\sqrt{x - 1}\right)^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the $x$-axis by using the Disk Method.

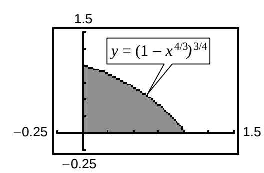
$$2\pi \int_0^2 y[5 - (y^2 + 1)]\,dy$$

represents this same volume by using the Shell Method.




Disk Method

**29.** (a)

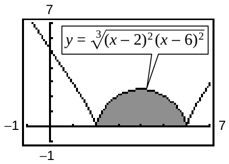



(b) $x^{4/3} + y^{4/3} = 1,\ x = 0,\ y = 0$

$$y = (1 - x^{4/3})^{3/4}$$

$$V = 2\pi \int_0^1 x(1 - x^{4/3})^{3/4}\,dx \approx 1.5056$$

**31.** (a)

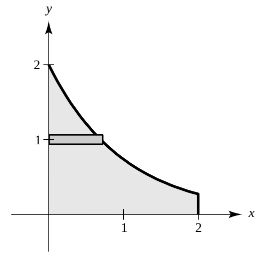



(b) $V = 2\pi \int_2^6 x\sqrt[3]{(x-2)^2(x-6)^2}\,dx \approx 187.249$

**33.** $y = 2e^{-x}$, $y = 0$, $x = 0$, $x = 2$

Volume $\approx 7.5$

Matches (d)




**35.** $p(x) = x$

$h(x) = 2 - \frac{1}{2}x^2$

$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2\right) dx = 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx = 2\pi\left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi$ (total volume)

Now find $x_0$ such that

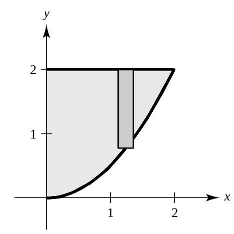
$$\pi = 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2\left[x^2 - \frac{1}{8}x^4\right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$

$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \quad \text{(Quadratic Formula)}$$




Take $x_0 = \sqrt{4 - 2\sqrt{3}}$ since the other root is too large.

Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

**37.** $V = 4\pi \int_{-1}^{1} (2 - x)\sqrt{1 - x^2}\,dx$

$= 8\pi \int_{-1}^{1} \sqrt{1 - x^2}\,dx - 4\pi \int_{-1}^{1} x\sqrt{1 - x^2}\,dx$

$= 8\pi\left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^{1} x(1 - x^2)^{1/2}(-2)\,dx$

$= 4\pi^2 + \left[2\pi\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_{-1}^{1} = 4\pi^2$

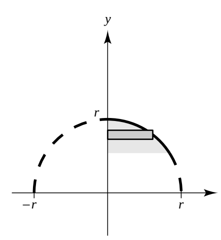
**39. Disk Method**

$R(y) = \sqrt{r^2 - y^2}$

$r(y) = 0$

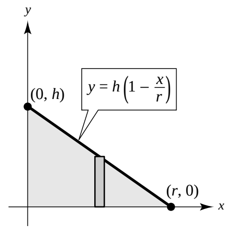
$V = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy$

$= \pi\left[r^2y - \frac{y^3}{3}\right]_{r-h}^{r} = \frac{1}{3}\pi h^2(3r - h)$



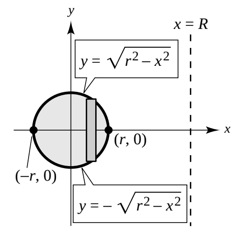
**41.** (a) $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as $r$ and height $h$.

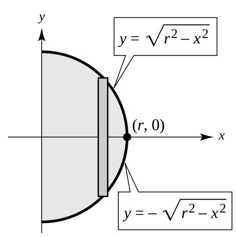



(b) $2\pi \int_{-r}^r (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$ (v)

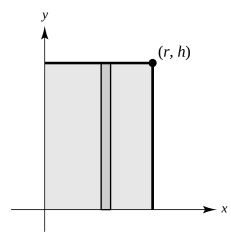
is the volume of a torus with the radius of its circular cross section as $r$ and the distance from the axis of the torus to the center of its cross section as $R$.




(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx$ (iii) is the volume of a sphere with radius $r$.

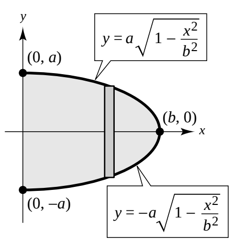



(d) $2\pi \int_0^r hx\, dx$ (i) is the volume of a right circular cylinder with a radius of $r$ and a height of $h$.




(e) $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)}\, dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.

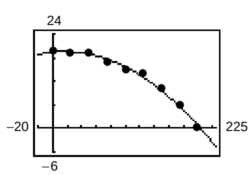



**43.** (a) $V = 2\pi \int_0^{200} x f(x)\, dx$

$$\approx \frac{2\pi(200)}{3(8)}[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$\approx$ 1,366,593 cubic feet

(b) $d = -0.000561x^2 + 0.0189x + 19.39$




(c) $V \approx 2\pi \int_0^{200} xd(x)\, dx \approx 2\pi(213{,}800) = 1{,}343{,}345$ cubic feet

(d) Number gallons $\approx V(7.48) = 10{,}048{,}221$ gallons

## Section 6.4 Arc Length and Surfaces of Revolution

**1.** $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2 + (12-0)^2} = 13$

(b) $y = \dfrac{12}{5}x$

$$y' = \frac{12}{5}$$

$$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2}\,dx = \left[\frac{13}{5}x\right]_0^5 = 13$$

**3.** $y = \dfrac{2}{3}x^{3/2} + 1$

$$y' = x^{1/2},\ [0, 1]$$

$$s = \int_0^1 \sqrt{1+x}\,dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}\left(\sqrt{8} - 1\right) \approx 1.219$$

**5.** $y = \dfrac{3}{2}x^{2/3}$

$$y' = \frac{1}{x^{1/3}},\ [1, 8]$$

$$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\,dx$$

$$= \int_1^8 \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx$$

$$= \frac{3}{2}\int_1^8 \sqrt{x^{2/3}+1}\left(\frac{2}{3x^{1/3}}\right)dx$$

$$= \frac{3}{2}\left[\frac{2}{3}(x^{2/3}+1)^{3/2}\right]_1^8$$

$$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$$

**7.** $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$

$$y' = \frac{1}{2}x^3 - \frac{1}{2x^3},\ [1, 2]$$

$$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2,\ [1, 2]$$

$$s = \int_a^b \sqrt{1 + (y')^2}\,dx$$

$$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)dx$$

$$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2 = \frac{33}{16} \approx 2.063$$

**9.** $y = \ln(\sin x),\ \left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$

$$y' = \frac{1}{\sin x}\cos x = \cot x$$

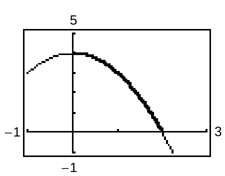
$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$

$$s = \int_{\pi/4}^{3\pi/4} \csc x\,dx$$

$$= \Big[\ln\left|\csc x - \cot x\right|\Big]_{\pi/4}^{3\pi/4}$$

$$= \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right) \approx 1.763$$

**11.** (a) $y = 4 - x^2,\ 0 \le x \le 2$



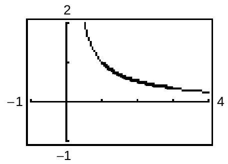

(b) $y' = -2x$

$$1 + (y')^2 = 1 + 4x^2$$

$$L = \int_0^2 \sqrt{1+4x^2}\,dx$$

(c) $L \approx 4.647$

**13.** (a) $y = \dfrac{1}{x}, 1 \le x \le 3$



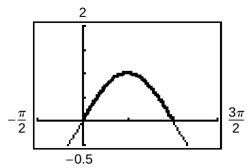

(b) $$y' = -\frac{1}{x^2}$$

$$1 + (y')^2 = 1 + \frac{1}{x^4}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

**15.** (a) $y = \sin x, 0 \le x \le \pi$




(b) $$y' = \cos x$$

$$1 + (y')^2 = 1 + \cos^2 x$$

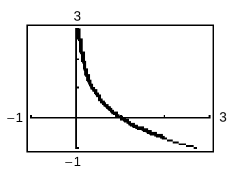
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

**17.** (a) $x = e^{-y}, 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$




(b) $$y' = -\frac{1}{x}$$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 2.221$

Alternatively, you can do all the computations with respect to $y$.

(a) $x = e^{-y} \quad 0 \le y \le 2$

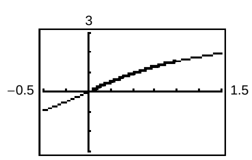
(b) $$\frac{dx}{dy} = -e^{-y}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$

$$L = \int_0^2 \sqrt{1 + e^{-2y}}\, dy$$

(c) $L \approx 2.221$

**19.** (a) $y = 2 \arctan x, 0 \le x \le 1$




(b) $y' = \dfrac{2}{1 + x^2}$

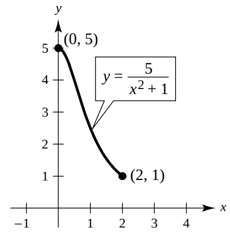
$$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}}\, dx$$

(c) $L \approx 1.871$

**21.** $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2+1}\right)\right]^2}\, dx$

$s \approx 5$

Matches (b)




**23.** $y = x^3$, $[0, 4]$

(a) $d = \sqrt{(4-0)^2 + (64-0)^2} \approx 64.125$

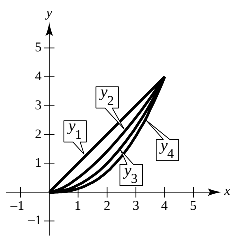
(b) $d = \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (8-1)^2} + \sqrt{(3-2)^2 + (27-8)^2} + \sqrt{(4-3)^2 + (64-27)^2}$

$\approx 64.525$

(c) $s = \displaystyle\int_0^4 \sqrt{1 + (3x^2)^2}\, dx = \int_0^4 \sqrt{1 + 9x^4}\, dx \approx 64.666$

(d) 64.672

**25.** (a)




(b) $y_1, y_2, y_3, y_4$

(c) $y_1' = 1,\ L_1 = \displaystyle\int_0^4 \sqrt{2}\, dx \approx 5.657$

$y_2' = \dfrac{3}{4}x^{1/2},\ L_2 = \displaystyle\int_0^4 \sqrt{1 + \frac{9x}{16}}\, dx \approx 5.759$

$y_3' = \dfrac{1}{2}x,\ L_3 = \displaystyle\int_0^4 \sqrt{1 + \frac{x^2}{4}}\, dx \approx 5.916$

$y_4' = \dfrac{5}{16}x^{3/2},\ L_4 = \displaystyle\int_0^4 \sqrt{1 + \frac{25}{256}x^3}\, dx \approx 6.063$

**27.** $y = \dfrac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$

When $x = 0$, $y = \frac{2}{3}$. Thus, the fleeting object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3}\left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right] = \left(\frac{1}{2}\right)\frac{x-1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x-1)^2}{4x} = \frac{(x+1)^2}{4x}$$

$$s = \int_0^1 \frac{x+1}{2x^{1/2}}\, dx = \frac{1}{2}\int_0^1 (x^{1/2} + x^{-1/2})\, dx = \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

**29.** $y = 20\cosh\dfrac{x}{20},\ -20 \le x \le 20$

$$y' = \sinh\frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2\frac{x}{20} = \cosh^2\frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh\frac{x}{20}\, dx = 2\int_0^{20} \cosh\frac{x}{20}\, dx = 2(20)\sinh\frac{x}{20}\Bigg]_0^{20}$$

$$= 40\sinh(1) \approx 47.008 \text{ m.}$$

**31.**
$$\begin{aligned}
y &= \sqrt{9 - x^2}\\
y' &= \frac{-x}{\sqrt{9 - x^2}}\\
1 + (y')^2 &= \frac{9}{9 - x^2}\\
s &= \int_0^2 \sqrt{\frac{9}{9 - x^2}}\,dx\\
&= \int_0^2 \frac{3}{\sqrt{9 - x^2}}\,dx\\
&= \left[3 \arcsin \frac{x}{3}\right]_0^2\\
&= 3\left(\arcsin \frac{2}{3} - \arcsin 0\right)\\
&= 3 \arcsin \frac{2}{3} \approx 2.1892
\end{aligned}$$

**33.**
$$\begin{aligned}
y &= \frac{x^3}{3}\\
y' &= x^2, [0, 3]\\
S &= 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\,dx\\
&= \frac{\pi}{6}\int_0^3 (1 + x^4)^{1/2}(4x^3)\,dx\\
&= \left[\frac{\pi}{9}(1 + x^4)^{3/2}\right]_0^3\\
&= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85
\end{aligned}$$

**35.**
$$\begin{aligned}
y &= \frac{x^3}{6} + \frac{1}{2x}\\
y' &= \frac{x^2}{2} - \frac{1}{2x^2}\\
1 + (y')^2 &= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, [1, 2]\\
S &= 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx\\
&= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx\\
&= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}
\end{aligned}$$

**37.**
$$\begin{aligned}
y &= \sqrt[3]{x} + 2\\
y' &= \frac{1}{3x^{2/3}}, [1, 8]\\
S &= 2\pi \int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\,dx\\
&= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\,dx\\
&= \frac{\pi}{18}\int_1^8 (9x^{4/3} + 1)^{1/2}(12x^{1/3})\,dx\\
&= \left[\frac{\pi}{27}(9x^{4/3} + 1)^{3/2}\right]_1^8\\
&= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48
\end{aligned}$$

**39.**
$$\begin{aligned}
y &= \sin x\\
y' &= \cos x, [0, \pi]\\
S &= 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\,dx\\
&\approx 14.4236
\end{aligned}$$

**41.** A rectifiable curve is one that has a finite arc length.

**43.** The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(di)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

**45.**
$$y = \frac{hx}{r}$$
$$y' = \frac{h}{r}$$
$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$
$$S = 2\pi \int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\,dx$$
$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

**47.**
$$y = \sqrt{9 - x^2}$$
$$y' = \frac{-x}{\sqrt{9 - x^2}}$$
$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$
$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}}\,dx$$
$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\,dx$$
$$= \left[-6\pi\sqrt{9 - x^2}\right]_0^2$$
$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 48.

**49.**
$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$
$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$
$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$
$$S = 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}(x^{-1/2} + 9^{1/2})^2}\,dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)(x^{-1/2} + 9x^{1/2})\,dx$$
$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right)dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in}^2$$

Amount of glass needed: $V = \dfrac{\pi}{27}\left(\dfrac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in}^3$

**51.** (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

(b) Area $= \displaystyle\int_0^{400} f(x)\,dx \approx 131{,}734.5$ square feet

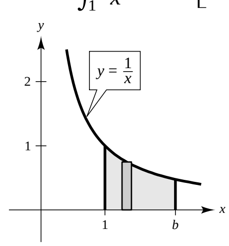
$\approx 3.0$ acres

(Answers will vary.)

(c) $L = \displaystyle\int_0^{400} \sqrt{1 + f'(x)^2}\,dx \approx 794.9$ feet

(Answers will vary.)

**53.** (a) $V = \pi\displaystyle\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$




(b)
$$S = 2\pi\int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$$
$$= 2\pi\int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$$
$$= 2\pi\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx$$

—CONTINUED—

**53. —CONTINUED—**

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Since

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \Big[\ln x\Big]_1^b = \ln b$$

and $\lim_{b\to\infty} \ln b \to \infty$. Thus,

$$\lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx = \infty.$$

**55.** (a) Area of circle with radius $L$: $A = \pi L^2$

Area of sector with central angle $\theta$ (in radians)

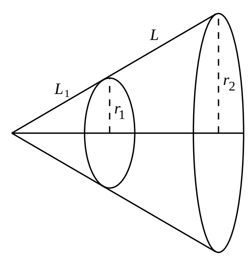
$$S = \frac{\theta}{2\pi}A = \frac{\theta}{2\pi}(\pi L^2) = \frac{1}{2}L^2\theta$$

(b) Let $s$ be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2}L^2\theta = \frac{1}{2}L^2\left(\frac{s}{L}\right) = \frac{1}{2}Ls = \frac{1}{2}L(2\pi r) = \pi rL$$

(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$S = \pi r_2(L + L_1) - \pi r_1 L_1$$
$$= \pi r_2 L + \pi L_1(r_2 - r_1)$$




By similar triangles, $\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow Lr_1 = L_1(r_2 - r_1)$

Hence,

$$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$$
$$= \pi L(r_1 + r_2).$$

# Section 6.5 Work

**1.** $W = Fd = (100)(10) = 1000$ ft · lb

**3.** $W = Fd = (112)(4) = 448$ joules (newton-meters)

**5.** Work equals force times distance, $W = FD$.

**7.** Since the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

**9.** $F(x) = kx$

$5 = k(4)$

$k = \frac{5}{4}$

$$W = \int_0^7 \frac{5}{4}x\,dx = \left[\frac{5}{8}x^2\right]_0^7$$
$$= \frac{245}{8} \text{ in} \cdot \text{lb}$$
$$= 30.625 \text{ in} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$$

**11.** $F(x) = kx$

$250 = k(30) \Rightarrow k = \frac{25}{3}$

$$W = \int_{20}^{50} F(x)\,dx = \int_{20}^{50} \frac{25}{3}x\,dx = \left.\frac{25x^2}{6}\right]_{20}^{50}$$
$$= 8750 \text{ n} \cdot \text{cm} = 87.5 \text{ joules or Nm}$$

**13.** $F(x) = kx$

$20 = k(9)$

$k = \dfrac{20}{9}$

$W = \displaystyle\int_0^{12} \frac{20}{9}x\,dx = \left[\frac{10}{9}x^2\right]_0^{12} = \frac{40}{3}$ ft · lb

**15.** $W = 18 = \displaystyle\int_0^{1/3} kx\,dx = \frac{kx^2}{2}\Bigg]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$W = \displaystyle\int_{1/3}^{7/12} 324x\,dx = 162x^2\Big]_{1/3}^{7/12} = 37.125$ ft · lbs

[**Note:** 4 inches $= \frac{1}{3}$ foot]

**17.** Assume that Earth has a radius of 4000 miles.

$F(x) = \dfrac{k}{x^2}$

$s = \dfrac{k}{(4000)^2}$

$k = 80{,}000{,}000$

$F(x) = \dfrac{80{,}000{,}000}{x^2}$

(a) $W = \displaystyle\int_{4000}^{4100} \frac{80{,}000{,}000}{x^2}\,dx = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{4100} \approx 487.8$ mi · tons

$\approx 5.15 \times 10^9$ ft · lb

(b) $W = \displaystyle\int_{4000}^{4300} \frac{80{,}000{,}000}{x^2}\,dx \approx 1395.3$ mi · ton

$\approx 1.47 \times 10^{10}$ ft · lb

**19.** Assume that the earth has a radius of 4000 miles.

$F(x) = \dfrac{k}{x^2}$

$10 = \dfrac{k}{(4000)^2}$

$k = 160{,}000{,}000$

$F(x) = \dfrac{160{,}000{,}000}{x^2}$

(a) $W = \displaystyle\int_{4000}^{15{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{15{,}000} \approx -10{,}666.667 + 40{,}000$

$= 29{,}333.333$ mi · ton

$\approx 2.93 \times 10^4$ mi · ton

$\approx 3.10 \times 10^{11}$ ft · lb

(b) $W = \displaystyle\int_{4000}^{26{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{26{,}000} \approx -6{,}153.846 + 40{,}000$

$= 33{,}846.154$ mi · ton

$\approx 3.38 \times 10^4$ mi · ton

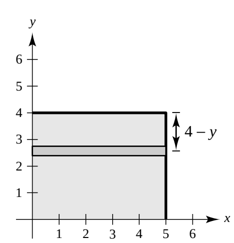
$\approx 3.57 \times 10^{11}$ ft · lb

**21.** Weight of each layer: $62.4(20)\,\Delta y$

Distance: $4 - y$

(a) $W = \displaystyle\int_2^4 62.4(20)(4 - y)\,dy = \Big[4992y - 624y^2\Big]_2^4 = 2496$ ft · lb

(b) $W = \displaystyle\int_0^4 62.4(20)(4 - y)\,dy = \Big[4992y - 624y^2\Big]_0^4 = 9984$ ft · lb




**23.** Volume of disk: $\pi(2)^2\,\Delta y = 4\pi\,\Delta y$

Weight of disk of water: $9800(4\pi)\,\Delta y$

Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5 - y)(9800)4\pi\,dy = 39{,}200\pi\int_0^4 (5 - y)\,dy$$

$$= 39{,}200\pi\left[5y - \frac{y^2}{2}\right]_0^4$$

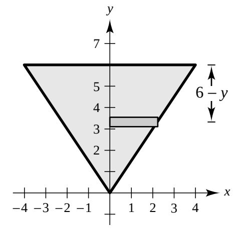
$$= 39{,}200\pi(12) = 470{,}400\pi \text{ newton–meters}$$

**25.** Volume of disk: $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9}\int_0^6 (6 - y)y^2\,dy = \frac{4}{9}(62.4)\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6 = 2995.2\pi \text{ ft} \cdot \text{lb}$$




**27.** Volume of disk: $\pi\left(\sqrt{36 - y^2}\right)^2 \Delta y$

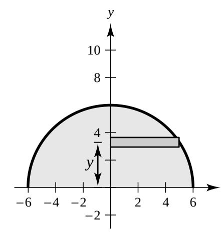
Weight of disk: $62.4\pi(36 - y^2)\,\Delta y$

Distance: $y$

$$W = 62.4\pi\int_0^6 y(36 - y^2)\,dy$$

$$= 62.4\pi\int_0^6 (36y - y^3)\,dy = 62.4\pi\left[18y^2 - \frac{1}{4}y^4\right]_0^6$$

$$= 20{,}217.6\pi \text{ ft} \cdot \text{lb}$$



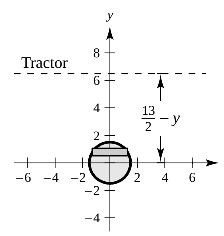

**29.** Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2}\,\Delta y$

Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2}\,\Delta y$

Distance: $\frac{13}{2} - y$

$$W = \int_{-1.5}^{1.5} 42(8)\sqrt{(9/4) - y^2}\left(\frac{13}{2} - y\right)dy$$

$$= 336\left[\frac{13}{2}\int_{-1.5}^{1.5}\sqrt{(9/4) - y^2}\,dy - \int_{-1.5}^{1.5}\sqrt{(9/4) - y^2}\,y\,dy\right]$$




The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. Thus, the work is

$$W = 336\left(\frac{13}{2}\right)\pi\left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right) = 2457\pi \text{ ft} \cdot \text{lb}$$

**31.** Weight of section of chain: $3\,\Delta y$

Distance: $15 - y$

$$W = 3\int_0^{15} (15 - y)\,dy$$

$$= \left[-\frac{3}{2}(15 - y)^2\right]_0^{15}$$

$$= 337.5 \text{ ft} \cdot \text{lb}$$

**33.** The lower 5 feet of chain are raised 10 feet with a constant force.

$$W_1 = 3(5)(10) = 150 \text{ ft} \cdot \text{lb}$$

The top 10 feet of chain are raised with a variable force.

Weight per section: $3\,\Delta y$

Distance: $10 - y$

$$W_2 = 3\int_0^{10} (10 - y)\,dy = \left[-\frac{3}{2}(10 - y)^2\right]_0^{10}$$

$$= 150 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

**35.** Weight of section of chain: $3\ \Delta y$

Distance: $15 - 2y$

$$W = 3\int_0^{7.5} (15 - 2y)\ dy = \left[-\frac{3}{4}(15 - 2y)^2\right]_0^{7.5}$$
$$= \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

**37.** Work to pull up the ball: $W_1 = 500(15) = 7500 \text{ ft} \cdot \text{lb}$

Work to wind up the top 15 feet of cable: force is variable

Weight per section: $1\ \Delta y$

Distance: $15 - x$

$$W_2 = \int_0^{15} (15 - x)\ dx = \left[-\frac{1}{2}(15 - x)^2\right]_0^{15}$$
$$= 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$$
$$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375$$
$$= 7987.5 \text{ ft} \cdot \text{lb}$$

**39.**
$$p = \frac{k}{V}$$
$$1000 = \frac{k}{2}$$
$$k = 2000$$
$$W = \int_2^3 \frac{2000}{V}\ dV = \Big[2000 \ln|V|\Big]_2^3$$
$$= 2000 \ln\left(\frac{3}{2}\right) \approx 810.93 \text{ ft} \cdot \text{lb}$$

**41.** $F(x) = \dfrac{k}{(2 - x)^2}$

$$W = \int_{-2}^{1} \frac{k}{(2 - x)^2}\ dx = \left[\frac{k}{2 - x}\right]_{-2}^{1} = k\left(1 - \frac{1}{4}\right)$$
$$= \frac{3k}{4}(\text{units of work})$$

**43.** $W = \displaystyle\int_0^5 1000[1.8 - \ln(x + 1)]\ dx \approx 3249.44 \text{ ft} \cdot \text{lb}$

**45.** $W = \displaystyle\int_0^5 100x\sqrt{125 - x^3}\ dx \approx 10{,}330.3 \text{ ft} \cdot \text{lb}$

## Section 6.6 Moments, Centers of Mass, and Centroids

**1.** $\bar{x} = \dfrac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\dfrac{6}{7}$

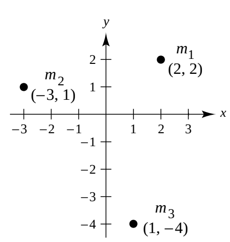
**3.** $\bar{x} = \dfrac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

**5.** (a) $\bar{x} = \dfrac{(7 + 5) + (8 + 5) + (12 + 5) + (15 + 5) + (18 + 5)}{5} = 17 = 12 + 5$

(b) $\bar{x} = \dfrac{12(-6 - 3) + 1(-4 - 3) + 6(-2 - 3) + 3(0 - 3) + 11(8 - 3)}{12 + 1 + 6 + 3 + 11} = \dfrac{-99}{33} = -3$

**7.**
$$50x = 75(L - x) = 75(10 - x)$$
$$50x = 750 - 75x$$
$$125x = 750$$
$$x = 6 \text{ feet}$$

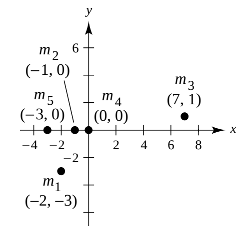
**9.**
$$\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$$
$$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$$
$$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$$



**11.** $$\bar{x} = \frac{3(-2) + 4(-1) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{8}$$

$$\bar{y} = \frac{3(-3) + 4(0) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{16}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{7}{8}, -\frac{7}{16}\right)$$




**13.** $$m = \rho\int_0^4 \sqrt{x}\,dx = \left[\frac{2\rho}{3}x^{3/2}\right]_0^4 = \frac{16\rho}{3}$$

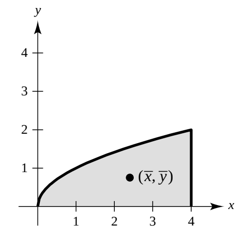
$$M_x = \rho\int_0^4 \frac{\sqrt{x}}{2}\left(\sqrt{x}\right)dx = \left[\rho\frac{x^2}{4}\right]_0^4 = 4\rho$$

$$\bar{y} = \frac{M_x}{m} = 4\rho\left(\frac{3}{16\rho}\right) = \frac{3}{4}$$

$$M_y = \rho\int_0^4 x\sqrt{x}\,dx = \left[\rho\frac{2}{5}x^{5/2}\right]_0^4 = \frac{64\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5}\left(\frac{3}{16\rho}\right) = \frac{12}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4}\right)$$




**15.** $$m = \rho\int_0^1 (x^2 - x^3)\,dx = \rho\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\rho}{12}$$

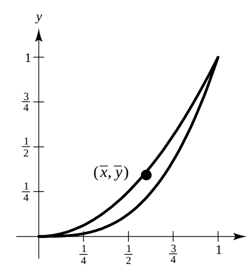
$$M_x = \rho\int_0^1 \frac{(x^2 + x^3)}{2}(x^2 - x^3)\,dx = \frac{\rho}{2}\int_0^1 (x^4 - x^6)\,dx = \frac{\rho}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{\rho}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35}\left(\frac{12}{\rho}\right) = \frac{12}{35}$$

$$M_y = \rho\int_0^1 x(x^2 - x^3)\,dx = \rho\int_0^1 (x^3 - x^4)\,dx = \rho\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{\rho}{20}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20}\left(\frac{12}{\rho}\right) = \frac{3}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$$




**17.** $$m = \rho\int_0^3 [(-x^2 + 4x + 2) - (x + 2)]\,dx = -\rho\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = \frac{9\rho}{2}$$

$$M_x = \rho\int_0^3 \left[\frac{(-x^2 + 4x + 2) + (x + 2)}{2}\right][(-x^2 + 4x + 2) - (x + 2)]\,dx$$

$$= \frac{\rho}{2}\int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x)\,dx = \frac{\rho}{2}\int_0^3 (x^4 - 8x^3 + 11x^2 + 12x)\,dx$$

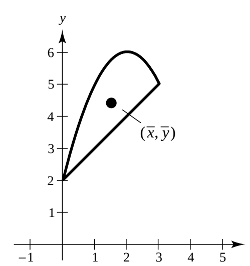
$$= \frac{\rho}{2}\left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2\right]_0^3 = \frac{99\rho}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{22}{5}$$

$$M_y = \rho\int_0^3 x[(-x^2 + 4x - 2) - (x + 2)]\,dx = \rho\int_0^3 (-x^3 + 3x^2)\,dx = \rho\left[-\frac{x^4}{4} + x^3\right]_0^3 = \frac{27\rho}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{22}{5}\right)$$



**19.** $m = \rho \int_0^8 x^{2/3}\,dx = \rho\left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{96\rho}{5}$

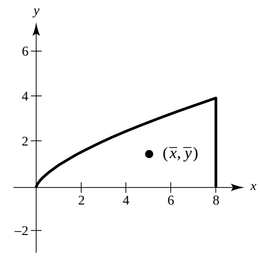
$M_x = \rho \int_0^8 \frac{x^{2/3}}{2}(x^{2/3})\,dx = \frac{\rho}{2}\left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{192\rho}{7}$

$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7}\left(\frac{5}{96\rho}\right) = \frac{10}{7}$

$M_y = \rho \int_0^8 x(x^{2/3})\,dx = \rho\left[\frac{3}{8}x^{8/3}\right]_0^8 = 96\rho$

$\bar{x} = \frac{M_y}{m} = 96\rho\left(\frac{5}{96\rho}\right) = 5$

$(\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$



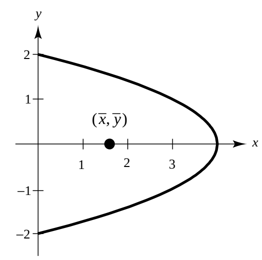

**21.** $m = 2\rho \int_0^2 (4 - y^2)\,dy = 2\rho\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32\rho}{3}$

$M_y = 2\rho \int_0^2 \left(\frac{4 - y^2}{2}\right)(4 - y^2)\,dy = \rho\left[16y - \frac{8}{3}y^3 + \frac{y^5}{5}\right]_0^2 = \frac{256\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15}\left(\frac{3}{32\rho}\right) = \frac{8}{5}$

By symmetry, $M_x$ and $\bar{y} = 0$.

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$




**23.** $m = \rho \int_0^3 [(2y - y^2) - (-y)]\,dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{9\rho}{2}$

$M_y = \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2}[(2y - y^2) - (-y)]\,dy = \frac{\rho}{2}\int_0^3 (y - y^2)(3y - y^2)\,dy$

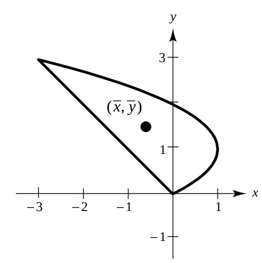
$= \frac{\rho}{2}\int_0^3 (y^4 - 4y^3 + 3y^2)\,dy = \frac{\rho}{2}\left[\frac{y^5}{5} - y^4 + y^3\right]_0^3 = -\frac{27\rho}{10}$

$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10}\left(\frac{2}{9\rho}\right) = -\frac{3}{5}$

$M_x = \rho \int_0^3 y[(2y - y^2) - (-y)]\,dy = \rho \int_0^3 (3y^2 - y^3)\,dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$




**25.** $A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

$M_x = \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}$

$M_y = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}$

**27.** $A = \int_0^3 (2x + 4)\,dx = \Big[x^2 + 4x\Big]_0^3 = 9 + 12 = 21$

$$M_x = \frac{1}{2}\int_0^3 (2x+4)^2\,dx = \int_0^3 (2x^2 + 8x + 8)\,dx = \left[\frac{2x^3}{3} + 4x^2 + 8x\right]_0^3 = 18 + 36 + 24 = 78$$

$$M_y = \int_0^3 (2x^2 + 4x)\,dx = \left[\frac{2x^3}{3} + 2x^2\right]_0^3 = 18 + 18 = 36$$

**29.** $m = \rho\int_0^5 10x\sqrt{125 - x^3}\,dx \approx 1033.0\rho$

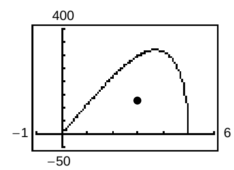
$$M_x = \rho\int_0^5 \left(\frac{10x\sqrt{125-x^3}}{2}\right)\left(10x\sqrt{125-x^3}\right)dx = 50\rho\int_0^5 x^2(125 - x^3)\,dx = \frac{3{,}124{,}375\rho}{24} \approx 130{,}208\rho$$

$$M_y = \rho\int_0^5 10x^2\sqrt{125 - x^3}\,dx = -\frac{10\rho}{3}\int_0^5 \sqrt{125 - x^3}\,(-3x^2)\,dx = \frac{12{,}500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is $(3.0, 126.0)$.



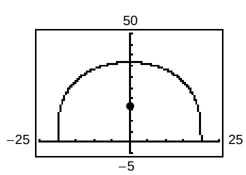

**31.** $m = \rho\int_{-20}^{20} 5\sqrt[3]{400 - x^2}\,dx \approx 1239.76\rho$

$$M_x = \rho\int_{-20}^{20} \frac{5\sqrt[3]{400-x^2}}{2}\left(5\sqrt[3]{400-x^2}\right)dx$$

$$= \frac{25\rho}{2}\int_{-20}^{20} (400 - x^2)^{2/3}\,dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 16.2)$.




**33.** $A = \frac{1}{2}(2a)c = ac$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left(\frac{1}{ac}\right)\frac{1}{2}\int_0^c \left[\left(\frac{b-a}{c}y + a\right)^2 - \left(\frac{b+a}{c}y - a\right)^2\right]dy$$

$$= \frac{1}{2ac}\int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2\right]dy$$

$$= \frac{1}{2ac}\left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3\right]_0^c = \frac{1}{2ac}\left(\frac{2}{3}abc\right) = \frac{b}{3}$$

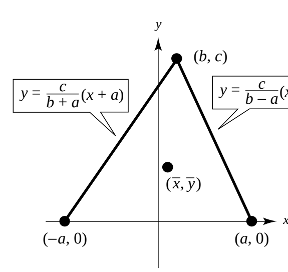
$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b-a}{c}y + a\right) - \left(\frac{b+a}{c}y - a\right)\right]dy$$

$$= \frac{1}{ac}\int_0^c y\left(-\frac{2a}{c}y + 2a\right)dy = \frac{2}{c}\int_0^c \left(y - \frac{y^2}{c}\right)dy$$

$$= \frac{2}{c}\left[\frac{y^2}{2} - \frac{y^3}{3c}\right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

In Exercise 566 of Section P.2, you found that $(b/3, c/3)$ is the point of intersection of the medians.



**35.** $A = \dfrac{c}{2}(a+b)$

$$\frac{1}{A} = \frac{2}{c(a+b)}$$

$$\bar{x} = \frac{2}{c(a+b)}\int_0^c x\left(\frac{b-a}{c}x + a\right)dx = \frac{2}{c(a+b)}\int_0^c \left(\frac{b-a}{c}x^2 + ax\right)dx = \frac{2}{c(a+b)}\left[\frac{b-a}{c}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^c$$

$$= \frac{2}{c(a+b)}\left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2}\right] = \frac{2}{c(a+b)}\left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6}\right] = \frac{c(2b+a)}{3(a+b)} = \frac{(a+2b)c}{3(a+b)}$$

$$\bar{y} = \frac{2}{c(a+b)}\frac{1}{2}\int_0^c \left(\frac{b-a}{c}x + a\right)^2 dx = \frac{1}{c(a+b)}\int_0^c \left[\left(\frac{b-a}{c}\right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2\right]dx$$

$$= \frac{1}{c(a+b)}\left[\left(\frac{b-a}{c}\right)^2\frac{x^3}{3} + \frac{2a(b-a)}{c}\frac{x^2}{2} + a^2x\right]_0^c = \frac{1}{c(a+b)}\left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c\right]$$

$$= \frac{1}{3c(a+b)}[(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c]$$

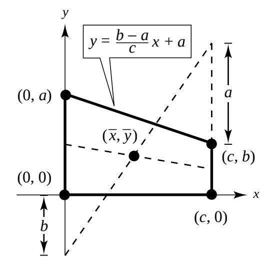
$$= \frac{1}{3(a+b)}[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a+b)}$$

Thus, $(\bar{x}, \bar{y}) = \left(\dfrac{(a+2b)c}{3(a+b)}, \dfrac{a^2+ab+b^2}{3(a+b)}\right)$.

The one line passes through $(0, a/2)$ and $(c, b/2)$. It's equation is $y = \dfrac{b-a}{2c}x + \dfrac{a}{2}$.

The other line passes through $(0, -b)$ and $(c, a+b)$. It's equation is $y = \dfrac{a+2b}{c}x - b$.

$(\bar{x}, \bar{y})$ is the point of intersection of these two lines.




**37.** $\bar{x} = 0$ by symmetry

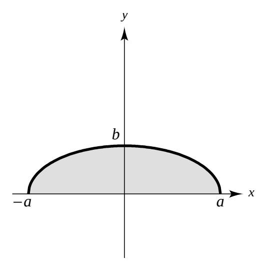
$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

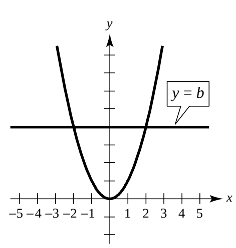
$$\bar{y} = \frac{2}{\pi ab}\frac{1}{2}\int_{-a}^{a}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi ab}\left(\frac{b^2}{a^2}\right)\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{b}{\pi a^3}\left[\frac{4a^3}{3}\right] = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$$




**39.** (a)




(b) $\bar{x} = 0$ by symmetry

(c) $M_y = \displaystyle\int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2)\,dx = 0$ because $bx - x^3$ is odd

(d) $\bar{y} > \dfrac{b}{2}$ since there is more area above $y = \dfrac{b}{2}$ than below

(e) $M_x = \displaystyle\int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b + x^2)(b - x^2)}{2}\,dx$

$$= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2}\,dx = \frac{1}{2}\left[b^2x - \frac{x^5}{5}\right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$$

$$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2)\,dx = \left[bx - \frac{x^3}{3}\right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= \left(b\sqrt{b} - \frac{b\sqrt{b}}{3}\right)2 = 4\frac{b\sqrt{b}}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b.$$

**41.** (a) $\bar{x} = 0$ by symmetry

$$A = 2\int_0^{40} f(x)\,dx = \frac{2(40)}{3(4)}[30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2}\,dx = \frac{40}{3(4)}[30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3}(7216) = \frac{72160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72160/3}{5560/3} = \frac{72160}{5560} \approx 12.98$$

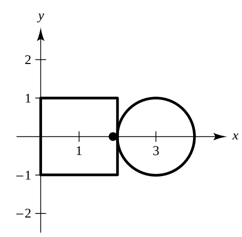
$(\bar{x}, \bar{y}) = (0, 12.98)$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{23697.68}{1843.54} \approx 12.85$

$(\bar{x}, \bar{y}) = (0, 12.85)$

**43.** Centroids of the given regions: $(1, 0)$ and $(3, 0)$



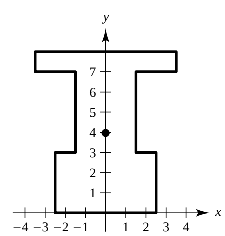
Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$$

**45.** Centroids of the given regions: $\left(0, \frac{3}{2}\right)$, $(0, 5)$, and $\left(0, \frac{15}{2}\right)$



Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right)$$

**47.** Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right) \approx (2.22, 0)$$

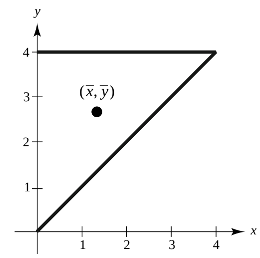
**49.** $V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14$

**51.** $A = \frac{1}{2}(4)(4) = 8$

$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4 + x)(4 - x)\,dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$

$r = \bar{y} = \frac{8}{3}$

$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$




**53.** $m = m_1 + \cdot\cdot\cdot + m_n$

$M_y = m_1x_1 + \cdot\cdot\cdot + m_nx_n$

$M_x = m_1y_1 + \cdot\cdot\cdot + m_ny_n$

$\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$

**55.** (a) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$

(b) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$

(c) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$

(d) No.

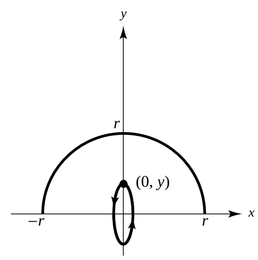
**57.** The surface area of the sphere is $S = 4\pi r^2$. The arc length of $C$ is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius $y$.

$$d = 2\pi y$$

Thus, $2\pi y = 4r$ and we have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.




**59.** $A = \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n + 1}\right]_0^1 = \frac{1}{n + 1}$

$m = \rho A = \frac{\rho}{n + 1}$

$M_x = \frac{\rho}{2}\int_0^1 (x^n)^2\,dx = \left[\frac{\rho}{2} \cdot \frac{x^{2n+1}}{2n + 1}\right]_0^1 = \frac{\rho}{2(2n + 1)}$

$M_y = \rho\int_0^1 x(x^n)\,dx = \left[\rho \cdot \frac{x^{n+2}}{n + 2}\right]_0^1 = \frac{\rho}{n + 2}$

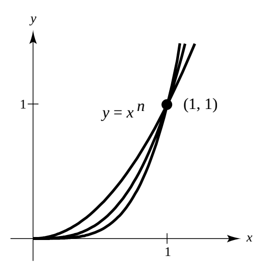
$\bar{x} = \frac{M_y}{m} = \frac{n + 1}{n + 2}$

$\bar{y} = \frac{M_x}{m} = \frac{n + 1}{2(2n + 1)} = \frac{n + 1}{4n + 2}$

Centroid: $\left(\frac{n + 1}{n + 2}, \frac{n + 1}{4n + 2}\right)$

As $n \to \infty$, $(\bar{x}, \bar{y}) \to \left(1, \frac{1}{4}\right)$.

The graph approaches the $x$-axis and the line $x = 1$ as $n \to \infty$.



## Section 6.7 Fluid Pressure and Fluid Force

**1.** $F = PA = [62.4(5)](3) = 936$ lb

**3.** $F = 62.4(h + 2)(6) - (62.4)(h)(6)$

$= 62.4(2)(6) = 748.8$ lb

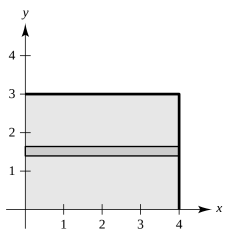
**5.** $h(y) = 3 - y$

$L(y) = 4$

$F = 62.4 \int_0^3 (3 - y)(4)\, dy$

$= 249.6 \int_0^3 (3 - y)\, dy$

$= 249.6\left[3y - \dfrac{y^2}{2}\right]_0^3 = 1123.2$ lb



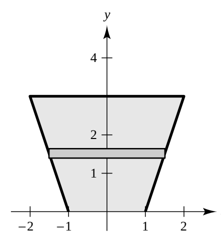
**7.** $h(y) = 3 - y$

$L(y) = 2\left(\dfrac{y}{3} + 1\right)$

$F = 2(62.4) \int_0^3 (3 - y)\left(\dfrac{y}{3} + 1\right) dy$

$= 124.8 \int_0^3 \left(3 - \dfrac{y^2}{3}\right) dy$

$= 124.8\left[3y - \dfrac{y^3}{9}\right]_0^3 = 748.8$ lb



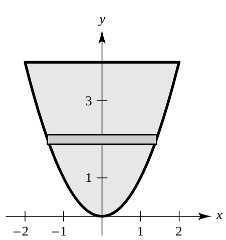
**9.** $h(y) = 4 - y$

$L(y) = 2\sqrt{y}$

$F = 2(62.4) \int_0^4 (4 - y)\sqrt{y}\, dy$

$= 124.8 \int_0^4 (4y^{1/2} - y^{3/2})\, dy$

$= 124.8\left[\dfrac{8y^{3/2}}{3} - \dfrac{2y^{5/2}}{5}\right]_0^4 = 1064.96$ lb

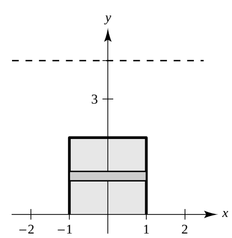


**11.** $h(y) = 4 - y$

$L(y) = 2$

$F = 9800 \int_0^2 2(4 - y)\, dy$

$= 9800\left[8y - y^2\right]_0^2 = 117{,}600$ Newtons

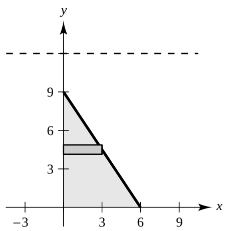


**13.** $h(y) = 12 - y$

$L(y) = 6 - \frac{2y}{3}$

$$F = 9800\int_0^9 (12 - y)\left(6 - \frac{2y}{3}\right) dy$$

$$= 9800\left[72y - 7y^2 + \frac{2y^3}{9}\right]_0^9 = 2{,}381{,}400 \text{ Newtons}$$



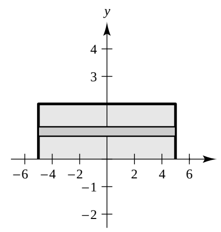

**15.** $h(y) = 2 - y$

$L(y) = 10$

$$F = 140.7\int_0^2 (2 - y)(10)\, dy$$

$$= 1407\int_0^2 (2 - y)\, dy$$

$$= 1407\left[2y - \frac{y^2}{2}\right]_0^2 = 2814 \text{ lb}$$



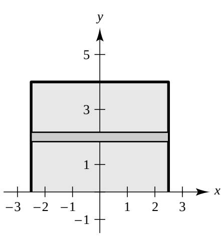

**17.** $h(y) = 4 - y$

$L(y) = 6$

$$F = 140.7\int_0^4 (4 - y)(6)\, dy$$

$$= 844.2\int_0^4 (4 - y)\, dy$$

$$= 844.2\left[4y - \frac{y^2}{2}\right]_0^4 = 6753.6 \text{ lb}$$



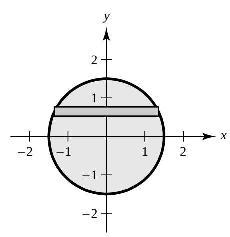

**19.** $h(y) = -y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42\int_{-3/2}^0 (-y)\sqrt{9 - 4y^2}\, dy$$

$$= \frac{42}{8}\int_{-3/2}^0 (9 - 4y^2)^{1/2}(-8y)\, dy$$

$$= \left[\left(\frac{21}{4}\right)\left(\frac{2}{3}\right)(9 - 4y^2)^{3/2}\right]_{-3/2}^0 = 94.5 \text{ lb}$$

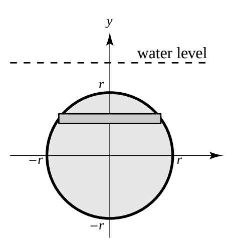



**21.** $h(y) = k - y$

$L(y) = 2\sqrt{r^2 - y^2}$

$$F = w\int_{-r}^r (k - y)\sqrt{r^2 - y^2}\,(2)\, dy$$

$$= w\left[2k\int_{-r}^r \sqrt{r^2 - y^2}\, dy + \int_{-r}^r \sqrt{r^2 - y^2}\,(-2y)\, dy\right]$$




The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius $r$.

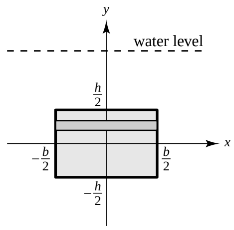
$$F = w\left[(2k)\frac{\pi r^2}{2} + 0\right] = wk\pi r^2$$

**23.** $h(y) = k - y$

$L(y) = b$

$$F = w\int_{-h/2}^{h/2} (k - y)b\,dy$$

$$= wb\left[ky - \frac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$$




**25.** From Exercise 22:

$F = 64(15)(1)(1) = 960$ lb

**27.** $h(y) = 4 - y$

$$F = 62.4\int_0^4 (4 - y)L(y)\,dy$$

Using Simpson's Rule with $n = 8$ we have:

$$F \approx 62.4\left(\frac{4 - 0}{3(8)}\right)[0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0]$$

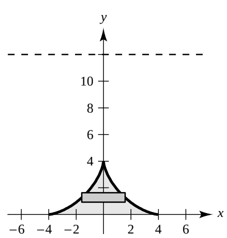
$$= 3010.8 \text{ lb}$$

**29.** $h(y) = 12 - y$

$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$

$$F = 62.4\int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2}\,dy$$

$$\approx 6448.73 \text{ lb}$$




**31.** (a) If the fluid force is one half of 1123.2 lb, and the height of the water is $b$, then

$h(y) = b - y$

$L(y) = 4$

$$F = 62.4\int_0^b (b - y)(4)\,dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y)\,dy = 2.25$$

$$\left[by - \frac{y^2}{2}\right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

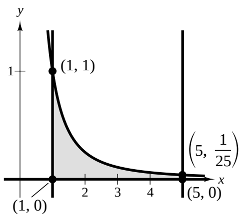
$$b^2 = 4.5 \implies b \approx 2.12 \text{ ft.}$$

(b) The pressure increases with increasing depth.

**33.** $F = Fw = w\displaystyle\int_c^d h(y)L(y)\,dy$, see page 471.

# Review Exercises for Chapter 6

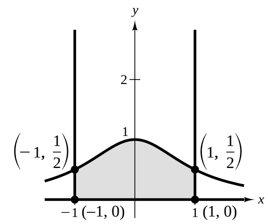
**1.** $A = \int_1^5 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$




**3.** $A = \int_{-1}^{1} \frac{1}{x^2+1}\,dx$

$= \left[\arctan x\right]_{-1}^{1}$

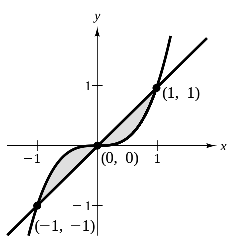
$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$




**5.** $A = 2\int_0^1 (x - x^3)\,dx$

$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

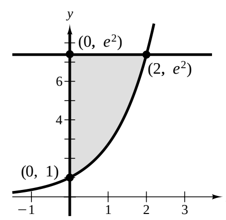
$= \frac{1}{2}$




**7.** $A = \int_0^2 (e^2 - e^x)\,dx$

$= \left[xe^2 - e^x\right]_0^2$

$= e^2 + 1$




**9.** $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$

$= \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$

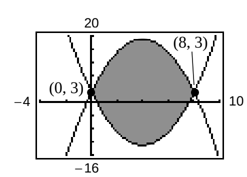
$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$

**11.** $A = \int_0^8 \left[(3 + 8x - x^2) - (x^2 - 8x + 3)\right] dx$

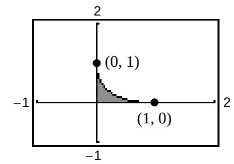
$= \int_0^8 (16x - 2x^2)\,dx$

$= \left[8x^2 - \frac{2}{3}x^3\right]_0^8 = \frac{512}{3} \approx 170.667$



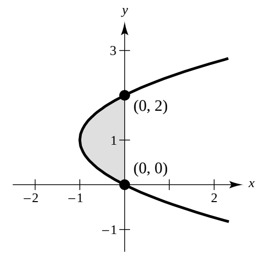
**13.** $y = (1 - \sqrt{x})^2$

$$A = \int_0^1 (1 - \sqrt{x})^2\,dx$$
$$= \int_0^1 (1 - 2x^{1/2} + x)\,dx$$
$$= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{6} \approx 0.1667$$




**15.** $x = y^2 - 2y \implies x + 1 = (y - 1)^2 \implies y = 1 \pm \sqrt{x + 1}$

$$A = \int_{-1}^{0} \left[\left(1 + \sqrt{x+1}\right) - \left(1 - \sqrt{x+1}\right)\right] dx = \int_{-1}^{0} 2\sqrt{x+1}\,dx$$
$$A = \int_0^2 [0 - (y^2 - 2y)]\,dy = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \frac{4}{3}$$

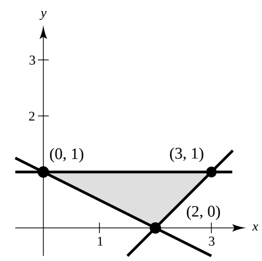



**17.** $$A = \int_0^2 \left[1 - \left(1 - \frac{x}{2}\right)\right] dx + \int_2^3 [1 - (x - 2)]\,dx$$
$$= \int_0^2 \frac{x}{2}\,dx + \int_2^3 (3 - x)\,dx$$

$y = 1 - \dfrac{x}{2} \implies x = 2 - 2y$

$y = x - 2 \implies x = y + 2, y = 1$

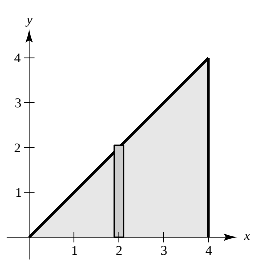
$$A = \int_0^1 [(y + 2) - (2 - 2y)]\,dy$$
$$= \int_0^1 3y\,dy = \left[\frac{3}{2}y^2\right]_0^1 = \frac{3}{2}$$




**19.** Job 1 is better. The salary for Job 1 is greater than the salary for Job 2 for all the years except the first and 10th years.

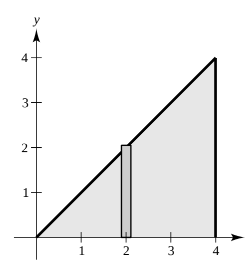
**21.** (a) **Disk**

$$V = \pi \int_0^4 x^2\,dx = \left[\frac{\pi x^3}{3}\right]_0^4 = \frac{64\pi}{3}$$




(b) **Shell**

$$V = 2\pi \int_0^4 x^2\,dx = \left[\frac{2\pi}{3}x^3\right]_0^4 = \frac{128\pi}{3}$$

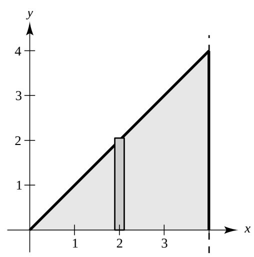



**—CONTINUED—**

**21. —CONTINUED—**

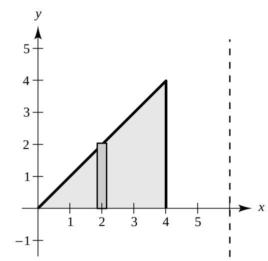
(c) **Shell**

$$V = 2\pi \int_0^4 (4 - x)x\, dx$$
$$= 2\pi \int_0^4 (4x - x^2)\, dx$$
$$= 2\pi \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \frac{64\pi}{3}$$



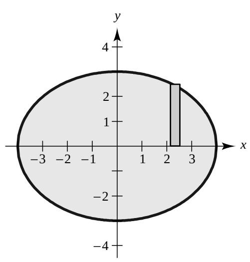
(d) **Shell**

$$V = 2\pi \int_0^4 (6 - x)x\, dx$$
$$= 2\pi \int_0^4 (6x - x^2)\, dx$$
$$= 2\pi \left[3x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{160\pi}{3}$$



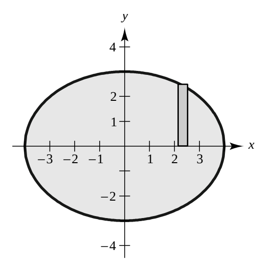
**23.** (a) **Shell**

$$V = 4\pi \int_0^4 x\left(\frac{3}{4}\right)\sqrt{16 - x^2}\, dx$$
$$= \left[3\pi\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(16 - x^2)^{3/2}\right]_0^4 = 64\pi$$



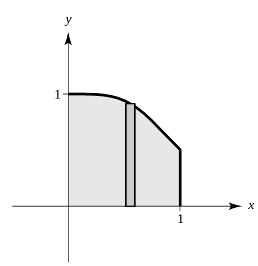
(b) **Disk**

$$V = 2\pi \int_0^4 \left[\frac{3}{4}\sqrt{16 - x^2}\right]^2 dx$$
$$= \frac{9\pi}{8}\left[16x - \frac{x^3}{3}\right]_0^4 = 48\pi$$

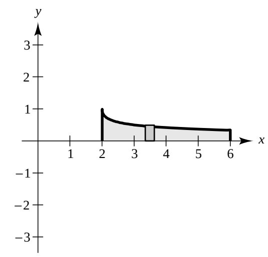


**25.** **Shell**

$$V = 2\pi \int_0^1 \frac{x}{x^4 + 1}\, dx$$
$$= \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1}\, dx$$
$$= \left[\pi \arctan(x^2)\right]_0^1$$
$$= \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$



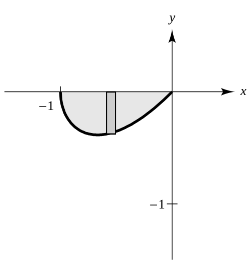
**27. Shell**




$$u = \sqrt{x-2}$$
$$x = u^2 + 2$$
$$dx = 2u\,du$$
$$V = 2\pi\int_2^6 \frac{x}{1+\sqrt{x-2}}\,dx = 4\pi\int_0^2 \frac{(u^2+2)u}{1+u}\,du$$
$$= 4\pi\int_0^2 \frac{u^3+2u}{1+u}\,du = 4\pi\int_0^2\left(u^2 - u + 3 - \frac{3}{1+u}\right)du$$
$$= 4\pi\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3\ln(1+u)\right]_0^2 = \frac{4\pi}{3}(20 - 9\ln 3) \approx 42.359$$

**29.** Since $y \le 0$, $A = -\int_{-1}^{0} x\sqrt{x+1}\,dx$.

$$u = x + 1$$
$$x = u - 1$$
$$dx = du$$
$$A = -\int_0^1 (u-1)\sqrt{u}\,du = -\int_0^1 (u^{3/2} - u^{1/2})\,du$$
$$= -\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^1 = \frac{4}{15}$$




**31.** From Exercise 23(a) we have: $V = 64\pi\text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

**Disk:** $$\pi\int_{-3}^{y_0}\frac{16}{9}(9 - y^2)\,dy = 16\pi$$
$$\frac{1}{9}\int_{-3}^{y_0}(9-y^2)\,dy = 1$$
$$\left[9y - \frac{1}{3}y^3\right]_{-3}^{y_0} = 9$$
$$\left(9y_0 - \frac{1}{3}y_0^3\right) - (-27 + 9) = 9$$
$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ ft.

**33.**

$$f(x) = \frac{4}{5}x^{5/4}$$
$$f'(x) = x^{1/4}$$
$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$
$$u = 1 + \sqrt{x}$$
$$x = (u-1)^2$$
$$dx = 2(u-1)\,du$$
$$s = \int_0^4 \sqrt{1+\sqrt{x}}\,dx = 2\int_1^3 \sqrt{u}(u-1)\,du$$
$$= 2\int_1^3 (u^{3/2} - u^{1/2})\,du$$
$$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^3 = \frac{4}{15}\left[u^{3/2}(3u-5)\right]_1^3$$
$$= \frac{8}{15}\left(1 + 6\sqrt{3}\right) \approx 6.076$$

**35.** $y = 300\cosh\left(\frac{x}{2000}\right) - 280,\ -2000 \le x \le 2000$

$$y' = \frac{3}{20}\sinh\left(\frac{x}{2000}\right)$$
$$s = \int_{-2000}^{2000}\sqrt{1 + \left[\frac{3}{20}\sinh\left(\frac{x}{2000}\right)\right]^2}\,dx$$
$$= \frac{1}{20}\int_{-2000}^{2000}\sqrt{400 + 9\sinh^2\left(\frac{x}{2000}\right)}\,dx$$

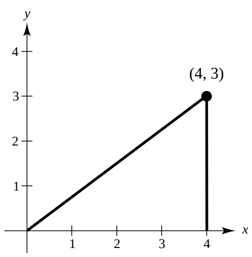
$\approx 4018.2$ ft (by Simpson's Rule or graphing utility)

**37.** $y = \frac{3}{4}x$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi\int_0^4 \left(\frac{3}{4}x\right)\sqrt{\frac{25}{16}}\,dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$




**39.** $F = kx$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x\,dx = \left[2x^2\right]_0^5$$

$$= 50 \text{ in} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

**41.** Volume of disk: $\pi\left(\frac{1}{3}\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $175 - y$

$$W = \frac{62.4\pi}{9}\int_0^{150} (175 - y)\,dy = \frac{62.4\pi}{9}\left[175y - \frac{y^2}{2}\right]_0^{150}$$

$$= 104{,}000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

**43.** Weight of section of chain: $5\,\Delta x$

Distance moved: $10 - x$

$$W = 5\int_0^{10} (10 - x)\,dx = \left[-\frac{5}{2}(10 - x)^2\right]_0^{10}$$

$$= 250 \text{ ft} \cdot \text{lb}$$

**45.** $W = \int_a^b F(x)\,dx$

$$80 = \int_0^4 ax^2 dx = \frac{ax^3}{3}\bigg]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

**47.** $A = \int_0^a \left(\sqrt{a} - \sqrt{x}\right)^2 dx = \int_0^a \left(a - 2\sqrt{a}\,x^{1/2} + x\right) dx = \left[ax - \frac{4}{3}\sqrt{a}\,x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$

$$\frac{1}{A} = \frac{6}{a^2}$$

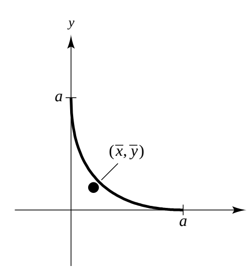
$$\bar{x} = \frac{6}{a^2}\int_0^a x\left(\sqrt{a} - \sqrt{x}\right)^2 dx = \frac{6}{a^2}\int_0^a \left(ax - 2\sqrt{a}\,x^{3/2} + x^2\right) dx$$

$$\bar{y} = \left(\frac{6}{a^2}\right)\frac{1}{2}\int_0^a \left(\sqrt{a} - \sqrt{x}\right)^4 dx$$

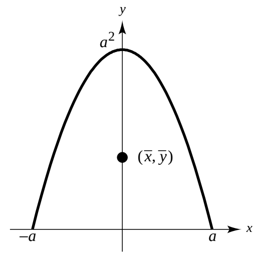
$$= \frac{3}{a^2}\int_0^a \left(a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2\right) dx$$

$$= \frac{3}{a^2}\left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



**49.** By symmetry, $x = 0$.




$$A = 2\int_0^1 (a^2 - x^2)\,dx = 2\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4a^3}{3}$$

$$\frac{1}{A} = \frac{3}{4a^3}$$

$$\bar{y} = \left(\frac{3}{4a^3}\right)\frac{1}{2}\int_{-a}^{a} (a^2 - x^2)^2\,dx$$

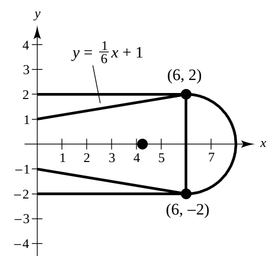
$$= \frac{6}{8a^3}\int_0^a (a^4 - 2a^2x^2 + x^4)\,dx$$

$$= \frac{6}{8a^3}\left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5\right]_0^a$$

$$= \frac{6}{8a^3}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5}\right)$$

**51.** $\bar{y} = 0$ by symmetry




For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$M_y = \rho\int_0^6 x\left[\left(\frac{1}{6}x + 1\right) - \left(-\frac{1}{6}x - 1\right)\right]dx$$

$$= \rho\int_0^6 \left(\frac{1}{3}x^2 + 2x\right)dx = \rho\left[\frac{x^3}{9} + x^2\right]_0^6 = 60\rho$$

For the semicircle:

$$m = \left(\frac{1}{2}\right)(\pi)(2)^2\rho = 2\pi\rho$$

$$M_y = \rho\int_6^8 x\left[\sqrt{4 - (x-6)^2} - \left(-\sqrt{4 - (x-6)^2}\right)\right]dx = 2\rho\int_6^8 x\sqrt{4 - (x-6)^2}\,dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6$, $u = 0$. When $x = 8$, $u = 2$.

$$M_y = 2\rho\int_0^2 (u + 6)\sqrt{4 - u^2}\,du = 2\rho\int_0^2 u\sqrt{4 - u^2}\,du + 12\rho\int_0^2 \sqrt{4 - u^2}\,du$$

$$= 2\rho\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(4 - u^2)^{3/2}\right]_0^2 + 12\rho\left[\frac{\pi(2)^2}{4}\right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4 + 9\pi)}{3}$$

Thus, we have:

$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4 + 9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4 + 9\pi)}{3} \cdot \frac{1}{2\rho(9 + \pi)} = \frac{2(9\pi + 49)}{3(\pi + 9)}$$

The centroid of the blade is $\left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$.

**53.** Let $D$ = surface of liquid; $\rho$ = weight per cubic volume.

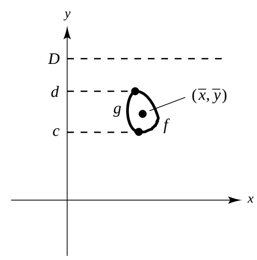
$$F = \rho\int_c^d (D-y)[f(y)-g(y)]\,dy$$

$$= \rho\left[\int_c^d D[f(y)-g(y)]\,dy - \int_c^d y[f(y)-g(y)]\,dy\right]$$

$$= \rho\left[\int_c^d [f(y)-g(y)]\,dy\right]\left[D - \frac{\int_c^d y[f(y)-g(y)]\,dy}{\int_c^d [f(y)-g(y)]\,dy}\right]$$

$$= \rho(\text{Area})(D-\bar{y})$$

$$= \rho(\text{Area})(\text{depth of centroid})$$




## Problem Solving for Chapter 6

**1.** $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2)\,dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c\to 0^+}\frac{T}{R} = \lim_{c\to 0^+}\frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

**3.** (a) $\frac{1}{2}V = \int_0^1 \left[\pi\left(2+\sqrt{1-y^2}\right)^2 - \pi\left(2-\sqrt{1-y^2}\right)^2\right]dy$

$$= \pi\int_0^1 \left[\left(4+4\sqrt{1-y^2}+(1-y^2)\right) - \left(4-4\sqrt{1-y^2}+(1-y^2)\right)\right]dy$$

$$= 8\pi\int_0^1 \sqrt{1-y^2}\,dy \quad \text{(Integral represents 1/4 (area of circle))}$$

$$= 8\pi\left(\frac{\pi}{4}\right) = 2\pi^2 \implies V = 4\pi^2$$

(b) $(x-R)^2 + y^2 = r^2 \implies x = R \pm \sqrt{r^2-y^2}$

$$\frac{1}{2}V = \int_0^r \left[\pi\left(R+\sqrt{r^2-y^2}\right)^2 - \pi\left(R-\sqrt{r^2-y^2}\right)^2\right]dy$$

$$= \pi\int_0^r 4R\sqrt{r^2-y^2}\,dy$$

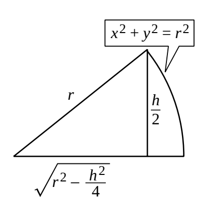
$$= \pi(4R)\frac{1}{4}\pi r^2 - \pi^2 r^2 R$$

$$V = 2\pi^2 r^2 R$$

**5.** $V = 2(2\pi)\int_{\sqrt{r^2-(h^2/4)}}^{r} x\sqrt{r^2-x^2}\,dx$

$$= -2\pi\left[\frac{2}{3}(r^2-x^2)^{3/2}\right]_{\sqrt{r^2-(h^2/4)}}^{r}$$

$$= \frac{-4\pi}{3}\left[-\frac{h^3}{8}\right] = \frac{\pi h^3}{6} \text{ which does not depend on } r!$$



**7.** (a) Tangent at $A$: $y = x^3, y' = 3x^2$

$$y - 1 = 3(x - 1)$$
$$y = 3x - 2$$

To find point $B$:
$$x^3 = 3x - 2$$
$$x^3 - 3x + 2 = 0$$
$$(x - 1)^2(x + 2) = 0 \implies B = (-2, -8)$$

Tangent at $B$: $y = x^3, y' = 3x^2$

$$y + 8 = 12(x + 2)$$
$$y = 12x + 16$$

To find point $C$:
$$x^3 = 12x + 16$$
$$x^3 - 12x - 16 = 0$$
$$(x + 2)^2(x - 4) = 0 \implies C = (4, 64)$$

$$\text{Area of } R = \int_{-2}^{1}(x^3 - 3x + 2)\,dx = \frac{27}{4}$$

$$\text{Area of } S = \int_{-2}^{4}(12x + 16 - x^3)\,dx = 108$$

Area of $S = 16$(area of $R$) $\left[\dfrac{\text{area } S}{\text{area } R} = 16\right]$

(b) Tangent at $A(a, a^3)$: $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point $B$:
$$x^3 - 3a^2x + 2a^3 = 0$$
$$(x - a)^2(x + 2a) = 0 \implies$$
$$B = (-2a, -8a^3)$$

Tangent at $B$:
$$y + 8a^3 = 12a^2(x + 2a)$$
$$y = 12a^2x + 16a^3$$

To find point $C$:
$$x^3 - 12a^2x - 16a^3 = 0$$
$$(x + 2a)^2(x - 4a) = 0 \implies$$
$$C = (4a, 64a^3)$$

$$\text{Area of } R = \int_{-2a}^{a}\left[x^3 - 3a^2x + 2a^3\right]dx = \frac{27}{4}a^4$$

$$\text{Area of } S = \int_{-2a}^{4a}\left[12a^2x + 16a^3 - x^3\right]dx = 108a^4$$

Area of $S = 16$(area of $R$)

**9.** $s(x) = \displaystyle\int_{\alpha}^{x}\sqrt{1 + f'(t)^2}\,dt$

(a) $s'(x) = \dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2}\,dx$

$$(ds)^2 = [1 + f'(x)^2](dx)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \displaystyle\int_{1}^{x}\sqrt{1 + \left(\frac{3}{2}t^{1/2}\right)^2}\,dt = \int_{1}^{x}\sqrt{1 + \frac{9}{4}t}\,dt$

(d) $s(2) = \displaystyle\int_{1}^{2}\sqrt{1 + \frac{9}{4}t}\,dt = \left[\frac{8}{27}\left(1 + \frac{9}{4}t\right)^{3/2}\right]_{1}^{2} = \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$

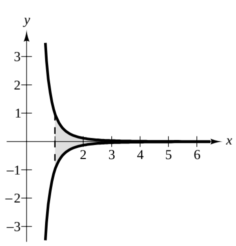
This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

**11.** (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_{1}^{6}x\left(\frac{1}{x^3} - \left(-\frac{1}{x^3}\right)\right)dx = \int_{1}^{6}\frac{2}{x^2}\,dx = \left[-2\frac{1}{x}\right]_{1}^{6} = \frac{5}{3}$$

$$m = 2\int_{1}^{6}\frac{1}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_{1}^{6} = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \qquad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0\right)$$




(b) $m = 2\displaystyle\int_{1}^{b}\frac{1}{x^3}\,dx = \frac{b^2 - 1}{b^2}$

$$M_y = 2\int_{1}^{6}\frac{1}{x^2}\,dx = \frac{2(b - 1)}{b}$$

$$\bar{x} = \frac{2(b - 1)/b}{(b^2 - 1)/b^2} = \frac{2b}{b + 1} \qquad (\bar{x}, \bar{y}) = \left(\frac{2b}{b + 1}, 0\right)$$

(c) $\displaystyle\lim_{b\to\infty}\bar{x} = \lim_{b\to\infty}\frac{2b}{b + 1} = 2 \qquad (\bar{x}, \bar{y}) = (2, 0)$

**13.** (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

**15.** Point of equilibrium: $50 - 0.5x = 0.125x$

$$x = 80,\ p = 10$$

$(P_0, x_0) = (10, 80)$

$$\text{Consumer surplus} = \int_0^{80} [(50 - 0.5x) - 10]\,dx = 1600$$

$$\text{Producer surplus} = \int_0^{80} [10 - 0.125x]\,dx = 400$$

**17.** (a) Wall at shallow end

From Exercise 22: $F = 62.4(2)(4)(20) = 9984$ lb

(b) Wall at deep end

From Exercise 22: $F = 62.4(4)(8)(20) = 39{,}936$ lb

(c) Side wall

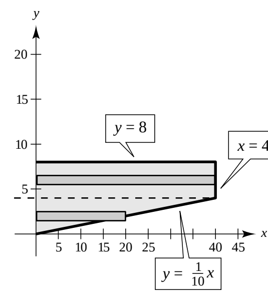
From Exercise 22: $F_1 = 62.4(2)(4)(40) = 19{,}968$ lb

$$F_2 = 62.4\int_0^4 (8 - y)(10y)\,dy$$

$$= 624\int_0^4 (8y - y^2)\,dy = 624\left[4y^2 - \frac{y^3}{3}\right]_0^4$$

$$= 26{,}624 \text{ lb}$$

Total force: $F_1 + F_2 = 46{,}592$ lb



# CHAPTER 7
# Integration Techniques, L'Hôpital's Rule, and Improper Integrals

# CHAPTER 7
# Integration Techniques, L'Hôpital's Rule, and Improper Integrals

## Section 7.1 Basic Integration Rules

Solutions to Odd-Numbered Exercises

**1.** (a) $\frac{d}{dx}\left[2\sqrt{x^2+1}+C\right] = 2\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2+1}}$

(b) $\frac{d}{dx}\left[\sqrt{x^2+1}+C\right] = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2+1}}$

(d) $\frac{d}{dx}\left[\ln(x^2+1)+C\right] = \frac{2x}{x^2+1}$

$\int \frac{x}{\sqrt{x^2+1}}\,dx$ matches (b).

**3.** (a) $\frac{d}{dx}\left[\ln\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$

(b) $\frac{d}{dx}\left[\frac{2x}{(x^2+1)^2}+C\right] = \frac{(x^2+1)^2(2)-(2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$

(c) $\frac{d}{dx}\left[\arctan x+C\right] = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}\left[\ln(x^2+1)+C\right] = \frac{2x}{x^2+1}$

$\int \frac{1}{x^2+1}\,dx$ matches (c).

**5.** $\int (3x-2)^4\,dx$

$u = 3x-2,\ du = 3\,dx,\ n = 4$

Use $\int u^n\,du$.

**7.** $\int \frac{1}{\sqrt{x}\left(1-2\sqrt{x}\right)}\,dx$

$u = 1-2\sqrt{x},\ du = -\frac{1}{\sqrt{x}}\,dx$

Use $\int \frac{du}{u}$.

**9.** $\int \frac{3}{\sqrt{1-t^2}}\,dt$

$u = t,\ du = dt,\ a = 1$

Use $\int \frac{du}{\sqrt{a^2-u^2}}$

**11.** $\int t\sin t^2\,dt$

$u = t^2,\ du = 2t\,dt$

Use $\int \sin u\,du$.

**13.** $\int \cos x e^{\sin x}\,dx$

$u = \sin x,\ du = \cos x\,dx$

Use $\int e^u\,du$.

**15.** Let $u = -2x + 5$, $du = -2\,dx$.

$$\int (-2x+5)^{3/2}\,dx = -\frac{1}{2}\int (-2x+5)^{3/2}(-2)\,dx$$
$$= -\frac{1}{5}(-2x+5)^{5/2} + C$$

**17.** Let $u = z - 4$, $du = dz$

$$\int \frac{5}{(z-4)^5}\,dz = 5\int (z-4)^{-5}\,dx = 5\,\frac{(z-4)^{-4}}{-4} + C$$
$$= \frac{-5}{4(z-4)^4} + C$$

**19.** Let $u = t^3 - 1$, $du = 3t^2\,dt$.

$$\int t^2\sqrt[3]{t^3-1}\,dt = \frac{1}{3}\int (t^3-1)^{1/3}(3t^2)\,dt$$
$$= \frac{1}{3}\,\frac{(t^3-1)^{4/3}}{4/3} + C$$
$$= \frac{(t^3-1)^{4/3}}{4} + C$$

**21.** $$\int \left[v + \frac{1}{(3v-1)^3}\right] dv = \int v\,dv + \frac{1}{3}\int (3v-1)^{-3}(3)dv$$
$$= \frac{1}{2}v^2 - \frac{1}{6(3v-1)^2} + C$$

**23.** Let $u = -t^3 + 9t + 1$, $du = (-3t^2 + 9)\,dt = -3(t^2 - 3)\,dt$.

$$\int \frac{t^2-3}{-t^3+9t+1}\,dt = -\frac{1}{3}\int \frac{-3(t^2-3)}{-t^3+9t+1}\,dt = -\frac{1}{3}\ln\left|-t^3+9t+1\right| + C$$

**25.** $$\int \frac{x^2}{x-1}\,dx = \int (x+1)\,dx + \int \frac{1}{x-1}\,dx$$
$$= \frac{1}{2}x^2 + x + \ln|x-1| + C$$

**27.** Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x) + C$$

**29.** $$\int (1+2x^2)^2\,dx = \int (4x^4 + 4x^2 + 1)dx = \frac{4}{5}x^5 + \frac{4}{3}x^3 + x + C = \frac{x}{15}(12x^4 + 20x^2 + 15) + C$$

**31.** Let $u = 2\pi x^2$, $du = 4\pi x\,dx$.

$$\int x(\cos 2\pi x^2)\,dx = \frac{1}{4\pi}\int (\cos 2\pi x^2)(4\pi x)\,dx$$
$$= \frac{1}{4\pi}\sin 2\pi x^2 + C$$

**33.** Let $u = \pi x$, $du = \pi\,dx$.

$$\int \csc(\pi x)\cot(\pi x)\,dx = \frac{1}{\pi}\int \csc(\pi x)\cot(\pi x)\pi\,dx = -\frac{1}{\pi}\csc(\pi x) + C$$

**35.** Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}\,dx = \frac{1}{5}\int e^{5x}(5)\,dx = \frac{1}{5}e^{5x} + C$$

**37.** Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int \frac{2}{e^{-x}+1}\,dx = 2\int \left(\frac{1}{e^{-x}+1}\right)\left(\frac{e^x}{e^x}\right)dx$$
$$= 2\int \frac{e^x}{1+e^x}\,dx = 2\ln(1+e^x) + C$$

**39.** $\displaystyle\int \frac{\ln x^2}{x}\,dx = 2\int (\ln x)\frac{1}{x}\,dx = 2\,\frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$

**41.** $\displaystyle\int \frac{1 + \sin x}{\cos x}\,dx = \int (\sec x + \tan x)\,dx = \ln|\sec x + \tan x| + \ln|\sec x| + C = \ln|\sec x(\sec x + \tan x)| + C$

**43.** $$\frac{1}{\cos\theta - 1} = \frac{1}{\cos\theta - 1}\cdot\frac{\cos\theta + 1}{\cos\theta + 1} = \frac{\cos\theta + 1}{\cos^2\theta - 1} = \frac{\cos\theta + 1}{-\sin^2\theta}$$
$$= -\csc\theta\cdot\cot\theta - \csc^2\theta$$
$$\int \frac{1}{\cos\theta - 1}\,d\theta = \int (-\csc\theta\cot\theta - \csc^2\theta)\,d\theta$$
$$= \csc\theta + \cot\theta + C$$
$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + C$$
$$= \frac{1 + \cos\theta}{\sin\theta} + C$$

**45.** $$\int \frac{3z + 2}{z^2 + 9}\,dz = \frac{3}{2}\int \frac{2z}{z^2 + 9}\,dz + 2\int \frac{dz}{z^2 + 9}$$
$$= \frac{3}{2}\ln(z^2 + 9) + \frac{2}{3}\arctan\left(\frac{z}{3}\right) + C$$

**47.** Let $u = 2t - 1$, $du = 2\,dt$.
$$\int \frac{-1}{\sqrt{1 - (2t - 1)^2}}\,dt = -\frac{1}{2}\int \frac{2}{\sqrt{1 - (2t - 1)^2}}\,dt$$
$$= -\frac{1}{2}\arcsin(2t - 1) + C$$

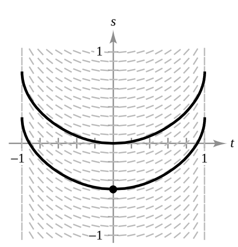
**49.** Let $u = \cos\left(\dfrac{2}{t}\right)$, $du = \dfrac{2\sin(2/t)}{t^2}\,dt$.
$$\int \frac{\tan(2/t)}{t^2}\,dt = \frac{1}{2}\int \frac{1}{\cos(2/t)}\left[\frac{2\sin(2/t)}{t^2}\right]dt$$
$$= \frac{1}{2}\ln\left|\cos\left(\frac{2}{t}\right)\right| + C$$

**51.** $\displaystyle\int \frac{3}{\sqrt{6x - x^2}}\,dx = 3\int \frac{1}{\sqrt{9 - (x - 3)^2}}\,dx = 3\arcsin\left(\frac{x - 3}{3}\right) + C$

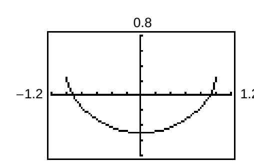
**53.** $\displaystyle\int \frac{4}{4x^2 + 4x + 65}\,dx = \int \frac{1}{[x + (1/2)]^2 + 16}\,dx = \frac{1}{4}\arctan\left[\frac{x + (1/2)}{4}\right] + C = \frac{1}{4}\arctan\left(\frac{2x + 1}{8}\right) + C$

**55.** $\dfrac{ds}{dt} = \dfrac{t}{\sqrt{1 - t^4}}$, $\left(0, -\dfrac{1}{2}\right)$

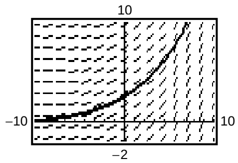
(a)




(b) $u = t^2$, $du = 2t\,dt$
$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{2t}{\sqrt{1 - (t^2)^2}}\,dt = \frac{1}{2}\arcsin t^2 + C$$
$$\left(0, -\frac{1}{2}\right):\ -\frac{1}{2} = \frac{1}{2}\arcsin 0 + C \implies C = -\frac{1}{2}$$
$$s = \frac{1}{2}\arcsin t^2 - \frac{1}{2}$$



**57.**




$y = 3e^{0.2x}$

**59.** $y = \int (1 + e^x)^2\,dx = \int (e^{2x} + 2e^x + 1)\,dx$

$= \frac{1}{2}e^{2x} + 2e^x + x + C$

**61.** $\frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$

Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$y = \int \frac{\sec^2 x}{4 + \tan^2 x}\,dx = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + C$$

**63.** Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\pi/4} \cos 2x\,dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x(2)\,dx$$

$$= \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \frac{1}{2}$$

**65.** Let $u = -x^2$, $du = -2x\,dx$.

$$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1$$

$$= \frac{1}{2}(1 - e^{-1}) \approx 0.316$$

**67.** Let $u = x^2 + 9$, $du = 2x\,dx$.

$$\int_0^4 \frac{2x}{\sqrt{x^2 + 9}}\,dx = \int_0^4 (x^2 + 9)^{-1/2}(2x)\,dx$$

$$= \left[2\sqrt{x^2 + 9}\right]_0^4 = 4$$

**69.** Let $u = 3x$, $du = 3\,dx$.

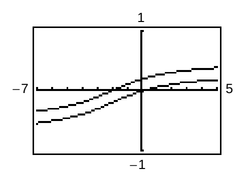
$$\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2}\,dx = \frac{1}{3}\int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2}\,dx$$

$$= \left[\frac{1}{6}\arctan\left(\frac{3x}{2}\right)\right]_0^{2/\sqrt{3}}$$

$$= \frac{\pi}{18} \approx 0.175$$

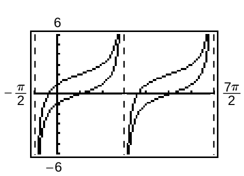
**71.** $\int \frac{1}{x^2 + 4x + 13}\,dx = \frac{1}{3}\arctan\left(\frac{x + 2}{3}\right) + C$

The antiderivatives are vertical translations of each other.




**73.** $\int \frac{1}{1 + \sin\theta}\,d\theta = \tan\theta - \sec\theta + C \left(\text{or } \frac{-2}{1 + \tan(\theta/2)}\right)$

The antiderivatives are vertical translations of each other.




**75.** Power Rule: $\int u^n\,du = \frac{u^{n+1}}{n + 1} + C, n \neq -1.$

$u = x^2 + 1, n = 3$

**77.** Log Rule: $\int \frac{du}{u} = \ln|u| + C, u = x^2 + 1.$

**79.** The are equivalent because

$e^{x + C_1} = e^x \cdot e^{C_1} = Ce^x, C = e^{C_1}$

**81.** $\sin x + \cos x = a\sin(x + b)$

$\sin x + \cos x = a\sin x\cos b + a\cos x\sin b$

$\sin x + \cos x = (a\cos b)\sin x + (a\sin b)\cos x$

Equate coefficients of like terms to obtain the following.

$$1 = a\cos b \quad \text{and} \quad 1 = a\sin b$$

Thus, $a = 1/\cos b$. Now, substitute for $a$ in $1 = a\sin b$.

$$1 = \left(\frac{1}{\cos b}\right)\sin b$$

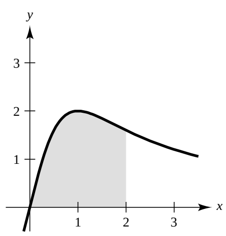
$$1 = \tan b \implies b = \frac{\pi}{4}$$

Since $b = \dfrac{\pi}{4}$, $a = \dfrac{1}{\cos(\pi/4)} = \sqrt{2}$. Thus, $\sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$.

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2}\sin(x + (\pi/4))} = \frac{1}{\sqrt{2}}\int \csc\left(x + \frac{\pi}{4}\right)dx = -\frac{1}{\sqrt{2}}\ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right| + C$$

**83.** $\displaystyle\int_0^2 \frac{4x}{x^2 + 1}\,dx \approx 3$

Matches (a).

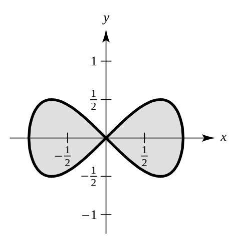



**85.** Let $u = 1 - x^2$, $du = -2x\,dx$.

$$A = 4\int_0^1 x\sqrt{1 - x^2}\,dx$$

$$= -2\int_0^1 (1 - x^2)^{1/2}(-2x)\,dx$$

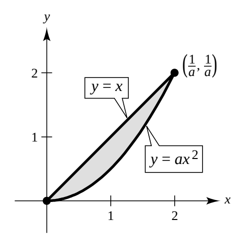
$$= \left[-\frac{4}{3}(1 - x^2)^{3/2}\right]_0^1 = \frac{4}{3}$$




**87.** $\displaystyle\int_0^{1/a} (x - ax^2)\,dx = \left[\frac{1}{2}x^2 - \frac{a}{3}x^3\right]_0^{1/a}$

$$= \frac{1}{6a^2}$$

Let $\dfrac{1}{6a^2} = \dfrac{2}{3}$, $12a^2 = 3$, $a = \dfrac{1}{2}$.




**89.** (a) **Shell Method:**

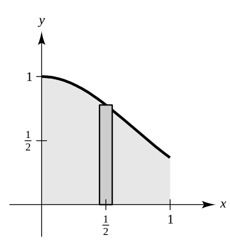
Let $u = -x^2$, $du = -2x\,dx$.

$$V = 2\pi\int_0^1 xe^{-x^2}\,dx$$

$$= -\pi\int_0^1 e^{-x^2}(-2x)\,dx$$

$$= \left[-\pi e^{-x^2}\right]_0^1$$

$$= \pi(1 - e^{-1}) \approx 1.986$$




(b) **Shell Method:**

$$V = 2\pi\int_0^b xe^{-x^2}\,dx$$

$$= \left[-\pi e^{-x^2}\right]_0^b$$

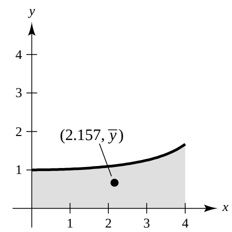
$$= \pi(1 - e^{-b^2}) = \frac{4}{3}$$

$$e^{-b^2} = \frac{3\pi - 4}{3\pi}$$

$$b = \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)}$$

$$\approx 0.743$$

**91.** $A = \int_0^4 \frac{5}{\sqrt{25 - x^2}}\,dx = \left[5 \arcsin \frac{x}{5}\right]_0^4 = 5 \arcsin \frac{4}{5}$




$$\begin{aligned}
\overline{x} &= \frac{1}{A}\int_0^4 x\left(\frac{5}{\sqrt{25 - x^2}}\right) dx \\
&= \frac{1}{5\arcsin(4/5)}\left(-\frac{5}{2}\right)\int_0^4 (25 - x^2)^{-1/2}(-2x)\,dx \\
&= \frac{1}{5\arcsin(4/5)}(-5)\Big[(25 - x^2)^{1/2}\Big]_0^4 \\
&= -\frac{1}{\arcsin(4/5)}[3 - 5] \\
&= \frac{2}{\arcsin(4/5)} \approx 2.157
\end{aligned}$$

**93.**
$$\begin{aligned}
y &= \tan(\pi x) \\
y' &= \pi \sec^2(\pi x) \\
1 + (y')^2 &= 1 + \pi^2 \sec^4(\pi x) \\
s &= \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)}\,dx \\
&\approx 1.0320
\end{aligned}$$

## Section 7.2 Integration by Parts

**1.** $\frac{d}{dx}[\sin x - x\cos x] = \cos x - (-x\sin x + \cos x) = x \sin x.$ Matches (b)

**3.** $\frac{d}{dx}[x^2e^x - 2xe^x + 2e^x] = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x.$ Matches (c)

**5.** $\int xe^{2x}\,dx$

$u = x,\ dv = e^{2x}\,dx$

**7.** $\int (\ln x)^2\,dx$

$u = (\ln x)^2,\ dv = dx$

**9.** $\int x\sec^2 x\,dx$

$u = x,\ dv = \sec^2 x\,dx$

**11.** $dv = e^{-2x}\,dx \implies v = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x}$

$u = x \implies du = dx$

$$\begin{aligned}
\int xe^{-2x}\,dx &= -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx \\
&= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C = \frac{-1}{4e^{2x}}(2x + 1) + C
\end{aligned}$$

**13.** Use integration by parts three times.

(1) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$ (2) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$ (3) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = x^3 \implies du = 3x^2\,dx$ $u = x^2 \implies du = 2x\,dx$ $u = x \implies du = dx$

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx = x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

**15.** $\int x^2\,e^{x^3}\,dx = \dfrac{1}{3}\int e^{x^3}(3x^2)dx = \dfrac{1}{3}e^{x^3} + C$

**17.** $dv = t\,dt \implies v = \int t\,dt = \dfrac{t^2}{2}$

$u = \ln(t+1) \implies du = \dfrac{1}{t+1}\,dt$

$$\int t\ln(t+1)\,dt = \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int \frac{t^2}{t+1}\,dt$$

$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int\left(t - 1 + \frac{1}{t+1}\right)dt$$

$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right] + C$$

$$= \frac{1}{4}\left[2(t^2 - 1)\ln|t+1| - t^2 + 2t\right] + C$$

**19.** Let $u = \ln x$, $du = \dfrac{1}{x}\,dx$.

$$\int \frac{(\ln x)^2}{x}\,dx = \int(\ln x)^2\left(\frac{1}{x}\right)dx = \frac{(\ln x)^3}{3} + C$$

**21.** $dv = \dfrac{1}{(2x+1)^2}\,dx \implies v = \int(2x+1)^{-2}\,dx = -\dfrac{1}{2(2x+1)}$

$u = xe^{2x} \implies du = (2xe^{2x} + e^{2x})\,dx = e^{2x}(2x+1)\,dx$

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \int\frac{e^{2x}}{2}\,dx$$

$$= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C$$

$$= \frac{e^{2x}}{4(2x+1)} + C$$

**23.** Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$ (2) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = x^2 \implies du = 2x\,dx$ $u = x \implies du = dx$

$$\int(x^2 - 1)e^x\,dx = \int x^2e^x\;dx - \int e^x dx = x^2e^x - 2\int xe^x\;dx - e^x$$

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] - e^x = x^2e^x - 2xe^x + e^x + C = (x-1)^2e^x + C$$

**25.** $dv = \sqrt{x-1}\,dx \implies v = \int (x-1)^{1/2} dx = \frac{2}{3}(x-1)^{3/2}$

$u = x \implies du = dx$

$$\int x\sqrt{x-1}\,dx = \frac{2}{3}x(x-1)^{3/2} - \frac{2}{3}\int (x-1)^{3/2}\,dx$$

$$= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C$$

$$= \frac{2(x-1)^{3/2}}{15}(3x+2) + C$$

**27.** $dv = \cos x\,dx \implies v = \int \cos x\,dx = \sin x$

$u = x \implies du = dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

**29.** Use integration by parts three times.

(1) $u = x^3, du = 3x^2, dv = \sin x\,dx, v = -\cos x$

$$\int x^3 \sin dx = -x^3\cos x + 3\int x^2\cos x\,dx$$

(2) $u = x^2, du = 2x\,dx, dv = \cos x\,dx, v = \sin x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3\left[x^2\sin x - 2\int x\sin x\,dx\right]$$

$$= -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$$

(3) $u = x, du = dx, dv = \sin x\,dx, v = -\cos x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3x^2\sin x - 6\left[-x\cos x + \int \cos x\,dx\right]$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

**31.** $u = t, du = dt, dv = \csc t\cot dt, v = -\csc t$

$$\int t\csc t\cot t\,dt = -t\csc t + \int \csc t\,dt$$

$$= -t\csc t - \ln\left|\csc t + \cot t\right| + C$$

**33.** $dv = dx \implies v = \int dx = x$

$u = \arctan x \implies du = \frac{1}{1+x^2}\,dx$

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

**35.** Use integration by parts twice.

(1) $dv = e^{2x}dx \implies v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \sin x \implies du = \cos x\,dx$

(2) $dv = e^{2x}\,dx \implies v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \cos x \implies du = -\sin x\,dx$

$$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx\right)$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$$

**37.** $y' = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

**39.** Use integration by parts twice.

(1) $dv = \dfrac{1}{\sqrt{2+3t}}\,dt \implies v = \displaystyle\int (2+3t)^{-1/2}\,dt = \frac{2}{3}\sqrt{2+3t}$

$u = t^2 \implies du = 2t\,dt$

(2) $dv = \sqrt{2+3t}\,dt \implies v = \displaystyle\int (2+3t)^{1/2}\,dt = \frac{2}{9}(2+3t)^{3/2}$

$u = t \implies du = dt$

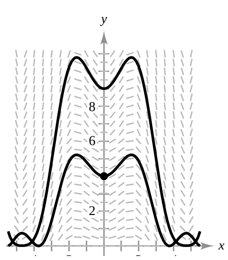
$$\begin{aligned} y = \int \frac{t^2}{\sqrt{2+3t}}\,dt &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\int t\sqrt{2+3t}\,dt \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\left[\frac{2t}{9}(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2}\,dt\right] \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{8t}{27}(2+3t)^{3/2} + \frac{16}{405}(2+3t)^{5/2} + C \\ &= \frac{2\sqrt{2+3t}}{405}(27t^2 - 24t + 32) + C \end{aligned}$$

**41.** $(\cos y)y' = 2x$

$$\int \cos y\,dy = \int 2x\,dx$$

$$\sin y = x^2 + C$$

**43.** (a)




(b) $\dfrac{dy}{dx} = x\sqrt{y}\cos x,\ (0, 4)$

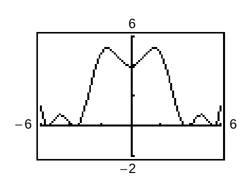
$$\int \frac{dy}{\sqrt{y}} = \int x\cos x\,dx$$

$$\int y^{-1/2}\,dy = \int x\cos x\,dx \qquad (u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x)$$

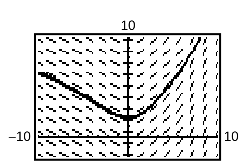
$$\begin{aligned} 2y^{1/2} &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C \end{aligned}$$

$(0, 4)$: $2(4)^{1/2} = 0 + 1 + C \implies C = 3$

$2\sqrt{y} = x\sin x + \cos x + 3$




**45.** $\dfrac{dy}{dx} = \dfrac{x}{y}e^{x/8},\ y(0) = 2$



**47.** $u = x,\ du = dx,\ dv = e^{-x/2}\,dx,\ v = -2e^{-x/2}$

$$\int xe^{-x/2}\,dx = -2xe^{-x/2} + \int 2e^{-x/2}\,dx = -2xe^{-x/2} - 4e^{-x/2} + C$$

Thus, $\displaystyle\int_0^4 xe^{-x/2}\,dx = \Big[-2xe^{-x/2} - 4e^{-x/2}\Big]_0^4$

$$= -8e^{-2} - 4e^{-2} + 4$$

$$= -12e^{-2} + 4 \approx 2.376.$$

**49.** See Exercise 27.

$$\int_0^{\pi/2} x\cos x\,dx = \Big[x\sin x + \cos x\Big]_0^{\pi/2} = \frac{\pi}{2} - 1$$

**51.** $u = \arccos x,\ du = -\dfrac{1}{\sqrt{1-x^2}}\,dx,\ dv = dx,\ v = x$

$$\int \arccos x\,dx = x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx = x\arccos x - \sqrt{1-x^2} + C$$

Thus, $\displaystyle\int_0^{1/2} \arccos x = \Big[x\arccos x - \sqrt{1-x^2}\Big]_0^{1/2}$

$$= \frac{1}{2}\arccos\!\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658.$$

**53.** Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \sin x \implies du = \cos x\,dx$

(2) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \cos x \implies du = -\sin x\,dx$

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$$

$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x)$$

$$\int e^x\sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Thus, $\displaystyle\int_0^1 e^x\sin x\,dx = \left[\frac{e^x}{2}(\sin x - \cos x)\right]_0^1 = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$

**55.** $dv = x^2\,dx,\ v = \dfrac{x^3}{3},\ u = \ln x,\ du = \dfrac{1}{x}\,dx$

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\left(\frac{1}{x}\right)dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

Hence, $\displaystyle\int_1^2 x^2\ln x\,dx = \left[\frac{x^3}{3}\ln x - \frac{1}{9}x^3\right]_1^2$

$$= \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{7}{9} \approx 1.071.$$

**57.** $dv = x\,dx,\ v = \dfrac{x^2}{2},\ u = \operatorname{arcsec} x,\ du = \dfrac{1}{x\sqrt{x^2-1}}\,dx$

$$\int x \operatorname{arcsec} x\,dx = \frac{x^2}{2}\operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2-1}}\,dx$$
$$= \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{4}\int \frac{2x}{\sqrt{x^2-1}}\,dx$$
$$= \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2-1} + C$$

Hence,

$$\int_2^4 x \operatorname{arcsec} x\,dx = \left[\frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2-1}\right]_2^4$$
$$= \left(8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2}\right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$
$$= 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3}$$
$$\approx 7.380.$$

**59.** $$\int x^2e^{2x}\,dx = x^2\left(\frac{1}{2}e^{2x}\right) - (2x)\left(\frac{1}{4}e^{2x}\right) + 2\left(\frac{1}{8}e^{2x}\right) + C$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$
$$= \frac{1}{4}e^{2x}(2x^2 - 2x + 1) + C$$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| $+$ | $x^2$ | $e^{2x}$ |
| $-$ | $2x$ | $\frac{1}{2}e^{2x}$ |
| $+$ | $2$ | $\frac{1}{4}e^{2x}$ |
| $-$ | $0$ | $\frac{1}{8}e^{2x}$ |

**61.** $$\int x^3 \sin x\,dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x\cos x - 6\sin x + C$$
$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$
$$= (3x^2 - 6)\sin x - (x^3 - 6x)\cos x + C$$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| $+$ | $x^3$ | $\sin x$ |
| $-$ | $3x^2$ | $-\cos x$ |
| $+$ | $6x$ | $-\sin x$ |
| $-$ | $6$ | $\cos x$ |
| $+$ | $0$ | $\sin x$ |

**63.** $$\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C$$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| $+$ | $x$ | $\sec^2 x$ |
| $-$ | $1$ | $\tan x$ |
| $+$ | $0$ | $-\ln\|\cos x\|$ |

**65.** Integration by parts is based on the product rule.

**67.** No. Substitution.

**69.** Yes. $u = x^2,\ dv = e^{2x}\,dx$

**71.** Yes. Let $u = x$ and $du = \dfrac{1}{\sqrt{x+1}},\ dx.$

(Substitution also works. Let $u = \sqrt{x+1}$)

**73.** $$\int t^3e^{-4t}\,dt = -\frac{e^{-4t}}{128}(32t^3 + 24t^2 + 12t + 3) + C$$

**75.** $$\int_0^{\pi/2} e^{-2x}\sin 3x\,dx = \left[\frac{e^{-2x}(-2\sin 3x - 3\cos 3x)}{13}\right]_0^{\pi/2} = \frac{1}{13}(2e^{-\pi} + 3) \approx 0.2374$$

**77.** (a) $dv = \sqrt{2x-3}\,dx \implies v = \int (2x-3)^{1/2}\,dx = \frac{1}{3}(2x-3)^{3/2}$

$u = 2x \implies du = 2\,dx$

$$\int 2x\sqrt{2x-3}\,dx = \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{3}\int (2x-3)^{3/2}\,dx$$

$$= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{15}(2x-3)^{5/2} + C$$

$$= \frac{2}{15}(2x-3)^{3/2}(3x+3) + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C$$

(b) $u = 2x - 3 \implies x = \dfrac{u+3}{2}$ and $dx = \dfrac{1}{2}\,du$

$$\int 2x\sqrt{2x-3}\,dx = \int 2\left(\frac{u+3}{2}\right)u^{1/2}\left(\frac{1}{2}\right)du = \frac{1}{2}\int (u^{3/2} + 3u^{1/2})\,du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right] + C$$

$$= \frac{1}{5}u^{3/2}(u+5) + C = \frac{1}{5}(2x-3)^{3/2}[(2x-3)+5] + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C$$

**79.** (a) $dv = \dfrac{x}{\sqrt{4+x^2}}\,dx \implies v = \int (4+x^2)^{-1/2}x\,dx = \sqrt{4+x^2}$

$u = x^2 \implies du = 2x\,dx$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - 2\int x\sqrt{4+x^2}\,dx$$

$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{1}{3}\sqrt{4+x^2}\,(x^2-8) + C$$

(b) $u = 4 + x^2 \implies x^2 = u - 4$ and $2x\,dx = du \implies x\,dx = \dfrac{1}{2}\,du$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{x^2}{\sqrt{4+x^2}}\,x\,dx = \int \frac{u-4}{\sqrt{u}}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int (u^{1/2} - 4u^{-1/2})\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 8u^{1/2}\right) + C$$

$$= \frac{1}{3}u^{1/2}(u-12) + C = \frac{1}{3}\sqrt{4+x^2}\,[(4+x^2) - 12] + C = \frac{1}{3}\sqrt{4+x^2}\,(x^2-8) + C$$

**81.** $n = 0$: $\displaystyle\int \ln x\,dx = x(\ln x - 1) + C$

$n = 1$: $\displaystyle\int x\ln x\,dx = \frac{x^2}{4}(2\ln x - 1) + C$

$n = 2$: $\displaystyle\int x^2\ln x\,dx = \frac{x^3}{9}(3\ln x - 1) + C$

$n = 3$: $\displaystyle\int x^3\ln x\,dx = \frac{x^4}{16}(4\ln x - 1) + C$

$n = 4$: $\displaystyle\int x^4\ln x\,dx = \frac{x^5}{25}(5\ln x - 1) + C$

In general, $\displaystyle\int x^n\ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$. (See Exercise 85.)

**83.** $dv = \sin x\,dx \implies v = -\cos x$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$$

**85.** $dv = x^n\,dx \implies v = \dfrac{x^{n+1}}{n+1}$

$u = \ln x \implies du = \dfrac{1}{x}\,dx$

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^n}{n+1}\,dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

$$= \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$$

**87.** Use integration by parts twice.

(1) $dv = e^{ax}\,dx \implies v = \dfrac{1}{a}e^{ax}$

$u = \sin bx \implies du = b\cos bx\,dx$

(2) $dv = e^{ax}\,dx \implies v = \dfrac{1}{a}e^{ax}$

$u = \cos bx \implies du = -b\sin bx\,dx$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx$$

$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx\right] = \frac{e^{ax}\sin bx}{a} - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

Therefore, $\left(1 + \dfrac{b^2}{a^2}\right)\displaystyle\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2}$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C.$$

**89.** $n = 3$ (Use formula in Exercise 85.)

$$\int x^3 \ln x\,dx = \frac{x^4}{16}[4\ln x - 1] + C$$

**91.** $a = 2,\ b = 3$ (Use formula in Exercise 88.)

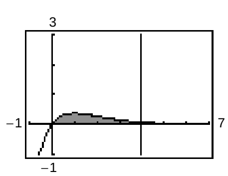
$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{13} + C$$

**93.** $dv = e^{-x}\,dx \implies v = -e^{-x}$

$u = x \implies du = dx$

$$A = \int_0^4 xe^{-x}\,dx = \Big[-xe^{-x}\Big]_0^4 + \int_0^4 e^{-x}\,dx = \frac{-4}{e^4} - \Big[e^{-x}\Big]_0^4$$

$$= 1 - \frac{5}{e^4} \approx 0.908$$

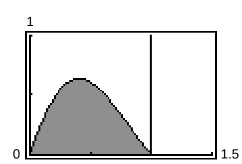



**95.** $A = \displaystyle\int_0^1 e^{-x}\sin(\pi x)\,dx$

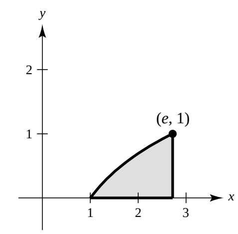
$$= \left[\frac{e^{-x}(-\sin \pi x - \pi\cos \pi x)}{1 + \pi^2}\right]_0^1$$

$$= \frac{1}{1 + \pi^2}\left(\frac{\pi}{e} + \pi\right) = \frac{\pi}{1 + \pi^2}\left(\frac{1}{e} + 1\right)$$

$\approx 0.395$ (See Exercise 87.)



**97.** (a) $A = \int_1^e \ln x \, dx = \left[-x + x \ln x\right]_1^e = 1$ (See Exercise 4.)




(b) $R(x) = \ln x,\ r(x) = 0$

$$V = \pi \int_1^e (\ln x)^2 \, dx$$

$$= \pi\left[x(\ln x)^2 - 2x \ln x + 2x\right]_1^e \text{ (Use integration by parts twice, see Exercise 7.)}$$

$$= \pi(e - 2) \approx 2.257$$

(c) $p(x) = x,\ h(x) = \ln x$

$$V = 2\pi \int_1^e x \ln x \, dx = 2\pi\left[\frac{x^2}{4}(-1 + 2\ln x)\right]_1^e$$

$$= \frac{(e^2 + 1)\pi}{2} \approx 13.177 \text{ (See Exercise 85.)}$$

(d)

$$\bar{x} = \frac{\int_1^e x \ln x \, dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$$

$$\bar{y} = \frac{\frac{1}{2}\int_1^e (\ln x)^2 dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2}\right) \approx (2.097, 0.359)$$

**99.** Average value $= \dfrac{1}{\pi}\displaystyle\int_0^\pi e^{-4t}(\cos 2t + 5 \sin 2t)\, dt$

$$= \frac{1}{\pi}\left[e^{-4t}\left(\frac{-4\cos 2t + 2\sin 2t}{20}\right) + 5e^{-4t}\left(\frac{-4\sin 2t - 2\cos 2t}{20}\right)\right]_0^\pi \text{(From Exercises 87 and 88)}$$

$$= \frac{7}{10\pi}(1 - e^{-4\pi}) \approx 0.223$$

**101.** $c(t) = 100{,}000 + 4000t,\ r = 5\%,\ t_1 = 10$

$$P = \int_0^{10} (100{,}000 + 4000t)e^{-0.05t} \, dt = 4000\int_0^{10} (25 + t)e^{-0.05t} \, dt$$

Let $u = 25 + t,\ dv = e^{-0.05t} dt,\ du = dt,\ v = -\dfrac{100}{5}e^{-0.05t}$

$$P = 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} + \frac{100}{5}\int_0^{10} e^{-0.05t} \, dt\right\}$$

$$= 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} - \left[\frac{10{,}000}{25}e^{-0.05t}\right]_0^{10}\right\} \approx \$931{,}265$$

**103.** $\displaystyle\int_{-\pi}^{\pi} x \sin nx \, dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$

$$= -\frac{\pi}{n}\cos \pi n - \frac{\pi}{n}\cos(-\pi n)$$

$$= -\frac{2\pi}{n}\cos \pi n$$

$$= \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$$

**105.** Let $u = x$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$I_1 = \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}x\right)\right]_0^1$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)$$

Let $u = (-x + 2)$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = -dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$I_2 = \int_1^2 (-x+2)\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2(-x+2)}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}x\right)\right]_1^2$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)$$

$$h(I_1 + I_2) = b_n = h\left[\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right] = \frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

**107. Shell Method:**

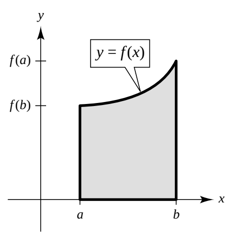
$$V = 2\pi\int_a^b x f(x)\,dx$$

$$dv = x\,dx \implies v = \frac{x^2}{2}$$

$$u = f(x) \implies du = f'(x)\,dx$$

$$V = 2\pi\left[\frac{x^2}{2}f(x) - \int \frac{x^2}{2}f'(x)dx\right]_a^b$$

$$= \pi\left[(b^2f(b) - a^2f(a)) - \int_a^b x^2 f'(x)\,dx\right]$$




**Disk Method:**

$$V = \pi\int_0^{f(a)} (b^2 - a^2)\,dy + \pi\int_{f(a)}^{f(b)} [b^2 - [f^{-1}(y)]^2]\,dy$$

$$= \pi(b^2 - a^2)f(a) + \pi b^2(f(b) - f(a)) - \pi\int_{f(a)}^{f(b)} [f^{-1}(y)]^2\,dy$$

$$= \pi\left[(b^2f(b) - a^2f(a)) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2\,dy\right]$$

Since $x = f^{-1}(y)$, we have $f(x) = y$ and $f'(x)dx = dy$. When $y = f(a)$, $x = a$. When $y = f(b)$, $x = b$. Thus,

$$\int_{f(a)}^{f(b)} [f^{-1}(y)]^2\,dy = \int_a^b x^2 f'(x)\,dx$$

and the volumes are the same.

**109.** $f'(x) = xe^{-x}$

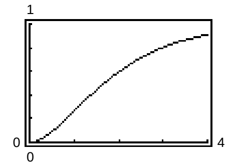
(a) $f(x) = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$

(Parts: $u = x,\ dv = e^{-x}\,dx$)

$f(0) = 0 = -1 + C \implies C = 1$

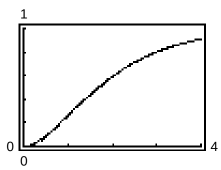
$f(x) = -xe^{-x} - e^{-x} + 1$

(b)



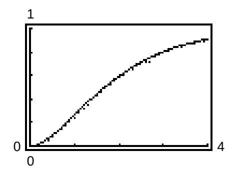

(c) You obtain the points

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.05 | 0 |
| 2 | 0.10 | $2.378 \times 10^{-3}$ |
| 3 | 0.15 | 0.0069 |
| 4 | 0.20 | 0.0134 |
| ⋮ | ⋮ | ⋮ |
| 80 | 4.0 | 0.9064 |




(d) You obtain the points

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0.1 | 0 |
| 2 | 0.2 | 0.0090484 |
| 3 | 0.3 | 0.025423 |
| 4 | 0.4 | 0.047648 |
| ⋮ | ⋮ | ⋮ |
| 40 | 4.0 | 0.9039 |




(e) $f(4) = 0.9084$

The approximations are tangent line approximations. The results in (c) are better because $\Delta x$ is smaller.

## Section 7.3 Trigonometric Integrals

**1.** $f(x) = \sin^4 x + \cos^4 x$

(a) $$\sin^4 x + \cos^4 x = \left(\frac{1 - \cos 2x}{2}\right)^2 + \left(\frac{1 + \cos 2x}{2}\right)^2$$

$$= \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x + 1 + 2\cos 2x + \cos^2 2x]$$

$$= \frac{1}{4}\left[2 + 2\frac{1 + \cos 4x}{2}\right]$$

$$= \frac{1}{4}[3 + \cos 4x]$$

(b) $$\sin^4 x + \cos^4 x = (\sin^2 x)^2 + \cos^4 x$$

$$= (1 - \cos^2 x)^2 + \cos^4 x$$

$$= 1 - 2\cos^2 x + 2\cos^4 x$$

(c) $$\sin^4 x + \cos^4 x = \sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x - 2\sin^2 x\cos^2 x$$

$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$$

$$= 1 - 2\sin^2 x\cos^2 x$$

**—CONTINUED—**

**1. —CONTINUED—**

(d) $1 - 2\sin^2 x\cos^2 x = 1 - (2\sin x\cos x)(\sin x\cos x)$

$$= 1 - (\sin 2x)\left(\frac{1}{2}\sin 2x\right)$$

$$= 1 - \frac{1}{2}\sin^2(2x)$$

(e) Four ways. There is often more than one way to rewrite a trigonometric expression.

**3.** Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \cos^3 x\sin x\,dx = -\int \cos^3 x(-\sin x)\,dx$$

$$= -\frac{1}{4}\cos^4 x + C$$

**5.** Let $u = \sin 2x$, $du = 2\cos 2x\,dx$.

$$\int \sin^5 2x\cos 2x\,dx = \frac{1}{2}\int \sin^5 2x(2\cos 2x)dx$$

$$= \frac{1}{12}\sin^6 2x + C$$

**7.** Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \sin^5 x\cos^2 x\,dx = \int \sin x(1 - \cos^2 x)^2\cos^2 x\,dx$$

$$= -\int(\cos^2 x - 2\cos^4 x + \cos^6 x)(-\sin x)\,dx = \frac{-1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

**9.** $$\int \cos^3\theta\sqrt{\sin\theta}\,d\theta = \int \cos\theta(1 = \sin^2\theta)(\sin\theta)^{1/2}\,d\theta$$

$$= \int[(\sin\theta)^{1/2} - (\sin\theta)^{5/2}]\cos\theta\,d\theta$$

$$= \frac{2}{3}(\sin\theta)^{3/2} - \frac{2}{7}(\sin\theta)^{7/2} + C$$

**11.** $$\int \cos^2 3x\,dx = \int \frac{1 + \cos 6x}{2}\,dx$$

$$= \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C$$

$$= \frac{1}{12}(6x + \sin 6x) + C$$

**13.** $$\int \sin^2\alpha \cdot \cos^2\alpha\,d\alpha = \int \frac{1 - \cos 2\alpha}{2}\cdot\frac{1 + \cos 2\alpha}{2}\,d\alpha$$

$$= \frac{1}{4}\int(1 - \cos^2 2\alpha)\,d\alpha$$

$$= \frac{1}{4}\int\left(1 - \frac{1 + \cos 4\alpha}{2}\right)d\alpha$$

$$= \frac{1}{8}\int(1 - \cos 4\alpha)\,d\alpha$$

$$= \frac{1}{8}\left[\alpha - \frac{1}{4}\sin 4\alpha\right] + C$$

$$= \frac{1}{32}[4\alpha - \sin 4\alpha] + C$$

**15.** Integration by parts.

$$dv = \sin^2 x\, dx = \frac{1 - \cos 2x}{2} \implies v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x \implies du = dx$$

$$\int x \sin^2 x\, dx = \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4}\int (2x - \sin 2x)\, dx$$

$$= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4}\left(x^2 + \frac{1}{2}\cos 2x\right) + C = \frac{1}{8}(2x^2 - 2x \sin 2x - \cos 2x) + C$$

**17.** Let $u = \sin x$, $du = \cos x\, dx$.

$$\int_0^{\pi/2} \cos^3 x\, dx = \int_0^{\pi/2} (1 - \sin^2 x) \cos x\, dx$$

$$= \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{2}{3}$$

**19.** Let $u = \sin x$, $du = \cos x\, dx$.

$$\int_0^{\pi/2} \cos^7 x\, dx = \int_0^{\pi/2} (1 - \sin^2 x)^3 \cos x\, dx = \int_0^{\pi/2} (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \cos x\, dx$$

$$= \left[\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right]_0^{\pi/2} = \frac{16}{35}$$

**21.** $\displaystyle\int \sec(3x)\, dx = \frac{1}{3}\ln\left|\sec 3x + \tan 3x\right| + C$

**23.** 
$$\int \sec^4 5x\, dx = \int (1 + \tan^2 5x) \sec^2 5x\, dx$$

$$= \frac{1}{5}\left(\tan 5x + \frac{\tan^3 5x}{3}\right) + C$$

$$= \frac{\tan 5x}{15}(3 + \tan^2 5x) + C$$

**25.** $dv = \sec^2 \pi x\, dx \implies v = \dfrac{1}{\pi}\tan \pi x$

$$u = \sec \pi x \implies du = \pi \sec \pi x \tan \pi x\, dx$$

$$\int \sec^3 \pi x\, dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x\, dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec \pi x(\sec^2 \pi x - 1)\, dx$$

$$2\int \sec^3 \pi x\, dx = \frac{1}{\pi}(\sec \pi x \tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right|) + C_1$$

$$\int \sec^3 \pi x\, dx = \frac{1}{2\pi}(\sec \pi x \tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right|) + C$$

**27.** 
$$\int \tan^5 \frac{x}{4}\, dx = \int \left(\sec^2 \frac{x}{4} - 1\right)\tan^3 \frac{x}{4}\, dx$$

$$= \int \tan^3 \frac{x}{4}\sec^2 \frac{x}{4}\, dx - \int \tan^3 \frac{x}{4}\, dx$$

$$= \tan^4 \frac{x}{4} - \int \left(\sec^2 \frac{x}{4} - 1\right) \tan \frac{x}{4}\, dx$$

$$= \tan^4 \frac{x}{4} - 2\tan^2 \frac{x}{4} - 4\ln\left|\cos \frac{x}{4}\right| + C$$

**29.** $u = \tan x$, $du = \sec^2 x\, dx$

$$\int \sec^2 x \tan x\, dx = \frac{1}{2}\tan^2 x + C$$

**31.** $\displaystyle\int \tan^2 x \sec^2 x\, dx = \frac{\tan^3 x}{3} + C$

**33.** $\displaystyle\int \sec^6 4x \tan 4x\, dx = \frac{1}{4}\int \sec^5 4x(4 \sec 4x \tan 4x)\, dx$

$$= \frac{\sec^6 4x}{24} + C$$

**35.** Let $u = \sec x$, $du = \sec x \tan x\, dx$.

$$\int \sec^3 x \tan x\, dx = \int \sec^2 x(\sec x \tan x)\, dx$$

$$= \frac{1}{3}\sec^3 x + C$$

**37.** $\displaystyle\int \frac{\tan^2 x}{\sec x}\, dx = \int \frac{(\sec^2 x - 1)}{\sec x}\, dx$

$$= \int (\sec x - \cos x)\, dx$$

$$= \ln|\sec x + \tan x| - \sin x + C$$

**39.** $\displaystyle r = \int \sin^4(\pi\theta)\, d\theta = \frac{1}{4}\int [1 - \cos(2\pi\theta)]^2\, d\theta$

$$= \frac{1}{4}\int [1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)]\, d\theta$$

$$= \frac{1}{4}\int \left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2}\right] d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{\pi}\sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi}\sin(4\pi\theta)\right] + C$$

$$= \frac{1}{32\pi}[12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)] + C$$

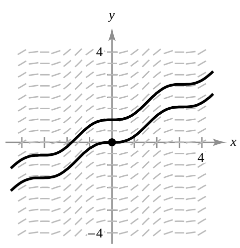
**41.** $\displaystyle y = \int \tan^3 3x \sec 3x\, dx$

$$= \int (\sec^2 3x - 1)\sec 3x \tan 3x\, dx$$

$$= \frac{1}{3}\int \sec^2 3x(3\sec 3x \tan 3x)\, dx - \frac{1}{3}\int 3\sec 3x \tan 3x\, dx$$

$$= \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$$

**43.** (a)

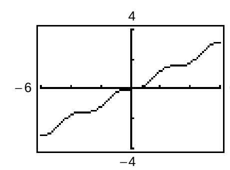



(b) $\displaystyle \frac{dy}{dx} = \sin^2 x,\ (0, 0)$

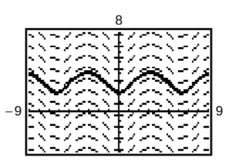
$$y = \int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

$(0, 0)$: $0 = C$, $\displaystyle y = \frac{1}{2}x - \frac{\sin 2x}{4}$




**45.** $\displaystyle \frac{dy}{dx} = \frac{3\sin x}{y},\ y(0) = 2$




**47.** $\displaystyle\int \sin 3x \cos 2x\, dx = \frac{1}{2}\int (\sin 5x + \sin x)\, dx$

$$= \frac{-1}{2}\left(\frac{1}{5}\cos 5x + \cos x\right) + C$$

$$= \frac{-1}{10}(\cos 5x + 5\cos x) + C$$

**49.** $\displaystyle\int \sin\theta \sin 3\theta\, d\theta = \frac{1}{2}\int (\cos 2\theta - \cos 4\theta)\, d\theta$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta\right) + C$$

$$= \frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$$

**51.** $\displaystyle\int \cot^3 2x\, dx = \int (\csc^2 2x - 1)\cot 2x\, dx$

$$= -\frac{1}{2}\int \cot 2x(-2\csc^2 2x)\, dx - \frac{1}{2}\int \frac{2\cos 2x}{\sin 2x}\, dx$$

$$= -\frac{1}{4}\cot^2 2x - \frac{1}{2}\ln|\sin 2x| + C$$

$$= \frac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$$

**53.** Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$.

$$\int \csc^4\theta\,d\theta = \int \csc^2\theta(1 + \cot^2\theta)\,d\theta$$
$$= \int \csc^2\theta\,d\theta + \int \csc^2\theta\cot^2\theta\,d\theta$$
$$= -\cot\theta - \frac{1}{3}\cot^3\theta + C$$

**55.** $$\int \frac{\cot^2 t}{\csc t}\,dt = \int \frac{\csc^2 t - 1}{\csc t}\,dt$$
$$= \int (\csc t - \sin t)dt$$
$$= \ln|\csc t - \cot t| + \cos t + C$$

**57.** $$\int \frac{1}{\sec x\tan x}\,dx = \int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1 - \sin^2 x}{\sin x}\,dx$$
$$= \int (\csc x - \sin x)\,dx$$
$$= \ln|\csc x - \cot x| + \cos x + C$$

**59.** $$\int (\tan^4 t - \sec^4 t)\,dt = \int (\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t)\,dt \qquad (\tan^2 t - \sec^2 t = -1)$$
$$= -\int (\tan^2 t + \sec^2 t)\,dt = -\int (2\sec^2 t - 1)\,dt = -2\tan t + t + C$$

**61.** $$\int_{-\pi}^{\pi} \sin^2 x\,dx = 2\int_0^{\pi} \frac{1 - \cos 2x}{2}\,dx$$
$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \pi$$

**63.** $$\int_0^{\pi/4} \tan^3 x\,dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan x\,dx$$
$$= \int_0^{\pi/4} \sec^2 x\tan x\,dx - \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$$
$$= \left[\frac{1}{2}\tan^2 x + \ln|\cos x|\right]_0^{\pi/4}$$
$$= \frac{1}{2}(1 - \ln 2)$$

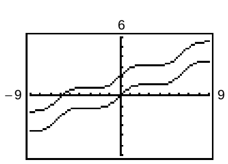
**65.** Let $u = 1 + \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t}\,dt = \Big[\ln|1 + \sin t|\Big]_0^{\pi/2} = \ln 2$$

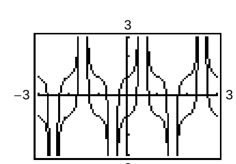
**67.** Let $u = \sin x$, $du = \cos x\,dx$.

$$\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx = 2\int_0^{\pi/2} (1 - \sin^2 x)\cos x\,dx$$
$$= 2\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{4}{3}$$

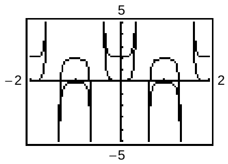
**69.** $$\int \cos^4\frac{x}{2}dx = \frac{1}{16}[6x + 8\sin x + \sin 2x] + C$$




**71.** $$\int \sec^5 \pi x\,dx = \frac{1}{4\pi}\left\{\sec^3 \pi x\tan \pi x + \frac{3}{2}[\sec \pi x\tan \pi x + \ln|\sec \pi x + \tan \pi x|]\right\} + C$$



**73.** $\displaystyle\int \sec^5 \pi x \tan \pi x \, dx = \frac{1}{5\pi} \sec^5 \pi x + C$




**75.** $\displaystyle\int_0^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta = \frac{1}{2}\left[\sin \theta - \frac{1}{5}\sin 5\theta\right]_0^{\pi/4} = \frac{3\sqrt{2}}{10}$

**77.** $\displaystyle\int_0^{\pi/2} \sin^4 x \, dx = \frac{1}{4}\left[\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right]_0^{\pi/2}$

$$= \frac{3\pi}{16}$$

**79.** (a) Save one sine factor and convert the remaining sine factors to cosine. Then expand and integrate.

(b) Save one cosine factor and convert the remaining cosine factors to sine. Then expand and integrate.

(c) Make repeated use of the power reducing formula to convert the integrand to odd powers of the cosine.

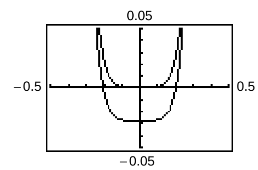
**81.** (a) Let $u = \tan 3x$, $du = 3\sec^2 3x \, dx$.

$$\begin{aligned}
\int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x \sec^2 3x \, dx \\
&= \frac{1}{3}\int (\tan^2 3x + 1)\tan^3 3x(3\sec^2 3x)\, dx \\
&= \frac{1}{3}\int (\tan^5 3x + \tan^3 3x)(3\sec^2 3x)\, dx \\
&= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1
\end{aligned}$$

Or let $u = \sec 3x$, $du = 3\sec 3x \tan 3x \, dx$.

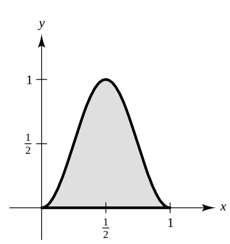
$$\begin{aligned}
\int \sec^4 3x \tan^3 3x \, dx &= \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x \, dx \\
&= \frac{1}{3}\int \sec^3 3x(\sec^2 3x - 1)(3\sec 3x \tan 3x)\, dx \\
&= \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C
\end{aligned}$$

(b)




(c) $$\begin{aligned}
\frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C &= \frac{(1 + \tan^2 3x)^3}{18} - \frac{(1 + \tan^2 3x)^2}{12} + C \\
&= \frac{1}{18}\tan^6 3x + \frac{1}{6}\tan^4 3x + \frac{1}{6}\tan^2 3x + \frac{1}{18} - \frac{1}{12}\tan^4 3x - \frac{1}{6}\tan^2 3x - \frac{1}{12} + C \\
&= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12}\right) + C \\
&= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2
\end{aligned}$$

**83.** $$\begin{aligned}
A &= \int_0^1 \sin^2(\pi x)\, dx \\
&= \int_0^1 \frac{1 - \cos(2\pi x)}{2}\, dx \\
&= \left[\frac{x}{2} - \frac{1}{4\pi}\sin(2\pi x)\right]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$



**85.** (a) $V = \pi\int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2}\int_0^{\pi}(1 - \cos 2x)\, dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{2}$

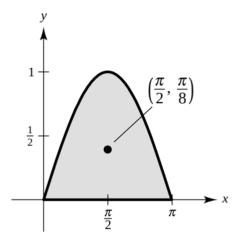
(b) $A = \int_0^{\pi} \sin x\, dx = \left[-\cos x\right]_0^{\pi} = 1 + 1 = 2$

Let $u = x$, $dv = \sin x\, dx$, $du = dx$, $v = -\cos x$.

$$\bar{x} = \frac{1}{A}\int_0^{\pi} x \sin x\, dx = \frac{1}{2}\left[\left[-x\cos x\right]_0^{\pi} + \int_0^{\pi}\cos x\, dx\right] = \frac{1}{2}\left[-x\cos x + \sin x\right]_0^{\pi} = \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A}\int_0^{\pi}\sin^2 x\, dx$$
$$= \frac{1}{8}\int_0^{\pi}(1 - \cos 2x)\, dx$$
$$= \frac{1}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$$




**87.** $dv = \sin x\, dx \implies v = -\cos x$

$u = \sin^{n-1} x \implies du = (n - 1)\sin^{n-2} x \cos x\, dx$

$$\int \sin^n x\, dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x \cos^2 x\, dx$$
$$= -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x(1 - \sin^2 x)\, dx$$
$$= -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\, dx - (n - 1)\int \sin^n x\, dx$$

Therefore, $n\int \sin^n x\, dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\, dx$

$$\int \sin^n x\, dx = \frac{-\sin^{n-1} x\cos x}{n} + \frac{n - 1}{n}\int \sin^{n-2} x\, dx.$$

**89.** Let $u = \sin^{n-1} x$, $du = (n - 1)\sin^{n-2} x\cos x\, dx$, $dv = \cos^m x\sin x\, dx$, $v = \frac{-\cos^{m+1} x}{m + 1}$.

$$\int \cos^m x\sin^n x\, dx = \frac{-\sin^{n-1} x\cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x\cos^{m+2} x\, dx$$
$$= \frac{-\sin^{n-1} x\cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x\cos^m x(1 - \sin^2 x)\, dx$$
$$= \frac{-\sin^{n-1} x\cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x\cos^m x\, dx - \frac{n - 1}{m + 1}\int \sin^n x\cos^m x\, dx$$

$$\frac{m + n}{m + 1}\int \cos^m x\sin^n x\, dx = \frac{-\sin^{n-1} x\cos^{m+1} x}{m + 1} + \frac{n - 1}{m + 1}\int \sin^{n-2} x\cos^m x\, dx$$

$$\int \cos^m x\sin^n x\, dx = \frac{-\cos^{m+1} x\sin^{n-1}}{m + n} + \frac{n - 1}{m + n}\int \cos^m x\sin^{n-2} x\, dx$$

**91.** $$\int \sin^5 x\,dx = -\frac{\sin^4 x\cos x}{5} + \frac{4}{5}\int \sin^3 x\,dx$$
$$= -\frac{\sin^4 x\cos x}{5} + \frac{4}{5}\left[-\frac{\sin^2 x\cos x}{3} + \frac{2}{3}\int \sin x\,dx\right]$$
$$= -\frac{1}{5}\sin^4 x\cos x - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x + C$$
$$= -\frac{\cos x}{15}[3\sin^4 x + 4\sin^2 x + 8] + C$$

**93.** $$\int \sec^4\left(\frac{2\pi x}{5}\right) dx = \frac{5}{2\pi}\int \sec^4\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx$$
$$= \frac{5}{2\pi}\left[\frac{1}{3}\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3}\int \sec^2\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx\right]$$
$$= \frac{5}{6\pi}\left[\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + 2\tan\left(\frac{2\pi x}{5}\right)\right] + C$$
$$= \frac{5}{6\pi}\tan\left(\frac{2\pi x}{5}\right)\left[\sec^2\left(\frac{2\pi x}{5}\right) + 2\right] + C$$

**95.** (a) $f(t) = a_0 + a_1\cos\frac{\pi t}{6} + b_1\sin\frac{\pi t}{6}$ where:

$$a_0 = \frac{1}{12}\int_0^{12} f(t)\,dt$$
$$a_1 = \frac{1}{6}\int_0^{12} f(t)\cos\frac{\pi t}{6}\,dt$$
$$b_1 = \frac{1}{6}\int_0^{12} f(t)\sin\frac{\pi t}{6}\,dt$$

$$a_0 \approx \frac{12-0}{3(12)^2}[30.9 + 4(32.2) + 2(41.1) + 4(53.7) + 2(64.6) + 4(74.0) + 2(78.2) + 4(77.0) + 2(71.0) + 4(60.1) + 2(47.1) + 4(35.7) + 30.9] \approx 55.46$$

$$a_1 \approx \frac{12-0}{6(3)(12)}\left[30.9\cos 0 + 4\left(32.2\cos\frac{\pi}{6}\right) + 2\left(41.1\cos\frac{\pi}{3}\right) + 4\left(53.7\cos\frac{\pi}{2}\right) + 2\left(64.6\cos\frac{2\pi}{3}\right) + 4\left(74.0\cos\frac{5\pi}{6}\right) + 2(78.2\cos\pi) + 4\left(77.0\cos\frac{7\pi}{6}\right) + 2\left(71.0\cos\frac{4\pi}{3}\right) + 4\left(60.1\cos\frac{3\pi}{2}\right) + 2\left(47.1\cos\frac{5\pi}{3}\right) + 4\left(35.7\cos\frac{11\pi}{6}\right) + 30.9\cos 2\pi\right] \approx -23.88$$

$$b_1 \approx \frac{12-0}{6(3)(12)}\left[30.9\sin 0 + 4\left(32.2\sin\frac{\pi}{6}\right) + 2\left(41.1\sin\frac{\pi}{3}\right) + 4\left(53.7\sin\frac{\pi}{2}\right) + 2\left(64.6\sin\frac{2\pi}{3}\right) + 4\left(74.0\sin\frac{5\pi}{6}\right) + 2(78.2\sin\pi) + 4\left(77.0\sin\frac{7\pi}{6}\right) + 2\left(71.0\sin\frac{4\pi}{3}\right) + 4\left(60.1\sin\frac{3\pi}{2}\right) + 2\left(47.1\sin\frac{5\pi}{3}\right) + 4\left(35.7\sin\frac{11\pi}{6}\right) + 30.9\sin 2\pi\right] \approx -3.34$$

$$H(t) \approx 55.46 - 23.88\cos\frac{\pi t}{6} - 3.34\sin\frac{\pi t}{6}$$

**—CONTINUED—**

**95. —CONTINUED—**

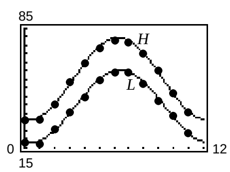
(b) $$a_0 \approx \frac{12-0}{3(12)^2}[18.0 + 4(17.7) + 2(25.8) + 4(36.1) + 2(45.4) + 4(55.2) + 2(59.9) + 4(59.4) + 2(53.1) + 4(43.2) + 2(34.3) + 4(24.2) + 18.0] \approx 39.34$$

$$a_1 \approx \frac{12-0}{6(3)(12)}\left[18.0\cos 0 + 4\left(17.7\cos\frac{\pi}{6}\right) + 2\left(25.8\cos\frac{\pi}{3}\right) + 4\left(36.1\cos\frac{\pi}{2}\right) + 2\left(45.4\cos\frac{2\pi}{3}\right) + 4\left(55.2\cos\frac{5\pi}{6}\right) + 2(59.9\cos\pi) + 4\left(59.4\cos\frac{7\pi}{6}\right) + 2\left(53.1\cos\frac{4\pi}{3}\right) + 4\left(43.2\cos\frac{3\pi}{2}\right) + 2\left(34.3\cos\frac{5\pi}{3}\right) + 4\left(24.2\cos\frac{11\pi}{6}\right) + 18\cos 2\pi\right] \approx -20.78$$

$$b_1 \approx \frac{12-0}{6(3)(12)}\left[18.0\sin 0 + 4\left(17.7\sin\frac{\pi}{6}\right) + 2\left(25.8\sin\frac{\pi}{3}\right) + 4\left(36.1\sin\frac{\pi}{2}\right) + 2\left(45.4\sin\frac{2\pi}{3}\right) + 4\left(55.2\sin\frac{5\pi}{6}\right) + 2(59.9\sin\pi) + 4\left(59.4\sin\frac{7\pi}{6}\right) + 2\left(53.1\sin\frac{4\pi}{3}\right) + 4\left(43.2\sin\frac{3\pi}{2}\right) + 2\left(34.3\sin\frac{5\pi}{3}\right) + 4\left(24.2\sin\frac{11\pi}{6}\right) + 18\sin 2\pi\right] \approx -4.33$$

$$L(t) \approx 39.34 - 20.78\cos\frac{\pi t}{6} - 4.33\sin\frac{\pi t}{6}$$

(c) The difference between the maximum and minimum temperatures is greatest in the summer.




**97.** $$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \ (m \neq n)$$

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m-n)x - \cos(m+n)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \ (m \neq n)$$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(m+n)x + \sin(m-n)x]\,dx$$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}, \ (m \neq n)$$

$$= -\frac{1}{2}\left[\left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right) - \left(\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n}\right)\right]$$

$$= 0, \text{ since } \cos(-\theta) = \cos\theta.$$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(mx)\,dx = \frac{1}{m}\left[\frac{\sin^2(mx)}{2}\right]_{-\pi}^{\pi} = 0$$

## Section 7.4 Trigonometric Substitution

**1.** $$\frac{d}{dx}\left[4\ln\left|\frac{\sqrt{x^2+16}-4}{x}\right| + \sqrt{x^2+16}+C\right] = \frac{d}{dx}\left[4\ln\left|\sqrt{x^2+16}-4\right| - 4\ln|x| + \sqrt{x^2+16}+C\right]$$

$$= 4\left[\frac{x/\sqrt{x^2+16}}{\sqrt{x^2+16}-4}\right] - \frac{4}{x} + \frac{x}{\sqrt{x^2+16}}$$

$$= \frac{4x}{\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)} - \frac{4}{x} + \frac{x}{\sqrt{x^2+16}}$$

$$= \frac{4x^2 - 4\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right) + x^2\left(\sqrt{x^2+16}-4\right)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$

$$= \frac{4x^2 - 4(x^2+16) + 16\sqrt{x^2+16} + x^2\sqrt{x^2+16} - 4x^2}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$

$$= \frac{\sqrt{x^2+16}(x^2+16) - 4(x^2+16)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$

$$= \frac{(x^2+16)\left(\sqrt{x^2+16}-4\right)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)} = \frac{\sqrt{x^2+16}}{x}$$

Indefinite integral: $\displaystyle\int \frac{\sqrt{x^2+16}}{x}\,dx$ Matches (b)

**3.** $$\frac{d}{dx}\left[8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + C\right] = 8\frac{1/4}{\sqrt{1-(x/4)^2}} - \frac{x(1/2)(16-x^2)^{-1/2}(-2x) + \sqrt{16-x^2}}{2}$$

$$= \frac{8}{\sqrt{16-x^2}} + \frac{x^2}{2\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2}$$

$$= \frac{16}{2\sqrt{16-x^2}} + \frac{x^2}{2\sqrt{16-x^2}} - \frac{(16-x^2)}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}}$$

Matches (a)

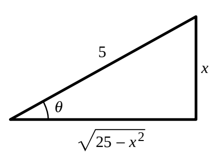
**5.** Let $x = 5\sin\theta$, $dx = 5\cos\theta\,d\theta$, $\sqrt{25-x^2} = 5\cos\theta$.

$$\int \frac{1}{(25-x^2)^{3/2}}\,dx = \int \frac{5\cos\theta}{(5\cos\theta)^3}\,d\theta$$

$$= \frac{1}{25}\int \sec^2\theta\,d\theta$$

$$= \frac{1}{25}\tan\theta + C$$

$$= \frac{x}{25\sqrt{25-x^2}} + C$$

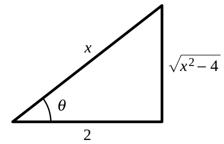



**7.** Same substitution as in Exercise 5

$$\int \frac{\sqrt{25-x^2}}{x}\,dx = \int \frac{25\cos^2\theta\,d\theta}{5\sin\theta} = 5\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 5\int(\csc\theta - \sin\theta)\,d\theta$$

$$= 5\left[\ln|\csc\theta - \cot\theta| + \cos\theta\right] + C = 5\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + C$$

**9.** Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2 - 4} = 2\tan\theta$.




$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} = \int \sec\theta\,d\theta = \ln\left|\sec\theta + \tan\theta\right| + C_1$$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2-4}\right| - \ln 2 + C_1 = \ln\left|x + \sqrt{x^2-4}\right| + C$$

**11.** Same substitution as in Exercise 9

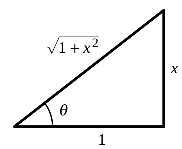
$$\int x^3\sqrt{x^2-4}\,dx = \int (8\sec^3\theta)(2\tan\theta)(2\sec\theta\tan\theta)\,d\theta = 32\int \tan^2\theta\sec^4\theta\,d\theta$$

$$= 32\int \tan^2\theta(1 + \tan^2\theta)\sec^2\theta\,d\theta = 32\left(\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right) + C$$

$$= \frac{32}{15}\tan^3\theta[5 + 3\tan^2\theta] + C = \frac{32}{15}\frac{(x^2-4)^{3/2}}{8}\left[5 + 3\frac{(x^2-4)}{4}\right] + C$$

$$= \frac{1}{15}(x^2-4)^{3/2}[20 + 3(x^2-4)] + C = \frac{1}{15}(x^2-4)^{3/2}(3x^2+8) + C$$

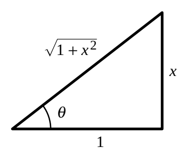
**13.** Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{1+x^2} = \sec\theta$.




$$\int x\sqrt{1+x^2}\,dx = \int \tan\theta(\sec\theta)\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C = \frac{1}{3}(1+x^2)^{3/2} + C$$

**Note:** This integral could have been evaluated with the Power Rule.

**15.** Same substitution as in Exercise 13




$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\left(\sqrt{1+x^2}\right)^4}\,dx$$

$$= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$= \int \cos^2\theta\,d\theta = \frac{1}{2}\int (1 + \cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left[\arctan x + \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C$$

$$= \frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right] + C$$

**17.** Let $u = 3x$, $a = 2$, and $du = 3\,dx$.

$$\int \sqrt{4 + 9x^2}\,dx = \frac{1}{3}\int \sqrt{(2)^2 + (3x)^2}\,3\,dx$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)\left(3x\sqrt{4+9x^2} + 4\ln\left|3x + \sqrt{4+9x^2}\right|\right) + C$$

$$= \frac{1}{2}x\sqrt{4+9x^2} + \frac{2}{3}\ln\left|3x + \sqrt{4+9x^2}\right| + C$$

**19.** $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int (x^2+9)^{-1/2}(2x)\,dx$

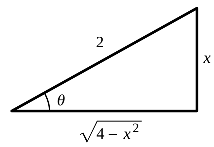
$\displaystyle = \sqrt{x^2+9} + C$

(Power Rule)

**21.** $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx = \arcsin\left(\frac{x}{4}\right) + C$

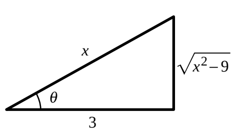
**23.** Let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$.

$$\begin{aligned}
\int \sqrt{16-4x^2}\,dx &= 2\int \sqrt{4-x^2}\,dx \\
&= 2\int 2\cos\theta(2\cos\theta\,d\theta) \\
&= 8\int \cos^2\theta\,d\theta \\
&= 4\int (1+\cos 2\theta)\,d\theta \\
&= 4\left[\theta + \frac{1}{2}\sin 2\theta\right] + C \\
&= 4\theta + 4\sin\theta\cos\theta + C \\
&= 4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2} + C
\end{aligned}$$



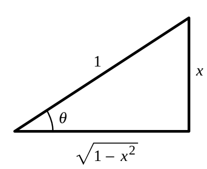

**25.** Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-9}}\,dx &= \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta} \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta + \tan\theta| + C_1 \\
&= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C_1 \\
&= \ln\left|x + \sqrt{x^2-9}\right| + C
\end{aligned}$$




**27.** Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-x^2} = \cos\theta$.

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^4}\,dx &= \int \frac{\cos\theta(\cos\theta\,d\theta)}{\sin^4\theta} \\
&= \int \cot^2\theta\csc^2\theta\,d\theta \\
&= -\frac{1}{3}\cot^3\theta + C \\
&= \frac{-(1-x^2)^{3/2}}{3x^3} + C
\end{aligned}$$

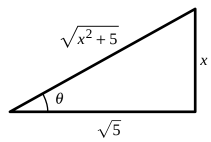



**29.** Same substitutions as in Exercise 28

$$\int \frac{1}{x\sqrt{4x^2+9}}\,dx = \int \frac{(3/2)\sec^2\theta\,d\theta}{(3/2)\tan\theta\,3\sec\theta}$$

$$= \frac{1}{3}\int \csc\theta\,d\theta = -\frac{1}{3}\ln|\csc\theta + \cot\theta| + C = -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C$$

**31.** Let $x = \sqrt{5}\tan\theta$, $dx = \sqrt{5}\sec^2\theta\,d\theta$, $x^2 + 5 = 5\sec^2\theta$.

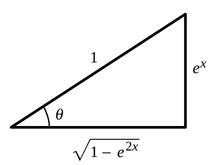



$$\begin{aligned}\int \frac{-5x}{(x^2+5)^{3/2}}\,dx &= \int \frac{-5\sqrt{5}\tan\theta}{(5\sec^2\theta)^{3/2}}\sqrt{5}\sec^2\theta\,d\theta\\ &= -\sqrt{5}\int \frac{\tan\theta}{\sec\theta}\,d\theta\\ &= -\sqrt{5}\int \sin\theta\,d\theta\\ &= \sqrt{5}\cos\theta + C\\ &= \sqrt{5}\,\frac{\sqrt{5}}{\sqrt{x^2+5}} + C\\ &= \frac{5}{\sqrt{x^2+5}} + C\end{aligned}$$

**33.** Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

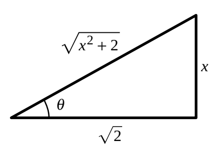
$$\int e^{2x}\sqrt{1+e^{2x}}\,dx = \frac{1}{2}\int (1+e^{2x})^{1/2}(2e^{2x})\,dx = \frac{1}{3}(1+e^{2x})^{3/2} + C$$

**35.** Let $e^x = \sin\theta$, $e^x\,dx = \cos\theta\,d\theta$, $\sqrt{1-e^{2x}} = \cos\theta$.




$$\begin{aligned}\int e^x\sqrt{1-e^{2x}}\,dx &= \int \cos^2\theta\,d\theta\\ &= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta\\ &= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]\\ &= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C = \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C\end{aligned}$$

**37.** Let $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$, $x^2 + 2 = 2\sec^2\theta$.




$$\begin{aligned}\int \frac{1}{4+4x^2+x^4}\,dx &= \int \frac{1}{(x^2+2)^2}\,dx\\ &= \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta}\\ &= \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta\\ &= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\right)\int (1+\cos 2\theta)\,d\theta\\ &= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C\\ &= \frac{\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{1}{4}\left[\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right] + C\end{aligned}$$

**39.** Since $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x, \implies du = \frac{1}{x\sqrt{4x^2 - 1}}\,dx,\ dv = dx \implies v = x$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2 - 1}}\,dx$$

$$2x = \sec\theta,\ dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2 - 1} = \tan\theta$$

$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{\tan\theta} = x \operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\,d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C = x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C.$$

**41.** $$\int \frac{1}{\sqrt{4x - x^2}}\,dx = \int \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx = \arcsin\left(\frac{x - 2}{2}\right) + C$$

**43.** Let $x + 2 = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, $\sqrt{(x + 2)^2 + 4} = 2\sec\theta$.

$$\int \frac{x}{\sqrt{x^2 + 4x + 8}}\,dx = \int \frac{x}{\sqrt{(x + 2)^2 + 4}}\,dx = \int \frac{(2\tan\theta - 2)(2\sec^2\theta)\,d\theta}{2\sec\theta}$$

$$= 2\int (\tan\theta - 1)(\sec\theta)\,d\theta$$

$$= 2\left[\sec\theta - \ln\left|\sec\theta + \tan\theta\right|\right] + C_1$$

$$= 2\left[\frac{\sqrt{(x + 2)^2 + 4}}{2} - \ln\left|\frac{\sqrt{(x + 2)^2 + 4}}{2} + \frac{x + 2}{2}\right|\right] + C_1$$

$$= \sqrt{x^2 + 4x + 8} - 2\left[\ln\left|\sqrt{x^2 + 4x + 8} + (x + 2)\right| - \ln 2\right] + C_1$$

$$= \sqrt{x^2 + 4x + 8} - 2\ln\left|\sqrt{x^2 + 4x + 8} + (x + 2)\right| + C$$

**45.** Let $t = \sin\theta$, $dt = \cos\theta\,d\theta$, $1 - t^2 = \cos^2\theta$.

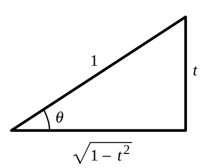
(a) $$\int \frac{t^2}{(1 - t^2)^{3/2}}\,dt = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= \frac{t}{\sqrt{1 - t^2}} - \arcsin t + C$$




Thus, $$\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt = \left[\frac{t}{\sqrt{1 - t^2}} - \arcsin t\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt = \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

**47.** (a) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{(27\tan^3\theta)(3\sec^2\theta\,d\theta)}{3\sec\theta}$$

$$= 27\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

$$= 27\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C = 9[\sec^3\theta - 3\sec\theta] + C$$

$$= 9\left[\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 3\left(\frac{\sqrt{x^2+9}}{3}\right)\right] + C = \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$$

Thus, $\displaystyle\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \left[\frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9}\right]_0^3$

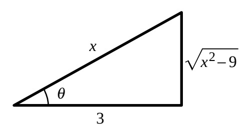
$$= \left(\frac{1}{3}\left(54\sqrt{2}\right) - 27\sqrt{2}\right) - (9 - 27)$$

$$= 18 - 9\sqrt{2} = 9\left(2 - \sqrt{2}\right) \approx 5.272.$$

(b) When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \pi/4$. Thus,

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\left[\sec^3\theta - 3\sec\theta\right]_0^{\pi/4} = 9\left(2\sqrt{2} - 3\sqrt{2}\right) - 9(1-3) = 9\left(2 - \sqrt{2}\right) \approx 5.272.$$

**49.** (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.




$$\int \frac{x^2}{\sqrt{x^2-9}}\,dx = \int \frac{9\sec^2\theta}{3\tan\theta}3\sec\theta\tan\theta\,d\theta$$

$$= 9\int \sec^3\theta\,d\theta$$

$$= 9\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta\right] \qquad \text{(7.3 Exercise 90)}$$

$$= \frac{9}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]$$

$$= \frac{9}{2}\left[\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]$$

Hence,

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \frac{9}{2}\left[\frac{x\sqrt{x^2-9}}{9} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]_4^6$$

$$= \frac{9}{2}\left[\left(\frac{6\sqrt{27}}{9} + \ln\left|2 + \frac{\sqrt{27}}{3}\right|\right) - \left(\frac{4\sqrt{7}}{9} + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right)\right]$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\left(\ln\left(\frac{6+\sqrt{27}}{3}\right) - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{\left(4-\sqrt{7}\right)\left(2+\sqrt{3}\right)}{3}\right) \approx 12.644.$$

—CONTINUED—

**49. —CONTINUED—**

(b) When $x = 4$, $\theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$.

When $x = 6$, $\theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$.

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \frac{9}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\operatorname{arcsec}(4/3)}^{\pi/3}$$

$$= \frac{9}{2}\left[2\cdot\sqrt{3} + \ln\left|2+\sqrt{3}\right|\right] - \frac{9}{2}\left[\frac{4}{3}\frac{\sqrt{7}}{3} + \ln\left|\frac{4}{3}+\frac{\sqrt{7}}{3}\right|\right]$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right) \approx 12.644$$

**51.** $\displaystyle\int \frac{x^2}{\sqrt{x^2+10x+9}}\,dx = \frac{1}{2}\sqrt{x^2+10x+9}\,(x-15) + 33\ln\left|(x+5)+\sqrt{x^2+10x+9}\right| + C$

**53.** $\displaystyle\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1} + \ln\left|x+\sqrt{x^2-1}\right|\right) + C$

**55.** (a) $u = a\sin\theta$ (b) $u = a\tan\theta$ (c) $u = a\sec\theta$

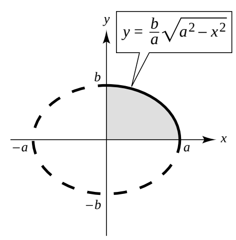
**57.** $\displaystyle A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \left[\frac{4b}{a}\left(\frac{1}{2}\right)\left(a^2\arcsin\frac{x}{a} + x\sqrt{a^2-x^2}\right)\right]_0^a$$

$$= \frac{2b}{a}\left(a^2\left(\frac{\pi}{2}\right)\right)$$

$$= \pi ab$$




**Note:** See Theorem 7.2 for $\int\sqrt{a^2-x^2}\,dx$.

**59.** $x^2 + y^2 = a^2$

$$x = \pm\sqrt{a^2-y^2}$$

$$A = 2\int_h^a \sqrt{a^2-y^2}\,dy = \left[a^2\arcsin\left(\frac{y}{a}\right) + y\sqrt{a^2-y^2}\right]_h^a \qquad \text{(Theorem 7.2)}$$

$$= \left(a^2\frac{\pi}{2}\right) - \left(a^2\arcsin\left(\frac{h}{a}\right) + h\sqrt{a^2-h^2}\right)$$

$$= \frac{a^2\pi}{2} - a^2\arcsin\left(\frac{h}{a}\right) - h\sqrt{a^2-h^2}$$

**61.** Let $x - 3 = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-(x-3)^2} = \cos\theta$.

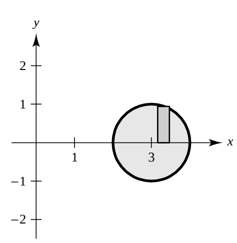
**Shell Method:**

$$V = 4\pi\int_2^4 x\sqrt{1-(x-3)^2}\,dx$$

$$= 4\pi\int_{-\pi/2}^{\pi/2}(3+\sin\theta)\cos^2\theta\,d\theta$$

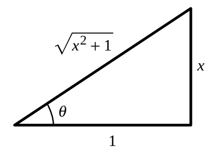
$$= 4\pi\left[\frac{3}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + \int_{-\pi/2}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\right]$$

$$= 4\pi\left[\frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2} = 6\pi^2$$



**63.** $y = \ln x, y' = \dfrac{1}{x}, 1 + (y')^2 = 1 + \dfrac{1}{x^2} = \dfrac{x^2+1}{x^2}$

Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$.




$$\begin{aligned}
s &= \int_1^5 \sqrt{\frac{x^2+1}{x^2}}\,dx = \int_1^5 \frac{\sqrt{x^2+1}}{x}\,dx\\
&= \int_a^b \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int_a^b \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta\\
&= \int_a^b (\csc\theta + \sec\theta\tan\theta)\,d\theta\\
&= \Big[-\ln|\csc\theta + \cot\theta| + \sec\theta\Big]_a^b\\
&= \left[-\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + \sqrt{x^2+1}\right]_1^5\\
&= \left[-\ln\left(\frac{\sqrt{26}+1}{5}\right) + \sqrt{26}\right] - \left[-\ln\left(\sqrt{2}+1\right) + \sqrt{2}\right]\\
&= \ln\left[\frac{5\left(\sqrt{2}+1\right)}{\sqrt{26}+1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln\left[\frac{\sqrt{26}-1}{5\left(\sqrt{2}-1\right)}\right] + \sqrt{26} - \sqrt{2}
\end{aligned}$$

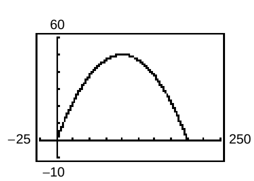
**65.** Length of one arch of sine curve: $y = \sin x, y' = \cos x$

$$L_1 = \int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$$

Length of one arch of cosine curve: $y = \cos x, y' = -\sin x$

$$\begin{aligned}
L_2 &= \int_{-\pi/2}^{\pi/2} \sqrt{1+\sin^2 x}\,dx\\
&= \int_{-\pi/2}^{\pi/2} \sqrt{1+\cos^2\left(x-\frac{\pi}{2}\right)}\,dx \qquad u = x - \frac{\pi}{2}, du = dx\\
&= \int_{-\pi}^{0} \sqrt{1+\cos^2 u}\,du\\
&= \int_0^{\pi} \sqrt{1+\cos^2 u}\,du = L_1
\end{aligned}$$

**67.** (a)




(b) $y = 0$ for $x = 200$ (range)

(c) $y = x - 0.005x^2, y' = 1 - 0.01x, 1 + (y')^2 = 1 + (1 - 0.01x)^2$

Let $u = 1 - 0.01x$, $du = -0.01\,dx$, $a = 1$. (See Theorem 7.2.)

$$\begin{aligned}
s &= \int_0^{200} \sqrt{1+(1-0.01x)^2}\,dx = -100\int_0^{200} \sqrt{(1-0.01x)^2+1}\,(-0.01)\,dx\\
&= -50\left[(1-0.01x)\sqrt{(1-0.01x)^2+1} + \ln\left|(1-0.01x) + \sqrt{(1-0.01x)^2+1}\right|\right]_0^{200}\\
&= -50\left[\left(-\sqrt{2} + \ln\left|-1+\sqrt{2}\right|\right) - \left(\sqrt{2} + \ln\left|1+\sqrt{2}\right|\right)\right]\\
&= 100\sqrt{2} + 50\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 229.559
\end{aligned}$$

**69.** Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.

$$A = 2\int_0^4 \frac{3}{\sqrt{x^2+9}}\,dx = 6\int_0^4 \frac{dx}{\sqrt{x^2+9}} = 6\int_a^b \frac{3\sec^2\theta\,d\theta}{3\sec\theta}$$

$$= 6\int_a^b \sec\theta\,d\theta = \Big[6\ln\left|\sec\theta + \tan\theta\right|\Big]_a^b = \left[6\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right|\right]_0^4 = 6\ln 3$$

$\bar{x} = 0$ (by symmetry)

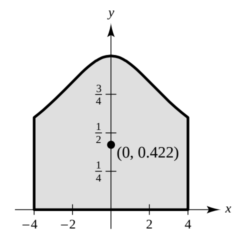
$$\bar{y} = \frac{1}{2}\left(\frac{1}{A}\right)\int_{-4}^4 \left(\frac{3}{\sqrt{x^2+9}}\right)^2 dx$$

$$= \frac{9}{12\ln 3}\int_{-4}^4 \frac{1}{x^2+9}\,dx$$

$$= \frac{3}{4\ln 3}\left[\frac{1}{3}\arctan\frac{x}{3}\right]_{-4}^4$$

$$= \frac{2}{4\ln 3}\arctan\frac{4}{3} \approx 0.422$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2\ln 3}\arctan\frac{4}{3}\right) \approx (0, 0.422)$$




**71.** $y = x^2$, $y' = 2x$, $1 + (y') = 1 + 4x^2$

$2x = \tan\theta$, $dx = \dfrac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{1+4x^2} = \sec\theta$

(For $\int\sec^5\theta\,d\theta$ and $\int\sec^3\theta\,d\theta$, see Exercise 80 in Section 7.3)

$$S = 2\pi\int_0^{\sqrt{2}} x^2\sqrt{1+4x^2}\,dx = 2\pi\int_a^b \left(\frac{\tan\theta}{2}\right)^2(\sec\theta)\left(\frac{1}{2}\sec^2\theta\right)d\theta$$

$$= \frac{\pi}{4}\int_a^b \sec^3\theta\tan^2\theta\,d\theta = \frac{\pi}{4}\left[\int_a^b \sec^5\theta\,d\theta - \int_a^b \sec^3\theta\,d\theta\right]$$

$$= \frac{\pi}{4}\left\{\frac{1}{4}\left[\sec^3\theta\tan\theta + \frac{3}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)\right] - \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)\right\}_a^b$$

$$= \frac{\pi}{4}\left[\frac{1}{4}\left[(1+4x^2)^{3/2}(2x)\right] - \frac{1}{8}\left[(1+4x^2)^{1/2}(2x) + \ln\left|\sqrt{1+4x^2}+2x\right|\right]\right]_0^{\sqrt{2}}$$

$$= \frac{\pi}{4}\left[\frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{6} = \frac{1}{8}\ln\left(3+2\sqrt{2}\right)\right]$$

$$= \frac{\pi}{4}\left(\frac{51\sqrt{2}}{4} - \frac{\ln\left(3+2\sqrt{2}\right)}{8}\right) = \frac{\pi}{32}\left[102\sqrt{2} - \ln\left(3+2\sqrt{2}\right)\right] \approx 13.989$$

**73.** (a) Area of representative rectangle: $2\sqrt{1-y^2}\,\Delta y$

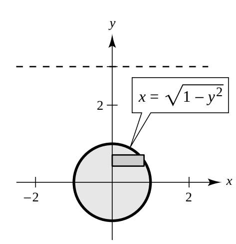
Pressure: $2(62.4)(3-y)\sqrt{1-y^2}\,\Delta y$

$$F = 124.8\int_{-1}^1 (3-y)\sqrt{1-y^2}\,dy$$

$$= 124.8\left[3+\int_{-1}^1 \sqrt{1-y^2}\,dy - \int_{-1}^1 y\sqrt{1-y^2}\,dy\right]$$

$$= 124.8\left[\frac{3}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{2}\left(\frac{2}{3}\right)(1-y^2)^{3/2}\right]_{-1}^1$$

$$= (62.4)3\left[\arcsin 1 - \arcsin(-1)\right] = 187.2\pi \text{ lb}$$

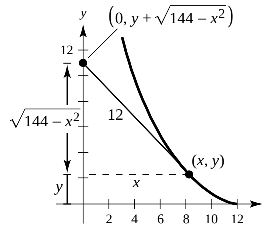



(b) $$F = 124.8\int_{-1}^1 (d-y)\sqrt{1-y^2}\,dy = 124.8d\int_{-1}^1 \sqrt{1-y^2}\,dy - 124.8\int_{-1}^1 y\sqrt{1-y^2}\,dy$$

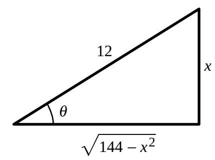
$$= 124.8\left(\frac{d}{2}\right)\left[\arcsin y + y\sqrt{1-y^2}\right]_{-1}^1 - 124.8(0) = 62.4\pi d \text{ lb}$$

**75.** (a) $m = \dfrac{dy}{dx} = \dfrac{y - \left(y + \sqrt{144 - x^2}\right)}{x - 0}$

$= -\dfrac{\sqrt{144 - x^2}}{x}$




(b) $y = -\displaystyle\int \frac{\sqrt{144 - x^2}}{x}\,dx$



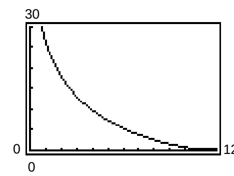

Let $x = 12 \sin\theta$, $dx = 12\cos\theta\,d\theta$, $\sqrt{144 - x^2} = 12\cos\theta$.

$$y = -\int \frac{12\cos\theta}{12\sin\theta}\,12\cos\theta\,d\theta = -12\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta$$

$$= -12\int (\csc\theta - \sin\theta)\,d\theta = -12\ln\left|\csc\theta - \cot\theta\right| - 12\cos\theta + C$$

$$= -12\ln\left|\frac{12}{x} - \frac{\sqrt{144 - x^2}}{x}\right| - 12\left(\frac{\sqrt{144 - x^2}}{12}\right) + C$$

$$= -12\ln\left|\frac{12 - \sqrt{144 - x^2}}{x}\right| - \sqrt{144 - x^2} + C$$




When $x = 12$, $y = 0 \implies C = 0$. Thus, $y = -12\ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$.

**Note:** $\dfrac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \le 12$

(c) Vertical asymptote: $x = 0$

(d) $y + \sqrt{144 - x^2} = 12 \implies y = 12 - \sqrt{144 - x^2}$

Thus,

$$12 - \sqrt{144 - x^2} = -12\ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$

$$-1 = \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right)$$

$$xe^{-1} = 12 - \sqrt{144 - x^2}$$

$$(xe^{-1} - 12)^2 = \left(-\sqrt{144 - x^2}\right)^2$$

$$x^2e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$

$$x^2(e^{-2} + 1) - 24xe^{-1} = 0$$

$$x\left[x(e^{-2} + 1) - 24e^{-1}\right] = 0$$

$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

Therefore,

$$s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x}\right)^2}\,dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + (144 - x^2)}{x^2}}\,dx$$

$$= \int_{7.77665}^{12} \frac{12}{x}\,dx = \Big[12\ln|x|\Big]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}$$

**77.** True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \int d\theta$$

**79.** False

$$\int_0^{\sqrt{3}} \frac{dx}{\left(\sqrt{1+x^2}\right)^3} = \int_0^{\pi/3} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int_0^{\pi/3} \cos\theta\,d\theta$$

**81.** Let $u = a\sin\theta$, $du = a\cos\theta\,d\theta$, $\sqrt{a^2-u^2} = a\cos\theta$.

$$\begin{aligned}\int \sqrt{a^2-u^2}\,du &= \int a^2\cos^2\theta\,d\theta = a^2\int \frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C\\ &= \frac{a^2}{2}\left[\arcsin\frac{u}{a} + \left(\frac{u}{a}\right)\left(\frac{\sqrt{a^2+u^2}}{a}\right)\right] + C = \frac{1}{2}\left[a^2\arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right] + C\end{aligned}$$

Let $u = a\sec\theta$, $du = a\sec\theta\tan\theta\,d\theta$, $\sqrt{u^2-a^2} = a\tan\theta$.

$$\begin{aligned}\int \sqrt{u^2-a^2}\,du &= \int a\tan\theta(a\sec\theta\tan\theta)\,d\theta = a^2\int \tan^2\theta\sec\theta\,d\theta\\ &= a^2\int(\sec^2\theta - 1)\sec\theta\,d\theta = a^2\int(\sec^3\theta - \sec\theta)\,d\theta\\ &= a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta\right] - a^2\int\sec\theta\,d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|\right]\\ &= \frac{a^2}{2}\left[\frac{u}{a}\cdot\frac{\sqrt{u^2-a^2}}{a} - \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right|\right] + C_1\\ &= \frac{1}{2}\left[u\sqrt{u^2-a^2} - a^2\ln\left|u + \sqrt{u^2-a^2}\right|\right] + C\end{aligned}$$

Let $u = a\tan\theta$, $du = a\sec^2\theta\,d\theta$, $\sqrt{u^2+a^2} = a\sec\theta\,d\theta$.

$$\begin{aligned}\int \sqrt{u^2+a^2}\,du &= \int(a\sec\theta)(a\sec^2\theta)\,d\theta\\ &= a^2\int\sec^3\theta\,d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|\right] + C_1\\ &= \frac{a^2}{2}\left[\frac{\sqrt{u^2+a^2}}{a}\cdot\frac{u}{a} + \ln\left|\frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a}\right|\right] + C_1 = \frac{1}{2}\left[u\sqrt{u^2+a^2} + a^2\ln\left|u+\sqrt{u^2+a^2}\right|\right] + C\end{aligned}$$

## Section 7.5 Partial Fractions

**1.** $\dfrac{5}{x^2-10x} = \dfrac{5}{x(x-10)} = \dfrac{A}{x} + \dfrac{B}{x-10}$

**3.** $\dfrac{2x-3}{x^3+10x} = \dfrac{2x-3}{x(x^2+10)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+10}$

**5.** $\dfrac{16x}{x^3-10x^2} = \dfrac{16x}{x^2(x-10)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-10}$

**7.** $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$

$$1 = A(x-1) + B(x+1)$$

When $x = -1$, $1 = -2A$, $A = -\frac{1}{2}$.
When $x = 1$, $1 = 2B$, $B = \frac{1}{2}$.

$$\begin{aligned}\int\frac{1}{x^2-1}\,dx &= -\frac{1}{2}\int\frac{1}{x+1}\,dx + \frac{1}{2}\int\frac{1}{x-1}\,dx\\ &= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C\\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C\end{aligned}$$

**9.** $\dfrac{3}{x^2+x-2} = \dfrac{3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

$$3 = (x+2) + B(x-1)$$

When $x = 1$, $3 = 3A$, $A = 1$.
When $x = -2$, $3 = -3B$, $B = -1$.

$$\int \frac{3}{x^2+x-2}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+2}\,dx$$
$$= \ln|x-1| - \ln|x+2| + C$$
$$= \ln\left|\frac{x-1}{x+2}\right| + C$$

**11.** $\dfrac{5-x}{2x^2+x-1} = \dfrac{5-x}{(2x-1)(x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{x+1}$

$$5 - x = A(x+1) + B(2x-1)$$

When $x = \frac{1}{2}$, $\frac{9}{2} = \frac{3}{2}A$, $A = 3$.

When $x = -1$, $6 = -3B$, $B = -2$.

$$\int \frac{5-x}{2x^2+x-1}\,dx = 3\int \frac{1}{2x-1}\,dx - 2\int \frac{1}{x+1}\,dx$$
$$= \frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$$

**13.** $\dfrac{x^2+12x+12}{x(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

$$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

When $x = 0$, $12 = -4A$, $A = -3$. When $x = -2$, $-8 = 8B$, $B = -1$. When $x = 2$, $40 = 8C$, $C = 5$.

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = 5\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx - 3\int \frac{1}{x}\,dx$$
$$= 5\ln|x-2| - \ln|x+2| - 3\ln|x| + C$$

**15.** $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \dfrac{x+5}{(x-4)(x+2)} = 2x + \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$$x + 5 = A(x+2) + B(x-4)$$

When $x = 4$, $9 = 6A$, $A = \frac{3}{2}$. When $x = -2$, $3 = -6B$, $B = -\frac{1}{2}$.

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx = \int \left[2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\right]dx$$
$$= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$$

**17.** $\dfrac{4x^2+2x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$$

When $x = 0$, $B = -1$. When $x = -1$, $C = 1$. When $x = 1$, $A = 3$.

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int \left[\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right]dx = 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$
$$= \frac{1}{x} + \ln|x^4 + x^3| + C$$

**19.** $\dfrac{x^2+3x-4}{x^3-4x^2+4x} = \dfrac{x^2+3x-4}{x(x-2)^2} = \dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

$$x^2 + 3x - 4 = A(x-2)^2 + Bx(x-2) + Cx$$

When $x = 0$, $-4 = -4A \implies A = -1$. When $x = 2$, $6 = 2C \implies C = 3$. When $x = 1$, $0 = -1 - B + 3 \implies B = 2$.

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{2}{(x-2)}\,dx + \int \frac{3}{(x-2)^2}\,dx$$
$$= -\ln|x| + 2\ln|x-2| - \frac{3}{(x-2)} + C$$

**21.** $\dfrac{x^2-1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$$x^2 - 1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = -1$. When $x = 1$, $0 = -2 + B + C$. When $x = -1$, $0 = -2 + B + C$. Solving these equations we have $A = -1$, $B = 2$, $C = 0$.

$$\begin{aligned}\int \frac{x^2-1}{x^3+x}\,dx &= -\int \frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx\\ &= \ln|x^2+1| - \ln|x| + C\\ &= \ln\left|\frac{x^2+1}{x}\right| + C\end{aligned}$$

**23.** $\dfrac{x^2}{x^4-2x^2-8} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+2}$

$$x^2 = A(x+2)(x^2+2) + B(x-2)(x^2+2) + (Cx+D)(x+2)(x-2)$$

When $x = 2$, $4 = 24A$. When $x = -2$, $4 = -24B$. When $x = 0$, $0 = 4A - 4B - 4D$, and when $x = 1$, $1 = 9A - 3B - 3C - 3D$. Solving these equations we have $A = \frac{1}{6}$, $B = -\frac{1}{6}$, $C = 0$, $D = \frac{1}{3}$.

$$\begin{aligned}\int \frac{x^2}{x^4-2x^2-8}\,dx &= \frac{1}{6}\left[\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx + 2\int \frac{1}{x^2+2}\,dx\right]\\ &= \frac{1}{6}\left[\ln\left|\frac{x-2}{x+2}\right| + \sqrt{2}\arctan\frac{x}{\sqrt{2}}\right] + C\end{aligned}$$

**25.** $\dfrac{x}{(2x-1)(2x+1)(4x^2+1)} = \dfrac{A}{2x-1} + \dfrac{B}{2x+1} + \dfrac{Cx+D}{4x^2+1}$

$$x = A(2x+1)(4x^2+1) + B(2x-1)(4x^2+1) + (Cx+D)(2x-1)(2x+1)$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 4A$. When $x = -\frac{1}{2}$, $-\frac{1}{2} = -4B$. When $x = 0$, $0 = A - B - D$, and when $x = 1$, $1 = 15A + 5B + 3C + 3D$. Solving these equations we have $A = \frac{1}{8}$, $B = \frac{1}{8}$, $C = -\frac{1}{2}$, $D = 0$.

$$\begin{aligned}\int \frac{x}{16x^4-1}\,dx &= \frac{1}{8}\left[\int \frac{1}{2x-1}\,dx + \int \frac{1}{2x+1}\,dx - 4\int \frac{x}{4x^2+1}\,dx\right]\\ &= \frac{1}{16}\ln\left|\frac{4x^2-1}{4x^2+1}\right| + C\end{aligned}$$

**27.** $\dfrac{x^2+5}{(x+1)(x^2-2x+3)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-2x+3}$

$$\begin{aligned}x^2+5 &= A(x^2-2x+3) + (Bx+C)(x+1)\\ &= (A+B)x^2 + (-2A+B+C)x + (3A+C)\end{aligned}$$

When $x = -1$, $A = 1$. By equating coefficients of like terms, we have $A + B = 1$, $-2A + B + C = 0$, $3A + C = 5$. Solving these equations we have $A = 1$, $B = 0$, $C = 2$.

$$\begin{aligned}\int \frac{x^2+5}{x^3-x^2+x+3}\,dx &= \int \frac{1}{x+1}\,dx + 2\int \frac{1}{(x-1)^2+2}\,dx\\ &= \ln|x+1| + \sqrt{2}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C\end{aligned}$$

**29.** $\dfrac{3}{(2x+1)(x+2)} = \dfrac{A}{2x+1} + \dfrac{B}{x+2}$

$$3 = A(x+2) + B(2x+1)$$

When $x = -\frac{1}{2}$, $A = 2$. When $x = -2$, $B = -1$.

$$\int_0^1 \frac{3}{2x^2+5x+2}\,dx = \int_0^1 \frac{2}{2x+1}\,dx - \int_0^1 \frac{1}{x+2}\,dx$$
$$= \Big[\ln|2x-1| - \ln|x+2|\Big]_0^1$$
$$= \ln 2$$

**31.** $\dfrac{x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

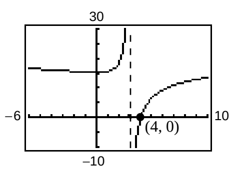
$$x + 1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = 1$. When $x = 1$, $2 = 2A + B + C$. When $x = -1$, $0 = 2A + B - C$. Solving these equations we have $A = 1$, $B = -1$, $C = 1$.

$$\int_1^2 \frac{x+1}{x(x^2+1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$$
$$= \left[\ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan x\right]_1^2$$
$$= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2$$
$$\approx 0.557$$

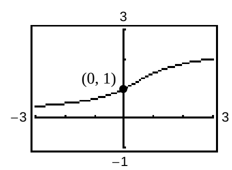
**33.** $\displaystyle\int \frac{3x\,dx}{x^2-6x+9} = 3\ln|x-3| - \frac{9}{x-3} + C$

$(4, 0)$: $\quad 3\ln|4-3| - \dfrac{9}{4-3} + C = 0 \Longrightarrow C = 9$



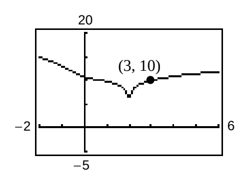

**35.** $\displaystyle\int \frac{x^2+x+2}{(x^2+2)^2}\,dx = \frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}} - \frac{1}{2(x^2+2)} + C$

$(0, 1)$: $\;0 - \dfrac{1}{4} + C = 1 \Longrightarrow C = \dfrac{5}{4}$



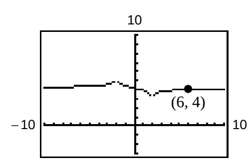

**37.** $\displaystyle\int \frac{2x^2-2x+3}{x^3-x^2-x-2}\,dx = \ln|x-2| + \frac{1}{2}\ln|x^2+x+1| - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$(3, 10)$: $\;0 + \dfrac{1}{2}\ln 13 - \sqrt{3}\arctan\dfrac{7}{\sqrt{3}} + C = 10 \Longrightarrow C = 10 - \dfrac{1}{2}\ln 13 + \sqrt{3}\arctan\dfrac{7}{\sqrt{3}}$




**39.** $\displaystyle\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

$(6, 4)$: $\;\dfrac{1}{4}\ln\left|\dfrac{4}{8}\right| + C = 4 \Longrightarrow C = 4 - \dfrac{1}{4}\ln\dfrac{1}{2} = 4 + \dfrac{1}{4}\ln 2$



**41.** Let $u = \cos x$ $du = \sin x\,dx$.

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

When $u = 0$, $A = -1$. When $u = 1$, $B = 1$, $u = \cos x$, $du = -\sin x\,dx$.

$$\begin{aligned}\int \frac{\sin x}{\cos x(\cos x - 1)}\,dx &= -\int \frac{1}{u(u-1)}\,du \\ &= \int \frac{1}{u}\,du - \int \frac{1}{u-1}\,du \\ &= \ln|u| - \ln|u-1| + C \\ &= \ln\left|\frac{u}{u-1}\right| + C \\ &= \ln\left|\frac{\cos x}{\cos x - 1}\right| + C\end{aligned}$$

**43.** $$\begin{aligned}\int \frac{3\cos x}{\sin^2 x + \sin x - 2}\,dx &= 3\int \frac{1}{u^2 + u - 2}\,du \\ &= \ln\left|\frac{u-1}{u+2}\right| + C \\ &= \ln\left|\frac{-1 + \sin x}{2 + \sin x}\right| + C\end{aligned}$$

(From Exercise 9 with $u = \sin x$, $du = \cos x\,dx$)

**45.** Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \frac{1}{5}$. When $u = -4$, $B = -\frac{1}{5}$, $u = e^x$, $du = e^x\,dx$.

$$\begin{aligned}\int \frac{e^x}{(e^x - 1)(e^x + 4)}\,dx &= \int \frac{1}{(u-1)(u+4)}\,du \\ &= \frac{1}{5}\left(\int \frac{1}{u-1}\,du - \int \frac{1}{u+4}\,du\right) \\ &= \frac{1}{5}\ln\left|\frac{u-1}{u+4}\right| + C \\ &= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C\end{aligned}$$

**47.** $$\frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$$

$$1 = A(a+bx) + Bx$$

When $x = 0$, $1 = aA \Longrightarrow A = 1/a$.
When $x = -a/b$, $1 = -(a/b)B \Longrightarrow B = -b/a$.

$$\begin{aligned}\int \frac{1}{x(a+bx)}\,dx &= \frac{1}{a}\int\left(\frac{1}{x} - \frac{b}{a+bx}\right)dx \\ &= \frac{1}{a}\big(\ln|x| - \ln|a+bx|\big) + C \\ &= \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C\end{aligned}$$

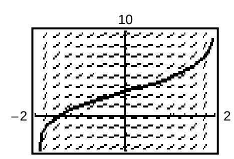
**49.** $$\frac{x}{(a+bx)^2} = \frac{A}{a+bx} + \frac{B}{(a+bx)^2}$$

$$x = A(a+bx) + B$$

When $x = -a/b$, $B = -a/b$.
When $x = 0$, $0 = aA + B \implies A = 1/b$.

$$\begin{aligned}\int \frac{x}{(a+bx)^2}\,dx &= \int\left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2}\right)dx \\ &= \frac{1}{b}\int \frac{1}{a+bx}\,dx - \frac{a}{b}\int \frac{1}{(a+bx)^2}\,dx \\ &= \frac{1}{b^2}\ln|a+bx| + \frac{a}{b^2}\left(\frac{1}{a+bx}\right) + C \\ &= \frac{1}{b^2}\left(\frac{a}{a+bx} + \ln|a+bx|\right) + C\end{aligned}$$

**51.** $\dfrac{dy}{dx} = \dfrac{6}{4 - x^2}$, $y(0) = 3$



**53.** Dividing $x^3$ by $x - 5$.

**55.** (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

(c) Trigonometric substitution (tan) or inverse tangent rule

**57.** Average Cost $= \dfrac{1}{80 - 75}\displaystyle\int_{75}^{80} \frac{124p}{(10 + p)(100 - p)}\, dp$

$$= \frac{1}{5}\int_{75}^{80}\left(\frac{-124}{(10 + p)11} + \frac{1240}{(100 - p)11}\right) dp$$

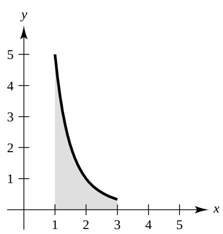
$$= \frac{1}{5}\left[\frac{-124}{11}\ln(10 + p) - \frac{1240}{11}\ln(100 - p)\right]_{75}^{80}$$

$$\approx \frac{1}{5}(24.51) = 4.9$$

Approximately \$490,000.

**59.** $A = \displaystyle\int_1^3 \frac{10}{x(x^2 + 1)}\, dx \approx 3$

Matches (c)




**61.** $$\frac{1}{(x + 1)(n - x)} = \frac{A}{x + 1} + \frac{B}{n - x}, A = B = \frac{1}{n + 1}$$

$$\frac{1}{n + 1}\int\left(\frac{1}{x + 1} + \frac{1}{n - x}\right) dx = kt + C$$

$$\frac{1}{n + 1}\ln\left|\frac{x + 1}{n - x}\right| = kt + C$$

When $t = 0$, $x = 0$, $C = \dfrac{1}{n + 1}\ln\dfrac{1}{n}$.

$$\frac{1}{n + 1}\ln\left|\frac{x + 1}{n - x}\right| = kt + \frac{1}{n + 1}\ln\frac{1}{n}$$

$$\frac{1}{n + 1}\left[\ln\left|\frac{x + 1}{n - x}\right| - \ln\frac{1}{n}\right] = kt$$

$$\ln\frac{nx + n}{n - x} = (n + 1)kt$$

$$\frac{nx + n}{n - x} = e^{(n + 1)kt}$$

$$x = \frac{n[e^{(n + 1)kt} - 1]}{n + e^{(n + 1)kt}}$$

**Note:** $\displaystyle\lim_{t \to \infty} x = n$

**63.** $\dfrac{x}{1+x^4} = \dfrac{Ax+B}{x^2+\sqrt{2}\,x+1} + \dfrac{Cx+D}{x^2-\sqrt{2}\,x+1}$

$$\begin{aligned} x &= (Ax+B)\left(x^2-\sqrt{2}\,x+1\right) + (Cx+D)\left(x^2+\sqrt{2}\,x+1\right) \\ &= (A+C)x^3 + \left(B+D-\sqrt{2}\,A+\sqrt{2}\,C\right)x^2 + \left(A+C-\sqrt{2}\,B+\sqrt{2}\,D\right)x + (B+D) \end{aligned}$$

$0 = A + C \implies C = -A$

$$\left.\begin{aligned} 0 &= B + D - \sqrt{2}\,A + \sqrt{2}\,C \\ 1 &= A + C - \sqrt{2}\,B + \sqrt{2}\,D \end{aligned}\right\} \begin{aligned} &-2\sqrt{2}\,A = 0 \implies A = 0 \text{ and } C = 0 \\ &-2\sqrt{2}\,B = 1 \implies B = -\frac{\sqrt{2}}{4} \text{ and } D = \frac{\sqrt{2}}{4} \end{aligned}$$

$0 = B + D \implies D = -B$

Thus,

$$\begin{aligned} \int_0^1 \frac{x}{1+x^4}\,dx &= \int_0^1 \left[\frac{-\sqrt{2}/4}{x^2+\sqrt{2}\,x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}\,x+1}\right]dx \\ &= \frac{\sqrt{2}}{4}\int_0^1 \left[\frac{-1}{\left[x+\left(\sqrt{2}/2\right)\right]^2+(1/2)} + \frac{1}{\left[x-\left(\sqrt{2}/2\right)\right]^2+(1/2)}\right]dx \\ &= \frac{\sqrt{2}}{4}\cdot\frac{1}{1/\sqrt{2}}\left[-\arctan\left(\frac{x+\left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x-\left(\sqrt{2}/2\right)}{1/\sqrt{2}}\right)\right]_0^1 \\ &= \frac{1}{2}\left[-\arctan\left(\sqrt{2}\,x+1\right) + \arctan\left(\sqrt{2}\,x-1\right)\right]_0^1 \\ &= \frac{1}{2}\left[\left(-\arctan\left(\sqrt{2}+1\right) + \arctan\left(\sqrt{2}-1\right)\right) - \left(-\arctan 1 + \arctan(-1)\right)\right] \\ &= \frac{1}{2}\left[\arctan\left(\sqrt{2}-1\right) - \arctan\left(\sqrt{2}+1\right) + \frac{\pi}{4} + \frac{\pi}{4}\right]. \end{aligned}$$

Since $\arctan x - \arctan y = \arctan[(x-y)/(1+xy)]$, we have:

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\left[\arctan\left(\frac{\left(\sqrt{2}-1\right)-\left(\sqrt{2}+1\right)}{1+\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[-\frac{\pi}{4} + \frac{\pi}{2}\right] = \frac{\pi}{8}$$

## Section 7.6 Integration by Tables and Other Integration Techniques

**1.** By Formula 6: $\displaystyle\int \frac{x^2}{1+x}\,dx = -\frac{x}{2}(2-x) + \ln|1+x| + C$

**3.** By Formula 26: $\displaystyle\int e^x\sqrt{1+e^{2x}}\,dx = \frac{1}{2}\left[e^x\sqrt{e^{2x}+1} + \ln\left|e^x+\sqrt{e^{2x}+1}\right|\right] + C$

$u = e^x,\ du = e^x\,dx$

**5.** By Formula 44: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = -\frac{\sqrt{1-x^2}}{x} + C$

**7.** By Formulas 50 and 48:
$$\int \sin^4(2x)\,dx = \frac{1}{2}\int \sin^4(2x)(2)\,dx$$
$$= \frac{1}{2}\left[\frac{-\sin^3(2x)\cos(2x)}{4} + \frac{3}{4}\int \sin^2(2x)(2)\,dx\right]$$
$$= \frac{1}{2}\left[\frac{-\sin^3(2x)\cos(2x)}{4} + \frac{3}{8}(2x - \sin 2x\cos 2x)\right] + C$$
$$= \frac{1}{16}(6x - 3\sin 2x\cos 2x - 2\sin^3 2x\cos 2x) + C$$

**9.** By Formula 57:
$$\int \frac{1}{\sqrt{x}\left(1 - \cos\sqrt{x}\right)}\,dx = 2\int \frac{1}{1 - \cos\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx$$
$$= -2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$$
$$u = \sqrt{x},\; du = \frac{1}{2\sqrt{x}}\,dx$$

**11.** By Formula 84:
$$\int \frac{1}{1 + e^{2x}}\,dx = x - \frac{1}{2}\ln(1 + e^{2x}) + C$$

**13.** By Formula 89:
$$\int x^3 \ln x\,dx = \frac{x^4}{16}\left(4\ln|x| - 1\right) + C$$

**15.** (a) By Formulas 83 and 82:
$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$$
$$= x^2e^x - 2[(x - 1)e^x + C_1]$$
$$= x^2e^x - 2xe^x + 2e^x + C$$

(b) Integration by parts: $u = x^2, du = 2x\,dx, dv = e^x\,dx, v = e^x$
$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$$
Parts again: $u = 2x, du = 2\,dx, dv = e^x\,dx, v = e^x$
$$\int x^2e^x\,dx = x^2e^x - \left[2xe^x - \int 2e^x\,dx\right] = x^2e^x - 2xe^x + 2e^x + C$$

**17.** (a) By Formula: 12, $a = b = 1, u = x$, and
$$\int \frac{1}{x^2(x + 1)}\,dx = \frac{-1}{1}\left(\frac{1}{x} + \frac{1}{1}\ln\left|\frac{x}{1 + x}\right|\right) + C$$
$$= \frac{-1}{x} - \ln\left|\frac{x}{1 + x}\right| + C$$
$$= \frac{-1}{x} + \ln\left|\frac{x + 1}{x}\right| + C$$

(b) Partial fractions:
$$\frac{1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$$
$$1 = Ax(x + 1) + B(x + 1) + Cx^2$$
$x = 0$: $1 = B$

$x = -1$: $1 = C$

$x = 1$: $1 = 2A + 2 + 1 \Longrightarrow A = -1$
$$\int \frac{1}{x^2(x + 1)}\,dx = \int\left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x + 1}\right]dx$$
$$= -\ln|x| - \frac{1}{x} + \ln|x + 1| + C$$
$$= -\frac{1}{x} - \ln\left|\frac{x}{x + 1}\right| + C$$

**19.** By Formula 81: $\displaystyle\int xe^{x^2} = \frac{1}{2}e^{x^2} + C$

**21.** By Formula 79: $\displaystyle\int x \operatorname{arcsec}(x^2 + 1)\,dx = \frac{1}{2}\int \operatorname{arcsec}(x^2 + 1)(2x)\,dx$

$$= \frac{1}{2}\Big[(x^2 + 1)\operatorname{arcsec}(x^2 + 1) - \ln\Big((x^2 + 1) + \sqrt{x^4 + 2x^2}\Big)\Big] + C$$

$u = x^2 + 1,\ du = 2x\,dx$

**23.** By Formula 89: $\displaystyle\int x^2 \ln x\,dx = \frac{x^3}{9}\big(-1 + 3\ln|x|\big) + C$

**25.** By Formula 35: $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - 4}}\,dx = \frac{\sqrt{x^2 - 4}}{4x} + C$

**27.** By Formula 4: $\displaystyle\int \frac{2x}{(1 - 3x)^2}\,dx = 2\int \frac{x}{(1 - 3x)^2}\,dx = \frac{2}{9}\left(\ln|1 - 3x| + \frac{1}{1 - 3x}\right) + C$

**29.** By Formula 76:

$$\int e^x \arccos e^x\,dx = e^x \arccos e^x - \sqrt{1 - e^{2x}} + C$$

$u = e^x,\ du = e^x\,dx$

**31.** By Formula 73:

$$\int \frac{x}{1 - \sec x^2}\,dx = \frac{1}{2}\int \frac{2x}{1 - \sec x^2}\,dx$$

$$= \frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$$

**33.** By Formula 23: $\displaystyle\int \frac{\cos x}{1 + \sin^2 x}\,dx = \arctan(\sin x) + C$

$u = \sin x,\ du = \cos x\,dx$

**35.** By Formula 14: $\displaystyle\int \frac{\cos\theta}{3 + 2\sin\theta + \sin^2\theta}\,d\theta = \frac{\sqrt{2}}{2}\arctan\left(\frac{1 + \sin\theta}{\sqrt{2}}\right) + C$

$u = \sin\theta,\ du = \cos\theta\,d\theta$

**37.** By Formula 35: $\displaystyle\int \frac{1}{x^2\sqrt{2 + 9x^2}}\,dx = 3\int \frac{3}{(3x)^2\sqrt{\left(\sqrt{2}\right)^2 + (3x)^2}}\,dx$

$$= -\frac{3\sqrt{2 + 9x^2}}{6x} + C$$

$$= -\frac{\sqrt{2 + 9x^2}}{2x} + C$$

**39.** By Formulas 54 and 55:

$$\int t^3 \cos t\,dt = t^3 \sin t - 3\int t^2 \sin t\,dt$$

$$= t^3 \sin t - 3\left[-t^2 \cos t + 2\int t \cos t\,dt\right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6\left[t \sin t - \int \sin t\,dt\right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6\cos t + C$$

**41.** By Formula 3: $\displaystyle\int \frac{\ln x}{x(3+2\ln x)}\,dx = \frac{1}{4}\left(2\ln|x| - 3\ln|3+2\ln|x||\right) + C$

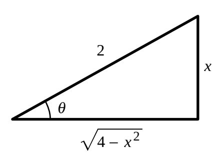
$u = \ln x,\ du = \dfrac{1}{x}\,dx$

**43.** By Formulas 1, 25, and 33: 
$$\int \frac{x}{(x^2-6x+10)^2}\,dx = \frac{1}{2}\int \frac{2x-6+6}{(x^2-6x+10)^2}\,dx$$
$$= \frac{1}{2}\int (x^2-6x+10)^{-2}(2x-6)\,dx + 3\int \frac{1}{[(x-3)^2+1]^2}\,dx$$
$$= -\frac{1}{2(x^2-6x+10)} + \frac{3}{2}\left[\frac{x-3}{x^2-6x+10} + \arctan(x-3)\right] + C$$
$$= \frac{3x-10}{2(x^2-6x+10)} + \frac{3}{2}\arctan(x-3) + C$$

**45.** By Formula 31: 
$$\int \frac{x}{\sqrt{x^4-6x^2+5}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{(x^2-3)^2-4}}\,dx$$
$$= \frac{1}{2}\ln\left|x^2-3+\sqrt{x^4-6x^2+5}\right| + C$$

$u = x^2-3,\ du = 2x\,dx$

**47.** 
$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int \frac{8\sin^3\theta(2\cos\theta\,d\theta)}{2\cos\theta}$$
$$= 8\int (1-\cos^2\theta)\sin\theta\,d\theta$$
$$= 8\int [\sin\theta - \cos^2\theta(\sin\theta)]\,d\theta$$
$$= -8\cos\theta + \frac{8\cos^3\theta}{3} + C$$
$$= \frac{-\sqrt{4-x^2}}{3}(x^2+8) + C$$




$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

**49.** By Formula 8: 
$$\int \frac{e^{3x}}{(1+e^x)^3}\,dx = \int \frac{(e^x)^2}{(1+e^x)^3}(e^x)\,dx$$
$$= \frac{2}{1+e^x} - \frac{1}{2(1+e^x)^2} + \ln|1+e^x| + C$$

$u = e^x,\ du = e^x\,dx$

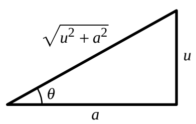
**51.** 
$$\frac{u^2}{(a+bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a+bu)^2} = \frac{1}{b^2} + \frac{A}{a+bu} + \frac{B}{(a+bu)^2}$$
$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a+bu) + B = (aA+B) + bAu$$

Equating the coefficients of like terms we have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations we have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^2}\int du - \frac{2a}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{a+bu}b\,du + \frac{a^2}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{(a+bu)^2}b\,du = \frac{1}{b^2}u - \frac{2a}{b^3}\ln|a+bu| - \frac{a^2}{b^3}\left(\frac{1}{a+bu}\right) + C$$
$$= \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

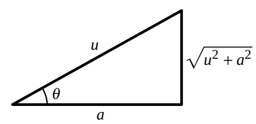
**53.** When we have $u^2 + a^2$:

$$u = a \tan\theta$$
$$du = a\sec^2\theta\,d\theta$$
$$u^2 + a^2 = a^2\sec^2\theta$$
$$\int \frac{1}{(u^2+a^2)^{3/2}}\,du = \int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta}$$
$$= \frac{1}{a^2}\int \cos\theta\,d\theta$$
$$= \frac{1}{a^2}\sin\theta + C$$
$$= \frac{u}{a^2\sqrt{u^2+a^2}} + C$$




When we have $u^2 - a^2$:

$$u = a\sec\theta$$
$$du = a\sec\theta\tan\theta\,d\theta$$
$$u^2 - a^2 = a^2\tan^2\theta$$
$$\int \frac{1}{(u^2-a^2)^{3/2}}\,du = \int \frac{a\sec\theta\tan\theta\,d\theta}{a^3\tan^3\theta}$$
$$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$
$$= -\frac{1}{a^2}\csc\theta + C$$
$$= \frac{-u}{a^2\sqrt{u^2-a^2}} + C$$



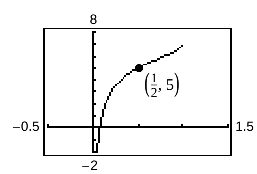

**55.** $$\int (\arctan u)\,du = u\arctan u - \frac{1}{2}\int \frac{2u}{1+u^2}\,du$$
$$= u\arctan u - \frac{1}{2}\ln(1+u^2) + C$$
$$= u\arctan u - \ln\sqrt{1+u^2} + C$$

$w = \arctan u,\ dv = du,\ dw = \dfrac{du}{1+u^2},\ v = u$

**57.** $$\int \frac{1}{x^{3/2}\sqrt{1-x}}\,dx = \frac{-2\sqrt{1-x}}{\sqrt{x}} + C$$

$\left(\dfrac{1}{2}, 5\right)$: $\dfrac{-2\sqrt{1/2}}{\sqrt{1/2}} + C = 5 \Rightarrow C = 7$

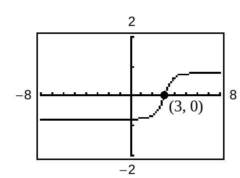
$$y = \frac{-2\sqrt{1-x}}{\sqrt{x}} + 7$$




**59.** $$\int \frac{1}{(x^2-6x+10)^2}\,dx = \frac{1}{2}\left[\tan^{-1}(x-3) + \frac{x-3}{x^2-6x+10}\right] + C$$

$(3, 0)$: $\dfrac{1}{2}\left[0 + \dfrac{0}{10}\right] + C = 0 \Rightarrow C = 0$

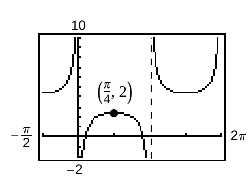
$$y = \frac{1}{2}\left[\tan^{-1}(x-3) + \frac{x-3}{x^2-6x+10}\right]$$




**61.** $$\int \frac{1}{\sin\theta\tan\theta}\,d\theta = -\csc\theta + C$$

$\left(\dfrac{\pi}{4}, 2\right)$: $-\dfrac{2}{\sqrt{2}} + C = 2 \Rightarrow C = 2 + \sqrt{2}$

$$y = -\csc\theta + 2 + \sqrt{2}$$



**63.** $$\int \frac{1}{2 - 3\sin\theta}\,d\theta = \int \left[\frac{\dfrac{2\,du}{1+u^2}}{2 - 3\left(\dfrac{2u}{1+u^2}\right)}\right]$$
$$= \int 2\frac{2}{(1+u^2) - 6u}\,du$$
$$= \int \frac{1}{u^2 - 3u + 1}\,du$$
$$= \int \frac{1}{\left(u - \frac{3}{2}\right)^2 - \frac{5}{4}}\,du$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\left(u - \frac{3}{2}\right) - \frac{\sqrt{5}}{2}}{\left(u - \frac{3}{2}\right) + \frac{\sqrt{5}}{2}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}}\right| + C$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2\tan\left(\frac{\theta}{2}\right) - 3 - \sqrt{5}}{2\tan\left(\frac{\theta}{2}\right) - 3 + \sqrt{5}}\right| + C$$

$u = \tan\dfrac{\theta}{2}$

**65.** $$\int_0^{\pi/2} \frac{1}{1 + \sin\theta + \cos\theta}\,d\theta = \int_0^1 \left[\frac{\dfrac{2\,du}{1+u^2}}{1 + \dfrac{2u}{1+u^2} + \dfrac{1-u^2}{1+u^2}}\right]$$
$$= \int_0^1 \frac{1}{1+u}\,du$$
$$= \Big[\ln|1+u|\Big]_0^1$$
$$= \ln 2$$

$u = \tan\dfrac{\theta}{2}$

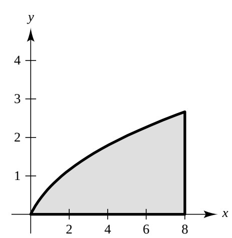
**67.** $$\int \frac{\sin\theta}{3 - 2\cos\theta}\,d\theta = \frac{1}{2}\int \frac{2\sin\theta}{3 - 2\cos\theta}\,d\theta$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln(3 - 2\cos\theta) + C$$

$u = 3 - 2\cos\theta,\ du = 2\sin\theta\,d\theta$

**69.** $$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = 2\int \cos\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$$
$$= 2\sin\sqrt{\theta} + C$$

$u = \sqrt{\theta},\ du = \dfrac{1}{2\sqrt{\theta}}\,d\theta$

**71.** $$A = \int_0^8 \frac{x}{\sqrt{x+1}}\,dx$$
$$= \left[\frac{-2(2-x)}{3}\sqrt{x+1}\right]_0^8$$
$$= 12 - \left(-\frac{4}{3}\right)$$
$$= \frac{40}{3} \approx 13.333 \text{ square units}$$




**73.** Arctangent Formula, Formula 23,

$$\int \frac{1}{u^2+1}\,du,\ u = e^x$$

**75.** Substitution: $u = x^2,\ du = 2x\,dx$

Then Formula 81.

**77.** Cannot be integrated.

**79.** Answers will vary. For example,

$$\int (2x)e^{2x}\,dx$$

can be integrated by first letting $u = 2x$ and then using Formula 82.

**81.** 
$$\begin{aligned} W &= \int_0^5 2000xe^{-x}\,dx \\ &= -2000\int_0^5 -xe^{-x}\,dx \\ &= 2000\int_0^5 (-x)e^{-x}(-1)\,dx \\ &= 2000\Big[(-x)e^{-x} - e^{-x}\Big]_0^5 \\ &= 2000\left(-\frac{6}{e^5} + 1\right) \\ &\approx 1919.145 \text{ ft} \cdot \text{lbs} \end{aligned}$$

**83.** (a) 
$$\begin{aligned} V &= 20(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy \\ &= \Big[80 \ln\left|y + \sqrt{1+y^2}\right|\Big]_0^3 \\ &= 80\ln\left(3+\sqrt{10}\right) \\ &\approx 145.5 \text{ cubic feet} \end{aligned}$$

$$\begin{aligned} W &= 148\left(80\ln\left(3+\sqrt{10}\right)\right) \\ &= 11{,}840\ln\left(3+\sqrt{10}\right) \\ &\approx 21{,}530.4 \text{ lb} \end{aligned}$$

(b) By symmetry, $\bar{x} = 0$.

$$M = \rho(2)\int_0^3 \frac{2}{\sqrt{1+y^2}}\,dy = \Big[4\rho\ln\left|y+\sqrt{1+y^2}\right|\Big]_0^3 = 4\rho\ln\left(3+\sqrt{10}\right)$$

$$M_x = 2\rho\int_0^3 \frac{2y}{\sqrt{1+y^2}}\,dy = \Big[4\rho\sqrt{1+y^2}\Big]_0^3 = 4\rho\left(\sqrt{10}-1\right)$$

$$\bar{y} = \frac{M_x}{M} = \frac{4\rho\left(\sqrt{10}-1\right)}{4\rho\ln\left(3+\sqrt{10}\right)} \approx 1.19$$

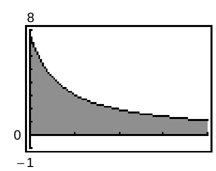
Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

**85.** (a) $\displaystyle\int_0^4 \frac{k}{2+3x}\,dx = 10$

$$k = \frac{10}{\displaystyle\int_0^4 \frac{1}{2+3x}\,dx} \approx \frac{10}{0.6486}$$

$$= 15.417 \quad \left(= \frac{30}{\ln 7}\right)$$

(b) $\displaystyle\int_0^4 \frac{15.417}{2+3x}\,dx$

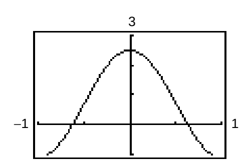



**87.** False. You might need to convert your integral using substitution or algebra.

## Section 7.7 Indeterminate Forms and L'Hôpital's Rule

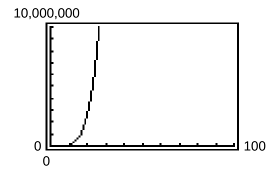
**1.** $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{\sin 2x} \approx 2.5\left(\text{exact: } \frac{5}{2}\right)$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 2.4132 | 2.4991 | 2.500 | 2.500 | 2.4991 | 2.4132 |



**3.** $\lim_{x \to \infty} x^5 e^{-x/100} \approx 0$

| $x$ | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.9901 | 90,484 | $3.7 \times 10^9$ | $4.5 \times 10^{10}$ | 0 | 0 |




**5.** (a) $\lim_{x \to 3} \frac{2(x-3)}{x^2-9} = \lim_{x \to 3} \frac{2(x-3)}{(x+3)(x-3)} = \lim_{x \to 3} \frac{2}{x+3} = \frac{1}{3}$

(b) $\lim_{x \to 3} \frac{2(x-3)}{x^2-9} = \lim_{x \to 3} \frac{(d/dx)[2(x-3)]}{(d/dx)[x^2-9]} = \lim_{x \to 3} \frac{2}{2x} = \frac{2}{6} = \frac{1}{3}$

**7.** (a) $\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x \to 3} \frac{(x+1)-4}{(x-3)\left[\sqrt{x+1}+2\right]} = \lim_{x \to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$

(b) $\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \to 3} \frac{(d/dx)\left[\sqrt{x+1}-2\right]}{(d/dx)[x-3]} = \lim_{x \to 3} \frac{1/\left(2\sqrt{x+1}\right)}{1} = \frac{1}{4}$

**9.** (a) $\lim_{x \to \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \to \infty} \frac{5-(3/x)+(1/x^2)}{3-(5/x^2)} = \frac{5}{3}$

(b) $\lim_{x \to \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \to \infty} \frac{(d/dx)[5x^2-3x+1]}{(d/dx)[3x^2-5]} = \lim_{x \to \infty} \frac{10x-3}{6x} = \lim_{x \to \infty} \frac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x \to \infty} \frac{10}{6} = \frac{5}{3}$

**11.** $\lim_{x \to 2} \frac{x^2-x-2}{x-2} = \lim_{x \to 2} \frac{2x-1}{1} = 3$

**13.** $\lim_{x \to 0} \frac{\sqrt{4-x^2}-2}{x} = \lim_{x \to 0} \frac{-x/\sqrt{4-x^2}}{1} = 0$

**15.** $\lim_{x \to 0} \frac{e^x-(1-x)}{x} = \lim_{x \to 0} \frac{e^x+1}{1} = 2$

**17.** Case 1: $n = 1$

$$\lim_{x \to 0^+} \frac{e^x-(1+x)}{x} = \lim_{x \to 0^+} \frac{e^x-1}{1} = 0$$

Case 2: $n = 2$

$$\lim_{x \to 0^+} \frac{e^x-(1+x)}{x^2} = \lim_{x \to 0^+} \frac{e^x-1}{2x} = \lim_{x \to 0^+} \frac{e^x}{2} = \frac{1}{2}$$

Case 3: $n \geq 3$

$$\lim_{x \to 0^+} \frac{e^x-(1+x)}{x^n} = \lim_{x \to 0^+} \frac{e^x-1}{nx^{n-1}} = \lim_{x \to 0^+} \frac{e^x}{n(n-1)x^{n-2}} = \infty$$

**19.** $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \to 0} \frac{2\cos 2x}{3\cos 3x} = \frac{2}{3}$

**21.** $\lim_{x \to 0} \frac{\arcsin x}{x} = \lim_{x \to 0} \frac{1/\sqrt{1-x^2}}{1} = 1$

**23.** $\lim_{x \to \infty} \frac{3x^2-2x+1}{2x^2+3} = \lim_{x \to \infty} \frac{6x-2}{4x}$

$= \lim_{x \to \infty} \frac{6}{4} = \frac{3}{2}$

**25.** $\lim_{x \to \infty} \frac{x^2+2x+3}{x-1} = \lim_{x \to \infty} \frac{2x+2}{1} = \infty$

**27.** $\lim_{x \to \infty} \frac{x^3}{e^{x/2}} = \lim_{x \to \infty} \frac{3x^2}{(1/2)e^{x/2}}$

$= \lim_{x \to \infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x \to \infty} \frac{6}{(1/8)e^{x/2}} = 0$

**29.** $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1+(1/x^2)}} = 1$

**Note:** L'Hôpital's Rule does not work on this limit. See Exercise 79.

**31.** $\lim_{x\to\infty} \dfrac{\cos x}{x} = 0$ by Squeeze Theorem

$\left(\dfrac{\cos x}{x} \le \dfrac{1}{x}\right)$

**33.** $\lim_{x\to\infty} \dfrac{\ln x}{x^2} = \lim_{x\to\infty} \dfrac{1/x}{2x} = \lim_{x\to\infty} \dfrac{1}{2x^2} = 0$

**35.** $\lim_{x\to\infty} \dfrac{e^x}{x^2} = \lim_{x\to\infty} \dfrac{e^x}{2x} = \lim_{x\to\infty} \dfrac{e^x}{2} = \infty$

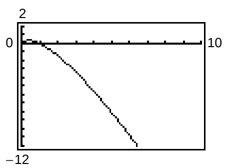
**37.** (a) $\lim_{x\to 0^+} (-x\ln x) = (-0)(-\infty) = (0)(\infty)$

(b) $\lim_{x\to 0^+} (-x\ln x) = \lim_{x\to 0^+} \dfrac{\ln x}{-1/x}$

$= \lim_{x\to 0^+} \dfrac{1/x}{1/x^2}$

$= \lim_{x\to 0^+} x = 0$

(c)



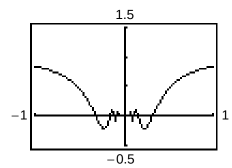

**39.** (a) $\lim_{x\to\infty} \left(x \sin\dfrac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x\sin\dfrac{1}{x} = \lim_{x\to\infty} \dfrac{\sin(1/x)}{1/x}$

$= \lim_{x\to\infty} \dfrac{(-1/x^2)\cos(1/x)}{-1/x^2}$

$= \lim_{x\to\infty} \cos\left(\dfrac{1}{x}\right) = 1$

(c)




**41.** (a) $\lim_{x\to 0^+} x^{1/x} = 0^\infty = 0$, not indeterminant (See Exercise 95)

(b) Let $y = x^{1/x}$

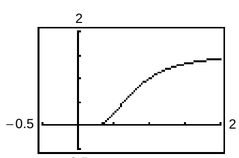
$\ln y = \ln x^{1/x} = \dfrac{1}{x}\ln x.$

Since $x \to 0^+$, $\dfrac{1}{x}\ln x \to (\infty)(-\infty) = -\infty$. Hence,

$\ln y \to -\infty \implies y \to 0^+.$

Therefore, $\lim_{x\to 0^+} x^{1/x} = 0.$

(c)




**43.** (a) $\lim_{x\to\infty} x^{1/x} = \infty^0$

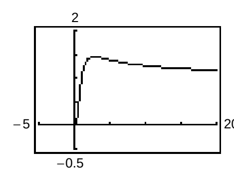
(b) Let $y = \lim_{x\to\infty} x^{1/x}$.

$\ln y = \lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \left(\dfrac{1/x}{1}\right) = 0$

Thus, $\ln y = 0 \implies y = e^0 = 1$. Therefore,

$\lim_{x\to\infty} x^{1/x} = 1.$

(c)




**45.** (a) $\lim_{x\to 0^+} (1+x)^{1/x} = 1^\infty$

(b) Let $y = \lim_{x\to 0^+} (1+x)^{1/x}$.

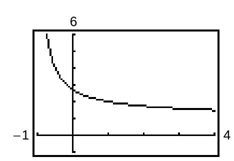
$\ln y = \lim_{x\to 0^+} \dfrac{\ln(1+x)}{x}$

$= \lim_{x\to 0^+} \left(\dfrac{1/(1+x)}{1}\right) = 1$

Thus, $\ln y = 1 \implies y = e^1 = e.$

Therefore, $\lim_{x\to 0^+} (1+x)^{1/x} = e.$

(c)



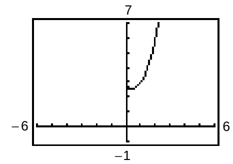
**47.** (a) $\lim_{x\to 0^+} [3(x)^{x/2}] = 0^0$

(b) Let $y = \lim_{x\to 0^+} 3(x)^{x/2}$.

$$\ln y = \lim_{x\to 0^+}\left[\ln 3 + \frac{x}{2}\ln x\right]$$
$$= \lim_{x\to 0^+}\left[\ln 3 + \frac{\ln x}{2/x}\right]$$
$$= \lim_{x\to 0^+} \ln 3 + \lim_{x\to 0^+} \frac{1/x}{-2/x^2}$$
$$= \lim_{x\to 0^+} \ln 3 - \lim_{x\to 0^+} \frac{x}{2}$$
$$= \ln 3$$

Hence, $\lim_{x\to 0^+} 3(x)^{x/2} = 3$.

(c)



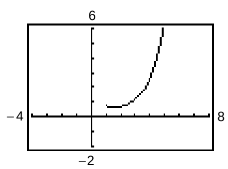

**49.** (a) $\lim_{x\to 1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = \lim_{x1^+} (\ln x)^{x-1}$

$$= \lim_{x\to 1^+} (x-1)\ln x = 0$$

Hence, $\lim_{x\to 1^+} (\ln x)^{x-1} = 1$

(c)

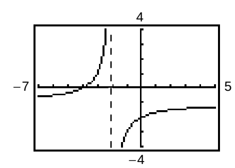



**51.** (a) $\lim_{x\to 2^+}\left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \infty - \infty$

(b) $\lim_{x\to 2^+}\left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \lim_{x\to 2^+}\frac{8-x(x+2)}{x^2-4}$

$$= \lim_{x\to 2^+}\frac{(2-x)(4+x)}{(x+2)(x-2)}$$
$$= \lim_{x\to 2^+}\frac{-(x+4)}{x+2} = \frac{-3}{2}$$

(c)

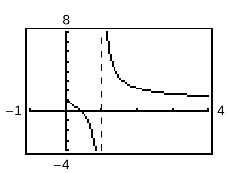



**53.** (a) $\lim_{x\to 1^+}\left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \infty - \infty$

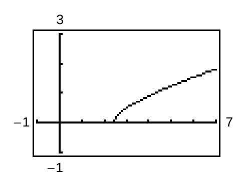
(b) $\lim_{x\to 1^+}\left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to 1^+}\frac{3x-3-2\ln x}{(x-1)\ln x}$

$$= \lim_{x\to 1^+}\frac{3-(2/x)}{[(x-1)/x]+\ln x} = \infty$$

(c)



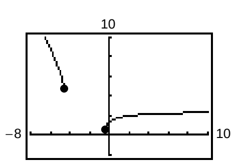

**55.** (a)




(b) $\lim_{x\to 3}\frac{x-3}{\ln(2x-5)} = \lim_{x\to 3}\frac{1}{2/(2x-5)}$

$$= \lim_{x\to 3}\frac{2x-5}{2} = \frac{1}{2}$$

**57.** (a)




(b) $\lim_{x\to\infty}\left(\sqrt{x^2+5x+2} - x\right) = \lim_{x\to\infty}\left(\sqrt{x^2+5x+2} - x\right)\frac{\left(\sqrt{x^2+5x+2} + x\right)}{\left(\sqrt{x^2+5x+2} + x\right)}$

$$= \lim_{x\to\infty}\frac{(x^2+5x+2)-x^2}{\sqrt{x^2+5x+2}+x}$$
$$= \lim_{x\to\infty}\frac{5x+2}{\sqrt{x^2+5x+2}+x}$$
$$= \lim_{x\to\infty}\frac{5+(2/x)}{\sqrt{1+(5/x)+(2/x^2)}+1} = \frac{5}{2}$$

**59.** $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^{\infty}, 0^0, \infty - \infty$

**61.** (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x - 5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x - 5)^3$.

**63.** $\lim_{x\to\infty} \frac{x^2}{e^{5x}} = \lim_{x\to\infty} \frac{2x}{5e^{5x}} = \lim_{x\to\infty} \frac{2}{25e^{5x}} = 0$

**65.**
$$\lim_{x\to\infty} \frac{(\ln x)^3}{x} = \lim_{x\to\infty} \frac{3(\ln x)^2(1/x)}{1}$$
$$= \lim_{x\to\infty} \frac{3(\ln x)^2}{x}$$
$$= \lim_{x\to\infty} \frac{6(\ln x)(1/x)}{1}$$
$$= \lim_{x\to\infty} \frac{6(\ln x)}{x} = \lim_{x\to\infty} \frac{6}{x} = 0$$

**67.**
$$\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}$$
$$= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m}$$
$$= \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m}$$
$$= \cdots = \lim_{x\to\infty} \frac{n!}{m^n x^m} = 0$$

**69.**

| $x$ | 10 | $10^2$ | $10^4$ | $10^6$ | $10^8$ | $10^{10}$ |
|---|---|---|---|---|---|---|
| $\frac{(\ln x)^4}{x}$ | 2.811 | 4.498 | 0.720 | 0.036 | 0.001 | 0.000 |

**71.** $y = x^{1/x}, x > 0$

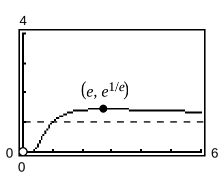
Horizontal asymptote: $y = 1$ (See Exercise 37)

$$\ln y = \frac{1}{x} \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right)$$
$$\frac{dy}{dx} = x^{1/x}\left(\frac{1}{x^2}\right)(1 - \ln x) = x^{(1/x)-2}(1 - \ln x) = 0$$

Critical number: $x = e$

| | | |
|---|---|---|
| Intervals: | $(0, e)$ | $(e, \infty)$ |
| Sign of $dy/dx$: | $+$ | $-$ |
| $y = f(x)$: | Increasing | Decreasing |

Relative maximum: $(e, e^{1/e})$




**73.** $y = 2xe^{-x}$

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

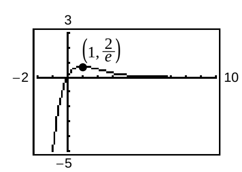
Horizontal asymptote: $y = 0$

$$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x}$$
$$= 2e^{-x}(1 - x) = 0$$

Critical number: $x = 1$

| | | |
|---|---|---|
| Intervals: | $(-\infty, 1)$ | $(1, \infty)$ |
| Sign of $dy/dx$: | $+$ | $-$ |
| $y = f(x)$: | Increasing | Decreasing |

Relative maximum: $\left(1, \frac{2}{e}\right)$




**75.** $\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or $\infty/\infty$.
L'Hôpital's Rule does not apply.

**77.** $\lim_{x\to\infty} x \cos \frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or $\infty/\infty$.
L'Hôpital's Rule does not apply.

**79.** (a) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/x}$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}/\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+(1/x^2)}}$$

$$= \frac{1}{\sqrt{1+0}} = 1$$

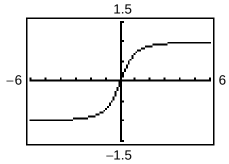
(b) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{x/\sqrt{x^2+1}}$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \frac{x/\sqrt{x^2+1}}{1}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

Applying L'Hôpital's rule twice results in the original limit, so L'Hôpital's rule fails.

(c)




**81.** $\displaystyle\lim_{k\to 0} \frac{32\left(1 - e^{-kt} + \dfrac{v_0 k e^{-kt}}{32}\right)}{k} = \lim_{k\to 0} \frac{32(1-e^{-kt})}{k} + \lim_{k\to 0} (v_0 e^{-kt})$

$$= \lim_{k\to 0} \frac{32(0 + te^{-kt})}{1} + \lim_{k\to 0} \left(\frac{v_0}{e^{kt}}\right) = 32t + v_0$$

**83.** Area of triangle: $\frac{1}{2}(2x)(1-\cos x) = x - x\cos x$

Shaded area: Area of rectangle − Area under curve

$$2x(1-\cos x) - 2\int_0^x (1-\cos t)\,dt = 2x(1-\cos x) - 2\Big[t - \sin t\Big]_0^x$$

$$= 2x(1-\cos x) - 2(x - \sin x) = 2\sin x - 2x\cos x$$

Ratio: $\displaystyle\lim_{x\to 0} \frac{x - x\cos x}{2\sin x - 2x\cos x} = \lim_{x\to 0} \frac{1 + x\sin x - \cos x}{2\cos x + 2x\sin x - 2\cos x}$

$$= \lim_{x\to 0} \frac{1 + x\sin x - \cos x}{2x\sin x}$$

$$= \lim_{x\to 0} \frac{x\cos x + \sin x + \sin x}{2x\cos x + 2\sin x}$$

$$= \lim_{x\to 0} \frac{x\cos x + 2\sin x}{2x\cos x + 2\sin x} \cdot \frac{1/\cos x}{1/\cos x}$$

$$= \lim_{x\to 0} \frac{x + 2\tan x}{2x + 2\tan x}$$

$$= \lim_{x\to 0} \frac{1 + 2\sec^2 x}{2 + 2\sec^2 x} = \frac{3}{4}$$

**85.** $f(x) = x^3,\ g(x) = x^2 + 1,\ [0, 1]$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{3c^2}{2c}$$

$$\frac{1}{1} = \frac{3c}{2}$$

$$c = \frac{2}{3}$$

**87.** $f(x) = \sin x,\ g(x) = \cos x,\ \left[0, \frac{\pi}{2}\right]$

$$\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{1}{-1} = \frac{\cos c}{-\sin c}$$

$$-1 = -\cot c$$

$$c = \frac{\pi}{4}$$

**89.** False. L'Hôpital's Rule does not apply since

$$\lim_{x\to 0}(x^2 + x + 1) \neq 0.$$

$$\lim_{x\to 0}\frac{x^2 + x + 1}{x} = \lim_{x\to 0}\left(x + 1 + \frac{1}{x}\right) = 1 + \infty = \infty$$

**91.** True

**93.** (a) $\sin\theta = BD$

$\cos\theta = DO \implies AD = 1 - \cos\theta$

$$\text{Area } \triangle ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos\theta)\sin\theta = \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta\cos\theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \triangle OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos\theta)(\sin\theta) = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$$

(c) $$R = \frac{(1/2)\sin\theta - (1/2)\sin\theta\cos\theta}{(1/2)\theta - (1/2)\sin\theta\cos\theta} = \frac{\sin\theta - \sin\theta\cos\theta}{\theta - \sin\theta\cos\theta}$$

(d) $$\lim_{\theta\to 0} R = \lim_{\theta\to 0}\frac{\sin\theta - (1/2)\sin 2\theta}{\theta - (1/2)\sin 2\theta}$$

$$= \lim_{\theta\to 0}\frac{\cos\theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta\to 0}\frac{-\sin\theta + 2\sin 2\theta}{2\sin 2\theta} = \lim_{\theta\to 0}\frac{-\cos\theta + 4\cos 2\theta}{4\cos 2\theta} = \frac{3}{4}$$

**95.** $\lim_{x\to a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x)\ln f(x)$$

$$\lim_{x\to a} g(x)\ln f(x) = (\infty)(-\infty) = -\infty$$

As $x \to a$, $\ln y \implies -\infty$, and hence $y = 0$. Thus,

$$\lim_{x\to a} f(x)^{g(x)} = 0.$$

**97.** $$f'(a)(b - a) - \int_a^b f''(t)(t - b)\,dt = f'(a)(b - a) - \left\{\Big[f'(t)(t - b)\Big]_a^b - \int_a^b f'(t)\,dt\right\}$$

$$= f'(a)(b - a) + f'(a)(a - b) + \Big[f(t)\Big]_a^b = f(b) - f(a)$$

$$dv = f''(t)dt \implies v = f'(t)$$

$$u = t - b \implies du = dt$$

## Section 7.8 Improper Integrals

**1.** Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^4 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{b\to 0^+}\Big[2\sqrt{x}\Big]_b^4$$

$$= \lim_{b\to 0^+}\left(4 - 2\sqrt{b}\right) = 4$$

Converges

**3.** Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b\to1^-}\int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{c\to1^+}\int_c^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b\to1^-}\left[-\frac{1}{x-1}\right]_0^b + \lim_{c\to1^+}\left[-\frac{1}{x-1}\right]_c^2 = (\infty - 1) + (-1 + \infty)$$

Diverges

**5.** Infinite limit of integration.

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 0 + 1 = 1$$

Converges

**7.** $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx \neq -2$

because the integrand is not defined at $x = 0$.
Diverges

**9.** $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1$$

**11.** $$\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx = \lim_{b\to\infty}\int_1^b 3x^{-1/3}\,dx$$

$$= \lim_{b\to\infty}\left[\frac{9}{2}x^{2/3}\right]_1^b = \infty$$

Diverges

**13.** $$\int_{-\infty}^0 xe^{-2x}\,dx = \lim_{b\to-\infty}\int_b^0 xe^{-2x}\,dx = \lim_{b\to-\infty}\frac{1}{4}\Big[(-2x-1)e^{-2x}\Big]_b^0 = \lim_{b\to-\infty}\frac{1}{4}\left[-1 + (2b+1)e^{-2b}\right] = -\infty \quad \text{(Integration by parts)}$$

Diverges

**15.** $$\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}(x^2+2x+2)\right]_0^b = \lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b} + 2\right) = 2$$

Since $\displaystyle\lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b}\right) = 0$ by L'Hôpital's Rule.

**17.** $$\int_0^\infty e^{-x}\cos x\,dx = \lim_{b\to\infty}\frac{1}{2}\Big[e^{-x}(-\cos x + \sin x)\Big]_0^b$$

$$= \frac{1}{2}[0 - (-1)] = \frac{1}{2}$$

**19.** $$\int_4^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_4^b (\ln x)^{-3}\frac{1}{x}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}(\ln x)^{-2}\right]_4^b$$

$$= -\frac{1}{2}(\ln b)^{-2} + \frac{1}{2}(\ln 4)^{-2}$$

$$= \frac{1}{2}\,\frac{1}{(2\ln 2)^2} = \frac{1}{8(\ln 2)^2}$$

**21.** $$\int_{-\infty}^\infty \frac{2}{4+x^2}\,dx = \int_{-\infty}^0 \frac{2}{4+x^2}\,dx + \int_0^\infty \frac{2}{4+x^2}\,dx$$

$$= \lim_{b\to-\infty}\int_b^0 \frac{2}{4+x^2}\,dx + \lim_{c\to\infty}\int_0^c \frac{2}{4+x^2}\,dx$$

$$= \lim_{b\to-\infty}\left[\arctan\left(\frac{x}{2}\right)\right]_b^0 + \lim_{c\to\infty}\left[\arctan\left(\frac{x}{2}\right)\right]_0^c$$

$$= \left(0 - \left(-\frac{\pi}{2}\right)\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$

**23.** $\displaystyle\int_0^{\infty} \frac{1}{e^x + e^{-x}}\,dx = \lim_{b\to\infty} \int_0^b \frac{e^x}{1 + e^{2x}}\,dx$

$$= \lim_{b\to\infty} \Big[\arctan(e^x)\Big]_0^b$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

**25.** $\displaystyle\int_0^{\infty} \cos \pi x\,dx = \lim_{b\to\infty}\left[\frac{1}{\pi}\sin \pi x\right]_0^b$

Diverges since $\sin \pi x$ does not approach a limit as $x \to \infty$.

**27.** $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+}\left[\frac{-1}{x}\right]_b^1 = -1 + \infty$

Diverges

**29.** $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{8 - x}}\,dx = \lim_{b\to 8^-}\int_0^b \frac{1}{\sqrt[3]{8 - x}}\,dx = \lim_{b\to 8^-}\left[\frac{-3}{2}(8 - x)^{2/3}\right]_0^b = 6$

**31.** $\displaystyle\int_0^1 x \ln x\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right]_b^1 = \lim_{b\to 0^+}\left[\frac{-1}{4} - \frac{b^2 \ln b}{2} + \frac{b^2}{4}\right] = \frac{-1}{4}$ since $\displaystyle\lim_{b\to 0^+}(b^2 \ln b) = 0$ by L'Hôpital's Rule.

**33.** $\displaystyle\int_0^{\pi/2} \tan \theta\,d\theta = \lim_{b\to(\pi/2)^-}\Big[\ln|\sec \theta|\Big]_0^b = \infty$

Diverges

**35.** $\displaystyle\int_2^4 \frac{2}{x\sqrt{x^2 - 4}}\,dx = \lim_{b\to 2^+}\int_b^4 \frac{2}{x\sqrt{x^2 - 4}}\,dx$

$$= \lim_{b\to 2^+}\left[\operatorname{arcsec}\left|\frac{x}{2}\right|\right]_b^4$$

$$= \lim_{b\to 2^+}\left(\operatorname{arcsec} 2 - \operatorname{arcsec}\left(\frac{b}{2}\right)\right)$$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

**37.** $\displaystyle\int_2^4 \frac{1}{\sqrt{x^2 - 4}} = \lim_{b\to 2^+}\Big[\ln\big|x + \sqrt{x^2 - 4}\big|\Big]_b^4$

$$= \ln\big(4 + 2\sqrt{3}\big) - \ln 2$$

$$= \ln\big(2 + \sqrt{3}\big) \approx 1.317$$

**39.** $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x - 1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x - 1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x - 1}}\,dx$

$$= \lim_{b\to 1^-}\left[\frac{3}{2}(x - 1)^{2/3}\right]_0^b + \lim_{c\to 1^+}\left[\frac{3}{2}(x - 1)^{2/3}\right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0$$

**41.** $\displaystyle\int_0^{\infty} \frac{4}{\sqrt{x}(x + 6)}\,dx = \int_0^1 \frac{4}{\sqrt{x}(x + 6)}\,dx + \int_1^{\infty} \frac{4}{\sqrt{x}(x + 6)}\,dx$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\int \frac{4}{\sqrt{x}(x + 6)}\,dx = \int \frac{4(2u\,du)}{u(u^2 + 6)} = 8\int \frac{du}{u^2 + 6} = \frac{8}{\sqrt{6}}\arctan\left(\frac{4}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

Thus, $\displaystyle\int_0^{\infty} \frac{4}{\sqrt{x}(x + 6)}\,dx = \lim_{b\to 0^+}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_b^1 + \lim_{c\to\infty}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_1^c$

$$= \left(\frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}}0\right) + \left(\frac{8}{\sqrt{6}}\frac{\pi}{2} - \frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right)\right)$$

$$= \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.$$

**43.** If $p = 1$, $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\Big]_1^b$.

Diverges. For $p \neq 1$,

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty}\left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right].$$

This converges to $\dfrac{1}{p-1}$ if $1 - p < 0$ or $p > 1$.

**45.** For $n = 1$ we have

$$\begin{aligned}
\int_0^{\infty} xe^{-x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\,dx \\
&= \lim_{b\to\infty}\Big[-e^{-x}x - e^{-x}\Big]_0^b \qquad \text{(Parts: } u = x,\ dv = e^{-x}\,dx) \\
&= \lim_{b\to\infty}\left[-e^{-b}b - e^{-b} + 1\right] \\
&= \lim_{b\to\infty}\left[\frac{-b}{e^b} - \frac{1}{e^b} + 1\right] = 1 \qquad \text{(L'Hôpital's Rule)}
\end{aligned}$$

Assume that $\displaystyle\int_0^{\infty} x^n e^{-x}\,dx$ converges. Then for $n + 1$ we have

$$\int x^{n+1}e^{-x}\,dx = -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\,dx$$

by parts ($u = x^{n+1}$, $du = (n+1)x^n\,dx$, $dv = e^{-x}\,dx$, $v = -e^{-x}$).

Thus,

$$\int_0^{\infty} x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\Big[-x^{n+1}e^{-x}\Big]_0^b + (n+1)\int_0^{\infty} x^n e^{-x}\,dx = 0 + (n+1)\int_0^{\infty} x^n e^{-x}\,dx, \text{ which converges.}$$

**47.** $\displaystyle\int_0^1 \frac{1}{x^3}\,dx$ diverges.

(See Exercise 44, $p = 3 \not< 1$.)

**49.** $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx = \frac{1}{3-1} = \frac{1}{2}$ converges.

(See Exercise 43, $p = 3$.)

**51.** Since $\dfrac{1}{x^2+5} \leq \dfrac{1}{x^2}$ on $[1, \infty)$ and $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges by Exercise 43, $\displaystyle\int_1^{\infty} \frac{1}{x^2+5}\,dx$ converges.

**53.** Since $\dfrac{1}{\sqrt[3]{x(x-1)}} \geq \dfrac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and $\displaystyle\int_2^{\infty} \frac{1}{\sqrt[3]{x^2}}\,dx$ diverges by Exercise 43, $\displaystyle\int_2^{\infty} \frac{1}{\sqrt[3]{x(x-1)}}\,dx$ diverges.

**55.** Since $e^{-x^2} \leq e^{-x}$ on $[1, \infty)$ and $\displaystyle\int_0^{\infty} e^{-x}\,dx$ converges (see Exercise 5), $\displaystyle\int_0^{\infty} e^{-x^2}\,dx$ converges.

**57.** Answers will vary. See pages 540, 543.

**59.** $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$

These two integrals diverge by Exercise 44.

**61.** $f(t) = 1$

$$F(s) = \int_0^{\infty} e^{-st}\,dx = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \frac{1}{s},\ s > 0$$

**63.** $f(t) = t^2$

$$\begin{aligned}
F(s) &= \int_0^{\infty} t^2 e^{-st}\,dx = \lim_{b\to\infty}\left[\frac{1}{s^3}(-s^2t^2 - 2st - 2)e^{-st}\right]_0^b \\
&= \frac{2}{s^3},\ s > 0
\end{aligned}$$

**65.** $f(t) = \cos at$

$$F(s) = \int_0^\infty e^{-st} \cos at \, dt$$
$$= \lim_{b\to\infty} \left[ \frac{e^{-st}}{s^2 + a^2}(-s \cos at + a \sin at) \right]_0^b$$
$$= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2}, \; s > 0$$

**67.** $f(t) = \cosh at$

$$F(s) = \int_0^\infty e^{-st} \cosh at \, dt = \int_0^\infty e^{-st}\left(\frac{e^{at} + e^{-at}}{2}\right) dt = \frac{1}{2}\int_0^\infty \left[ e^{t(-s+a)} + e^{t(-s-a)} \right] dt$$
$$= \lim_{b\to\infty} \frac{1}{2}\left[ \frac{1}{(-s+a)} e^{t(-s+a)} + \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2}\left[ \frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right]$$
$$= \frac{-1}{2}\left[ \frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] = \frac{s}{s^2 - a^2}, \; s > |a|$$

**69.** (a) $A = \displaystyle\int_0^\infty e^{-x} \, dx$

$$= \lim_{b\to\infty} \left[ -e^{-x} \right]_0^b = 0 - (-1) = 1$$

(b) **Disk:**

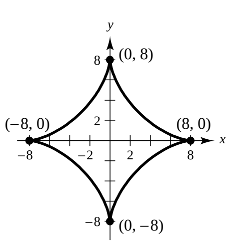
$$V = \pi \int_0^\infty (e^{-x})^2 \, dx$$
$$= \lim_{b\to\infty} \pi \left[ -\frac{1}{2} e^{-2x} \right]_0^b = \frac{\pi}{2}$$

(c) **Shell:**

$$V = 2\pi \int_0^\infty x e^{-x} \, dx$$
$$= \lim_{b\to\infty} \left\{ 2\pi \left[ -e^{-x}(x+1) \right]_0^b \right\} = 2\pi$$

**71.**

$$x^{2/3} + y^{2/3} = 4$$
$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$
$$y' = \frac{-y^{1/3}}{x^{1/3}}$$
$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}$$
$$s = 4\int_0^8 \frac{2}{x^{1/3}} \, dx = \lim_{b\to 0^+} \left[ 8 \cdot \frac{3}{2} x^{2/3} \right]_b^8 = 48$$



**73.** $\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\,dx$

(a) $\Gamma(1) = \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = 1$

$\Gamma(2) = \int_0^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}(x+1)\Big]_0^b = 1$

$\Gamma(3) = \int_0^{\infty} x^2e^{-x}\,dx = \lim_{b\to\infty}\Big[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big]_0^b = 2$

(b) $\Gamma(n+1) = \int_0^{\infty} x^ne^{-x}\,dx = \lim_{b\to\infty}\Big[-x^ne^{-x}\Big]_0^b + \lim_{b\to\infty} n\int_0^b x^{n-1}e^{-x}\,dx = 0 + n\Gamma(n) \qquad (u = x^n,\, dv = e^{-x}\,dx)$

(c) $\Gamma(n) = (n-1)!$

**75.** (a) $\int_{-\infty}^{\infty} \frac{1}{7}e^{-t/7}\,dt = \int_0^{\infty}\frac{1}{7}e^{-t/7}\,dt = \lim_{b\to\infty}\Big[-e^{-t/7}\Big]_0^b = 1$

(b) $\int_0^4 \frac{1}{7}e^{-t/7}\,dt = \Big[-e^{-t/7}\Big]_0^4 = -e^{-4/7} + 1$

$\approx 0.4353 = 43.53\%$

(c) $\int_0^{\infty} t\left[\frac{1}{7}e^{-t/7}\right]dt = \lim_{b\to\infty}\Big[-te^{-t/7} - 7e^{-t/7}\Big]_0^b$

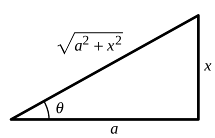
$= 0 + 7 = 7$

**77.** (a) $C = 650{,}000 + \int_0^5 25{,}000\,e^{-0.06t}\,dt = 650{,}000 - \left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^5 \approx \$757{,}992.41$

(b) $C = 650{,}000 + \int_0^{10} 25{,}000e^{-0.06t}\,dt \approx \$837{,}995.15$

(c) $C = 650{,}000 + \int_0^{\infty} 25{,}000e^{-0.06t}\,dt = 650{,}000 - \lim_{b\to\infty}\left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^b \approx \$1{,}066{,}666.67$

**79.** Let $x = a\tan\theta,\ dx = a\sec^2\theta\,d\theta,\ \sqrt{a^2+x^2} = a\sec\theta.$




$$\int \frac{1}{(a^2+x^2)^{3/2}}\,dx = \int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int\cos\theta\,d\theta$$

$$= \frac{1}{a^2}\sin\theta = \frac{1}{a^2}\frac{x}{\sqrt{a^2+x^2}}$$

Hence,

$$P = k\int_1^{\infty}\frac{1}{(a^2+x^2)^{3/2}}\,dx = \frac{k}{a^2}\lim_{b\to\infty}\left[\frac{x}{\sqrt{a^2+x^2}}\right]_1^b$$

$$= \frac{k}{a^2}\left[1 - \frac{1}{\sqrt{a^2+1}}\right] = \frac{k\left(\sqrt{a^2+1}-1\right)}{a^2\sqrt{a^2+1}}.$$

**81.** $\frac{10}{x^2-2x} = \frac{10}{x(x-2)} \implies x = 0, 2.$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

**83.** For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_0^b (x^2+1)^{-4}(2x\,dx) = \lim_{b\to\infty}\left[-\frac{1}{6}\frac{1}{(x^2+1)^3}\right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2+1)^{n+3}}\,dx = \lim_{b\to\infty}\left[\frac{-x^{2n-2}}{2(n+2)(x^2+1)^n} + 2\right]_0^b + \frac{n-1}{n+2}\int_0^\infty \frac{x^{2n-3}}{(x^2+1)^{n+2}}\,dx = 0 + \frac{n-1}{n+2}(I_{n-1})$$

$$u = x^{2n-2},\ du = (2n-2)x^{2n-3}\,dx,\ dv = \frac{x}{(x^2+1)^{n+3}}\,dx,\ v = \frac{-1}{2(n+2)(x^2+1)^{n+2}}$$

(a) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty}\left[-\frac{1}{6(x^2+1)^3}\right]_0^b = \frac{1}{6}$

(b) $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{1}{4}\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \frac{1}{4}\left(\frac{1}{6}\right) = \frac{1}{24}$

(c) $\displaystyle\int_0^\infty \frac{x^5}{(x^2+1)^6} = \frac{2}{5}\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{2}{5}\left(\frac{1}{24}\right) = \frac{1}{60}$

**85.** False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x\to\infty} 1/(x+1) = 0$, but $\displaystyle\int_0^\infty \frac{1}{x+1}\,dx = \lim_{b\to\infty}\Big[\ln|x+1|\Big]_0^b = \infty.$

Diverges

**87.** True

# Review Exercises for Chapter 7

**1.**
$$\int x\sqrt{x^2-1}\,dx = \frac{1}{2}\int (x^2-1)^{1/2}(2x)\,dx$$
$$= \frac{1}{2}\frac{(x^2-1)^{3/2}}{3/2} + C$$
$$= \frac{1}{3}(x^2-1)^{3/2} + C$$

**3.**
$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx$$
$$= \frac{1}{2}\ln|x^2-1| + C$$

**5.** $\displaystyle\int \frac{\ln(2x)}{x}\,dx = \frac{(\ln 2x)^2}{2} + C$

**7.** $\displaystyle\int \frac{16}{\sqrt{16-x^2}}\,dx = 16\arcsin\left(\frac{x}{4}\right) + C$

**9.**
$$\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$
$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)$$
$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$
$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

(1) $dv = \sin 3x\,dx \implies v = -\frac{1}{3}\cos 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

(2) $dv = \cos 3x\,dx \implies v = \frac{1}{3}\sin 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

**11.** $u = x,\ du = dx,\ dv = (x-5)^{1/2}\,dx,\ v = \frac{2}{3}(x-5)^{3/2}$

$$\begin{aligned}\int x\sqrt{x-5}\,dx &= \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2}\,dx\\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C\\ &= (x-5)^{3/2}\left[\frac{2}{3}x - \frac{4}{15}(x-5)\right] + C\\ &= (x-5)^{3/2}\left[\frac{6}{15}x + \frac{4}{3}\right] + C\\ &= \frac{2}{15}(x-5)^{3/2}[3x+10] + C\end{aligned}$$

**13.** $$\begin{aligned}\int x^2\sin 2x\,dx &= -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx\\ &= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx\\ &= -\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C\end{aligned}$$

(1) $dv = \sin 2x\,dx \implies v = -\frac{1}{2}\cos 2x$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = \cos 2x\,dx \implies v = \frac{1}{2}\sin 2x$

$u = x \implies du = dx$

**15.** $$\begin{aligned}\int x\arcsin 2x\,dx &= \frac{x^2}{2}\arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}}\,dx\\ &= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\int \frac{2(2x)^2}{\sqrt{1-(2x)^2}}\,dx\\ &= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\left(\frac{1}{2}\right)\left[-(2x)\sqrt{1-4x^2} + \arcsin 2x\right] + C \quad \text{(by Formula 43 of Integration Tables)}\\ &= \frac{1}{16}\left[(8x^2-1)\arcsin 2x + 2x\sqrt{1-4x^2}\right] + C\end{aligned}$$

$dv = x\,dx \implies v = \frac{x^2}{2}$

$u = \arcsin 2x \implies du = \frac{2}{\sqrt{1-4x^2}}\,dx$

**17.** $$\begin{aligned}\int \cos^3(\pi x - 1)\,dx &= \int [1 - \sin^2(\pi x - 1)]\cos(\pi x - 1)\,dx\\ &= \frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{1}{3}\sin^3(\pi x - 1)\right] + C\\ &= \frac{1}{3\pi}\sin(\pi x - 1)[3 - \sin^2(\pi x - 1)] + C\\ &= \frac{1}{3\pi}\sin(\pi x - 1)[3 - (1 - \cos^2(\pi x - 1))] + C\\ &= \frac{1}{3\pi}\sin(\pi x - 1)[2 + \cos^2(\pi x - 1)] + C\end{aligned}$$

**19.** $$\begin{aligned}\int \sec^4\left(\frac{x}{2}\right)dx &= \int \left[\tan^2\left(\frac{x}{2}\right) + 1\right]\sec^2\left(\frac{x}{2}\right)dx\\ &= \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx + \int \sec^2\left(\frac{x}{2}\right)dx\\ &= \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) + C = \frac{2}{3}\left[\tan^3\left(\frac{x}{2}\right) + 3\tan\left(\frac{x}{2}\right)\right] + C\end{aligned}$$

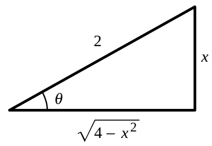
**21.** $$\int \frac{1}{1-\sin\theta}\,d\theta = \int \frac{1+\sin\theta}{\cos^2\theta}\,d\theta = \int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$$

**23.** $\displaystyle\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx = \int \frac{-24\cos\theta\,d\theta}{(4\sin^2\theta)(2\cos\theta)}$

$$= -3\int \csc^2\theta\,d\theta$$

$$= 3\cot\theta + C$$

$$= \frac{3\sqrt{4-x^2}}{x} + C$$

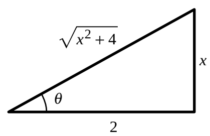



$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

**25.** $x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$4 + x^2 = 4\sec^2\theta$




$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta$$

$$= 8\int \tan^3\theta\sec\theta\,d\theta$$

$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$

$$= 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

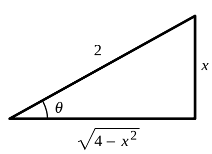
$$= 8\left[\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2}\right] + C$$

$$= \sqrt{x^2+4}\left[\frac{1}{3}(x^2+4) - 4\right] + C$$

$$= \frac{1}{3}x^2\sqrt{x^2+4} - \frac{8}{3}\sqrt{x^2+4} + C$$

$$= \frac{1}{3}(x^2+4)^{1/2}(x^2-8) + C$$

**27.** $\displaystyle\int \sqrt{4-x^2}\,dx = \int (2\cos\theta)(2\cos\theta)\,d\theta$




$$= 2\int (1 + \cos 2\theta)\,d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= 2(\theta + \sin\theta\cos\theta) + C$$

$$= 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] + C$$

$$= \frac{1}{2}\left[4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2}\right] + C$$

$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

**29.** (a) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta$

$$= 8\int(\cos^{-4}\theta - \cos^{-2}\theta)\sin\theta\,d\theta$$

$$= \frac{8}{3}\sec\theta(\sec^2\theta - 3) + C$$

$$= \frac{\sqrt{4+x^2}}{3}(x^2-8) + C$$

$x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta$

(b) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int (u^2-4)\,du$

$$= \frac{1}{3}u^3 - 4u + C$$

$$= \frac{u}{3}(u^2-12) + C$$

$$= \frac{\sqrt{4+x^2}}{3}(x^2-8) + C$$

$u^2 = 4 + x^2,\ 2u\,du = 2x\,dx$

(c) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - \int 2x\sqrt{4+x^2}\,dx$

$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3}(x^2-8) + C$$

$dv = \dfrac{x}{\sqrt{4+x^2}}\,dx \implies v = \sqrt{4+x^2}$

$u = x^2 \implies du = 2x\,dx$

**31.** $\dfrac{x-28}{x^2-x-6} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

$x - 28 = A(x+2) + B(x-3)$

$x = -2 \implies -30 = B(-5) \implies B = 6$

$x = 3 \implies -25 = A(5) \implies A = -5$

$$\int \frac{x-28}{x^2-6-6}\,dx = \int\left(\frac{-5}{x-3} + \frac{6}{x+2}\right)dx = -5\ln|x-3| + 6\ln|x+2| + C$$

**33.** $\dfrac{x^2+2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$x^2 + 2x = A(x^2+1) + (Bx+C)(x-1)$

Let $x = 1$: $3 = 2A \implies A = \dfrac{3}{2}$

Let $x = 0$: $0 = A - C \implies C = \dfrac{3}{2}$

Let $x = 2$: $8 = 5A + 2B + C \implies B = -\dfrac{1}{2}$

$$\int \frac{x^2+2x}{x^3-x^2+x-1}\,dx = \frac{3}{2}\int\frac{1}{x-1}\,dx - \frac{1}{2}\int\frac{x-3}{x^2+1}\,dx$$

$$= \frac{3}{2}\int\frac{1}{x-1}\,dx - \frac{1}{4}\int\frac{2x}{x^2+1}\,dx + \frac{3}{2}\int\frac{1}{x^2+1}\,dx$$

$$= \frac{3}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\arctan x + C$$

$$= \frac{1}{4}[6\ln|x-1| - \ln(x^2+1) + 6\arctan x] + C$$

**35.** $\dfrac{x^2}{x^2+2x-15} = 1 + \dfrac{15-2x}{x^2+2x-15}$

$$\frac{15-2x}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

$$15 - 2x = A(x+5) + B(x-3)$$

Let $x = 3$: $9 = 8A \implies A = \dfrac{9}{8}$

Let $x = -5$: $25 = -8B \implies B = -\dfrac{25}{8}$

$$\int \frac{x^2}{x^2+2x-15}\,dx = \int dx + \frac{9}{8}\int \frac{1}{x-3}\,dx - \frac{25}{8}\int \frac{1}{x+5}\,dx$$

$$= x + \frac{9}{8}\ln|x-3| - \frac{25}{8}\ln|x+5| + C$$

**37.** $\displaystyle\int \frac{x}{(2+3x)^2}\,dx = \frac{1}{9}\left[\frac{2}{2+3x} + \ln|2+3x|\right] + C$

(Formula 4)

**39.** $\displaystyle\int \frac{x}{1+\sin x^2}\,dx = \frac{1}{2}\int \frac{1}{1+\sin u}\,du$ $\quad(u = x^2)$

$$= \frac{1}{2}[\tan u - \sec u] + C \quad \text{(Formula 56)}$$

$$= \frac{1}{2}[\tan x^2 - \sec x^2] + C$$

**41.** $\displaystyle\int \frac{x}{x^2+4x+8}\,dx = \frac{1}{2}\left[\ln|x^2+4x+8| - 4\int \frac{1}{x^2+4x+8}\,dx\right]$ (Formula 15)

$$= \frac{1}{2}[\ln|x^2+4x+8|] - 2\left[\frac{2}{\sqrt{32-16}}\arctan\left(\frac{2x+4}{\sqrt{32-16}}\right)\right] + C \quad \text{(Formula 14)}$$

$$= \frac{1}{2}\ln|x^2+4x+8| - \arctan\left(1+\frac{x}{2}\right) + C$$

**43.** $\displaystyle\int \frac{1}{\sin \pi x \cos \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{\sin \pi x \cos \pi x}(\pi)\,dx$ $\quad(u = \pi x)$

$$= \frac{1}{\pi}\ln|\tan \pi x| + C \quad \text{(Formula 58)}$$

**45.** $dv = dx \implies v = x$

$u = (\ln x)^n \implies du = n(\ln x)^{n-1}\dfrac{1}{x}dx$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

**47.** $\displaystyle\int \theta \sin\theta \cos\theta\,d\theta = \frac{1}{2}\int \theta \sin 2\theta\,d\theta$

$$= -\frac{1}{4}\theta\cos 2\theta + \frac{1}{4}\int \cos 2\theta\,d\theta = -\frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$$

$dv = \sin 2\theta\,d\theta \implies v = -\dfrac{1}{2}\cos 2\theta$

$u = \theta \implies du = d\theta$

**49.** $\displaystyle\int \frac{x^{1/4}}{1 + x^{1/2}}\,dx = 4\int \frac{u(u^3)}{1 + u^2}\,du$

$\displaystyle = 4\int \left(u^2 - 1 + \frac{1}{u^2 + 1}\right) du$

$\displaystyle = 4\left(\frac{1}{3}u^3 - u + \arctan u\right) + C$

$\displaystyle = \frac{4}{3}[x^{3/4} - 3x^{1/4} + 3\arctan(x^{1/4})] + C$

$y = \sqrt[4]{x},\ x = u^4,\ dx = 4u^3\,du$

**51.** $\displaystyle\int \sqrt{1 + \cos x}\,dx = \int \frac{\sin x}{\sqrt{1 - \cos x}}\,dx$

$\displaystyle = \int (1 - \cos x)^{-1/2}(\sin x)\,dx$

$= 2\sqrt{1 - \cos x} + C$

$u = 1 - \cos x,\ du = \sin x\,dx$

**53.** $\displaystyle\int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx$

$= \sin x \ln(\sin x) - \sin x + C$

$dv = \cos x\,dx \implies v = \sin x$

$\displaystyle u = \ln(\sin x) \implies du = \frac{\cos x}{\sin x}\,dx$

**55.** $\displaystyle y = \int \frac{9}{x^2 - 9}\,dx = \frac{3}{2}\ln\left|\frac{x - 3}{x + 3}\right| + C$

(by Formula 24 of Integration Tables)

**57.** $\displaystyle y = \int \ln(x^2 + x)dx = x\ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x}\,dx$

$\displaystyle = x\ln|x^2 + x| - \int \frac{2x + 1}{x + 1}\,dx$

$\displaystyle = x\ln|x^2 + x| - \int 2dx + \int \frac{1}{x + 1}\,dx$

$= x\ln|x^2 + x| - 2x + \ln|x + 1| + C$

$dv = dx \implies v = x$

$\displaystyle u = \ln(x^2 + x) \implies du = \frac{2x + 1}{x^2 + x}\,dx$

**59.** $\displaystyle\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2}\,dx = \left[\frac{1}{5}(x^2 - 4)^{5/2}\right]_2^{\sqrt{5}} = \frac{1}{5}$

**61.** $\displaystyle\int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}(\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$

**63.** $\displaystyle\int_0^{\pi} x\sin x\,dx = \Big[-x\cos x + \sin x\Big]_0^{\pi} = \pi$

**65.** $\displaystyle A = \int_0^4 x\sqrt{4 - x}\,dx = \int_2^0 (4 - u^2)u(-2u)\,du$

$\displaystyle = \int_2^0 2(u^4 - 4u^2)\,du$

$\displaystyle = \left[2\left(\frac{u^5}{5} - \frac{4u^3}{3}\right)\right]_2^0 = \frac{128}{15}$

$u = \sqrt{4 - x},\ x = 4 - u^2,\ dx = -2u\,du$

**67.** By symmetry, $\bar{x} = 0$, $A = \dfrac{1}{2}\pi$.

$\displaystyle \bar{y} = \frac{2}{\pi}\left(\frac{1}{2}\right)\int_{-1}^{1} \left(\sqrt{1 - x^2}\right)^2 dx = \frac{1}{\pi}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3\pi}$

$\displaystyle (\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$

**69.** $\displaystyle s = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx \approx 3.82$

**71.** $\displaystyle\lim_{x\to 1}\left[\frac{(\ln x)^2}{x - 1}\right] = \lim_{x\to 1}\left[\frac{2(1/x)\ln x}{1}\right] = 0$

**73.** $\displaystyle\lim_{x\to\infty} \frac{e^{2x}}{x^2} = \lim_{x\to\infty} \frac{2e^{2x}}{2x} = \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty$

**75.** $\displaystyle y = \lim_{x\to\infty} (\ln x)^{2/x}$

$\displaystyle \ln y = \lim_{x\to\infty} \frac{2\ln(\ln x)}{x} = \lim_{x\to\infty}\left[\frac{2/(x\ln x)}{1}\right] = 0$

Since $\ln y = 0$, $y = 1$.

**77.** $\lim_{n\to\infty} 1000\left(1 + \frac{0.09}{n}\right)^n = 1000 \lim_{n\to\infty}\left(1 + \frac{0.09}{n}\right)^n$

Let $y = \lim_{n\to\infty}\left(1 + \frac{0.09}{n}\right)^n$.

$$\ln y = \lim_{n\to\infty} n \ln\left(1 + \frac{0.09}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1 + \frac{0.09}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}}\right) = \lim_{n\to\infty} \frac{0.09}{1 + \left(\frac{0.09}{n}\right)} = 0.09$$

Thus, $\ln y = 0.09 \implies y = e^{0.09}$ and $\lim_{n\to\infty} 1000\left(1 + \frac{0.09}{n}\right)^n = 1000e^{0.09} \approx 1094.17$.

**79.** $\int_0^{16} \frac{1}{\sqrt[4]{x}}\, dx = \lim_{b\to 0^+}\left[\frac{4}{3}x^{3/4}\right]_b^{16} = \frac{32}{3}$

Converges

**81.** $\int_1^{\infty} x^2 \ln x\, dx = \lim_{b\to\infty}\left[\frac{x^3}{9}(-1 + 3\ln x)\right]_1^b = \infty$

Diverges

**83.** $\int_0^{t_0} 500{,}000e^{-0.05t}\, dt = \left[\frac{500{,}000}{-0.05}e^{-0.05t}\right]_0^{t_0}$

$$= \frac{-500{,}000}{0.05}(e^{-0.05t_0} - 1)$$

$$= 10{,}000{,}000(1 - e^{-0.05t_0})$$

(a) $t_0 = 20$: \$6,321,205.59

(b) $t_0 \to \infty$: \$10,000,000

**85.** (a) $P(13 \le x < \infty) = \frac{1}{0.95\sqrt{2\pi}}\int_{13}^{\infty} e^{-(x-12.9)^2/2(0.95)^2}\, dx \approx 0.4581$

(b) $P(15 \le x < 20) = \frac{1}{0.95\sqrt{2\pi}}\int_{15}^{\infty} e^{-(x-12.9)^2/2(0.95)^2}\, dx \approx 0.0135$

## Problem Solving for Chapter 7

**1.** (a) $\int_{-1}^{1}(1 - x^2)\, dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$

$$\int_{-1}^{1}(1 - x^2)^2\, dx = \int_{-1}^{1}(1 - 2x^2 + x^4)\, dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}$$

(b) Let $x = \sin u$, $dx = \cos u\, du$, $1 - x^2 = 1 - \sin^2 u = \cos^2 u$.

$$\int_{-1}^{1}(1 - x^2)^n\, dx = \int_{-\pi/2}^{\pi/2}(\cos^2 u)^n \cos u\, du$$

$$= \int_{-\pi/2}^{\pi/2}\cos^{2n+1} u\, du$$

$$= 2\left[\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{(2n)}{(2n+1)}\right] \qquad \text{(Wallis's Formula)}$$

$$= 2\left[\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{2\cdot 3\cdot 4\cdot 5\cdots(2n)(2n+1)}\right]$$

$$= \frac{2(2^{2n})(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

**3.**
$$\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 9$$
$$\lim_{x\to\infty} x\ln\left(\frac{x+c}{x-c}\right) = \ln 9$$
$$\lim_{x\to\infty}\frac{\ln(x+c)-\ln(x-c)}{1/x} = \ln 9$$
$$\lim_{x\to\infty}\frac{\dfrac{1}{x+c}-\dfrac{1}{x-c}}{-\dfrac{1}{x^2}} = \ln 9$$
$$\lim_{x\to\infty}\frac{-2c}{(x+c)(x-c)}(-x^2) = \ln 9$$
$$\lim_{x\to\infty}\left(\frac{2cx^2}{x^2-c^2}\right) = \ln 9$$
$$2c = \ln 9$$
$$2c = 2\ln 3$$
$$c = \ln 3$$

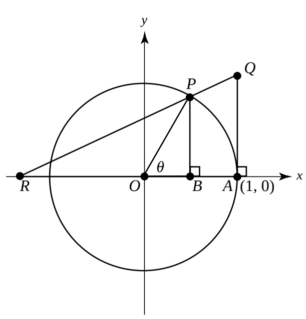
**5.** $\sin\theta = \dfrac{PB}{OP} = PB$, $\cos\theta = OB$

$AQ = \widehat{AP} = \theta$

$BR = OR + OB = OR + \cos\theta$

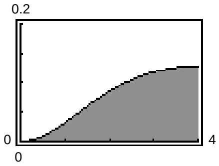
The triangles $\triangle AQR$ and $\triangle BPR$ are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \implies \frac{OR+1}{\theta} = \frac{OR+\cos\theta}{\sin\theta}$$
$$\sin\theta(OR) + \sin\theta = (OR)\theta + \theta\cos\theta$$
$$OR = \frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$



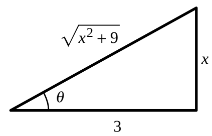

$$\lim_{\theta\to 0^+} OR = \lim_{\theta\to 0^+}\frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$
$$= \lim_{\theta\to 0^+}\frac{-\theta\sin\theta + \cos\theta - \cos\theta}{\cos\theta - 1}$$
$$= \lim_{\theta\to 0^+}\frac{-\theta\sin\theta}{\cos\theta - 1}$$
$$= \lim_{\theta\to 0^+}\frac{-\sin\theta - \theta\cos\theta}{-\sin\theta}$$
$$= \lim_{\theta\to 0}\frac{\cos\theta + \cos\theta - \theta\sin\theta}{\cos\theta}$$
$$= 2$$

**7.** (a)




Area $\approx 0.2986$

(b) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $x^2 + 9 = 9\sec^2\theta$.




$$\begin{aligned}
\int \frac{x^2}{(x^2+9)}\,dx &= \int \frac{9\tan^2\theta}{(9\sec^2\theta)^{3/2}}(3\sec^2\theta\,d\theta)\\
&= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta\\
&= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta\\
&= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta\\
&= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C
\end{aligned}$$

$$\begin{aligned}
\text{Area} = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx &= \Big[\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\Big]_0^{\tan^{-1}(4/3)}\\
&= \left[\ln\left(\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right) - \frac{x}{\sqrt{x^2+9}}\right]_0^4\\
&= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \frac{4}{5} = \ln 3 - \frac{4}{5}
\end{aligned}$$

(c) $x = 3\sinh u$, $dx = 3\cosh u\,du$, $x^2 + 9 = 9\sinh^2 u + 9 = 9\cosh^2 u$

$$\begin{aligned}
A = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx &= \int_0^{\sin^{-1}(4/3)} \frac{9\sinh^2 u}{(9\cosh^2 u)^{3/2}}(3\cosh u\,du)\\
&= \int_0^{\sinh^{-1}(4/3)} \tanh^2 u\,du\\
&= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u)\,du\\
&= \Big[u - \tanh u\Big]_0^{\sinh^{-1}(4/3)}\\
&= \sinh^{-1}\left(\frac{4}{3}\right) - \tanh\left(\sinh^{-1}\left(\frac{4}{3}\right)\right)\\
&= \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}\right) - \tanh\left[\ln\left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}\right)\right]\\
&= \ln\left(\frac{4}{3} + \frac{5}{3}\right) - \tanh\left(\ln\left(\frac{4}{3} + \frac{5}{3}\right)\right)\\
&= \ln 3 - \tanh(\ln 3)\\
&= \ln 3 - \frac{3 - (1/3)}{3 + (1/3)}\\
&= \ln 3 - \frac{4}{5}
\end{aligned}$$

**9.** $y = \ln(1 - x^2),\, y' = \dfrac{-2x}{1 - x^2}$

$$1 + (y')^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \left(\frac{1 + x^2}{1 - x^2}\right)^2$$

$$\begin{aligned}
\text{Arc length} &= \int_0^{1/2} \sqrt{1 + (y')^2}\, dx \\
&= \int_0^{1/2} \left(\frac{1 + x^2}{1 - x^2}\right) dx \\
&= \int_0^{1/2} \left(-1 + \frac{2}{1 - x^2}\right) dx \\
&= \int_0^{1/2} \left(-1 + \frac{1}{x + 1} + \frac{1}{1 - x}\right) dx \\
&= \Big[-x + \ln(1 + x) - \ln(1 - x)\Big]_0^{1/2} \\
&= \left(-\frac{1}{2} + \ln\frac{3}{2} - \ln\frac{1}{2}\right) \\
&= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 \\
&= \ln 3 - \frac{1}{2} \approx 0.5986
\end{aligned}$$

**11.** Consider $\displaystyle\int \frac{1}{\ln x}\, dx.$

Let $u = \ln x,\, du = \dfrac{1}{dx}\, dx,\, x = e^u$. Then $\displaystyle\int \frac{1}{\ln x}\, dx = \int \frac{1}{u} e^u\, du = \int \frac{e^u}{u}\, du.$

If $\displaystyle\int \frac{1}{\ln x}\, dx$ were elementary, then $\displaystyle\int \frac{e^u}{u}\, du$ would be too, which is false.

Hence, $\displaystyle\int \frac{1}{\ln x}\, dx$ is not elementary.

**13.** $x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$

$$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$a = -c,\, b = d = 1,\, a = \sqrt{2}$

$$x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$$

$$\begin{aligned}
\int_0^1 \frac{1}{x^4 + 1}\, dx &= \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1}\, dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1}\, dx \\
&= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\, dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\, dx \\
&= \frac{\sqrt{2}}{4}\Big[\arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right)\Big]_0^1 + \frac{\sqrt{2}}{8}\Big[\ln\left(x^2 + \sqrt{2}x + 1\right) - \ln\left(x^2 - \sqrt{2}x + 1\right)\Big]_0^1 \\
&= \frac{\sqrt{2}}{4}\left[\arctan\left(\sqrt{2} + 1\right) + \arctan\left(\sqrt{2} - 1\right)\right] + \frac{\sqrt{2}}{8}\left[\ln\left(2 + \sqrt{2}\right) - \ln\left(2 - \sqrt{2}\right)\right] - \frac{\sqrt{2}}{4}\left[\frac{\pi}{4} - \frac{\pi}{4}\right] - \frac{\sqrt{2}}{8}[0] \\
&\approx 0.5554 + 0.3116 \\
&\approx 0.8670
\end{aligned}$$

**15.** Using a graphing utility,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty$

(b) $\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = 0$

(c) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) \approx -\frac{2}{3}.$

Analytically,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty + \infty = \infty$

(b) $$\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x\cot x - 1}{x} = \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x}$$

$$= \lim_{x\to 0^+}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{-x\sin x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+}\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0.$$

(c) $$\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \cot^2 x - \frac{1}{x^2}$$

$$= \frac{x^2\cot^2 x - 1}{x^2}$$

$$\lim_{x\to 0^+}\frac{x^2\cot^2 x - 1}{x^2} = \lim_{x\to 0^+}\frac{2x\cot^2 x - 2x^2\cot x\csc^2 x}{2x}$$

$$= \lim_{x\to 0^+}\frac{\cot^2 x - x\cot x\csc^2 x}{1}$$

$$= \lim_{x\to 0^+}\frac{\cos^2 x\sin x - x\cos x}{\sin^3 x}$$

$$= \lim_{x\to 0^+}\frac{(1 - \sin^2 x)\sin x - x\cos x}{\sin^3 x}$$

$$= \lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} - 1$$

Now, $$\lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} = \lim_{x\to 0^+}\frac{\cos x - \cos x + x\sin x}{3\sin^2 x\cos x}$$

$$= \lim_{x\to 0^+}\frac{x}{3\sin x \cdot \cos x}$$

$$= \lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)\frac{1}{3\cos x} = \frac{1}{3}.$$

Thus, $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \frac{1}{3} - 1 = -\frac{2}{3}.$

The form $0 \cdot \infty$ is indeterminant.

**17.** $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{P_1}{x} + \dfrac{P_2}{x-1} + \dfrac{P_3}{x+4} + \dfrac{P_4}{x-3} \Longrightarrow c_1 = 0,\ c_2 = 1,\ c_3 = -4,\ c_4 = 3$

$N(x) = x^3 - 3x^2 + 1$

$D'(x) = 4x^3 - 26x + 12$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

Thus, $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{1/12}{x} + \dfrac{1/10}{x-1} + \dfrac{111/140}{x+4} + \dfrac{1/42}{x-3}$.

**19.** By parts,

$$\begin{aligned}\int_a^b f(x)g''(x)\,dx &= \Big[f(x)g'(x)\Big]_a^b - \int f'(x)g'(x)\,dx\\ &= -\int_a^b f'(x)g'(x)\,dx\\ &= \Big[-f'(x)g(x)\Big]_a^b + \int_a^b g(x)f''(x)\,dx\\ &= \int_a^b f''(x)g(x)\,dx.\end{aligned}$$

# CHAPTER 8
# Infinite Series

# CHAPTER 8
# Infinite Series

## Section 8.1 Sequences

Solutions to Odd-Numbered Exercises

**1.** $a_n = 2^n$

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

**3.** $a_n = \left(-\frac{1}{2}\right)^n$

$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$

$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$

$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$

$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$

**5.** $a_n = \sin \frac{n\pi}{2}$

$a_1 = \sin \frac{\pi}{2} = 1$

$a_2 = \sin \pi = 0$

$a_3 = \sin \frac{3\pi}{2} = -1$

$a_4 = \sin 2\pi = 0$

$a_5 = \sin \frac{5\pi}{2} = 1$

**7.** $a_n = \frac{(-1)^{n(n+1)/2}}{n^2}$

$a_1 = \frac{(-1)^1}{1^2} = -1$

$a_2 = \frac{(-1)^3}{2^2} = -\frac{1}{4}$

$a_3 = \frac{(-1)^6}{3^2} = \frac{1}{9}$

$a_4 = \frac{(-1)^{10}}{4^2} = \frac{1}{16}$

$a_5 = \frac{(-1)^{15}}{5^2} = -\frac{1}{25}$

**9.** $a_n = 5 - \frac{1}{n} + \frac{1}{n^2}$

$a_1 = 5 - 1 + 1 = 5$

$a_2 = 5 - \frac{1}{2} + \frac{1}{4} = \frac{19}{4}$

$a_3 = 5 - \frac{1}{3} + \frac{1}{9} = \frac{43}{9}$

$a_4 = 5 - \frac{1}{4} + \frac{1}{16} = \frac{77}{16}$

$a_5 = 5 - \frac{1}{5} + \frac{1}{25} = \frac{121}{25}$

**11.** $a_n = \frac{3^n}{n!}$

$a_1 = \frac{3}{1!} = 3$

$a_2 = \frac{3^2}{2!} = \frac{9}{2}$

$a_3 = \frac{3^3}{3!} = \frac{27}{6}$

$a_4 = \frac{3^4}{4!} = \frac{81}{24}$

$a_5 = \frac{3^5}{5!} = \frac{243}{120}$

**13.** $a_1 = 3, a_{k+1} = 2(a_k - 1)$

$a_2 = 2(a_1 - 1)$

$= 2(3 - 1) = 4$

$a_3 = 2(a_2 - 1)$

$= 2(4 - 1) = 6$

$a_4 = 2(a_3 - 1)$

$= 2(6 - 1) = 10$

$a_5 = 2(a_4 - 1)$

$= 2(10 - 1) = 18$

**15.** $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$

$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$

$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$

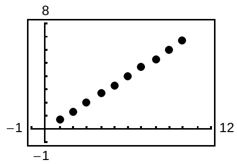
$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$

$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$

**17.** Because $a_1 = 8/(1 + 1) = 4$ and $a_2 = 8/(2 + 1) = \frac{8}{3}$, the sequence matches graph (d).

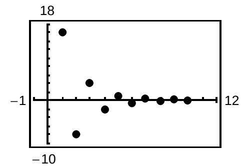
**19.** This sequence decreases and $a_1 = 4$, $a_2 = 4(0.5) = 2$. Matches (c).

**21.**



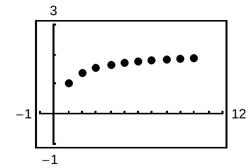
$a_n = \frac{2}{3}n, n = 1, . . . , 10$

**23.**



$a_n = 16(-0.5)^{n-1}, n = 1, . . . , 10$

**25.**



$a_n = \frac{2n}{n + 1}, n = 1, 2, . . . , 10$

**27.** $a_n = 3n - 1$

$a_5 = 3(5) - 1 = 14$

$a_6 = 3(6) - 1 = 17$

Add 3 to preceeding term.

**29.** $a_n = \dfrac{3}{(-2)^{n-1}}$

$a_n = \dfrac{3}{(-2)^4} = \dfrac{3}{16}$

$a_6 = \dfrac{3}{(-2)^5} = -\dfrac{3}{32}$

Multiply the preceeding term by $-\frac{1}{2}$.

**31.** $\dfrac{10!}{8!} = \dfrac{8!(9)(10)}{8!}$

$= (9)(10) = 90$

**33.** $\dfrac{(n + 1)!}{n!} = \dfrac{n!(n + 1)}{n!}$

$= n + 1$

**35.** $\dfrac{(2n - 1)!}{(2n + 1)!} = \dfrac{(2n - 1)!}{(2n - 1)!(2n)(2n + 1)}$

$= \dfrac{1}{2n(2n + 1)}$

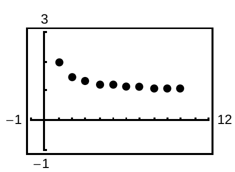
**37.** $\lim\limits_{n\to\infty} \dfrac{5n^2}{n^2 + 2} = 5$

**39.** $\lim\limits_{n\to\infty} \dfrac{2n}{\sqrt{n^2 + 1}} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{1 + (1/n^2)}}$

$= \dfrac{2}{1} = 2$

**41.** $\lim\limits_{n\to\infty} \sin\left(\dfrac{1}{n}\right) = 0$

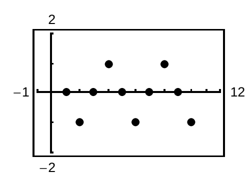
**43.**



The graph seems to indicate that the sequence converges to 1. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n + 1}{n} = \lim_{x\to\infty} \frac{x + 1}{x} = \lim_{x\to\infty} 1 = 1.$$

**45.**



The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$$\{a_n\} = \{0, -1, 0, 1, 0, -1, . . .\}.$$

Hence, $\lim\limits_{n\to\infty} a_n$ does not exist.

**47.** $\lim\limits_{n\to\infty} (-1)^n\left(\dfrac{n}{n + 1}\right)$

does not exist (oscillates between $-1$ and 1), diverges.

**49.** $\lim\limits_{n\to\infty} \dfrac{3n^2 - n + 4}{2n^2 + 1} = \dfrac{3}{2}$, converges

**51.** $\lim\limits_{n\to\infty} \dfrac{1 + (-1)^n}{n} = 0$, converges

**53.** $\lim\limits_{n\to\infty} \dfrac{\ln(n^3)}{2n} = \lim\limits_{n\to\infty} \dfrac{3}{2}\dfrac{\ln(n)}{n}$

$= \lim\limits_{n\to\infty} \dfrac{3}{2}\left(\dfrac{1}{n}\right) = 0$, converges

(L'Hôpital's Rule)

**55.** $\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$, converges

**57.** $\lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$, diverges

**59.** $\lim_{n\to\infty}\left(\frac{n-1}{n} - \frac{n}{n-1}\right) = \lim_{n\to\infty}\frac{(n-1)^2 - n^2}{n(n-1)}$

$= \lim_{n\to\infty}\frac{1-2n}{n^2-n} = 0$, converges

**61.** $\lim_{n\to\infty}\frac{n^p}{e^n} = 0$, converges

$(p > 0, n \geq 2)$

**63.** $a_n = \left(1 + \frac{k}{n}\right)^n$

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = \lim_{u\to 0}\left[(1+u)^{1/u}\right]^k = e^k$$

where $u = \frac{k}{n}$, converges

**65.** $\lim_{n\to\infty}\frac{\sin n}{n} = \lim_{n\to\infty}(\sin n)\frac{1}{n} = 0$, converges

**67.** $a_n = 3n - 2$

**69.** $a_n = n^2 - 2$

**71.** $a_n = \frac{n+1}{n+2}$

**73.** $a_n = \frac{(-1)^{n-1}}{2^{n-2}}$

**75.** $a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

**77.** $a_n = \frac{n}{(n+1)(n+2)}$

**79.** $a_n = \frac{(-1)^{n-1}}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{(-1)^{n-1}2^n n!}{(2n)!}$

**81.** $a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1}$,

monotonic; $|a_n| < 4$ bounded.

**83.** $\frac{n}{2^{n+2}} \overset{?}{\geq} \frac{n+1}{2^{(n+1)+2}}$

$2^{n+3}n \overset{?}{\geq} 2^{n+2}(n+1)$

$2n \overset{?}{\geq} n+1$

$n \geq 1$

Hence, $n \geq 1$

$2n \geq n+1$

$2^{n+3}n \geq 2^{n+2}(n+1)$

$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$

$a_n \geq a_{n+1}$

True; monotonic; $|a_n| \leq \frac{1}{8}$, bounded

**85.** $a_n = (-1)^n\left(\frac{1}{n}\right)$

$a_1 = -1$

$a_2 = \frac{1}{2}$

$a_3 = -\frac{1}{3}$

Not monotonic; $|a_n| \leq 1$, bounded

**87.** $a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$

Monotonic; $|a_n| \leq \frac{2}{3}$, bounded

**89.** $a_n = \sin\left(\frac{n\pi}{6}\right)$

$a_1 = 0.500$

$a_2 = 0.8660$

$a_3 = 1.000$

$a_4 = 0.8660$

Not monotonic; $|a_n| \leq 1$, bounded

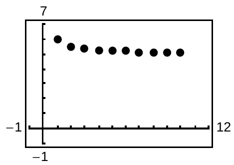
**91.** (a) $a_n = 5 + \dfrac{1}{n}$

$$\left|5 + \frac{1}{n}\right| \le 6 \implies \{a_n\} \text{ bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1}$$

$$= a_{n+1} \implies \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.

(b) 


$$\lim_{n\to\infty}\left(5 + \frac{1}{n}\right) = 5$$

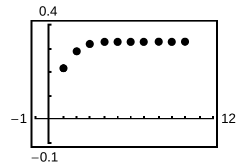
**93.** (a) $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)$

$$\left|\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right| < \frac{1}{3} \implies \{a_n\} \text{ bounded}$$

$$a_n = \frac{1}{3}\left(1 - \frac{1}{3^n}\right) < \frac{1}{3}\left(1 - \frac{1}{3^{n+1}}\right)$$

$$= a_{n+1} \implies \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.

(b) 


$$\lim_{n\to\infty}\left[\frac{1}{3}\left(1 - \frac{1}{3^n}\right)\right] = \frac{1}{3}$$

**95.** $A_n = P\left[1 + \dfrac{r}{12}\right]^n$

(a) $\lim_{n\to\infty} A_n = \infty$, divergent. The amount will grow arbitrarily large over time.

(b) $A_n = 9000\left[1 + \dfrac{0.115}{12}\right]^n$

$A_1 = \$9086.25$ $A_6 = \$9530.06$

$A_2 = \$9173.33$ $A_7 = \$9621.39$

$A_3 = \$9261.24$ $A_8 = \$9713.59$

$A_4 = \$9349.99$ $A_9 = \$9806.68$

$A_5 = \$9439.60$ $A_{10} = \$9900.66$

**97.** (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit.

(c) A bounded monotonic sequence is a sequence that has nondecreasing or nonincreasing terms, and an upper and lower bound.

**99.** $a_n = 10 - \dfrac{1}{n}$

**101.** $a_n = \dfrac{3n}{4n + 1}$

**103.** (a) $A_n = (0.8)^n(2.5)$ billion

(b) $A_1 = \$2$ billion

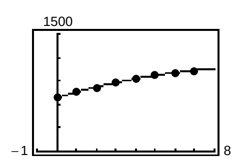
$A_2 = \$1.6$ billion

$A_3 = \$1.28$ billion

$A_4 = \$1.024$ billion

(c) $\lim_{n\to\infty}(0.8)^n(2.5) = 0$

**105.** (a) $a_n = -3.7262n^2 + 75.9167n + 684.25$




(b) For 2004, $n = 14$ and $a_{14} \approx 1017$, or \$1017.

**107.** $a_n = \dfrac{10^n}{n!}$

(a) $a_9 = a_{10} = \dfrac{10^9}{9!}$

$= \dfrac{1{,}000{,}000{,}000}{362{,}880}$

$= \dfrac{1{,}562{,}500}{567}$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

**109.** $\{a_n\} = \left\{\sqrt[n]{n}\right\} = \{n^{1/n}\}$

$a_1 = 1^{1/1} = 1$

$a_2 = \sqrt{2} \approx 1.4142$

$a_3 = \sqrt[3]{3} \approx 1.4422$

$a_4 = \sqrt[4]{4} \approx 1.4142$

$a_5 = \sqrt[5]{5} \approx 1.3797$

$a_6 = \sqrt[6]{6} \approx 1.3480$

Let $y = \lim\limits_{n\to\infty} n^{1/n}$.

$$\ln y = \lim_{n\to\infty}\left(\frac{1}{n}\ln n\right)$$

$$= \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0$$

Since $\ln y = 0$, we have $y = e^0 = 1$. Therefore, $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$.

**111.** $a_{n+2} = a_n + a_{n+1}$

(a)

| | |
|---|---|
| $a_1 = 1$ | $a_7 = 8 + 5 = 13$ |
| $a_2 = 1$ | $a_8 = 13 + 8 = 21$ |
| $a_3 = 1 + 1 = 2$ | $a_9 = 21 + 13 = 34$ |
| $a_4 = 2 + 1 = 3$ | $a_{10} = 34 + 21 = 55$ |
| $a_5 = 3 + 2 = 5$ | $a_{11} = 55 + 34 = 89$ |
| $a_6 = 5 + 3 = 8$ | $a_{12} = 89 + 55 = 144$ |

(b) $b_n = \dfrac{a_{n+1}}{a_n}, n \geq 1$

| | |
|---|---|
| $b_1 = \dfrac{1}{1} = 1$ | $b_6 = \dfrac{13}{8}$ |
| $b_2 = \dfrac{2}{1} = 2$ | $b_7 = \dfrac{21}{13}$ |
| $b_3 = \dfrac{3}{2}$ | $b_8 = \dfrac{34}{21}$ |
| $b_4 = \dfrac{5}{3}$ | $b_9 = \dfrac{55}{34}$ |
| $b_5 = \dfrac{8}{5}$ | $b_{10} = \dfrac{89}{55}$ |

(c) $1 + \dfrac{1}{b_{n-1}} = 1 + \dfrac{1}{a_n/a_{n-1}}$

$= 1 + \dfrac{a_{n-1}}{a_n}$

$= \dfrac{a_n + a_{n-1}}{a_n} = \dfrac{a_{n+1}}{a_n} = b_n$

(d) If $\lim\limits_{n\to\infty} b_n = \rho$, then $\lim\limits_{n\to\infty}\left(1 + \dfrac{1}{b_{n-1}}\right) = \rho$.

Since $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_{n-1}$ we have,

$1 + (1/\rho) = \rho$.

$\rho + 1 = \rho^2$

$0 = \rho^2 - \rho - 1$

$\rho = \dfrac{1 \pm \sqrt{1 + 4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Since $a_n$, and thus $b_n$, is positive,

$\rho = \left(1 + \sqrt{5}\right)/2 \approx 1.6180$.

**113.** True

**115.** True

**117.** $a_1 = \sqrt{2} \approx 1.4142$

$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$

$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$

$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$

$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$

$\{a_n\}$ is increasing and bounded by 2, and hence converges to $L$. Letting $\lim\limits_{n\to\infty} a_n = L$ implies that $\sqrt{2 + L} = L \implies L = 2$. Hence, $\lim\limits_{n\to\infty} a_n = 2$.

## Section 8.2 Series and Convergence

**1.** $S_1 = 1$

$S_2 = 1 + \frac{1}{4} = 1.2500$

$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$

$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$

$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$

**3.** $S_1 = 3$

$S_2 = 3 - \frac{9}{2} = -1.5$

$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$

$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$

$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$

**5.** $S_1 = 3$

$S_2 = 3 + \frac{3}{2} = 4.5$

$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$

$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$

$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$

**7.** $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$ Geometric series

$r = \frac{3}{2} > 1$

Diverges by Theorem 8.6

**9.** $\sum_{n=0}^{\infty} 1000(1.055)^n$ Geometric series

$r = 1.055 > 1$

Diverges by Theorem 8.6

**11.** $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

Diverges by Theorem 8.9

**13.** $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$

Diverges by Theorem 8.9

**15.** $\sum_{n=0}^{\infty} \frac{2^n+1}{2^{n+1}}$

$\lim_{n\to\infty} \frac{2^n+1}{2^{n+1}} = \lim_{n\to\infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$

Diverges by Theorem 8.9

**17.** $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n = \frac{9}{4}\left[1 + \frac{1}{4} + \frac{1}{16} + \cdots\right]$

$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \ldots$

Matches graph (c).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right)\left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$$

**19.** $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15}{4}\left[1 - \frac{1}{4} + \frac{1}{16} - \cdots\right]$

$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \ldots$

Matches graph (a).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15/4}{1-(-1/4)} = \frac{15/4}{5/4} = 3$$

**21.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

**23.** $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

Geometric series with $r = \frac{3}{4} < 1$.

Converges by Theorem 8.6

**25.** $\sum_{n=0} (0.9)^n$

Geometric series with $r = 0.9 < 1$.

Converges by Theorem 8.6

**27.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

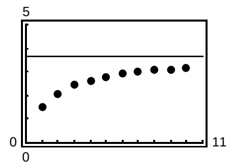
$$= 2\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots\right]$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3}\right] = \frac{11}{3} \approx 3.667$$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 2.7976 | 3.1643 | 3.3936 | 3.5513 | 3.6078 |

(c)



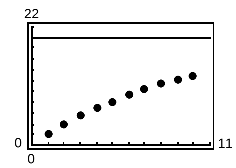

(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

**29.** (a) $\displaystyle\sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 8.1902 | 13.0264 | 17.5685 | 19.8969 | 19.9995 |

(c)



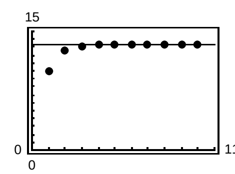

(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

**31.** (a) $\displaystyle\sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 13.3203 | 13.3333 | 13.3333 | 13.3333 | 13.3333 |

(c)




(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

**33.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots\right]$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

**35.** $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 8\left[\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots\right] = 8\left(\frac{1}{2}\right) = 4$

**37.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-(1/2)} = 2$

**39.** $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$

**41.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = \frac{1}{1-(1/10)} = \frac{10}{9}$

**43.** $\displaystyle\sum_{n=0}^{\infty} 3\left(-\frac{1}{3}\right)^n = \frac{3}{1-(-1/3)} = \frac{9}{4}$

**45.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$$=\frac{1}{1-(1/2)}-\frac{1}{1-(-1/3)}$$

$$=2-\frac{3}{2}=\frac{1}{2}$$

**47.** $0.\overline{4}=\displaystyle\sum_{n=0}^{\infty}\frac{4}{10}\left(\frac{1}{10}\right)^n$

Geometric series with $a=\frac{4}{10}$ and $r=\frac{1}{10}$

$$S=\frac{a}{1-r}=\frac{4/10}{1-(1/10)}=\frac{4}{9}$$

**49.** $0.075\overline{75}=\displaystyle\sum_{n=0}^{\infty}\frac{3}{40}\left(\frac{1}{100}\right)^n$

Geometric series with $a=\frac{3}{40}$ and $r=\frac{1}{100}$

$$S=\frac{a}{1-r}=\frac{3/40}{99/100}=\frac{5}{66}$$

**51.** $\displaystyle\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$

$$\lim_{n\to\infty}\frac{n+10}{10n+1}=\frac{1}{10}\neq 0$$

Diverges by Theorem 8.9

**53.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\cdots=1+\frac{1}{2}=\frac{3}{2}$, converges

**55.** $\displaystyle\sum_{n=1}^{\infty}\frac{3n-1}{2n+1}$

$$\lim_{n\to\infty}\frac{3n-1}{2n+1}=\frac{3}{2}\neq 0$$

Diverges by Theorem 8.9

**57.** $\displaystyle\sum_{n=0}^{\infty}\frac{4}{2^n}=4\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

Geometric series with $r=\frac{1}{2}$

Converges by Theorem 8.6

**59.** $\displaystyle\sum_{n=0}^{\infty}(1.075)^n$

Geometric series with $r=1.075$

Diverges by Theorem 8.6

**61.** $\displaystyle\sum_{n=2}^{\infty}\frac{n}{\ln n}$

$$\lim_{n\to\infty}\frac{n}{\ln n}=\lim_{n\to\infty}\frac{1}{1/n}=\infty$$

(by L'Hôpital's Rule) Diverges by Theorem 8.9

**63.** See definition, page 567.

**65.** The series given by

$$\sum_{n=0}^{\infty}ar^n=a+ar+ar^2+\cdots+ar^n+\cdots,\ a\neq 0$$

is a geometric series with ratio $r$. When $0<|r|<1$, the series converges to $\dfrac{a}{1-r}$. The series diverges if $|r|\geq 1$.

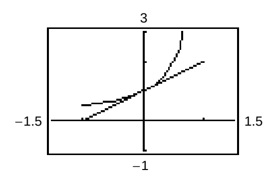
**67.** (a) $x$ is the common ratio.

(b) $1+x+x^2+\cdots=\displaystyle\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$, $|x|<1$

Geometric series: $a=1$, $r=x$, $|x|<1$

(c) $y_1=\dfrac{1}{1-x}$

$y_2=1+x$

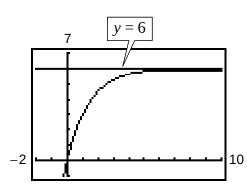



**69.** $f(x)=3\left[\dfrac{1-0.5^x}{1-0.5}\right]$

Horizontal asymptote: $y=6$

$$\sum_{n=0}^{\infty}3\left(\frac{1}{2}\right)^n$$

$$S=\frac{3}{1-(1/2)}=6$$




The horizontal asymptote is the sum of the series. $f(n)$ is the $n^{\text{th}}$ partial sum.

**71.** $\dfrac{1}{n(n+1)} < 0.001$

$$10{,}000 < n^2 + n$$

$$0 < n^2 + n - 10{,}000$$

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10{,}000)}}{2}$$

Choosing the positive value for $n$ we have $n \approx 99.5012$. The first *term* that is less than 0.001 is $n = 100$.

$$\left(\frac{1}{8}\right)^n < 0.001$$

$$10{,}000 < 8^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

**73.** $\displaystyle\sum_{i=0}^{n-1} 8000(0.9)^i = \frac{8000\left[1 - (0.9)^{(n-1)+1}\right]}{1 - 0.9}$

$$= 80{,}000(1 - 0.9^n),\ n > 0$$

**75.** $\displaystyle\sum_{i=0}^{n-1} 100(0.75)^i = \frac{100\left[1 - 0.75^{(n-1)+1}\right]}{1 - 0.75}$

$= 400(1 - 0.75^n)$ million dollars.

Sum $= 400$ million dollars

**77.** $D_1 = 16$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$

$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$

$$\vdots$$

$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots = -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81} = 152.42 \text{ ft}$$

**79.** $P(n) = \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n$

$$P(2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - (1/2)} = 1$$

**81.** (a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1}{2}\,\frac{1}{(1 - (1/2))} = 1$

(b) No, the series is not geometric.

(c) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$

**83.** Present Value $\displaystyle= \sum_{n=1}^{19} 50{,}000\left(\frac{1}{1.06}\right)^n$

$$= \sum_{n=0}^{18} \frac{50{,}000}{1.06}\left(\frac{1}{1.06}\right)^n, \quad r = \frac{1}{1.06}$$

$$= \frac{50{,}000}{1.06}\left(\frac{1 - 1.06^{-19}}{1 - 1.06^{-1}}\right)$$

$$\approx \$557{,}905.82$$

The present value is less than \$1,000,000. After accruing interest over 20 years, it attains its full value.

**85.** $\displaystyle w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1 - 2^n)}{1 - 2} = 0.01(2^n - 1)$

(a) When $n = 29$: $w = \$5{,}368{,}709.11$

(b) When $n = 30$: $w = \$10{,}737{,}418.23$

(c) When $n = 31$: $w = \$21{,}474{,}836.47$

**87.** $P = 50, r = 0.03, t = 20$

(a) $A = 50\left(\frac{12}{0.03}\right)\left[\left(1 + \frac{0.03}{12}\right)^{12(20)} - 1\right] \approx \$16{,}415.10$

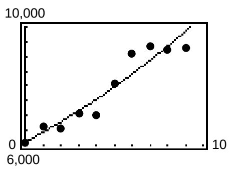
(b) $A = \frac{50 - (e^{0.03(20)} - 1)}{e^{0.03/12} - 1} \approx \$16{,}421.83$

**89.** $P = 100, r = 0.04, t = 40$

(a) $A = 100\left(\frac{12}{0.04}\right)\left[\left(1 + \frac{0.04}{12}\right)^{12(40)} - 1\right] \approx \$118{,}196.13$

(b) $A = \frac{100(e^{0.04(40)} - 1)}{e^{0.04/12} - 1} \approx \$118{,}393.43$

**91.** (a) $a_n = 6110.1832(1.0544)^x = 6110.1832e^{0.05297n}$




(b) 78,530 or \$78,530,000,000

(c) Total $= \sum_{n=0}^{9} a_n \approx 78{,}449$ or \$78,449,000,000

**93.** $x = 0.749999\ldots = 0.74 + \sum_{n=0}^{\infty} 0.009(0.1)^n$

$= 0.74 + \frac{0.009}{1 - 0.1}$

$= 0.74 + 0.01 = 0.75$

**95.** By letting $S_0 = 0$, we have $a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}$. Thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (S_n - S_{n-1}) = \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c) = \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)].$$

**97.** Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum(a_n + b_n) = \sum_{n=0}^{\infty}[1 + (-1)] = \sum_{n=0}^{\infty}[1 - 1] = 0$$

**99.** False. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**101.** False

$$\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1 - r}\right) - a$$

The formula requires that the geometric series begins with $n = 0$.

**103.** Let $H$ represent the half-life of the drug. If a patient receives $n$ equal doses of $P$ units each of this drug, administered at equal time interval of length $t$, the total amount of the drug in the patient's system at the time the last dose is administered is given by

$$T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt}$$

where $k = -(\ln 2)/H$. One time interval *after* the last dose is administered is given by

$$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}.$$

Two time intervals *after* the last dose is administered is given by

$$T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \cdots + Pe^{(n+1)kt}$$

and so on. Since $k < 0$, $T_{n+s} \to 0$ as $s \to \infty$, where $s$ is an integer.

## Section 8.3 The Integral Test and $p$-Series

**1.** $\sum_{n=1}^{\infty} \frac{1}{n+1}$

Let $f(x) = \frac{1}{x+1}$.

$f$ is positive, continuous and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x+1}\,dx = \Big[\ln(x+1)\Big]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**3.** $\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x}\,dx = \Big[-e^{-x}\Big]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 8.10

**5.** $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \frac{1}{x^2+1}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 8.10

**7.** $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let $f(x) = \frac{\ln(x+1)}{x+1}$

$f$ is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{1-\ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \left[\frac{\ln^2(x+1)}{2}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**9.** $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+c}$

Let $f(x) = \frac{x^{k-1}}{x^k+c}$.

$f$ is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$ since

$$f'(x) = \frac{x^{k-2}[c(k-1)-x^k]}{(x^k+c)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k+c}\,dx = \left[\frac{1}{k}\ln(x^k+c)\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**11.** $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \frac{1}{x^3}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 8.10

**13.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent $p$-series with $p = \frac{1}{5} < 1$

**15.** $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent $p$-series with $p = \frac{1}{2} < 1$

**17.** $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent $p$-series with $p = \frac{3}{2} > 1$

**19.** $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent $p$-series with $p = 1.04 > 1$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}} = \frac{2}{1} + \frac{2}{2^{3/4}} + \frac{2}{3^{3/4}} + \cdots$

$S_1 = 2$

$S_2 \approx 3.189$

$S_3 \approx 4.067$

Matches (a)

Diverges—$p$-series with $p = \frac{3}{4} < 1$

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = 2 + 2/2^{3/2} + 2/3^{3/2} + \cdots$

$S_1 = 2$

$S_2 \approx 2.707$

$S_3 \approx 3.092$

Matches (b)

Converges—$p$-series with $p = 3/2 > 1$

**25.** No. Theorem 8.9 says that if the series converges, then the terms $a_n$ tend to zero. Some of the series in Exercises 21-24 converge because the terms tend to 0 very rapidly.

**27.** $\displaystyle\sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$

(a)

| $M$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $N$ | 4 | 31 | 227 | 1674 |

(b) No. Since the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

**29.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_2^{\infty} (\ln x)^{-p}\frac{1}{x}\,dx = \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$.

**31.** Let $f$ be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge (Theorem 8.10). See Example 1, page 578.

**33.** Your friend is not correct. The series

$$\sum_{n=10{,}000}^{\infty} \frac{1}{n} = \frac{1}{10{,}000} + \frac{1}{10{,}001} + \cdots$$

is the harmonic series, starting with the 10,000th term, and hence diverges.

**35.** Since $f$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, we have,

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n > 0.$$

Also, $\displaystyle R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n \leq a_{N+1} + \int_{N+1}^{\infty} f(x)\,dx \leq \int_N^{\infty} f(x)\,dx$. Thus,

$$0 \leq R_N \leq \int_N^{\infty} f(x)\,dx.$$

**37.** $\displaystyle S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$

$$R_6 \leq \int_6^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_6^{\infty} \approx 0.0015$$

$$1.0811 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.0811 + 0.0015 = 1.0826$$

**39.** $S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$

$$R_{10} = \leq \int_{10}^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq 0.9818 + 0.0997 = 1.0815$$

**41.** $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$$R_4 \leq \int_4^{\infty} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_4^{\infty} = 5.6 \times 10^{-8}$$

$$0.4049 \leq \sum_{n=1}^{\infty} ne^{-n^2} \leq 0.4049 + 5.6 \times 10^{-8}$$

**43.** $0 \leq R_N \leq \int_N^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_N^{\infty} = \frac{1}{3N^3} < 0.001$

$$\frac{1}{N^3} < 0.003$$

$$N^3 > 333.33$$

$$N > 6.93$$

$$N \geq 7$$

**45.** $R_N \leq \int_N^{\infty} e^{-5x}\,dx = \left[-\frac{1}{5}e^{-5x}\right]_N^{\infty} = \frac{e^{-5N}}{5} < 0.001$

$$\frac{1}{e^{5N}} < 0.005$$

$$e^{5N} > 200$$

$$5N > \ln 200$$

$$N > \frac{\ln 200}{5}$$

$$N > 1.0597$$

$$N \geq 2$$

**47.** $R_N \leq \int_N^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_N^{\infty}$

$$= \frac{\pi}{2} - \arctan N < 0.001$$

$$-\arctan N < -1.5698$$

$$\arctan N > 1.5698$$

$$N > \tan 1.5698$$

$$N \geq 1004$$

**49.** (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$. This is a convergent $p$-series with $p = 1.1 > 1$.

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is a divergent series. Use the Integral Test.

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \Big[\ln|\ln x|\Big]_2^{\infty} = \infty$$

(b) $\sum_{n=2}^{6} \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$

$\sum_{n=2}^{6} \frac{1}{n \ln n} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \frac{1}{5\ln 5} + \frac{1}{6\ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$

The terms of the convergent series **seem** to be larger than those of the divergent series!

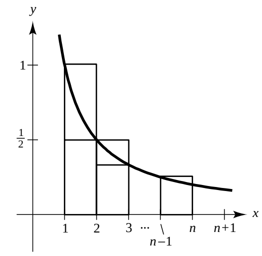
(c) $\frac{1}{n^{1.1}} < \frac{1}{n \ln n}$

$$n \ln n < n^{1.1}$$

$$\ln n < n^{0.1}$$

This inequality holds when $n \geq 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$.

**51.** (a) Let $f(x) = 1/x$. $f$ is positive, continuous, and decreasing on $[1, \infty)$.




$$S_n - 1 \leq \int_1^n \frac{1}{x}\,dx$$

$$S_n - 1 \leq \ln n$$

Hence, $S_n \leq 1 + \ln n$. Similarly,

$$S_n \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1).$$

Thus, $\ln(n+1) \leq S_n \leq 1 + \ln n$.

(b) Since $\ln(n+1) \leq S_n \leq 1 + \ln n$, we have $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$. Also, since $\ln x$ is an increasing function, $\ln(n+1) - \ln n > 0$ for $n \geq 1$. Thus, $0 \leq S_n - \ln n \leq 1$ and the sequence $\{a_n\}$ is bounded.

(c) $a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \displaystyle\int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \geq 0$

Thus, $a_n \geq a_{n+1}$ and the sequence is decreasing.

(d) Since the sequence is bounded and monotonic, it converges to a limit, $\gamma$.

(e) $a_{100} = S_{100} - \ln 100 \approx 0.5822$ (Actually $\gamma \approx 0.577216$.)

**53.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Let $f(x) = \dfrac{1}{2x-1}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x-1}\,dx = \Big[\ln \sqrt{2x-1}\Big]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**55.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

$p$-series with $p = \frac{5}{4}$

Converges by Theorem 8.11

**57.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 8.6

**59.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 8.9

**61.** $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

Fails $n$th Term Test

Diverges by Theorem 8.9

**63.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

Let $f(x) = \dfrac{1}{x(\ln x)^3}$.

$f$ is positive, continuous and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_2^{\infty} (\ln x)^{-3}\frac{1}{x}\,dx = \left[\frac{(\ln x)^{-2}}{-2}\right]_2^{\infty} = \left[-\frac{1}{2(\ln x)^2}\right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

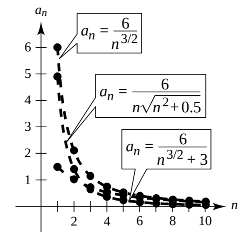
Converges by Theorem 8.10. See Exercise 13.

## Section 8.4 Comparisons of Series

**1.** (a) $\displaystyle\sum_{n=1}^{\infty}\frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots \quad S_1 = 6$

$$\sum_{n=1}^{\infty}\frac{6}{n^{3/2}+3} = \frac{6}{4} + \frac{6}{2^{3/2}+3} + \cdots \quad S_1 = \frac{3}{2}$$

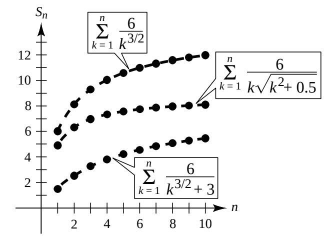
$$\sum_{n=1}^{\infty}\frac{6}{n\sqrt{n^2+0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots \quad S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$$




(b) The first series is a $p$-series. It converges $(p = 3/2 > 1)$.

(c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent $p$-series. Hence, the other two series converge.

(d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.




**3.** $\dfrac{1}{n^2+1} < \dfrac{1}{n^2}$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

converges by comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

**5.** $\dfrac{1}{n-1} > \dfrac{1}{n}$ for $n \geq 2$

Therefore,

$$\sum_{n=2}^{\infty}\frac{1}{n-1}$$

diverges by comparison with the divergent $p$-series

$$\sum_{n=2}^{\infty}\frac{1}{n}.$$

**7.** $\dfrac{1}{3^n+1} < \dfrac{1}{3^n}$

Therefore,

$$\sum_{n=0}^{\infty}\frac{1}{3^n+1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n.$$

**9.** For $n \geq 3$, $\dfrac{\ln n}{n+1} > \dfrac{1}{n+1}$.

Therefore,

$$\sum_{n=1}^{\infty}\frac{\ln n}{n+1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty}\frac{1}{n+1}.$$

**Note:** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+1}$ diverges by the integral test.

**11.** For $n > 3$, $\dfrac{1}{n^2} > \dfrac{1}{n!}$.

Therefore,

$$\sum_{n=0}^{\infty}\frac{1}{n!}$$

converges by comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

**13.** $\dfrac{1}{e^{n^2}} \leq \dfrac{1}{e^n}$

Therefore,

$$\sum_{n=0}^{\infty}\frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n.$$

**15.** $\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**17.** $\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**19.** $\lim_{n\to\infty} \frac{\frac{2n^2-1}{3n^5+2n+1}}{1/n^3} = \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1} = \frac{2}{3}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$$

converges by a limit comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

**21.** $\lim_{n\to\infty} \frac{\frac{n+3}{n(n+2)}}{1/n} = \lim_{n\to\infty} \frac{n^2+3n}{n^2+2n} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n+3}{n(n+2)}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**23.** $\lim_{n\to\infty} \frac{1/\left(n\sqrt{n^2+1}\right)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2+1}} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$$

converges by a limit comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

**25.** $\lim_{n\to\infty} \frac{(n^{k-1})/(n^k+1)}{1/n} = \lim_{n\to\infty} \frac{n^k}{n^k+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**27.** $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1/n^2)\cos(1/n)}{-1/n^2}$

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**29.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Diverges

$p$-series with $p = \frac{1}{2}$

**31.** $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

Converges

Direct comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

**33.** $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

Diverges; $n$th Term Test

$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

**35.** $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

Converges; integral test

**37.** $\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} na_n$ by given conditions $\lim_{n\to\infty} na_n$ is finite and nonzero.

Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**39.** $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$,

which diverges since the degree of the numerator is only one less than the degree of the denominator.

**41.** $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

converges since the degree of the numerator is three less than the degree of the denominator.

**43.** $\lim_{n\to\infty} n\left(\frac{n^3}{5n^4+3}\right) = \lim_{n\to\infty} \frac{n^4}{5n^4+3} = \frac{1}{5} \neq 0$

Therefore,

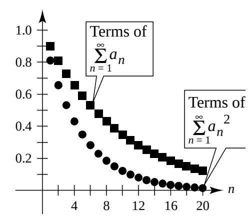
$$\sum_{n=1}^{\infty} \frac{n^3}{5n^4+3} \text{ diverges.}$$

**45.** See Theorem 8.12, page 583. One example is $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges because

$$\frac{1}{n^2+1} < \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges ($p$-series).

**47.**




For $0 < a_n < 1$, $0 < a_n^2 < a_n < 1$. Hence, the lower terms are those of $\Sigma\, a_n^2$.

**49.** $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$, diverges

**51.** $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200+n^2}$, converges

**53.** Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

**55.** False. Let $a_n = 1/n^3$ and $b_n = 1/n^2$. $0 < a_n \le b_n$ and both

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge.

**57.** True

**59.** Since $\sum_{n=1}^{\infty} b_n$ converges, $\lim_{n\to\infty} b_n = 0$. There exists $N$ such that $b_n < 1$ for $n > N$. Thus,

$$a_n b_n < a_n \text{ for } n > N \text{ and } \sum_{n=1}^{\infty} a_n b_n$$

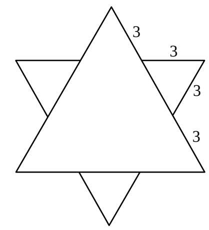
converges by comparison to the convergent series $\sum_{i=1}^{\infty} a_n$.

**61.** $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both converge, and hence so does $\sum\left(\frac{1}{n^2}\right)\left(\frac{1}{n^3}\right) = \sum \frac{1}{n^5}$.

**63.** (a) Suppose $\Sigma b_n$ converges and $\Sigma a_n$ diverges. Then there exists $N$ such that $0 < b_n < a_n$ for $n \geq N$. This means that $1 < a_n/b_n$ for $n \geq N$. Therefore, $\lim_{n\to\infty} a_n/b_n \neq 0$. Thus, $\Sigma a_n$ must also converge.

(b) Suppose $\Sigma b_n$ diverges and $\Sigma a_n$ converges. Then there exists $N$ such that $0 < a_n < b_n$ for $n \geq N$. This means that $0 < a_n/b_n < 1$ for $n \geq N$. Therefore, $\lim_{n\to\infty} a_n/b_n \neq \infty$. Thus, $\Sigma a_n$ must also diverge.

**65.** Start with one triangle whose sides have length 9. At the $n$th step, each side is replaced by four smaller line segments each having $\frac{1}{3}$ the length of the original side.




| #Sides | Length of sides |
|---|---|
| $3$ | $9$ |
| $3 \cdot 4$ | $9\left(\frac{1}{3}\right)$ |
| $3 \cdot 4^2$ | $9\left(\frac{1}{3}\right)^2$ |
| $\vdots$ | |
| $3 \cdot 4^n$ | $9\left(\frac{1}{3}\right)^n$ |

At the $n$th step there are $3 \cdot 4^n$ sides, each of length $9\left(\frac{1}{3}\right)^n$. At the next step, there are $3 \cdot 4^n$ new triangles of side $9\left(\frac{1}{3}\right)^{n+1}$. The area of an equilateral triangle of side $x$ is $\frac{1}{4}\sqrt{3}\,x^2$. Thus, the new triangles each have area

$$9\frac{\sqrt{3}}{4}\left(\frac{1}{3^{n+1}}\right)^2 = \frac{\sqrt{3}}{4}\frac{1}{3^{2n}}.$$

The area of the $3 \cdot 4^n$ new triangles is

$$(3 \cdot 4^n)\left(\frac{\sqrt{3}}{4}\frac{1}{3^{2n}}\right) = \frac{3\sqrt{3}}{4}\left(\frac{4}{9}\right)^n.$$

The total area is the infinite sum

$$\frac{9\sqrt{3}}{4} + \sum_{n=0}^{\infty}\frac{3\sqrt{3}}{4}\left(\frac{4}{9}\right)^n = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{1}{1-4/9}\right) = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4}\left(\frac{9}{5}\right) = \frac{18\sqrt{3}}{5}.$$

The perimeter is infinite, since at step $n$ there are $3 \cdot 4^n$ sides of length $9\left(\frac{1}{3}\right)^n$. Thus, the perimeter at step $n$ is $27\left(\frac{4}{3}\right)^n \to \infty$.

## Section 8.5 Alternating Series

**1.** $\displaystyle\sum_{n=1}^{\infty}\frac{6}{n^2} = \frac{6}{1} + \frac{6}{4} + \frac{6}{9} + \cdots$

$S_1 = 6,\ S_2 = 7.5$

Matches (b)

**3.** $\displaystyle\sum_{n=1}^{\infty}\frac{10}{n2^n} = \frac{10}{2} + \frac{10}{8} + \cdots$

$S_1 = 5,\ S_2 = 6.25$

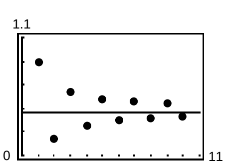
Matches (c)

**5.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0.6667 | 0.8667 | 0.7238 | 0.8349 | 0.7440 | 0.8209 | 0.7543 | 0.8131 | 0.7605 |

(b)




(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

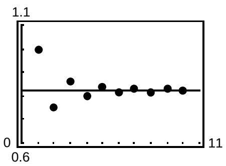
(d) The distance in part (c) is always less than the magnitude of the next term of the series.

**7.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0.75 | 0.8611 | 0.7986 | 0.8386 | 0.8108 | 0.8312 | 0.8156 | 0.8280 | 0.8180 |

(b)




(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

**9.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Converges by Theorem 8.14.

**11.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

$$a_{n+1} = \frac{1}{2(n+1)-1} < \frac{1}{2n-1} = a_n$$

$$\lim_{n\to\infty} \frac{1}{2n-1} = 0$$

Converges by Theorem 8.14

**13.** $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Diverges by the $n$th Term Test

**15.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Theorem 8.14

**17.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

$$\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1/(n+1)} = \lim_{n\to\infty} (n+1) = \infty$$

Diverges by the $n$th Term Test

**19.** $\sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$

Diverges by the $n$th Term Test

**21.** $\sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$

Diverges by the $n$th Term Test

**23.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n!} = 0$$

Converges by Theorem 8.14

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

$a_{n+1} = \dfrac{\sqrt{n+1}}{(n+1)+2} < \dfrac{\sqrt{n}}{n+2}$ for $n \geq 2$

$\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$

Converges by Theorem 8.14

**27.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} - 1}$

Let $f(x) = \dfrac{2e^x}{e^{2x} - 1}$. Then

$$f'(x) = \frac{-2e^x(e^{2x} + 1)}{(e^{2x} - 1)^2} < 0.$$

Thus, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n} - 1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 8.14.

**29.** $S_6 = \displaystyle\sum_{n=1}^{6} \frac{3(-1)^{n+1}}{n^2} = 2.4325$

$|R_6| = |S - S_6| \leq a_7 = \dfrac{3}{49} \approx 0.0612;\ 2.3713 \leq S \leq 2.4937$

**31.** $S_6 = \displaystyle\sum_{n=0}^{5} \frac{2(-1)^n}{n!} \approx 0.7333$

$|R_6| = |S - S_6| \leq a_7 = \dfrac{2}{6!} = 0.002778;\ 0.7305 \leq S \leq 0.7361$

**33.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when $N = 6$.

(b) We may approximate the series by

$$\sum_{n=0}^{6} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368.$$

(7 terms. Note that the sum begins with $n = 0$.)

**35.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when $N = 2$.

(b) We may approximate the series by

$$\sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with $n = 0$.)

**37.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when $N = 1000$.

(b) We may approximate the series by

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} \approx 0.693.$$

(1000 terms)

**39.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$

By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when $N = 7$.

**41.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by comparison to the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converge absolutely.

**43.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

The given series converges by the Alternating Series Test, but does not converge absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent $p$-series. Therefore, the series converges conditionally.

**45.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the $n$th Term Test.

**47.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges by comparison to the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, the series converges conditionally.

**49.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3-1}$

$$\sum_{n=2}^{\infty} \frac{n}{n^3-1}$$

converges by a limit comparison to the convergent $p$-series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converges absolutely.

**51.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

since

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

**53.** $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$\displaystyle\lim_{n\to\infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1$, therefore the series converges conditionally.

**55.** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p$-series. Therefore, the given series converges absolutely.

**57.** An alternating series is a series whose terms alternate in sign. See Theorem 8.14.

**59.** $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges.

**61.** (b). The partial sums alternate above and below the horizontal line representing the sum.

**63.** Since $\sum_{n=1}^{\infty} |a_n|$ converges we have

$$\lim_{n\to\infty} |a_n| = 0.$$

Thus, there must exist an $N > 0$ such that $|a_N| < 1$ for all $n > N$ and it follows that $a_n^2 \le |a_n|$ for all $n > N$. Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let $a_n = 1/n$ to see that the converse is false.

**65.** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, hence so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

**67.** False

Let $a_n = \dfrac{(-1)^n}{n}$.

**69.** $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergent $p$-series

**71.** Diverges by $n$th Term Test. $\lim_{n\to\infty} a_n = \infty$

**73.** Convergent Geometric Series $\left(r = \frac{7}{8} < 1\right)$

**75.** Convergent Geometric Series $\left(r = \dfrac{1}{\sqrt{e}}\right)$ or Integral Test

**77.** Converges (absolutely) by Alternating Series Test

**79.** The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

## Section 8.6 The Ratio and Root Tests

**1.** 
$$\frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!}$$
$$= (n+1)(n)(n-1)$$

**3.** Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid since $1 = \dfrac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n-1)(2n+1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}.$$

**—CONTINUED—**

**3. —CONTINUED—**

To do this, note that:

$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) &= [1 \cdot 3 \cdot 5 \cdots (2n-1)](2n+1) \\ &= \frac{(2n)!}{2^n n!} \cdot (2n+1) \\ &= \frac{(2n)!(2n+1)}{2^n n!} \cdot \frac{(2n+2)}{2(n+1)} \\ &= \frac{(2n)!(2n+1)(2n+2)}{2^{n+1} n!(n+1)} \\ &= \frac{(2n+2)!}{2^{n+1}(n+1)} \end{aligned}$$

The formula is valid for all $n \geq 1$.

**5.** $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots$

$S_1 = \frac{3}{4}, S_2 \approx 1.875$

Matches (d)

**7.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \cdots$

$S_1 = 9$

Matches (f)

**9.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots$

$S_1 = 2$

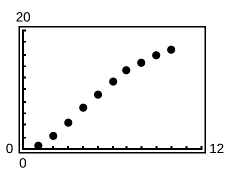
Matches (a)

**11.** (a) Ratio Test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2(5/8)^{n+1}}{n^2(5/8)^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{5}{8} = \frac{5}{8} < 1$. Converges

(b)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 9.2104 | 16.7598 | 18.8016 | 19.1878 | 19.2491 |

(c)




(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

**13.** $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{3} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

**15.** $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_n + 1}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)(3/4)^{n+1}}{n(3/4)^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{3(n+1)}{4n}\right| = \frac{3}{4} \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

**17.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n}\right| \\ &= \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = 2 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

**21.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|$$
$$= \lim_{n\to\infty}\frac{2}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)3^{n+1}}\cdot\frac{n3^n}{n!}\right|$$
$$= \lim_{n\to\infty}\frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

**25.** $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)!}\cdot\frac{n!}{4^n}\right|$$
$$= \lim_{n\to\infty}\frac{4}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

**27.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{3^n}\right| = \lim_{n\to\infty}\frac{3(n+1)^n}{(n+2)^{n+1}} = \lim_{n\to\infty}\frac{3}{n+2}\left(\frac{n+1}{n+2}\right)^n = (0)\left(\frac{1}{e}\right) = 0$$

To find $\displaystyle\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$, let $\displaystyle y = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$. Then,

$$\ln y = \lim_{n\to\infty} n\ln\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty}\frac{\ln[(n+1)/(n+2)]}{1/n} = \frac{0}{0}$$

$$\ln y = \lim_{n\to\infty}\frac{[(1)/(n+1)] - [(1)/(n+2)]}{-(1/n^2)} = -1 \text{ by L'Hôpital's Rule}$$

$$y = e^{-1} = \frac{1}{e}.$$

Therefore, by the Ratio Test, the series converges.

**29.** $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{3^{n+1}+1}\cdot\frac{3^n+1}{4^n}\right| = \lim_{n\to\infty}\frac{4(3^n+1)}{3^{n+1}+1} = \lim_{n\to\infty}\frac{4(1+1/3^n)}{3+1/3^n} = \frac{4}{3}$$

Therefore, by the Ratio Test, the series diverges.

**31.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n+1)}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are $\displaystyle -1 + \frac{1}{1\cdot 3} - \frac{2!}{1\cdot 3\cdot 5} + \frac{3!}{1\cdot 3\cdot 5\cdot 7} - \cdots$

**33.** (a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{3/2} = 1$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{1/2} = 1$$

**35.** $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

**37.** $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|}$$

$$= \lim_{n\to\infty} \frac{1}{|\ln n|} = 0$$

Therefore, by the Root Test, the series converges.

**39.** $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(2\sqrt[n]{n}+1\right)^n} = \lim_{n\to\infty} \left(2\sqrt[n]{n}+1\right)$$

To find $\lim_{n\to\infty} \sqrt[n]{n}$, let $y = \lim_{n\to\infty} \sqrt[x]{x}$. Then

$$\ln y = \lim_{n\to\infty} \left(\ln \sqrt[x]{x}\right) = \lim_{n\to\infty} \frac{1}{x} \ln x = \lim_{n\to\infty} \frac{\ln x}{x} = \lim_{n\to\infty} \frac{1/x}{1} = 0.$$

Thus, $\ln y = 0$, so $y = e^0 = 1$ and $\lim_{n\to\infty} \left(2\sqrt[n]{n}+1\right) = 2(1) + 1 = 3$. Therefore, by the Root Test, the series diverges.

**41.** $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

**43.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n}$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n\to\infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

**45.** $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is convergent $p$-series.

**47.** $\sum_{n=1}^{\infty} \frac{2n}{n+1}$

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2 \neq 0$$

This diverges by the $n$th Term Test for Divergence.

**49.** $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^n$

Since $|r| = \frac{3}{2} > 1$, this is a divergent geometric series.

**51.** $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

$$\lim_{n\to\infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n\to\infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a limit comparison test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

**53.** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$

$$\left|\frac{\cos(n)}{2^n}\right| \le \frac{1}{2^n}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{2^n}\right|$$

converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

**55.** $\displaystyle\sum_{n=1}^{\infty} \frac{n7^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n}\right| = \lim_{n\to\infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

**57.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 3^{n-1}}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}}\right| = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

**59.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n}\right| = \lim_{n\to\infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

**61.** (a) and (c)

$$\sum_{n=1}^{\infty} \frac{n5^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!}$$

$$= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots$$

**63.** (a) and (b) are the same.

**65.** Replace $n$ with $n+1$.

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}}$$

**67.** Since

$$\frac{3^{10}}{2^{10}\, 10!} = 1.59 \times 10^{-5},$$

use 9 terms.

$$\sum_{k=1}^{9} \frac{(-3)^k}{2^k\, k!} \approx -0.7769$$

**69.** See Theorem 8.17, page 597.

**71.** No. Let $a_n = \dfrac{1}{n+10{,}000}$.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+10{,}000}$ diverges.

**73.** The series converges absolutely. See Theorem 8.17.

**75.** First, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$$

and choose $R$ such that $0 \le r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. Thus, for $n > N$, we $|a_n| < R^n$ and since the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, we can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for all $n > M$. Thus, for $n > M$, we have $|a_n| > R_n > 1$ which implies that $\lim_{n\to\infty} a_n \neq 0$ which in turn implies that

$\sum_{n=1}^{\infty} a_n$ diverges.

## Section 8.7 Taylor Polynomials and Approximations

**1.** $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

**3.** $y = e^{-1/2}[(x + 1) + 1]$

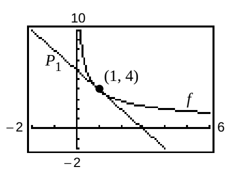
Linear

Matches (a)

**5.** $f(x) = \dfrac{4}{\sqrt{x}} = 4x^{-1/2}$ $\qquad f(1) = 4$

$f'(x) = -2x^{-3/2}$ $\qquad f'(1) = -2$

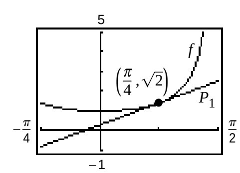
$$P_1(x) = f(1) + f'(1)(x - 1)$$
$$= 4 + (-2)(x - 1)$$
$$P_1(x) = -2x + 6$$




**7.** $f(x) = \sec x$ $\qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x$ $\qquad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$
$$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

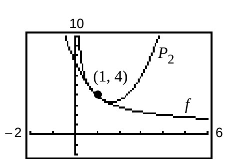



**9.** $f(x) = \dfrac{4}{\sqrt{x}} = 4x^{-1/2}$ $\qquad f(1) = 4$

$f'(x) = -2x^{-3/2}$ $\qquad f'(1) = -2$

$f''(x) = 3x^{-5/2}$ $\qquad f''(1) = 3$

$$P_2 = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2$$
$$= 4 - 2(x - 1) + \frac{3}{2}(x - 1)^2$$




| $x$ | 0 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 2 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | Error | 4.4721 | 4.2164 | 4.0 | 3.8139 | 3.6515 | 2.8284 |
| $P_2(x)$ | 7.5 | 4.46 | 4.215 | 4.0 | 3.815 | 3.66 | 3.5 |

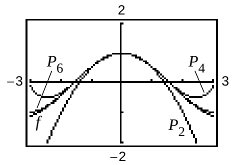
**11.** $f(x) = \cos x$

$P_2(x) = 1 - \frac{1}{2}x^2$

$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

(a)




(b) $f'(x) = -\sin x \qquad P_2'(x) = -x$

$f''(x) = -\cos x \qquad P_2''(x) = -1$

$f''(0) = P_2''(0) = -1$

$f'''(x) = \sin x \qquad P_4'''(x) = x$

$f^{(4)}(x) = \cos x \qquad P_4^{(4)}(x) = 1$

$f^{(4)}(0) = 1 = P_4^{(4)}(0)$

$f^{(5)}(x) = -\sin x \qquad P_6^{(5)}(x) = -x$

$f^{(6)}(x) = -\cos x \qquad P^{(6)}(x) = -1$

$f^{(6)}(0) = -1 = P_6^{(6)}(0)$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all $n$.

**13.** $f(x) = e^{-x} \qquad f(0) = 1$

$f'(x) = -e^{-x} \qquad f'(0) = -1$

$f''(x) = e^{-x} \qquad f''(0) = 1$

$f'''(x) = -e^{-x} \qquad f'''(0) = -1$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

**15.** $f(x) = e^{2x} \qquad f(0) = 1$

$f'(x) = 2e^{2x} \qquad f'(0) = 2$

$f''(x) = 4e^{2x} \qquad f''(0) = 4$

$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$

$f^{(4)}(x) = 16^{2x} \qquad f^{(4)}(0) = 16$

$$P_4(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

**17.** $f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

$$P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

**19.** $f(x) = xe^x \qquad f(0) = 0$

$f'(x) = xe^x + e^x \qquad f'(0) = 1$

$f''(x) = xe^x + 2e^x \qquad f''(0) = 2$

$f'''(x) = xe^x + 3e^x \qquad f'''(0) = 3$

$f^{(4)}(x) = xe^x + 4e^x \qquad f^{(4)}(0) = 4$

$$P_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

**21.** $f(x) = \dfrac{1}{x+1} \qquad f(0) = 1$

$f'(x) = -\dfrac{1}{(x+1)^2} \qquad f'(0) = -1$

$f''(x) = \dfrac{2}{(x+1)^2} \qquad f''(0) = 2$

$f'''(x) = \dfrac{-6}{(x+1)^4} \qquad f'''(0) = -6$

$f^{(4)}(x) = \dfrac{24}{(x+1)^5} \qquad f^{(4)}(0) = 24$

$$P_4(x) = 1 - x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4$$

$$= 1 - x + x^2 - x^3 + x^4$$

**23.** $f(x) = \sec x \qquad f(0) = 1$

$f'(x) = \sec x \tan x \qquad f'(0) = 0$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''(0) = 1$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

**25.** $f(x) = \dfrac{1}{x}$ $\quad f(1) = 1$

$f'(x) = -\dfrac{1}{x^2}$ $\quad f'(1) = -1$

$f''(x) = \dfrac{2}{x^3}$ $\quad f''(1) = 2$

$f'''(x) = -\dfrac{6}{x^4}$ $\quad f'''(1) = -6$

$f^{(4)}(x) = \dfrac{24}{x^5}$ $\quad f^{(4)}(1) = 24$

$$P_4(x) = 1 - (x - 1) + \frac{2}{2!}(x - 1)^2 + \frac{-6}{3!}(x - 1)^3 + \frac{24}{4!}(x - 1)^4$$

$$= 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4$$

**27.** $f(x) = \sqrt{x}$ $\quad f(1) = 1$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $\quad f'(1) = \dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4x\sqrt{x}}$ $\quad f''(1) = -\dfrac{1}{4}$

$f'''(x) = \dfrac{3}{8x^2\sqrt{x}}$ $\quad f'''(1) = \dfrac{3}{8}$

$f^{(4)}(x) = -\dfrac{15}{16x^3\sqrt{x}}$ $\quad f^{(4)}(1) = -\dfrac{15}{16}$

$$P_4(x) = 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3 - \frac{5}{128}(x - 1)^4$$

**29.** $f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = \dfrac{1}{x}$ $\quad f'(1) = 1$

$f''(x) = -\dfrac{1}{x^2}$ $\quad f''(1) = -1$

$f'''(x) = \dfrac{2}{x^3}$ $\quad f'''(1) = 2$

$f^{(4)}(x) = -\dfrac{6}{x^4}$ $\quad f^{(4)}(1) = -6$

$$P_4(x) = 0 + (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$$

**31.** $f(x) = \tan x$

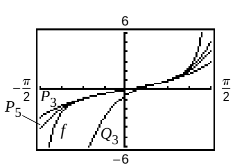
$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f^{(4)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$f^{(5)}(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x$




(a) $n = 3, c = 0$

$$P_3(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$

(b) $n = 5, c = 0$

$$P_5(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5$$

(c) $n = 3, c = \dfrac{\pi}{4}$

$$Q_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

**33.** $f(x) = \sin x$

$P_1(x) = x$

$P_3(x) = x - \frac{1}{6}x^3$

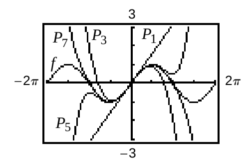
$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$P_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$

(a)

| $x$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $\sin x$ | 0.0000 | 0.2474 | 0.4794 | 0.6816 | 0.8415 |
| $P_1(x)$ | 0.0000 | 0.2500 | 0.5000 | 0.7500 | 1.0000 |
| $P_3(x)$ | 0.0000 | 0.2474 | 0.4792 | 0.6797 | 0.8333 |
| $P_5(x)$ | 0.0000 | 0.2474 | 0.4794 | 0.6817 | 0.8417 |
| $P_7(x)$ | 0.0000 | 0.2474 | 0.4794 | 0.6816 | 0.8415 |

(b)




(c) As the distance increases, the accuracy decreases

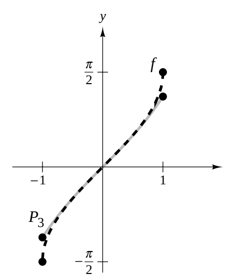
**35.** $f(x) = \arcsin x$

(a) $P_3(x) = x + \dfrac{x^3}{6}$

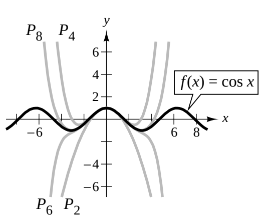
(b)

| $x$ | $-0.75$ | $-0.50$ | $-0.25$ | 0 | 0.25 | 0.50 | 0.75 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-0.848$ | $-0.524$ | $-0.253$ | 0 | 0.253 | 0.524 | 0.848 |
| $P_3(x)$ | $-0.820$ | $-0.521$ | $-0.253$ | 0 | 0.253 | 0.521 | 0.820 |

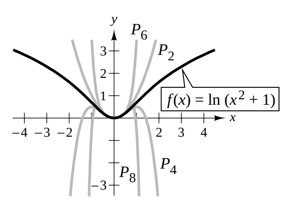
(c)




**37.** $f(x) = \cos x$




**39.** $f(x) = \ln(x^2 + 1)$




**41.** $f(x) = e^{-x} \approx 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$

$f\left(\dfrac{1}{2}\right) \approx 0.6042$

**43.** $f(x) = \ln x \approx (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

$f(1.2) \approx 0.1823$

**45.** $f(x) = \cos x$; $f^{(5)}(x) = -\sin x \implies$ Max on $[0, 0.3]$ is 1.

$R_4(x) \leq \dfrac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

**47.** $f(x) = \arcsin x;\ f^{(4)}(x) = \dfrac{x(6x^2+9)}{(1-x^2)^{7/2}} \Longrightarrow$ Max on $[0, 0.4]$ is $f^{(4)}(0.4) \approx 7.3340$.

$$R_3(x) \le \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$$

**49.** $g(x) = \sin x$

$g^{(n+1)}(x) \le 1$ for all $x$

$$R_n(x) \le \frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error, $n = 3$.

**51.** $f(x) = \ln(x+1)$

$f^{(n+1)}(x) = \dfrac{(-1)^{n+1}n!}{(x+1)^{n+1}} \Longrightarrow$ Max on $[0, 0.5]$ is $n!$.

$$R_n \le \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

**53.** $f(x) = e^x \approx 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6},\ x < 0$

$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$

$$e^z x^4 < 0.024$$

$$xe^{z/4} < 0.3936$$

$$x < \frac{0.3936}{e^{z/4}} < 0.3936,\ z < 0$$

$$-0.3936 < x < 0$$

**55.** The graph of the approximating polynomial $P$ and the elementary function $f$ both pass through the point $(c, f(c))$ and the slopes of $P$ and $f$ agree at $(c, f(c))$. Depending on the degree of $P$, the $n$th derivatives of $P$ and $f$ agree at $(c, f(c))$.

**57.** See definition on page 607.

**59.** The accuracy increases as the degree increases (for values within the interval of convergence).

**61.** (a) $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = x\,P_4(x)$$

(b) $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = x\,P_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c) $g(x) = \dfrac{\sin x}{x} = \dfrac{1}{x}P_5(x) = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!}$

**63.** (a) $Q_2(x) = -1 + \dfrac{\pi^2(x+2)^2}{32}$

(b) $R_2(x) = -1 + \dfrac{\pi^2(x-6)^2}{32}$

(c) No. The polynomial will be linear.
Translations are possible at $x = -2 + 8n$.

**65.** Let $f$ be an even function and $P_n$ be the $n$th Maclaurin polynomial for $f$. Since $f$ is even, $f'$ is odd, $f''$ is even, $f'''$ is odd, etc. (see Exercise 45). All of the odd derivatives of $f$ are odd and thus, all of the odd powers of $x$ will have coefficients of zero. $P_n$ will only have terms with even powers of $x$.

**67.** As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

## Section 8.8 Power Series

**1.** Centered at 0

**3.** Centered at 2

**5.** $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}$

$$L=\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^nx^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right||x|=|x|$$

$|x|<1\Longrightarrow R=1$

**7.** $\displaystyle\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2}$

$$L=\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(2x)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{2n^2x}{(n+1)^2}\right|=2|x|$$

$2|x|<1\Longrightarrow R=\frac{1}{2}$

**9.** $\displaystyle\sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!}$

$$L=\lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(2x)^{2n+2}/(2n+2)!}{(2x)^{2n}/(2n)!}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(2x)^2}{(2n+2)(2n+1)}\right|=0$$

Thus, the series converges for all $x$. $R=\infty$.

**11.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

Since the series is geometric, it converges only if $|x/2|<1$ or $-2<x<2$.

**13.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^nx^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\cdot\frac{n}{(-1)^nx^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{nx}{n+1}\right|=|x|$$

Interval: $-1<x<1$

When $x=1$, the alternating series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.

When $x=-1$, the $p$-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Therefore, the interval of convergence is $-1<x\le 1$.

**15.** $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0$$

The series converges for all $x$. Therefore, the interval of convergence is $-\infty<x<\infty$.

**17.** $\displaystyle\sum_{n=0}^{\infty}(2n)!\left(\frac{x}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(2n+2)!x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(2n)!x^n}\right|=\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x}{2}\right|=\infty$$

Therefore, the series converges only for $x=0$.

**19.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{4^n}$

Since the series is geometric, it converges only if $|x/4|<1$ or $-4<x<4$.

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-5)}{5(n+1)}\right| = \frac{1}{5}|x-5|$$

$R = 5$

Center: $x = 5$

Interval: $-5 < x - 5 < 5$ or $0 < x < 10$

When $x = 0$, the $p$-series $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 10$, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $0 < x \le 10$.

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-1)}{n+2}\right| = |x-1|$$

$R = 1$

Center: $x = 1$

Interval: $-1 < x - 1 < 1$ or $0 < x < 2$

When $x = 0$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $0 < x \le 2$.

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges.

When $x = 2c$, the series $\displaystyle\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $0 < x < 2c$.

**27.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-2x)(n+1)^2}{n(n+2)}\right| = 2|x|$$

$R = \dfrac{1}{2}$

Interval: $-\dfrac{1}{2} < x < \dfrac{1}{2}$

When $x = -\dfrac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the $n$th Term Test.

When $x = \dfrac{1}{2}$, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ diverges.

Therefore, the interval of convergence is $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

**29.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+3)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{k(k+1)\cdots(k+n-1)x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{k(k+1)\cdots(k+n-1)(k+n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k+1)\cdots(k+n-1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(k+n)x}{n+1}\right| = |x|$$

$R = 1$

When $x = \pm 1$, the series diverges and the interval of convergence is $-1 < x < 1$.

$$\left[\frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n} \ge 1\right]$$

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(-1)^{n+1}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(4n+3)(x-3)}{4}\right| = \infty$$

$R = 0$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

**35.** (a) $f(x) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n, -2 < x < 2$ (Geometric)

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{2}\right)\left(\frac{x}{2}\right)^{n-1}, -2 < x < 2$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty}\left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{x}{2}\right)^{n-2}, -2 < x < 2$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=0}^{\infty}\frac{2}{n+1}\left(\frac{x}{2}\right)^{n+1}, -2 \le x < 2$

**37.** (a) $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, 0 < x \le 2$

(b) $f'(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}(x-1)^n,\ 0 < x < 2$

(c) $f''(x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n(x-1)^{n-1}, 0 < x < 2$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, 0 \le x \le 2$

**39.** $g(1) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

$S_1 = 1, S_2 = 1.33$. Matches (c)

**41.** $g(3.1) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{3.1}{3}\right)^n$ diverges. Matches (b)

**43.** A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series centered at $c$.

**45.** A single point, $a_n$ interval, or the entire real line.

**47.** (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty$ (See Exercise 29.)

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty$

(b) $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$

(d) $f(x) = \sin x$ and $g(x) = \cos x$

**49.**

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!}$$

$$y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$$

$$\begin{aligned}
y'' - xy' - y &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\
&= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \left[\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)}\right] \\
&= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n}[(2n+1) - (2n+1)]}{2^{n+1}(n+1)!} = 0
\end{aligned}$$

**51.** $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$

(a) $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2}[(k+1)!]^2} \cdot \frac{2^{2k}(k!)^2}{(-1)^k x^{2k}}\right| = \lim_{k\to\infty} \left|\frac{(-1)x^2}{2^2(k+1)^2}\right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b)

$$J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k(k!)^2}$$

$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k(k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1}[(k+1)!]^2}$$

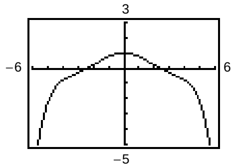
$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k(k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1}[(k+1)!]^2}$$

$$\begin{aligned}
x^2J_0'' + xJ_0' + x^2J_0 &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k(k!)^2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k(k!)^2}\left[(-1)\frac{2(2k+1)}{4(k+1)} + (-1)\frac{2}{4(k+1)} + 1\right] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k(k!)^2}\left[\frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4}\right] = 0
\end{aligned}$$

**—CONTINUED—**

**51. —CONTINUED—**

(c) $P_6(x) = 1 - \dfrac{x^2}{4} + \dfrac{x^4}{64} - \dfrac{x^6}{2304}$




(d) $\displaystyle\int_0^1 J_0\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2}\,dx$

$$= \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)}\right]_0^1$$

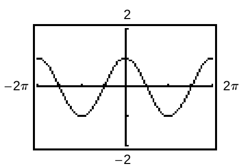
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)}$$

$$= 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92$$

(exact integral is 0.9197304101)

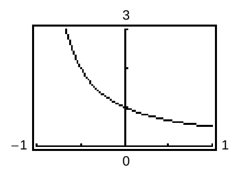
**53.** $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$

(See Exercise 47.)




**55.** $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$

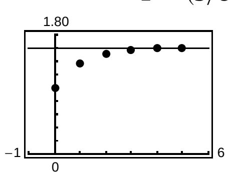
$$= \frac{1}{1-(-x)} = \frac{1}{1+x} \text{ for } -1 < x < 1$$




**57.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

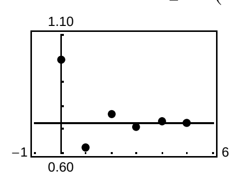
(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n$

$$= \frac{1}{1-(3/8)} = \frac{8}{5} = 1.6$$




(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$

$$= \frac{1}{1-(-3/8)} = \frac{8}{11} \approx 0.7272$$




(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d) $\displaystyle\sum_{n=0}^{N} \left(\frac{3}{2}\right)^n > M$

| $M$ | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|
| $N$ | 4 | 9 | 15 | 21 |

**59.** False;

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n 2^n}$$

converges for $x = 2$ but diverges for $x = -2$.

**61.** True; the radius of convergence is $R = 1$ for both series.

## Section 8.9 Representation of Functions by Power Series

**1.** (a) $\dfrac{1}{2-x} = \dfrac{1/2}{1-(x/2)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b)
$$\begin{array}{r l}
 & \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots \\ \cline{2-2}
2 - x \,\big) & 1 \\
 & \underline{1 - \frac{x}{2}} \\
 & \qquad \frac{x}{2} \\
 & \qquad \underline{\frac{x}{2} - \frac{x^2}{4}} \\
 & \qquad\qquad \frac{x^2}{4} \\
 & \qquad\qquad \underline{\frac{x^2}{4} - \frac{x^3}{8}} \\
 & \qquad\qquad\qquad \frac{x^3}{8} \\
 & \qquad\qquad\qquad \underline{\frac{x^3}{8} - \frac{x^4}{16}} \\
 & \qquad\qquad\qquad\qquad \vdots
\end{array}$$

**3.** (a) $\dfrac{1}{2+x} = \dfrac{1/2}{1-(-x/2)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b)
$$\begin{array}{r l}
 & \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\ \cline{2-2}
2 + x \,\big) & 1 \\
 & \underline{1 + \frac{x}{2}} \\
 & \qquad -\frac{x}{2} \\
 & \qquad \underline{-\frac{x}{2} - \frac{x^2}{4}} \\
 & \qquad\qquad \frac{x^2}{4} \\
 & \qquad\qquad \underline{\frac{x^2}{4} + \frac{x^3}{8}} \\
 & \qquad\qquad\qquad -\frac{x^3}{8} \\
 & \qquad\qquad\qquad \underline{-\frac{x^3}{8} - \frac{x^4}{16}} \\
 & \qquad\qquad\qquad\qquad \vdots
\end{array}$$

**5.** Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that $a = -1/3$ and $r = (-1/3)(x-5)$.

Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3}\left[-\frac{1}{3}(x-5)\right]^n$$

$$= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}},\ |x-5| < 3 \text{ or } 2 < x < 8.$$

**7.** Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that $a = -3$ and $r = 2x$.

Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n$$

$$= -3\sum_{n=0}^{\infty} (2x)^n,\ |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

**9.** Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{-1}{11-2(x+3)}$$

$$= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r}$$

which implies that $a = -1/11$ and $r = (2/11)(x+3)$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{11}\right)\left[\frac{2}{11}(x+3)\right]^n$$

$$= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}},$$

$|x+3| < \dfrac{11}{2}$ or $-\dfrac{17}{2} < x < \dfrac{5}{2}$.

**11.** Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that $a = 3/2$ and $r = (-1/2)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{1}{2}x\right)^n$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n,$$

$|x| < 2$ or $-2 < x < 2$.

**13.** $\dfrac{3x}{x^2+x-2} = \dfrac{2}{x+2} + \dfrac{1}{x-1} = \dfrac{2}{2+x} + \dfrac{1}{-1+x} = \dfrac{1}{1+(1/2)x} + \dfrac{-1}{1-x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2+x-2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}(-1)(x)^n = \sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$$

The interval of convergence is $-1 < x < 1$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1-(-2)^n)x^n}\right| = \lim_{n\to\infty}\left|\frac{(1-(-2)^{n+1})x}{-2-(-2)^{n+1}}\right| = |x|.$$

**15.** $\dfrac{2}{1-x^2} = \dfrac{1}{1-x} + \dfrac{1}{1+x}$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1-x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(1+(-1)^n)x^n = \sum_{n=0}^{\infty} 2x^{2n}.$$

The interval of convergence is $|x^2| < 1$ or $-1 < x < 1$ since $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2x^{2n+2}}{2x^2}\right| = |x^2|$.

**17.** $\dfrac{1}{1+x} = \sum\limits_{n=0}^{\infty}(-1)^n x^n$

$\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty}(-1)^n(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^{2n}x^n = \sum\limits_{n=0}^{\infty} x^n$

$h(x) = \dfrac{-2}{x^2-1} = \dfrac{1}{1+x} + \dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty}(-1)^n x^n + \sum\limits_{n=0}^{\infty} x^n = \sum\limits_{n=0}^{\infty}[(-1)^n + 1]x^n$

$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum\limits_{n=0}^{\infty} 2x^{2n},\ -1 < x < 1$ (See Exercise 15.)

**19.** By taking the first derivative, we have $\dfrac{d}{dx}\left[\dfrac{1}{x+1}\right] = \dfrac{-1}{(x+1)^2}$. Therefore,

$$\frac{-1}{(x+1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n,\ -1 < x < 1.$$

**21.** By integrating, we have $\displaystyle\int \frac{1}{x+1}\,dx = \ln(x+1)$. Therefore,

$$\ln(x+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},\ -1 < x \le 1.$$

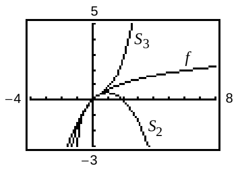
To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},\ -1 < x \le 1.$$

**23.** $\dfrac{1}{x^2+1} = \sum\limits_{n=0}^{\infty}(-1)^n(x^2)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^{2n},\ -1 < x < 1$

**25.** Since, $\dfrac{1}{x+1} = \sum\limits_{n=0}^{\infty}(-1)^n x^n$, we have $\dfrac{1}{4x^2+1} = \sum\limits_{n=0}^{\infty}(-1)^n(4x^2)^n = \sum\limits_{n=0}^{\infty}(-1)^n 4^n x^{2n} = \sum\limits_{n=0}^{\infty}(-1)^n(2x)^{2n},\ -\dfrac{1}{2} < x < \dfrac{1}{2}$.

**27.** $x - \dfrac{x^2}{2} \le \ln(x+1) \le x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$




| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $x - \dfrac{x^2}{2}$ | 0.000 | 0.180 | 0.320 | 0.420 | 0.480 | 0.500 |
| $\ln(x+1)$ | 0.000 | 0.180 | 0.336 | 0.470 | 0.588 | 0.693 |
| $x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ | 0.000 | 0.183 | 0.341 | 0.492 | 0.651 | 0.833 |

**29.** $g(x) = x$, line, Matches (c)

**31.** $g(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$, Matches (a)

**33.** $f(x) = \arctan x$ is an odd function (symmetric to the origin)

**In Exercises 35 and 37, arctan $x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.**

**35.** $\arctan \dfrac{1}{4} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \cdots$

Since $\frac{1}{5120} < 0.001$, we can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

**37.**

$$\frac{\arctan x^2}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)}$$

$$\int_0^{1/2} \frac{\arctan x^2}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \cdots$$

Since $\dfrac{1}{1152} < 0.001$, we can approximate the series by its first term: $\displaystyle\int_0^{1/2} \frac{\arctan x^2}{x}\,dx \approx 0.125$

**In Exercises 39 and 41, use $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n,\ |x| < 1$.**

**39.** (a) $\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty} x^n\right] = \displaystyle\sum_{n=1}^{\infty} nx^{n-1},\ |x| < 1$

(b) $\dfrac{x}{(1-x)^2} = x\displaystyle\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n,\ |x| < 1$

(c) $\dfrac{1+x}{(1-x)^2} = \dfrac{1}{(1-x)^2} + \dfrac{x}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} n(x^{n-1} + x^n),\ |x| < 1$

$= \displaystyle\sum_{n=0}^{\infty} (2n+1)x^n,\ |x| < 1$

(d) $\dfrac{x(1+x)}{(1-x)^2} = x\displaystyle\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1},\ |x| < 1$

**41.** $P(n) = \left(\dfrac{1}{2}\right)^n$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1}{2}\,\frac{1}{[1-(1/2)]^2} = 2$$

Since the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

**43.** Replace $x$ with $(-x)$.

**45.** Replace $x$ with $(-x)$ and multiply the series by 5.

**47.** Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan\theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \tan\theta$$

$$\frac{x+y}{1-xy} = \tan\theta$$

$$\arctan\left(\frac{x+y}{1-xy}\right) = \theta. \text{ Therefore, } \arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \text{ for } xy \neq 1.$$

**49.** (a) $2\arctan\frac{1}{2} = \arctan\frac{1}{2} + \arctan\frac{1}{2} = \arctan\left[\frac{2(1/2)}{1-(1/2)^2}\right] = \arctan\frac{4}{3}$

$$2\arctan\frac{1}{2} - \arctan\frac{1}{7} = \arctan\frac{4}{3} + \arctan\left(-\frac{1}{7}\right) = \arctan\left[\frac{(4/3)-(1/7)}{1+(4/3)(1/7)}\right] = \arctan\frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

(b) $\pi = 8\arctan\frac{1}{2} - 4\arctan\frac{1}{7} \approx 8\left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7}\right] - 4\left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7}\right] \approx 3.14$

**51.** From Exercise 21, we have

$$\ln(x+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}.$$

Thus, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/2)^n}{n}$

$$= \ln\left(\frac{1}{2}+1\right) = \ln\frac{3}{2} \approx 0.4055$$

**53.** From Exercise 51, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{5^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2/5)^n}{n}$$

$$= \ln\left(\frac{2}{5}+1\right) = \ln\frac{7}{5} \approx 0.3365.$$

**55.** From Exercise 54, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1} = \arctan\frac{1}{2} \approx 0.4636.$$

**57.** The series in Exercise 54 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

**59.** $f(x) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n},\ 0 < x \le 2$

$$f(0.5) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-0.5)^n}{n} = \sum_{n=1}^{\infty} -\frac{(1/2)^n}{n}$$

$$\sum_{n=1}^{\infty} -\frac{(1/2)^n}{n} = -0.6931$$

## Section 8.10 Taylor and Maclaurin Series

**1.** For $c = 0$, we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Longrightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$$

**3.** For $c = \pi/4$, we have:

$$f(x) = \cos(x) \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \qquad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \qquad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, we have:

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\
&= \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \cdots\right] \\
&= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (\pi/4)]^n}{n!}.
\end{aligned}$$

[**Note:** $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \ldots$]

**5.** For $c = 1$, we have,

$$f(x) = \ln x \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(1) = -6$$

$$f^{(5)}(x) = \frac{24}{x^5} \qquad f^{(5)}(1) = 24$$

and so on. Therefore, we have:

$$\begin{aligned}
\ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x - 1)^n}{n!} \\
&= 0 + (x - 1) - \frac{(x - 1)^2}{2!} + \frac{2(x - 1)^3}{3!} - \frac{6(x - 1)^4}{4!} + \frac{24(x - 1)^5}{5!} - \cdots \\
&= (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \frac{(x - 1)^5}{5} - \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(x - 1)^{n+1}}{n + 1}
\end{aligned}$$

**7.** For $c = 0$, we have:

$$\begin{aligned} f(x) &= \sin 2x & f(0) &= 0 \\ f'(x) &= 2\cos 2x & f'(0) &= 2 \\ f''(x) &= -4\sin 2x & f''(0) &= 0 \\ f'''(x) &= -8\cos 2x & f'''(0) &= -8 \\ f^{(4)}(x) &= 16\sin 2x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= 32\cos 2x & f^{(5)}(0) &= 32 \\ f^{(6)}(x) &= -64\sin 2x & f^{(6)}(0) &= 0 \\ f^{(7)}(x) &= -128\cos 2x & f^{(7)}(0) &= -128 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned} \sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots \\ &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n+1}}{(2n+1)!} \end{aligned}$$

**9.** For $c = 0$, we have:

$$\begin{aligned} f(x) &= \sec(x) & f(0) &= 1 \\ f'(x) &= \sec(x)\tan(x) & f'(0) &= 0 \\ f''(x) &= \sec^3(x) + \sec(x)\tan^2(x) & f''(0) &= 1 \\ f'''(x) &= 5\sec^3(x)\tan(x) + \sec(x)\tan^3(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= 5\sec^5(x) + 18\sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) & f^{(4)}(0) &= 5 \end{aligned}$$

$$\sec(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots$$

**11.** The Maclaurin series for $f(x) = \cos x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, we have $\left|f^{(n+1)}(z)\right| \le 1$ for all $z$. Hence by Taylor's Theorem,

$$0 \le |Rn(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $Rn(x) \to 0$ as $n \to \infty$. Hence, the Maclaurin series for $\cos x$ converges to $\cos x$ for all $x$.

**13.** Since $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\begin{aligned} (1+x)^{-2} &= 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{5!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n(n+1)x^n. \end{aligned}$$

**15.** $\dfrac{1}{\sqrt{4+x^2}} = \left(\dfrac{1}{2}\right)\left[1 + \left(\dfrac{x}{2}\right)^2\right]^{-1/2}$ and since $(1+x)^{-1/2} = 1 + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, we have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n\, n!}\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

**17.** Since $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n\, n!}$ (Exercise 14)

we have $(1+x^2)^{1/2} = 1 + \dfrac{x^2}{2} + \displaystyle\sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{2n}}{2^n\, n!}.$

**19.**
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$
$$e^{x^2/2} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

**21.**
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

**23.**
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!} = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

**25.**
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$
$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$
$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

**27.**
$$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$$
$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} - \cdots\right]$$
$$= \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right]$$

**29.**
$$x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$$

**31.**
$$\frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!},\ x \neq 0$$

**33.** $$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$

$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$

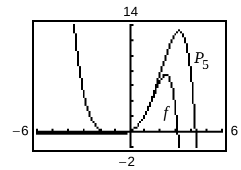
$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

**35.** $f(x) = e^x \sin x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

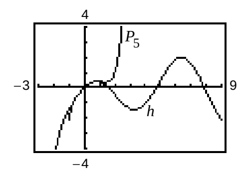



**37.** $h(x) = \cos x \ln(1 + x)$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots$$

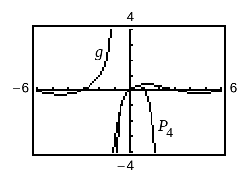
$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$




**39.** $g(x) = \dfrac{\sin x}{1 + x}$. Divide the series for $\sin x$ by $(1 + x)$.

$$\begin{array}{r l}
 & x - x^2 + \frac{5x^2}{6} - \frac{5x^4}{6} + \\
1 + x \,\big) & x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots \\
 & \underline{x + x^2} \\
 & -x^2 - \frac{x^3}{6} \\
 & \underline{-x^2 - x^3} \\
 & \frac{5x^3}{6} + 0x^4 \\
 & \underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \\
 & -\frac{5x^4}{6} + \frac{x^5}{120} \\
 & \underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \\
 & \vdots
\end{array}$$

$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$




**41.** $y = x^2 - \dfrac{x^4}{3!} = x\left(x - \dfrac{x^3}{3!}\right) \approx x \sin x.$

Matches (a)

**43.** $y = x + x^2 + \dfrac{x^3}{2!} = x\left(1 + x + \dfrac{x^2}{2!}\right) \approx xe^x.$

Matches (c)

**45.** $\displaystyle\int_0^x (e^{-t^2} - 1)\,dt = \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right) - 1\right] dt$

$$= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!}\right] dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}$$

**47.** Since $\displaystyle \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$

we have $\displaystyle \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931.$ (10,001 terms)

**49.** Since $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$

we have $\displaystyle e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891.$ (12 terms)

**51.** Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$\frac{1 - \cos}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

we have $\displaystyle \lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{(2n+2)!} = 0.$

**53.** $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$

Since $1/(7 \cdot 7!) < 0.0001$, we have

$$\int_0^1 \frac{\sin x}{x}\,dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \cdots \approx 0.9461.$$

**Note:** We are using $\displaystyle \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$

**55.** $\displaystyle\int_0^{\pi/2} \sqrt{x}\cos x\,dx = \int_0^{\pi/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\dfrac{4n+3}{2}\right)(2n)!}\right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!}\right]_0^{\pi/2}$

Since $(\pi/2)^{19/2}/766{,}080 < 0.0001$, we have

$$\int_0^1 \sqrt{x}\cos x\,dx = 2\left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10{,}800} + \frac{(\pi/2)^{19/2}}{766{,}080}\right] \approx 0.7040.$$

**57.** $\displaystyle\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \int_{0.1}^{0.3} \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \cdots\right) dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \cdots\right]_{0.1}^{0.3}$

Since $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$, we have

$$\int_{0.1}^{0.3} \sqrt{1 + x^3}\,dx = \left[(0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4) - \frac{1}{56}(0.3^7 - 0.1^7)\right] \approx 0.2010.$$

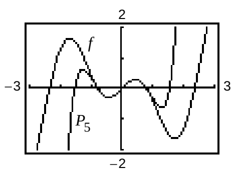
**59.** From Exercise 19, we have

$$\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_0^1 = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!(2n+1)}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left[1 - \frac{1}{2\cdot 1\cdot 3} + \frac{1}{2^2\cdot 2!\cdot 5} - \frac{1}{2^3\cdot 3!\cdot 7}\right] \approx 0.3414.$$

**61.** $f(x) = x\cos 2x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

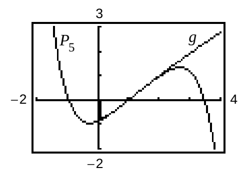
$P_5(x) = x - 2x^3 + \dfrac{2x^5}{3}$




The polynomial is a reasonable approximation on the interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

**63.** $f(x) = \sqrt{x}\ln x,\ c = 1$

$P_5(x) = (x-1) - \dfrac{(x-1)^3}{24} + \dfrac{(x-1)^4}{24} - \dfrac{71(x-1)^5}{1920}$




The polynomial is a reasonable approximation on the interval $\left[\frac{1}{4}, 2\right]$.

**65.** See Guidelines, page 636.

**67.** (a) Replace $x$ with $(-x)$.

(b) Replace $x$ with $3x$.

(c) Multiply series by $x$.

(d) Replace $x$ with $2x$, then replace $x$ with $-2x$, and add the two together.

**69.** $y = \left(\tan\theta - \dfrac{g}{kv_0\cos\theta}\right)x - \dfrac{g}{k^2}\ln\left(1 - \dfrac{kx}{v_0\cos\theta}\right)$

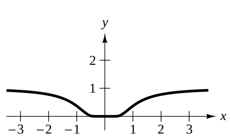
$$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} - \frac{g}{k^2}\left[-\frac{kx}{v_0\cos\theta} - \frac{1}{2}\left(\frac{kx}{v_0\cos\theta}\right)^2 - \frac{1}{3}\left(\frac{kx}{v_0\cos\theta}\right)^3 - \frac{1}{4}\left(\frac{kx}{v_o\cos\theta}\right)^4 - \cdots\right]$$

$$= (\tan\theta)x - \frac{gx}{kv_0\cos\theta} + \frac{gx}{kv_0\cos\theta} + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{gkx^3}{3v_0^3\cos^3\theta} + \frac{gk^2x^4}{4v_0^4\cos^4\theta} + \cdots\Bigg]$$

$$= (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2gx^4}{4v_0^4\cos^4\theta} + \cdots$$

**71.** $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)




(b) $f'(0) = \displaystyle\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{e^{-1/x^2} - 0}{x}$

Let $y = \displaystyle\lim_{x\to 0}\frac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x\to 0}\ln\left(\frac{e^{-1/x^2}}{x}\right) = \lim_{x\to 0^+}\left[-\frac{1}{x^2} - \ln x\right] = \lim_{x\to 0^+}\left[\frac{-1 - x^2\ln x}{x^2}\right] = -\infty.$$

Thus, $y = e^{-\infty} = 0$ and we have $f'(0) = 0$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots = 0 \neq f(x)$

This series converges to $f$ at $x = 0$ only.

**73.** By the Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$ which shows that $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all $x$.

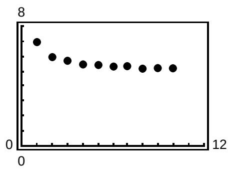
# Review Exercises for Chapter 8

**1.** $a_n = \dfrac{1}{n!}$

**3.** $a_n = 4 + \dfrac{2}{n}$: 6, 5, 4.67, . . .
Matches (a)

**5.** $a_n = 10(0.3)^{n-1}$: 10, 3, . . .
Matches (d)

**7.** $a_n = \dfrac{5n + 2}{n}$




The sequence seems to converge to 5.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n+2}{n}$$
$$= \lim_{n\to\infty}\left(5 + \frac{2}{n}\right) = 5$$

**9.** $\displaystyle\lim_{n\to\infty} \frac{n+1}{n^2} = 0$
Converges

**11.** $\displaystyle\lim_{n\to\infty} \frac{n^3}{n^2+1} = \infty$

**13.** $\displaystyle\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$ Converges

**15.** $\displaystyle\lim_{n\to\infty} \frac{\sin(n)}{\sqrt{n}} = 0$
Converges

**17.** $A_n = 5000\left(1 + \dfrac{0.05}{4}\right)^n = 5000(1.0125)^n$
$n = 1, 2, 3$

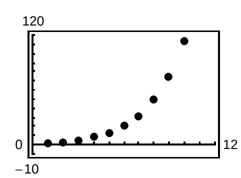
(a) $A_1 = 5062.50$ $\quad A_5 \approx 5320.41$
$A_2 \approx 5125.78$ $\quad A_6 \approx 5386.92$
$A_3 \approx 5189.85$ $\quad A_7 \approx 5454.25$
$A_4 \approx 5254.73$ $\quad A_8 \approx 5522.43$

(b) $A_{40} \approx 8218.10$

**19.** (a)

| $k$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_k$ | 13.2 | 113.3 | 873.8 | 6448.5 | 50,500.3 |

(b)




(c) The series diverges $\left(\text{geometric } r = \frac{3}{2} > 1\right)$

**21.** (a)

| $k$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_k$ | 0.4597 | 0.4597 | 0.4597 | 0.4597 | 0.4597 |

(b)


(c) The series converges by the Alternating Series Test.

**23.** Converges. Geometric series, $r = 0.82$, $|r| < 1$.

**25.** Diverges. $n$th Term Test. $\displaystyle\lim_{n\to\infty} a_n \neq 0$.

**27.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$

Geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$S = \frac{a}{1-r} = \frac{1}{1-(2/3)} = \frac{1}{1/3} = 3$$

**29.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

**31.** $0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \displaystyle\sum_{n=0}^{\infty}(0.09)(0.01)^n = \frac{0.09}{1-0.01} = \frac{1}{11}$

**33.** $D_1 = 8$

$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$

$\vdots$

$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$

$\displaystyle = -8 + \sum_{n=0}^{\infty}16(0.7)^n = -8 + \frac{16}{1-0.7} = 45\tfrac{1}{3}$ meters

**35.** See Exercise 86 in Section 8.2.

$$A = \frac{P(e^{rt}-1)}{e^{r/12}-1}$$

$$= \frac{200(e^{(0.06)(2)}-1)}{e^{0.06/12}-1}$$

$$\approx \$5087.14$$

**37.** $\displaystyle\int_1^{\infty} x^{-4}\ln(x)\,dx = \lim_{b\to\infty}\left[-\frac{\ln x}{3x^3}-\frac{1}{9x^3}\right]_1^b$

$$= 0 + \frac{1}{9} = \frac{1}{9}$$

By the Integral Test, the series converges.

**39.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n}$

Since the second series is a divergent $p$-series while the first series is a convergent $p$-series, the difference diverges.

**41.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2n}}$

$$\lim_{n\to\infty}\frac{1/\sqrt{n^3+2n}}{1/(n^{3/2})} = \lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+2n}} = 1$$

By a limit comparison test with the convergent $p$-series

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, the series converges.

**43.** $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$

$$a_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$$

$$= \left(\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2n-1}{2n-2}\right)\frac{1}{2n} > \frac{1}{2n}$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series), so does the original series.

**45.** Converges by the Alternating Series Test (Conditional convergence)

**47.** Diverges by the $n$th Term Test

**49.** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{1}{e^{2n+1}}\right)\left(\frac{n+1}{n}\right)$$

$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

**51.** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^3}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^3}\cdot\frac{n^3}{2^n}\right|$$

$$= \lim_{n\to\infty}\frac{2n^3}{(n+1)^3} = 2$$

Therefore, by the Ratio Test, the series diverges.

**53.** (a) Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(3/5)^{n+1}}{n(3/5)^n}$

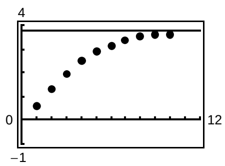
$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{3}{5}\right) = \frac{3}{5} < 1$$

Converges

(b)

| $x$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 2.8752 | 3.6366 | 3.7377 | 3.7488 | 3.7499 |

(c)




(d) The sum is approximately 3.75.

**55.** (a) $\int_N^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_N^\infty = \frac{1}{N}$

| $N$ | 5 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| $\sum_{n=1}^{N}\frac{1}{n^2}$ | 1.4636 | 1.5498 | 1.5962 | 1.6122 | 1.6202 |
| $\int_N^\infty \frac{1}{x^2}\,dx$ | 0.2000 | 0.1000 | 0.0500 | 0.0333 | 0.0250 |

(b) $\int_N^\infty \frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_N^\infty = \frac{1}{4N^4}$

| $N$ | 5 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| $\sum_{n=1}^{N}\frac{1}{n^5}$ | 1.0367 | 1.0369 | 1.0369 | 1.0369 | 1.0369 |
| $\int_N^\infty \frac{1}{x^5}\,dx$ | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

**57.** $f(x) = e^{-x/2}$ $\qquad f(0) = 1$

$f'(x) = -\frac{1}{2}e^{-x/2}$ $\qquad f'(0) = -\frac{1}{2}$

$f''(x) = \frac{1}{4}e^{-x/2}$ $\qquad f''(0) = \frac{1}{4}$

$f'''(x) = -\frac{1}{8}e^{-x/2}$ $\qquad f'''(0) = -\frac{1}{8}$

$$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{4}\frac{x^2}{2!} - \frac{1}{8}\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$$

**59.** $\sin(95^\circ) = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} + \frac{(95\pi)^9}{180^9 9!} \approx 0.996$

**61.** $\ln(1.75) \approx (0.75) - \frac{(0.75)^2}{2} + \frac{(0.75)^3}{3} - \frac{(0.75)^4}{4} + \frac{(0.75)^5}{5} - \frac{(0.75)^6}{6} + \cdots + \frac{(0.75)^{15}}{15} \approx 0.560$

**63.** $f(x) = \cos x,\ c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$$

$$\left|f^{(n+1)}(z)\right| \le 1 \implies R_n(x) \le \frac{x^{n+1}}{(n+1)!}$$

(a) $R_n(x) \le \dfrac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 4$.

(b) $R_n(x) \le \dfrac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 6$.

(c) $R_n(x) \le \dfrac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 5$.

(d) $R_n(x) \le \dfrac{2^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 10$.

**65.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

**67.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{(-1)^n(x-2)^n}\right| = |x-2|$$

$R = 1$
Center: 2

Since the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $1 \le x \le 3$.

**69.** $\displaystyle\sum_{n=0}^{\infty} n!(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n}\right| = \infty$$

which implies that the series converges only at the center $x = 2$.

**71.**

$$y = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^n(n!)^2}$$

$$y' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{4^n(n!)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)x^{2n+1}}{4^{n+1}[(n+1)!]^2}$$

$$y'' = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n}}{4^{n+1}[(n+1)!]^2}$$

$$\begin{aligned}
x^2y'' + xy' + x^2y &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+2)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{4^n(n!)^2}\\
&= \sum_{n=0}^{\infty}\left[(-1)^{n+1}\frac{(2n+2)(2n+1)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^{n+1}(2n+2)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^n}{4^n(n!)^2}\right]x^{2n+2}\\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}(2n+2)(2n+1+1)}{4^{n+1}[(n+1)!]^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2}\\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}4(n+1)^2}{4^{n+1}[(n+1)!]^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2}\\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}1}{4^n(n!)^2} + (-1)^n\frac{1}{4^n(n!)^2}\right]x^{2n+2} = 0
\end{aligned}$$

**73.** $\dfrac{2}{3-x} = \dfrac{2/3}{1-(x/3)} = \dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty}\frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2x^n}{3^{n+1}}$$

**75.** Derivative: $\displaystyle\sum_{n=1}^{\infty}\frac{2nx^{n-1}}{3^{n+1}}$

**77.** $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n = \frac{1}{1-(2x/3)} = \frac{3}{3-2x}, \quad -\frac{3}{2} < x < \frac{3}{2}$

**79.**
$$f(x) = \sin(x)$$
$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x), \cdots$$
$$\sin(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x)[x-(3\pi/4)]^n}{n!}$$
$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n+1)/2}[x-(3\pi/4)]^n}{n!}$$

**81.** $3^x = (e^{\ln(3)})^x = e^{x\ln(3)}$ and since $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, we have
$$3^x = \sum_{n=0}^{\infty}\frac{(x\ln 3)^n}{n!}$$
$$= 1 + x\ln 3 + \frac{x^2\ln^2 3}{2!} + \frac{x^3\ln^3 3}{3!} + \frac{x^4\ln^4 3}{4!} + \cdots.$$

**83.**
$$f(x) = \frac{1}{x}$$
$$f'(x) = -\frac{1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$
$$f'''(x) = -\frac{6}{x^4}, \cdots$$
$$\frac{1}{x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(-1)(x+1)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty}(x+1)^n$$

**85.**
$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$
$$(1+x)^{1/5} = 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots$$
$$= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \cdots$$
$$= 1 + \frac{x}{5} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!}$$
$$= 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots$$

**87.**
$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}, \qquad 0 < x \le 2$$
$$\ln\left(\frac{5}{4}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{(5/4)-1}{n}\right)^n$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{4^n n} \approx 0.2231$$

**89.**
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty$$
$$e^{1/2} = \sum_{n=0}^{\infty}\frac{1}{2^n n!} \approx 1.6487$$

**91.** $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

**93.** The series for Exercise 41 converges very slowly because the terms approach 0 at a slow rate.

**95.** (a)

$$f(x) = e^{2x} \qquad f(0) = 1$$
$$f'(x) = 2e^{2x} \qquad f'(0) = 2$$
$$f''(x) = 4e^{2x} \qquad f''(0) = 4$$
$$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$$
$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

(b)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

(c)

$$e^{2x} = e^x \cdot e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)$$
$$= 1 + (x + x) + \left(x^2 + \frac{x^2}{2} + \frac{x^2}{2}\right) + \left(\frac{x^3}{6} + \frac{x^3}{6} + \frac{x^3}{2} + \frac{x^3}{2}\right) + \cdots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

**97.**

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$
$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$
$$\int_0^x \frac{\sin t}{t}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!}\right]_0^x$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

**99.**

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$
$$\ln(1+t) = \int \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$
$$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$
$$\int_0^x \frac{\ln(t+1)}{t}\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

**101.**

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$
$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$
$$\lim_{x\to 0} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule, $\lim_{x\to 0} \frac{\arctan x}{\sqrt{x}} = \lim_{x\to 0} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x\to 0} \frac{2\sqrt{x}}{1+x^2} = 0.$

## Problem Solving for Chapter 8

**1.** (a) $1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$

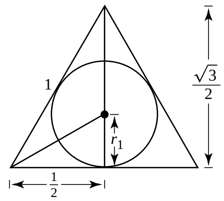
(b) $0, \frac{1}{3}, \frac{2}{3}, 1,$ etc.

(c) $\lim_{n\to\infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - 1 = 0$

**3.** If there are $n$ rows, then $a_n = \dfrac{n(n+1)}{2}$.

For one circle,

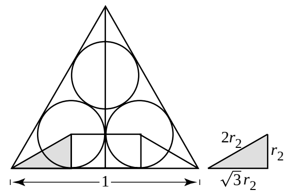
$$a_1 = 1 \text{ and } r_1 = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$$




For three circles,

$$a_2 = 3 \text{ and } 1 = 2\sqrt{3}r_2 + 2r_2$$

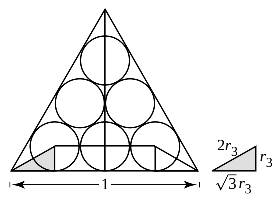
$$r_2 = \frac{1}{2 + 2\sqrt{3}}$$




For six circles,

$$a_3 = 6 \text{ and } 1 = 2\sqrt{3}r_3 + 4r_3$$

$$r_3 = \frac{1}{2\sqrt{3} + 4}$$




Continuing this pattern, $r_n = \dfrac{1}{2\sqrt{3} + 2(n-1)}$.

$$\text{Total Area} = (\pi r_n^{\,2})a_n = \pi\left(\frac{1}{2\sqrt{3} + 2(n-1)}\right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi}{2}\,\frac{n(n+1)}{\left[2\sqrt{3} + 2(n+1)^2\right]}$$

$$\lim_{n\to\infty} A_n = \frac{\pi}{2}\cdot\frac{1}{4} = \frac{\pi}{8}$$

**5.** (a) $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \cdots$

$$= (1 + x^3 + x^6 + \cdots) + 2(x + x^4 + x^7 + \cdots) + 3(x^2 + x^5 + x^8 + \cdots)$$

$$= (1 + x^3 + x^6 + \cdots)[1 + 2x + 3x^2]$$

$$= (1 + 2x + 3x^2)\frac{1}{1 - x^3}$$

$R = 1$ because each series in the second line has $R = 1$.

(b) $\sum a_n x^n = (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}) + (a_0 x^p + a_1 x^{p+1} + \cdots) + \cdots$

$$= a_0(1 + x^p + \cdots) + a_1 x(1 + x^p + \cdots) + \cdots + a_{p-1}x^{p-1}(1 + x^p + \cdots)$$

$$= (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})(1 + x^p + \cdots)$$

$$= (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})\frac{1}{1 - x^p}.$$

$R = 1$

**7.** $$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x\,dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting $x = 0$, $C = 1$. Letting $x = 1$,

$$1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}$.

**9.** Let $a_1 = \displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx$, $a_2 = -\displaystyle\int_{\pi}^{2\pi} \frac{\sin x}{x}\,dx$, $a_3 = \displaystyle\int_{2\pi}^{3\pi} \frac{\sin x}{x}\,dx$, etc.

Then,

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = a_1 - a_2 + a_3 - a_4 + \cdots.$$

Since $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$, this series converges.

**11.** (a) $a_1 = 3.0$

$a_2 \approx 1.73205$

$a_3 \approx 2.17533$

$a_4 \approx 2.27493$

$a_5 \approx 2.29672$

$a_6 \approx 2.30146$

$\lim_{n\to\infty} a_n = \dfrac{1 + \sqrt{13}}{2}$ [See part (b) for proof.]

(b) Use mathematical induction to show the sequence is increasing. Clearly, $a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1$.

Now assume $a_n > a_{n-1}$. Then

$$a_n + a > a_{n-1} + a$$
$$\sqrt{a_n + a} > \sqrt{a_{n-1} + a}$$
$$a_{n+1} > a_n.$$

Use mathematical induction to show that the sequence is bounded above by $a$. Clearly, $a_1 = \sqrt{a} < a$.

Now assume $a_n < a$. Then $a > a_n$ and $a - 1 > 1$ implies

$$a(a-1) > a_n(1)$$
$$a^2 - a > a_n$$
$$a^2 > a_n + a$$
$$a > \sqrt{a_n + a} = a_{n+1}.$$

Hence, the sequence converges to some number $L$. To find $L$, assume $a_{n+1} \approx a_n \approx L$:

$$L = \sqrt{a + L} \implies L^2 = a + L \implies L^2 - L - a = 0$$

$$L = \frac{1 \pm \sqrt{1 + 4a}}{2}.$$

Hence, $L = \dfrac{1 + \sqrt{1 + 4a}}{2}$.

**13.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \cdots$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_1 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

(b) $\displaystyle\frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2^{(-1)^n}}{2^{1+(-1)^{n+1}}}$

This sequence is $\frac{1}{8}, 2, \frac{1}{8}, 2, \ldots$ which diverges.

(c) $\displaystyle\sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left(\frac{1}{2^n \cdot 2^{(-1)^n}}\right)^{1/n}$

$\displaystyle = \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \rightarrow \frac{1}{2} < 1$ converges because $\{2^{(-1)^n}\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \ldots$ and $\sqrt[n]{1/2} \rightarrow 1$ and $\sqrt[n]{2} \rightarrow 1$.

**15.** $S_6 = 130 + 70 + 40 = 240$

$S_7 = 240 + 130 + 70 = 440$

$S_8 = 440 + 240 + 130 = 810$

$S_9 = 810 + 440 + 240 = 1490$

$S_{10} = 1490 + 810 + 440 = 2740$

# CHAPTER 9
# Conics, Parametric Equations, and Polar Coordinates

# CHAPTER 9
# Conics, Parametric Equations, and Polar Coordinates

## Section 9.1 Conics and Calculus

Solutions to Odd-Numbered Exercises

**1.** $y^2 = 4x$

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right
Matches graph (h).

**3.** $(x + 3)^2 = -2(y - 2)$

Vertex: $(-3, 2)$

$p = -\frac{1}{2} < 0$

Opens downward
Matches graph (e).

**5.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Center: $(0, 0)$
Ellipse
Matches (f)

**7.** $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$

Hyperbola

Center: $(0, 0)$

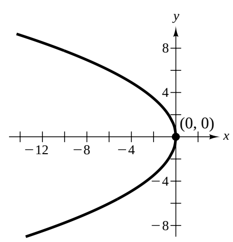
Vertical transverse axis.
Matches (c)

**9.** $y^2 = -6x = 4\left(-\frac{3}{2}\right)x$

Vertex: $(0, 0)$

Focus: $\left(-\frac{3}{2}, 0\right)$

Directrix: $x = \frac{3}{2}$



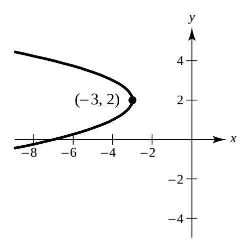

**11.** $(x + 3) + (y - 2)^2 = 0$

$$(y - 2)^2 = 4\left(-\tfrac{1}{4}\right)(x + 3)$$

Vertex: $(-3, 2)$

Focus: $(-3.25, 2)$

Directrix: $x = -2.75$




**13.** $y^2 - 4y - 4x = 0$

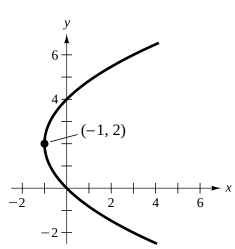
$$y^2 - 4y + 4 = 4x + 4$$

$$(y - 2)^2 = 4(1)(x + 1)$$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$




**15.** $x^2 + 4x + 4y - 4 = 0$

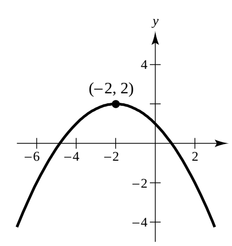
$$x^2 + 4x + 4 = -4y + 4 + 4$$

$$(x + 2)^2 = 4(-1)(y - 2)$$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$



**17.** $y^2 + x + y = 0$

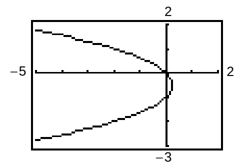
$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$

$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$

Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$




**19.** $y^2 - 4x - 4 = 0$

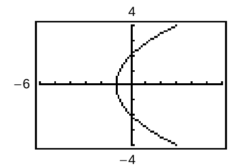
$y^2 = 4x + 4$

$= 4(1)(x + 1)$

Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$




**21.** $(y - 2)^2 = 4(-2)(x - 3)$

$y^2 - 4y + 8x - 20 = 0$

**23.** $(x - h)^2 = 4p(y - k)$

$x^2 = 4(6)(y - 4)$

$x^2 - 24y + 96 = 0$

**25.** $y = 4 - x^2$

$x^2 + y - 4 = 0$

**27.** Since the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, we obtain

$$3 = c,\ 4 = 9a + 3b + c,\ 11 = 16a + 4b + c.$$

Solving this system, we have $a = \frac{5}{3}, b = -\frac{14}{3}, c = 3$. Therefore,

$$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3 \text{ or } 5x^2 - 14x - 3y + 9 = 0.$$

**29.** $x^2 + 4y^2 = 4$

$\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

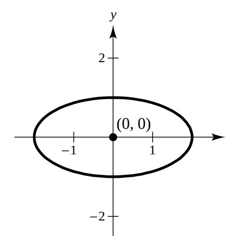
$a^2 = 4, b^2 = 1, c^2 = 3$

Center: $(0, 0)$

Foci: $\left(\pm\sqrt{3}, 0\right)$

Vertices: $(\pm 2, 0)$

$e = \dfrac{\sqrt{3}}{2}$




**31.** $\dfrac{(x - 1)^2}{9} + \dfrac{(y - 5)^2}{25} = 1$

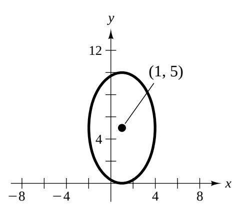
$a^2 = 25, b^2 = 9, c^2 = 16$

Center: $(1, 5)$

Foci: $(1, 9), (1, 1)$

Vertices: $(1, 10), (1, 0)$

$e = \dfrac{4}{5}$




**33.** $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$

$= 36$

$\dfrac{(x + 2)^2}{4} + \dfrac{(y - 3)^2}{9} = 1$

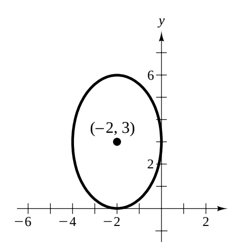
$a^2 = 9, b^2 = 4, c^2 = 5$

Center: $(-2, 3)$

Foci: $\left(-2, 3 \pm \sqrt{5}\right)$

Vertices: $(-2, 6), (-2, 0)$

$e = \dfrac{\sqrt{5}}{3}$



**35.** $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20$$
$$= 60$$
$$\frac{[x - (1/2)]^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$a^2 = 5, b^2 = 3, c^2 = 2$

Center: $\left(\frac{1}{2}, -1\right)$

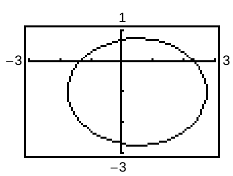
Foci: $\left(\frac{1}{2} \pm \sqrt{2}, -1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{5}, -1\right)$

Solve for $y$:

$$20(y^2 + 2y + 1) = -12x^2 + 12x + 37 + 20$$
$$(y + 1)^2 = \frac{57 + 12x - 12x^2}{20}$$
$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)




**37.** $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2(y^2 + 2y + 1) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$
$$\frac{[x - (3/2)]^2}{4} + \frac{(y + 1)^2}{2} = 1$$

$a^2 = 4, b^2 = 2, c^2 = 2$

Center: $\left(\frac{3}{2}, -1\right)$

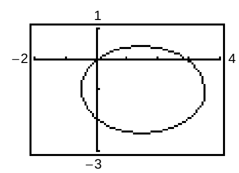
Foci: $\left(\frac{3}{2} \pm \sqrt{2}, -1\right)$

Vertices: $\left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$

Solve for $y$: $2(y^2 + 2y + 1) = -x^2 + 3x - \frac{1}{4} + 2$

$$(y + 1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$
$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)




**39.** Center: $(0, 0)$
Focus: $(2, 0)$
Vertex: $(3, 0)$
Horizontal major axis

$a = 3, c = 2 \implies b = \sqrt{5}$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

**41.** Vertices: $(3, 1), (3, 9)$
Minor axis length: 6
Vertical major axis
Center: $(3, 5)$

$a = 4, b = 3$

$$\frac{(x - 3)^2}{9} + \frac{(y - 5)^2}{16} = 1$$

**43.** Center: $(0, 0)$
Horizontal major axis
Points on ellipse: $(3, 1), (4, 0)$

Since the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, we have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16, b^2 = 16/7$.

Therefore,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

**45.** $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$

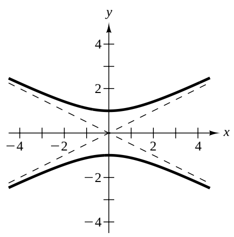
$a = 1, b = 2, c = \sqrt{5}$

Center: $(0, 0)$

Vertices: $(0, \pm 1)$

Foci: $(0, \pm\sqrt{5})$

Asymptotes: $y = \pm\frac{1}{2}x$




**47.** $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

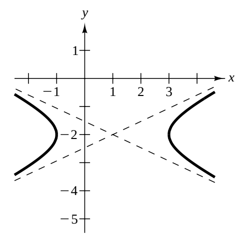
$a = 2, b = 1, c = \sqrt{5}$

Center: $(1, -2)$

Vertices: $(-1, -2), (3, -2)$

Foci: $(1 \pm \sqrt{5}, -2)$

Asymptotes: $y = -2 \pm \frac{1}{2}(x - 1)$




**49.** $9x^2 - y^2 - 36x - 6y + 18 = 0$

$$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

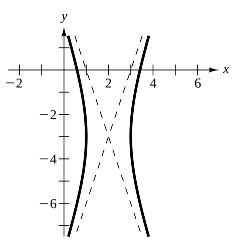
$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $(2 \pm \sqrt{10}, -3)$

Asymptotes: $y = -3 \pm 3(x - 2)$



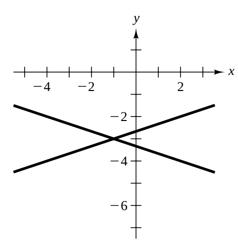

**51.** $x^2 - 9y^2 + 2x - 54y - 80 = 0$

$$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$$

$$(x + 1)^2 - 9(y + 3)^2 = 0$$

$$y + 3 = \pm\frac{1}{3}(x + 1)$$

Degenerate hyperbola is two lines intersecting at $(-1, -3)$.




**53.** $9y^2 - x^2 + 2x + 54y + 62 = 0$

$$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$$

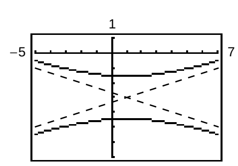
$$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: $(1, -3)$

Vertices: $(1, -3 \pm \sqrt{2})$

Foci: $(1, -3 \pm 2\sqrt{5})$




Solve for $y$:

$$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$$

$$(y + 3)^2 = \frac{x^2 - 2x + 19}{9}$$

$$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$$

(Graph each curve separately.)

**55.** $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

$$3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$$

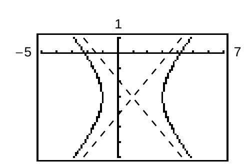
$$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$$

$a = 2, b = \sqrt{6}, c = \sqrt{10}$

Center: $(1, -3)$

Vertices: $(-1, -3), (3, -3)$

Foci: $(1 \pm \sqrt{10}, -3)$




Solve for $y$:

$$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$$

$$(y + 3)^2 = \frac{3x^2 - 6x - 9}{2}$$

$$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$$

(Graph each curve separately.)

**57.** Vertices: $(\pm 1, 0)$
Asymptotes: $y = \pm 3x$
Horizontal transverse axis
Center: $(0, 0)$

$a = 1, \pm\frac{b}{a} = \pm\frac{b}{1} = \pm 3 \implies b = 3$

Therefore, $\frac{x^2}{1} - \frac{y^2}{9} = 1.$

**59.** Vertices: $(2, \pm 3)$
Point on graph: $(0, 5)$
Vertical transverse axis
Center: $(2, 0)$
$a = 3$
Therefore, the equation is of the form

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, we have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

Therefore, the equation is $\frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1.$

**61.** Center: $(0, 0)$
Vertex: $(0, 2)$
Focus: $(0, 4)$
Vertical transverse axis

$a = 2, c = 4, b^2 = c^2 - a^2 = 12$

Therefore, $\frac{y^2}{4} - \frac{x^2}{12} = 1.$

**63.** Vertices: $(0, 2), (6, 2)$

Asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

Horizontal transverse axis
Center: $(3, 2)$
$a = 3$

Slopes of asymptotes: $\pm\frac{b}{a} = \pm\frac{2}{3}$

Thus, $b = 2$. Therefore,

$$\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1.$$

**65.** (a) $\frac{x^2}{9} - y^2 = 1, \frac{2x}{9} - 2yy' = 0, \frac{x}{9y} = y'$

At $x = 6$: $y = \pm\sqrt{3}, y' = \frac{\pm 6}{9\sqrt{3}} = \frac{\pm 2\sqrt{3}}{9}$

At $(6, \sqrt{3})$: $y - \sqrt{3} = \frac{2\sqrt{3}}{9}(x - 6)$

or $2x - 3\sqrt{3}y - 3 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \frac{-2\sqrt{3}}{9}(x - 6)$

or $2x + 3\sqrt{3}y - 3 = 0$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 9/(2\sqrt{3})$.

At $(6, \sqrt{3})$: $y - \sqrt{3} = -\frac{9}{2\sqrt{3}}(x - 6)$

or $9x + 2\sqrt{3}y - 60 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \frac{9}{2\sqrt{3}}(x - 6)$

or $9x - 2\sqrt{3}y - 60 = 0$

**67.** $x^2 + 4y^2 - 6x + 16y + 21 = 0$
$A = 1, C = 4$
$AC = 4 > 0$
Ellipse

**69.** $y^2 - 4y - 4x = 0$
$A = 0, C = 1$
Parabola

**71.** $4x^2 + 4y^2 - 16y + 15 = 0$
$A = C = 4$
Circle

**73.** $9x^2 + 9y^2 - 36x + 6y + 34 = 0$
$A = C = 9$
Circle

**75.** $3x^2 - 6x + 3 = 6 + 2y^2 + 4y + 2$

$3x^2 - 2y^2 - 6x - 4y - 5 = 0$

$A = 3, C = -2, AC < 0$

Hyperbola

**77.** (a) A parabola is the set of all points $(x, y)$ that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$

(c) See Theorem 9.2.

**79.** (a) A hyperbola is the set of all points $(x, y)$ for which the absolute value of the difference between the distances from two distance fixed points (foci) is constant.

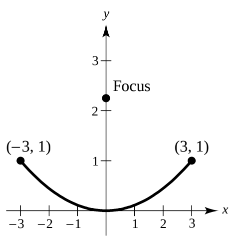
(b) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ or $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

(c) $y = k \pm \dfrac{b}{a}(x - h)$ or $y = k \pm \dfrac{a}{b}(x - h)$

**81.** Assume that the vertex is at the origin.

$$x^2 = 4py$$
$$(3)^2 = 4p(1)$$
$$\frac{9}{4} = p$$

The pipe is located $\frac{9}{4}$ meters from the vertex.




**83.** $y = ax^2$

$y' = 2ax$

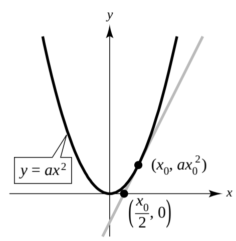
The equation of the tangent line is

$$y - ax_0^2 = 2ax_0(x - x_0) \text{ or } y = 2ax_0x - ax_0^2.$$

Let $y = 0$. Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$
$$ax_0^2 = 2ax_0x$$

Therefore, $\dfrac{x_0}{2} = x$ is the $x$-intercept.




**85.** (a) Consider the parabola $x^2 = 4py$. Let $m_0$ be the slope of the one tangent line at $(x_1, y_1)$ and therefore, $-1/m_0$ is the slope of the second at $(x_2, y_2)$. From the derivative given in Exercise 32 we have:

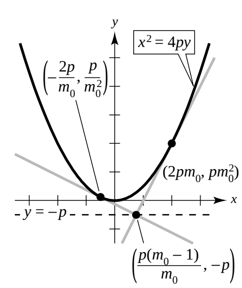
$$m_0 = \frac{1}{2p}x_1 \text{ or } x_1 = 2pm_0$$
$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \text{ or } x_2 = \frac{-2p}{m_0}$$

Substituting these values of $x$ into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \quad \text{and} \quad \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is

$$\left(\frac{p(m_0^2 - 1)}{m_0}, -p\right) \text{ and is on the directrix, } y = -p.$$




**—CONTINUED—**

**85. —CONTINUED—**

(b) $x^2 - 4x - 4y + 8 = 0$

$$(x-2)^2 = 4(y-1). \text{ Vertex } (2, 1)$$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(-2, 5)$, $dy/dx = -2$. At $\left(3, \frac{5}{4}\right)$, $dy/dx = \frac{1}{2}$.

Tangent line at $(-2, 5)$: $y - 5 = -2(x + 2) \implies 2x + y - 1 = 0$.

Tangent line at $\left(3, \frac{5}{4}\right)$: $y - \frac{5}{4} = \frac{1}{2}(x - 3) \implies 2x - 4y - 1 = 0$.

Since $m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$, the lines are perpendicular.

Point of intersection: $-2x + 1 = \frac{1}{2}x - \frac{1}{4}$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $\left(\frac{1}{2}, 0\right)$ lies on this line.

**87.** $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

At $(x_1, y_1)$ on the mountain, $m = 1 - 2x_1$. Also, $m = \dfrac{y_1 - 1}{x_1 + 1}$.

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

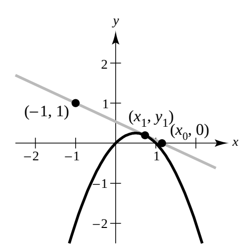
$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

Choosing the positive value for $x_1$, we have $x_1 = -1 + \sqrt{3}$.

$$m = 1 - 2\left(-1 + \sqrt{3}\right) = 3 - 2\sqrt{3}$$

$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$




Thus, $-\dfrac{1}{x_0 + 1} = 3 - 2\sqrt{3}$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

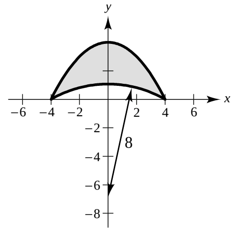
$$\frac{2\sqrt{3}}{3} = x_0.$$

The closest the receiver can be to the hill is $\left(2\sqrt{3}/3\right) - 1 \approx 0.155$.

**89.** Parabola

Vertex: $(0, 4)$

$$x^2 = 4p(y - 4)$$
$$4^2 = 4p(0 - 4)$$
$$p = -1$$
$$x^2 = -4(y - 4)$$
$$y = 4 - \frac{x^2}{4}$$




Circle

Center: $(0, k)$

Radius: 8

$$x^2 + (y - k)^2 = 64$$
$$4^2 + (0 - k)^2 = 64$$
$$k^2 = 48$$
$$k = -4\sqrt{3} \quad \text{(Center is on the negative } y\text{-axis.)}$$
$$x^2 + \left(y + 4\sqrt{3}\right)^2 = 64$$
$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

Since the $y$-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

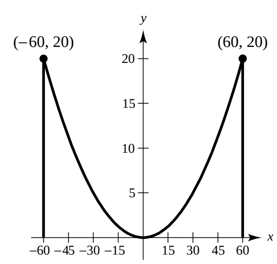
$$A = 2\int_0^4 \left[\left(4 - \frac{x^2}{4}\right) - \left(-4\sqrt{3} + \sqrt{64 - x^2}\right)\right] dx$$
$$= 2\left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2}\left(x\sqrt{64 - x^2} + 64 \arcsin\frac{x}{8}\right)\right]_0^4$$
$$= 2\left[16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin\frac{1}{2}\right]$$
$$= \frac{16\left(4 + 3\sqrt{3} - 2\pi\right)}{3} \approx 15.536 \text{ square feet}$$

**91.** (a) Assume that $y = ax^2$.

$$20 = a(60)^2 \implies a = \frac{2}{360} = \frac{1}{180} \implies y = \frac{1}{180}x^2$$

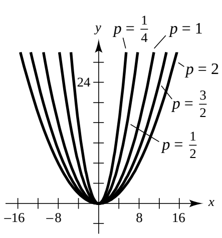
(b) $f(x) = \frac{1}{180}x^2, f'(x) = \frac{1}{90}x$

$$S = 2\int_0^{60} \sqrt{1 + \left(\frac{1}{90}x\right)^2}\, dx = \frac{2}{90}\int_0^{60} \sqrt{90^2 + x^2}\, dx$$
$$= \frac{2}{90}\frac{1}{2}\left[x\sqrt{90^2 + x^2} + 90^2 \ln\left|x + \sqrt{90^2 + x^2}\right|\right]_0^{60} \quad \text{(formula 26)}$$
$$= \frac{1}{90}\left[60\sqrt{11{,}700} + 90^2 \ln\left(60 + \sqrt{11{,}700}\right) - 90^2 \ln 90\right]$$
$$= \frac{1}{90}\left[1800\sqrt{13} + 90^2 \ln\left(60 + 30\sqrt{13}\right) - 90^2 \ln 90\right]$$
$$= 20\sqrt{13} + 90 \ln\left(\frac{60 + 30\sqrt{13}}{90}\right)$$
$$= 10\left[2\sqrt{13} + 9 \ln\left(\frac{2 + \sqrt{13}}{3}\right)\right] \approx 128.4 \text{ m}$$

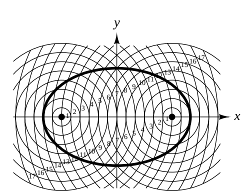



**93.** $x^2 = 4py, p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As $p$ increases, the graph becomes wider.




**95.**



**97.** $a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$

The tacks should be placed 1.5 feet from the center. The string should be $2a = 5$ feet long.

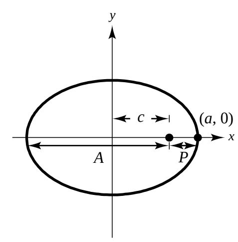
**99.** $$e = \frac{c}{a}$$

$$A + P = 2a$$

$$a = \frac{A+P}{2}$$

$$c = a - P = \frac{A+P}{2} - P = \frac{A-P}{2}$$

$$e = \frac{c}{a} = \frac{(A-P)/2}{(A+P)/2} = \frac{A-P}{A+P}$$




**101.** $e = \frac{A-P}{A+P} = \frac{35.34au - 0.59au}{35.34au + 0.59au} \approx 0.9672$

**103.** $$\frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$$

$$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$$

$$y' = \frac{-5^2x}{10^2y} = \frac{-x}{4y}$$

At $(-8, 3)$: $y' = \frac{8}{12} = \frac{2}{3}$

The equation of the tangent line is $y - 3 = \frac{2}{3}(x + 8)$. It will cross the $y$-axis when $x = 0$ and $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$.

**105.** $$16x^2 + 9y^2 + 96x + 36y + 36 = 0$$

$$32x + 18yy' + 96 + 36y' = 0$$

$$y'(18y + 36) = -(32x + 96)$$

$$y' = \frac{-(32x + 96)}{18y + 36}$$

$y' = 0$ when $x = -3$. $y'$ is undefined when $y = -2$.

At $x = -3$, $y = 2$ or $-6$.

Endpoints of major axis: $(-3, 2), (-3, -6)$

At $y = -2$, $x = 0$ or $-6$.

Endpoints of minor axis: $(0, -2), (-6, -2)$

**Note:** Equation of ellipse is $\frac{(x+3)^2}{9} + \frac{(y+2)^2}{16} = 1$

**107.** (a) $A = 4\int_0^2 \frac{1}{2}\sqrt{4 - x^2}\,dx = \left[x\sqrt{4 - x^2} + 4\arcsin\left(\frac{x}{2}\right)\right]_0^2 = 2\pi$ [or, $A = \pi ab = \pi(2)(1) = 2\pi$]

(b) **Disk:** $$V = 2\pi\int_0^2 \frac{1}{4}(4 - x^2)\,dx = \frac{1}{2}\pi\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{8\pi}{3}$$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi)\int_0^2 y\left(\frac{\sqrt{16 - 3x^2}}{4y}\right)dx = \frac{\pi}{2\sqrt{3}}\left[\sqrt{3}x\sqrt{16 - 3x^2} + 16\arcsin\left(\frac{\sqrt{3}x}{4}\right)\right]_0^2 = \frac{2\pi}{9}\left(9 + 4\sqrt{3}\pi\right) \approx 21.48$$

**—CONTINUED—**

**107. —CONTINUED—**

(c) **Shell:** $V = 2\pi\int_0^2 x\sqrt{4 - x^2}\,dx = -\pi\int_0^2 -2x(4 - x^2)^{1/2}\,dx = -\frac{2\pi}{3}\left[(4 - x^2)^{3/2}\right]_0^2 = \frac{16\pi}{3}$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi)\int_0^1 2\sqrt{1 - y^2}\,\frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}\,dy = 8\pi\int_0^1 \sqrt{1 + 3y^2}\,dy$$

$$= \frac{8\pi}{2\sqrt{3}}\left[\sqrt{3}y\sqrt{1 + 3y^2} + \ln\left|\sqrt{3}y + \sqrt{1 + 3y^2}\right|\right]_0^1 = \frac{4\pi}{3}\left|6 + \sqrt{3}\ln\left(2 + \sqrt{3}\right)\right| \approx 34.69$$

**109.** From Example 5,

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\,d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, we have

$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}.$

$$C = 4(7)\int_0^{\pi/2} \sqrt{1 - \frac{24}{49}\sin^2\theta}\,d\theta$$

$$\approx 28(1.3558) \approx 37.9614$$

**111.** Area circle $= \pi r^2 = 100\pi$

Area ellipse $= \pi ab = \pi a(10)$

$$2(100\pi) = 10\pi a \implies a = 20$$

Hence, the length of the major axis is $2a = 40$.

**113.** The transverse axis is horizontal since $(2, 2)$ and $(10, 2)$ are the foci (see definition of hyperbola).

Center: $(6, 2)$

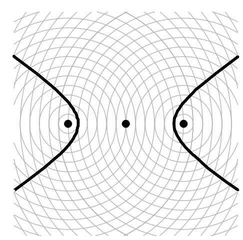
$c = 4, 2a = 6, b^2 = c^2 - a^2 = 7$

Therefore, the equation is

$$\frac{(x - 6)^2}{9} - \frac{(y - 2)^2}{7} = 1.$$

**115.** $2a = 10 \implies a = 5$

$c = 6 \implies b = \sqrt{11}$



**117.** Time for sound of bullet hitting target to reach $(x, y)$: $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s}$

Time for sound of rifle to reach $(x, y)$: $\frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

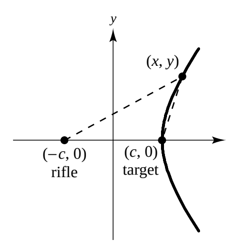
Since the times are the same, we have: $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s} = \frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

$$\frac{4c^2}{{v_m}^2} + \frac{4c}{v_m v_s}\sqrt{(x - c)^2 + y^2} + \frac{(x - c)^2 + y^2}{{v_s}^2} = \frac{(x + c)^2 + y^2}{{v_s}^2}$$

$$\sqrt{(x - c)^2 + y^2} = \frac{{v_m}^2 x - {v_s}^2 c}{v_s v_m}$$

$$\left(1 - \frac{{v_m}^2}{{v_s}^2}\right)x^2 + y^2 = \left(\frac{{v_s}^2}{{v_m}^2} - 1\right)c^2$$

$$\frac{x^2}{c^2{v_s}^2/{v_m}^2} - \frac{y^2}{c^2({v_m}^2 - {v_s}^2)/{v_m}^2} = 1$$



**119.** The point $(x, y)$ lies on the line between $(0, 10)$ and $(10, 0)$. Thus, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, we have:

$$\frac{x^2}{36} - \frac{(10 - x)^2}{64} = 1$$

$$16x^2 - 9(10 - x)^2 = 576$$

$$7x^2 + 180x - 1476 = 0$$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)} = \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}$$

Choosing the positive value for $x$ we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and } y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

**121.** $$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \implies \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2},\ c^2 = a^2 - b^2$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \implies \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \implies 2 = x^2\left(\frac{1}{a^2} + \frac{1}{a^2 - b^2}\right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \implies x = \pm\frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2}\left(\frac{2a^2c^2}{2a^2 - b^2}\right) \implies \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \implies y = \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection: $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, $\left(-\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \implies \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \implies y'_e = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \implies \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \implies y'_h = \frac{b^2x}{2c^2y}$$

At $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, the slopes of the tangent lines are:

$$y'_e = \frac{-b^2\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2a^2\left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = -\frac{c}{a} \quad \text{and} \quad y'_h = \frac{b^2\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2c^2\left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = \frac{a}{c}$$

Since the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

**123.** False. See the definition of a parabola.

**125.** True

**127.** False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines.

**129.** True

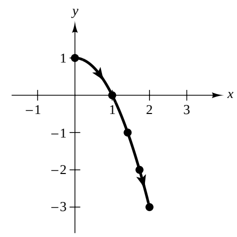
## Section 9.2 Plane Curves and Parametric Equations

**1.** $x = \sqrt{t},\ y = 1 - t$

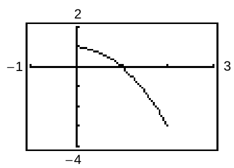
(a)

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | $\sqrt{2}$ | $\sqrt{3}$ | 2 |
| $y$ | 1 | 0 | $-1$ | $-2$ | $-3$ |

(b)



(c)



(d) $x^2 = t$

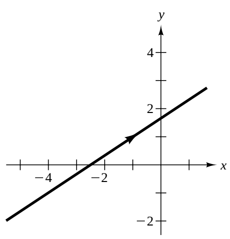
$y = 1 - x^2, x \geq 0$

**3.** $x = 3t - 1$

$y = 2t + 1$

$y = 2\left(\dfrac{x+1}{3}\right) + 1$

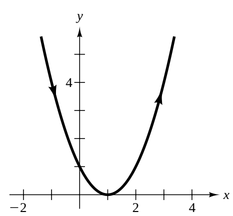
$2x - 3y + 5 = 0$



**5.** $x = t + 1$

$y = t^2$

$y = (x - 1)^2$

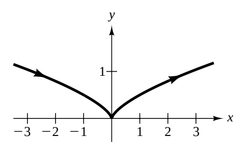


**7.** $x = t^3$

$y = \frac{1}{2}t^2$

$x = t^3$ implies $t = x^{1/3}$

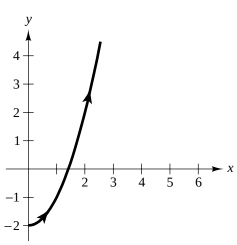
$y = \frac{1}{2}x^{2/3}$



**9.** $x = \sqrt{t}, t \geq 0$

$y = t - 2$

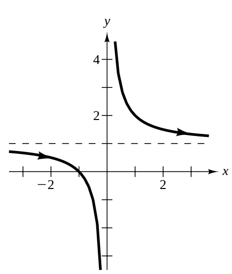
$y = x^2 - 2, x \geq 0$



**11.** $x = t - 1$

$y = \dfrac{t}{t-1}$

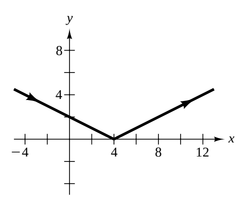
$y = \dfrac{x+1}{x}$



**13.** $x = 2t$

$y = |t - 2|$

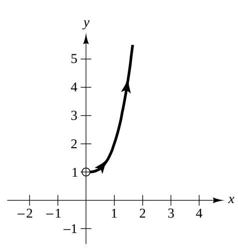
$y = \left|\dfrac{x}{2} - 2\right| = \dfrac{|x-4|}{2}$



**15.** $x = e^t, x > 0$

$y = e^{3t} + 1$

$y = x^3 + 1, x > 0$



**17.** $x = \sec\theta$

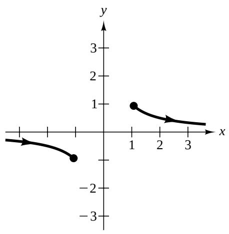
$y = \cos\theta$

$0 \le \theta < \frac{\pi}{2},\ \frac{\pi}{2} < \theta \le \pi$

$xy = 1$

$y = \frac{1}{x}$

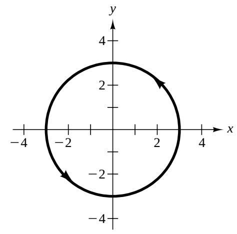
$|x| \ge 1,\ |y| \le 1$




**19.** $x = 3\cos\theta,\ y = 3\sin\theta$

Squaring both equations and adding, we have

$x^2 + y^2 = 9.$



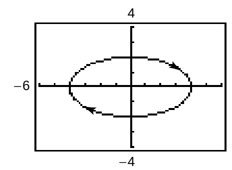

**21.** $x = 4\sin 2\theta$

$y = 2\cos 2\theta$

$\frac{x^2}{16} = \sin^2 2\theta$

$\frac{y^2}{4} = \cos^2 2\theta$

$\frac{x^2}{16} + \frac{y^2}{4} = 1$



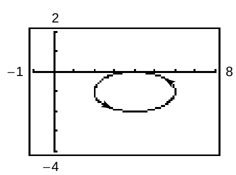

**23.** $x = 4 + 2\cos\theta$

$y = -1 + \sin\theta$

$\frac{(x-4)^2}{4} = \cos^2\theta$

$\frac{(y+1)^2}{1} = \sin^2\theta$

$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$



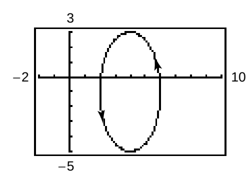

**25.** $x = 4 + 2\cos\theta$

$y = -1 + 4\sin\theta$

$\frac{(x-4)^2}{4} = \cos^2\theta$

$\frac{(y+1)^2}{16} = \sin^2\theta$

$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} = 1$



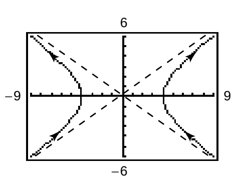

**27.** $x = 4\sec\theta$

$y = 3\tan\theta$

$\frac{x^2}{16} = \sec^2\theta$

$\frac{y^2}{9} = \tan^2\theta$

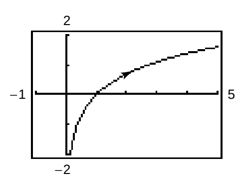
$\frac{x^2}{16} - \frac{y^2}{9} = 1$




**29.** $x = t^3$

$y = 3\ln t$

$y = 3\ln\sqrt[3]{x} = \ln x$




**31.** $x = e^{-t}$

$y = e^{3t}$

$e^t = \frac{1}{x}$

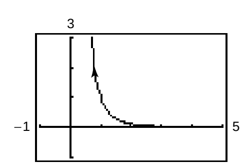
$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \frac{1}{x}$

$y = \frac{1}{x^3}$

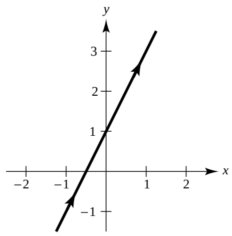
$x > 0$

$y > 0$



**33.** By eliminating the parameters in (a) – (d), we get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

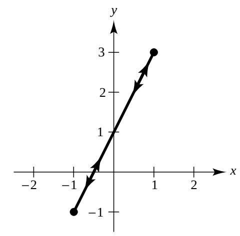
(a) $x = t,\ y = 2t + 1$




(b) $x = \cos\theta \qquad y = 2\cos\theta + 1$

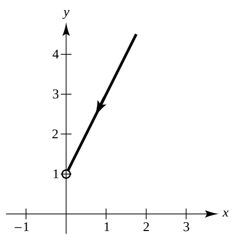
$-1 \le x \le 1 \qquad -1 \le y \le 3$

$\dfrac{dx}{d\theta} = \dfrac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$



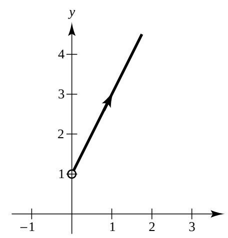

(c) $x = e^{-t} \qquad y = 2e^{-t} + 1$

$x > 0 \qquad y > 1$



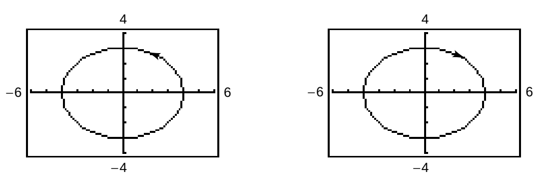

(d) $x = e^{t} \qquad y = 2e^{t} + 1$

$x > 0 \qquad y > 1$




**35.** The curves are identical on $0 < \theta < \pi$. They are both smooth. Represent $y = 2(1 - x^2)$

**37.** (a)




(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Many answers possible. For example, $x = 1 + t$, $y = 1 + 2t$, and $x = 1 - t$, $x = 1 - 2t$.

**39.**

$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$
$$\frac{x - x_1}{x_2 - x_1} = t$$
$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - y_1 = m(x - x_1)$$

**41.**

$$x = h + a\cos\theta$$
$$y = k + b\sin\theta$$
$$\frac{x - h}{a} = \cos\theta$$
$$\frac{y - k}{b} = \sin\theta$$
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

**43.** From Exercise 39 we have

$x = 5t$

$y = -2t.$

Solution not unique

**45.** From Exercise 40 we have

$x = 2 + 4\cos\theta$

$y = 1 + 4\sin\theta.$

Solution not unique

**47.** From Exercise 41 we have

$a = 5, c = 4 \implies b = 3$

$x = 5\cos\theta$

$y = 3\sin\theta.$

Center: $(0, 0)$
Solution not unique

**49.** From Exercise 42 we have

$a = 4, c = 5 \implies b = 3$

$x = 4 \sec \theta$

$y = 3 \tan \theta.$

Center: $(0, 0)$
Solution not unique

**51.** $y = 3x - 2$

<u>Example</u>

$x = t, \qquad y = 3t - 2$

$x = t - 3, \qquad y = 3t - 11$

**53.** $y = x^3$

<u>Example</u>

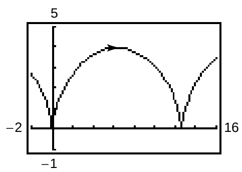
$x = t, \qquad y = t^3$

$x = \sqrt[3]{t}, \qquad y = t$

$x = \tan t, \qquad y = \tan^3 t$

**55.** $x = 2(\theta - \sin \theta)$

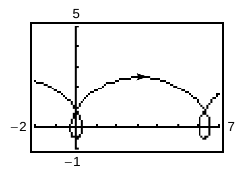
$y = 2(1 - \cos \theta)$




Not smooth at $\theta = 2n\pi$

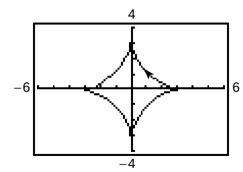
**57.** $x = \theta - \frac{3}{2} \sin \theta$

$y = 1 - \frac{3}{2} \cos \theta$




**59.** $x = 3 \cos^3 \theta$

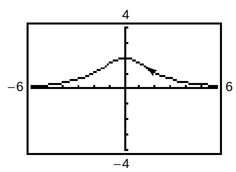
$y = 3 \sin^3 \theta$




Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

**61.** $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$




Smooth everywhere

**63.** See definition on page 665.

**65.** A plane curve $C$, represented by $x = f(t)$, $y = g(t)$, is smooth if $f'$ and $g'$ are continuous and not simultaneously 0. See page 670.

**67.** $x = 4 \cos \theta$

$y = 2 \sin 2\theta$

Matches (d)

**69.** $x = \cos \theta + \theta \sin \theta$

$y = \sin \theta - \theta \cos \theta$

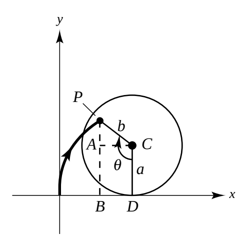
Matches (b)

**71.** When the circle has rolled $\theta$ radians, we know that the center is at $(a\theta, a)$.

$$\sin \theta = \sin(180° - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \quad \text{or} \quad |BD| = b \sin \theta$$

$$\cos \theta = -\cos(180° - \theta) = \frac{|AP|}{-b} \quad \text{or} \quad |AP| = -b \cos \theta$$

Therefore, $x = a\theta - b \sin \theta$ and $y = a - b \cos \theta$.




**73.** False

$x = t^2 \implies x \geq 0$

$x = t^2 \implies y \geq 0$

The graph of the parametric equations is only a portion of the line $y = x$.

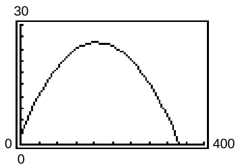
**75.** (a) $100 \text{ mi/hr} = \dfrac{(100)(5280)}{3600} = \dfrac{440}{3} \text{ ft/sec}$

$$x = (v_0 \cos\theta)t = \left(\frac{440}{3}\cos\theta\right)t$$

$$y = h + (v_0 \sin\theta)t - 16t^2$$

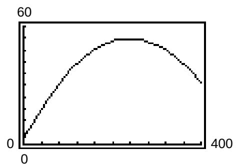
$$= 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2$$

(b)




It is not a home run—when $x = 400$, $y \le 20$.

(c)




Yes, it's a home run when $x = 400$, $y > 10$.

(d) We need to find the angle $\theta$ (and time $t$) such that

$$x = \left(\frac{440}{3}\cos\theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440\cos\theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{1200}{440\cos\theta}\right) - 16\left(\frac{1200}{440\cos\theta}\right)^2$$

$$7 = 400\tan\theta - 16\left(\frac{120}{44}\right)^2\sec^2\theta$$

$$= 400\tan\theta - 16\left(\frac{120}{44}\right)^2(\tan^2\theta + 1).$$

We now solve the quadratic for $\tan\theta$:

$$16\left(\frac{120}{44}\right)^2\tan^2\theta - 400\tan\theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0$$

$\tan\theta \approx 0.35185 \implies \theta \approx 19.4°$

## Section 9.3 Parametric Equations and Calculus

**1.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-4}{2t} = \dfrac{-2}{t}$

**3.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\cos t\sin t}{2\sin t\cos t} = -1$

$\left[\text{Note: } x + y = 1 \implies y = 1 - x \text{ and } \dfrac{dy}{dt} = -1\right]$

**5.** $x = 2t,\ y = 3t - 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$$

$\dfrac{d^2y}{dx^2} = 0$ Line

**7.** $x = t + 1,\ y = t^2 + 3t$

$\dfrac{dy}{dx} = \dfrac{2t+3}{1} = 1$ when $t = -1$.

$\dfrac{d^2y}{dx^2} = 2$ concave upwards

**9.** $x = 2\cos\theta,\ y = 2\sin\theta$

$\dfrac{dy}{dx} = \dfrac{2\cos\theta}{-2\sin\theta} = -\cot\theta = -1$ when $\theta = \dfrac{\pi}{4}$.

$\dfrac{d^2y}{dx^2} = \dfrac{\csc^2\theta}{-2\sin\theta} = \dfrac{-\csc^3\theta}{2} = -\sqrt{2}$ when $\theta = \dfrac{\pi}{4}$.

concave downward

**11.** $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta}$$

$= \dfrac{2\sec\theta}{\tan\theta} = 2\csc\theta = 4$ when $\theta = \dfrac{\pi}{6}$.

$$\frac{d^2y}{dx^2} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta}$$

$= -2\cot^3\theta = -6\sqrt{3}$ when $\theta = \dfrac{\pi}{6}$.

concave downward

**13.** $x = \cos^3\theta,\ y = \sin^3\theta$

$$\frac{dy}{dx} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}$$

$$= -\tan\theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3\cos^2\theta\sin\theta} = \frac{1}{3\cos^4\theta\sin\theta}$$

$$= \frac{\sec^4\theta\csc\theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

concave upward

**15.** $x = 2\cot\theta, y = 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{-2\csc^2\theta} = -2\sin^3\theta\cos\theta$$

At $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$, $\theta = \frac{2\pi}{3}$, and $\frac{dy}{dx} = \frac{3\sqrt{3}}{8}$.

Tangent line: $y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$

$$3\sqrt{3}x - 8y + 18 = 0$$

At $(0, 2)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

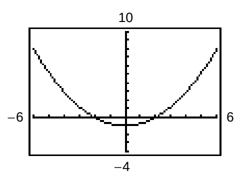
Tangent line: $y - 2 = 0$

At $\left(2\sqrt{3}, \frac{1}{2}\right)$, $\theta = \frac{\pi}{6}$, and $\frac{dy}{dx} = -\frac{\sqrt{3}}{8}$.

Tangent line: $y - \frac{1}{2} = -\frac{\sqrt{3}}{8}\left(x - 2\sqrt{3}\right)$

$$\sqrt{3}x + 8y - 10 = 0$$

**17.** $x = 2t,\ y = t^2 - 1,\ t = 2$

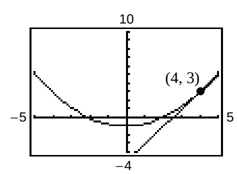
(a) 


(b) At $t = 2$, $(x, y) = (4, 3)$, and

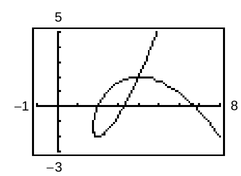
$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 4,\ \frac{dy}{dx} = 2$$

(c) $\frac{dy}{dx} = 2$. At $(4, 3)$, $y - 3 = 2(x - 4)$

$$y = 2x - 5$$

(d) 


**19.** $x = t^2 - t + 2,\ y = t^3 - 3t,\ t = -1$

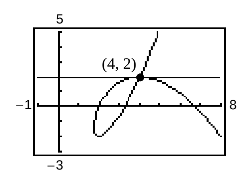
(a) 


(b) At $t = -1$, $(x, y) = (4, 2)$, and

$$\frac{dx}{dt} = -3,\ \frac{dy}{dt} = 0,\ \frac{dy}{dx} = 0$$

(c) $\frac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$

$$y = 2$$

(d) 


**21.** $x = 2\sin 2t, y = 3\sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3\cos t}{4\cos 2t}$$

At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = \frac{-3}{4}x$ Tangent Line

**23.** $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \theta\sin\theta = 0$ when $\theta = 0,\ \pi,\ 2\pi,\ 3\pi, \ldots.$

Points: $(-1, [2n-1]\pi),\ (1, 2n\pi)$ where $n$ is an integer.

Points shown: $(1, 0),\ (-1, \pi),\ (1, -2\pi)$

Vertical tangents: $\dfrac{dx}{d\theta} = \theta\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots.$

Points: $\left(\dfrac{(-1)^{n+1}(2n-1)\pi}{2}, (-1)^{n+1}\right)$

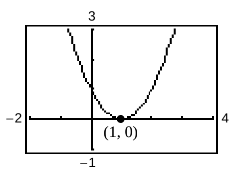
Points shown: $\left(\dfrac{\pi}{2}, 1\right),\ \left(-\dfrac{3\pi}{2}, -1\right),\ \left(\dfrac{5\pi}{2}, 1\right)$

**25.** $x = 1 - t,\ y = t^2$

Horizontal tangents: $\dfrac{dy}{dt} = 2t = 0$ when $t = 0$.

Point: $(1, 0)$

Vertical tangents: $\dfrac{dx}{dt} = -1 \neq 0$; none

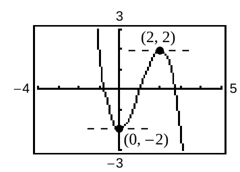



**27.** $x = 1 - t,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(0, -2), (2, 2)$

Vertical tangents: $\dfrac{dx}{dt} = -1 \neq 0$; none



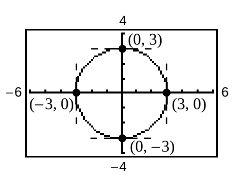

**29.** $x = 3\cos\theta, y = 3\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 3\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Points: $(0, 3), (0, -3)$

Vertical tangents: $\dfrac{dx}{d\theta} = -3\sin\theta = 0$ when $\theta = 0, \pi.$

Points: $(3, 0), (-3, 0)$



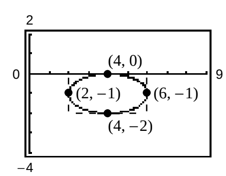

**31.** $x = 4 + 2\cos\theta,\ y = -1 + \sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Points: $(4, 0), (4, -2)$

Vertical tangents: $\dfrac{dx}{d\theta} = -2\sin\theta = 0$ when $x = 0, \pi.$

Points: $(6, -1), (2, -1)$

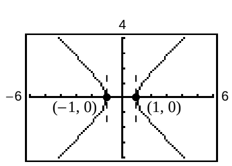



**33.** $x = \sec\theta,\ y = \tan\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \sec^2\theta \neq 0$; none

Vertical tangents: $\dfrac{dx}{d\theta} = \sec\theta\tan\theta = 0$ when $x = 0, \pi.$

Points: $(1, 0), (-1, 0)$




**35.** $x = t^2, y = 2t, 0 \le t \le 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2+1)$$

$$s = 2\int_0^2 \sqrt{t^2+1}\,dt$$

$$= \left[t\sqrt{t^2+1} + \ln\left|t + \sqrt{t^2+1}\right|\right]_0^2$$

$$= 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

**37.** $x = e^{-t}\cos t,\ y = e^{-t}\sin t,\ 0 \le t \le \dfrac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t),\ \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

$$s = \int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2}\sqrt{2e^{-2t}}\,dt = -\sqrt{2}\int_0^{\pi/2} e^{-t}(-1)\,dt$$

$$= \Big[-\sqrt{2}e^{-t}\Big]_0^{\pi/2} = \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12$$

**39.** $x = \sqrt{t},\ y = 3t - 1,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}},\ \dfrac{dy}{dt} = 3$

$$S = \int_0^1 \sqrt{\frac{1}{4t} + 9}\,dt = \frac{1}{2}\int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}}\,dt$$

$$= \frac{1}{6}\int_0^6 \sqrt{1 + u^2}\,du$$

$$= \frac{1}{12}\Big[\ln\left(\sqrt{1 + u^2} + u\right) + u\sqrt{1 + u^2}\Big]_0^6$$

$$= \frac{1}{12}\Big[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\Big] \approx 3.249$$

$$u = 6\sqrt{t},\ du = \frac{3}{\sqrt{t}}\,dt$$

**41.** $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$S = 4\int_0^{\pi/2}\sqrt{9a^2\cos^4\theta\sin^2 + 9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$= 12a\int_0^{\pi/2}\sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 6a\int_0^{\pi/2}\sin 2\theta\,d\theta = \Big[-3a\cos 2\theta\Big]_0^{\pi/2} = 6a$$

**43.** $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),$

$$\frac{dx}{d\theta} = a(1 - \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

$$S = 2\int_0^{\pi}\sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

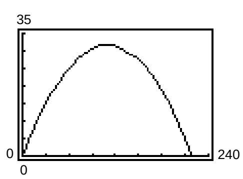
$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{1 - \cos\theta}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\frac{\sin\theta}{\sqrt{1 + \cos\theta}}\,d\theta$$

$$= \Big[-4\sqrt{2}a\sqrt{1 + \cos\theta}\Big]_0^{\pi} = 8a$$

**45.** $x = (90\cos 30°)t,\ y = (90\sin 30°)t - 16t^2$

(a)




(b) Range: 219.2 ft

(c) $\dfrac{dx}{dt} = 90\cos 30°,\ \dfrac{dy}{dt} = 90\sin 30° - 32t.$

$$y = 0 \text{ for } t = \frac{45}{16}.$$

$$s = \int_0^{45/16}\sqrt{(90\cos 30°)^2 + (90\sin 30° - 32t)^2}\,dt$$

$$= 230.8 \text{ ft}$$

(d) $y = 0 \implies (90\sin\theta)t = 16t^2 \implies t = \dfrac{90}{16}\sin\theta$

$$x = (90\cos\theta)t = \frac{90^2}{16}\cos\theta\sin\theta = \frac{90^2}{32}\sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32}2\cos 2\theta = 0 \implies \theta = 45°$$

By the First Derivative Test, $\theta = 45°\left(\dfrac{\pi}{4}\right)$ maximizes the range.

$$\frac{dx}{dt} = 90\cos\theta,$$

$$\frac{dy}{dt} = 90\sin\theta - 32 = 90\sin\theta - 32\left(\frac{90}{16}\sin\theta\right) = -90\sin\theta$$

$$s = \int_0^{(90/16)\sin\theta}\sqrt{(90\cos\theta)^2 + (-90\sin\theta)^2}\,dt$$

$$= \int_0^{(90/16)\sin\theta} 90\,dt = 90t\Big]_0^{(90/16)\sin\theta}$$

$$= \frac{90^2}{16}\sin\theta$$

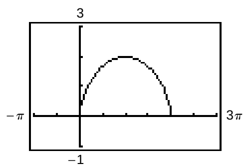
$$\frac{ds}{d\theta} = \frac{90^2}{16}\cos\theta = 0 \implies \theta = \frac{\pi}{2}$$

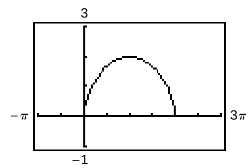
By the First Derivative Test, $\theta = 90°$ maximizes the arc length.

**47.** (a) $x = t - \sin t$ $\quad x = 2t - \sin(2t)$

$y = 1 - \cos t$ $\quad y = 1 - \cos(2t)$

$0 \le t \le 2\pi$ $\quad 0 \le t \le \pi$




(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right)$

$y = 1 - \cos\left(\frac{1}{2}t\right)$

The time required for the particle to traverse the same path is $t = 4\pi$.

**49.** $x = t,\ y = 2t,\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = 2$

(a) $S = 2\pi\displaystyle\int_0^4 2t\sqrt{1+4}\,dt = 4\sqrt{5}\pi\int_0^4 t\,dt$

$= \left[2\sqrt{5}\pi t^2\right]_0^4 = 32\pi\sqrt{5}$

(b) $S = 2\pi\displaystyle\int_0^4 t\sqrt{1+4}\,dt = 2\sqrt{5}\pi\int_0^4 t\,dt$

$= \left[\sqrt{5}\pi t^2\right]_0^4 = 16\pi\sqrt{5}$

**51.** $x = 4\cos\theta,\ y = 4\sin\theta,\ \dfrac{dx}{d\theta} = -4\sin\theta,\ \dfrac{dy}{d\theta} = 4\cos\theta$

$$S = 2\pi\int_0^{\pi/2} 4\cos\theta\sqrt{(-4\sin\theta)^2 + (4\cos\theta)^2}\,d\theta$$

$$= 32\pi\int_0^{\pi/2}\cos\theta\,d\theta = \Big[32\pi\sin\theta\Big]_0^{\pi/2} = 32\pi$$

**53.** $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

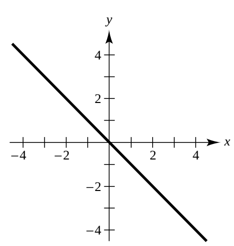
$$S = 4\pi\int_0^{\pi/2} a\sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta = 12a^2\pi\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta = \frac{12\pi a^2}{5}\Big[\sin^5\theta\Big]_0^{\pi/2} = \frac{12}{5}\pi a^2$$

**55.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

See Theorem 9.7, page 675.

**57.** One possible answer is the graph given by

$x = t, y = -t.$




**59.** $s = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

See Theorem 9.8, page 678.

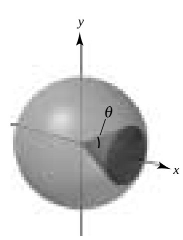
**61.** $x = r\cos\phi,\ y = r\sin\phi$

$$S = 2\pi\int_0^{\theta} r\sin\phi\sqrt{r^2\sin^2\phi + r^2\cos^2\phi}\,d\phi$$

$$= 2\pi r^2\int_0^{\theta}\sin\phi\,d\phi$$

$$= \Big[-2\pi r^2\cos\phi\Big]_0^{\theta}$$

$$= 2\pi r^2(1 - \cos\theta)$$



**63.** $x = \sqrt{t},\ y = 4 - t,\ 0 \le t \le 4$

$$A = \int_0^4 (4-t)\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^4 (4t^{-1/2} - t^{1/2})\,dt = \left[\frac{1}{2}\left(8\sqrt{t} - \frac{2}{3}t\sqrt{t}\right)\right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{3}{16}\int_0^4 (4-t)\sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)dt = \frac{3}{32}\int_0^4 (4-t)\,dt = \left[\frac{3}{32}\left(4t - \frac{t^2}{2}\right)\right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{3}{32}\int_0^4 (4-t)^2\frac{1}{2\sqrt{t}}\,dt = \frac{3}{64}\int_0^4 [16t^{-1/2} - 8t^{1/2} + t^{3/2}]\,dt = \frac{3}{64}\left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t}\right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5}\right)$$

**65.** $x = 3\cos\theta,\ y = 3\sin\theta,\ \dfrac{dx}{d\theta} = -3\sin\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-3\sin\theta)\,d\theta \\ &= -54\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -54\pi\int_{\pi/2}^0 (1-\cos^2\theta)\sin\theta\,d\theta \\ &= -54\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 36\pi \end{aligned}$$

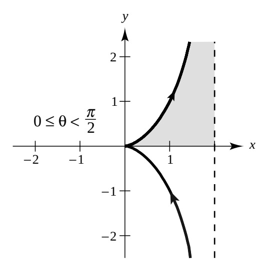
**67.** $x = 2\sin^2\theta$

$y = 2\sin^2\theta\tan\theta$

$\dfrac{dx}{d\theta} = 4\sin\theta\cos\theta$

$$A = \int_0^{\pi/2} 2\sin^2\theta\tan\theta(4\sin\theta\cos\theta)\,d\theta = 8\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= 8\left[\frac{-\sin^3\theta\cos\theta}{4} - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$

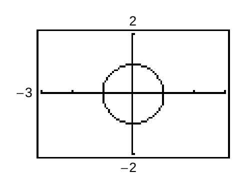



**69.** $\pi ab$ is area of ellipse (d).

**71.** $6\pi a^2$ is area of cardioid (f).

**73.** $\frac{8}{3}ab$ is area of hourglass (a).

**75.** (a) $x = \dfrac{1-t^2}{1+t^2},\ y = \dfrac{2t}{1+t^2},\ -20 \le t \le 20$




The graph is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

Verify: $x^2 + y^2 = \left(\dfrac{1-t^2}{1+t^2}\right)^2 + \left(\dfrac{2t}{1+t^2}\right)^2 = \dfrac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2} = \dfrac{(1+t^2)^2}{(1+t^2)^2} = 1$

(b) As $t$ increases from $-20$ to 0, the speed increases, and as $t$ increases from 0 to 20, the speed decreases.

**77.** False

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

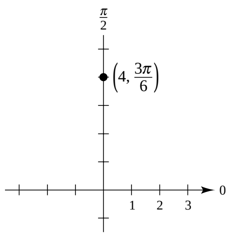
## Section 9.4 Polar Coordinates and Polar Graphs

**1.** $\left(4, \dfrac{\pi}{2}\right)$

$x = 4\cos\left(\dfrac{\pi}{2}\right) = 0$

$y = 4\sin\left(\dfrac{\pi}{2}\right) = 4$

$(x, y) = (0, 4)$

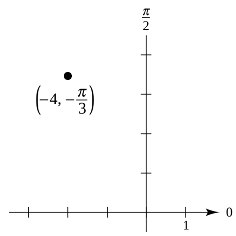



**3.** $\left(-4, -\dfrac{\pi}{3}\right)$

$x = -4\cos\left(-\dfrac{\pi}{3}\right) = -2$

$y = -4\sin\left(-\dfrac{\pi}{3}\right) = 2\sqrt{3}$

$(x, y) = \left(-2, 2\sqrt{3}\right)$

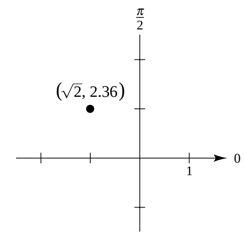



**5.** $\left(\sqrt{2}, 2.36\right)$

$x = \sqrt{2}\cos(2.36) \approx -1.004$

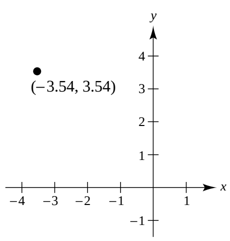
$y = \sqrt{2}\sin(2.36) \approx 0.996$

$(x, y) = (-1.004, 0.996)$



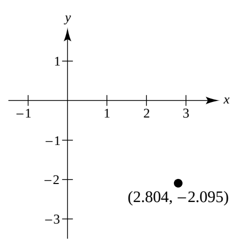

**7.** $(r, \theta) = \left(5, \dfrac{3\pi}{4}\right)$

$(x, y) = (-3.5355, 3.5355)$




**9.** $(r, \theta) = (-3.5, 2.5)$

$(x, y) = (2.804, -2.095)$

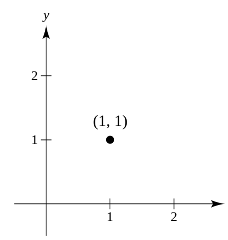



**11.** $(x, y) = (1, 1)$

$r = \pm\sqrt{2}$

$\tan\theta = 1$

$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \left(\sqrt{2}, \dfrac{\pi}{4}\right), \left(-\sqrt{2}, \dfrac{5\pi}{4}\right)$

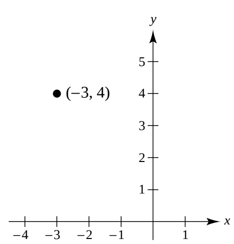



**13.** $(x, y) = (-3, 4)$

$r = \pm\sqrt{9 + 16} = \pm 5$

$\tan\theta = -\frac{4}{3}$

$\theta \approx 2.214, 5.356, (5, 2.214), (-5, 5.356)$



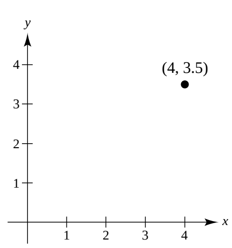

**15.** $(x, y) = (3, -2)$

$(r, \theta) = (3.606, -0.588)$

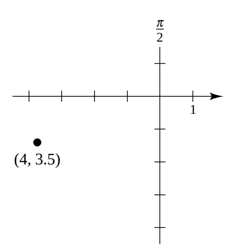
**17.** $(x, y) = \left(\frac{5}{2}, \frac{4}{3}\right)$

$(r, \theta) = (2.833, 0.490)$

**19.** (a) $(x, y) = (4, 3.5)$

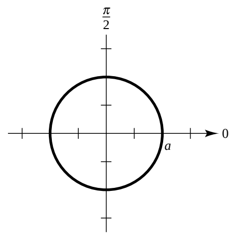



(b) $(r, \theta) = (4, 3.5)$



**21.** $x^2 + y^2 = a^2$

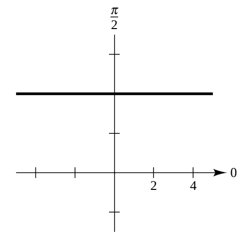
$r = a$




**23.** $y = 4$

$r \sin \theta = 4$

$r = 4 \csc \theta$

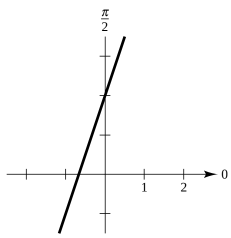



**25.** $3x - y + 2 = 0$

$3r \cos \theta - r \sin \theta + 2 = 0$

$r(3 \cos \theta - \sin \theta) = -2$

$$r = \frac{-2}{3 \cos \theta - \sin \theta}$$

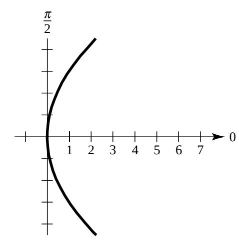



**27.** $y^2 = 9x$

$r^2 \sin^2 \theta = 9r \cos \theta$

$$r = \frac{9 \cos \theta}{\sin^2 \theta}$$

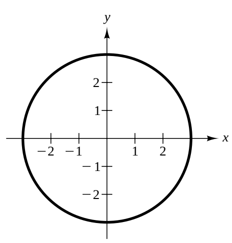
$r = 9 \csc^2 \theta \cos \theta$




**29.** $r = 3$

$r^2 = 9$

$x^2 + y^2 = 9$



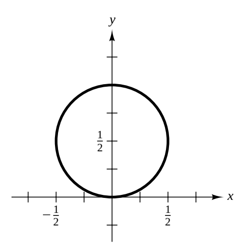

**31.** $r = \sin \theta$

$r^2 = r \sin \theta$

$x^2 + y^2 = y$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$x^2 + y^2 - y = 0$

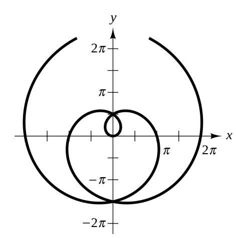



**33.** $r = \theta$

$\tan r = \tan \theta$

$$\tan \sqrt{x^2 + y^2} = \frac{y}{x}$$

$$\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$$

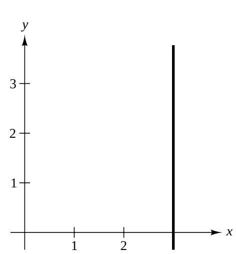



**35.** $r = 3 \sec \theta$

$r \cos \theta = 3$

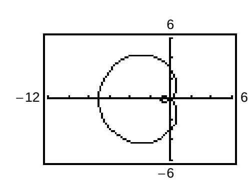
$x = 3$

$x - 3 = 0$



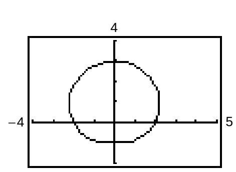

**37.** $r = 3 - 4 \cos \theta$

$0 \le \theta < 2\pi$



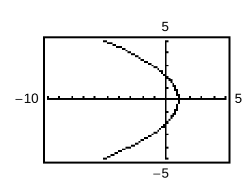

**39.** $r = 2 + \sin \theta$

$0 \le \theta < 2\pi$



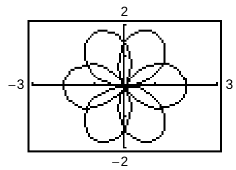

**41.** $r = \dfrac{2}{1 + \cos \theta}$

Traced out once on $-\pi < \theta < \pi$



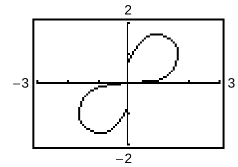
**43.** $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$



**45.** $r^2 = 4\sin 2\theta$

$0 \le \theta < \dfrac{\pi}{2}$



**47.**

$$\begin{aligned} r &= 2(h\cos\theta + k\sin\theta) \\ r^2 &= 2r(h\cos\theta + k\sin\theta) \\ r^2 &= 2[h(r\cos\theta) + k(r\sin\theta)] \\ x^2 + y^2 &= 2(hx + ky) \\ x^2 + y^2 - 2hx - 2ky &= 0 \\ (x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) &= 0 + h^2 + k^2 \\ (x-h)^2 + (y-k)^2 &= h^2 + k^2 \end{aligned}$$

Radius: $\sqrt{h^2 + k^2}$

Center: $(h, k)$

**49.** $\left(4, \dfrac{2\pi}{3}\right), \left(2, \dfrac{\pi}{6}\right)$

$$d = \sqrt{4^2 + 2^2 - 2(4)(2)\cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)}$$

$$= \sqrt{20 - 16\cos\frac{\pi}{2}} = 2\sqrt{5} \approx 4.5$$

**51.** $(2, 0.5)$, $(7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7)\cos(0.5 - 1.2)}$$

$$= \sqrt{53 - 28\cos(-0.7)} \approx 5.6$$

**53.** $r = 2 + 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + \cos\theta(2 + 3\sin\theta)}{3\cos\theta\cos\theta - \sin\theta(2 + 3\sin\theta)}$$

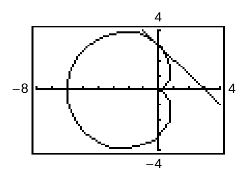
$$= \frac{2\cos\theta(3\sin\theta + 1)}{3\cos 2\theta - 2\sin\theta} = \frac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3}$$

At $\left(5, \dfrac{\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

At $(2, \pi)$, $\dfrac{dy}{dx} = -\dfrac{2}{3}$.

At $\left(-1, \dfrac{3\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

**55.** (a), (b) $r = 3(1 - \cos\theta)$



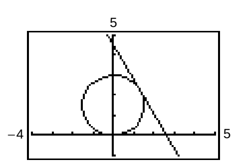
$(r, \theta) = \left(3, \dfrac{\pi}{2}\right) \implies (x, y) = (0, 3)$

Tangent line: $y - 3 = -1(x - 0)$

$y = -x + 3$

(c) At $\theta = \dfrac{\pi}{2}, \dfrac{dy}{dx} = -1.0$.

**57.** (a), (b) $r = 3\sin\theta$



$(r, \theta) = \left(\dfrac{3\sqrt{3}}{2}, \dfrac{\pi}{3}\right) \implies (x, y) = \left(\dfrac{3\sqrt{3}}{4}, \dfrac{9}{4}\right)$

Tangent line: $y - \dfrac{9}{4} = -\sqrt{3}\left(x - \dfrac{3\sqrt{3}}{4}\right)$

$$y = -\sqrt{3}x + \frac{9}{2}$$

(c) At $\theta = \dfrac{\pi}{3}, \dfrac{dy}{dx} = -\sqrt{3} \approx -1.732$.

**59.** $r = 1 - \sin\theta$

$$\frac{dy}{d\theta} = (1 - \sin\theta)\cos\theta - \cos\theta\sin\theta$$
$$= \cos\theta(1 - 2\sin\theta) = 0$$

$\cos\theta = 0, \sin\theta = \dfrac{1}{2} \implies \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Horizontal tangents: $\left(2, \dfrac{3\pi}{2}\right), \left(\dfrac{1}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{1}{2}, \dfrac{5\pi}{6}\right)$

$$\frac{dx}{d\theta} = (-1 + \sin\theta)\sin\theta - \cos\theta\cos\theta$$
$$= -\sin\theta + \sin^2\theta + \sin^2\theta - 1$$
$$= 2\sin^2\theta - \sin\theta - 1$$
$$= (2\sin\theta + 1)(\sin\theta - 1) = 0$$

$\sin\theta = 1, \sin\theta = -\dfrac{1}{2} \implies \theta = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Vertical tangents: $\left(\dfrac{3}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3}{2}, \dfrac{11\pi}{6}\right)$

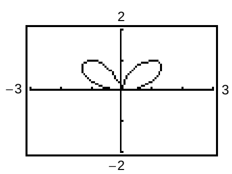
**61.** $r = 2\csc\theta + 3$

$$\frac{dy}{d\theta} = (2\csc\theta + 3)\cos\theta + (-2\csc\theta\cot\theta)\sin\theta$$
$$= 3\cos\theta = 0$$

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Horizontal: $\left(5, \dfrac{\pi}{2}\right), \left(1, \dfrac{3\pi}{2}\right)$

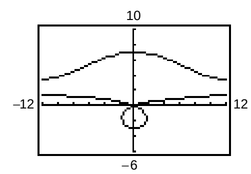
**63.** $r = 4\sin\theta\cos^2\theta$




Horizontal tangents:

(0, 0), (1.4142, 0.7854), (1.4142, 2.3562)

**65.** $r = 2\csc\theta + 5$



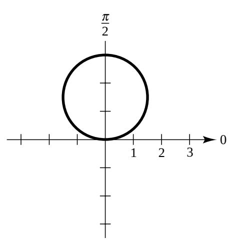

Horizontal tangents: $\left(7, \dfrac{\pi}{2}\right), \left(3, \dfrac{3\pi}{2}\right)$

**67.**
$$r = 3\sin\theta$$
$$r^2 = 3r\sin\theta$$
$$x^2 + y^2 = 3y$$
$$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

Circle $r = \dfrac{3}{2}$

Center: $\left(0, \dfrac{3}{2}\right)$

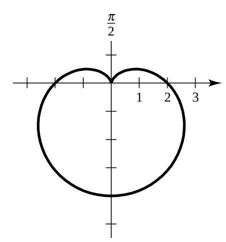
Tangent at the pole: $\theta = 0$




**69.** $r = 2(1 - \sin\theta)$

Cardioid

Symmetric to $y$-axis, $\theta = \dfrac{\pi}{2}$




**71.** $r = 2\cos(3\theta)$

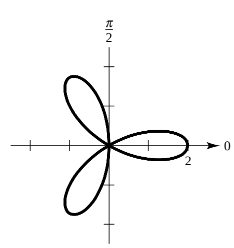
Rose curve with three petals

Symmetric to the polar axis

Relative extrema: (2, 0), $\left(-2, \dfrac{\pi}{3}\right), \left(2, \dfrac{2\pi}{3}\right)$

| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|
| $r$ | 2 | 0 | $-\sqrt{2}$ | $-2$ | 0 | 2 | 0 | $-2$ |

Tangents at the pole: $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$



**73.** $r = 3\sin 2\theta$

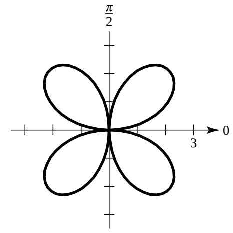
Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 3, \dfrac{\pi}{4}\right), \left(\pm 3, \dfrac{5\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$

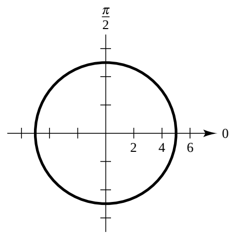
($\theta = \pi,\ 3\pi/2$ give the same tangents.)




**75.** $r = 5$

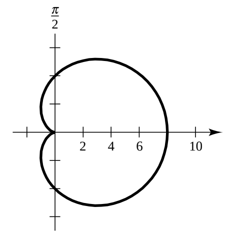
Circle radius: 5

$x^2 + y^2 = 25$




**77.** $r = 4(1 + \cos\theta)$

Cardioid

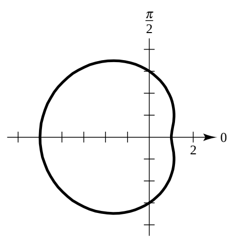



**79.** $r = 3 - 2\cos\theta$

Limaçon

Symmetric to polar axis

| $\theta$ | 0 | $\dfrac{\pi}{3}$ | $\dfrac{\pi}{2}$ | $\dfrac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | 1 | 2 | 3 | 4 | 5 |

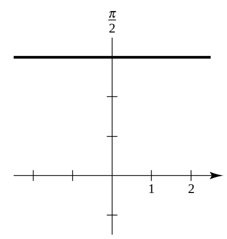



**81.** $r = 3\csc\theta$

$r\sin\theta = 3$

$y = 3$

Horizontal line

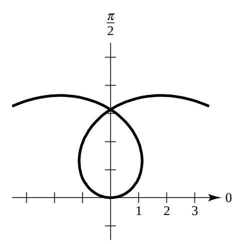



**83.** $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \dfrac{\pi}{2}$

| $\theta$ | 0 | $\dfrac{\pi}{4}$ | $\dfrac{\pi}{2}$ | $\dfrac{3\pi}{4}$ | $\pi$ | $\dfrac{5\pi}{4}$ | $\dfrac{3\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $r$ | 0 | $\dfrac{\pi}{2}$ | $\pi$ | $\dfrac{3\pi}{2}$ | $2\pi$ | $\dfrac{5\pi}{2}$ | $3\pi$ |




Tangent at the pole: $\theta = 0$

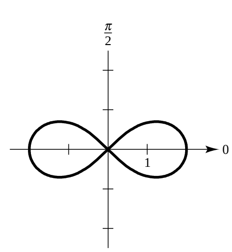
**85.** $r^2 = 4\cos(2\theta)$

Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(\pm 2, 0)$

| $\theta$ | 0 | $\dfrac{\pi}{6}$ | $\dfrac{\pi}{4}$ |
|---|---|---|---|
| $r$ | $\pm 2$ | $\pm\sqrt{2}$ | 0 |



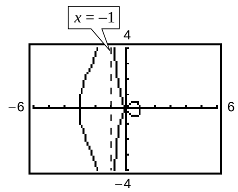

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

**87.** Since

$$r = 2 - \sec\theta = 2 - \frac{1}{\cos\theta},$$

the graph has polar axis symmetry and the lengths at the pole are

$$\theta = \frac{\pi}{3}, \frac{-\pi}{3}.$$




Furthermore,

$$r \Longrightarrow -\infty \text{ as } \theta \Longrightarrow \frac{\pi}{2^-}$$

$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow -\frac{\pi}{2^+}.$$

Also, $r = 2 - \dfrac{1}{\cos\theta} = 2 - \dfrac{r}{r\cos\theta} = 2 - \dfrac{r}{x}$

$$rx = 2x - r$$

$$r = \frac{2x}{1+x}.$$

Thus, $r \Longrightarrow \pm\infty$ as $x \Longrightarrow -1$.

**89.** $r = \dfrac{2}{\theta}$

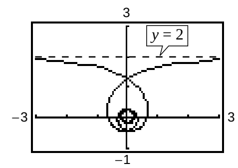
Hyperbolic spiral

$r \Longrightarrow \infty$ as $\theta \Longrightarrow 0$

$$r = \frac{2}{\theta} \Longrightarrow \theta = \frac{2}{r} = \frac{2\sin\theta}{r\sin\theta} = \frac{2\sin\theta}{y}$$

$$y = \frac{2\sin\theta}{\theta}$$

$$\lim_{\theta\to 0}\frac{2\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{2\cos\theta}{1} = 2$$




**91.** The rectangular coordinate system consists of all points of the form $(x, y)$ where $x$ is the directed distance from the $y$-axis to the point, and $y$ is the directed distance from the $x$-axis to the point. Every point has a unique representation.

The polar coordinate system uses $(r, \theta)$ to designate the location of a point.

$r$ is the directed distance to the origin and $\theta$ is the angle the point makes with the positive $x$-axis, measured clockwise.

Point do not have a unique polar representation.

**93.** $r = a$ circle

$\theta = b$ line

**95.** $r = 2\sin\theta$ circle

Matches (c)

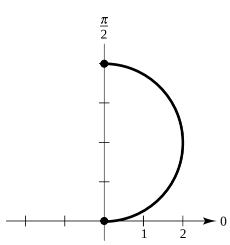
**97.** $r = 3(1 + \cos\theta)$

Cardioid

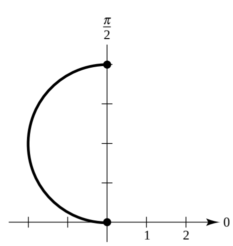
Matches (a)

**99.** $r = 4\sin\theta$

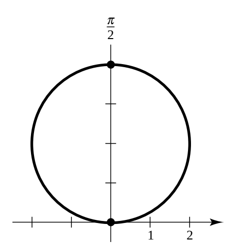
(a) $0 \le \theta \le \dfrac{\pi}{2}$




(b) $\dfrac{\pi}{2} \le \theta \le \pi$




(c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

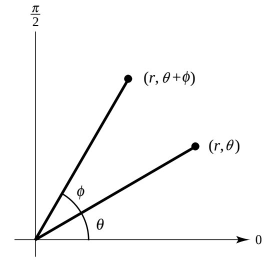


**101.** Let the curve $r = f(\theta)$ be rotated by $\phi$ to form the curve $r = g(\theta)$. If $(r_1, \theta_1)$ is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$$g(\theta_1 + \phi) = r_1 = f(\theta_1).$$

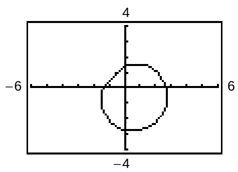
Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, we see that

$$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi).$$

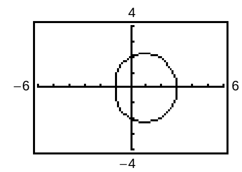



**103.** $r = 2 - \sin\theta$

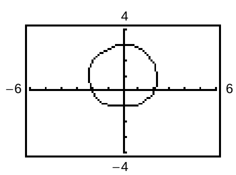
(a) $r = 2 - \sin\left(\theta - \dfrac{\pi}{4}\right) = 2 - \dfrac{\sqrt{2}}{2}(\sin\theta - \cos\theta)$



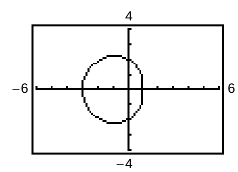

(b) $r = 2 - (-\cos\theta) = 2 + \cos\theta$



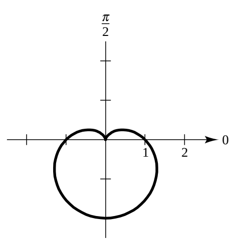

(c) $r = 2 - (-\sin\theta) = 2 + \sin\theta$




(d) $r = 2 - \cos\theta$




**105.** (a) $r = 1 - \sin\theta$

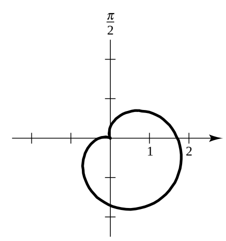



(b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

Rotate the graph of

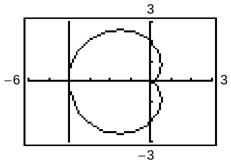
$r = 1 - \sin\theta$

through the angle $\pi/4$.



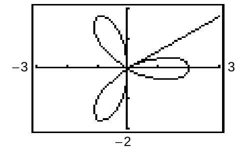

**107.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2(1 - \cos\theta)}{2\sin\theta}$

At $\theta = \pi$, $\tan\psi$ is undefined $\Longrightarrow \psi = \dfrac{\pi}{2}$.

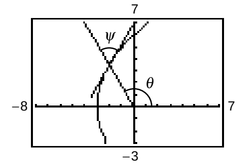



**109.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2\cos 3\theta}{-6\sin 3\theta}$

At $\theta = \dfrac{\pi}{6}$, $\tan\psi = 0 \Longrightarrow \psi = 0$.



**111.** $r = \dfrac{6}{1-\cos\theta} = 6(1-\cos\theta)^{-1} \Rightarrow \dfrac{dr}{d\theta} = \dfrac{6\sin\theta}{(1-\cos\theta)^2}$




$$\tan\psi = \frac{r}{\dfrac{dr}{d\theta}} = \frac{\dfrac{6}{(1-\cos\theta)}}{\dfrac{6\sin\theta}{(1-\cos\theta)^2}} = \frac{1-\cos\theta}{\sin\theta}$$

At $\theta = \dfrac{2\pi}{3}$, $\tan\psi = \dfrac{1-\left(-\dfrac{1}{2}\right)}{\dfrac{\sqrt{3}}{2}} = \sqrt{3}.$

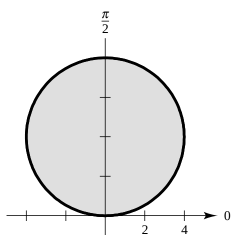
$\psi = \dfrac{\pi}{3}$, $(60°)$

**113.** True

**115.** True

## Section 9.5 Area and Arc Length in Polar Coordinates

**1.** (a) $r = 8\sin\theta$




$A = \pi(4)^2 = 16\pi$

(b) $A = 2\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/2}\left[8\sin\theta\right]^2 d\theta$

$$= 64\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= 32\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= 32\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 16\pi$$

**3.** $A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/6}(2\cos 3\theta)^2\,d\theta\right] = 2\left[\theta + \dfrac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \dfrac{\pi}{3}$

**5.** $A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/4}(\cos 2\theta)^2\,d\theta\right]$

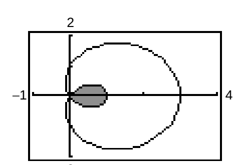
$$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8}$$

**7.** $A = 2\left[\dfrac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/2}(1-\sin\theta)^2\,d\theta\right]$

$$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

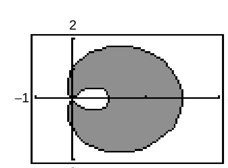
**9.** $A = 2\left[\dfrac{1}{2}\displaystyle\int_{2\pi/3}^{\pi}(1+2\cos\theta)^2\,d\theta\right]$

$$= \left[3\theta + 4\sin\theta + \sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$$




**11.** The area inside the outer loop is

$$2\left[\frac{1}{2}\int_0^{2\pi/3}(1+2\cos\theta)^2\,d\theta\right] = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}.$$




From the result of Exercise 9, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2}\right) - \left(\frac{2\pi - 3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$

**13.** $r = 1 + \cos\theta$

$r = 1 - \cos\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \cos\theta$$
$$2\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing $r$ by $-r$ and $\theta$ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \cos\theta$, $\cos\theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right), (0, 0)$

**15.** $r = 1 + \cos\theta$

$r = 1 - \sin\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \sin\theta$$
$$\cos\theta = -\sin\theta$$
$$\tan\theta = -1$$
$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing $r$ by $-r$ and $\theta$ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \sin\theta$, $\sin\theta + \cos\theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{2-\sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(\frac{2+\sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$

**17.** $r = 4 - 5\sin\theta$

$r = 3\sin\theta$

Solving simultaneously,

$$4 - 5\sin\theta = 3\sin\theta$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$

**19.** $r = \dfrac{\theta}{2}$

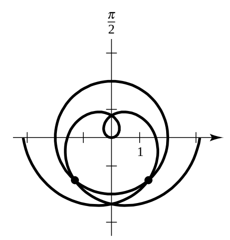
$r = 2$

Solving simultaneously, we have

$\theta/2 = 2$, $\theta = 4$.

Points of intersection:

$(2, 4), (-2, -4)$

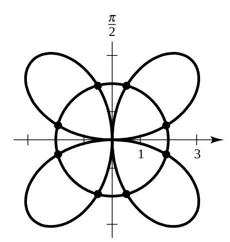



**21.** $r = 4\sin 2\theta$

$r = 2$

$r = 4\sin 2\theta$ is the equation of a rose curve with four petals and is symmetric to the polar axis, $\theta = \pi/2$, and the pole. Also, $r = 2$ is the equation of a circle of radius 2 centered at the pole. Solving simultaneously,

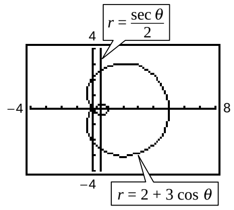
$$4\sin 2\theta = 2$$
$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}.$$




Therefore, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. By symmetry, the other points of intersection are $(2, 7\pi/12)$, $(2, 11\pi/12)$, $(2, 13\pi/12)$, $(2, 17\pi/12)$, $(2, 19\pi/12)$, and $(2, 23\pi/12)$.

**23.** $r = 2 + 3\cos\theta$

$r = \dfrac{\sec\theta}{2}$




The graph of $r = 2 + 3\cos\theta$ is a limaçon with an inner loop $(b > a)$ and is symmetric to the polar axis. The graph of $r = (\sec\theta)/2$ is the vertical line $x = 1/2$. Therefore, there are four points of intersection. Solving simultaneously,

$$2 + 3\cos\theta = \frac{\sec\theta}{2}$$

$$6\cos^2\theta + 4\cos\theta - 1 = 0$$

$$\cos\theta = \frac{-2 \pm \sqrt{10}}{6}$$

$$\theta = \arccos\left(\frac{-2 + \sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2 - \sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection: $(-0.581, \pm 2.607)$, $(2.581, \pm 1.376)$

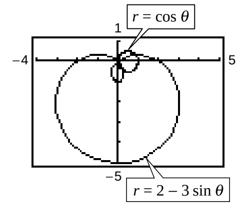
**25.** $r = \cos\theta$

$r = 2 - 3\sin\theta$

Points of intersection:

$(0, 0)$, $(0.935, 0.363)$, $(0.535, -1.006)$

The graphs reach the pole at different times ($\theta$ values).

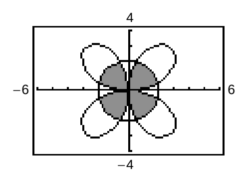



**27.** From Exercise 21, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2}\int_0^{\pi/12} (4\sin 2\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} (4\sin 2\theta)^2\,d\theta$$

$$= 16\int_0^{\pi/12} \sin^2(2\theta)\,d\theta + 2\int_{\pi/12}^{5\pi/12} d\theta \text{ (by symmetry of the petal)}$$

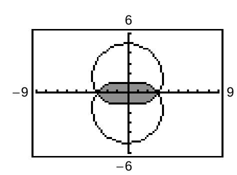
$$= 8\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$




$$\text{Total area} = 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}\left(4\pi - 3\sqrt{3}\right)$$

**29.** $$A = 4\left[\frac{1}{2}\int_0^{\pi/2} (3 - 2\sin\theta)^2\,d\theta\right]$$

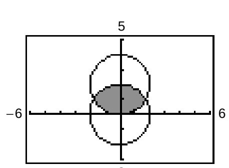
$$= 2\left[11\theta + 12\cos\theta - \sin(2\theta)\right]_0^{\pi/2} = 11\pi - 24$$




**31.** $$A = 2\left[\frac{1}{2}\int_0^{\pi/6} (4\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (2)^2\,d\theta\right]$$

$$= 16\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/6} + \left[4\theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}\left(4\pi - 3\sqrt{3}\right)$$




**33.** $$A = 2\left[\frac{1}{2}\int_0^{\pi} [a(1 + \cos\theta)]^2\,d\theta\right] - \frac{a^2\pi}{4}$$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} - \frac{a^2\pi}{4}$$

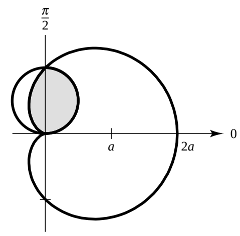
$$= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4}$$

**35.** $A = \dfrac{\pi a^2}{8} + \dfrac{1}{2}\displaystyle\int_{\pi/2}^{\pi} [a(1 + \cos\theta)]^2\, d\theta$

$= \dfrac{\pi a^2}{8} + \dfrac{a^2}{2}\displaystyle\int_{\pi/2}^{\pi}\left(\dfrac{3}{2} + 2\cos\theta + \dfrac{\cos 2\theta}{2}\right) d\theta$

$= \dfrac{\pi a^2}{8} + \dfrac{a^2}{2}\left[\dfrac{3}{2}\theta + 2\sin\theta + \dfrac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$

$= \dfrac{\pi a^2}{8} + \dfrac{a^2}{2}\left[\dfrac{3\pi}{2} - \dfrac{3\pi}{4} - 2\right] = \dfrac{a^2}{2}[\pi - 2]$

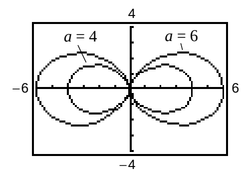



**37.** (a) $r = a\cos^2\theta$

$r^3 = ar^2\cos^2\theta$

$(x^2 + y^2)^{3/2} = ax^2$

(b)




(c) $A = 4\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/2} [(6\cos^2\theta)^2 - (4\cos^2\theta)^2]\, d\theta = 40\int_0^{\pi/2} \cos^4\theta\, d\theta = 10\int_0^{\pi/2}(1 + \cos 2\theta)^2\, d\theta$

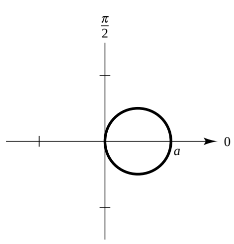
$= 10\displaystyle\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \dfrac{1 - \cos 4\theta}{2}\right) d\theta = 10\left[\dfrac{3}{2}\theta + \sin 2\theta + \dfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \dfrac{15\pi}{2}$

**39.** $r = a\cos(n\theta)$

For $n = 1$:

$r = a\cos\theta$

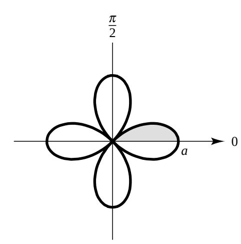
$A = \pi\left(\dfrac{a}{2}\right)^2 = \dfrac{\pi a^2}{4}$




For $n = 2$:

$r = a\cos 2\theta$

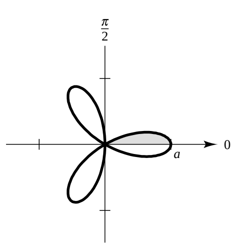
$A = 8\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/4}(a\cos 2\theta)^2\, d\theta = \dfrac{\pi a^2}{2}$




For $n = 3$:

$r = a\cos 3\theta$

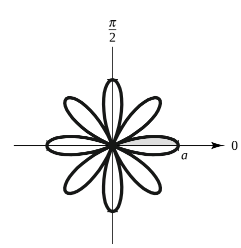
$A = 6\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/6}(a\cos 3\theta)^2\, d\theta = \dfrac{\pi a^2}{4}$




For $n = 4$:

$r = a\cos 4\theta$

$A = 16\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/8}(a\cos 4\theta)^2\, d\theta = \dfrac{\pi a^2}{2}$




In general, the area of the region enclosed by $r = a\cos(n\theta)$ for $n = 1, 2, 3, \ldots$ is $(\pi a^2)/4$ if $n$ is odd and is $(\pi a^2)/2$ if $n$ is even.

**41.** $r = a$

$r' = 0$

$$s = \int_0^{2\pi} \sqrt{a^2 + 0^2}\, d\theta = \Big[a\theta\Big]_0^{2\pi} = 2\pi a$$

(circumference of circle of radius $a$)

**43.** $r = 1 + \sin\theta$

$r' = \cos\theta$

$$s = 2\int_{\pi/2}^{3\pi/2} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

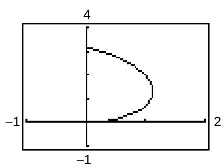
$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \sqrt{1+\sin\theta}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta$$

$$= \Big[4\sqrt{2}\sqrt{1-\sin\theta}\Big]_{\pi/2}^{3\pi/2}$$

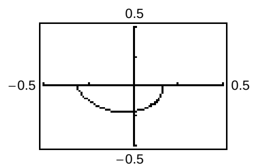
$$= 4\sqrt{2}\left(\sqrt{2} - 0\right) = 8$$

**45.** $r = 2\theta,\ 0 \le \theta \le \dfrac{\pi}{2}$



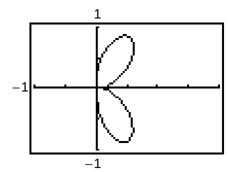

Length $\approx 4.16$

**47.** $r = \dfrac{1}{\theta},\ \pi \le \theta \le 2\pi$




Length $\approx 0.71$

**49.** $r = \sin(3\cos\theta),\ 0 \le \theta \le \pi$




Length $\approx 4.39$

**51.** $r = 6\cos\theta$

$r' = -6\sin\theta$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta\sqrt{36\cos^2\theta + 36\sin^2\theta}\ d\theta$$

$$= 72\pi\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$= \Big[36\pi\sin^2\theta\Big]_0^{\pi/2}$$

$$= 36\pi$$

**53.** $r = e^{a\theta}$

$r' = ae^{a\theta}$

$$S = 2\pi\int_0^{\pi/2} e^{a\theta}\cos\theta\sqrt{(e^{a\theta})^2 + (ae^{a\theta})^2}\ d\theta$$

$$= 2\pi\sqrt{1+a^2}\int_0^{\pi/2} e^{2a\theta}\cos\theta\, d\theta$$

$$= 2\pi\sqrt{1+a^2}\left[\frac{e^{2a\theta}}{4a^2+1}(2a\cos\theta + \sin\theta)\right]_0^{\pi/2}$$

$$= \frac{2\pi\sqrt{1+a^2}}{4a^2+1}(e^{\pi a} - 2a)$$

**55.** $r = 4\cos 2\theta$

$r' = -8\sin 2\theta$

$$S = 2\pi\int_0^{\pi/4} 4\cos 2\theta\sin\theta\sqrt{16\cos^2 2\theta + 64\sin^2 2\theta}\ d\theta$$

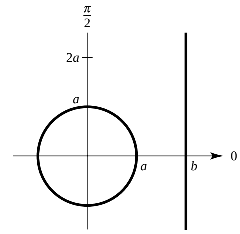
$$= 32\pi\int_0^{\pi/4} \cos 2\theta\sin\theta\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\ d\theta \approx 21.87$$

**57.** $$\text{Area} = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$$\text{Arc length} = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

**59.** (a) is correct: $s \approx 33.124$.

**61.** Revolve $r = a$ about the line $r = b \sec \theta$ where $b > a > 0$.




$$f(\theta) = a$$

$$f'(\theta) = 0$$

$$S = 2\pi \int_0^{2\pi} [b - a \cos \theta]\sqrt{a^2 + 0^2}\, d\theta$$

$$= 2\pi a \Big[ b\theta - a \sin \theta \Big]_0^{2\pi}$$

$$= 2\pi a(2\pi b) = 4\pi^2 ab$$

**63.** False. $f(\theta) = 1$ and $g(\theta) = -1$ have the same graphs.

**65.** In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Using $\theta$ instead of $t$, we have $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$. Thus,

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \text{ and } \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

Therefore, $s = \displaystyle\int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$

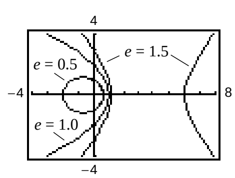
## Section 9.6 Polar Equations of Conics and Kepler's Laws

**1.** $r = \dfrac{2e}{1 + e \cos \theta}$

(a) $e = 1$, $r = \dfrac{2}{1 + \cos \theta}$, parabola

(b) $e = 0.5$, $r = \dfrac{1}{1 + 0.5 \cos \theta} = \dfrac{2}{2 + \cos \theta}$, ellipse

(c) $e = 1.5$, $r = \dfrac{3}{1 + 1.5 \cos \theta} = \dfrac{6}{2 + 3 \cos \theta}$, hyperbola

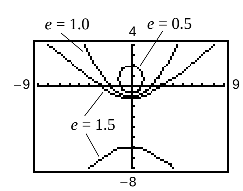



**3.** $r = \dfrac{2e}{1 - e \sin \theta}$

(a) $e = 1$, $r = \dfrac{2}{1 - \sin \theta}$, parabola

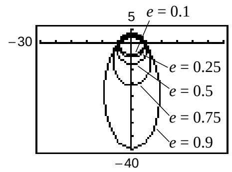
(b) $e = 0.5$, $r = \dfrac{1}{1 - 0.5 \sin \theta} = \dfrac{2}{2 - \sin \theta}$, ellipse

(c) $e = 1.5$, $r = \dfrac{3}{1 - 1.5 \sin \theta} = \dfrac{6}{2 - 3 \sin \theta}$, hyperbola



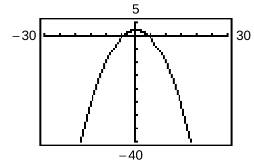
**5.** $r = \dfrac{4}{1 + e \sin \theta}$

(a)



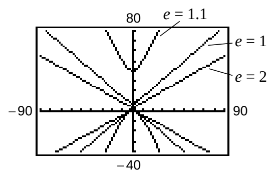

The conic is an ellipse. As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.

(b)




The conic is a parabola.

(c)




The conic is a hyperbola. As $e \to 1^+$, the hyperbolas opens more slowly, and as $e \to \infty$, they open more rapidly.

**7.** Parabola; Matches (c)

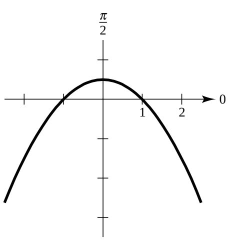
**9.** Hyperbola; Matches (a)

**11.** Ellipse; Matches (b)

**13.** $r = \dfrac{-1}{1 - \sin \theta}$

Parabola since $e = 1$

Vertex: $\left(-\dfrac{1}{2}, \dfrac{3\pi}{2}\right)$

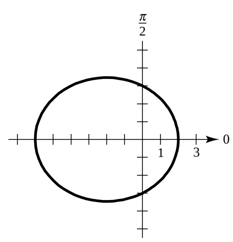



**15.** $r = \dfrac{6}{2 + \cos \theta}$

$= \dfrac{3}{1 + (1/2) \cos \theta}$

Ellipse since $e = \dfrac{1}{2} < 1$

Vertices: $(2, 0), (6, \pi)$



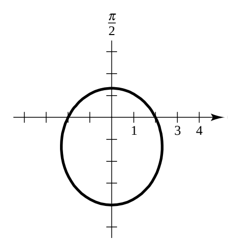

**17.** $r(2 + \sin \theta) = 4$

$r = \dfrac{4}{2 + \sin \theta}$

$= \dfrac{2}{1 + (1/2) \sin \theta}$

Ellipse since $e = \dfrac{1}{2} < 1$

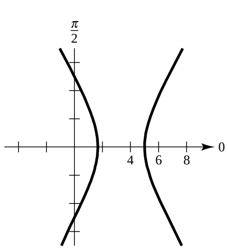
Vertices: $\left(\dfrac{4}{3}, \dfrac{\pi}{2}\right), \left(4, \dfrac{3\pi}{2}\right)$




**19.** $r = \dfrac{5}{-1 + 2 \cos \theta} = \dfrac{-5}{1 - 2 \cos \theta}$

Hyperbola since $e = 2 > 1$

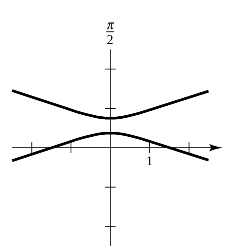
Vertices: $(5, 0), \left(-\dfrac{5}{3}, \pi\right)$

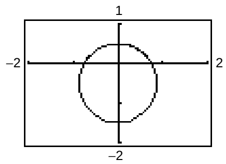



**21.** $r = \dfrac{3}{2 + 6 \sin \theta} = \dfrac{3/2}{1 + 3 \sin \theta}$

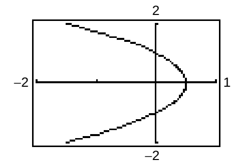
Hyperbola since $e = 3 > 1$

Vertices: $\left(\dfrac{3}{8}, \dfrac{\pi}{2}\right), \left(-\dfrac{3}{4}, \dfrac{3\pi}{2}\right)$



**23.** 


Ellipse

**25.** 

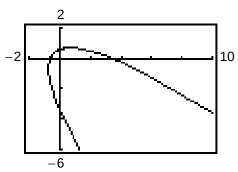

Parabola

**27.** $r = \dfrac{-1}{1 - \sin\left(\theta - \dfrac{\pi}{4}\right)}$

Rotate the graph of

$$r = \frac{-1}{1 - \sin\theta}$$

counterclockwise through the angle $\dfrac{\pi}{4}$.

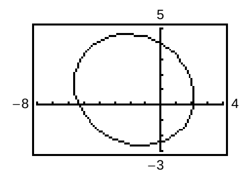



**29.** $r = \dfrac{6}{2 + \cos\left(\theta + \dfrac{\pi}{6}\right)}$

Rotate the graph of

$$r = \frac{6}{2 + \cos\theta}$$

clockwise through the angle $\dfrac{\pi}{6}$.




**31.** Change $\theta$ to $\theta + \dfrac{\pi}{4}$: $r = \dfrac{5}{5 + 3\cos\left(\theta + \dfrac{\pi}{4}\right)}$.

**33.** Parabola

$e = 1, x = -1, d = 1$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{1}{1 - \cos\theta}$$

**35.** Ellipse

$e = \dfrac{1}{2}, y = 1, d = 1$

$$r = \frac{ed}{1 + e\sin\theta}$$

$$= \frac{1/2}{1 + (1/2)\sin\theta}$$

$$= \frac{1}{2 + \sin\theta}$$

**37.** Hyperbola

$e = 2, x = 1, d = 1$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{2}{1 + 2\cos\theta}$$

**39.** Parabola

Vertex: $\left(1, -\dfrac{\pi}{2}\right)$

$e = 1, d = 2, r = \dfrac{2}{1 - \sin\theta}$

**41.** Ellipse

Vertices: $(2, 0), (8, \pi)$

$e = \dfrac{3}{5}, d = \dfrac{16}{3}$

$$r = \frac{ed}{1 + e\cos\theta}$$

$$= \frac{16/5}{1 + (3/5)\cos\theta}$$

$$= \frac{16}{5 + 3\cos\theta}$$

**43.** Hyperbola

Vertices: $\left(1, \dfrac{3\pi}{2}\right), \left(9, \dfrac{3\pi}{2}\right)$

$e = \dfrac{5}{4}, d = \dfrac{9}{5}$

$$r = \frac{ed}{1 - e\sin\theta}$$

$$= \frac{9/4}{1 - (5/4)\sin\theta}$$

$$= \frac{9}{4 - 5\sin\theta}$$

**45.** Ellipse if $0 < e < 1$, parabola if $e = 1$, hyperbola if $e > 1$.

**47.** (a) Hyperbola $(e = 2 > 1)$

(b) Ellipse $\left(e = \dfrac{1}{2} < 1\right)$

(c) Parabola $(e = 1)$

(d) Rotated hyperbola $(e = 3)$

**49.** $a = 5, c = 4, e = \dfrac{4}{5}, b = 3$

$$r^2 = \frac{9}{1 - (16/25)\cos^2\theta}$$

**51.** $a = 3, b = 4, c = 5, e = \dfrac{5}{3}$

$$r^2 = \frac{-16}{1 - (25/9)\cos^2\theta}$$

**53.** $A = 2\left[\dfrac{1}{2}\displaystyle\int_0^{\pi}\left(\dfrac{3}{2 - \cos\theta}\right)^2 d\theta\right]$

$$= 9\int_0^{\pi}\frac{1}{(2 - \cos\theta)^2}\,d\theta \approx 10.88$$

**55.** Vertices: $(126{,}000, 0), (4119, \pi)$

$$a = \frac{126{,}000 + 4119}{2} = 65{,}059.5,\ c = 65{,}059.5 - 4119 = 60{,}940.5,\ e = \frac{c}{a} = \frac{40{,}627}{43{,}373},\ d = 4119\left(\frac{84{,}000}{40{,}627}\right)$$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{4119(84{,}000/43{,}373)}{1 - (40{,}627/43{,}373)\cos\theta} = \frac{345{,}996{,}000}{43{,}373 - 40{,}627\cos\theta}$$

When $\theta = 60^\circ$, $r = \dfrac{345{,}996{,}000}{23{,}059.5} \approx 15{,}004.49$.

Distance between the surface of the earth and the satellite is $r - 4000 = 11{,}004.49$ miles.

**57.** $a = 92.957 \times 10^6$ mi, $e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{92{,}931{,}075.2223}{1 - 0.0167\cos\theta}$$

Perihelion distance: $a(1 - e) \approx 91{,}404{,}618$ mi

Aphelion distance: $a(1 + e) \approx 94{,}509{,}382$ mi

**59.** $a = 5.900 \times 10^9$ km, $e = 0.2481$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} \approx \frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}$$

Perihelion distance: $a(1 - e) = 4.436 \times 10^9$ km

Aphelion distance: $a(1 + e) = 7.364 \times 10^9$ km

**61.** $r = \dfrac{5.537 \times 10^9}{1 - 0.2481\cos\theta}$

(a) $A = \dfrac{1}{2}\displaystyle\int_0^{\pi/9}\left[\dfrac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta \approx 9.341 \times 10^{18}\text{ km}^2$

$$248\left[\frac{\dfrac{1}{2}\displaystyle\int_0^{\pi/9}\left[\dfrac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta}{\dfrac{1}{2}\displaystyle\int_0^{2\pi}\left[\dfrac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta}\right] \approx 21.867\text{ yr}$$

(b) $\dfrac{1}{2}\displaystyle\int_{\pi}^{\alpha - \pi}\left[\dfrac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right]^2 d\theta = 9.341 \times 10^{18}$

$\alpha \approx \pi + 0.8995$ rad

In part (a) the ray swept through a smaller angle to generate the same area since the length of the ray is longer than in part (b).

(c) $r' = \dfrac{(-5.537 \times 10^9)(0.2481\sin\theta)}{(1 - 0.2481\cos\theta)^2}$

$$s = \int_0^{\pi/9}\sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right)^2 + \left[\frac{-1.3737297 \times 10^9\sin\theta}{(1 - 0.2481\cos\theta)^2}\right]^2}\,d\theta \approx 2.559 \times 10^9\text{ km}$$

$$\frac{2.559 \times 10^9\text{ km}}{21.867\text{ yr}} \approx 1.17 \times 10^8\text{ km/yr}$$

$$s = \int_{\pi}^{\pi + 0.899}\sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481\cos\theta}\right)^2 + \left[\frac{-1.3737297 \times 10^9\sin\theta}{(1 - 0.2481\cos\theta)^2}\right]^2}\,d\theta \approx 4.119 \times 10^9\text{ km}$$

$$\frac{4.119 \times 10^9\text{ km}}{21.867\text{ yr}} \approx 1.88 \times 10^8\text{ km/yr}$$

**63.** $r_1 = \dfrac{ed}{1 + \sin\theta}$ and $r_2 = \dfrac{ed}{1 - \sin\theta}$

Points of intersection: $(ed, 0)$, $(ed, \pi)$

$$r_1\text{:}\ \frac{dy}{dx} = \frac{\left(\dfrac{ed}{1+\sin\theta}\right)(\cos\theta) + \left(\dfrac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\sin\theta)}{\left(\dfrac{-ed}{1+\sin\theta}\right)(\sin\theta) + \left(\dfrac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = -1$. At $(ed, \pi)$, $\dfrac{dy}{dx} = 1$.

$$r_2\text{:}\ \frac{dy}{dx} = \frac{\left(\dfrac{ed}{1-\sin\theta}\right)(\cos\theta) + \left(\dfrac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\sin\theta)}{\left(\dfrac{-ed}{1-\sin\theta}\right)(\sin\theta) + \left(\dfrac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = 1$. At $(ed, \pi)$, $\dfrac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1m_2 = (-1)(1) = -1$, and at $(ed, \pi)$ we have $m_1m_2 = 1(-1) = -1$. The curves intersect at right angles.

# Review Exercises for Chapter 9

**1.** Matches (d) - ellipse

**3.** Matches (a) - parabola

**5.** $$16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

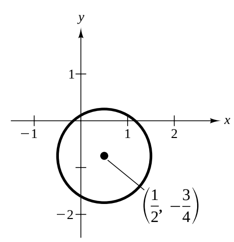
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1




**7.** $$3x^2 - 2y^2 + 24x + 12y + 24 = 0$$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

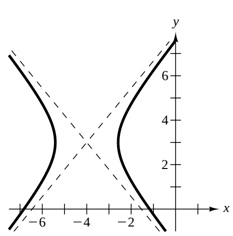
$$\frac{(x+4)^2}{2} - \frac{(y-3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $\left(-4 \pm \sqrt{2}, 3\right)$

Asymptotes: $y = 3 \pm \sqrt{\dfrac{3}{2}}(x + 4)$




**9.** $$3x^2 + 2y^2 - 12x + 12y + 29 = 0$$

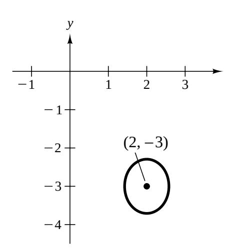
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x-2)^2}{1/3} + \frac{(y+3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \dfrac{\sqrt{2}}{2}\right)$



**11.** Vertex: $(0, 2)$

Directrix: $x = -3$

Parabola opens to the right

$p = 3$

$(y - 2)^2 = 4(3)(x - 0)$

$y^2 - 4y - 12x + 4 = 0$

**13.** Vertices: $(-3, 0), (7, 0)$

Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$a = 5, c = 2, b = \sqrt{21}$

$\dfrac{(x-2)^2}{25} + \dfrac{y^2}{21} = 1$

**15.** Vertices: $(\pm 4, 0)$

Foci: $(\pm 6, 0)$

Center: $(0, 0)$

Horizontal transverse axis

$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$

$\dfrac{x^2}{16} - \dfrac{y^2}{20} = 1$

**17.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \dfrac{\sqrt{5}}{3}$

By Example 5 of Section 9.1,

$$C = 12\int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right)\sin^2\theta}\, d\theta \approx 15.87.$$

**19.** $y = x - 2$ has a slope of 1. The perpendicular slope is $-1$.

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

Perpendicular line: $\quad y - \dfrac{5}{4} = -1\left(x - \dfrac{1}{2}\right)$

$$4x + 4y - 7 = 0$$

**21.** (a) $V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$

(b) $$F = 2(62.4)\int_{-3}^{3}(3 - y)\frac{4}{3}\sqrt{9 - y^2}\, dy = \frac{8}{3}(62.4)\left[3\int_{-3}^{3}\sqrt{9 - y^2}\, dy - \int_{-3}^{3} y\sqrt{9 - y^2}\, dy\right]$$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(y\sqrt{9 - y^2} + 9\arcsin\frac{y}{3}\right) + \frac{1}{3}(9 - y^2)^{3/2}\right]_{-3}^{3}$$

$$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(\frac{9\pi}{2}\right) - \frac{3}{2}\left(-\frac{9\pi}{2}\right)\right] = \frac{8}{3}(62.4)\left(\frac{27\pi}{2}\right) \approx 7057.274$$

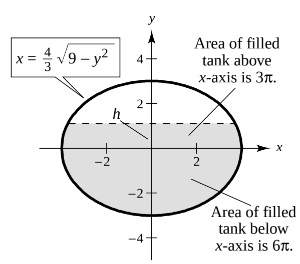
(c) You want $\frac{3}{4}$ of the total area of $12\pi$ covered. Find $h$ so that

$$2\int_0^h \frac{4}{3}\sqrt{9 - y^2}\, dy = 3\pi$$

$$\int_0^h \sqrt{9 - y^2}\, dy = \frac{9\pi}{8}$$

$$\frac{1}{2}\left[y\sqrt{9 - y^2} + 9\arcsin\left(\frac{y}{3}\right)\right]_0^h = \frac{9\pi}{8}$$

$$h\sqrt{9 - h^2} + 9\arcsin\left(\frac{h}{3}\right) = \frac{9\pi}{4}.$$




By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

(d) Area of ends $= 2(12\pi) = 24\pi$

Area of sides $=$ (Perimeter)(Length)

$$= 16\int_0^{\pi/2}\left(\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\theta}\right) d\theta(16) \quad \text{[from Example 5 of Section 9.1]}$$

$$\approx 256\left(\frac{\pi/2}{12}\right)\left[\sqrt{1 - \left(\frac{7}{16}\right)\sin^2(0)} + 4\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{4}\right)}\right.$$

$$\left. + 4\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right)\sin^2\left(\frac{\pi}{2}\right)}\right] \approx 353.65$$

Total area $= 24\pi + 353.65 \approx 429.05$

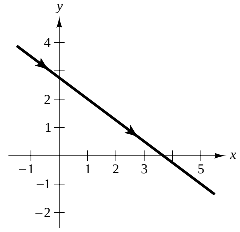
**23.** $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$$4y + 3x - 11 = 0$$

Line

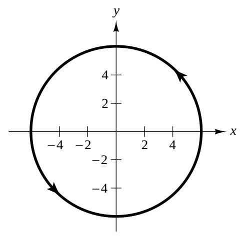


**25.** $x = 6\cos\theta, y = 6\sin\theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

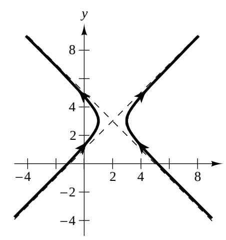


**27.** $x = 2 + \sec\theta, y = 3 + \tan\theta$

$$(x-2)^2 = \sec^2\theta = 1 + \tan^2\theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola



**29.** $x = 3 + (3 - (-2))t = 3 + 5t$

$y = 2 + (2 - 6)t = 2 - 4t$

(other answers possible)

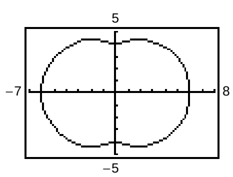
**31.** $\dfrac{(x+3)^2}{16} + \dfrac{(y-4)^2}{9} = 1$

Let $\dfrac{(x+3)^2}{16} = \cos^2\theta$ and $\dfrac{(y-4)^2}{9} = \sin^2\theta.$

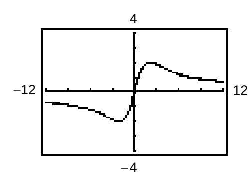
Then $x = -3 + 4\cos\theta$ and $y = 4 + 3\sin\theta.$

**33.** $x = \cos 3\theta + 5\cos\theta$

$y = \sin 3\theta + 5\sin\theta$



**35.** (a) $x = 2\cot\theta, y = 4\sin\theta\cos\theta, 0 < \theta < \pi$



(b) 
$$\begin{aligned}(4 + x^2)y &= (4 + 4\cot^2\theta)4\sin\theta\cos\theta \\ &= 16\csc^2\theta \cdot \sin\theta \cdot \cos\theta \\ &= 16\frac{\cos\theta}{\sin\theta} \\ &= 8(2\cot\theta) \\ &= 8x\end{aligned}$$

**37.** $x = 1 + 4t$

$y = 2 - 3t$

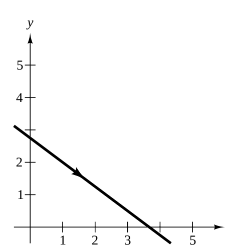
(a) $\dfrac{dy}{dx} = -\dfrac{3}{4}$

No horizontal tangents

(b) $t = \dfrac{x-1}{4}$

$$y = 2 - \frac{3}{4}(x-1) = \frac{-3x+11}{4}$$

(c)



**39.** $x = \dfrac{1}{t}$

$y = 2t + 3$

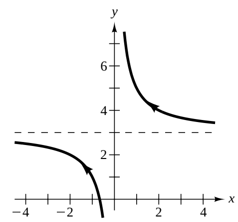
(a) $\dfrac{dy}{dx} = \dfrac{2}{-1/t^2} = -2t^2$

No horizontal tangents $(t \neq 0)$

(b) $t = \dfrac{1}{x}$

$y = \dfrac{2}{x} + 3$

(c)



**41.** $x = \dfrac{1}{2t + 1}$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{\dfrac{-(2t-2)}{(t^2-2t)^2}}{\dfrac{-2}{(2t+1)^2}} = \dfrac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

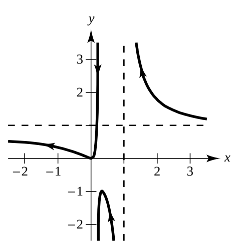
Point of horizontal tangency: $\left(\frac{1}{3}, -1\right)$

(b) $2t + 1 = \dfrac{1}{x} \implies t = \dfrac{1}{2}\left(\dfrac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1-x}{x}\right)\left[\frac{1}{2}\left(\frac{1-x}{x}\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}$$

(c)



**43.** $x = 3 + 2\cos\theta$

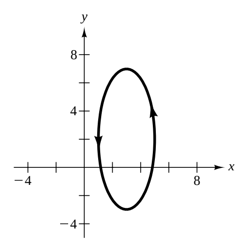
$y = 2 + 5\sin\theta$

(a) $\dfrac{dy}{dx} = \dfrac{5\cos\theta}{-2\sin\theta} = -2.5\cot\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

(b) $\dfrac{(x-3)^2}{4} + \dfrac{(y-2)^2}{25} = 1$

(c)



**45.** $x = \cos^3\theta$

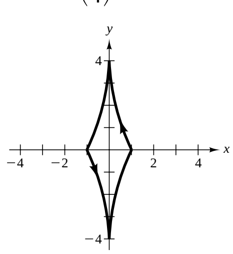
$y = 4\sin^3\theta$

(a) $\dfrac{dy}{dx} = \dfrac{12\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)} = \dfrac{-4\sin\theta}{\cos\theta} = -4\tan\theta = 0$ when $\theta = 0, \pi$.

But, $\dfrac{dy}{dt} = \dfrac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\dfrac{y}{4}\right)^{2/3} = 1$

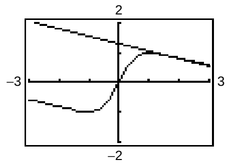
(c)



**47.** $x = \cot\theta$

$y = \sin 2\theta = 2\sin\theta\cos\theta$

(a), (c)




(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} = -4, \dfrac{dy}{d\theta} = 1$, and $\dfrac{dy}{dx} = -\dfrac{1}{4}$

**49.** $x = r(\cos\theta + \theta\sin\theta)$

$y = r(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = r\theta\cos\theta$$

$$\frac{dy}{d\theta} = r\theta\sin\theta$$

$$s = r\int_0^{\pi}\sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

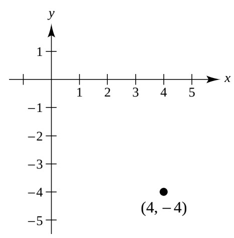
$$= r\int_0^{\pi}\theta\,d\theta = \frac{r}{2}\Big[\theta^2\Big]_0^{\pi} = \frac{1}{2}\pi^2 r$$

**51.** $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = 7\frac{\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$




**53.**
$$r = 3\cos\theta$$
$$r^2 = 3r\cos\theta$$
$$x^2 + y^2 = 3x$$
$$x^2 + y^2 - 3x = 0$$

**55.**
$$r = -2(1 + \cos\theta)$$
$$r^2 = -2r(1 + \cos\theta)$$
$$x^2 + y^2 = -2\left(\pm\sqrt{x^2 + y^2}\right) - 2x$$
$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

**57.**
$$r^2 = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$(x^2 + y^2)^2 = x^2 - y^2$$

**59.**
$$r = 4\cos 2\theta\sec\theta$$
$$= 4(2\cos^2\theta - 1)\left(\frac{1}{\cos\theta}\right)$$
$$r\cos\theta = 8\cos^2\theta - 4$$
$$x = 8\left(\frac{x^2}{x^2 + y^2}\right) - 4$$
$$x^3 + xy^2 = 4x^2 - 4y^2$$
$$y^2 = x^2\left(\frac{4 - x}{4 + x}\right)$$

**61.** $(x^2 + y^2)^2 = ax^2y$
$$r^4 = a(r^2\cos^2\theta)(r\sin\theta)$$
$$r = a\cos^2\theta\sin\theta$$

**63.** $x^2 + y^2 = a^2\left(\arctan\dfrac{y}{x}\right)^2$
$$r^2 = a^2\theta^2$$

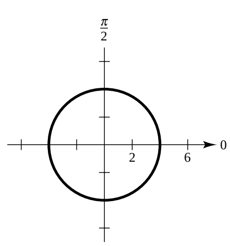
**65.** $r = 4$

Circle of radius 4

Centered at the pole

Symmetric to polar axis,

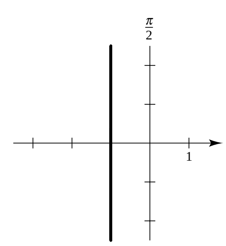
$\theta = \pi/2$, and pole




**67.** $r = -\sec\theta = \dfrac{-1}{\cos\theta}$

$r\cos\theta = -1, x = -1$

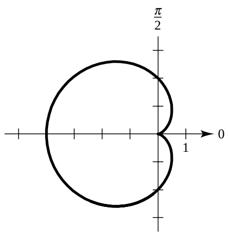
Vertical line



**69.** $r = -2(1 + \cos\theta)$

Cardioid

Symmetric to polar axis

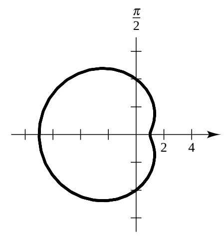



| $\theta$ | $0$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |

**71.** $r = 4 - 3\cos\theta$

Limaçon

Symmetric to polar axis




| $\theta$ | $0$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $1$ | $\frac{5}{2}$ | $4$ | $\frac{11}{2}$ | $7$ |

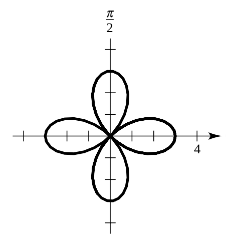
**73.** $r = -3\cos(2\theta)$

Rose curve with four petals

Symmetric to polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(-3, 0), \left(3, \dfrac{\pi}{2}\right), (-3, \pi), \left(3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$




**75.** $r^2 = 4\sin^2(2\theta)$

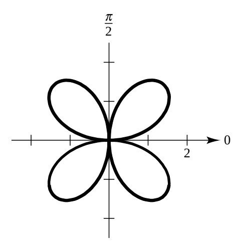
$r = \pm 2\sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

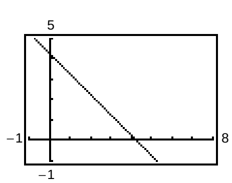
Relative extrema: $\left(\pm 2, \dfrac{\pi}{4}\right), \left(\pm 2, \dfrac{3\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$




**77.** $r = \dfrac{3}{\cos[\theta - (\pi/4)]}$

Graph of $r = 3\sec\theta$ rotated through an angle of $\pi/4$



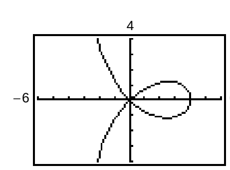

**79.** $r = 4\cos 2\theta \sec\theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$



**81.** $r = 1 - 2\cos\theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

(b) $\dfrac{dy}{dx} = \dfrac{2\sin^2\theta + (1 - 2\cos\theta)\cos\theta}{2\sin\theta\cos\theta - (1 - 2\cos\theta)\sin\theta}$

Horizontal tangents: $-4\cos^2\theta + \cos\theta + 2 = 0, \cos\theta = \dfrac{-1 \pm \sqrt{1 + 32}}{-8} = \dfrac{1 \pm \sqrt{33}}{8}$

When $\cos\theta = \dfrac{1 \pm \sqrt{33}}{8}, r = 1 - 2\left(\dfrac{1 + \sqrt{33}}{8}\right) = \dfrac{3 \mp \sqrt{33}}{4},$

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, -2.206).$$

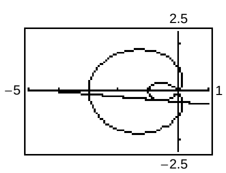
Vertical tangents:

$$\sin\theta(4\cos\theta - 1) = 0, \sin\theta = 0, \cos\theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm\arccos\left(\frac{1}{4}\right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm\arccos\frac{1}{4}\right) \approx (0.5, \pm 1.318)$$

(c)




**83.** Circle: $r = 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + 3\sin\theta\cos\theta}{3\cos\theta\cos\theta - 3\sin\theta\sin\theta} = \frac{\sin 2\theta}{\cos^2\theta - \sin^2\theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5\sin\theta$

$$\frac{dy}{dx} = \frac{-5\cos\theta\sin\theta + (4 - 5\sin\theta)\cos\theta}{-5\cos\theta\cos\theta - (4 - 5\sin\theta)\sin\theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let $\alpha$ be the angle between the curves:

$$\tan\alpha = \frac{\sqrt{3} - \left(\sqrt{3}/9\right)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

Therefore, $\alpha = \arctan\left(\dfrac{2\sqrt{3}}{3}\right) \approx 49.1°.$

**85.** $r = 1 + \cos\theta, r = 1 - \cos\theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos\theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sin^2\theta + \cos\theta(1 + \cos\theta)}{-\sin\theta\cos\theta - \sin\theta(1 + \cos\theta)}.$$

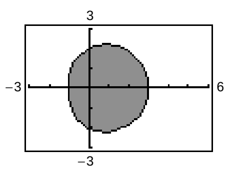
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos\theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2\theta + \cos\theta(1 - \cos\theta)}{\sin\theta\cos\theta - \sin\theta(1 - \cos\theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

**87.** $r = 2 + \cos\theta$

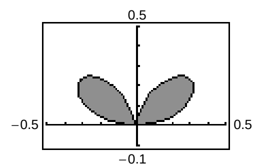
$$A = 2\left[\frac{1}{2}\int_0^{\pi} (2 + \cos\theta)^2\,d\theta\right] \approx 14.14 \quad \left(\frac{9\pi}{2}\right)$$




**89.** $r = \sin\theta \cdot \cos^2\theta$

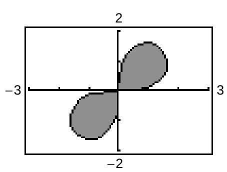
$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} (\sin\theta\cos^2\theta)^2\,d\theta\right]$$

$$\approx 0.10 \left(\frac{\pi}{32}\right)$$



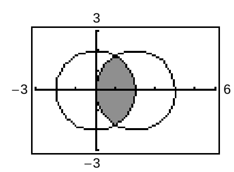

**91.** $r^2 = 4\sin 2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta\,d\theta\right] = 4$$




**93.** $r = 4\cos\theta, r = 2$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3} 4\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\,d\theta\right] \approx 4.91$$

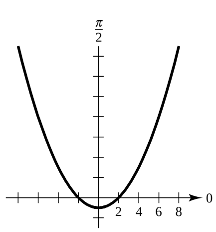



**95.** $s = 2\displaystyle\int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\,d\theta$

$$= 2\sqrt{2}\,a\int_0^{\pi} \sqrt{1 - \cos\theta}\,d\theta = 2\sqrt{2}\,a\int_0^{\pi} \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\,d\theta = \left[-4\sqrt{2}\,a(1 + \cos\theta)^{1/2}\right]_0^{\pi} = 8a$$

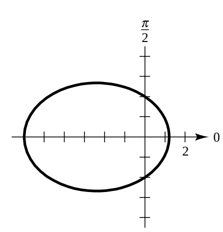
**97.** $r = \dfrac{2}{1 - \sin\theta}, e = 1$

Parabola



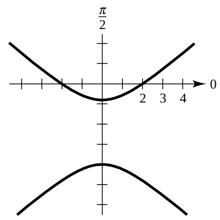

**99.** $r = \dfrac{6}{3 + 2\cos\theta} = \dfrac{2}{1 + (2/3)\cos\theta}, e = \dfrac{2}{3}$

Ellipse



**101.** $r = \dfrac{4}{2 - 3\sin\theta} = \dfrac{2}{1 - (3/2)\sin\theta}, e = \dfrac{3}{2}$

Hyperbola




**103.** Circle

Center: $\left(5, \dfrac{\pi}{2}\right) = (0, 5)$ in rectangular coordinates

Solution point: $(0, 0)$

$$x^2 + (y - 5)^5 = 25$$
$$x^2 + y^2 - 10y = 0$$
$$r^2 - 10r\sin\theta = 0$$
$$r = 10\sin\theta$$

**105.** Parabola

Vertex: $(2, \pi)$

Focus: $(0, 0)$

$e = 1, d = 4$

$r = \dfrac{4}{1 - \cos\theta}$

**107.** Ellipse

Vertices: $(5, 0), (1, \pi)$

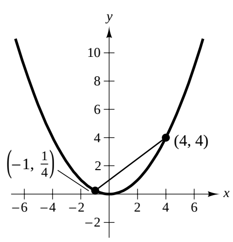
Focus: $(0, 0)$

$a = 3, c = 2, e = \dfrac{2}{3}, d = \dfrac{5}{2}$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right)\cos\theta} = \frac{5}{3 - 2\cos\theta}$$

# Problem Solving for Chapter 9

**1.** (a)




(b)
$$x^2 = 4y$$
$$2x = 4y'$$
$$y' = \frac{1}{2}x$$

$y - 4 = 2(x - 4) \implies y = 2x - 4$ Tangent line at $(4, 4)$

$y - \dfrac{1}{4} = -\dfrac{1}{2}(x + 1) \implies y = -\dfrac{1}{2}x - \dfrac{1}{4}$ Tangent line at $\left(-1, \dfrac{1}{4}\right)$

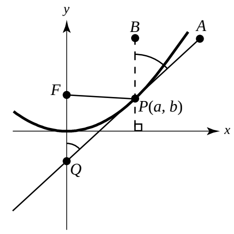
Tangent lines have slopes of 2 and $-1/2 \implies$ perpendicular.

(c) Intersection:

$$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$$
$$8x - 16 = -2x - 1$$
$$10x = 15$$
$$x = \frac{3}{2} \implies \left(\frac{3}{2}, -1\right)$$

Point of intersection, $(3/2, -1)$, is on directrix $y = -1$.

**3.** Consider $x^2 = 4py$ with focus $(0, p)$.




Let $p(a, b)$ be point on parabola.

$zx = 4py' \implies y' = \dfrac{x}{2p}$

$y - b = \dfrac{a}{2p}(x - a)$ Tangent line

For $x = 0$, $y = b + \dfrac{a}{2p}(-a) = b - \dfrac{a^2}{2p} = b - \dfrac{4pb}{2p} = -b.$

Thus, $Q = (0, -b)$.

$\triangle FQP$ is isosceles because

$$|FQ| = p + b$$

$$\begin{aligned}|FP| = \sqrt{(a-0)^2 + (b-p)^2} &= \sqrt{a^2 + b^2 - 2bp + p^2}\\ &= \sqrt{4pb + b^2 - 2bp + p^2}\\ &= \sqrt{(b+p)^2}\\ &= b + p.\end{aligned}$$

Thus, $\measuredangle FQP = \measuredangle BPA = \measuredangle FPQ$.

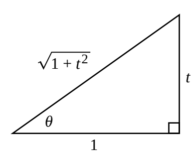
**5.** (a) In $\triangle OCB$, $\cos\theta = \dfrac{2a}{OB} \implies OB = 2a \cdot \sec\theta.$

In $\triangle OAC$, $\cos\theta = \dfrac{OA}{2a} \implies OA = 2a \cdot \cos\theta.$

$$\begin{aligned}r = OP = AB = OB - OA &= 2a(\sec\theta - \cos\theta)\\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right)\\ &= 2a \cdot \frac{\sin^2\theta}{\cos\theta}\\ &= 2a \cdot \tan\theta\sin\theta\end{aligned}$$

(b) $x = r\cos\theta = (2a\tan\theta\sin\theta)\cos\theta = 2a\sin^2\theta$

$y = r\sin\theta = (2a\tan\theta\sin\theta)\sin\theta = 2a\tan\theta \cdot \sin^2\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$




Let $t = \tan\theta$, $-\infty < t < \infty$.

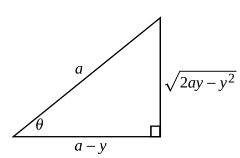
Then $\sin^2\theta = \dfrac{t^2}{1+t^2}$ and $x = 2a\dfrac{t^2}{1+t^2}$, $y = 2a\dfrac{t^3}{1+t^2}$.

(c)

$$\begin{aligned}r &= 2a\tan\theta\sin\theta\\ r\cos\theta &= 2a\sin^2\theta\\ r^3\cos\theta &= 2a\,r^2\sin^2\theta\\ (x^2 + y^2)x &= 2ay^2\\ y^2 &= \frac{x^3}{(2a - x)}\end{aligned}$$

**7.** $y = a(1 - \cos\theta) \implies \cos\theta = \dfrac{a - y}{a}$

$\theta = \arccos\left(\dfrac{a-y}{a}\right)$




$$\begin{aligned}x &= a(\theta - \sin\theta)\\ &= a\left(\arccos\left(\frac{a-y}{a}\right) - \sin\left(\arccos\left(\frac{a-y}{a}\right)\right)\right)\\ &= a\left(\arccos\left(\frac{a-y}{a}\right) - \frac{\sqrt{2ay - y^2}}{a}\right)\end{aligned}$$

$x = a \cdot \arccos\left(\dfrac{a-y}{a}\right) - \sqrt{2ay - y^2},\ 0 \le y \le 2a$

**9.** For $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

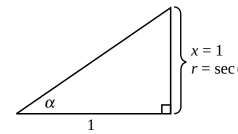
$$y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \ldots$$

Hence, the curve has length greater that

$$\begin{aligned} S &= \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \cdots \\ &= \frac{2}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) \\ &> \frac{2}{\pi}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right) \\ &= \infty. \end{aligned}$$

**11.** (a) Area $= \int_0^{\alpha} \frac{1}{2} r^2 \, d\theta$

$$= \frac{1}{2}\int_0^{\alpha} \sec^2 \theta \, d\theta$$




(b) $\tan \alpha = \frac{h}{1} \implies$ Area $= \frac{1}{2}(1)\tan \alpha$

$$\implies \tan \alpha = \int_0^{\alpha} \sec^2 \theta \, d\theta$$

(c) Differentiating, $\frac{d}{d\alpha}(\tan \alpha) = \sec^2 \alpha$.

**13.** If a dog is located at $(r, \theta)$, then its neighbor is at $\left(r, \theta + \frac{\pi}{2}\right)$:

$$(x, y) = (r \cos \theta, r \sin \theta) \text{ and } (x, y) = (-r \sin \theta, r \cos \theta).$$

The slope joining these points is

$$\frac{r \cos \theta - r \sin \theta}{-r \sin \theta - r \cos \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \text{ slope of tangent line at } (r, \theta).$$

$$\frac{\frac{dr}{d\theta}\sin \theta + r \cos \theta}{\frac{dr}{d\theta}\cos \theta - r \sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$$

$$\begin{aligned} \implies \frac{dr}{d\theta} &= -r \\ \frac{dr}{r} &= -d\theta \\ \ln r &= -\theta + C_1 \\ r &= e^{-\theta + C_1} \\ r &= Ce^{-\theta} \end{aligned}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \implies r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \implies C = \frac{d}{\sqrt{2}}e^{\pi/4}$$

Finally, $r = \frac{d}{\sqrt{2}} e^{((\pi/4) - \theta)}$.

**15.** (a) The first plane makes an angle of 70° with the positive $x$-axis, and is 150 miles from $P$:

$$x_1 = \cos 70°(150 - 375t)$$

$$y_1 = \sin 70°(150 - 375t)$$

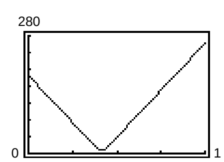
Similarly for the second plane,

$$\begin{aligned} x_2 &= \cos 135°(190 - 450t) \\ &= \cos 45°(-190 + 450t) \\ y_2 &= \sin 135°(190 - 450t) \\ &= \sin 45°(190 - 450t) \end{aligned}$$

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

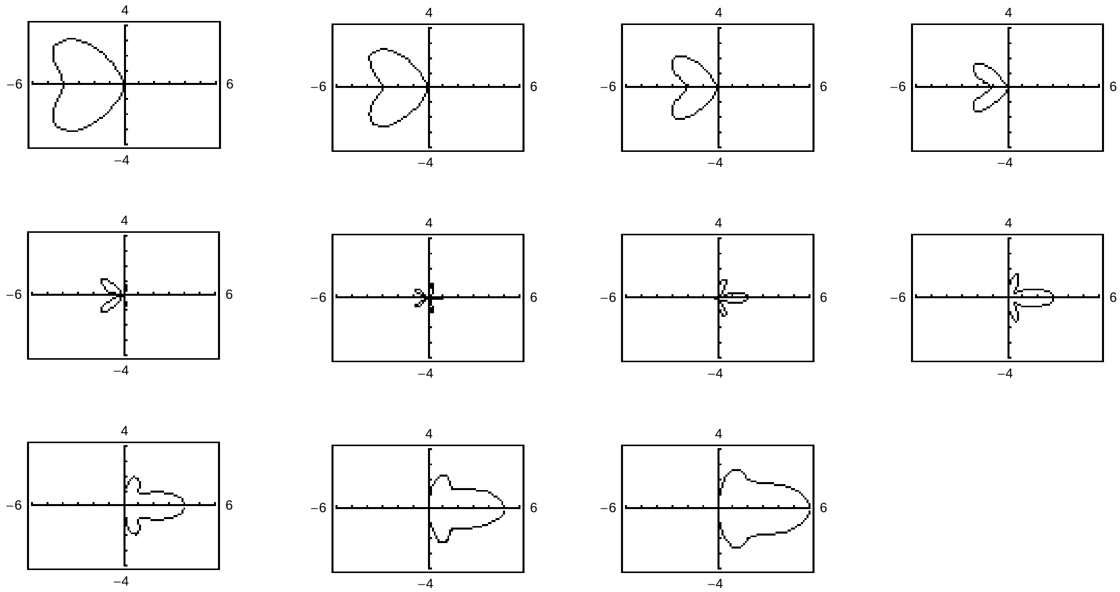
$$= [[\cos 45(-190 + 450t) - \cos 70(150 - 375t)]^2 + [\sin 45(190 - 450t) - \sin 70(150 - 375t)]^2]^{1/2}$$

(c)




The minimum distance is 7.59 miles when $t = 0.4145$.

**17.**



$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts".

# CHAPTER 10
# Vectors and the Geometry of Space

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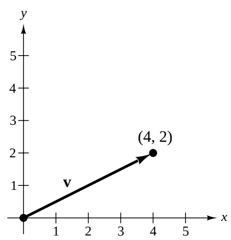
# Vectors and the Geometry of Space

## Section 10.1 Vectors in the Plane

Solutions to Odd-Numbered Exercises

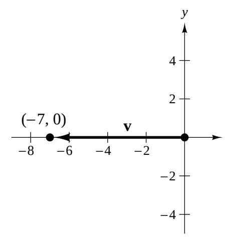
**1.** (a) $\mathbf{v} = \langle 5-1, 3-1 \rangle = \langle 4, 2 \rangle$

(b)




**3.** (a) $\mathbf{v} = \langle -4-3, -2-(-2) \rangle = \langle -7, 0 \rangle$

(b)




**5.** $\mathbf{u} = \langle 5-3, 6-2 \rangle = \langle 2, 4 \rangle$

$\mathbf{v} = \langle 1-(-1), 8-4 \rangle = \langle 2, 4 \rangle$

$\mathbf{u} = \mathbf{v}$

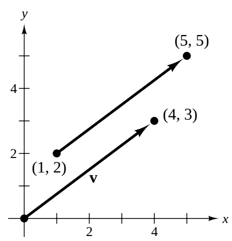
**7.** $\mathbf{u} = \langle 6-0, -2-3 \rangle = \langle 6, -5 \rangle$

$\mathbf{v} = \langle 9-3, 5-10 \rangle = \langle 6, -5 \rangle$

$\mathbf{u} = \mathbf{v}$

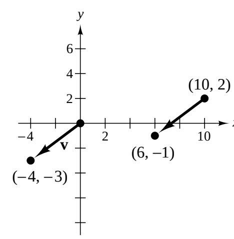
**9.** (b) $\mathbf{v} = \langle 5-1, 5-2 \rangle = \langle 4, 3 \rangle$

(a) and (c).



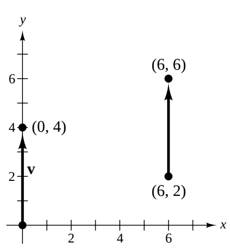

**11.** (b) $\mathbf{v} = \langle 6-10, -1-2 \rangle = \langle -4, -3 \rangle$

(a) and (c).



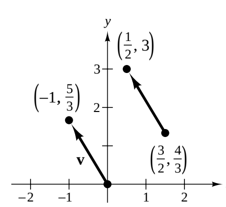

**13.** (b) $\mathbf{v} = \langle 6-6, 6-2 \rangle = \langle 0, 4 \rangle$

(a) and (c).

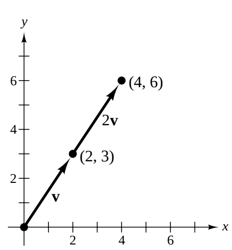



**15.** (b) $\mathbf{v} = \left\langle \frac{1}{2} - \frac{3}{2}, 3 - \frac{4}{3} \right\rangle = \left\langle -1, \frac{5}{3} \right\rangle$

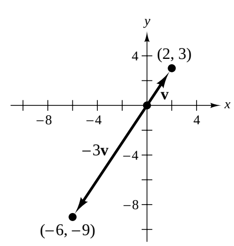
(a) and (c).




**17.** (a) $2\mathbf{v} = \langle 4, 6 \rangle$




(b) $-3\mathbf{v} = \langle -6, -9 \rangle$

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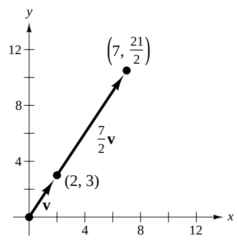
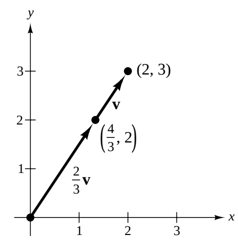
**17. —CONTINUED—**

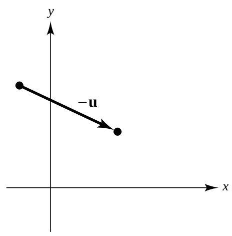
(c) $\frac{7}{2}\mathbf{v} = \left\langle 7, \frac{21}{2} \right\rangle$



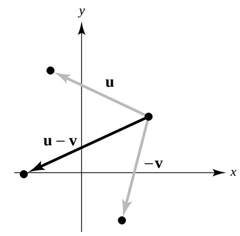

(d) $\frac{2}{3}\mathbf{v} = \left\langle \frac{4}{3}, 2 \right\rangle$




**19.**




**21.**



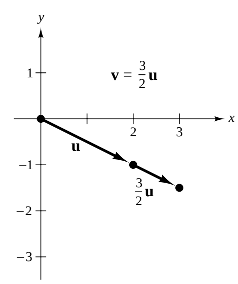

**23.** (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9\rangle = \left\langle \frac{8}{3}, 6 \right\rangle$

(b) $\mathbf{v} - \mathbf{u} = \langle 2, -5\rangle - \langle 4, 9\rangle = \langle -2, -14\rangle$

(c) $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9\rangle + 5\langle 2, -5\rangle = \langle 18, -7\rangle$

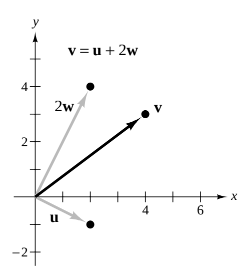
**25.** $\mathbf{v} = \frac{3}{2}(2\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - \frac{3}{2}\mathbf{j}$

$= \left\langle 3, -\frac{3}{2} \right\rangle$




**27.** $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + 2(\mathbf{i} + 2\mathbf{j})$

$= 4\mathbf{i} + 3\mathbf{j} = \langle 4, 3\rangle$




**29.** $u_1 - 4 = -1$

$u_2 - 2 = 3$

$u_1 = 3$

$u_2 = 5$

$Q = (3, 5)$

**31.** $\|\mathbf{v}\| = \sqrt{16 + 9} = 5$

**33.** $\|\mathbf{v}\| = \sqrt{36 + 25} = \sqrt{61}$

**35.** $\|\mathbf{v}\| = \sqrt{0 + 16} = 4$

**37.** $\|\mathbf{u}\| = \sqrt{3^2 + 12^2} = \sqrt{153}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 3, 12\rangle}{\sqrt{153}} = \left\langle \frac{3}{\sqrt{153}}, \frac{12}{\sqrt{153}} \right\rangle$$

$$= \left\langle \frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17} \right\rangle \text{ unit vector}$$

**39.** $\|\mathbf{u}\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle (3/2), (5/2)\rangle}{\sqrt{34}/2} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

$$= \left\langle \frac{3\sqrt{34}}{34}, \frac{5\sqrt{34}}{34} \right\rangle \text{ unit vector}$$

**41.** $\|\mathbf{u}\| = \langle 1, -1\rangle, \mathbf{v} = \langle -1, 2\rangle$

(a) $\|\mathbf{u}\| = \sqrt{1+1} = \sqrt{2}$

(b) $\|\mathbf{v}\| = \sqrt{1+4} = \sqrt{5}$

(c) $\mathbf{u} + \mathbf{v} = \langle 0, 1\rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{0+1} = 1$

(d) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{\sqrt{2}}\langle 1, -1\rangle$

$\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$

(e) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{5}}\langle -1, 2\rangle$

$\left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$

(f) $\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|} = \langle 0, 1\rangle$

$\left\|\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$

**43.** $\mathbf{u} = \left\langle 1, \dfrac{1}{2}\right\rangle, \mathbf{v} = \langle 2, 3\rangle$

(a) $\|\mathbf{u}\| = \sqrt{1 + \dfrac{1}{4}} = \dfrac{\sqrt{5}}{2}$

(b) $\|\mathbf{v}\| = \sqrt{4+9} = \sqrt{13}$

(c) $\mathbf{u} + \mathbf{v} = \left\langle 3, \dfrac{7}{2}\right\rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \dfrac{49}{4}} = \dfrac{\sqrt{85}}{2}$

(d) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{2}{\sqrt{5}}\left\langle 1, \dfrac{1}{2}\right\rangle$

$\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = 1$

(e) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{13}}\langle 2, 3\rangle$

$\left\|\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = 1$

(f) $\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|} = \dfrac{2}{\sqrt{85}}\left\langle 3, \dfrac{7}{2}\right\rangle$

$\left\|\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = 1$

**45.**
$$\begin{aligned}
\mathbf{u} &= \langle 2, 1\rangle \\
\|\mathbf{u}\| &= \sqrt{5} \approx 2.236 \\
\mathbf{v} &= \langle 5, 4\rangle \\
\|\mathbf{v}\| &= \sqrt{41} \approx 6.403 \\
\mathbf{u}+\mathbf{v} &= \langle 7, 5\rangle \\
\|\mathbf{u}+\mathbf{v}\| &= \sqrt{74} \approx 8.602 \\
\|\mathbf{u}+\mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\|
\end{aligned}$$

**47.**
$$\begin{aligned}
\frac{\mathbf{u}}{\|\mathbf{u}\|} &= \frac{1}{\sqrt{2}}\langle 1, 1\rangle \\
4\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) &= 2\sqrt{2}\langle 1, 1\rangle \\
\mathbf{v} &= \left\langle 2\sqrt{2}, 2\sqrt{2}\right\rangle
\end{aligned}$$

**49.**
$$\begin{aligned}
\frac{\mathbf{u}}{\|\mathbf{u}\|} &= \frac{1}{2\sqrt{3}}\left\langle \sqrt{3}, 3\right\rangle \\
2\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) &= \frac{1}{\sqrt{3}}\left\langle \sqrt{3}, 3\right\rangle \\
\mathbf{v} &= \left\langle 1, \sqrt{3}\right\rangle
\end{aligned}$$

**51.** $\mathbf{v} = 3[(\cos 0°)\mathbf{i} + (\sin 0°)\mathbf{j}] = 3\mathbf{i} = \langle 3, 0\rangle$

**53.**
$$\begin{aligned}
\mathbf{v} &= 2[(\cos 150°)\mathbf{i} + (\sin 150°)\mathbf{j}] \\
&= -\sqrt{3}\mathbf{i} + \mathbf{j} = \left\langle -\sqrt{3}, 1\right\rangle
\end{aligned}$$

**55.**
$$\begin{aligned}
\mathbf{u} &= \mathbf{i} \\
\mathbf{v} &= \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} \\
\mathbf{u}+\mathbf{v} &= \left(\frac{2+3\sqrt{2}}{2}\right)\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}
\end{aligned}$$

**57.** $$\mathbf{u} = 2(\cos 4)\mathbf{i} + 2(\sin 4)\mathbf{j}$$
$$\mathbf{v} = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j}$$
$$\mathbf{u} + \mathbf{v} = (2\cos 4 + \cos 2)\mathbf{i} + (2\sin 4 + \sin 2)\mathbf{j}$$

**59.** A scalar is a real number. A vector is represented by a directed line segment. A vector has both length and direction.

**61.** To normalize $\mathbf{v}$, you find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

**For Exercises 63–67, $a\mathbf{u} + b\mathbf{w} = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}$.**

**63.** $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$. Therefore, $a + b = 2$, $2a - b = 1$. Solving simultaneously, we have $a = 1$, $b = 1$.

**65.** $\mathbf{v} = 3\mathbf{i}$. Therefore, $a + b = 3$, $2a - b = 0$. Solving simultaneously, we have $a = 1$, $b = 2$.

**67.** $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Therefore, $a + b = 1$, $2a - b = 1$. Solving simultaneously, we have $a = \frac{2}{3}$, $b = \frac{1}{3}$.

**69.** $y = x^3$, $y' = 3x^2 = 3$ at $x = 1$.

(a) $m = 3$. Let $\mathbf{w} = \langle 1, 3\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{\sqrt{10}}\langle 1, 3\rangle.$$

(b) $m = -\frac{1}{3}$. Let $\mathbf{w} = \langle 3, -1\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{\sqrt{10}}\langle 3, -1\rangle.$$

**71.** $f(x) = \sqrt{25 - x^2}$

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}} = \frac{-3}{4} \text{ at } x = 3.$$

(a) $m = -\frac{3}{4}$. Let $\mathbf{w} = \langle -4, 3\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{5}\langle -4, 3\rangle.$$

(b) $m = \frac{4}{3}$. Let $\mathbf{w} = \langle 3, 4\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{5}\langle 3, 4\rangle$$

**73.** $$\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$
$$\mathbf{u} + \mathbf{v} = \sqrt{2}\mathbf{j}$$
$$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

**75.** Programs will vary.

**77.** $\|\mathbf{F}_1\| = 2$, $\theta_{\mathbf{F}_1} = 33°$

$\|\mathbf{F}_2\| = 3$, $\theta_{\mathbf{F}_2} = -125°$

$\|\mathbf{F}_3\| = 2.5$, $\theta_{\mathbf{F}_3} = 110°$

$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 1.33$

$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 132.5°$

**79.** (a) $180(\cos 30\mathbf{i} + \sin 30\mathbf{j}) + 275\mathbf{i} = 430.88\mathbf{i} + 90\mathbf{j}$

Direction: $\alpha = \arctan\left(\frac{90}{430.88}\right) = 0.206(= 11.8°)$

Magnitude: $\sqrt{430.88^2 + 90^2} = 440.18$ newtons

**—CONTINUED—**

(b) $M = \sqrt{(275 + 180\cos\theta)^2 + (180\sin\theta)^2}$

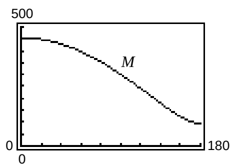
$$\alpha = \arctan\left[\frac{180\sin\theta}{275 + 180\cos\theta}\right]$$

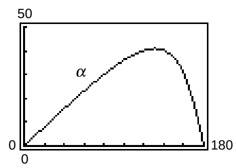
**79. —CONTINUED—**

(c)

| $\theta$ | $0°$ | $30°$ | $60°$ | $90°$ | $120°$ | $150°$ | $180°$ |
|---|---|---|---|---|---|---|---|
| $M$ | 455 | 440.2 | 396.9 | 328.7 | 241.9 | 149.3 | 95 |
| $\alpha$ | $0°$ | $11.8°$ | $23.1°$ | $33.2°$ | $40.1°$ | $37.1°$ | 0 |

(d)




(e) $M$ decreases because the forces change from acting in the same direction to acting in the opposite direction as $\theta$ increases from $0°$ to $180°$.

**81.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (75\cos 30°\mathbf{i} + 75\sin 30°\mathbf{j}) + (100\cos 45°\mathbf{i} + 100\sin 45°\mathbf{j}) + (125\cos 120°\mathbf{i} + 125\sin 120°\mathbf{j})$

$$= \left(\frac{75}{2}\sqrt{3} + 50\sqrt{2} - \frac{125}{2}\right)\mathbf{i} + \left(\frac{75}{2} + 50\sqrt{2} + \frac{125}{2}\sqrt{3}\right)\mathbf{j}$$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 228.5 \text{ lb}$$

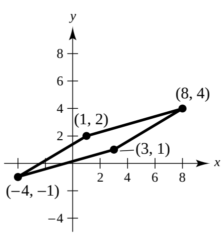
$$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 71.3°$$

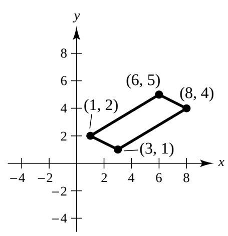
**83.** (a) The forces act along the same direction. $\theta = 0°$.

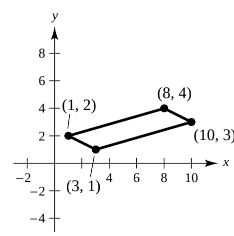
(b) The forces cancel out each other. $\theta = 180°$.

(c) No, the magnitude of the resultant can not be greater than the sum.

**85.** $(-4, -1)$, $(6, 5)$, $(10, 3)$




**87.** $\mathbf{u} = \overrightarrow{CB} = \|\mathbf{u}\|(\cos 30°\,\mathbf{i} + \sin 30°\,\mathbf{j})$

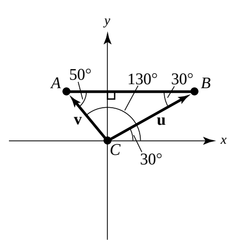
$\mathbf{v} = \overrightarrow{CA} = \|\mathbf{v}\|(\cos 130°\,\mathbf{i} + \sin 130°\,\mathbf{j})$

Vertical components: $\|\mathbf{u}\|\sin 30° + \|\mathbf{v}\|\sin 130° = 2000$

Horizontal components: $\|\mathbf{u}\|\cos 30° + \|\mathbf{v}\|\cos 130° = 0$

Solving this system, you obtain

$\|\mathbf{u}\| \approx 1305.5$ and $\|\mathbf{v}\| \approx 1758.8$.




**89.** Horizontal component $= \|\mathbf{v}\|\cos\theta = 1200\cos 6° \approx 1193.43$ ft/sec

Vertical component $= \|\mathbf{v}\|\sin\theta = 1200\sin 6° \approx 125.43$ ft/sec

**91.** $\mathbf{u} = 900[\cos 148° \, \mathbf{i} + \sin 148° \, \mathbf{j}]$

$\mathbf{v} = 100[\cos 45° \, \mathbf{i} + \sin 45° \, \mathbf{j}]$

$$\mathbf{u} + \mathbf{v} = [900 \cos 148° + 100 \cos 45°]\mathbf{i} + [900 \sin 148° + 100 \sin 45°]\mathbf{j}$$

$$\approx -692.53 \, \mathbf{i} + 547.64 \, \mathbf{j}$$

$$\theta \approx \arctan\left(\frac{547.64}{-692.53}\right) \approx -38.34°. \quad 38.34° \text{ North of West.}$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(-692.53)^2 + (547.64)^2} \approx 882.9 \text{ km/hr.}$$

**93.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$

$-3600\mathbf{j} + T_2(\cos 35°\mathbf{i} - \sin 35° \, \mathbf{j}) + T_3(\cos 92°\mathbf{i} + \sin 92°\mathbf{j}) = 0$

$T_2 \cos 35° + T_3 \cos 92° = 0$

$-T_2 \cos 35° + T_3 \sin 92° = 3600$

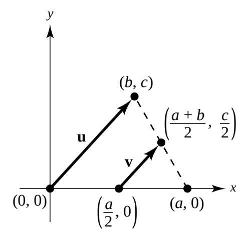
$$T_2 = \frac{-T_3 \cos 92°}{\cos 35°} \implies \frac{T_3 \cos 92°}{\cos 35°} \sin 35° + T_3 \sin 92° = 3600 \text{ and } T_3(0.97495) = 3600 \implies T_3 \approx 3692.48$$

Finally, $T_2 = 157.32$

**95.** Let the triangle have vertices at $(0, 0)$, $(a, 0)$, and $(b, c)$. Let $\mathbf{u}$ be the vector joining $(0, 0)$ and $(b, c)$, as indicated in the figure. Then $\mathbf{v}$, the vector joining the midpoints, is

$$\mathbf{v} = \left(\frac{a + b}{2} - \frac{a}{2}\right)\mathbf{i} + \frac{c}{2}\mathbf{j}$$

$$= \frac{b}{2}\mathbf{i} + \frac{c}{2}\mathbf{j} = \frac{1}{2}(b\mathbf{i} + c\mathbf{j}) = \frac{1}{2}\mathbf{u}$$




**97.** $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$

$$= \|\mathbf{u}\|\big[\|\mathbf{v}\| \cos \theta_{\mathbf{v}} \mathbf{i} + \|\mathbf{v}\| \sin \theta_{\mathbf{v}} \mathbf{j}\big] + \|\mathbf{v}\|\big[\|\mathbf{u}\| \cos \theta_{\mathbf{u}} \mathbf{i} + \|\mathbf{u}\| \sin \theta_{\mathbf{u}} \mathbf{j}\big] = \|\mathbf{u}\| \, \|\mathbf{v}\|\big[(\cos \theta_{\mathbf{u}} + \cos \theta_{\mathbf{v}})\mathbf{i} + (\sin \theta_{\mathbf{u}} + \sin \theta_{\mathbf{v}})\mathbf{j}\big]$$

$$= 2\|\mathbf{u}\| \, \|\mathbf{v}\|\left[\cos\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)\mathbf{i} + \sin\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)\mathbf{j}\right]$$

$$\tan \theta_{\mathbf{w}} = \frac{\sin\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)}{\cos\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)\cos\left(\frac{\theta_{\mathbf{u}} - \theta_{\mathbf{v}}}{2}\right)} = \tan\left(\frac{\theta_{\mathbf{u}} + \theta_{\mathbf{v}}}{2}\right)$$

Thus, $\theta_{\mathbf{w}} = (\theta_{\mathbf{u}} + \theta_{\mathbf{v}})/2$ and $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

**99.** True

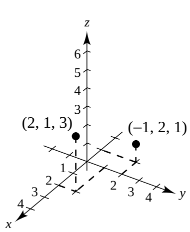
**101.** True

**103.** False

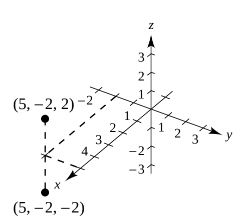
$\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}\,|a|$

## Section 10.2 Space Coordinates and Vectors in Space

**1.**




**3.**



**5.** $A(2, 3, 4)$

$B(-1, -2, 2)$

**7.** $x = -3, y = 4, z = 5$: $(-3, 4, 5)$

**9.** $y = z = 0, x = 10$: $(10, 0, 0)$

**11.** The $z$-coordinate is 0.

**13.** The point is 6 units above the $xy$-plane.

**15.** The point is on the plane parallel to the $yz$-plane that passes through $x = 4$.

**17.** The point is to the left of the $xz$-plane.

**19.** The point is on or between the planes $y = 3$ and $y = -3$.

**21.** The point $(x, y, z)$ is 3 units below the $xy$-plane, and below either quadrant I or III.

**23.** The point could be above the $xy$-plane and thus above quadrants II or IV, or below the $xy$-plane, and thus below quadrants I or III.

**25.** $$d = \sqrt{(5-0)^2 + (2-0)^2 + (6-0)^2} = \sqrt{25 + 4 + 36} = \sqrt{65}$$

**27.** $$d = \sqrt{(6-1)^2 + (-2-(-2))^2 + (-2-4)^2} = \sqrt{25 + 0 + 36} = \sqrt{61}$$

**29.** $A(0, 0, 0), B(2, 2, 1), C(2, -4, 4)$

$|AB| = \sqrt{4 + 4 + 1} = 3$

$|AC| = \sqrt{4 + 16 + 16} = 6$

$|BC| = \sqrt{0 + 36 + 9} = 3\sqrt{5}$

$|BC|^2 = |AB|^2 + |AC|^2$

Right triangle

**31.** $A(1, -3, -2), B(5, -1, 2), C(-1, 1, 2)$

$|AB| = \sqrt{16 + 4 + 16} = 6$

$|AC| = \sqrt{4 + 16 + 16} = 6$

$|BC| = \sqrt{36 + 4 + 0} = 2\sqrt{10}$

Since $|AB| = |AC|$, the triangle is isosceles.

**33.** The $z$-coordinate is changed by 5 units:

$(0, 0, 5), (2, 2, 6), (2, -4, 9)$

**35.** $$\left(\frac{5 + (-2)}{2}, \frac{-9 + 3}{2}, \frac{7 + 3}{2}\right) = \left(\frac{3}{2}, -3, 5\right)$$

**37.** Center: $(0, 2, 5)$

Radius: 2

$(x - 0)^2 + (y - 2)^2 + (z - 5)^2 = 4$

$x^2 + y^2 + z^2 - 4y - 10z + 25 = 0$

**39.** Center: $\dfrac{(2, 0, 0) + (0, 6, 0)}{2} = (1, 3, 0)$

Radius: $\sqrt{10}$

$(x - 1)^2 + (y - 3)^2 + (z - 0)^2 = 10$

$x^2 + y^2 + z^2 - 2x - 6y = 0$

**41.** $$\begin{aligned} x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 &= 0 \\ (x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 + 8z + 16) &= -1 + 1 + 9 + 16 \\ (x - 1)^2 + (y + 3)^2 + (z + 4)^2 &= 25 \end{aligned}$$

Center: $(1, -3, -4)$

Radius: 5

**43.** $$9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$$

$$x^2 + y^2 + z^2 - \frac{2}{3}x + 2y + \frac{1}{9} = 0$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + (y^2 + 2y + 1) + z^2 = -\frac{1}{9} + \frac{1}{9} + 1$$

$$\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + (z - 0)^2 = 1$$

Center: $\left(\frac{1}{3}, -1, 0\right)$

Radius: 1

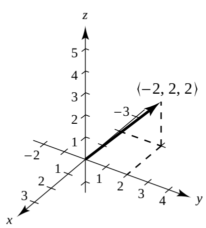
**45.** $x^2 + y^2 + z^2 \le 36$

Solid ball of radius 6 centered at origin.

**47.** (a) $\mathbf{v} = (2 - 4)\mathbf{i} + (4 - 2)\mathbf{j} + (3 - 1)\mathbf{k}$

$= -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \langle -2, 2, 2\rangle$

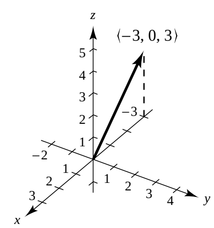
(b)




**49.** (a) $\mathbf{v} = (0 - 3)\mathbf{i} + (3 - 3)\mathbf{j} + (3 - 0)\mathbf{k}$

$= -3\mathbf{i} + 3\mathbf{k} = \langle -3, 0, 3\rangle$

(b)




**51.** $\langle 4 - 3, 1 - 2, 6 - 0\rangle = \langle 1, -1, 6\rangle$

$\|\langle 1, -1, 6\rangle\| = \sqrt{1 + 1 + 36} = \sqrt{38}$

Unit vector: $\dfrac{\langle 1, -1, 6\rangle}{\sqrt{38}} = \left\langle \dfrac{1}{\sqrt{38}}, \dfrac{-1}{\sqrt{38}}, \dfrac{6}{\sqrt{38}}\right\rangle$

**53.** $\langle -5 - (-4), 3 - 3, 0 - 1\rangle = \langle -1, 0, -1\rangle$

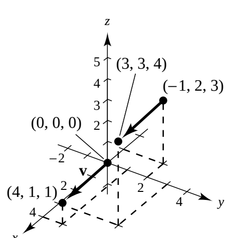
$\|\langle -1, 0, -1\rangle\| = \sqrt{1 + 1} = \sqrt{2}$

Unit vector: $\left\langle \dfrac{-1}{\sqrt{2}}, 0, \dfrac{-1}{\sqrt{2}}\right\rangle$

**55.** (b) $\mathbf{v} = (3 + 1)\mathbf{i} + (3 - 2)\mathbf{j} + (4 - 3)\mathbf{k}$

$= 4\mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 4, 1, 1\rangle$

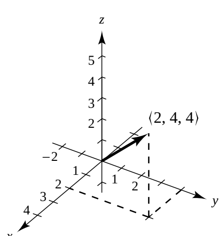
(a) and (c).



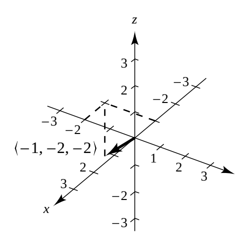

**57.** $(q_1, q_2, q_3) - (0, 6, 2) = (3, -5, 6)$

$Q = (3, 1, 8)$

**59.** (a) $2\mathbf{v} = \langle 2, 4, 4\rangle$




(b) $-\mathbf{v} = \langle -1, -2, -2\rangle$

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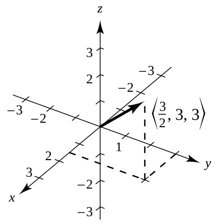
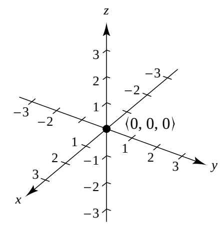
**59. —CONTINUED—**

(c) $\frac{3}{2}\mathbf{v} = \left\langle \frac{3}{2}, 3, 3 \right\rangle$




(d) $0\mathbf{v} = \langle 0, 0, 0 \rangle$




**61.** $\mathbf{z} = \mathbf{u} - \mathbf{v} = \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle = \langle -1, 0, 4 \rangle$

**63.** $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w} = \langle 2, 4, 6 \rangle + \langle 8, 8, -4 \rangle - \langle 4, 0, -4 \rangle = \langle 6, 12, 6 \rangle$

**65.** $2\mathbf{z} - 3\mathbf{u} = 2\langle z_1, z_2, z_3 \rangle - 3\langle 1, 2, 3 \rangle = \langle 4, 0, -4 \rangle$

$$2z_1 - 3 = 4 \implies z_1 = \frac{7}{2}$$

$$2z_2 - 6 = 0 \implies z_2 = 3$$

$$2z_3 - 9 = -4 \implies z_3 = \frac{5}{2}$$

$$\mathbf{z} = \left\langle \frac{7}{2}, 3, \frac{5}{2} \right\rangle$$

**67.** (a) and (b) are parallel since $\langle -6, -4, 10 \rangle = -2\langle 3, 2, -5 \rangle$ and $\left\langle 2, \frac{4}{3}, -\frac{10}{3} \right\rangle = \frac{2}{3}\langle 3, 2, -5 \rangle$.

**69.** $\mathbf{z} = -3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(a) is parallel since $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k} = 2\mathbf{z}$.

**71.** $P(0, -2, -5)$, $Q(3, 4, 4)$, $R(2, 2, 1)$

$$\overrightarrow{PQ} = \langle 3, 6, 9 \rangle$$

$$\overrightarrow{PR} = \langle 2, 4, 6 \rangle$$

$$\langle 3, 6, 9 \rangle = \frac{3}{2}\langle 2, 4, 6 \rangle$$

Therefore, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel. The points are collinear.

**73.** $P(1, 2, 4)$, $Q(2, 5, 0)$, $R(0, 1, 5)$

$$\overrightarrow{PQ} = \langle 1, 3, -4 \rangle$$

$$\overrightarrow{PR} = \langle -1, -1, 1 \rangle$$

Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not parallel, the points are not collinear.

**75.** $A(2, 9, 1)$, $B(3, 11, 4)$, $C(0, 10, 2)$, $D(1, 12, 5)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{CD} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{AC} = \langle -2, 1, 1 \rangle$$

$$\overrightarrow{BD} = \langle -2, 1, 1 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the given points form the vertices of a parallelogram.

**77.** $\|\mathbf{v}\| = 0$

**79.** $\mathbf{v} = \langle 1, -2, -3 \rangle$

$$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

**81.** $\mathbf{v} = \langle 0, 3, -5 \rangle$

$$\|\mathbf{v}\| = \sqrt{0 + 9 + 25} = \sqrt{34}$$

**83.** $\mathbf{u} = \langle 2, -1, 2 \rangle$

$$\|\mathbf{u}\| = \sqrt{4 + 1 + 4} = 3$$

(a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{3}\langle 2, -1, 2 \rangle$

(b) $-\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = -\dfrac{1}{3}\langle 2, -1, 2 \rangle$

**85.** $\mathbf{u} = \langle 3, 2, -5 \rangle$

$$\|\mathbf{u}\| = \sqrt{9 + 4 + 25} = \sqrt{38}$$

(a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

(b) $-\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = -\dfrac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

**87.** Programs will vary.

**89.** $c\mathbf{v} = \langle 2c, 2c, -c\rangle$

$\|c\mathbf{v}\| = \sqrt{4c^2 + 4c^2 + c^2} = 5$

$9c^2 = 25$

$c = \pm\dfrac{5}{3}$

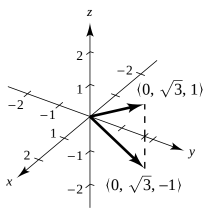
**91.** $\mathbf{v} = 10\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = 10\left\langle 0, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right\rangle$

$= \left\langle 0, \dfrac{10}{\sqrt{2}}, \dfrac{10}{\sqrt{2}}\right\rangle$

**93.** $\mathbf{v} = \dfrac{3}{2}\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{3}{2}\left\langle\dfrac{2}{3}, \dfrac{-2}{3}, \dfrac{1}{3}\right\rangle = \left\langle 1, -1, \dfrac{1}{2}\right\rangle$

**95.** $\mathbf{v} = 2[\cos(\pm 30°)\mathbf{j} + \sin(\pm 30°)\mathbf{k}]$

$= \sqrt{3}\mathbf{j} \pm \mathbf{k} = \left\langle 0, \sqrt{3}, \pm 1\right\rangle$

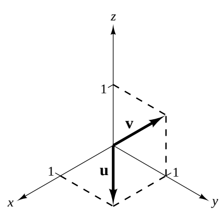



**97.** $\mathbf{v} = \langle -3, -6, 3\rangle$

$\frac{2}{3}\mathbf{v} = \langle -2, -4, 2\rangle$

$(4, 3, 0) + (-2, -4, 2) = (2, -1, 2)$

**99.** (a)




(b) $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\mathbf{i} + (a + b)\mathbf{j} + b\mathbf{k} = \mathbf{0}$

$a = 0, a + b = 0, b = 0$

Thus, a and b are both zero.

(c) $a\mathbf{i} + (a + b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$a = 1, b = 1$

$\mathbf{w} = \mathbf{u} + \mathbf{v}$

(d) $a\mathbf{i} + (a + b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$a = 1, a + b = 2, b = 3$

Not possible

**101.** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

**103.** Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

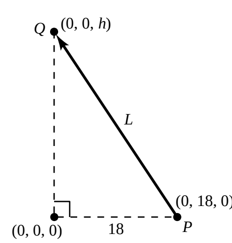
**105.** (a) The height of the right triangle is $h = \sqrt{L^2 - 18^2}$. The vector $\overrightarrow{PQ}$ is given by

$\overrightarrow{PQ} = \langle 0, -18, h\rangle.$

The tension vector $\mathbf{T}$ in each wire is

$\mathbf{T} = c\langle 0, -18, h\rangle$ where $ch = \dfrac{24}{3} = 8.$

Hence, $\mathbf{T} = \dfrac{8}{h}\langle 0, -18, h\rangle$ and




$$T = \|\mathbf{T}\| = \frac{8}{h}\sqrt{18^2 + h^2} = \frac{8}{\sqrt{L^2 - 18^2}}\sqrt{18^2 + (L^2 - 18^2)} = \frac{8L}{\sqrt{L^2 - 18^2}}$$

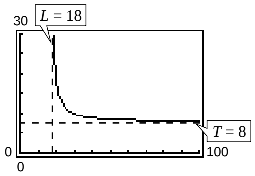
(b)

| $L$ | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|
| $T$ | 18.4 | 11.5 | 10 | 9.3 | 9.0 | 8.7 | 8.6 |

**—CONTINUED—**

**105.** **—CONTINUED—**

(c)




$x = 18$ is a vertical asymptote and $y = 8$ is a horizontal asymptote.

(d) $\lim_{L \to 18^+} \dfrac{8L}{\sqrt{L^2 - 18^2}} = \infty$

$$\lim_{L \to \infty} \frac{8L}{\sqrt{L^2 - 18^2}} = \lim_{L \to \infty} \frac{8}{\sqrt{1 - (18/L)^2}} = 8$$

(e) From the table, $T = 10$ implies $L = 30$ inches.

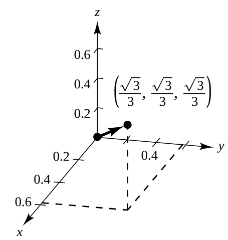
**107.** Let $\alpha$ be the angle between $\mathbf{v}$ and the coordinate axes.

$$\mathbf{v} = (\cos \alpha)\mathbf{i} + (\cos \alpha)\mathbf{j} + (\cos \alpha)\mathbf{k}$$

$$\|\mathbf{v}\| = \sqrt{3} \cos \alpha = 1$$

$$\cos \alpha = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}\langle 1, 1, 1 \rangle$$




**109.** $\overrightarrow{AB} = \langle 0, 70, 115 \rangle$, $\mathbf{F}_1 = C_1\langle 0, 70, 115 \rangle$

$\overrightarrow{AC} = \langle -60, 0, 115 \rangle$, $\mathbf{F}_2 = C_2\langle -60, 0, 115 \rangle$

$\overrightarrow{AD} = \langle 45, -65, 115 \rangle$, $\mathbf{F}_3 = C_3\langle 45, -65, 115 \rangle$

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 0, 0, 500 \rangle$

Thus:

$$\begin{aligned} -60C_2 + 45C_3 &= 0 \\ 70C_1 \qquad\quad - 65C_3 &= 0 \\ 115(C_1 + C_2 + C_3) &= 500 \end{aligned}$$

Solving this system yields $C_1 = \frac{104}{69}$, $C_2 = \frac{28}{23}$, and $C_3 = -\frac{112}{69}$. Thus:

$\|\mathbf{F}_1\| \approx 202.919N$

$\|\mathbf{F}_2\| \approx 157.909N$

$\|\mathbf{F}_3\| \approx 226.521N$

**111.** $d(AP) = 2d(BP)$

$$\sqrt{x^2 + (y + 1)^2 + (z - 1)^2} = 2\sqrt{(x - 1)^2 + (y - 2)^2 + z^2}$$

$$x^2 + y^2 + z^2 + 2y - 2z + 2 = 4(x^2 + y^2 + z^2 - 2x - 4y + 5)$$

$$0 = 3x^2 + 3y^2 + 3z^2 - 8x - 18y + 2z + 18$$

$$-6 + \frac{16}{9} + 9 + \frac{1}{9} = \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + (y^2 - 6y + 9) + \left(z^2 + \frac{2}{3}z + \frac{1}{9}\right)$$

$$\frac{44}{9} = \left(x - \frac{4}{3}\right)^2 + (y - 3)^2 + \left(z + \frac{1}{3}\right)^2$$

Sphere; center: $\left(\frac{4}{3}, 3, -\frac{1}{3}\right)$, radius: $\dfrac{2\sqrt{11}}{3}$

## Section 10.3 The Dot Product of Two Vectors

**1.** $\mathbf{u} = \langle 3, 4\rangle, \mathbf{v} = \langle 2, -3\rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 3(2) + 4(-3) = -6$

(b) $\mathbf{u} \cdot \mathbf{u} = 3(3) + 4(4) = 25$

(c) $\|\mathbf{u}\|^2 = 25$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -6\langle 2, -3\rangle = \langle -12, 18\rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$

**3.** $\mathbf{u} = \langle 2, -3, 4\rangle, \mathbf{v} = \langle 0, 6, 5\rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(0) + (-3)(6) + (4)(5) = 2$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-3)(-3) + 4(4) = 29$

(c) $\|\mathbf{u}\|^2 = 29$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 2\langle 0, 6, 5\rangle = \langle 0, 12, 10\rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(2) = 4$

**5.** $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - \mathbf{k}$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1) = 1$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1) = 6$

(c) $\|\mathbf{u}\|^2 = 6$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v} = \mathbf{i} - \mathbf{k}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

**7.** $\mathbf{u} = \langle 3240, 1450, 2235\rangle$

$\mathbf{v} = \langle 2.22, 1.85, 3.25\rangle$

$\mathbf{u} \cdot \mathbf{v} = \$17{,}139.05$

This gives the total amount that the person earned on his products.

**9.** $\dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \cos\theta$

$$\mathbf{u} \cdot \mathbf{v} = (8)(5)\cos\frac{\pi}{3} = 20$$

**11.** $\mathbf{u} = \langle 1, 1\rangle, \mathbf{v} = \langle 2, -2\rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{0}{\sqrt{2}\sqrt{8}} = 0$$

$$\theta = \frac{\pi}{2}$$

**13.** $\mathbf{u} = 3\mathbf{i} + \mathbf{j}, \mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-2}{\sqrt{10}\sqrt{20}} = \frac{-1}{5\sqrt{2}}$$

$$\theta = \arccos\left(-\frac{1}{5\sqrt{2}}\right) \approx 98.1^\circ$$

**15.** $\mathbf{u} = \langle 1, 1, 1\rangle, \mathbf{v} = \langle 2, 1, -1\rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\theta = \arccos\frac{\sqrt{2}}{3} \approx 61.9^\circ$$

**17.** $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-8}{5\sqrt{13}} = \frac{-8\sqrt{13}}{65}$$

$$\theta = \arccos\left(-\frac{8\sqrt{13}}{65}\right) \approx 116.3^\circ$$

**19.** $\mathbf{u} = \langle 4, 0\rangle, \mathbf{v} = \langle 1, 1\rangle$

$\mathbf{u} \neq c\mathbf{v} \implies$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 4 \neq 0 \implies$ not orthogonal

Neither

**21.** $\mathbf{u} = \langle 4, 3\rangle, \mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3}\right\rangle$

$\mathbf{u} \neq c\mathbf{v} \implies$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \implies$ orthogonal

**23.** $\mathbf{u} = \mathbf{j} + 6\mathbf{k}, \mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \implies$ not parallel

$\mathbf{u} \cdot \mathbf{v} = -8 \neq 0 \implies$ not orthogonal

Neither

**25.** $\mathbf{u} = \langle 2, -3, 1\rangle, \mathbf{v} = \langle -1, -1, -1\rangle$

$\mathbf{u} \neq c\mathbf{v} \implies$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \implies$ orthogonal

**27.** $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \|\mathbf{u}\| = 3$

$$\cos\alpha = \frac{1}{3}$$

$$\cos\beta = \frac{2}{3}$$

$$\cos\gamma = \frac{2}{3}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

**29.** $\mathbf{u} = \langle 0, 6, -4\rangle, \|\mathbf{u}\| = \sqrt{52} = 2\sqrt{13}$

$$\cos\alpha = 0$$

$$\cos\beta = \frac{3}{\sqrt{13}}$$

$$\cos\gamma = -\frac{2}{\sqrt{13}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0 + \frac{9}{13} + \frac{4}{13} = 1$$

**31.** $\mathbf{u} = \langle 3, 2, -2\rangle \quad \|\mathbf{u}\| = \sqrt{17}$

$$\cos\alpha = \frac{3}{\sqrt{17}} \implies \alpha \approx 0.7560 \text{ or } 43.3°$$

$$\cos\beta = \frac{2}{\sqrt{17}} \implies \beta \approx 1.0644 \text{ or } 61.0°$$

$$\cos\gamma = \frac{-2}{\sqrt{17}} \implies \gamma \approx 2.0772 \text{ or } 119.0°$$

**33.** $\mathbf{u} = \langle -1, 5, 2\rangle \quad \|\mathbf{u}\| = \sqrt{30}$

$$\cos\alpha = \frac{-1}{\sqrt{30}} \implies \alpha \approx 1.7544 \text{ or } 100.5°$$

$$\cos\beta = \frac{5}{\sqrt{30}} \implies \beta \approx 0.4205 \text{ or } 24.1°$$

$$\cos\gamma = \frac{2}{\sqrt{30}} \implies \gamma \approx 1.1970 \text{ or } 68.6°$$

**35.** $\mathbf{F}_1\colon C_1 = \dfrac{50}{\|\mathbf{F}_1\|} \approx 4.3193$

$$\mathbf{F}_2\colon C_2 = \frac{80}{\|\mathbf{F}_2\|} \approx 5.4183$$

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\\ &\approx 4.3193\langle 10, 5, 3\rangle + 5.4183\langle 12, 7, -5\rangle\\ &= \langle 108.2126, 59.5246, -14.1336\rangle\end{aligned}$$

$$\|\mathbf{F}\| \approx 124.310 \text{ lb}$$

$$\cos\alpha \approx \frac{108.2126}{\|\mathbf{F}\|} \implies \alpha \approx 29.48°$$

$$\cos\beta \approx \frac{59.5246}{\|\mathbf{F}\|} \implies \beta \approx 61.39°$$

$$\cos\gamma \approx \frac{-14.1336}{\|\mathbf{F}\|} \implies \gamma \approx 96.53°$$

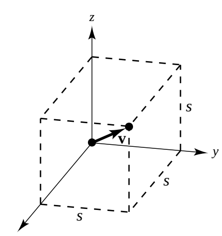
**37.** Let $s$ = length of a side.

$$\mathbf{v} = \langle s, s, s\rangle$$

$$\|\mathbf{v}\| = s\sqrt{3}$$

$$\cos\alpha = \cos\beta = \cos\gamma = \frac{s}{s\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \beta = \gamma = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$$




**39.** $\overrightarrow{OA} = \langle 0, 10, 10\rangle$

$$\cos\alpha = \frac{0}{\sqrt{0^2 + 10^2 + 10^2}} = 0 \implies \alpha = 90°$$

$$\cos\beta = \cos\gamma = \frac{10}{\sqrt{0^2 + 10^2 + 10^2}}$$

$$= \frac{1}{\sqrt{2}} \implies \beta = \gamma = 45°$$

**41.** $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 6, 7\rangle - \langle 2, 8\rangle = \langle 4, -1\rangle$

**43.** $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 0, 3, 3\rangle - \langle -2, 2, 2\rangle = \langle 2, 1, 1\rangle$

**45.** $\mathbf{u} = \langle 2, 3\rangle, \mathbf{v} = \langle 5, 1\rangle$

(a) $\mathbf{w}_1 = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \dfrac{13}{26}\langle 5, 1\rangle = \left\langle \dfrac{5}{2}, \dfrac{1}{2}\right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \left\langle -\dfrac{1}{2}, \dfrac{5}{2}\right\rangle$

**47.** $\mathbf{u} = \langle 2, 1, 2\rangle, \mathbf{v} = \langle 0, 3, 4\rangle$

(a) $\mathbf{w}_1 = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$= \dfrac{11}{25}\langle 0, 3, 4\rangle = \left\langle 0, \dfrac{33}{25}, \dfrac{44}{25}\right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \left\langle 2, -\dfrac{8}{25}, \dfrac{6}{25}\right\rangle$

**49.** $\mathbf{u}\cdot\mathbf{v} = \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$

**51.** (a) Orthogonal, $\theta = \dfrac{\pi}{2}$ (b) Acute, $0 < \theta < \dfrac{\pi}{2}$ (c) Obtuse, $\dfrac{\pi}{2} < \theta < \pi$

**53.** See page 738. Direction cosines of $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ are

$\cos\alpha = \dfrac{v_1}{\|\mathbf{v}\|}, \cos\beta = \dfrac{v_2}{\|\mathbf{v}\|}, \cos\gamma = \dfrac{v_3}{\|\mathbf{v}\|}.$

$\alpha$, $\beta$, and $\gamma$ are the direction angles. See Figure 10.26.

**55.** (a) $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \mathbf{u} \implies \mathbf{u} = c\mathbf{v} \implies \mathbf{u}$ and $\mathbf{v}$ are parallel.

(b) $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \mathbf{0} \implies \mathbf{u}\cdot\mathbf{v} = 0 \implies \mathbf{u}$ and $\mathbf{v}$ are orthogonal.

**57.** Programs will vary.

**59.** Programs will vary.

**61.** Because **u** appears to be perpendicular to **v**, the projection of **u** onto **v** is **0**. Analytically,

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\langle 2, -3\rangle\cdot\langle 6, 4\rangle}{\|\langle 6, 4\rangle\|^2}\langle 6, 4\rangle = 0\langle 6, 4\rangle = \mathbf{0}.$$

**63.** $\mathbf{u} = \dfrac{1}{2}\mathbf{i} - \dfrac{2}{3}\mathbf{j}$. Want $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$ and $-\mathbf{v} = -8\mathbf{i} - 6\mathbf{j}$ are orthogonal to **u**.

**65.** $\mathbf{u} = \langle 3, 1, -2\rangle$. Want $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{v} = \langle 0, 2, 1\rangle$ and $-\mathbf{v} = \langle 0, -2, -1\rangle$ are orthogonal to **u**.

**67.** (a) Gravitational Force $\mathbf{F} = -48{,}000\,\mathbf{j}$

$\mathbf{v} = \cos 10^\circ\,\mathbf{i} + \sin 10^\circ\,\mathbf{j}$

$\mathbf{w}_1 = \dfrac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v} = (-48{,}000)(\sin 10^\circ)\mathbf{v}$

$\approx -8335.1(\cos 10^\circ\,\mathbf{i} + \sin 10^\circ\,\mathbf{j})$

$\|\mathbf{w}_1\| \approx 8335.1$ lb

(b) $\mathbf{w}_2 = \mathbf{F} \cdot \mathbf{w}_1 = -48{,}000\,\mathbf{j} + 8335.1(\cos 10^\circ\,\mathbf{i} + \sin 10^\circ\,\mathbf{j})$

$= 8208.5\,\mathbf{i} - 46{,}552.6\,\mathbf{j}$

$\|\mathbf{w}_2\| \approx 47{,}270.8$ lb

**69.** $\mathbf{F} = 85\left(\dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}\right)$

$\mathbf{v} = 10\mathbf{i}$

$W = \mathbf{F}\cdot\mathbf{v} = 425$ ft · lb

**71.** $\overrightarrow{PQ} = \langle 4, 7, 5\rangle$

$\mathbf{v} = \langle 1, 4, 8\rangle$

$W = \overrightarrow{PQ}\cdot\mathbf{v} = 72$

**73.** False. Let $\mathbf{u} = \langle 2, 4\rangle$, $\mathbf{v} = \langle 1, 7\rangle$ and $\mathbf{w} = \langle 5, 5\rangle$. Then $\mathbf{u}\cdot\mathbf{v} = 2 + 28 = 30$ and $\mathbf{u}\cdot\mathbf{w} = 10 + 20 = 30$.

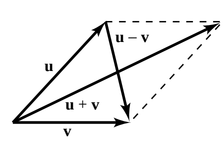
**75.** In a rhombus, $\|\mathbf{u}\| = \|\mathbf{v}\|$. The diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

$(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) = (\mathbf{u} + \mathbf{v})\cdot\mathbf{u} - (\mathbf{u} + \mathbf{v})\cdot\mathbf{v}$

$= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v}$

$= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$

Therefore, the diagonals are orthogonal.



**77.** $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle, \mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$

The angle between $\mathbf{u}$ and $\mathbf{v}$ is $\alpha - \beta$. (Assuming that $\alpha > \beta$). Also,

$$\cos(\alpha - \beta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{(1)(1)} = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

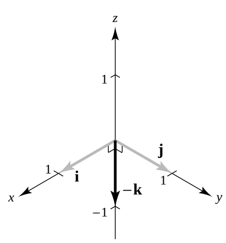
**79.**
$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= (\mathbf{u}-\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}-\mathbf{v})\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 - \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2\\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}\end{aligned}$$

**81.**
$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2\\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ from Exercise 66}\\ &\le (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

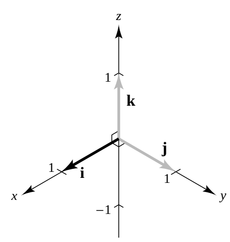
Therefore, $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

## Section 10.4 The Cross Product of Two Vectors in Space

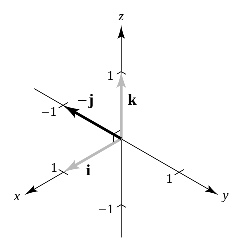
**1.** $\mathbf{j}\times\mathbf{i} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 0\\ 1 & 0 & 0\end{vmatrix} = -\mathbf{k}$




**3.** $\mathbf{j}\times\mathbf{k} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = \mathbf{i}$




**5.** $\mathbf{i}\times\mathbf{k} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix} = -\mathbf{j}$




**7.** (a) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 3 & 4\\ 3 & 7 & 2\end{vmatrix} = \langle -22, 16, -23\rangle$

(b) $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \langle 22, -16, 23\rangle$

(c) $\mathbf{v}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 7 & 2\\ 3 & 7 & 2\end{vmatrix} = 0$

**9.** (a) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 7 & 3 & 2\\ 1 & -1 & 5\end{vmatrix} = \langle 17, -33, -10\rangle$

(b) $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \langle -17, 33, 10\rangle$

(c) $\mathbf{v}\times\mathbf{v} = 0$

**11.** $\mathbf{u} = \langle 2, -3, 1\rangle, \mathbf{v} = \langle 1, -2, 1\rangle$

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -3 & 1\\ 1 & -2 & 1\end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1\rangle$

$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 2(-1) + (-3)(-1) + (1)(-1) = 0 \implies \mathbf{u}\perp\mathbf{u}\times\mathbf{v}$

$\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 1(-1) + (-2)(-1) + (1)(-1) = 0 \implies \mathbf{v}\perp\mathbf{u}\times\mathbf{v}$

**13.** $\mathbf{u} = \langle 12, -3, 0\rangle$, $\mathbf{v} = \langle -2, 5, 0\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = 54\mathbf{k} = \langle 0, 0, 54\rangle$$

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= 12(0) + (-3)(0) + 0(54) \\ &= 0 \implies \mathbf{u} \perp \mathbf{u}\times\mathbf{v}\end{aligned}$$

$$\begin{aligned}\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) &= -2(0) + 5(0) + 0(54) \\ &= 0 \implies \mathbf{v} \perp \mathbf{u}\times\mathbf{v}\end{aligned}$$

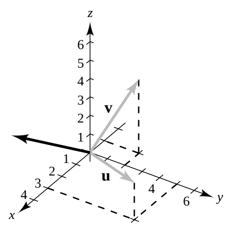
**15.** $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = \langle -2, 3, -1\rangle$$

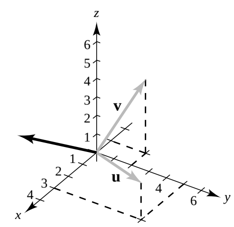
$$\begin{aligned}\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= 1(-2) + 1(3) + 1(-1) \\ &= 0 \implies \mathbf{u} \perp \mathbf{u}\times\mathbf{v}\end{aligned}$$

$$\begin{aligned}\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) &= 2(-2) + 1(3) + (-1)(-1) \\ &= 0 \implies \mathbf{v} \perp \mathbf{u}\times\mathbf{v}\end{aligned}$$

**17.**




**19.**




**21.** $\mathbf{u} = \langle 4, -3.5, 7\rangle$

$\mathbf{v} = \langle -1, 8, 4\rangle$

$$\mathbf{u}\times\mathbf{v} = \left\langle -70, -23, \frac{57}{2}\right\rangle$$

$$\frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|} = \left\langle \frac{-140}{\sqrt{24{,}965}}, \frac{-46}{\sqrt{24{,}965}}, \frac{57}{\sqrt{24{,}965}}\right\rangle$$

**23.** $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$

$$\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$$

$$\mathbf{u}\times\mathbf{v} = \left\langle -\frac{71}{20}, -\frac{11}{5}, \frac{5}{4}\right\rangle$$

$$\begin{aligned}\frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|} &= \frac{20}{\sqrt{7602}}\left\langle -\frac{71}{20}, -\frac{11}{5}, \frac{5}{4}\right\rangle \\ &= \left\langle -\frac{71}{\sqrt{7602}}, -\frac{44}{\sqrt{7602}}, \frac{25}{\sqrt{7602}}\right\rangle\end{aligned}$$

**25.** Programs will vary.

**27.** $\mathbf{u} = \mathbf{j}$

$\mathbf{v} = \mathbf{j} + \mathbf{k}$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}$$

$$A = \|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{i}\| = 1$$

**29.** $\mathbf{u} = \langle 3, 2, -1\rangle$

$\mathbf{v} = \langle 1, 2, 3\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 8, -10, 4\rangle$$

$$A = \|\mathbf{u}\times\mathbf{v}\| = \|\langle 8, -10, 4\rangle\| = \sqrt{180} = 6\sqrt{5}$$

**31.** $A(1, 1, 1,)$, $B(2, 3, 4)$, $C(6, 5, 2)$, $D(7, 7, 5)$

$\overrightarrow{AB} = \langle 1, 2, 3\rangle$, $\overrightarrow{AC} = \langle 5, 4, 1\rangle$, $\overrightarrow{CD} = \langle 1, 2, 3\rangle$, $\overrightarrow{BD} = \langle 5, 4, 1\rangle$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the figure is a parallelogram. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are adjacent sides and

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{vmatrix} = -10\mathbf{i} + 14\mathbf{j} - 6\mathbf{k}.$$

$$A = \|\overrightarrow{AB}\times\overrightarrow{AC}\| = \sqrt{332} = 2\sqrt{83}$$

**33.** $A(0, 0, 0)$, $B(1, 2, 3)$, $C(-3, 0, 0)$

$\overrightarrow{AB} = \langle 1, 2, 3\rangle$, $\overrightarrow{AC} = \langle -3, 0, 0\rangle$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -3 & 0 & 0 \end{vmatrix} = -9\mathbf{j} + 6\mathbf{k}$$

$$A = \frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = \frac{1}{2}\sqrt{117} = \frac{3}{2}\sqrt{13}$$

**35.** $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$

$\overrightarrow{AB} = \langle -3, 12, 5\rangle, \overrightarrow{AC} = \langle 2, 13, -4\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix} = \langle -113, -2, -63\rangle$$

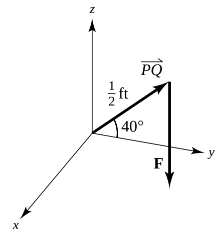
$$\text{Area} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{16{,}742}$$

**37.** $\mathbf{F} = -20\mathbf{k}$

$$\overrightarrow{PQ} = \frac{1}{2}(\cos 40°\,\mathbf{j} + \sin 40°\,\mathbf{k})$$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos 40°/2 & \sin 40°/2 \\ 0 & 0 & -20 \end{vmatrix} = -10\cos 40°\,\mathbf{i}$$

$$\|\overrightarrow{PQ} \times \mathbf{F}\| = 10\cos 40° \approx 7.66 \text{ ft} \cdot \text{lb}$$

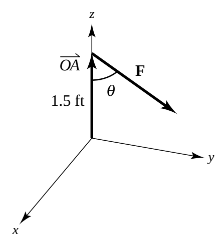


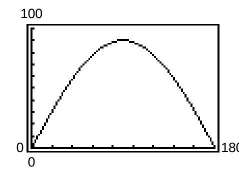

**39.** (a) $\overrightarrow{OA} = \frac{3}{2}\mathbf{k}$

$$\mathbf{F} = -60(\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k})$$

$$\overrightarrow{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3/2 \\ 0 & -60\sin\theta & -60\cos\theta \end{vmatrix} = 90\sin\theta\,\mathbf{i}$$

$$\|\overrightarrow{OA} \times \mathbf{F}\| = 90\sin\theta$$




(b) When $\theta = 45°$: $\|\overrightarrow{OA} \times \mathbf{F}\| = 90\left(\frac{\sqrt{2}}{2}\right) = 45\sqrt{2} \approx 63.64$.

(c) Let $T = 90\sin\theta$.

$$\frac{dT}{d\theta} = 90\cos\theta = 0 \text{ when } \theta = 90°.$$

This is what we expected. When $\theta = 90°$ the pipe wrench is horizontal.

**41.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

**43.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$

**45.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$

$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 2$

**47.** $\mathbf{u} = \langle 3, 0, 0\rangle$

$\mathbf{v} = \langle 0, 5, 1\rangle$

$\mathbf{w} = \langle 2, 0, 5\rangle$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 75$$

$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 75$

**49.** $\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

**51.** The magnitude of the cross product will increase by a factor of 4.

**53.** If the vectors are ordered pairs, then the cross product does not exist. False.

**55.** True

**57.** $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $\mathbf{w} = \langle w_1, w_2, w_3\rangle$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix}$$

$$= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\mathbf{i} - [u_1(v_3 + w_3) - u_3(v_1 + w_1)]\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} + (u_2w_3 - u_3w_2)\mathbf{i} - (u_1w_3 - u_3w_1)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k}$$

$$= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$$

**59.** $\mathbf{u} = \langle u_1, u_2, u_3\rangle$

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$$

**61.** $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = 0$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (u_2v_3 - u_3v_2)v_1 + (u_3v_1 - u_1v_3)v_2 + (u_1v_2 - u_2v_1)v_3 = 0$$

Thus, $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$.

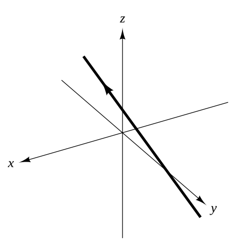
**63.** $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\theta = \pi/2$ and $\sin\theta = 1$. Therefore, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|$.

## Section 10.5 Lines and Planes in Space

**1.** $x = 1 + 3t, y = 2 - t, z = 2 + 5t$

(a)




(b) When $t = 0$ we have $P = (1, 2, 2)$. When $t = 3$ we have $Q = (10, -1, 17)$.

$$\overrightarrow{PQ} = \langle 9, -3, 15\rangle$$

The components of the vector and the coefficients of $t$ are proportional since the line is parallel to $\overrightarrow{PQ}$.

(c) $y = 0$ when $t = 2$. Thus, $x = 7$ and $z = 12$.
Point: $(7, 0, 12)$

$x = 0$ when $t = -\frac{1}{3}$. Point: $\left(0, \frac{7}{3}, \frac{1}{3}\right)$

$z = 0$ when $t = -\frac{2}{5}$. Point: $\left(-\frac{1}{5}, \frac{12}{5}, 0\right)$

**3.** Point: $(0, 0, 0)$

Direction vector: $\mathbf{v} = \langle 1, 2, 3\rangle$

Direction numbers: 1, 2, 3

(a) Parametric: $x = t, y = 2t, z = 3t$

(b) Symmetric: $x = \dfrac{y}{2} = \dfrac{z}{3}$

**5.** Point: $(-2, 0, 3)$

Direction vector: $\mathbf{v} = \langle 2, 4, -2\rangle$

Direction numbers: $2, 4, -2$

(a) Parametric: $x = -2 + 2t, y = 4t, z = 3 - 2t$

(b) Symmetric: $\dfrac{x + 2}{2} = \dfrac{y}{4} = \dfrac{z - 3}{-2}$

**7.** Point: $(1, 0, 1)$

Direction vector: $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Direction numbers: $3, -2, 1$

(a) Parametric: $x = 1 + 3t, y = -2t, z = 1 + t$

(b) Symmetric: $\dfrac{x-1}{3} = \dfrac{y}{-2} = \dfrac{z-1}{1}$

**9.** Points: $(5, -3, -2), \left(\dfrac{-2}{3}, \dfrac{2}{3}, 1\right)$

Direction vector: $\mathbf{v} = \dfrac{17}{3}\mathbf{i} - \dfrac{11}{3}\mathbf{j} - 3\mathbf{k}$

Direction numbers: $17, -11, -9$

(a) Parametric: $x = 5 + 17t, y = -3 - 11t, z = -2 - 9t$

(b) Symmetric: $\dfrac{x-5}{17} = \dfrac{y+3}{-11} = \dfrac{z+2}{-9}$

**11.** Points: $(2, 3, 0), (10, 8, 12)$

Direction vector: $\langle 8, 5, 12 \rangle$

Direction numbers: $8, 5, 12$

(a) Parametric: $x = 2 + 8t, y = 3 + 5t, z = 12t$

(b) Symmetric: $\dfrac{x-2}{8} = \dfrac{y-3}{5} = \dfrac{z}{12}$

**13.** Point: $(2, 3, 4)$

Direction vector: $\mathbf{v} = \mathbf{k}$

Direction numbers: $0, 0, 1$

Parametric: $x = 2, y = 3, z = 4 + t$

**15.** Point: $(-2, 3, 1)$

Direction vector: $\mathbf{v} = 4\mathbf{i} - \mathbf{k}$

Direction numbers: $4, 0, -1$

Parametric: $x = -2 + 4t, y = 3, z = 1 - t$

Symmetric: $\dfrac{x+2}{4} = \dfrac{z-1}{-1}, y = 3$

(a) On line

(b) On line

(c) Not on line $(y \neq 3)$

(d) Not on line $\left(\dfrac{6+2}{4} \neq \dfrac{-2-1}{-1}\right)$

**17.** $L_i$: $\mathbf{v} = \langle -3, 2, 4 \rangle$ $(6, -2, 5)$ on line

$L_2$: $\mathbf{v} = \langle 6, -4, -8 \rangle$ $(6, -2, 5)$ on line

$L_3$: $\mathbf{v} = \langle -6, 4, 8 \rangle$ $(6, -2, 5)$ not on line

$L_4$: $\mathbf{v} = \langle 6, 4, -6 \rangle$ not parallel to $L_1, L_2$, nor $L_3$

Hence, $L_1$ and $L_2$ are identical.

$L_1 = L_2$ and $L_3$ are parallel.

**19.** At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. Thus,

(i) $4t + 2 = 2s + 2$, (ii) $3 = 2s + 3$, and (iii) $-t + 1 = s + 1$.

From (ii), we find that $s = 0$ and consequently, from (iii), $t = 0$. Letting $s = t = 0$, we see that equation (i) is satisfied and therefore the two lines intersect. Substituting zero for $s$ or for $t$, we obtain the point $(2, 3, 1)$.

$\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ (First line)

$\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Second line)

$$\cos\theta = \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{8-1}{\sqrt{17}\sqrt{9}} = \frac{7}{3\sqrt{17}} = \frac{7\sqrt{17}}{51}$$

**21.** Writing the equations of the lines in parametric form we have

$x = 3t$ $y = 2 - t$ $z = -1 + t$

$x = 1 + 4s$ $y = -2 + s$ $z = -3 - 3s.$

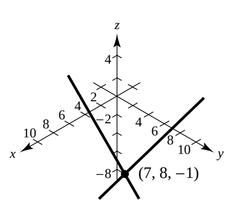
For the coordinates to be equal, $3t = 1 + 4s$ and $2 - t = -2 + s$. Solving this system yields $t = \frac{17}{7}$ and $s = \frac{11}{7}$. When using these values for $s$ and $t$, the $z$ coordinates are not equal. The lines do not intersect.

**23.** $x = 2t + 3$ $x = -2s + 7$

$y = 5t - 2$ $y = s + 8$

$z = -t + 1$ $z = 2s - 1$

Point of intersection: $(7, 8, -1)$



**25.** $4x - 3y - 6z = 6$

(a) $P = (0, 0, -1), Q = (0, -2, 0), R = (3, 4, -1)$

$\overrightarrow{PQ} = \langle 0, -2, 1\rangle, \overrightarrow{PR} = \langle 3, 4, 0\rangle$

(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 1 \\ 3 & 4 & 0 \end{vmatrix} = \langle -4, 3, 6\rangle$

The components of the cross product are proportional to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

**27.** Point: $(2, 1, 2)$

$\mathbf{n} = \mathbf{i} = \langle 1, 0, 0\rangle$

$1(x - 2) + 0(y - 1) + 0(z - 2) = 0$

$x - 2 = 0$

**29.** Point: $(3, 2, 2)$

Normal vector: $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$2(x - 3) + 3(y - 2) - 1(z - 2) = 0$

$2x + 3y - z = 10$

**31.** Point: $(0, 0, 6)$

Normal vector: $\mathbf{n} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$-1(x - 0) + 1(y - 0) - 2(z - 6) = 0$

$-x + y - 2z + 12 = 0$

$x - y + 2z = 12$

**33.** Let $\mathbf{u}$ be the vector from $(0, 0, 0)$ to $(1, 2, 3)$:

$\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Let $\mathbf{v}$ be the vector from $(0, 0, 0)$ to $(-2, 3, 3)$:

$\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix}$

$= -3\mathbf{i} + (-9)\mathbf{j} + 7\mathbf{k}$

$-3(x - 0) - 9(y - 0) + 7(z - 0) = 0$

$3x + 9y - 7z = 0$

**35.** Let $\mathbf{u}$ be the vector from $(1, 2, 3)$ to $(3, 2, 1)$: $\mathbf{u} = 2\mathbf{i} - 2\mathbf{k}$

Let $\mathbf{v}$ be the vector from $(1, 2, 3)$ to $(-1, -2, 2)$: $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

Normal vector: $\left(\frac{1}{2}\mathbf{u}\right) \times (-\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 4 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

$4(x - 1) - 3(y - 2) + 4(z - 3) = 0$

$4x - 3y + 4z = 10$

**37.** $(1, 2, 3)$, Normal vector: $\mathbf{v} = \mathbf{k}$, $1(z - 3) = 0, z = 3$

**39.** The direction vectors for the lines are $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Point of intersection of the lines: $(-1, 5, 1)$

$(x + 1) + (y - 5) + (z - 1) = 0$

$x + y + z = 5$

**41.** Let $\mathbf{v}$ be the vector from $(-1, 1, -1)$ to $(2, 2, 1)$: $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Let $\mathbf{n}$ be a vector normal to the plane $2x - 3y + z = 3$: $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Since $v$ and $n$ both lie in the plane $p$, the normal vector to $p$ is

$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 7\mathbf{i} + \mathbf{j} - 11\mathbf{k}$

$7(x - 2) + 1(y - 2) - 11(z - 1) = 0$

$7x + y - 11z = 5$

**43.** Let $\mathbf{u} = \mathbf{i}$ and let $\mathbf{v}$ be the vector from $(1, -2, -1)$ to $(2, 5, 6)$: $\mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

Since $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane $P$, the normal vector to $P$ is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix} = -7\mathbf{j} + 7\mathbf{k} = -7(\mathbf{j} - \mathbf{k})$$

$$[y - (-2)] - [z - (-1)] = 0$$

$$y - z = -1$$

**45.** The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 5, -3, 1\rangle, \mathbf{n}_2 = \langle 1, 4, 7\rangle, \cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|} = 0.$$

Thus, $\theta = \pi/2$ and the planes are orthogonal.

**47.** The normal vectors to the planes are

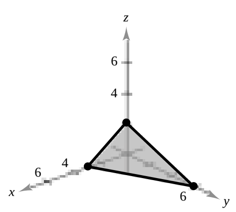
$$\mathbf{n}_1 = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}, \ \mathbf{n}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|} = \frac{|5 - 3 - 6|}{\sqrt{46}\sqrt{27}} = \frac{4\sqrt{138}}{414}.$$

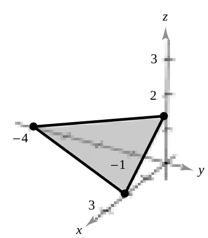
Therefore, $\theta = \arccos\left(\dfrac{4\sqrt{138}}{414}\right) \approx 83.5°$.

**49.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -5, -1\rangle$ and $\mathbf{n}_2 = \langle 5, -25, -5\rangle$. Since $\mathbf{n}_2 = 5\mathbf{n}_1$, the planes are parallel, but not equal.

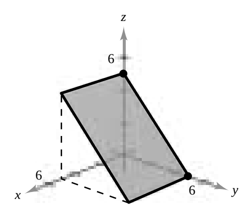
**51.** $4x + 2y + 6z = 12$



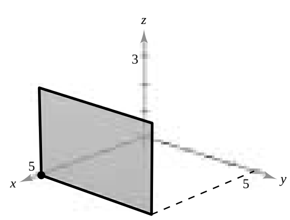

**53.** $2x - y + 3z = 4$



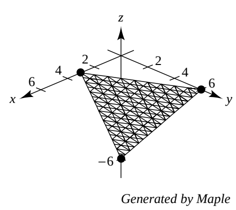

**55.** $y + z = 5$



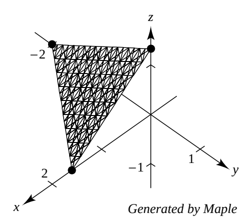

**57.** $x = 5$




**59.** $2x + y - z = 6$




**61.** $-5x + 4y - 6z + 8 = 0$




**63.** $P_1$: $\mathbf{n} = \langle 3, -2, 5\rangle$ $\quad (1, -1, 1)$ on plane

$P_2$: $\mathbf{n} = \langle -6, 4, -10\rangle$ $\quad (1, -1, 1)$ not on plane

$P_3$: $\mathbf{n} = \langle -3, 2, 5\rangle$

$P_4$: $\mathbf{n} = \langle 75, -50, 125\rangle$ $\quad (1, -1, 1)$ on plane

$P_1$ and $P_4$ are identical.

$P_1 = P_4$ is parallel to $P_2$.

**65.** Each plane passes through the points

$$(c, 0, 0), (0, c, 0), \text{ and } (0, 0, c).$$

**67.** The normals to the planes are $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 7(\mathbf{j} + 2\mathbf{k}).$$

Now find a point of intersection of the planes.

$$\begin{aligned} 6x + 4y - 2y &= 14 \\ x - 4y + 2z &= 0 \\ 7x \qquad\qquad &= 14 \\ x &= 2 \end{aligned}$$

Substituting 2 for $x$ in the second equation, we have $-4y + 2z = -2$ or $z = 2y - 1$. Letting $y = 1$, a point of intersection is $(2, 1, 1)$.

$$x = 2, y = 1 + t, z = 1 + 2t$$

**69.** Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = \frac{1}{2} + t,\ y = \frac{-3}{2} - t,\ z = -1 + 2t$$

$$2\left(\frac{1}{2} + t\right) - 2\left(\frac{-3}{2} - t\right) + (-1 + 2t) = 12,\ t = \frac{3}{2}$$

Substituting $t = 3/2$ into the parametric equations for the line we have the point of intersection $(2, -3, 2)$. The line does not lie in the plane.

**71.** Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = 1 + 3t,\ y = -1 - 2t,\ z = 3 + t$$

$$2(1 + 3t) + 3(-1 - 2t) = 10,\ -1 = 10, \text{ contradiction}$$

Therefore, the line does not intersect the plane.

**73.** Point: $Q(0, 0, 0)$

Plane: $2x + 3y + z - 12 = 0$

Normal to plane: $\mathbf{n} = \langle 2, 3, 1\rangle$

Point in plane: $P(6, 0, 0)$

Vector $\overrightarrow{PQ} = \langle -6, 0\ 0\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-12|}{\sqrt{14}} = \frac{6\sqrt{14}}{7}$$

**75.** Point: $Q(2, 8, 4)$

Plane: $2x + y + z = 5$

Normal to plane: $\mathbf{n} = \langle 2, 1, 1\rangle$

Point in plane: $P\langle 0, 0, 5\rangle$

Vector: $\overrightarrow{PQ} = \langle 2, 8, -1\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{11}{\sqrt{6}} = \frac{11\sqrt{6}}{6}$$

**77.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -3, 4\rangle$ and $\mathbf{n}_2 = \langle 1, -3, 4\rangle$. Since $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$P = (10, 0, 0)$ is a point in $x - 3y + 4z = 10$.
$Q = (6, 0, 0)$ is a point in $x - 3y + 4z = 6$.

$$\overrightarrow{PQ} = \langle -4, 0, 0\rangle, D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{4}{\sqrt{26}} = \frac{2\sqrt{26}}{13}$$

**79.** The normal vectors to the planes are $\mathbf{n}_1 = \langle -3, 6, 7\rangle$ and $\mathbf{n}_2 = \langle 6, -12, -14\rangle$. Since $\mathbf{n}_2 = -2\mathbf{n}_1$, the planes are parallel. Choose a point in each plane.

$P = (0, -1, 1)$ is a point in $-3x + 6y + 7z = 1$.

$Q = \left(\frac{25}{6}, 0, 0\right)$ is a point in $6x - 12y - 14z = 25$.

$$\overrightarrow{PQ} = \left\langle \frac{25}{6}, 1, -1\right\rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{|-27/2|}{\sqrt{94}} = \frac{27}{2\sqrt{94}} = \frac{27\sqrt{94}}{188}$$

**81.** $\mathbf{u} = \langle 4, 0, -1\rangle$ is the direction vector for the line. $Q(1, 5, -2)$ is the given point, and $P(-2, 3, 1)$ is on the line. Hence, $\overrightarrow{PQ} = \langle 3, 2, -3\rangle$ and

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 4 & 0 & -1 \end{vmatrix} = \langle -2, -9, -8\rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{149}}{\sqrt{17}} = \frac{\sqrt{2533}}{17}$$

**83.** The parametric equations of a line $L$ parallel to $\mathbf{v} = \langle a, b, c,\rangle$ and passing through the point $P(x_1, y_1, z_1)$ are

$$x = x_1 + at, y = y_1 + bt, z = z_1 + ct.$$

The symmetric equations are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

**85.** Solve the two linear equations representing the planes to find two points of intersection. Then find the line determined by the two points.

**87.** (a) Sphere

$(x-3)^2 + (y+2)^2 + (z-5)^2 = 16$

$x^2 + y^2 + z^2 - 6x + 4y - 10z + 22 = 0$

(b) Parallel planes

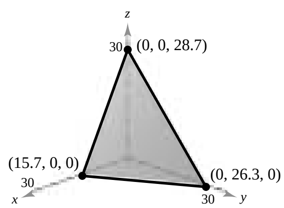
$4x - 3y + z = 10 \pm 4\|\mathbf{n}\| = 10 \pm 4\sqrt{26}$

**89.** (a) $z = 28.7 - 1.83x - 1.09y$

| Year | 1980 | 1985 | 1990 | 1994 | 1995 | 1996 | 1997 |
|---|---|---|---|---|---|---|---|
| $z$ (approx.) | 16.16 | 14.23 | 9.81 | 8.60 | 8.42 | 8.27 | 8.23 |

(b) An increase in $x$ or $y$ will cause a decrease in $z$. In fact, any increase in two variables will cause a decrease in the third.

(c)




**91.** True

## Section 10.6 Surfaces in Space

**1.** Ellipsoid

Matches graph (c)

**3.** Hyperboloid of one sheet

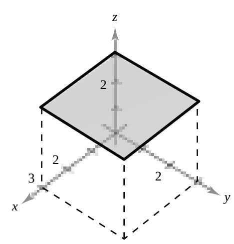
Matches graph (f)

**5.** Elliptic paraboloid

Matches graph (d)

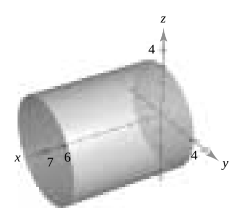
**7.** $z = 3$

Plane parallel to the $xy$-coordinate plane



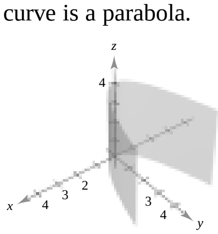

**9.** $y^2 + z^2 = 9$

The $x$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is a circle.




**11.** $y = x^2$

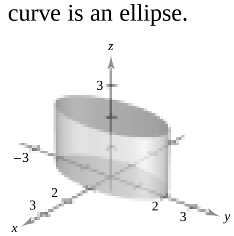
The $z$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $z$-axis. The generating curve is a parabola.




**13.** $4x^2 + y^2 = 4$

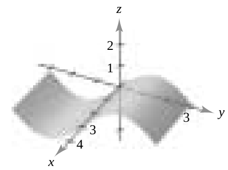
$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

The $z$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $z$-axis. The generating curve is an ellipse.



**15.** $z = \sin y$

The $x$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is the sine curve.




**17.** $x = x^2 + y^2$

(a) You are viewing the paraboloid from the $x$-axis: $(20, 0, 0)$

(b) You are viewing the paraboloid from above, but not on the $z$-axis: $(10, 10, 20)$

(c) You are viewing the paraboloid from the $z$-axis: $(0, 0, 20)$

(d) You are viewing the paraboloid from the $y$-axis: $(0, 20, 0)$

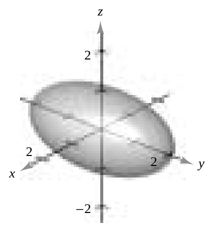
**19.** $\dfrac{x^2}{1} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$

Ellipsoid

$xy$-trace: $\dfrac{x^2}{1} + \dfrac{y^2}{4} = 1$ ellipse

$xz$-trace: $x^2 + z^2 = 1$ circle

$yz$-trace: $\dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$ ellipse




**21.** $16x^2 - y^2 + 16z^2 = 4$

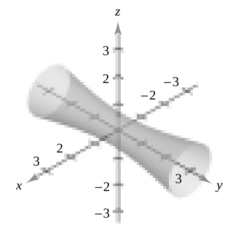
$4x^2 - \dfrac{y^2}{4} + 4z^2 = 1$

Hyperboloid on one sheet

$xy$-trace: $4x^2 - \dfrac{y^2}{4} = 1$ hyperbola

$xz$-trace: $4(x^2 + z^2) = 1$ circle

$yz$-trace: $\dfrac{-y^2}{4} + 4z^2 = 1$ hyperbola




**23.** $x^2 - y + z^2 = 0$

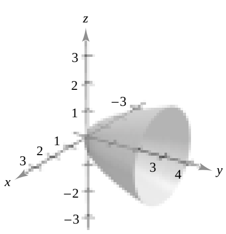
Elliptic paraboloid

$xy$-trace: $y = x^2$

$xz$-trace: $x^2 + z^2 = 0$,
point $(0, 0, 0)$

$yz$-trace: $y = z^2$

$y = 1$: $x^2 + z^2 = 1$




**25.** $x^2 - y^2 + z = 0$

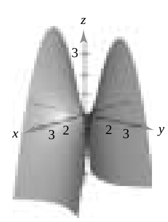
Hyperbolic paraboloid

$xy$-trace: $y = \pm x$

$xz$-trace: $z = -x^2$

$yz$-trace: $z = y^2$

$y = \pm 1$: $z = 1 - x^2$




**27.** $z^2 = x^2 + \dfrac{y^2}{4}$

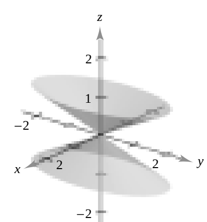
Elliptic Cone

$xy$-trace: point $(0, 0, 0)$

$xz$-trace: $z = \pm x$

$yz$-trace: $z = \dfrac{\pm 1}{2} y$

$z = \pm 1$: $x^2 + \dfrac{y^2}{4} = 1$




**29.**

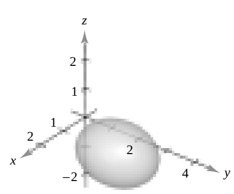
$$16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$$

$$16(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 16z^2 = -36 + 16 + 36$$

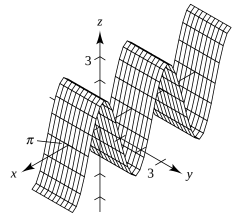
$$16(x - 1)^2 + 9(y - 2)^2 + 16z^2 = 16$$

$$\frac{(x - 1)^2}{1} + \frac{(y - 2)^2}{16/9} + \frac{z^2}{1} = 1$$

Ellipsoid with center $(1, 2, 0)$.

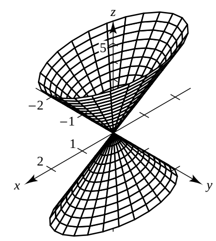


**31.** $z = 2 \sin x$



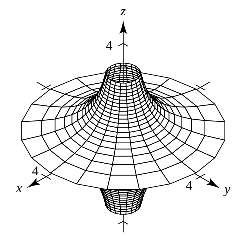
**33.** $z^2 = x^2 + 4y^2$

$z = \pm\sqrt{x^2 + 4y^2}$

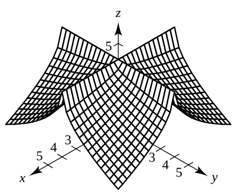


**35.** $x^2 + y^2 = \left(\dfrac{2}{z}\right)^2$

$y = \pm\sqrt{\dfrac{4}{z^2} - x^2}$

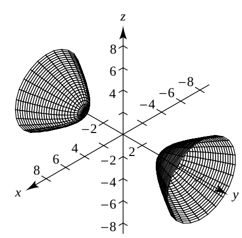


**37.** $z = 4 - \sqrt{|xy|}$



**39.** $4x^2 - y^2 + 4z^2 = -16$

$z = \pm\sqrt{\dfrac{y^2}{4} - x^2 - 4}$

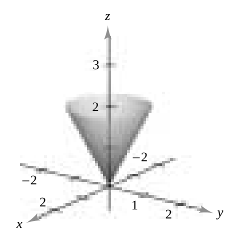


**41.** $z = 2\sqrt{x^2 + y^2}$

$z = 2$

$2\sqrt{x^2 + y^2} = 2$

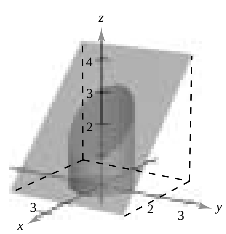
$x^2 + y^2 = 1$



**43.** $x^2 + y^2 = 1$

$x + z = 2$

$z = 0$



**45.** $x^2 + z^2 = [r(y)]^2$ and $z = r(y) = \pm 2\sqrt{y}$; therefore,

$x^2 + z^2 = 4y.$

**47.** $x^2 + y^2 = [r(z)]^2$ and $y = r(z) = \dfrac{z}{2}$; therefore,

$x^2 + y^2 = \dfrac{z^2}{4},\ 4x^2 + 4y^2 = z^2.$

**49.** $y^2 + z^2 = [r(x)]^2$ and $y = r(x) = \dfrac{2}{x}$; therefore,

$y^2 + z^2 = \left(\dfrac{2}{x}\right)^2,\ y^2 + z^2 = \dfrac{4}{x^2}.$

**51.** $x^2 + y^2 - 2z = 0$

$x^2 + y^2 = \left(\sqrt{2z}\right)^2$

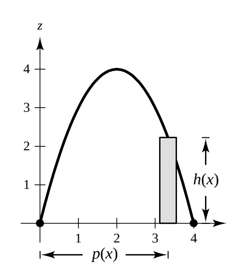
Equation of generating curve: $y = \sqrt{2z}$ or $x = \sqrt{2z}$

**53.** Let $C$ be a curve in a plane and let $L$ be a line not in a parallel plane. The set of all lines parallel to $L$ and intersecting $C$ is called a cylinder.

**55.** See pages 765 and 766.

**57.** $V = 2\pi\displaystyle\int_0^4 x(4x - x^2)\,dx$

$= 2\pi\left[\dfrac{4x^3}{3} - \dfrac{x^4}{4}\right]_0^4 = \dfrac{128\pi}{3}$



**59.** $z = \dfrac{x^2}{2} + \dfrac{y^2}{4}$

(a) When $z = 2$ we have $2 = \dfrac{x^2}{2} + \dfrac{y^2}{4}$, or $1 = \dfrac{x^2}{4} + \dfrac{y^2}{8}$

Major axis: $2\sqrt{8} = 4\sqrt{2}$

Minor axis: $2\sqrt{4} = 4$

$c^2 = a^2 - b^2, c^2 = 4, c = 2$

Foci: $(0, \pm 2, 2)$

(b) When $z = 8$ we have $8 = \dfrac{x^2}{2} + \dfrac{y^2}{4}$, or $1 = \dfrac{x^2}{16} + \dfrac{y^2}{32}$.

Major axis: $2\sqrt{32} = 8\sqrt{2}$

Minor axis: $2\sqrt{16} = 8$

$c^2 = 32 - 16 = 16, c = 4$

Foci: $(0, \pm 4, 8)$

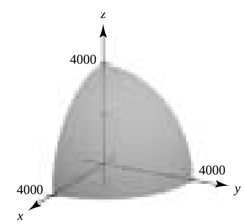
**61.** If $(x, y, z)$ is on the surface, then

$$\begin{aligned}(y + 2)^2 &= x^2 + (y - 2)^2 + z^2\\ y^2 + 4y + 4 &= x^2 + y^2 - 4y + 4 + z^2\\ x^2 + z^2 &= 8y\end{aligned}$$

Elliptic paraboloid

Traces parallel to $xz$-plane are circles.

**63.** $\dfrac{x^2}{3963^2} + \dfrac{y^2}{3963^2} + \dfrac{z^2}{3942^2} = 1$




**65.** $z = \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2}, z = bx + ay$

$$bx + ay = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$$\frac{1}{a^2}\left(x^2 + a^2bx + \frac{a^4b^2}{4}\right) = \frac{1}{b^2}\left(y^2 - ab^2y + \frac{a^2b^4}{4}\right)$$

$$\frac{\left(x + \frac{a^2b}{2}\right)^2}{a^2} = \frac{\left(y - \frac{ab^2}{2}\right)^2}{b^2}$$

$$y = \pm\frac{b}{a}\left(x + \frac{a^2b}{2}\right) + \frac{ab^2}{2}$$

Letting $x = at$, you obtain the two intersecting lines
$x = at, y = -bt, z = 0$ and $x = at, y = bt + ab^2$
$z = 2abt + a^2b^2$.

**67.** The Klein bottle *does not* have both an "inside" and an "outside." It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

## Section 10.7 Cylindrical and Spherical Coordinates

**1.** $(5, 0, 2)$, cylindrical

$x = 5\cos 0 = 5$

$y = 5\sin 0 = 0$

$z = 2$

$(5, 0, 2)$, rectangular

**3.** $\left(2, \dfrac{\pi}{3}, 2\right)$, cylindrical

$x = 2\cos\dfrac{\pi}{3} = 1$

$y = 2\sin\dfrac{\pi}{3} = \sqrt{3}$

$z = 2$

$\left(1, \sqrt{3}, 2\right)$, rectangular

**5.** $\left(4, \dfrac{7\pi}{6}, 3\right)$, cylindrical

$x = 4\cos\dfrac{7\pi}{6} = -2\sqrt{3}$

$y = 4\sin\dfrac{7\pi}{6} = -2$

$z = 3$

$\left(-2\sqrt{3}, -2, 3\right)$, rectangular

**7.** $(0, 5, 1)$, rectangular

$r = \sqrt{(0)^2 + (5)^2} = 5$

$\theta = \arctan \frac{5}{0} = \frac{\pi}{2}$

$z = 1$

$\left(5, \frac{\pi}{2}, 1\right)$, cylindrical

**9.** $\left(1, \sqrt{3}, 4\right)$, rectangular

$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

$\theta = \arctan \sqrt{3} = \frac{\pi}{3}$

$z = 4$

$\left(2, \frac{\pi}{3}, 4\right)$, cylindrical

**11.** $(2, -2, -4)$, rectangular

$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

$\theta = \arctan(-1) = -\frac{\pi}{4}$

$z = -4$

$\left(2\sqrt{2}, \frac{-\pi}{4}, -4\right)$, cylindrical

**13.** $x^2 + y^2 + z^2 = 10$ rectangular equation

$r^2 + z^2 = 10$ cylindrical equation

**15.** $y = x^2$ rectangular equation

$r \sin \theta = (r \cos \theta)^2$

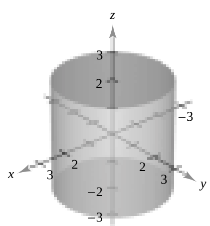
$\sin \theta = r \cos^2 \theta$

$r = \sec \theta \cdot \tan \theta$ cylindrical equation

**17.** $r = 2$

$\sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4$



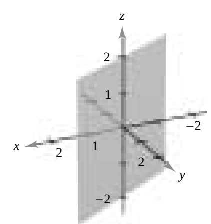
**19.** $\theta = \frac{\pi}{6}$

$\tan \frac{\pi}{6} = \frac{y}{x}$

$\frac{1}{\sqrt{3}} = \frac{y}{x}$

$x = \sqrt{3}y$

$x - \sqrt{3}y = 0$



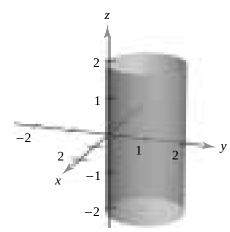
**21.** $r = 2 \sin \theta$

$r^2 = 2r \sin \theta$

$x^2 + y^2 = 2y$

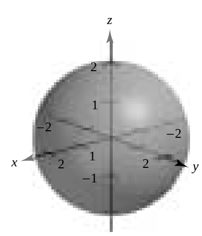
$x^2 + y^2 - 2y = 0$

$x^2 + (y - 1)^2 = 1$



**23.** $r^2 + z^2 = 4$

$x^2 + y^2 + z^2 = 4$



**25.** $(4, 0, 0)$, rectangular

$\rho = \sqrt{4^2 + 0^2 + 0^2} = 4$

$\theta = \arctan 0 = 0$

$\phi = \arccos 0 = \frac{\pi}{2}$

$\left(4, 0, \frac{\pi}{2}\right)$, spherical

**27.** $\left(-2, 2\sqrt{3}, 4\right)$, rectangular

$\rho = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2 + 4^2} = 4\sqrt{2}$

$\theta = \arctan\left(-\sqrt{3}\right) = \frac{2\pi}{3}$

$\phi = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$, spherical

**29.** $\left(\sqrt{3}, 1, 2\sqrt{3}\right)$, rectangular

$\rho = \sqrt{3 + 1 + 12} = 4$

$\theta = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\phi = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\left(4, \frac{\pi}{6}, \frac{\pi}{6}\right)$, spherical

**31.** $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$, spherical

$x = 4 \sin \frac{\pi}{4} \cos \frac{\pi}{6} = \sqrt{6}$

$y = 4 \sin \frac{\pi}{4} \sin \frac{\pi}{6} = \sqrt{2}$

$z = 4 \cos \frac{\pi}{4} = 2\sqrt{2}$

$\left(\sqrt{6}, \sqrt{2}, 2\sqrt{2}\right)$, rectangular

**33.** $\left(12, \frac{-\pi}{4}, 0\right)$, spherical

$x = 12 \sin 0 \cos\left(\frac{-\pi}{4}\right) = 0$

$y = 12 \sin 0 \sin\left(\frac{-\pi}{4}\right) = 0$

$z = 12 \cos 0 = 12$

$(0, 0, 12)$, rectangular

**35.** $\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$x = 5 \sin \frac{3\pi}{4} \cos \frac{\pi}{4} = \frac{5}{2}$

$y = 5 \sin \frac{3\pi}{4} \sin \frac{\pi}{4} = \frac{5}{2}$

$z = 5 \cos \frac{3\pi}{4} = -\frac{5\sqrt{2}}{2}$

$\left(\frac{5}{2}, \frac{5}{2}, -\frac{5\sqrt{2}}{2}\right)$, rectangular

**37.** (a) Programs will vary.

(b) $(x, y, z) = (3, -4, 2)$

$(\rho, \theta, \phi) = (5.385, -0.927, 1.190)$

**39.** $x^2 + y^2 + z^2 = 36$ rectangular equation

$\rho^2 = 36$ spherical equation

**41.** $x^2 + y^2 = 9$ rectangular equation

$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 9$

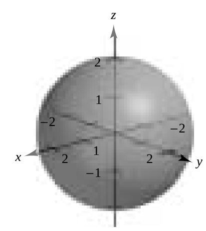
$\rho^2 \sin^2 \phi = 9$

$\rho \sin \phi = 3$

$\rho = 3 \csc \phi$ spherical equation

**43.** $\rho = 2$

$x^2 + y^2 + z^2 = 4$



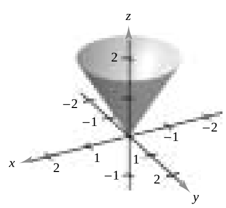

**45.** $\phi = \frac{\pi}{6}$

$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

$\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

$\frac{3}{4} = \frac{z^2}{x^2 + y^2 + z^2}$

$3x^2 + 3y^2 - z^2 = 0$

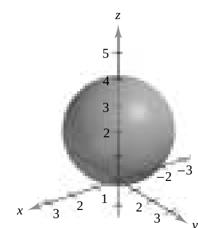



**47.** $\rho = 4 \cos \phi$

$\sqrt{x^2 + y^2 + z^2} = \frac{4z}{\sqrt{x^2 + y^2 + z^2}}$

$x^2 + y^2 + z^2 - 4z = 0$

$x^2 + y^2 + (z - 2)^2 = 4$

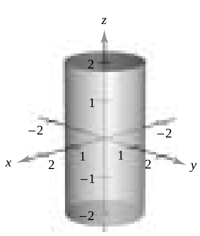



**49.** $\rho = \csc \phi$

$\rho \sin \phi = 1$

$\sqrt{x^2 + y^2} = 1$

$x^2 + y^2 = 1$




**51.** $\left(4, \frac{\pi}{4}, 0\right)$, cylindrical

$\rho = \sqrt{4^2 + 0^2} = 4$

$\theta = \frac{\pi}{4}$

$\phi = \arccos 0 = \frac{\pi}{2}$

$\left(4, \frac{\pi}{4}, \frac{\pi}{2}\right)$, spherical

**53.** $\left(4, \frac{\pi}{2}, 4\right)$, cylindrical

$\rho = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

$\theta = \frac{\pi}{2}$

$\phi = \arccos\left(\frac{4}{4\sqrt{2}}\right) = \frac{\pi}{4}$

$\left(4\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$, spherical

**55.** $\left(4, \frac{-\pi}{6}, 6\right)$, cylindrical

$\rho = \sqrt{4^2 + 6^2} = 2\sqrt{13}$

$\theta = \frac{-\pi}{6}$

$\phi = \arccos \frac{3}{\sqrt{13}}$

$\left(2\sqrt{13}, \frac{-\pi}{6}, \arccos \frac{3}{\sqrt{13}}\right)$, spherical

**57.** $(12, \pi, 5)$, cylindrical

$\rho = \sqrt{12^2 + 5^2} = 13$

$\theta = \pi$

$\phi = \arccos \frac{5}{13}$

$\left(13, \pi, \arccos \frac{5}{13}\right)$, spherical

**59.** $\left(10, \frac{\pi}{6}, \frac{\pi}{2}\right)$, spherical

$r = 10 \sin \frac{\pi}{2} = 10$

$\theta = \frac{\pi}{6}$

$z = 10 \cos \frac{\pi}{2} = 0$

$\left(10, \frac{\pi}{6}, 0\right)$, cylindrical

**61.** $\left(36, \pi, \frac{\pi}{2}\right)$, spherical

$r = \rho \sin \phi = 36 \sin \frac{\pi}{2} = 36$

$\theta = \pi$

$z = \rho \cos \phi = 36 \cos \frac{\pi}{2} = 0$

$(36, \pi, 0)$, cylindrical

**63.** $\left(6, -\frac{\pi}{6}, \frac{\pi}{3}\right)$, spherical

$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}$

$\theta = -\frac{\pi}{6}$

$z = 6 \cos \frac{\pi}{3} = 3$

$\left(3\sqrt{3}, -\frac{\pi}{6}, 3\right)$, cylindrical

**65.** $\left(8, \frac{7\pi}{6}, \frac{\pi}{6}\right)$, spherical

$r = 8 \sin \frac{\pi}{6} = 4$

$\theta = \frac{7\pi}{6}$

$z = 8 \cos \frac{\pi}{6} = \frac{8\sqrt{3}}{2}$

$\left(4, \frac{7\pi}{6}, 4\sqrt{3}\right)$, cylindrical

| | Rectangular | Cylindrical | Spherical |
|---|---|---|---|
| **67.** | $(4, 6, 3)$ | $(7.211, 0.983, 3)$ | $(7.810, 0.983, 1.177)$ |
| **69.** | $(4.698, 1.710, 8)$ | $\left(5, \frac{\pi}{9}, 8\right)$ | $(9.434, 0.349, 0.559)$ |
| **71.** | $(-7.071, 12.247, 14.142)$ | $(14.142, 2.094, 14.142)$ | $\left(20, \frac{2\pi}{3}, \frac{\pi}{4}\right)$ |
| **73.** | $(3, -2, 2)$ | $(3.606, -0.588, 2)$ | $(4.123, -0.588, 1.064)$ |
| **75.** | $\left(\frac{5}{2}, \frac{4}{3}, \frac{-3}{2}\right)$ | $(2.833, 0.490, -1.5)$ | $(3.206, 0.490, 2.058)$ |
| **77.** | $(-3.536, 3.536, -5)$ | $\left(5, \frac{3\pi}{4}, -5\right)$ | $(7.071, 2.356, 2.356)$ |
| **79.** | $(2.804, -2.095, 6)$ | $(-3.5, 2.5, 6)$ | $(6.946, 5.642, 0.528)$ |

[Note: Use the cylindrical coordinates $(3.5, 5.642, 6)$]

**81.** $r = 5$

Cylinder

Matches graph (d)

**83.** $\rho = 5$

Sphere

Matches graph (c)

**85.** $r^2 = z, x^2 + y^2 = z$

Paraboloid

Matches graph (f)

**87.** Rectangular to cylindrical: $r^2 = x^2 + y^2$

$\tan \theta = \frac{y}{x}$

$z = z$

Cylindrical to rectangular: $x = r \cos \theta$

$y = r \sin \theta$

$z = z$

**89.** Rectangular to spherical: $\rho^2 = x^2 + y^2 + z^2$

$\tan \theta = \frac{y}{x}$

$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

Spherical to rectangular: $x = \rho \sin \phi \cos \theta$

$y = \rho \sin \phi \sin \theta$

$z = \rho \cos \phi$

**91.** $x^2 + y^2 + z^2 = 16$

(a) $r^2 + z^2 = 16$

(b) $\rho^2 = 16, \rho = 4$

**93.** $x^2 + y^2 + z^2 - 2z = 0$

(a) $r^2 + z^2 - 2z = 0, r^2 + (z - 1)^2 = 1$

(b) $\rho^2 - 2\rho\cos\phi = 0, \rho(\rho - 2\cos\phi) = 0,$

$\rho = 2\cos\phi$

**95.** $x^2 + y^2 = 4y$

(a) $r^2 = 4r\sin\theta,\ r = 4\sin\theta$

(b) $\rho^2\sin^2\phi = 4\rho\sin\phi\sin\theta,$

$\rho\sin\phi(\rho\sin\phi - 4\sin\theta) = 0,$

$\rho = \dfrac{4\sin\theta}{\sin\phi},\ \rho = 4\sin\theta\csc\phi$

**97.** $x^2 - y^2 = 9$

(a) $r^2\cos^2\theta - r^2\sin^2\theta = 9,$

$r^2 = \dfrac{9}{\cos^2\theta - \sin^2\theta}$

(b) $\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta = 9,$

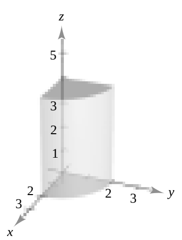
$\rho^2\sin^2\phi = \dfrac{9}{\cos^2\theta - \sin^2\theta},$

$\rho^2 = \dfrac{9\csc^2\phi}{\cos^2\theta - \sin^2\theta}$

**99.** $0 \le \theta \le \dfrac{\pi}{2}$

$0 \le r \le 2$

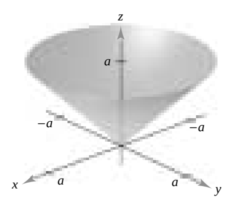
$0 \le z \le 4$




**101.** $0 \le \theta \le 2\pi$

$0 \le r \le a$

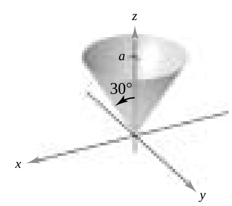
$r \le z \le a$




**103.** $0 \le \theta \le 2\pi$

$0 \le \phi \le \dfrac{\pi}{6}$

$0 \le \rho \le a\sec\phi$

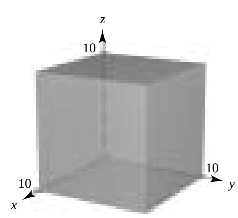



**105.** Rectangular

$0 \le x \le 10$

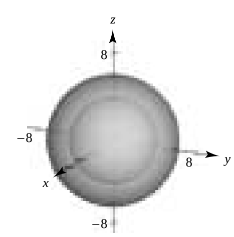
$0 \le y \le 10$

$0 \le z \le 10$




**107.** Spherical

$4 \le \rho \le 6$




**109.** $z = \sin\theta, r = 1$

$z = \dfrac{y}{r} = \dfrac{y}{1} = y$

The curve of intersection is the ellipse formed by the intersection of the plane $z = y$ and the cylinder $r = 1$.

## Review Exercises for Chapter 10

**1.** $P = (1, 2),\ Q = (4, 1),\ R = (5, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 3, -1\rangle = 3\mathbf{i} - \mathbf{j},$

$\mathbf{v} = \overrightarrow{PR} = \langle 4, 2\rangle = 4\mathbf{i} + 2\mathbf{j}$

(b) $\|\mathbf{v}\| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

(c) $2\mathbf{u} + \mathbf{v} = \langle 6, -2\rangle + \langle 4, 2\rangle = \langle 10, 0\rangle = 10\mathbf{i}$

**3.** $\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j} = 8\cos 120°\,\mathbf{i} + 8\sin 120°\,\mathbf{j}$

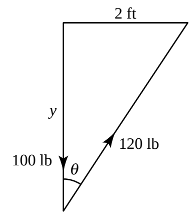
$= -4\mathbf{i} + 4\sqrt{3}\mathbf{j}$

**5.** $120 \cos \theta = 100$

$$\theta = \arccos\left(\frac{5}{6}\right)$$

$$\tan \theta = \frac{2}{y} \Rightarrow y = \frac{2}{\tan \theta}$$

$$y = \frac{2}{\tan[\arccos(5/6)]} = \frac{2}{\sqrt{11}/5} = \frac{10}{\sqrt{11}} \approx 3.015 \text{ ft}$$




**7.** $z = 0,\ y = 4,\ x = -5\colon\ (-5, 4, 0)$

**9.** Looking down from the positive $x$-axis towards the $yz$-plane, the point is either in the first quadrant $(y > 0, z > 0)$ or in the third quadrant $(y < 0, z < 0)$. The $x$-coordinate can be any number.

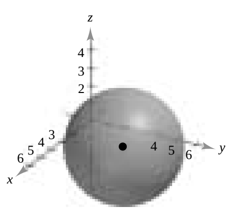
**11.** $(x - 3)^2 + (y + 2)^2 + (z - 6)^2 = \left(\frac{15}{2}\right)^2$

**13.** $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$

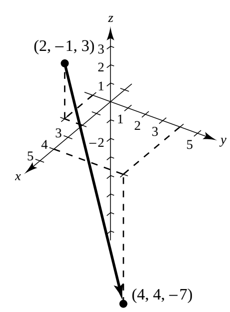
$(x - 2)^2 + (y - 3)^2 + z^2 = 9$

Center: $(2, 3, 0)$

Radius: 3




**15.** $\mathbf{v} = \langle 4 - 2, 4 + 1, -7 - 3\rangle = \langle 2, 5, -10\rangle$




**17.** $\mathbf{v} = \langle -1 - 3, 6 - 4, 9 + 1\rangle = \langle -4, 2, 10\rangle$

$\mathbf{w} = \langle 5 - 3, 3 - 4, -6 + 1\rangle = \langle 2, -1, -5\rangle$

Since $-2\mathbf{w} = \mathbf{v}$, the points lie in a straight line.

**19.** Unit vector: $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{\langle 2, 3, 5\rangle}{\sqrt{38}} = \left\langle \dfrac{2}{\sqrt{38}}, \dfrac{3}{\sqrt{38}}, \dfrac{5}{\sqrt{38}} \right\rangle$

**21.** $P = (5, 0, 0),\ Q = (4, 4, 0),\ R = (2, 0, 6)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -1, 4, 0\rangle = -\mathbf{i} + 4\mathbf{j}$,

$\mathbf{v} = \overrightarrow{PR} = \langle -3, 0, 6\rangle = -3\mathbf{i} + 6\mathbf{k}$

(b) $\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + 4(0) + 0(6) = 3$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 = 45$

**23.** $\mathbf{u} = \langle 7, -2, 3\rangle,\ \mathbf{v} = \langle -1, 4, 5\rangle$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are orthogonal.

**25.** $\mathbf{u} = 5\left(\cos \frac{3\pi}{4}\mathbf{i} + \sin \frac{3\pi}{4}\mathbf{j}\right) = \frac{5\sqrt{2}}{2}[-\mathbf{i} + \mathbf{j}]$

$$\mathbf{v} = 2\left(\cos \frac{2\pi}{3}\mathbf{i} + \sin \frac{2\pi}{3}\mathbf{j}\right) = -\mathbf{i} + \sqrt{3}\mathbf{j}$$

$$\mathbf{u} \cdot \mathbf{v} = \frac{5\sqrt{2}}{2}\left(1 + \sqrt{3}\right)$$

$\|\mathbf{u}\| = 5$

$\|\mathbf{v}\| = 2$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\left(5\sqrt{2}/2\right)\left(1 + \sqrt{3}\right)}{5(2)} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\theta = \arccos \frac{\sqrt{2} + \sqrt{6}}{4} = 15°$$

**27.** $\mathbf{u} = \langle 10, -5, 15\rangle,\ \mathbf{v} = \langle -2, 1, -3\rangle$

$\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$ is parallel to $\mathbf{v}$ and in the opposite direction.

$\theta = \pi$

**29.** There are many correct answers. For example: $v = \pm\langle 6, -5, 0\rangle$.

**In Exercises 31–39, $\mathbf{u} = \langle 3, -2, 1\rangle$, $\mathbf{v} = \langle 2, -4, -3\rangle$, $\mathbf{w} = \langle -1, 2, 2\rangle$.**

**31.** $$\mathbf{u}\cdot\mathbf{u} = 3(3) + (-2)(-2) + (1)(1) = 14 = \left(\sqrt{14}\right)^2 = \|\mathbf{u}\|^2$$

**33.** $$\text{proj}_{\mathbf{u}}\mathbf{w} = \left(\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|^2}\right)\mathbf{u} = -\frac{5}{14}\langle 3, -2, 1\rangle = \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14}\right\rangle = \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14}\right\rangle$$

**35.** $$\mathbf{n} = \mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ -1 & 2 & 2 \end{vmatrix} = -2\mathbf{i} - \mathbf{j}$$

$$\|\mathbf{n}\| = \sqrt{5}$$

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} - \mathbf{j})$$

**37.** $$V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |\langle 3, -2, 1\rangle\cdot\langle -2, -1, 0\rangle| = |-4| = 4$$

**39.** Area parallelogram $= \|\mathbf{u}\times\mathbf{v}\| = \sqrt{10^2 + 11^2 + (-8)^2}$ (See Exercises 36, 38) $= \sqrt{285}$

**41.** $$\mathbf{F} = c(\cos 20°\mathbf{j} + \sin 20°\mathbf{k})$$

$$\overrightarrow{PQ} = 2\mathbf{k}$$

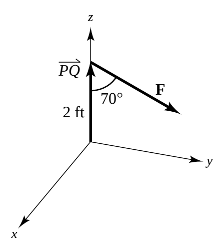
$$\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & c\cos 20° & c\sin 20° \end{vmatrix} = -2c\cos 20°\mathbf{i}$$

$$200 = \|\overrightarrow{PQ}\times\mathbf{F}\| = 2c\cos 20°$$

$$c = \frac{100}{\cos 20°}$$

$$\mathbf{F} = \frac{100}{\cos 20°}(\cos 20°\mathbf{j} + \sin 20°\mathbf{k}) = 100(\mathbf{j} + \tan 20°\mathbf{k})$$

$$\|\mathbf{F}\| = 100\sqrt{1 + \tan^2 20°} = 100\sec 20° \approx 106.4 \text{ lb}$$




**43.** $\mathbf{v} = \mathbf{j}$

(a) $x = 1,\ y = 2 + t,\ z = 3$

(b) None

**45.** $3x - 3y - 7z = -4,\ x - y + 2z = 3$

Solving simultaneously, we have $z = 1$. Substituting $z = 1$ into the second equation we have $y = x - 1$. Substituting for $x$ in this equation we obtain two points on the line of intersection, $(0, -1, 1)$, $(1, 0, 1)$. The direction vector of the line of intersection is $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

(a) $x = t,\ y = -1 + t,\ z = 1$

(b) $x = y + 1,\ z = 1$

**47.** The two lines are parallel as they have the same direction numbers, $-2, 1, 1$. Therefore, a vector parallel to the plane is $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. A point on the first line is $(1, 0, -1)$ and a point on the second line is $(-1, 1, 2)$. The vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ connecting these two points is also parallel to the plane. Therefore, a normal to the plane is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= -2\mathbf{i} - 4\mathbf{j} = -2(\mathbf{i} + 2\mathbf{j}).$$

Equation of the plane: $(x - 1) + 2y = 0$

$$x + 2y = 1$$

**49.** $Q = (1, 0, 2)$

$2x - 3y + 6z = 6$

A point $P$ on the plane is $(3, 0, 0)$.

$\overrightarrow{PQ} = \langle -2, 0, 2 \rangle$

$\mathbf{n} = \langle 2, -3, 6 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{7}$$

**51.** $Q(3, -2, 4)$ point

$P(5, 0, 0)$ point on plane

$\mathbf{n} = \langle 2, -5, 1 \rangle$ normal to plane

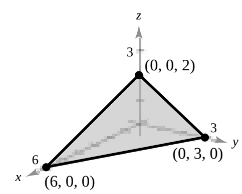
$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{10}{\sqrt{30}} = \frac{\sqrt{30}}{3}$$

**53.** $x + 2y + 3z = 6$

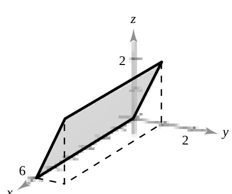
Plane

Intercepts: $(6, 0, 0)$, $(0, 3, 0)$, $(0, 0, 2)$




**55.** $y = \frac{1}{2}z$

Plane with rulings parallel to the $x$-axis



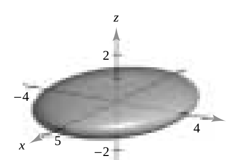

**57.** $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

$xy$-trace: $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$xz$-trace: $\frac{x^2}{16} + z^2 = 1$

$yz$-trace: $\frac{y^2}{9} + z^2 = 1$



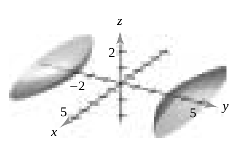

**59.** $\frac{x^2}{16} - \frac{y^2}{9} + z^2 = -1$

Hyperboloid of two sheets

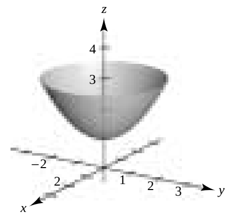
$xy$-trace: $\frac{y^2}{4} - \frac{x^2}{16} = 1$

$xz$-trace: None

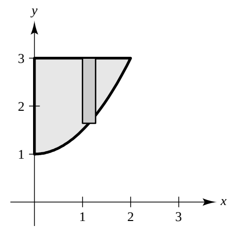
$yz$-trace: $\frac{y^2}{9} - z^2 = 1$



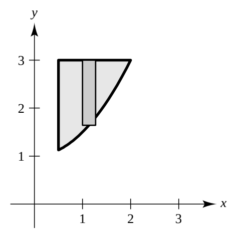
**61.** (a)
$$x^2 + y^2 = [r(z)]^2$$
$$= \left[\sqrt{2(z-1)}\right]^2$$
$$x^2 + y^2 - 2z + 2 = 0$$




(b)
$$V = 2\pi\int_0^2 x\left[3 - \left(\frac{1}{2}x^2 + 1\right)\right]dx$$
$$= 2\pi\int_0^2\left(2x - \frac{1}{2}x^3\right)dx$$
$$= 2\pi\left[x^2 - \frac{x^4}{8}\right]_0^2$$
$$= 4\pi \approx 12.6 \text{ cm}^3$$




(c)
$$V = 2\pi\int_{1/2}^2 x\left[3 - \left(\frac{1}{2}x^2 + 1\right)\right]dx$$
$$= 2\pi\int_{1/2}^2\left(2x - \frac{1}{2}x^3\right)dx$$
$$= 2\pi\left[x^2 - \frac{x^4}{8}\right]_{1/2}^2$$
$$= 4\pi - \frac{31\pi}{64} = \frac{225\pi}{64} \approx 11.04 \text{ cm}^3$$




**63.** $\left(-2\sqrt{2}, 2\sqrt{2}, 2\right)$, rectangular

(a) $r = \sqrt{\left(-2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2} = 4,\ \theta = \arctan(-1) = \dfrac{3\pi}{4},\ z = 2,\ \left(4, \dfrac{3\pi}{4}, 2\right)$, cylindrical

(b) $\rho = \sqrt{\left(-2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2 + (2)^2} = 2\sqrt{5},\ \theta = \dfrac{3\pi}{4},\ \phi = \arccos\dfrac{2}{2\sqrt{5}} = \arccos\dfrac{1}{\sqrt{5}},\ \left(2\sqrt{5}, \dfrac{3\pi}{4}, \arccos\dfrac{\sqrt{5}}{5}\right)$, spherical

**65.** $\left(100, -\dfrac{\pi}{6}, 50\right)$, cylindrical

$$\rho = \sqrt{100^2 + 50^2} = 50\sqrt{5}$$
$$\theta = -\frac{\pi}{6}$$
$$\phi = \arccos\left(\frac{50}{50\sqrt{5}}\right) = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 63.4°$$

$\left(50\sqrt{5}, -\dfrac{\pi}{6}, 63.4°\right)$, spherical

**67.** $\left(25, -\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$, spherical

$$r^2 = \left(25\sin\left(\frac{3\pi}{4}\right)\right)^2 \implies r = 25\frac{\sqrt{2}}{2}$$
$$\theta = -\frac{\pi}{4}$$
$$z = \rho\cos\phi - 25\cos\frac{3\pi}{4} = -25\frac{\sqrt{2}}{2}$$

$\left(25\dfrac{\sqrt{2}}{2}, -\dfrac{\pi}{4}, -\dfrac{25\sqrt{2}}{2}\right)$, cylindrical

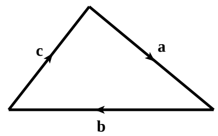
**69.** $x^2 - y^2 = 2z$

(a) Cylindrical: $r^2\cos^2\theta - r^2\sin^2\theta = 2z,\ r^2\cos 2\theta = 2z$

(b) Spherical: $\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta = 2\rho\cos\phi,\ \rho\sin^2\phi\cos 2\theta - 2\cos\phi = 0,\ \rho = 2\sec 2\theta\cos\phi\csc^2\phi$

# Problem Solving for Chapter 10

**1.**
$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$
$$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$$
$$(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$$
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{c}\|$$
$$\|\mathbf{b} \times \mathbf{c}\| = \|\mathbf{b}\|\,\|\mathbf{c}\| \sin A$$
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin C$$



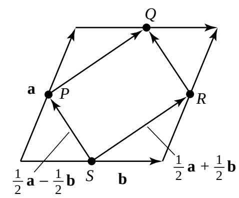

Then,
$$\frac{\sin A}{\|\mathbf{a}\|} = \frac{\|\mathbf{b} \times \mathbf{c}\|}{\|\mathbf{a}\|\,\|\mathbf{b}\|\,\|\mathbf{c}\|} = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\|\,\|\mathbf{b}\|\,\|\mathbf{c}\|} = \frac{\sin C}{\|\mathbf{c}\|}.$$

The other case, $\dfrac{\sin A}{\|\mathbf{a}\|} = \dfrac{\sin B}{\|\mathbf{b}\|}$ is similar.

**3.** Label the figure as indicated.

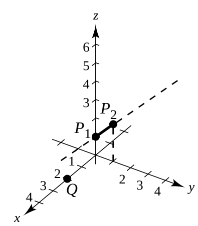
From the figure, you see that
$$\overrightarrow{SP} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \overrightarrow{RQ} \text{ and}$$
$$\overrightarrow{SR} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \overrightarrow{PQ}.$$

Since $\overrightarrow{SP} = \overrightarrow{RQ}$ and $\overrightarrow{SR} = \overrightarrow{PQ}$, $PSRQ$ is a parallelogram.




**5.** (a) $\mathbf{u} = \langle 0, 1, 1 \rangle$ direction vector of line determined by $P_1$ and $P_2$.

$$D = \frac{\|\overrightarrow{P_1Q} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\|\langle 2, 0, -1 \rangle \times \langle 0, 1, 1 \rangle\|}{\sqrt{2}} = \frac{\|\langle 1, -2, 2 \rangle\|}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$




(b) The shortest distance to the line **segment** is $\|P_1Q\| = \|\langle 2, 0, -1 \rangle\| = \sqrt{5}$.

**7.** (a) $V = \pi \displaystyle\int_0^1 \left(\sqrt{2}\right)^2 dz = \left[\pi \frac{z^2}{2}\right]_0^1 = \frac{1}{2}\pi$

Note: $\frac{1}{2}$(base)(altitude) $= \frac{1}{2}\pi(1) = \frac{1}{2}\pi$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z$: (slice at $z = c$)

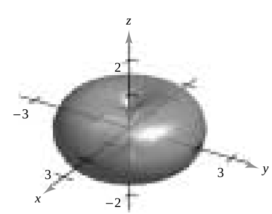
$$\frac{x^2}{\left(\sqrt{ca}\right)^2} + \frac{y^2}{\left(\sqrt{cb}\right)^2} = 1$$

At $z = c$, figure is ellipse of area
$$\pi\left(\sqrt{ca}\right)\left(\sqrt{cb}\right) = \pi abc.$$
$$V = \int_0^k \pi abc \cdot dc = \left[\frac{\pi abc^2}{2}\right]_0^k = \frac{\pi abk^2}{2}$$

(c) $V = \frac{1}{2}(\pi abk)k = \frac{1}{2}$(base)(height)

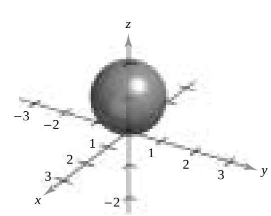
**9.** (a) $\rho = 2 \sin \phi$

Torus




(b) $\rho = 2 \cos \phi$

Sphere




**11.** From Exercise 64, Section 10.4, $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}$.

**13.** (a) $\mathbf{u} = \|\mathbf{u}\|(\cos 0\,\mathbf{i} + \sin 0\,\mathbf{j}) = \|\mathbf{u}\|\mathbf{i}$

Downward force $\mathbf{w} = -\mathbf{j}$

$\mathbf{T} = \|\mathbf{T}\|(\cos(90° + \theta)\mathbf{i} + \sin(90° + \theta)\mathbf{j})$

$= \|\mathbf{T}\|(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$

$\mathbf{0} = \mathbf{u} + \mathbf{w} + \mathbf{T} = \|\mathbf{u}\|\mathbf{i} - \mathbf{j} + \|\mathbf{T}\|(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$

$\|\mathbf{u}\| = \sin\theta\,\|\mathbf{T}\|$

$1 = \cos\theta\,\|\mathbf{T}\|$

If $\theta = 30°$, $\|\mathbf{u}\| = (1/2)\|\mathbf{T}\|$ and $1 = \left(\sqrt{3}/2\right)\|\mathbf{T}\|$

$$\Longrightarrow \|\mathbf{T}\| = \frac{2}{\sqrt{3}} \approx 1.1547 \text{ lb}$$

and

$$\|\mathbf{u}\| = \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) \approx 0.5774 \text{ lb}$$

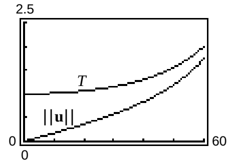
(b) From part (a), $\|\mathbf{u}\| = \tan\theta$ and $\|\mathbf{T}\| = \sec\theta$.

Domain: $0 \le \theta \le 90°$

(c)

| $\theta$ | 0° | 10° | 20° | 30° | 40° | 50° | 60° |
|---|---|---|---|---|---|---|---|
| T | 1 | 1.0154 | 1.0642 | 1.1547 | 1.3054 | 1.5557 | 2 |
| $\|\mathbf{u}\|$ | 0 | 0.1763 | 0.3640 | 0.5774 | 0.8391 | 1.1918 | 1.7321 |

(d)




(e) Both are increasing functions.

(f) $\lim_{\theta \to \pi/2^-} T = \infty$ and $\lim_{\theta \to \pi/2^-} \|\mathbf{u}\| = \infty$.

**15.** Let $\theta = \alpha - \beta$, the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\sin(\alpha - \beta) = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{u}\|}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

For $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$ and $\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$, $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\mathbf{k}.$$

Thus, $\sin(\alpha - \beta) = \|\mathbf{v} \times \mathbf{u}\| = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

**17.** From Theorem 10.13 and Theorem 10.7 (6) we have

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

$$= \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{u} \times \mathbf{v}\|}.$$

**19.** $a_1, b_1, c_1$, and $a_2, b_2, c_2$ are two sets of direction numbers for the same line. The line is parallel to both $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Therefore, $\mathbf{u}$ and $\mathbf{v}$ are parallel, and there exists a scalar $d$ such that $\mathbf{u} = d\mathbf{v}$, $a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} = d(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$, $a_1 = a_2 d$, $b_1 = b_2 d$, $c_1 = c_2 d$.

# CHAPTER 11
# Vector-Valued Functions

# CHAPTER 11
# Vector-Valued Functions

## Section 11.1 Vector-Valued Functions

Solutions to Odd-Numbered Exercises

**1.** $\mathbf{r}(t) = 5t\mathbf{i} - 4t\mathbf{j} - \dfrac{1}{t}\mathbf{k}$

Component functions: $f(t) = 5t$

$g(t) = -4t$

$h(t) = -\dfrac{1}{t}$

Domain: $(-\infty, 0) \cup (0, \infty)$

**3.** $\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

Component functions: $f(t) = \ln t$

$g(t) = -e^t$

$h(t) = -t$

Domain: $(0, \infty)$

**5.** $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = \left(\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}\right) + (\cos t\mathbf{i} + \sin t\mathbf{j}) = 2\cos t\mathbf{i} + \sqrt{t}\mathbf{k}$

Domain: $[0, \infty)$

**7.** $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t\cos t\mathbf{j} + \sin^2 t\mathbf{k}$

Domain: $(-\infty, \infty)$

**9.** $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t - 1)\mathbf{j}$

(a) $\mathbf{r}(1) = \frac{1}{2}\mathbf{i}$

(b) $\mathbf{r}(0) = \mathbf{j}$

(c) $\mathbf{r}(s + 1) = \frac{1}{2}(s + 1)^2\mathbf{i} - (s + 1 - 1)\mathbf{j} = \frac{1}{2}(s + 1)^2\mathbf{i} - s\mathbf{j}$

(d)
$$\begin{aligned} \mathbf{r}(2 + \Delta t) - \mathbf{r}(2) &= \tfrac{1}{2}(2 + \Delta t)^2\mathbf{i} - (2 + \Delta t - 1)\mathbf{j} - (2\mathbf{i} - \mathbf{j}) \\ &= \left(2 + 2\Delta t + \tfrac{1}{2}(\Delta t)^2\right)\mathbf{i} - (1 + \Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j} \\ &= \left(2\Delta t + \tfrac{1}{2}(\Delta t)^2\right)\mathbf{i} - (\Delta t)\mathbf{j} \end{aligned}$$

**11.** $\mathbf{r}(t) = \ln t\mathbf{i} + \dfrac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2) = \ln 2\mathbf{i} + \dfrac{1}{2}\mathbf{j} + 6\mathbf{k}$

(b) $\mathbf{r}(-3)$ is not defined. ($\ln(-3)$ does not exist.)

(c) $\mathbf{r}(t - 4) = \ln(t - 4)\mathbf{i} + \dfrac{1}{t - 4}\mathbf{j} + 3(t - 4)\mathbf{k}$

(d)
$$\begin{aligned} \mathbf{r}(1 + \Delta t) - \mathbf{r}(1) &= \ln(1 + \Delta t)\mathbf{i} + \frac{1}{1 + \Delta t}\mathbf{j} + 3(1 + \Delta t)\mathbf{k} - (0\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= \ln(1 + \Delta t)\mathbf{i} + \left(\frac{1}{1 + \Delta t} - 1\right)\mathbf{j} + (3\Delta t)\mathbf{k} \end{aligned}$$

**13.** $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{r}(t)\| = \sqrt{(\sin 3t)^2 + (\cos 3t)^2 + t^2} = \sqrt{1 + t^2}$

**15.** $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$

$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2$, a scalar.

The dot product is a scalar-valued function.

**17.** $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, -2 \le t \le 2$

$x = t, y = 2t, z = t^2$

Thus, $z = x^2$. Matches (b)

**19.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \le t \le 2$

$x = t, y = t^2, z = e^{0.75t}$

Thus, $y = x^2$. Matches (d)

**21.** (a) View from the negative $x$-axis: $(-20, 0, 0)$

(b) View from above the first octant: $(10, 20, 10)$

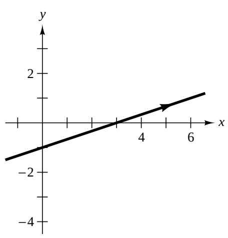
(c) View from the $z$-axis: $(0, 0, 20)$

(d) View from the positive $x$-axis: $(20, 0, 0)$

**23.** $x = 3t$

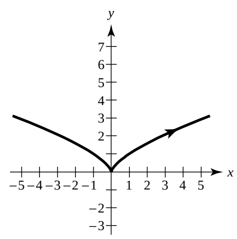
$y = t - 1$

$y = \frac{x}{3} - 1$



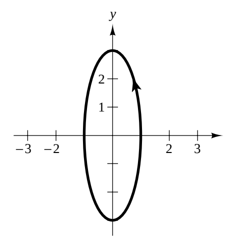
**25.** $x = t^3, y = t^2$

$y = x^{2/3}$



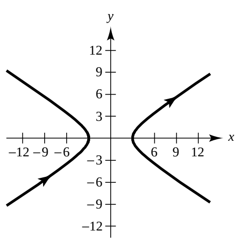
**27.** $x = \cos \theta, y = 3 \sin \theta$

$x^2 + \frac{y^2}{9} = 1$ Ellipse



**29.** $x = 3 \sec \theta, y = 2 \tan \theta$

$\frac{x^2}{9} = \frac{y^2}{4} + 1$ Hyperbola



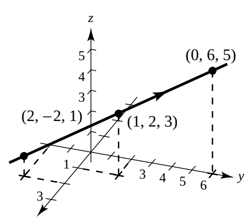
**31.** $x = -t + 1$

$y = 4t + 2$

$z = 2t + 3$

Line passing through the points:

$(0, 6, 5)$, $(1, 2, 3)$

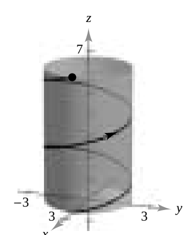



**33.** $x = 2 \cos t,\ y = 2 \sin t,\ z = t$

$\frac{x^2}{4} + \frac{y^2}{4} = 1$

$z = t$

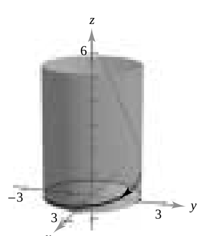
Circular helix



**35.** $x = 2 \sin t,\ y = 2 \cos t,\ z = e^{-t}$

$x^2 + y^2 = 4$

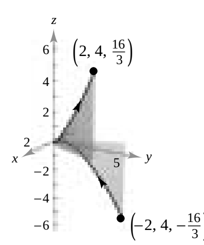
$z = e^{-t}$



**37.** $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

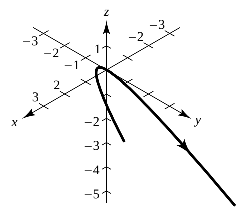
$y = x^2,\ z = \frac{2}{3}x^3$

| | | | | | |
|---|---|---|---|---|---|
| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $y$ | $4$ | $1$ | $0$ | $1$ | $4$ |
| $z$ | $-\frac{16}{3}$ | $-\frac{2}{3}$ | $0$ | $\frac{2}{3}$ | $\frac{16}{3}$ |



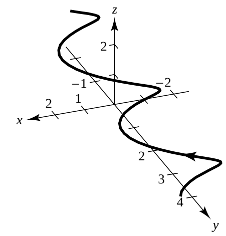
**39.** $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola

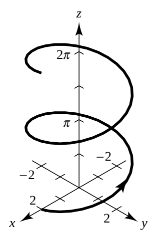


**41.** $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

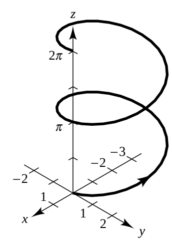
Helix



**43.**

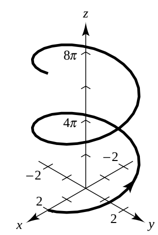


(a)



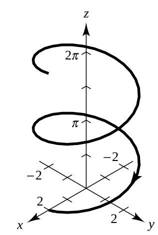
The helix is translated 2 units back on the $x$-axis.

(b)



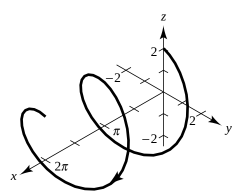
The height of the helix increases at a faster rate.

(c)



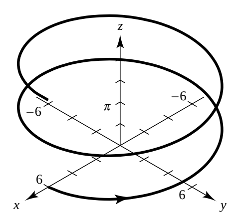
The orientation of the helix is reversed.

(d)



The axis of the helix is the $x$-axis.

(e)



The radius of the helix is increased from 2 to 6.

**45.** $y = 4 - x$

Let $x = t$, then $y = 4 - t$.

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t)\mathbf{j}$$

**47.** $y = (x - 2)^2$

Let $x = t$, then $y = (t - 2)^2$.

$$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\,\mathbf{j}$$

**49.** $x^2 + y^2 = 25$

Let $x = 5\cos t$, then $y = 5\sin t$.

$$\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}$$

**51.** $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Let $x = 4\sec t$, $y = 2\tan t$.

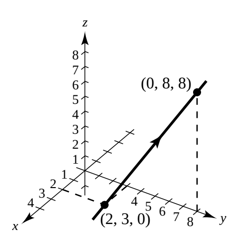
$$\mathbf{r}(t) = 4\sec t\mathbf{i} + 2\tan t\mathbf{j}$$

**53.** The parametric equations for the line are

$$x = 2 - 2t,\ y = 3 + 5t,\ z = 8t.$$

One possible answer is

$$\mathbf{r}(t) = (2 - 2t)\mathbf{i} + (3 + 5t)\mathbf{j} + 8t\mathbf{k}.$$




**55.** $\mathbf{r}_1(t) = t\mathbf{i}$, $\quad 0 \le t \le 4 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(4) = 4\mathbf{i})$

$\mathbf{r}_2(t) = (4 - 4t)\mathbf{i} + 6t\mathbf{j}$, $\quad 0 \le t \le 1 \quad (\mathbf{r}_2(0) = 4\mathbf{i}, \mathbf{r}_2(1) = 6\mathbf{j})$

$\mathbf{r}_3(t) = (6 - t)\mathbf{j}$, $\quad 0 \le t \le 6 \quad (\mathbf{r}_3(0) = 6\mathbf{j}, \mathbf{r}_3(6) = \mathbf{0})$

(Other answers possible)

**57.** $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2 \ \ (y = x^2)$

$\mathbf{r}_2(t) = (2 - t)\mathbf{i}, \quad 0 \le t \le 2$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \le t \le 4$

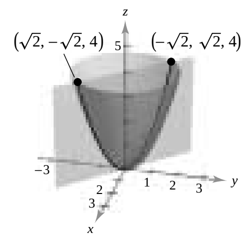
(Other answers possible)

**59.** $z = x^2 + y^2,\ x + y = 0$

Let $x = t$, then $y = -x = -t$ and $z = x^2 + y^2 = 2t^2$. Therefore,

$$x = t,\ y = -t,\ z = 2t^2.$$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

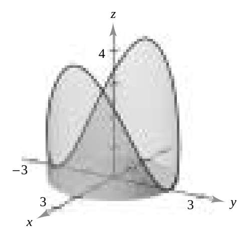



**61.** $x^2 + y^2 = 4,\ z = x^2$

$x = 2\sin t,\ y = 2\cos t$

$z = x^2 = 4\sin^2 t$

| $t$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $x$ | $0$ | $1$ | $\sqrt{2}$ | $2$ | $\sqrt{2}$ | $0$ |
| $y$ | $2$ | $\sqrt{3}$ | $\sqrt{2}$ | $0$ | $-\sqrt{2}$ | $-2$ |
| $z$ | $0$ | $1$ | $2$ | $4$ | $2$ | $0$ |




$\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 4\sin^2 t\,\mathbf{k}$

**63.** $x^2 + y^2 + z^2 = 4,\ x + z = 2$

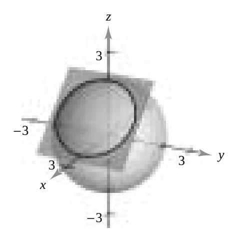
Let $x = 1 + \sin t$, then $z = 2 - x = 1 - \sin t$ and $x^2 + y^2 + z^2 = 4$.

$$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2\sin^2 t + y^2 = 4$$

$y^2 = 2\cos^2 t, \quad y = \pm\sqrt{2}\cos t$

$x = 1 + \sin t,\ y = \pm\sqrt{2}\cos t$

$z = 1 - \sin t$




| $t$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{6}$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $x$ | $0$ | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ | $2$ |
| $y$ | $0$ | $\pm\frac{\sqrt{6}}{2}$ | $\pm\sqrt{2}$ | $\pm\frac{\sqrt{6}}{2}$ | $0$ |
| $z$ | $2$ | $\frac{3}{2}$ | $1$ | $\frac{1}{2}$ | $0$ |

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2}\cos t\mathbf{j} + (1 + \sin t)\mathbf{k}$ and

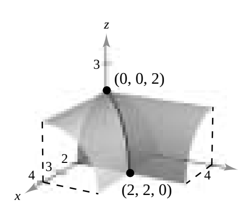
$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

**65.** $x^2 + z^2 = 4,\ y^2 + z^2 = 4$

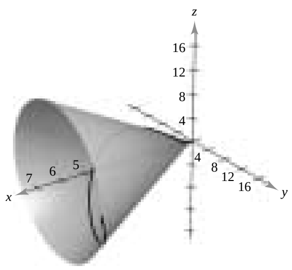
Subtracting, we have $x^2 - y^2 = 0$ or $y = \pm x$.

Therefore, in the first octant, if we let $x = t$, then $x = t,\ y = t,\ z = \sqrt{4 - t^2}$.

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$



**67.** $y^2 + z^2 = (2t\cos t)^2 + (2t\sin t)^2 = 4t^2 = 4x^2$




**69.** $\lim_{t\to 2}\left[t\mathbf{i} + \frac{t^2-4}{t^2-2t}\mathbf{j} + \frac{1}{t}\mathbf{k}\right] = 2\mathbf{i} + 2\mathbf{j} + \frac{1}{2}\mathbf{k}$

since

$$\lim_{t\to 2}\frac{t^2-4}{t^2-2t} = \lim_{t\to 2}\frac{2t}{2t-2} = 2. \quad \text{(L'Hôpital's Rule)}$$

**71.** $\lim_{t\to 0}\left[t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k}\right] = \mathbf{0}$

since

$$\lim_{t\to 0}\frac{1-\cos t}{t} = \lim_{t\to 0}\frac{\sin t}{1} = 0. \quad \text{(L'Hôpital's Rule)}$$

**73.** $\lim_{t\to 0}\left[\frac{1}{t}\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}\right]$

does not exist since $\lim_{t\to 0}\frac{1}{t}$ does not exist.

**75.** $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

Continuous on $(-\infty, 0),\ (0, \infty)$

**77.** $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

Continuous on $[-1, 1]$

**79.** $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

Discontinuous at $t = \frac{\pi}{2} + n\pi$

Continuous on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

**81.** See the definition on page 786.

**83.** $\mathbf{r}(t) = t^2\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(a) $\mathbf{s}(t) = \mathbf{r}(t) + 2\mathbf{k} = t^2\mathbf{i} + (t-3)\mathbf{j} + (t+3)\mathbf{k}$

(b) $\mathbf{s}(t) = \mathbf{r}(t) - 2\mathbf{i} = (t^2-2)\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(c) $\mathbf{s}(t) = \mathbf{r}(t) + 5\mathbf{j} = t^2\mathbf{i} + (t+2)\mathbf{j} + t\mathbf{k}$

**85.** Let $\mathbf{r}(t) = x_1(t) + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}
\lim_{t\to c}[\mathbf{r}(t)\times\mathbf{u}(t)] &= \lim_{t\to c}\{[y_1(t)z_2(t) - y_2(t)z_1(t)]\mathbf{i} - [x_1(t)z_2(t) - x_2(t)z_1(t)]\mathbf{j} + [x_1(t)y_2(t) - x_2(t)y_1(t)]\mathbf{k}\}\\
&= \left[\lim_{t\to c} y_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} y_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{i} - \left[\lim_{t\to c} x_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{j}\\
&\qquad + \left[\lim_{t\to c} x_1(t)\lim_{t\to c} y_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} y_1(t)\right]\mathbf{k}\\
&= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\times\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\
&= \lim_{t\to c}\mathbf{r}(t)\times\lim_{t\to c}\mathbf{u}(t)
\end{aligned}$$

**87.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Since $\mathbf{r}$ is continuous at $t = c$, then $\lim_{t\to c}\mathbf{r}(t) = \mathbf{r}(c)$.

$$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \implies x(c),\ y(c),\ z(c)$$

are defined at $c$.

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t\to c}\|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

Therefore, $\|\mathbf{r}\|$ is continuous at $c$.

**89.** True

## Section 11.2 Differentiation and Integration of Vector-Valued Functions

**1.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j},\ t_0 = 2$

$x(t) = t^2,\ y(t) = t$

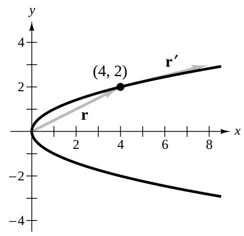
$x = y^2$

$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.




**3.** $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ t_0 = \dfrac{\pi}{2}$

$x(t) = \cos t,\ y(t) = \sin t$

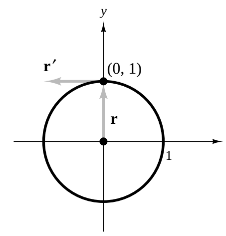
$x^2 + y^2 = 1$

$\mathbf{r}\left(\dfrac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

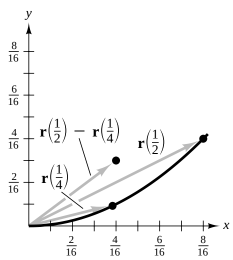
$\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -\mathbf{i}$

$\mathbf{r}'(t_0)$ is tangent to the curve.




**5.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

(a)




(b) $\mathbf{r}\left(\dfrac{1}{4}\right) = \dfrac{1}{4}\mathbf{i} + \dfrac{1}{16}\mathbf{j}$

$\mathbf{r}\left(\dfrac{1}{2}\right) = \dfrac{1}{2}\mathbf{i} + \dfrac{1}{4}\mathbf{j}$

$\mathbf{r}\left(\dfrac{1}{2}\right) - \mathbf{r}\left(\dfrac{1}{4}\right) = \dfrac{1}{4}\mathbf{i} + \dfrac{3}{16}\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}'\left(\dfrac{1}{4}\right) = \mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{(1/2) - (1/4)} = \frac{(1/4)\mathbf{i} + (3/16)\mathbf{j}}{1/4} = \mathbf{i} + \frac{3}{4}\mathbf{j}$$

This vector approximates $\mathbf{r}'\left(\frac{1}{4}\right)$.

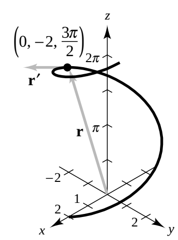
**7.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ t_0 = \dfrac{3\pi}{2}$

$x^2 + y^2 = 4,\ z = t$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \dfrac{3\pi}{2}\mathbf{k}$

$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$




**9.** $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$

**11.** $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'(t) = -3a\cos^2 t\sin t\mathbf{i} + 3a\sin^2 t\cos t\mathbf{j}$

**13.** $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i}$

**15.** $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$

$\mathbf{r}'(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$

**17.** $\mathbf{r}(t) = t^3\mathbf{i} + \dfrac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

**19.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-4\sin t)(-4\cos t) + 4\cos t(-4\sin t)$

$= 0$

**21.** $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{i} + t\mathbf{k}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = t(1) - 1(0) + \frac{1}{2}t^2(t) = t + \frac{t^3}{2}$

**23.** $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

(a) $\mathbf{r}'(t) = \langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 1\rangle$

$= \langle t\cos t, t\sin t, 1\rangle$

$\mathbf{r}''(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (t\cos t)(\cos t - t\sin t) + (t\sin t)(\sin t + t\cos t) = t$

**25.**

$$\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k},\ t_0 = -\frac{1}{4}$$

$$\mathbf{r}'(t) = -\pi\sin(\pi t)\mathbf{i} + \pi\cos(\pi t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{r}'\left(-\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{2}\mathbf{i} + \frac{\sqrt{2}\pi}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

$$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{\left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\pi^2 + \frac{1}{4}} = \frac{\sqrt{4\pi^2 + 1}}{2}$$

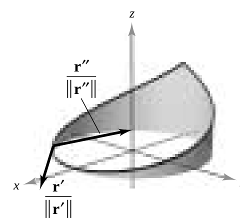
$$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2 + 1}}\left(\sqrt{2}\pi\mathbf{i} + \sqrt{2}\pi\mathbf{j} - \mathbf{k}\right)$$

$$\mathbf{r}''(t) = -\pi^2\cos(\pi t)\mathbf{i} - \pi^2\sin(\pi t)\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}''\left(-\frac{1}{4}\right) = -\frac{\sqrt{2}\pi^2}{2}\mathbf{i} + \frac{\sqrt{2}\pi^2}{2}\mathbf{j} + 2\mathbf{k}$$

$$\left\|\mathbf{r}''\left(-\frac{1}{4}\right)\right\| = \sqrt{\left(-\frac{\sqrt{2}\pi^2}{2}\right)^2 + \left(\frac{\sqrt{2}\pi^2}{2}\right)^2 + (2)^2} = \sqrt{\pi^4 + 4}$$

$$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{2\sqrt{\pi^4 + 4}}\left(-\sqrt{2}\pi^2\mathbf{i} + \sqrt{2}\pi^2\mathbf{j} + 4\mathbf{k}\right)$$




**27.** $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{0}$

Smooth on $(-\infty, 0)$, $(0, \infty)$

**29.** $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

$\mathbf{r}'(\theta) = -6\cos^2\theta\sin\theta\mathbf{i} + 9\sin^2\theta\cos\theta\mathbf{j}$

$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$

Smooth on $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)$, $n$ any integer.

**31.** $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 - 2\cos\theta)\mathbf{i} + (1 + 2\sin\theta)\mathbf{j}$

$\mathbf{r}'(\theta) \neq \mathbf{0}$ for any value of $\theta$

Smooth on $(-\infty, \infty)$

**33.** $\mathbf{r}(t) = (t - 1)\mathbf{i} + \frac{1}{t}\mathbf{j} - t^2\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j} - 2t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq 0$: $(-\infty, 0), \cup (0, \infty)$

**35.** $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq \dfrac{\pi}{2} + n\pi = \dfrac{2n+1}{2}\pi$.

Smooth on intervals of form $\left(-\dfrac{\pi}{2} + n\pi, \dfrac{\pi}{2} + n\pi\right)$

**37.** $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k},\ \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b) $\mathbf{r}''(t) = 2\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 4t^2 + 3t^3 + t^5$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 8t + 9t^2 + 5t^4$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = -t\mathbf{i} + (9t - t^2)\mathbf{j} + (3t^2 - t^3)\mathbf{k}$

$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = -\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$

(e) $\mathbf{r}(t) \times \mathbf{u}(t) = 2t^4\mathbf{i} - (t^4 - 4t^3)\mathbf{j} + (t^3 - 12t^2)\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$

(f) $\|\mathbf{r}(t)\| = \sqrt{10t^2 + t^4} = t\sqrt{10 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{10 + 2t^2}{\sqrt{10 + t^2}}$

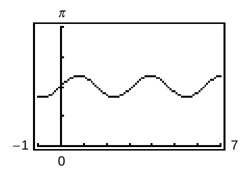
**39.** $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 9\sin t\cos t - 16\cos t\sin t = -7\sin t\cos t$

$$\cos\theta = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|\,\|\mathbf{r}'(t)\|} = \frac{-7\sin t\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}\sqrt{9\cos^2 t + 16\sin^2 t}}$$

$$\theta = \arccos\left[\frac{-7\sin t\cos t}{\sqrt{(9\sin^2 t + 16\cos^2 t)(9\cos^2 t + 16\sin^2 t)}}\right]$$




$\theta = 1.855$ maximum at $t = 3.927\left(\dfrac{5\pi}{4}\right)$ and $t = 0.785\left(\dfrac{\pi}{4}\right)$.

$\theta = 1.287$ minimum at $t = 2.356\left(\dfrac{3\pi}{4}\right)$ and $t = 5.498\left(\dfrac{7\pi}{4}\right)$.

$\theta = \dfrac{\pi}{2}(1.571)$ for $t = n\dfrac{\pi}{2}$, $n = 0, 1, 2, 3, \ldots$

**41.**
$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{[3(t + \Delta t) + 2]\mathbf{i} + [1 - (t + \Delta t)^2]\mathbf{j} - (3t + 2)\mathbf{i} - (1 - t^2)\mathbf{j}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{(3\Delta t)\mathbf{i} - (2t(\Delta t) + (\Delta t)^2)\mathbf{j}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} 3\mathbf{i} - (2t + \Delta t)\mathbf{j} = 3\mathbf{i} - 2t\mathbf{j}$$

**43.** $\displaystyle\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

**45.** $\displaystyle\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right) dt = \ln t\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

**47.** $\displaystyle\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}\right] dt = (t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

**49.** $\displaystyle\int \left[\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right] dt = \tan t\mathbf{i} + \arctan t\mathbf{j} + \mathbf{C}$

**51.** $\displaystyle\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt = \Big[4t^2\mathbf{i}\Big]_0^1 + \left[\frac{t^2}{2}\mathbf{j}\right]_0^1 - \Big[t\mathbf{k}\Big]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

**53.** $\displaystyle\int_0^{\pi/2} [(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}]\,dt = \Big[a\sin t\mathbf{i}\Big]_0^{\pi/2} - \Big[a\cos t\mathbf{j}\Big]_0^{\pi/2} + \Big[t\mathbf{k}\Big]_0^{\pi/2} = a\mathbf{i} + a\mathbf{j} + \frac{\pi}{2}\mathbf{k}$

**55.** $\displaystyle\mathbf{r}(t) = \int (4e^{2t}\mathbf{i} + 3e^t\mathbf{j})\,dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}$

$\mathbf{r}(t) = 2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$

**57.** $\displaystyle\mathbf{r}'(t) = \int -32\mathbf{j}\,dt = -32t\mathbf{j} + \mathbf{C}_1$

$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$

$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}$

$$\begin{aligned}\mathbf{r}(t) &= \int \left[600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}\right] dt\\ &= 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j} + \mathbf{C}\end{aligned}$$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j}$

**59.** $\displaystyle\mathbf{r}(t) = \int (te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k})\,dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\displaystyle\mathbf{r}(t) = \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t+1)\mathbf{k} = \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t+1)\mathbf{k}$

**61.** See "Definition of the Derivative of a Vector-Valued Function" and Figure 11.8 on page 794.

**63.** At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the $x$, $y$, and $z$ directions simultaneously.

**65.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$ and

$$\begin{aligned}D_t[c\mathbf{r}(t)] &= cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k}\\ &= c[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = c\mathbf{r}'(t).\end{aligned}$$

**67.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $f(t)\mathbf{r}(t) = f(t)x(t)\mathbf{i} + f(t)y(t)\mathbf{j} + f(t)z(t)\mathbf{k}$.

$$\begin{aligned}D_t[f(t)\mathbf{r}(t)] &= [f(t)x'(t) + f'(t)x(t)]\mathbf{i} + [f(t)y'(t) + f'(t)y(t)]\mathbf{j} + [f(t)z'(t) + f'(t)z(t)]\mathbf{k}\\ &= f(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] + f'(t)[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]\\ &= f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)\end{aligned}$$

**69.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(f(t)) = x(f(t))\mathbf{i} + y(f(t))\mathbf{j} + z(f(t))\mathbf{k}$ and

$$\begin{aligned}D_t[\mathbf{r}(f(t))] &= x'(f(t))f'(t)\mathbf{i} + y'(f(t))f'(t)\mathbf{j} + z'(f(t))f'(t)\mathbf{k} \qquad \text{(Chain Rule)}\\ &= f'(t)[x'(f(t))i + y'(f(t))\mathbf{j} + z'(f(t))\mathbf{k}] = f'(t)\mathbf{r}'(f(t)).\end{aligned}$$

**71.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] = x_1(t)[y_2(t)z_3(t) - z_2(t)y_3(t)] - y_1(t)[x_2(t)z_3(t) - z_2(t)x_3(t)] + z_1(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]$$

$$\begin{aligned}
D_t[\mathbf{r}(t) \cdot (\mathbf{u}(t) \times \mathbf{v}(t))] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t) - \\
&\quad x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1''(t)x_2(t)z_3(t) + \\
&\quad y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t) + \\
&\quad z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t) \\
&= \{x_1'(t)[y_2(t)z_3(t) - y_3(t)z_2(t)] + y_1'(t)[-x_2(t)z_3(t) + z_2(t)x_3(t)] + z_1'(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]\} + \\
&\quad \{x_1(t)[y_2'(t)z_3(t) - y_3(t)z_2'(t)] + y_1(t)[-x_2'(t)z_3(t) + z_2'(t)x_3(t)] + z_1(t)[x_2'(t)y_3(t) - y_2'(t)x_3(t)]\} + \\
&\quad \{x_1(t)[y_2(t)z_3'(t) - y_3'(t)z_2(t)] + y_1(t)[-x_2(t)z_3'(t) + z_2(t)x_3'(t)] + z_1(t)[x_2(t)y_3'(t) - y_2(t)x_3'(t)]\} \\
&= \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]
\end{aligned}$$

**73.** False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$$\begin{aligned}
\|\mathbf{r}(t)\| &= \sqrt{2} \\
\frac{d}{dt}[\|\mathbf{r}(t)\|] &= 0 \\
\mathbf{r}'(t) &= -\sin t\mathbf{i} + \cos t\mathbf{j} \\
\|\mathbf{r}'(t)\| &= 1
\end{aligned}$$

## Section 11.3 Velocity and Acceleration

**1.** $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$

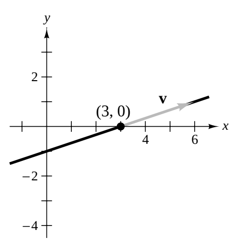
$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 3t,\ y = t - 1,\ y = \dfrac{x}{3} - 1$

At $(3, 0)$, $t = 1$.

$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}$, $\mathbf{a}(1) = \mathbf{0}$




**3.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

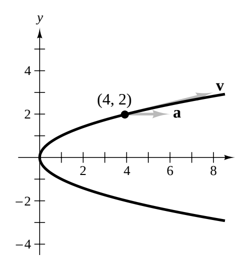
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$

$x = t^2,\ y = t,\ x = y^2$

At $(4, 2)$, $t = 2$.

$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{a}(2) = 2\mathbf{i}$




**5.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

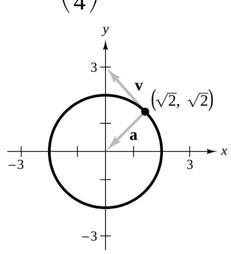
$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$

$x = 2\cos t,\ y = 2\sin t,\ x^2 + y^2 = 4$

At $\left(\sqrt{2}, \sqrt{2}\right)$, $t = \dfrac{\pi}{4}$.

$$\mathbf{v}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$$




**7.** $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - \cos t, \sin t\rangle$

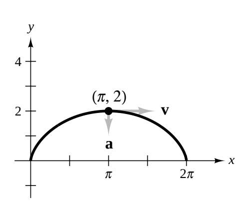
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle \sin t, \cos t\rangle$

$x = t - \sin t,\ y = 1 - \cos t$ (cycloid)

At $(\pi, 2)$, $t = \pi$.

$\mathbf{v}(\pi) = \langle 2, 0\rangle = 2\mathbf{i}$

$\mathbf{a}(\pi) = \langle 0, -1\rangle = -\mathbf{j}$



**9.** $\mathbf{r}(t) = t\mathbf{i} + (2t - 5)\mathbf{j} + 3t\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\mathbf{a}(t) = \mathbf{0}$

**11.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$s(t) = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

**13.** $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\,\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \dfrac{t}{\sqrt{9 - t^2}}\mathbf{k}$

$s(t) = \sqrt{1 + 1 + \dfrac{t^2}{9 - t^2}} = \sqrt{\dfrac{18 - t^2}{9 - t^2}}$

$\mathbf{a}(t) = -\dfrac{9}{(9 - t^2)^{3/2}}\mathbf{k}$

**15.** $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$

$\mathbf{v}(t) = \langle 4, -3\sin t, 3\cos t\rangle = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$s(t) = \sqrt{16 + 9\sin^2 t + 9\cos^2 t} = 5$

$\mathbf{a}(t) = \langle 0, -3\cos t, -3\sin t\rangle = -3\cos t\mathbf{j} - 3\sin t\mathbf{k}$

**17.** (a) $\mathbf{r}(t) = \left\langle t, -t^2, \dfrac{t^3}{4}\right\rangle, t_0 = 1$

$\mathbf{r}'(t) = \left\langle 1, -2t, \dfrac{3t^2}{4}\right\rangle$

$\mathbf{r}'(1) = \left\langle 1, -2, \dfrac{3}{4}\right\rangle$

$x = 1 + t,\ y = -1 - 2t,\ z = \dfrac{1}{4} + \dfrac{3}{4}t$

(b) $\mathbf{r}(1 + 0.1) \approx \left\langle 1 + 0.1, -1 - 2(0.1), \dfrac{1}{4} + \dfrac{3}{4}(0.1)\right\rangle$

$= \langle 1.100, -1.200, 0.325\rangle$

**19.** $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \displaystyle\int(\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(t) = \displaystyle\int(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\,dt = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$

$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

**21.** $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \mathbf{v}(1) = 5\mathbf{j}, \mathbf{r}(1) = \mathbf{0}$

$\mathbf{v}(t) = \displaystyle\int(t\mathbf{j} + t\mathbf{k})\,dt = \dfrac{t^2}{2}\mathbf{j} + \dfrac{t^2}{2}\mathbf{k} + \mathbf{C}$

$\mathbf{v}(1) = \dfrac{1}{2}\mathbf{j} + \dfrac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \implies \mathbf{C} = \dfrac{9}{2}\mathbf{j} - \dfrac{1}{2}\mathbf{k}$

$\mathbf{v}(t) = \left(\dfrac{t^2}{2} + \dfrac{9}{2}\right)\mathbf{j} + \left(\dfrac{t^2}{2} - \dfrac{1}{2}\right)\mathbf{k}$

$\mathbf{r}(t) = \displaystyle\int\left[\left(\dfrac{t^2}{2} + \dfrac{9}{2}\right)\mathbf{j} + \left(\dfrac{t^2}{2} - \dfrac{1}{2}\right)\mathbf{k}\right]dt$

$= \left(\dfrac{t^3}{6} + \dfrac{9}{2}t\right)\mathbf{j} + \left(\dfrac{t^3}{6} - \dfrac{1}{2}t\right)\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \dfrac{14}{3}\mathbf{j} - \dfrac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \implies \mathbf{C} = -\dfrac{14}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$

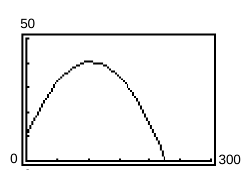
$\mathbf{r}(t) = \left(\dfrac{t^3}{6} + \dfrac{9}{2}t - \dfrac{14}{3}\right)\mathbf{j} + \left(\dfrac{t^3}{6} - \dfrac{1}{2}t + \dfrac{1}{3}\right)\mathbf{k}$

$\mathbf{r}(2) = \dfrac{17}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

**23.** The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

**25.** $\mathbf{r}(t) = (88\cos 30°)t\mathbf{i} + [10 + (88\sin 30°)t - 16t^2]\mathbf{j}$

$= 44\sqrt{3}\,t\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$



**27.** $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{v_0}{\sqrt{2}}t\mathbf{i} + \left(3 + \frac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j}$

$\frac{v_0}{\sqrt{2}}t = 300$ when $3 + \frac{v_0}{\sqrt{2}}t - 16t^2 = 3.$

$$t = \frac{300\sqrt{2}}{v_0},\ \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0,\ 300 - \frac{300^2(32)}{{v_0}^2} = 0$$

$${v_0}^2 = 300(32),\ v_0 = \sqrt{9600} = 40\sqrt{6},\ v_0 = 40\sqrt{6} \approx 97.98 \text{ ft/sec}$$

The maximum height is reached when the derivative of the vertical component is zero.

$$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$$

$$y'(t) = 40\sqrt{3} - 32t = 0$$

$$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$$

Maximum height: $y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78$ feet

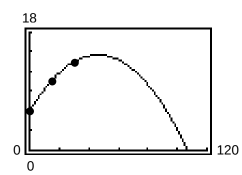
**29.** $x(t) = t(v_0 \cos\theta)$ or $t = \frac{x}{v_0 \cos\theta}$

$$y(t) = t(v_0 \sin\theta) - 16t^2 + h$$

$$y = \frac{x}{v_0 \cos\theta}(v_0 \sin\theta) - 16\left(\frac{x^2}{{v_0}^2 \cos^2\theta}\right) + h = (\tan\theta)x - \left(\frac{16}{{v_0}^2}\sec^2\theta\right)x^2 + h$$

**31.** $\mathbf{r}(t) = t\mathbf{i} + (-0.004t^2 + 0.3667t + 6)\mathbf{j}$, or

(a) $y = -0.004x^2 + 0.3667x + 6$

(b) 


(c) $y' = -0.008x + 0.3667 = 0 \implies x = 45.8375$ and

$y(45.8375) \approx 14.4$ feet.

(d) From Exercise 29,

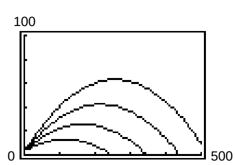
$\tan\theta = 0.3667 \implies \theta \approx 20.14°$

$\frac{16\sec^2\theta}{{v_0}^2} = 0.004 \implies {v_0}^2 = \frac{16\sec^2\theta}{0.004} = \frac{4000}{\cos^2\theta}$

$\implies v_0 \approx 67.4$ ft/sec.

**33.** $100 \text{ mph} = \left(100\,\frac{\text{miles}}{\text{hr}}\right)\left(5280\,\frac{\text{feet}}{\text{mile}}\right)/(3600 \text{ sec/hour}) = \frac{440}{3}$ ft/sec

(a) $\mathbf{r}(t) = \left(\frac{440}{3}\cos\theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3}\sin\theta_0\right)t - 16t^2\right]\mathbf{j}$

(b) 


Graphing these curves together with $y = 10$ shows that $\theta_0 = 20°$.

**—CONTINUED—**

**33. —CONTINUED—**

(c) We want

$$x(t) = \left(\frac{440}{3}\cos\theta\right)t \geq 400 \quad \text{and} \quad y(t) = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 \geq 10.$$

From $x(t)$, the minimum angle occurs when $t = 30/(11\cos\theta)$. Substituting this for $t$ in $y(t)$ yields:

$$3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{30}{11\cos\theta}\right) - 16\left(\frac{30}{11\cos\theta}\right)^2 = 10$$

$$400\tan\theta - \frac{14{,}400}{121}\sec^2\theta = 7$$

$$\frac{14{,}400}{121}(1 + \tan^2\theta) - 400\tan\theta + 7 = 0$$

$$14{,}400\tan^2\theta - 48{,}400\tan\theta + 15{,}247 = 0$$

$$\tan\theta = \frac{48{,}400 \pm \sqrt{48{,}400^2 - 4(14{,}400)(15{,}247)}}{2(14{,}400)}$$

$$\theta = \tan^{-1}\left(\frac{48{,}400 - \sqrt{1{,}464{,}332{,}800}}{28{,}800}\right) \approx 19.38^\circ$$

**35.** $\mathbf{r}(t) = (v\cos\theta)t\,\mathbf{i} + [(v\sin\theta)t - 16t^2]\mathbf{j}$

(a) We want to find the minimum initial speed $v$ as a function of the angle $\theta$. Since the bale must be thrown to the position (16, 8), we have

$$16 = (v\cos\theta)t$$

$$8 = (v\sin\theta)t - 16t^2.$$

$t = 16/(v\cos\theta)$ from the first equation. Substituting into the second equation and solving for $v$, we obtain:

$$8 = (v\sin\theta)\left(\frac{16}{v\cos\theta}\right) - 16\left(\frac{16}{v\cos\theta}\right)^2$$

$$1 = 2\frac{\sin\theta}{\cos\theta} - 512\left(\frac{1}{v^2\cos^2\theta}\right)$$

$$512\frac{1}{v^2\cos^2\theta} = 2\frac{\sin\theta}{\cos\theta} - 1$$

$$\frac{1}{v^2} = \left(2\frac{\sin\theta}{\cos\theta} - 1\right)\frac{\cos^2\theta}{512} = \frac{2\sin\theta\cos\theta - \cos^2\theta}{512}$$

$$v^2 = \frac{512}{2\sin\theta\cos\theta - \cos^2\theta}$$

We minimize $f(\theta) = \dfrac{512}{2\sin\theta\cos\theta - \cos^2\theta}$.

$$f'(\theta) = -512\frac{2\cos^2\theta - 2\sin^2\theta + 2\sin\theta\cos\theta}{(2\sin\theta\cos\theta - \cos^2\theta)^2}$$

$$f'(\theta) = 0 \implies 2\cos(2\theta) + \sin(2\theta) = 0$$

$$\tan(2\theta) = -2$$

$$\theta \approx 1.01722 \approx 58.28^\circ$$

Substituting into the equation for $v$, $v \approx 28.78$ feet per second.

(b) If $\theta = 45^\circ$,

$$16 = (v\cos\theta)t = v\frac{\sqrt{2}}{2}t$$

$$8 = (v\sin\theta)t - 16t^2 = v\frac{\sqrt{2}}{2}t - 16t^2$$

From part (a), $v^2 = \dfrac{512}{2\left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right) - \left(\sqrt{2}/2\right)^2} = \dfrac{512}{1/2} = 1024 \implies v = 32$ ft/sec.

**37.** $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - 16t^2]\mathbf{j}$

$(v_0 \sin \theta)t - 16t^2 = 0$ when $t = 0$ and $t = \dfrac{v_0 \sin \theta}{16}$.

The range is

$$x = (v_0 \cos \theta)t = (v_0 \cos \theta)\frac{v_0 \sin \theta}{16} = \frac{v_0^2}{32} \sin 2\theta.$$

Hence,

$$x = \frac{1200^2}{32} \sin(2\theta) = 3000 \implies \sin 2\theta = \frac{1}{15} \implies \theta \approx 1.91°.$$

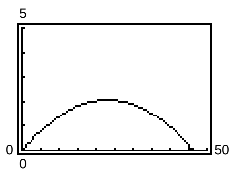
**39.** (a) $\theta = 10°$, $v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 10°)t\mathbf{i} + [0 + (66 \sin 10°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (65t)\mathbf{i} + (11.46t - 16t^2)\mathbf{j}$

Maximum height: 2.052 feet

Range: 46.557 feet



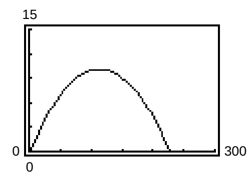

(b) $\theta = 10°$, $v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 10°)t\mathbf{i} + [0 + (146 \sin 10°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + (25.35t - 16t^2)\mathbf{j}$

Maximum height: 10.043 feet

Range: 227.828 feet



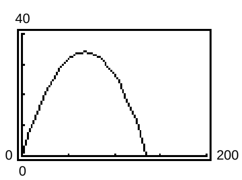

(c) $\theta = 45°$, $v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 45°)t\mathbf{i} + [0 + (66 \sin 45°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + (46.67t - 16t^2)\mathbf{j}$

Maximum height: 34.031 feet

Range: 136.125 feet



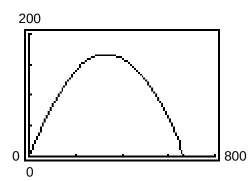

(d) $\theta = 45°$, $v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 45°)t\mathbf{i} + [0 + (146 \sin 45°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + (103.24t - 16t^2)\mathbf{j}$

Maximum height: 166.531 feet

Range: 666.125 feet



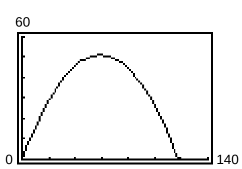

(e) $\theta = 60°$, $v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 60°)t\mathbf{i} + [0 + (66 \sin 60°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (33t)\mathbf{i} + (57.16t - 16t^2)\mathbf{j}$

Maximum height: 51.074 feet

Range: 117.888 feet



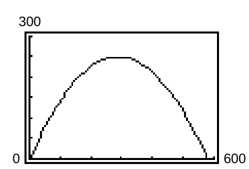

(f) $\theta = 60°$, $v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 60°)t\mathbf{i} + [0 + (146 \sin 60°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (73t)\mathbf{i} + (126.44t - 16t^2)\mathbf{j}$

Maximum height: 249.797 feet

Range: 576.881 feet



**41.** $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j}$

$= (100 \cos 30°)t\mathbf{i} + [1.5 + (100 \sin 30°)t - 4.9t^2]\mathbf{j}$

The projectile hits the ground when $-4.9t^2 + 100\left(\frac{1}{2}\right)t + 1.5 = 0 \implies t \approx 10.234$ seconds.

The range is therefore $(100 \cos 30°)(10.234) \approx 886.3$ meters.

The maximum height occurs when $dy/dt = 0$.

$100 \sin 30 = 9.8t \implies t \approx 5.102$ sec

The maximum height is

$y = 1.5 + (100 \sin 30°)(5.102) - 4.9(5.102)^2 \approx 129.1$ meters.

**43.** $\mathbf{r}(t) = b(\omega t - \sin \omega t)\,\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$\mathbf{v}(t) = b(\omega - \omega \cos \omega t)\mathbf{i} + b\omega \sin \omega t\,\mathbf{j} = b\omega(1 - \cos \omega t)\mathbf{i} + b\omega \sin \omega t\mathbf{j}$

$\mathbf{a}(t) = (b\omega^2 \sin \omega t)\mathbf{i} + (b\omega^2 \cos \omega t)\mathbf{j} = b\omega^2[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}]$

$\|\mathbf{v}(t)\| = \sqrt{2}\,b\omega\sqrt{1 - \cos(\omega t)}$

$\|\mathbf{a}(t)\| = b\omega^2$

(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$

(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots,$ then $\|\mathbf{v}(t)\| = 2b\omega$.

**45.** $\mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t) \cos(\omega t) + b^2\omega \sin(\omega t) \cos(\omega t) = 0$

Therefore, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

**47.** $\mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2\mathbf{r}(t)$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and thus $\mathbf{a}(t)$ is directed toward the origin.

**49.** $\|\mathbf{a}(t)\| = \omega^2 b$

$1 = m(32)$

$F = m(\omega^2 b) = \dfrac{1}{32}(2\omega^2) = 10$

$\omega = 4\sqrt{10}$ rad/sec

$\|\mathbf{v}(t)\| = b\omega = 8\sqrt{10}$ ft/sec

**51.** To find the range, set $y(t) = h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 = 0$ then $0 = \left(\frac{1}{2}g\right)t^2 - (v_0 \sin \theta)t - h$.
By the Quadratic Formula, (discount the negative value)

$$t = \frac{v_0 \sin \theta + \sqrt{(-v_0 \sin \theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]} = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}.$$

At this time,

$$x(t) = v_0 \cos \theta\left(\frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}\right) = \frac{v_0 \cos \theta}{g}\left(v_0 \sin \theta + \sqrt{v_0^2\left(\sin^2 \theta + \frac{2gh}{v_0^2}\right)}\right)$$

$$= \frac{v_0^2 \cos \theta}{g}\left(\sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}}\right).$$

**53.** $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ Position vector

$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ Velocity vector

$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ Acceleration vector

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2}$$
$$= C, \ C \text{ is a constant.}$$
$$\frac{d}{dt}[x'(t)^2 + y'(t)^2 + z'(t)^2] = 0$$
$$2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 0$$
$$2[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)] = 0$$
$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$$

Orthogonal

**55.** $\mathbf{r}(t) = 6\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -6\sin t\mathbf{i} + 3\cos t\mathbf{j}$

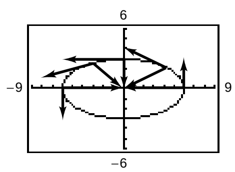
$$\|\mathbf{v}(t)\| = \sqrt{36\sin^2 t + 9\cos^2 t}$$
$$= 3\sqrt{4\sin^2 t + \cos^2 t}$$
$$= 3\sqrt{3\sin^2 t + 1}$$

$\mathbf{a}(t) = \mathbf{v}'(t) = -6\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b)

| $t$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| Speed | $3$ | $\frac{3}{2}\sqrt{10}$ | $6$ | $\frac{3}{2}\sqrt{13}$ | $3$ |

(c)




(d) The speed is increasing when the angle between $\mathbf{v}$ and $\mathbf{a}$ is in the interval

$$\left[0, \frac{\pi}{2}\right).$$

The speed is decreasing when the angle is in the interval

$$\left(\frac{\pi}{2}, \pi\right].$$

## Section 11.4 Tangent Vectors and Normal Vectors

**1.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2t\mathbf{i} + 2\mathbf{j}}{2\sqrt{t^2 + 1}} = \frac{1}{\sqrt{t^2 + 1}}(t\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

**3.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

**5.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$

When $t = 0$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, $[t = 0 \text{ at } (0, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = 1$, $b = 0$, $c = 1$

Parametric equations: $x = t$, $y = 0$, $z = t$

**7.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

When $t = 0$, $\mathbf{r}'(0) = 2\mathbf{j} + \mathbf{k}$, $[t = 0 \text{ at } (2, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{5}}{5}(2\mathbf{j} + \mathbf{k})$$

Direction numbers: $a = 0$, $b = 2$, $c = 1$

Parametric equations: $x = 2$, $y = 2t$, $z = t$

**9.** $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle$

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

When $t = \frac{\pi}{4}$, $\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$, $\left[t = \frac{\pi}{4} \text{ at } \left(\sqrt{2}, \sqrt{2}, 4\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2}\left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$$

Direction numbers: $a = -\sqrt{2}$, $b = \sqrt{2}$, $c = 0$

Parametric equations: $x = -\sqrt{2}t + \sqrt{2}$, $y = \sqrt{2}t + \sqrt{2}$, $z = 4$

**11.** $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3\right\rangle$

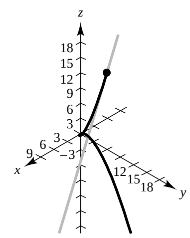
$\mathbf{r}'(t) = \langle 1, 2t, 2t^2\rangle$

When $t = 3$, $\mathbf{r}'(3) = \langle 1, 6, 18\rangle$, $[t = 3 \text{ at } (3, 9, 18)]$.

$$\mathbf{T}(3) = \frac{\mathbf{r}'(3)}{\|\mathbf{r}'(3)\|} = \frac{1}{19}\langle 1, 6, 18\rangle$$

Direction numbers: $a = 1$, $b = 6$, $c = 18$

Parametric equations: $x = t + 3$, $y = 6t + 9$, $z = 18t + 18$




**13.** $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \sqrt{t}\mathbf{k}$, $\quad t_0 = 1$

$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k} \cdot \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + \mathbf{j} + (1/2)\mathbf{k}}{\sqrt{1 + 1 + (1/4)}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

Tangent line: $x = 1 + t$, $y = t$, $z = 1 + \frac{1}{2}t$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(1.1) \approx 1.1\mathbf{i} + 0.1\mathbf{j} + 1.05\mathbf{k}$$

$$= \langle 1.1, 0.1, 1.05\rangle$$

**15.** $\mathbf{r}(4) = \langle 2, 16, 2\rangle$

$\mathbf{u}(8) = \langle 2, 16, 2\rangle$

Hence the curves intersect.

$$\mathbf{r}'(t) = \left\langle 1, 2t, \frac{1}{2}\right\rangle,\ \mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2}\right\rangle$$

$$\mathbf{u}'(s) = \left\langle \frac{1}{4}, 2, \frac{1}{3}s^{-2/3}\right\rangle,\ \mathbf{u}'(8) = \left\langle \frac{1}{4}, 2, \frac{1}{12}\right\rangle$$

$$\cos\theta = \frac{\mathbf{r}'(4)\cdot\mathbf{u}'(8)}{\|\mathbf{r}'(4)\|\,\|\mathbf{u}'(8)\|} \approx \frac{16.29167}{16.29513} \implies \theta \approx 1.2^\circ$$

**17.** $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}, t = 2$

$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1 + t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{(t^2 + 1)^{3/2}}\mathbf{i} + \frac{1}{(t^2 + 1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

**19.** $\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \mathbf{k}, t = \frac{3\pi}{4}$

$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 6\cos t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{T}(t)\| = 1$

$$\mathbf{N}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

**21.** $\mathbf{r}(t) = 4t\mathbf{i}$

$\mathbf{v}(t) = 4\mathbf{i}$

$\mathbf{a}(t) = \mathbf{O}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{4\mathbf{i}}{4} = \mathbf{i}$$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

**23.** $\mathbf{r}(t) = 4t^2\mathbf{i}$

$\mathbf{v}(t) = 8t\mathbf{i}$

$\mathbf{a}(t) = 8\mathbf{i}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{8t\mathbf{i}}{8t} = \mathbf{i}$$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

**25.** $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ \mathbf{v}(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j},\ \mathbf{v}(1) = \mathbf{i} - \mathbf{j},$

$\mathbf{a}(t) = \dfrac{2}{t^3}\mathbf{j},\ \mathbf{a}(1) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{t^2}{\sqrt{t^4+1}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) = \frac{1}{\sqrt{t^4+1}}(t^2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{2t}{(t^4+1)^{3/2}}\mathbf{i} + \dfrac{2t^3}{(t^4+1)^{3/2}}\mathbf{j}}{\dfrac{2t}{(t^4+1)}}$$

$$= \frac{1}{\sqrt{t^4+1}}(\mathbf{i} + t^2\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = -\sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

**27.** $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$

$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$

$\mathbf{a}(t) = e^t(-2 \sin t)\mathbf{i} + e^t(2 \cos t)\mathbf{j}$

At $t = \dfrac{\pi}{2},\ \mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j}).$

Motion along $\mathbf{r}$ is counterclockwise. Therefore,

$$\mathbf{N} = \frac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}).$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}e^{\pi/2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}e^{\pi/2}$

**29.** $\mathbf{r}(t_0) = (\cos \omega t_0 + \omega t_0 \sin \omega t_0)\mathbf{i} + (\sin \omega t_0 - \omega t_0 \cos \omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = (\omega^2 t_0 \cos \omega t_0)\mathbf{i} + (\omega^2 t_0 \sin \omega t_0)\mathbf{j}$

$\mathbf{a}(t_0) = \omega^2[(\cos \omega t_0 - \omega t_0 \sin \omega t_0)\mathbf{i} + (\omega t_0 \cos \omega t_0 + \sin \omega t_0)\mathbf{j}]$

$$\mathbf{T}(t_0) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$$

Motion along $\mathbf{r}$ is counterclockwise. Therefore

$$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}.$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \omega^2$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \omega^2(\omega t_0) = \omega^3 t_0$

**31.** $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$

$\mathbf{v}(t) = -a\omega\sin(\omega t)\mathbf{i} + a\omega\cos(\omega t)\mathbf{j}$

$\mathbf{a}(t) = -a\omega^2\cos(\omega t)\mathbf{i} - a\omega^2\sin(\omega t)\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos(\omega t)\mathbf{i} - \sin(\omega t)\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = a\omega^2$

**33.** Speed: $\|\mathbf{v}(t)\| = a\omega$

The speed is constant since $a_{\mathbf{T}} = 0$.

**35.** $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j},\ t_0 = 2$

$$x = t,\ y = \frac{1}{t} \Rightarrow xy = 1$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

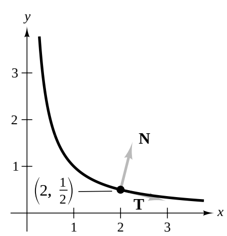
$$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4+1}}$$

$$\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{T}(2) = \frac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(2) = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$$




**37.** $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

$\mathbf{a}(t) = \mathbf{0}$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \frac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_{\mathbf{T}}, a_{\mathbf{N}}$ are not defined.

**39.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1+5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(1) = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1+5t^2)^{3/2}}}{\dfrac{\sqrt{5}}{1+5t^2}} = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$$

$$\mathbf{N}(1) = \frac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{5\sqrt{6}}{6}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{\sqrt{30}}{6}$$

**41.** $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

$\mathbf{v}(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$$\mathbf{v}\left(\frac{\pi}{2}\right) = 4\mathbf{i} - 3\mathbf{j}$$

$\mathbf{a}(t) = -3\cos t\mathbf{j} - 3\sin t\mathbf{k}$

$$\mathbf{a}\left(\frac{\pi}{2}\right) = -3\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5}(4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k})$$

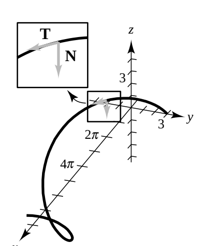
$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = -\cos t\mathbf{j} - \sin t\mathbf{k}$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{k}$$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = 3$



**43.** $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$

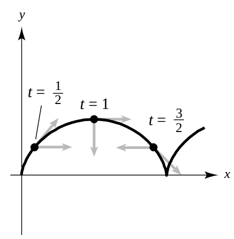
If $a(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$, then $a_{\mathbf{T}}$ is the tangential component of acceleration and $a_{\mathbf{N}}$ is the normal component of acceleration.

**45.** If $a_{\mathbf{N}} = 0$, then the motion is in a straight line.

**47.** $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t\rangle$

The graph is a cycloid.

(a) $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t\rangle$




$\mathbf{v}(t) = \langle \pi - \pi \cos \pi t, \pi \sin \pi t\rangle$

$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t\rangle$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}\langle 1 - \cos \pi t, \sin \pi t\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}\langle \sin \pi t, -1 + \cos \pi t\rangle$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}[\pi^2 \sin \pi t(1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t] = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}[\pi^2 \sin^2 \pi t + \pi^2 \cos \pi t(-1 + \cos \pi t)] = \frac{\pi^2(1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \frac{\pi^2\sqrt{2(1 - \cos \pi t)}}{2}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \frac{\pi^2}{\sqrt{2}} = \frac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$

When $t = 1$: $a_{\mathbf{T}} = 0$, $a_{\mathbf{N}} = \pi^2$

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$

(b) Speed: $s = \|\mathbf{v}(t)\| = \pi\sqrt{2(1 - \cos \pi t)}$

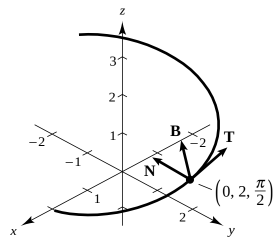
$$\frac{ds}{dt} = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_{\mathbf{T}}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \frac{\sqrt{2}\pi^2}{2} > 0 \implies$ the speed in increasing.

When $t = 1$: $a_{\mathbf{T}} = 0 \implies$ the height is maximum.

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2} < 0 \implies$ the speed is decreasing.

**49.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \dfrac{t}{2}\mathbf{k},\ t_0 = \dfrac{\pi}{2}$




$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{2\sqrt{17}}{17}\left(-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$$

$$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = 2\mathbf{j} + \frac{\pi}{4}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{2\sqrt{17}}{17}\left(-2\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{\sqrt{17}}{17}(-4\mathbf{i} + \mathbf{k})$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17}\mathbf{i} + \frac{4\sqrt{17}}{17}\mathbf{k} = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{k})$$

**51.** From Theorem 11.3 we have:

$$\mathbf{r}(t) = (v_0 t\cos\theta)\mathbf{i} + (h + v_0 t\sin\theta - 16t^2)\mathbf{j}$$

$$\mathbf{v}(t) = v_0\cos\theta\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(v_0\cos\theta)\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0\sin\theta - 32t)\mathbf{i} - v_0\cos\theta\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}} \qquad \text{(Motion is clockwise.)}$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{-32(v_0\sin\theta - 32t)}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{32v_0\cos\theta}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

Maximum height when $v_0\sin\theta - 32t = 0$; (vertical component of velocity)

At maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

**53.** $\mathbf{r}(t) = \langle 10\cos 10\pi t, 10\sin 10\pi t, 4 + 4t\rangle,\ 0 \le t \le \frac{1}{20}$

(a) $\mathbf{r}'(t) = \langle -100\pi\sin(10\pi t), 100\pi\cos(10\pi t), 4\rangle$

$$\|\mathbf{r}'(t)\| = \sqrt{(100\pi)^2\sin^2(10\pi t) + (100\pi)^2\cos^2(10\pi t) + 16}$$

$$= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314 \text{ mi/hr}$$

(b) $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 1000\pi^2$

$a_{\mathbf{T}} = 0$ because the speed is constant.

**55.** $\mathbf{r}(t) = (a\cos\omega t)\mathbf{i} + (a\sin\omega t)\mathbf{j}$

From Exercise 31, we know $\mathbf{a}\cdot\mathbf{T} = 0$ and $\mathbf{a}\cdot\mathbf{N} = a\omega^2$.

(a) Let $\omega_0 = 2\omega$. Then

$$\mathbf{a}\cdot\mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let $a_0 = a/2$. Then

$$\mathbf{a}\cdot\mathbf{N} = a_0\omega^2 = \left(\frac{a}{2}\right)\omega^2 = \left(\frac{1}{2}\right)a\omega^2$$

or the centripetal acceleration is halved when the radius is halved.

**57.** $v = \sqrt{\dfrac{9.56 \times 10^4}{4100}} \approx 4.83$ mi/sec

**59.** $v = \sqrt{\dfrac{9.56 \times 10^4}{4385}} \approx 4.67$ mi/sec

**61.** Let $\mathbf{T}(t) = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$ be the unit tangent vector. Then

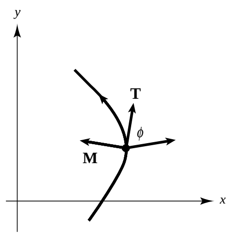
$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = -(\sin\phi\mathbf{i} + \cos\phi\mathbf{j})\frac{d\phi}{dt} = \mathbf{M}\frac{d\phi}{dt}.$$

$\mathbf{M} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} = \cos[\phi + (\pi/2)]\mathbf{i} + \sin[\phi + (\pi/2)]\mathbf{j}$ and is rotated counterclockwise through an angle of $\pi/2$ from $\mathbf{T}$.

If $d\phi/dt > 0$, then the curve bends to the left and $\mathbf{M}$ has the same direction as $\mathbf{T}'$. Thus, $\mathbf{M}$ has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

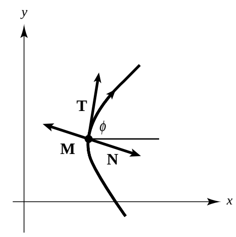
which is toward the concave side of the curve.




If $d\phi/dt < 0$, then the curve bends to the right and $\mathbf{M}$ has the opposite direction as $\mathbf{T}'$. Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.




**63.** Using $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, $\mathbf{T} \times \mathbf{T} = \mathbf{O}$, and $\|\mathbf{T} \times \mathbf{N}\| = 1$, we have:

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\|\mathbf{T} \times (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \\
&= \|\mathbf{v}\|a_{\mathbf{T}}(\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \\
&= \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \\
\|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\|a_{\mathbf{N}}\|\mathbf{T} \times \mathbf{N}\| \\
&= \|\mathbf{v}\|a_{\mathbf{N}}
\end{aligned}$$

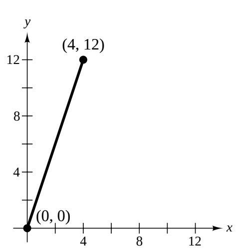
Thus, $a_{\mathbf{N}} = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$.

## Section 11.5 Arc Length and Curvature

**1.** $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 3,\ \frac{dz}{dt} = 0$$

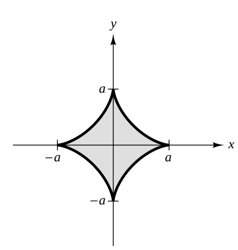
$$\begin{aligned}
s &= \int_0^4 \sqrt{1 + 9}\,dt \\
&= \sqrt{10}\int_0^4 dt \\
&= \Big[\sqrt{10}t\Big]_0^4 = 4\sqrt{10}
\end{aligned}$$




**3.** $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\begin{aligned}
s &= 4\int_0^{\pi/2} \sqrt{[3a\cos^2 t(-\sin t)]^2 + [3a\sin^2 t\cos t]^2}\,dt \\
&= 12a\int_0^{\pi/2} \sin t\cos t\,dt \\
&= 3a\int_0^{\pi/2} 2\sin 2t\,dt = \Big[-3a\cos 2t\Big]_0^{\pi/2} = 6a
\end{aligned}$$



**5.** (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$= (100 \cos 45°)t\mathbf{i} + \left[3 + (100 \sin 45°)t - \frac{1}{2}(32)t^2\right]\mathbf{j}$$

$$= 50\sqrt{2}t\mathbf{i} + \left[3 + 50\sqrt{2}t - 16t^2\right]\mathbf{j}$$

(b) $\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}$

$50\sqrt{2} - 32t = 0 \implies t = \dfrac{25\sqrt{2}}{16}$

Maximum height: $3 + 50\sqrt{2}\left(\dfrac{25\sqrt{2}}{16}\right) - 16\left(\dfrac{25\sqrt{2}}{16}\right)^2 = 81.125$ ft

(c) $3 + 50\sqrt{2}t - 16t^2 = 0 \implies t \approx 4.4614$

Range: $50\sqrt{2}(4.4614) \approx 315.5$ feet

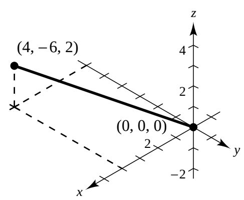
(d) $s = \displaystyle\int_0^{4.4614} \sqrt{\left(50\sqrt{2}\right)^2 + \left(50\sqrt{2} - 32t\right)^2}\,dt \approx 362.9$ feet

**7.** $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j} + t\mathbf{k}$

$\dfrac{dx}{dt} = 2\ \dfrac{dy}{dt} = -3,\ \dfrac{dz}{dt} = 1$

$$s = \int_0^2 \sqrt{2^2 + (-3)^2 + 1^2}\,dt$$

$$= \int_0^2 \sqrt{14}\,dt = \Big[\sqrt{14}t\Big]_0^2 = 2\sqrt{14}$$

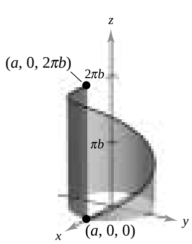



**9.** $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$

$\dfrac{dx}{dt} = -a \sin t,\ \dfrac{dy}{dt} = a \cos t,\ \dfrac{dz}{dt} = b$

$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2}\,dt = \Big[\sqrt{a^2 + b^2}t\Big]_0^{2\pi} = 2\pi\sqrt{a^2 + b^2}$$




**11.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \ln t\mathbf{k}$

$\dfrac{dx}{dt} = 2t,\ \dfrac{dy}{dt} = 1,\ \dfrac{dz}{dt} = \dfrac{1}{t}$

$$s = \int_1^3 \sqrt{(2t)^2 + (1)^2 + \left(\frac{1}{t}\right)^2}\,dt$$

$$= \int_1^3 \sqrt{\frac{4t^4 + t^2 + 1}{t^2}}\,dt$$

$$= \int_1^3 \frac{\sqrt{4t^4 + t^2 + 1}}{t}\,dt \approx 8.37$$

**13.** $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k},\quad 0 \le t \le 2$

(a) $\mathbf{r}(0) = \langle 0, 4, 0\rangle,\ \mathbf{r}(2) = \langle 2, 0, 8\rangle$

distance $= \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$

**—CONTINUED—**

**13. —CONTINUED—**

(b) $\mathbf{r}(0) = \langle 0, 4, 0\rangle$

$\mathbf{r}(0.5) = \langle 0.5, 3.75, .125\rangle$

$\mathbf{r}(1) = \langle 1, 3, 1\rangle$

$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375\rangle$

$\mathbf{r}(2) = \langle 2, 0, 8\rangle$

$$\text{distance} \approx \sqrt{(0.5)^2 + (.25)^2 + (.125)^2} + \sqrt{(.5)^2 + (.75)^2 + (.875)^2} + \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2} + \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2}$$

$$\approx 0.5728 + 1.2562 + 2.7300 + 4.9702 \approx 9.529$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

**15.** $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

(a) $$s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du = \int_0^t \sqrt{(-2\sin u)^2 + (2\cos u)^2 + (1)^2}\,du = \int_0^t \sqrt{5}\,du = \Big[\sqrt{5}u\Big]_0^t = \sqrt{5}t$$

(b) $$\frac{s}{\sqrt{5}} = t$$

$$x = 2\cos\left(\frac{s}{\sqrt{5}}\right),\ y = 2\sin\left(\frac{s}{\sqrt{5}}\right),\ z = \frac{s}{\sqrt{5}}$$

$$\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

(c) When $s = \sqrt{5}$:

$x = 2\cos 1 \approx 1.081$

$y = 2\sin 1 \approx 1.683$

$z = 1$

$(1.081, 1.683, 1.000)$

When $s = 4$:

$x = 2\cos\dfrac{4}{\sqrt{5}} \approx -0.433$

$y = 2\sin\dfrac{4}{\sqrt{5}} \approx 1.953$

$z = \dfrac{4}{\sqrt{5}} \approx 1.789$

$(-0.433, 1.953, 1.789)$

(d) $$\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$$

**17.** $$\mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$$

$$\mathbf{r}'(s) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$$

$\mathbf{T}'(s) = \mathbf{0} \implies K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

**19.** $$\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T}(s) = \mathbf{r}'(s) = -\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + \frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T}'(s) = -\frac{2}{5}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} - \frac{2}{5}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{2}{5}$$

**21.** $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

$\mathbf{T}'(t) = 0$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$ (The curve is a line.)

**23.** $\mathbf{r}(t) = t\,\mathbf{i} + \dfrac{1}{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = \dfrac{2}{t^3}\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \dfrac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \dfrac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$

$\mathbf{N}(1) = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$

$K = \dfrac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \dfrac{\sqrt{2}}{2}$

**25.** $\mathbf{r}(t) = 4\cos(2\pi t)\mathbf{i} + 4\sin(2\pi t)\mathbf{j}$

$\mathbf{r}'(t) = -8\pi\sin(2\pi t)\mathbf{i} + 8\pi\cos(2\pi t)\mathbf{j}$

$\mathbf{T}(t) = -\sin(2\pi t)\mathbf{i} + \cos(2\pi t)\mathbf{j}$

$\mathbf{T}'(t) = -2\pi\cos(2\pi t)\mathbf{i} - 2\pi\sin(2\pi t)\mathbf{j}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{2\pi}{8\pi} = \dfrac{1}{4}$

**27.** $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$

$\mathbf{r}'(t) = -a\omega\sin(\omega t)\mathbf{i} + a\omega\cos(\omega t)\mathbf{j}$

$\mathbf{T}(t) = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$

$\mathbf{T}'(t) = -\omega\cos(\omega t)\mathbf{i} - \omega\sin(\omega t)\mathbf{j}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{\omega}{a\omega} = \dfrac{1}{a}$

**29.** $\mathbf{r}(t) = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j}$

$\mathbf{r}'(t) = (-e^t\sin t + e^t\cos t)\mathbf{i} + (e^t\cos t + e^t\sin t)\mathbf{j}$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j}]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{2}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{2}e^t} = \dfrac{\sqrt{2}}{2}e^{-t}$

**31.** $\mathbf{r}(t) = \langle \cos\omega t + \omega t\sin\omega t,\ \sin\omega t - \omega t\cos\omega t\rangle$

From Exercise 21, Section 11.4, we have:

$\mathbf{a} \cdot \mathbf{N} = \omega^3 t$

$K = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}\|^2} = \dfrac{\omega^3 t}{\omega^4 t^2} = \dfrac{1}{\omega t}$

**33.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$

$\mathbf{T}'(t) = \dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

$= \dfrac{\dfrac{\sqrt{5}}{(1 + 5t^2)}}{\sqrt{1 + 5t^2}} = \dfrac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$

**35.** $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

$\mathbf{r}'(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$\mathbf{T}(t) = \dfrac{1}{5}[4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}]$

$\mathbf{T}'(t) = \dfrac{1}{5}[-3\cos t\mathbf{j} - 3\sin t\mathbf{k}]$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{3/5}{5} = \dfrac{3}{25}$

**37.** $y = 3x - 2$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

**39.** $y = 2x^2 + 3$

$y' = 4x$

$y'' = 4$

$$K = \frac{4}{[1 + (-4)^2]^{3/2}} = \frac{4}{17^{3/2}} \approx 0.057$$

$$\frac{1}{K} = \frac{17^{3/2}}{4} \approx 17.523 \qquad \text{(radius of curvature)}$$

**41.** $y = \sqrt{a^2 - x^2}$

$$y' = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$y'' = \frac{-(2x^2 - a^2)}{(a^2 - x^2)^{3/2}}$$

At $x = 0$: $\quad y' = 0$

$$y'' = \frac{1}{a}$$

$$K = \frac{1/a}{(1 + 0^2)^{3/2}} = \frac{1}{a}$$

$$\frac{1}{K} = a \qquad \text{(radius of curvature)}$$

**43.** (a) Point on circle: $\left(\frac{\pi}{2}, 1\right)$

Center: $\left(\frac{\pi}{2}, 0\right)$

Equation: $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

(b) The circles have different radii since the curvature is different and

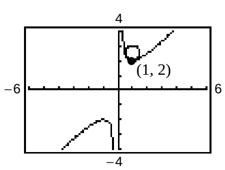
$$r = \frac{1}{K}.$$

**45.** $y = x + \frac{1}{x}, y' = 1 - \frac{1}{x^2}, y'' = \frac{2}{x^3}$

$$K = \frac{2}{(1 + 0^2)^{3/2}} = 2$$

Radius of curvature $= 1/2$. Since the tangent line is horizontal at $(1, 2)$, the normal line is vertical. The center of the circle is $1/2$ unit above the point $(1, 2)$ at $(1, 5/2)$.

Circle: $(x - 1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{1}{4}$




**47.** $y = e^x, \qquad x = 0$

$y' = e^x, \qquad y'' = e^x$

$y'(0) = 1, \qquad y''(0) = 1$

$$K = \frac{1}{(1 + 1^2)^{3/2}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}, \quad r = \frac{1}{K} = 2\sqrt{2}$$

The slope of the tangent line at $(0, 1)$ is $y'(0) = 1$.

The slope of the normal line is $-1$.

Equation of normal line: $y - 1 = -x$ or $y = -x + 1$

The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(0, 1)$.

$$\sqrt{(0 - x)^2 + (1 - y)^2} = 2\sqrt{2}$$

$$x^2 + x^2 = 8$$

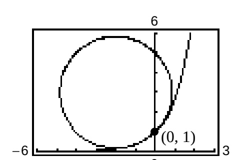
$$x^2 = 4$$

$$x = \pm 2$$

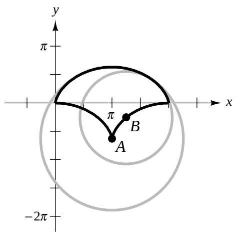
Since the circle is above the curve, $x = -2$ and $y = 3$.

Center of circle: $(-2, 3)$

Equation of circle: $(x + 2)^2 + (y - 3)^2 = 8$



**49.**




**51.** $y = (x-1)^2 + 3,\ y' = 2(x-1),\ y'' = 2$

$$K = \frac{2}{(1 + [2(x-1)]^2)^{3/2}} = \frac{2}{[1 + 4(x-1)^2]^{3/2}}$$

(a) $K$ is maximum when $x = 1$ or at the vertex $(1, 3)$.

(b) $\lim_{x\to\infty} K = 0$

**53.** $y = x^{2/3},\ y' = \frac{2}{3}x^{-1/3},\ y'' = -\frac{2}{9}x^{-4/3}$

$$K = \left|\frac{(-2/9)x^{-4/3}}{[1 + (4/9)x^{-2/3}]^{3/2}}\right| = \left|\frac{6}{x^{1/3}(9x^{2/3} + 4)^{3/2}}\right|$$

(a) $K \Rightarrow \infty$ as $x \Rightarrow 0$. No maximum

(b) $\lim_{x\to\infty} K = 0$

**55.** $y = (x-1)^3 + 3$

$y' = 3(x-1)^2$

$y'' = 6(x-1)$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|6(x-1)|}{[1 + 9(x-1)^4]^{3/2}} = 0 \text{ at } x = 1.$$

Curvature is 0 at $(1, 3)$.

**57.** $K = \dfrac{|y''|}{[1 + (y')^2]^{3/2}}$

The curvature is zero when $y'' = 0$.

**59.** $s = \displaystyle\int_a^b \|\mathbf{r}'(t)\|\, dt$

**61.** The curve is a line.

**63.** Endpoints of the major axis: $(\pm 2, 0)$

Endpoints of the minor axis: $(0, \pm 1)$

$x^2 + 4y^2 = 4$

$2x + 8yy' = 0$

$$y' = -\frac{x}{4y}$$

$$y'' = \frac{(4y)(-1) - (-x)(4y')}{16y^2} = \frac{-4y - (x^2/y)}{16y^2} = \frac{-(4y^2 + x^2)}{16y^3} = \frac{-1}{4y^3}$$

$$K = \frac{|-1/4y^3|}{[1 + (-x/4y)^2]^{3/2}} = \frac{|-16|}{(16y^2 + x^2)^{3/2}} = \frac{16}{(12y^2 + 4)^{3/2}} = \frac{16}{(16 - 3x^2)^{3/2}}$$

Therefore, since $-2 \le x \le 2$, $K$ is largest when $x = \pm 2$ and smallest when $x = 0$.

**65.** $f(x) = x^4 - x^2$

(a) $K = \dfrac{2|6x^2 - 1|}{[16x^6 - 16x^4 + 4x^2 + 1]^{3/2}}$

(b) For $x = 0$, $K = 2$. $f(0) = 0$. At $(0, 0)$, the circle of curvature has radius $\frac{1}{2}$. Using the symmetry of the graph of $f$, you obtain

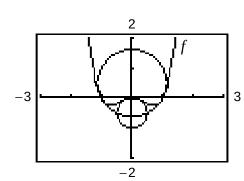
$$x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}.$$

For $x = 1$, $K = (2\sqrt{5})/5$. $f(1) = 0$. At $(1, 0)$, the circle of curvature has radius

$$\frac{\sqrt{5}}{2} = \frac{1}{K}.$$

Using the graph of $f$, you see that the center of curvature is $\left(0, \frac{1}{2}\right)$. Thus,

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$



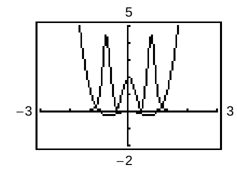

To graph these circles, use

$$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2} \quad \text{and} \quad y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}.$$

**—CONTINUED—**

**65. —CONTINUED—**

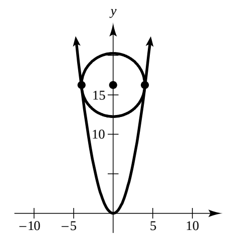
(c) The curvature tends to be greatest near the extrema of $f$, and $K$ decreases as $x \to \pm\infty$. However, $f$ and $K$ do not have the same critical numbers.




Critical numbers of $f$: $x = 0, \pm\frac{\sqrt{2}}{2} \approx \pm 0.7071$

Critical numbers of $K$: $x = 0, \pm .7647, \pm 0.4082$

**67.** (a) Imagine dropping the circle $x^2 + (y - k)^2 = 16$ into the parabola $y = x^2$. The circle will drop to the point where the tangents to the circle and parabola are equal.




$$y = x^2 \quad \text{and} \quad x^2 + (y - k)^2 = 16 \implies x^2 + (x^2 - k)^2 = 16$$

Taking derivatives, $2x + 2(y - k)y' = 0$ and $y' = 2x$. Hence,

$$(y - k)y' = -x \implies y' = \frac{-x}{y - k}.$$

Thus,

$$\frac{-x}{y - k} = 2x \implies -x = 2x(y - k) \implies -1 = 2(x^2 - k) \implies x^2 - k = -\frac{1}{2}.$$

Thus,

$$x^2 + (x^2 - k)^2 = x^2 + \left(-\frac{1}{2}\right)^2 = 16 \implies x^2 = 15.75.$$

Finally, $k = x^2 + \frac{1}{2} = 16.25$, and the center of the circle is 16.25 units from the vertex of the parabola. Since the radius of the circle is 4, the circle is 12.25 units from the vertex.

(b) In 2-space, the parabola $z = y^2$ (or $z = x^2$) has a curvature of $K = 2$ at $(0, 0)$. The radius of the largest sphere that will touch the vertex has radius $= 1/K = \frac{1}{2}$.

**69.** Given $y = f(x)$: $K = \dfrac{|y''|}{(1 + [y']^2)^{3/2}}$

$$R = \frac{1}{K}$$

The center of the circle is on the normal line at a distance of $R$ from $(x, y)$.

Equation of normal line: $y - y_0 = -\dfrac{1}{y'}(x - x_0)$

$$\sqrt{(x - x_0)^2 + \left[-\frac{1}{y'}(x - x_0)\right]^2} = \frac{(1 + [y']^2)^{3/2}}{|y''|}$$

$$(x - x_0)^2\left[1 + \frac{1}{(y')^2}\right] = \frac{(1 + [y']^2)^3}{(y'')^2}$$

$$(x - x_0)^2 = \frac{(y')^2(1 + [y']^2)^2}{(y'')^2}$$

$$x - x_0 = \frac{y'(1 + [y']^2)}{y''} = y'z$$

$$x_0 = x - y'z$$

$$y - y_0 = -\frac{1}{y'}(x - (x - y'z)) = -z$$

$$y_0 = y + z$$

Thus, $(x_0, y_0) = (x - y'z, y + z)$.

For $y = e^x$, $y' = e^x$, $y'' = e^x$, $z = \dfrac{1 + e^{2x}}{e^x} = e^{-x} + e^x$.

When $x = 0$: $x_0 = x - y'z = 0 - (1)(2) = -2$

$y_0 = y + z = 1 + 2 = 3$

Center of curvature: $(-2, 3)$

(See Exercise 47)

**71.** $r = 1 + \sin\theta$

$r' = \cos\theta$

$r'' = -\sin\theta$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2\cos^2\theta - (1 + \sin\theta)(-\sin\theta) + (1 + \sin\theta)^2|}{\sqrt{[\cos^2\theta + (1 + \sin\theta)^2]^3}}$$

$$= \frac{3(1 + \sin\theta)}{\sqrt{8(1 + \sin\theta)^3}} = \frac{3}{2\sqrt{2(1 + \sin\theta)}}$$

**73.** $r = a\sin\theta$

$r' = a\cos\theta$

$r'' = -a\sin\theta$

$$K = \frac{|2(r\omega)^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2a^2\cos^2\theta + a^2\sin^2\theta + a^2\sin^2\theta|}{\sqrt{[a^2\cos^2\theta + a^2\sin^2\theta]^3}}$$

$$= \frac{2a^2}{a^3} = \frac{2}{a}, \; a > 0$$

**75.** $r = e^{a\theta}, a > 0$

$r' = ae^{a\theta}$

$r'' = a^2e^{a\theta}$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2a^2e^{2a\theta} - a^2e^{2a\theta} + e^{2a\theta}|}{[a^2e^{2a\theta} + e^{2a\theta}]^{3/2}}$$

$$= \frac{1}{e^{a\theta}\sqrt{a^2 + 1}}$$

(a) As $\theta \Rightarrow \infty$, $K \Rightarrow 0$.

(b) As $a \Rightarrow \infty$, $K \Rightarrow 0$.

**77.** $r = 4\sin 2\theta$

$r' = 8\cos 2\theta$

At the pole: $K = \dfrac{2}{|r'(0)|} = \dfrac{2}{8} = \dfrac{1}{4}$

**79.** $x = f(t)$

$y = g(t)$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$y'' = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{\frac{dx}{dt}} = \frac{\frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}}{f'(t)}$$

$$= \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\left|\frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\left[1 + \left(\frac{g'(t)}{f'(t)}\right)^2\right]^{3/2}}$$

$$= \frac{\left|\frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\sqrt{\left\{\frac{[f'(t)]^2 + [g'(t)]^2}{[f'(t)]^2}\right\}^3}}$$

$$= \frac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}$$

**81.** $x(\theta) = a(\theta - \sin\theta) \qquad y(\theta) = a(1 - \cos\theta)$

$x'(\theta) = a(1 - \cos\theta) \qquad y'(\theta) = a\sin\theta$

$x''(\theta) = a\sin\theta \qquad y''(\theta) = a\cos\theta$

$$K = \frac{|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)|}{[x'(\theta)^2 + y'(\theta)^2]^{3/2}}$$

$$= \frac{|a^2(1 - \cos\theta)\cos\theta - a^2\sin^2\theta|}{[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta]^{3/2}}$$

$$= \frac{1}{a}\,\frac{|\cos\theta - 1|}{[2 - 2\cos\theta]^{3/2}}$$

$$= \frac{1}{a}\,\frac{1 - \cos\theta}{2\sqrt{2}[1 - \cos\theta]^{3/2}} \quad (1 - \cos \geq 0)$$

$$= \frac{1}{2a\sqrt{2 - 2\cos\theta}} = \frac{1}{4a}\csc\left(\frac{\theta}{2}\right)$$

Minimum: $\dfrac{1}{4a}$ $\qquad (\theta = \pi)$

Maximum: none $\qquad (K \to \infty$ as $\theta \to 0)$

**83.** $$a_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2 = \left(\frac{5500 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{100 \text{ ft}}\right)\left(\frac{30(5280) \text{ ft}}{3600 \text{ sec}}\right)^2 = 3327.5 \text{ lb}$$

**85.** Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $r' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$r\left(\frac{dr}{dt}\right) = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}\left[\frac{1}{2}\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\}^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t))\right]$$

$$= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'.$$

**87.** Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $x$, $y$, and $z$ are functions of $t$, and $r = \|\mathbf{r}\|$.

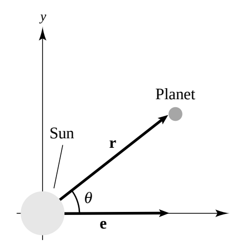
$$\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r} \cdot \mathbf{r}')/r]}{r^2} = \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \quad \text{(using Exercise 77)}$$

$$= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3}$$

$$= \frac{1}{r^3}[(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}]$$

$$= \frac{1}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$$

**89.** From Exercise 86, we have concluded that planetary motion is planar. Assume that the planet moves in the $xy$-plane with the sun at the origin. From Exercise 88, we have




$$\mathbf{r}' \times \mathbf{L} = GM\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right).$$

Since $\mathbf{r}' \times \mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$. Thus, $\mathbf{e}$ lies in the $xy$-plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive $x$-axis and $\theta$ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\|\,\|\mathbf{e}\| \cos\theta = re\cos\theta$. Also,

$$\|\mathbf{L}\|^2 = \mathbf{L} \cdot \mathbf{L} = (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L}$$

$$= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) = \mathbf{r} \cdot \left[GM\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)\right] = GM\left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r}\right] = GM[re\cos\theta + r]$$

Thus,

$$\frac{\|\mathbf{L}\|^2/GM}{1 + e\cos\theta} = r$$

and the planetary motion is a conic section. Since the planet returns to its initial position periodically, the conic is an ellipse.

**91.** $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$

Thus,

$$\frac{dA}{dt} = \frac{dA}{d\theta}\,\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

## Review Exercises for Chapter 11

**1.** $\mathbf{r}(t) = t\mathbf{i} + \csc t\mathbf{k}$

(a) Domain: $t \neq n\pi$, $n$ an integer

(b) Continuous except at $t = n\pi$, $n$ an integer

**3.** $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

**5.** (a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j} + \frac{8}{3}\mathbf{k}$

(c) $\mathbf{r}(c - 1) = (2(c - 1) + 1)\mathbf{i} + (c - 1)^2\mathbf{j} - \frac{1}{3}(c - 1)^3\mathbf{k}$

$\qquad = (2c - 1)\mathbf{i} + (c - 1)^2\mathbf{j} - \frac{1}{3}(c - 1)^3\mathbf{k}$

(d) $\mathbf{r}(1 + \Delta t) - \mathbf{r}(1) = ([2(1 + \Delta t) + 1]\mathbf{i} + [1 + \Delta t]^2\mathbf{j} - \frac{1}{3}[1 + \Delta t]^3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k})$

$\qquad = 2\Delta t\,\mathbf{i} + \Delta t(\Delta t + 2)\mathbf{j} - \frac{1}{3}(\Delta t^3 + 3\Delta t^2 + 3\Delta t)\mathbf{k}$

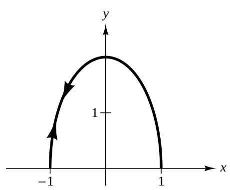
**7.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin^2 t\,\mathbf{j}$

$x(t) = \cos t,\ y(t) = 2\sin^2 t$

$$x^2 + \frac{y}{2} = 1$$

$$y = 2(1 - x^2)$$

$-1 \le x \le 1$

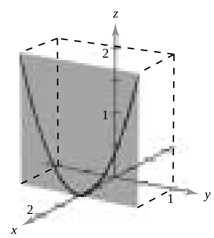



**9.** $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

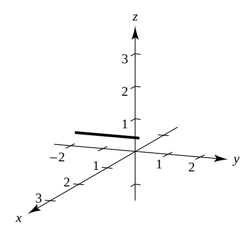
$z = t^2 \Rightarrow z = y^2$



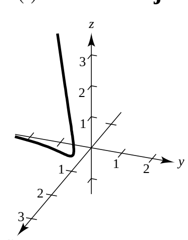

**11.** $\mathbf{r}(t) = \mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$

$x = 1,\ y = \sin t,\ z = 1$

| $t$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|
| $x$ | 1 | 1 | 1 | 1 |
| $y$ | 0 | 1 | 0 | −1 |
| $z$ | 1 | 1 | 1 | 1 |




**13.** $\mathbf{r}(t) = t\mathbf{i} + \ln t\,\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$




**15.** One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\mathbf{i} + (3 - t)\mathbf{j}, \quad 0 \le t \le 3$

$\mathbf{r}_3(t) = (4 - t)\mathbf{i}, \quad 0 \le t \le 4$

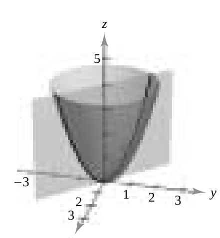
**17.** The vector joining the points is $\langle 7, 4, -10\rangle$. One path is

$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t\rangle$.

**19.** $z = x^2 + y^2,\ x + y = 0,\ t = x$

$x = t,\ y = -t,\ z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$




**21.** $\displaystyle\lim_{t\to 2^-} (t^2\mathbf{i} + \sqrt{4 - t^2}\,\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + \mathbf{k}$

**23.** $\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j},\ \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}''(t) = \mathbf{0}$

(c) $\mathbf{r}(t)\cdot\mathbf{u}(t) = 3t^2 + t^2(t-1) = t^3 + 2t^2$

$D_t[\mathbf{r}(t)\cdot\mathbf{u}(t)] = 3t^2 + 4t$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t-2)\mathbf{j} + 2t^2\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$

(f) $\mathbf{r}(t)\times\mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$

$D_t[\mathbf{r}(t)\times\mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$

**25.** $x(t)$ and $y(t)$ are increasing functions at $t = t_0$, and $z(t)$ is a decreasing function at $t = t_0$.

**27.** $\displaystyle\int (\cos t\mathbf{i} + t\cos t\mathbf{j})\,dt = \sin t\mathbf{i} + (t\sin t + \cos t)\mathbf{j} + \mathbf{C}$

**29.** $\displaystyle\int \|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\|\,dt = \int \sqrt{1+t^2}\,dt = \frac{1}{2}\left[t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right] + \mathbf{C}$

**31.** $\displaystyle\mathbf{r}(t) = \int (2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k})\,dt = t^2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Longrightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$

**33.** $\displaystyle\int_{-2}^{2} (3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k})\,dt = \left[\frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k}\right]_{-2}^{2} = \frac{32}{3}\mathbf{j}$

**35.** $\displaystyle\int_0^2 (e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k})\,dt = \left[2e^{t/2}\mathbf{i} - t^3\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

**37.** $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t, 3\rangle$

$$\begin{aligned}\|\mathbf{v}(t)\| &= \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t + 9}\\ &= 3\sqrt{\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) + 1}\\ &= 3\sqrt{\cos^2 t\sin^2 t + 1}\end{aligned}$$

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t\cos^2 t + 3\sin^2 t(-\sin t), 0\rangle\\ &= \langle 3\cos t(2\sin^2 t - \cos^2 t),\ 3\sin t(2\cos^2 t - \sin^2 t), 0\rangle\end{aligned}$$

**39.** $\mathbf{r}(t) = \left\langle \ln(t-3), t^2, \frac{1}{2}t\right\rangle,\ t_0 = 4$

$\mathbf{r}'(t) = \left\langle \dfrac{1}{t-3}, 2t, \dfrac{1}{2}\right\rangle$

$\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2}\right\rangle$ direction numbers

Since $\mathbf{r}(4) = \langle 0, 16, 2\rangle$, the parametric equations are $x = t, y = 16 + 8t, z = 2 + \frac{1}{2}t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05\rangle$

**41.** Range $= x = \dfrac{v_0^2}{32}\sin 2\theta = \dfrac{(75)^2}{32}\sin 60° \approx 152$ feet

**43.** Range $= x = \dfrac{v_0^2}{9.8}\sin 2\theta = 80 \implies v_0 = \sqrt{\dfrac{(80)(9.8)}{\sin 40°}} \approx 34.9$ m/sec

**45.** $\mathbf{r}(t) = 5t\mathbf{i}$

$\mathbf{v}(t) = 5\mathbf{i}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{N}(t)$ does not exist

$\mathbf{a} \cdot \mathbf{T} = 0$

$\mathbf{a} \cdot \mathbf{N}$ does not exist

(The curve is a line.)

**47.** $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j}$

$\|\mathbf{v}(t)\| = \dfrac{\sqrt{4t+1}}{2\sqrt{t}}$

$\mathbf{a}(t) = -\dfrac{1}{4t\sqrt{t}}\mathbf{j}$

$\mathbf{T}(t) = \dfrac{\mathbf{i} + \left(1/2\sqrt{t}\right)\mathbf{j}}{\left(\sqrt{4t+1}\right)/2\sqrt{t}} = \dfrac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{N}(t) = \dfrac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{T} = \dfrac{-1}{4t\sqrt{t}\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{N} = \dfrac{1}{2t\sqrt{4t+1}}$

**49.** $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{T}(t) = \dfrac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{N}(t) = \dfrac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{T} = \dfrac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{N} = \dfrac{2}{\sqrt{e^{2t} + e^{-2t}}}$

**51.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \dfrac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1+5t^2}}$

$\mathbf{N}(t) = \dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$

$\mathbf{a} \cdot \mathbf{T} = \dfrac{5t}{\sqrt{1+5t^2}}$

$\mathbf{a} \cdot \mathbf{N} = \dfrac{5}{\sqrt{5}\sqrt{1+5t^2}} = \dfrac{\sqrt{5}}{\sqrt{1+5t^2}}$

**53.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$, $x = 2\cos t$, $y = 2\sin t$, $z = t$

When $t = \dfrac{3\pi}{4}$, $x = -\sqrt{2}$, $y = \sqrt{2}$, $z = \dfrac{3\pi}{4}$.

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

Direction numbers when $t = \dfrac{3\pi}{4}$, $a = -\sqrt{2}$, $b = -\sqrt{2}$, $c = 1$

$x = -\sqrt{2}t - \sqrt{2}$, $y = -\sqrt{2}t + \sqrt{2}$, $z = t + \dfrac{3\pi}{4}$

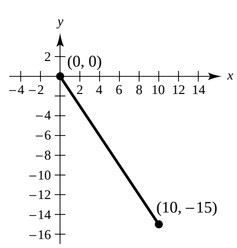
**55.** $v = \sqrt{\dfrac{9.56 \times 10^4}{4600}} \approx 4.56$ mi/sec

**57.** $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}$, $0 \le t \le 5$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^5 \sqrt{4+9}\,dt$$

$$= \sqrt{13}t\Big]_0^5 = 5\sqrt{13}$$



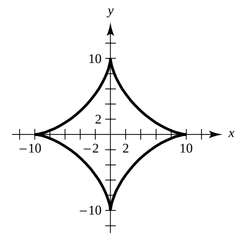
**59.** $\mathbf{r}(t) = 10\cos^3 t\mathbf{i} + 10\sin^3 t\mathbf{j}$

$\mathbf{r}'(t) = -30\cos^2 t\sin t\mathbf{i} + 30\sin^2 t\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 30\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}$

$= 30|\cos t\sin t|$

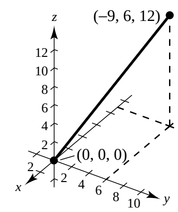
$$s = 4\int_0^{\pi/2} 30\cos t \cdot \sin t\,dt = \left[120\frac{\sin^2 t}{2}\right]_0^{\pi/2} = 60$$




**61.** $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k},\ 0 \le t \le 3$

$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

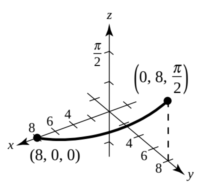
$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{9 + 4 + 16}\,dt = \int_0^3 \sqrt{29}\,dt = 3\sqrt{29}$$




**63.** $\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t\rangle,\ 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{r}'(t) = \langle -8\sin t, 8\cos t, 1\rangle,\ \|\mathbf{r}'(t)\| = \sqrt{65}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{65}\,dt = \frac{\pi\sqrt{65}}{2}$$




**65.** $\mathbf{r}(t) = \dfrac{1}{2}t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\quad 0 \le t \le \pi$

$\mathbf{r}'(t) = \dfrac{1}{2}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$

$$s = \int_0^\pi \|\mathbf{r}'(t)\|\,dt$$

$$= \int_0^\pi \sqrt{\frac{1}{4} + \cos^2 t + \sin^2 t}\,dt$$

$$= \frac{\sqrt{5}}{2}\int_0^\pi dt = \left[\frac{\sqrt{5}}{2}t\right]_0^\pi = \frac{\sqrt{5}}{2}\pi$$

**67.** $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

Line

$k = 0$

**69.** $\mathbf{r}(t) = 2t\mathbf{i} + \dfrac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k},\ \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$

$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k},\ \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

**71.** $y = \dfrac{1}{2}x^2 + 2$

$y' = x$

$y'' = 1$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

At $x = 4$, $K = \dfrac{1}{17^{3/2}}$ and $r = 17^{3/2} = 17\sqrt{17}$.

**73.** $y = \ln x$

$y' = \dfrac{1}{x},\ y'' = -\dfrac{1}{x^2}$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1/x^2}{[1 + (1/x)^2]^{3/2}}$$

At $x = 1$, $K = \dfrac{1}{2^{3/2}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$ and $r = 2\sqrt{2}$.

**75.** The curvature changes abruptly from zero to a nonzero constant at the points $B$ and $C$.

## Problem Solving for Chapter 11

**1.** $x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$

$x'(t) = \cos\left(\frac{\pi t^2}{2}\right), y'(t) = \sin\left(\frac{\pi t^2}{2}\right)$

(a) $s = \int_0^a \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^a dt = a$

(b) $x''(t) = -\pi t \sin\left(\frac{\pi t^2}{2}\right), y''(t) = \pi t \cos\left(\frac{\pi t^2}{2}\right)$

$$K = \frac{\left|\pi t \cos^2\left(\frac{\pi t^2}{2}\right) + \pi t \sin^2\left(\frac{\pi t^2}{2}\right)\right|}{1} = \pi t$$

At $t = a, K = \pi a$.

(c) $K = \pi a = \pi(\text{length})$

**3.** Bomb: $\mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2\rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos \theta)t, (v_0 \sin \theta)t - 16t^2\rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \implies t = 10$ seconds.

Projectile will travel 5 seconds:

$$5(v_0 \sin \theta) - 16(25) = 1600$$
$$v_0 \sin \theta = 400.$$

Horizontal position:

At $t = 10$, bomb is at $5000 - 400(10) = 1000$.

At $t = 5$, projectile is at $5v_0 \cos \theta$.

Thus, $v_0 \cos \theta = 200$.

Combining, $\frac{v_0 \sin \theta}{v_0 \cos \theta} = \frac{400}{200} \implies \tan \theta = 2 \implies \theta \approx 63.4°$.

$v_0 = \frac{200}{\cos \theta} \approx 447.2$ ft/sec

**5.** $x'(\theta) = 1 - \cos \theta, y'(\theta) = \sin \theta, 0 \le \theta \le 2\pi$

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta}$$
$$= \sqrt{2 - 2\cos \theta} = \sqrt{4 \sin^2 \frac{\theta}{2}}$$

$$s(t) = \int_\pi^t 2 \sin \frac{\theta}{2}\, d\theta = \left[-4 \cos \frac{\theta}{2}\right]_\pi^t = -4 \cos \frac{t}{2}$$

$x''(\theta) = \sin \theta, y''(\theta) = \cos \theta$

$$K = \frac{1\left|(1 - \cos \theta)\cos \theta - \sin \theta \sin \theta\right|}{\left(2 \sin \frac{\theta}{2}\right)^3} = \frac{\left|1 \cos \theta - 1\right|}{8 \sin^3 \frac{\theta}{2}}$$
$$= \frac{1}{4 \sin \frac{\theta}{2}}$$

Thus, $\rho = \frac{1}{K} = 4 \sin \frac{t}{2}$ and

$$s^2 + \rho^2 = 16 \cos^2\left(\frac{t}{2}\right) + 16 \sin^2\left(\frac{t}{2}\right) = 16.$$

**7.** $\|\mathbf{r}^2(t)\| = \mathbf{r}(t) \cdot \mathbf{r}(t)$

$$\frac{d}{dt}(\|\mathbf{r}(t)\|)^2 = 2\|\mathbf{r}(t)\|\frac{d}{dt}\|\mathbf{r}(t)\|$$
$$= \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) \implies \frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

**9.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k},\ t = \dfrac{\pi}{2}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 3\mathbf{k},\ \|\mathbf{r}'(t)\| = 5$

$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$$\mathbf{T} = -\frac{4}{5}\sin t\mathbf{i} + \frac{4}{5}\cos t\mathbf{j} + \frac{3}{5}\mathbf{k}$$

$$\mathbf{T}' = -\frac{4}{5}\cos t\mathbf{i} - \frac{4}{5}\sin t\mathbf{j}$$

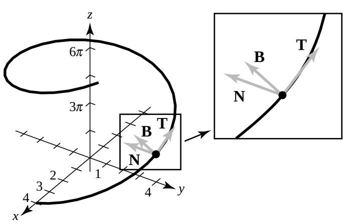
$$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{3}{5}\sin t\mathbf{i} - \frac{3}{5}\cos t\mathbf{j} + \frac{4}{5}\mathbf{k}$$

At $t = \dfrac{\pi}{2}$, $\mathbf{T}\left(\dfrac{\pi}{2}\right) = -\dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{k}$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$$




**11.** (a) $\|\mathbf{B}\| = \|\mathbf{T}\times\mathbf{N}\| = 1$ constant length $\Longrightarrow \dfrac{dB}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = (\mathbf{T}\times\mathbf{N}') + (\mathbf{T}'\times\mathbf{N})$$

$$\mathbf{T}\cdot\frac{d\mathbf{B}}{ds} = \mathbf{T}\cdot(\mathbf{T}\times\mathbf{N}') + \mathbf{T}\cdot(\mathbf{T}'\times\mathbf{N})$$

$$= (\mathbf{T}\times\mathbf{T})\cdot\mathbf{N}' + \mathbf{T}\cdot\left(\mathbf{T}'\times\frac{\mathbf{T}'}{\|\mathbf{T}'\|}\right) = 0$$

Hence, $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$ and $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{T} \Longrightarrow \dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ for some scalar $\tau$.

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. Using Exercise 10.3, number 64,

$$\begin{aligned}\mathbf{B}\times\mathbf{N} = (\mathbf{T}\times\mathbf{N})\times\mathbf{N} &= -\mathbf{N}\times(\mathbf{T}\times\mathbf{N})\\ &= -[(\mathbf{N}\cdot\mathbf{N})\mathbf{T} - (\mathbf{N}\cdot\mathbf{T})\mathbf{N}]\\ &= -\mathbf{T}\end{aligned}$$

$$\begin{aligned}\mathbf{B}\times\mathbf{T} = (\mathbf{T}\times\mathbf{N})\times\mathbf{T} &= -\mathbf{T}\times(\mathbf{T}\times\mathbf{N})\\ &= -[(\mathbf{T}\cdot\mathbf{N})\mathbf{T} - (\mathbf{T}\cdot\mathbf{T})\mathbf{N}]\\ &= \mathbf{N}.\end{aligned}$$

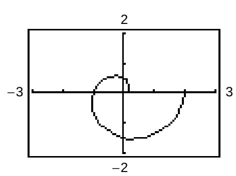
Now, $K\mathbf{N} = \left\|\dfrac{d\mathbf{T}}{ds}\right\|\dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \dfrac{d\mathbf{T}}{ds}$.

Finally,

$$\begin{aligned}\mathbf{N}'(s) = \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) &= (\mathbf{B}\times\mathbf{T}') + (\mathbf{B}'\times\mathbf{T})\\ &= (\mathbf{B}\times K\mathbf{N}) + (-\tau\mathbf{N}\times\mathbf{T})\\ &= -K\mathbf{T} + \tau\mathbf{B}.\end{aligned}$$

**13.** $\mathbf{r}(t) = \langle t\cos\pi t, t\sin\pi t\rangle,\ 0 \le t \le 2$

(a)




(b) Length $= \displaystyle\int_0^2 \|\mathbf{r}'(t)\|\,dt$

$$= \int_0^2 \sqrt{\pi^2t^2 + 1}\,dt \approx 6.766 \quad \text{(graphing utility)}$$

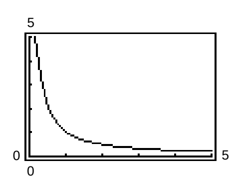
(c) $$K = \frac{\pi(\pi^2t^2 + 2)}{[\pi^2t^2 + 1]^{3/2}}$$

$$K(0) = 2\pi$$

$$K(1) = \frac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$$

$$K(2) \approx 0.51$$

(d)




(e) $\displaystyle\lim_{t\to\infty} K = 0$

(f) As $t \to \infty$, the graph spirals outward and the curvature decreases.

# CHAPTER 12
# Functions of Several Variables

# CHAPTER 12
# Functions of Several Variables

## Section 12.1 Introduction to Functions of Several Variables

Solutions to Odd-Numbered Exercises

**1.** $x^2z + yz - xy = 10$

$z(x^2 + y) = 10 + xy$

$$z = \frac{10 + xy}{x^2 + y}$$

Yes, $z$ is a function of $x$ and $y$.

**3.** $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

No, $z$ is not a function of $x$ and $y$. For example, $(x, y) = (0, 0)$ corresponds to both $z = \pm 1$.

**5.** $f(x, y) = \dfrac{x}{y}$

(a) $f(3, 2) = \dfrac{3}{2}$

(b) $f(-1, 4) = -\dfrac{1}{4}$

(c) $f(30, 5) = \dfrac{30}{5} = 6$

(d) $f(5, y) = \dfrac{5}{y}$

(e) $f(x, 2) = \dfrac{x}{2}$

(f) $f(5, t) = \dfrac{5}{t}$

**7.** $f(x, y) = xe^y$

(a) $f(5, 0) = 5e^0 = 5$

(b) $f(3, 2) = 3e^2$

(c) $f(2, -1) = 2e^{-1} = \dfrac{2}{e}$

(d) $f(5, y) = 5e^y$

(e) $f(x, 2) = xe^2$

(f) $f(t, t) = te^t$

**9.** $h(x, y, z) = \dfrac{xy}{z}$

(a) $h(2, 3, 9) = \dfrac{(2)(3)}{9} = \dfrac{2}{3}$

(b) $h(1, 0, 1) = \dfrac{(1)(0)}{1} = 0$

**11.** $f(x, y) = x \sin y$

(a) $f\left(2, \dfrac{\pi}{4}\right) = 2 \sin \dfrac{\pi}{4} = \sqrt{2}$

(b) $f(3, 1) = 3 \sin 1$

**13.** $g(x, y) = \displaystyle\int_x^y (2t - 3)\, dt$

(a) $g(0, 4) = \displaystyle\int_0^4 (2t - 3)\, dt = \Big[t^2 - 3t\Big]_0^4 = 4$

(b) $g(1, 4) = \displaystyle\int_1^4 (2t - 3)\, dt = \Big[t^2 - 3t\Big]_1^4 = 6$

**15.** $f(x, y) = x^2 - 2y$

(a)
$$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{[(x + \Delta x)^2 - 2y] - (x^2 - 2y)}{\Delta x}$$
$$= \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 2y - x^2 + 2y}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x + \Delta x,\ \Delta x \neq 0$$

(b)
$$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[x^2 - 2(y + \Delta y)] - (x^2 - 2y)}{\Delta y} = \frac{x^2 - 2y - 2\Delta y - x^2 + 2y}{\Delta y} = \frac{-2\Delta y}{\Delta y} = -2,\ \Delta y \neq 0$$

**17.** $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain: $4 - x^2 - y^2 \geq 0$

$$x^2 + y^2 \leq 4$$

$$\{(x, y): x^2 + y^2 \leq 4\}$$

Range: $0 \leq z \leq 2$

**19.** $f(x, y) = \arcsin(x + y)$

Domain:
$\{(x, y): -1 \leq x + y \leq 1\}$

Range: $-\dfrac{\pi}{2} \leq z \leq \dfrac{\pi}{2}$

**21.** $f(x, y) = \ln(4 - x - y)$

Domain: $4 - x - y > 0$

$$x + y < 4$$

$$\{(x, y): y < -x + 4\}$$

Range: all real numbers

**23.** $z = \dfrac{x + y}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers

**25.** $f(x, y) = e^{x/y}$

Domain: $\{(x, y): y \neq 0\}$

Range: $z > 0$

**27.** $g(x, y) = \dfrac{1}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers except zero

**29.** $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$

(a) View from the positive $x$-axis: $(20, 0, 0)$

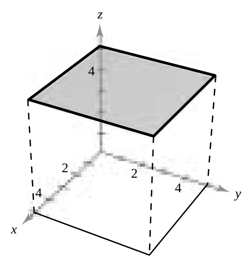
(b) View where $x$ is negative, $y$ and $z$ are positive: $(-15, 10, 20)$

(c) View from the first octant: $(20, 15, 25)$

(d) View from the line $y = x$ in the $xy$-plane: $(20, 20, 0)$

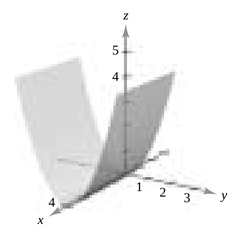
**31.** $f(x, y) = 5$

Plane: $z = 5$




**33.** $f(x, y) = y^2$

Since the variable $x$ is missing, the surface is a cylinder with rulings parallel to the $x$-axis. The generating curve is $z = y^2$. The domain is the entire $xy$-plane and the range is $z \geq 0$.

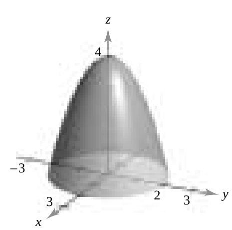



**35.** $z = 4 - x^2 - y^2$

Paraboloid

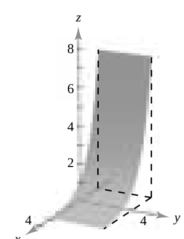
Domain: entire $xy$-plane

Range: $z \leq 4$




**37.** $f(x, y) = e^{-x}$

Since the variable $y$ is missing, the surface is a cylinder with rulings parallel to the $y$-axis. The generating curve is $z = e^{-x}$. The domain is the entire $xy$-plane and the range is $z > 0$.

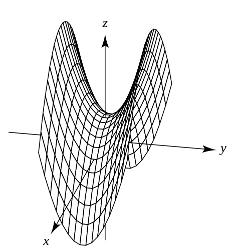



**39.** $z = y^2 - x^2 + 1$

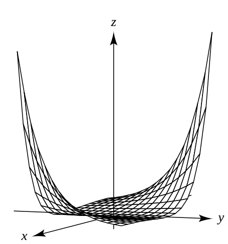
Hyperbolic paraboloid

Domain: entire $xy$-plane

Range: $-\infty < z < \infty$

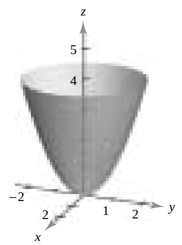



**41.** $f(x, y) = x^2 e^{(-xy/2)}$



**43.** $f(x, y) = x^2 + y^2$

(a)

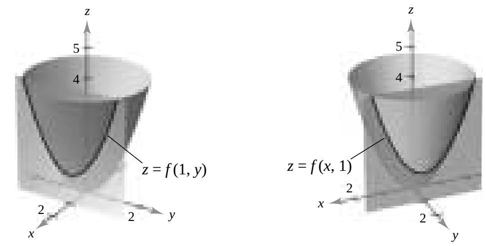



(b) $g$ is a vertical translation of $f$ two units upward

(c) $g$ is a horizontal translation of $f$ two units to the right. The vertex moves from $(0, 0, 0)$ to $(0, 2, 0)$.

(d) $g$ is a reflection of $f$ in the $xy$-plane followed by a vertical translation 4 units upward.

(e)




**45.** $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$
$$\ln c = 1 - x^2 - y^2$$
$$x^2 + y^2 = 1 - \ln c$$

Circles centered at $(0, 0)$

Matches (c)

**47.** $z = \ln|y - x^2|$

Level curves:

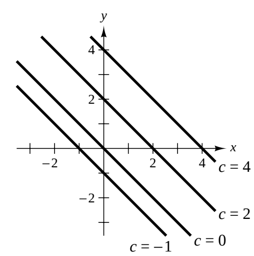
$$c = \ln|y - x^2|$$
$$\pm e^c = y - x^2$$
$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

**49.** $z = x + y$

Level curves are parallel lines of the form $x + y = c$.

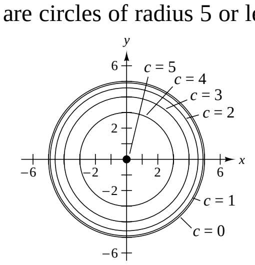



**51.** $f(x, y) = \sqrt{25 - x^2 - y^2}$

The level curves are of the form

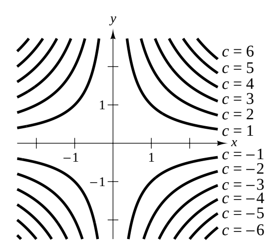
$$c = \sqrt{25 - x^2 - y^2},$$
$$x^2 + y^2 = 25 - c^2.$$

Thus, the level curves are circles of radius 5 or less, centered at the origin.




**53.** $f(x, y) = xy$

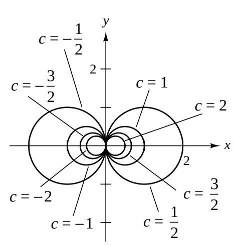
The level curves are hyperbolas of the form $xy = c$.




**55.** $f(x, y) = \dfrac{x}{x^2 + y^2}$

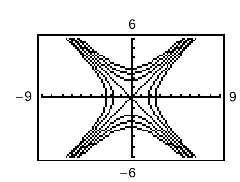
The level curves are of the form

$$c = \frac{x}{x^2 + y^2}$$
$$x^2 - \frac{x}{c} + y^2 = 0$$
$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2$$

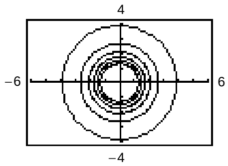



Thus, the level curves are circles passing through the origin and centered at $(1/2c, 0)$.

**57.** $f(x, y) = x^2 - y^2 + 2$



**59.** $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$




**61.** See Definition, page 838.

**63.** No, The following graphs are not hemispheres.

$z = e^{-(x^2+y^2)}$

$z = x^2 + y^2$

**65.** The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.
One possible function is

$$f(x, y) = x^2 - y^2.$$

**67.** $V(I, R) = 1000\left[\dfrac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$

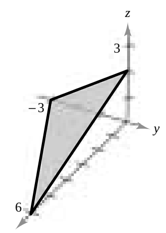
| | Inflation Rate | | |
|---|---|---|---|
| Tax Rate | 0 | 0.03 | 0.05 |
| 0 | 2593.74 | 1929.99 | 1592.33 |
| 0.28 | 2004.23 | 1491.34 | 1230.42 |
| 0.35 | 1877.14 | 1396.77 | 1152.40 |

**69.** $f(x, y, z) = x - 2y + 3z$

$c = 6$

$6 = x - 2y + 3z$

Plane

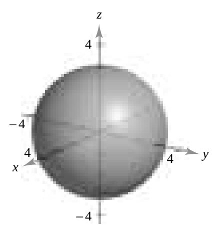



**71.** $f(x, y, z) = x^2 + y^2 + z^2$

$c = 9$

$9 = x^2 + y^2 + z^2$

Sphere

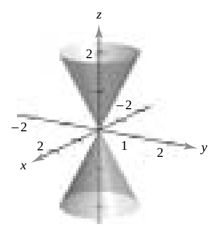



**73.** $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$c = 0$

$0 = 4x^2 + 4y^2 - z^2$

Elliptic cone




**75.** $N(d, L) = \left(\dfrac{d - 4}{4}\right)^2 L$

(a) $N(22, 12) = \left(\dfrac{22 - 4}{4}\right)^2(12) = 243$ board-feet

(b) $N(30, 12) = \left(\dfrac{30 - 4}{4}\right)^2(12) = 507$ board-feet

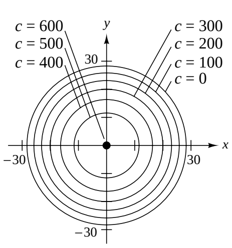
**77.** $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

$$c = 600 - 0.75x^2 - 0.75y^2$$

$$x^2 + y^2 = \frac{600 - c}{0.75}.$$

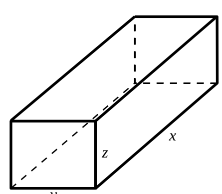
The level curves are circles centered at the origin.




**79.** $C = 0.75xy + 2(0.40)xz + 2(0.40)yz$

base + front & back + two ends

$= 0.75xy + 0.80(xz + yz)$



**81.** $PV = kT,\ 20(2600) = k(300)$

(a) $k = \dfrac{20(2600)}{300} = \dfrac{520}{3}$

(b) $P = \dfrac{kT}{V} = \dfrac{520}{3}\left(\dfrac{T}{V}\right)$

The level curves are of the form: $c = \left(\dfrac{520}{3}\right)\left(\dfrac{T}{V}\right)$

$$V = \frac{520}{3c}T$$

Thus, the level curves are lines through the origin with slope $\dfrac{520}{3c}$.

**83.** (a) Highest pressure at $C$

(b) Lowest pressure at $A$

(c) Highest wind velocity at $B$

**85.** (a) The boundaries between colors represent level curves

(b) No, the colors represent intervals of different lengths, as indicated in the box

(c) You could use more colors, which means using smaller intervals

**87.** False. Let

$f(x, y) = 2xy$

$f(1, 2) = f(2, 1)$, but $1 \neq 2$

**89.** False. Let

$f(x, y) = 5.$

Then, $f(2x, 2y) = 5 \neq 2^2 f(x, y)$.

## Section 12.2 Limits and Continuity

**1.** Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |y - b| < \varepsilon$ whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta = \varepsilon$, we have

$$\sqrt{(y - b)^2} < \varepsilon$$
$$|y - b| < \varepsilon.$$

**3.** $\displaystyle\lim_{(x, y)\to(a, b)}[f(x, y) - g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) - \lim_{(x, y)\to(a, b)} g(x, y) = 5 - 3 = 2$

**5.** $\displaystyle\lim_{(x, y)\to(a, b)}[f(x, y)g(x, y)] = \left[\lim_{(x, y)\to(a, b)} f(x, y)\right]\left[\lim_{(x, y)\to(a, b)} g(x, y)\right] = 5(3) = 15$

**7.** $\displaystyle\lim_{(x, y)\to(2, 1)}(x + 3y^2) = 2 + 3(1)^2 = 5$

Continuous everywhere

**9.** $\displaystyle\lim_{(x, y)\to(2, 4)} \frac{x + y}{x - y} = \frac{2 + 4}{2 - 4} = -3$

Continuous for $x \neq y$

**11.** $\displaystyle\lim_{(x, y)\to(0, 1)} \frac{\arcsin(x/y)}{1 + xy} = \arcsin 0 = 0$

Continuous for $xy \neq -1, y \neq 0, |x/y| \leq 1$

**13.** $\displaystyle\lim_{(x, y)\to(-1, 2)} e^{xy} = e^{-2} = \frac{1}{e^2}$

Continuous everywhere

**15.** $\displaystyle\lim_{(x, y, z)\to(1, 2, 5)} \sqrt{x + y + z} = \sqrt{8} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

**17.** $\displaystyle\lim_{(x, y)\to(0, 0)} e^{xy} = 1$

Continuous everywhere

**19.** $\lim_{(x, y)\to(0, 0)} \ln(x^2 + y^2) = \ln(0) = -\infty$

The limit does not exist.

Continuous except at $(0, 0)$

**21.** $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

| $(x, y)$ | $(1, 0)$ | $(0.5, 0)$ | $(0.1, 0)$ | $(0.01, 0)$ | $(0.001, 0)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | 0 | 0 | 0 | 0 | 0 |

Path: $y = x$

| $(x, y)$ | $(1, 1)$ | $(0.5, 0.5)$ | $(0.1, 0.1)$ | $(0.01, 0.01)$ | $(0.001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function equals $\frac{1}{2}$.

**23.** $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$

Continuous except at $(0, 0)$

Path: $x = y^2$

| $(x, y)$ | $(1, 1)$ | $(0.25, 0.5)$ | $(0.01, 0.1)$ | $(0.0001, 0.01)$ | $(0.000001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

Path: $x = -y^2$

| $(x, y)$ | $(-1, 1)$ | $(-0.25, 0.5)$ | $(-0.01, 0.1)$ | $(-0.0001, 0.01)$ | $(-0.000001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

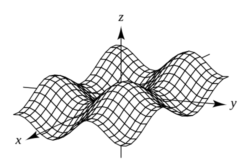
The limit does not exist because along the path $x = y^2$ the function equals $-\frac{1}{2}$, whereas along the path $x = -y^2$ the function equals $\frac{1}{2}$.

**25.** $$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \left(\frac{x^2 + 2xy^2 + y^2}{x^2 + y^2}\right)$$
$$= \lim_{(x, y)\to(0, 0)} \left(1 + \frac{2xy^2}{x^2 + y^2}\right) = 1$$

(same limit for $g$)

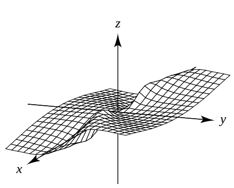
Thus, $f$ is not continuous at $(0, 0)$, whereas $g$ is continuous at $(0, 0)$.

**27.** $\lim_{(x, y)\to(0, 0)} (\sin x + \sin y) = 0$



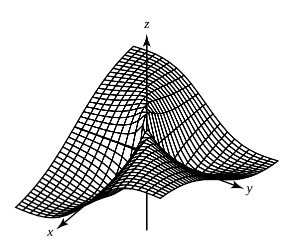

**29.** $\lim_{(x, y)\to(0, 0)} \dfrac{x^2y}{x^4 + 4y^2}$

Does not exist




**31.** $f(x, y) = \dfrac{10xy}{2x^2 + 3y^2}$

The limit does not exist. Use the paths $x = 0$ and $x = y$.



**33.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0}\frac{\sin r^2}{r^2} = \lim_{r\to 0}\frac{2r\cos r^2}{2r} = \lim_{r\to 0}\cos r^2 = 1$

**35.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0}\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = \lim_{r\to 0} r(\cos^3\theta+\sin^3\theta) = 0$

**37.** $f(x, y, z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Continuous except at $(0, 0, 0)$

**39.** $f(x, y, z) = \dfrac{\sin z}{e^x+e^y}$

Continuous everywhere

**41.**

$$\begin{aligned} f(t) &= t^2 \\ g(x, y) &= 3x - 2y \\ f(g(x, y)) &= f(3x - 2y) \\ &= (3x - 2y)^2 \\ &= 9x^2 - 12xy + 4y^2 \end{aligned}$$

Continuous everywhere

**43.**

$$\begin{aligned} f(t) &= \frac{1}{t} \\ g(x, y) &= 3x - 2y \\ f(g(x, y)) &= f(3x - 2y) = \frac{1}{3x-2y} \end{aligned}$$

Continuous for $y \neq \dfrac{3x}{2}$

**45.** $f(x, y) = x^2 - 4y$

(a) $\displaystyle\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0}\frac{[(x+\Delta x)^2 - 4y] - (x^2 - 4y)}{\Delta x}$

$\displaystyle= \lim_{\Delta x\to 0}\frac{2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0}(2x - \Delta x) = 2x$

(b) $\displaystyle\lim_{\Delta y\to 0}\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0}\frac{[x^2 - 4(y+\Delta y)] - (x^2 - 4y)}{\Delta y}$

$\displaystyle= \lim_{\Delta y\to 0}\frac{-4\Delta y}{\Delta y} = \lim_{\Delta y\to 0}(-4) = -4$

**47.** $f(x, y) = 2x + xy - 3y$

(a) $\displaystyle\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0}\frac{[2(x+\Delta x) + (x+\Delta x)y - 3y] - (2x + xy - 3y)}{\Delta x}$

$\displaystyle= \lim_{\Delta x\to 0}\frac{2\Delta x + \Delta xy}{\Delta x} = \lim_{\Delta x\to 0}(2 + y) = 2 + y$

(b) $\displaystyle\lim_{\Delta y\to 0}\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0}\frac{[2x + x(y+\Delta y) - 3(y+\Delta y)] - (2x + xy - 3y)}{\Delta y}$

$\displaystyle= \lim_{\Delta y\to 0}\frac{x\Delta y - 3\Delta y}{\Delta y} = \lim_{\Delta y\to 0}(x - 3) = x - 3$

**49.** See the definition on page 851.

Show that the value of $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ is not the same for two different paths to $(x_0, y_0)$.

**51.** No.

The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y)\to(2, 3)$.

**53.** Since $\lim_{(x, y)\to(a, b)} f(x, y) = L_1$, then for $\varepsilon/2 > 0$, there corresponds $\delta_1 > 0$ such that $|f(x, y) - L_1| < \varepsilon/2$ whenever

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1$.

Since $\lim_{(x, y)\to(a, b)} g(x, y) = L_2$, then for $\varepsilon/2 > 0$, there corresponds $\delta_2 > 0$ such that $|g(x, y) - L_2| < \varepsilon/2$ whenever

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_2$.

Let $\delta$ be the smaller of $\delta_1$ and $\delta_2$. By the triangle inequality, whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have

$$|f(x, y) + g(x, y) - (L_1 + L_2)| = |(f(x, y) - L_1) + (g(x, y) - L_2)| \le |f(x, y) - L_1| + |g(x, y) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = L_1 + L_2$.

**55.** True

**57.** False. Let

$$f(x, y) = \begin{cases} \ln(x^2 + y^2), & (x, y) \ne (0, 0) \\ 0, & x = 0, y = 0 \end{cases}$$

See Exercise 19.

## Section 12.3 Partial Derivatives

**1.** $f_x(4, 1) < 0$

**3.** $f_y(4, 1) > 0$

**5.** $f(x, y) = 2x - 3y + 5$

$f_x(x, y) = 2$

$f_y(x, y) = -3$

**7.** $z = x\sqrt{y}$

$$\frac{\partial z}{\partial x} = \sqrt{y}$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}}$$

**9.** $z = x^2 - 5xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 5y$$

$$\frac{\partial z}{\partial y} = -5x + 6y$$

**11.** $z = x^2e^{2y}$

$$\frac{\partial z}{\partial x} = 2xe^{2y}$$

$$\frac{\partial z}{\partial y} = 2x^2e^{2y}$$

**13.** $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

**15.** $z = \ln\left(\frac{x+y}{x-y}\right) = \ln(x + y) - \ln(x - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x+y} - \frac{1}{x-y} = -\frac{2y}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+y} + \frac{1}{x-y} = \frac{2x}{x^2 - y^2}$$

**17.** $z = \frac{x^2}{2y} + \frac{4y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{4y^2}{x^2} = \frac{x^3 - 4y^3}{x^2y}$$

$$\frac{\partial z}{\partial y} = -\frac{x^2}{2y^2} + \frac{8y}{x} = \frac{-x^3 + 16y^3}{2xy^2}$$

**19.** $h(x, y) = e^{-(x^2+y^2)}$

$h_x(x, y) = -2xe^{-(x^2+y^2)}$

$h_y(x, y) = -2ye^{-(x^2+y^2)}$

**21.** $f(x, y) = \sqrt{x^2 + y^2}$

$$f_x(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

**23.** $z = \tan(2x - y)$

$$\frac{\partial z}{\partial x} = 2\sec^2(2x - y)$$

$$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

**25.** $z = e^y \sin xy$

$$\frac{\partial z}{\partial x} = ye^y \cos xy$$

$$\frac{\partial z}{\partial y} = e^y \sin xy + xe^y \cos xy$$

$$= e^y(x \cos xy + \sin xy)$$

**27.** $f(x, y) = \int_x^y (t^2 - 1)\, dt$

$$= \left[\frac{t^3}{3} - t\right]_x^y = \left(\frac{y^3}{3} - y\right) - \left(\frac{x^3}{3} - x\right)$$

$$f_x(x, y) = -x^2 + 1 = 1 - x^2$$

$$f_y(x, y) = y^2 - 1$$

[You could also use the Second Fundamental Theorem of Calculus.]

**29.** $f(x, y) = 2x + 3y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2(x + \Delta x) + 3y - 2x - 3y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2x + 3(y + \Delta y) - 2x - 3y}{\Delta y} = \lim_{\Delta y \to 0} \frac{3\Delta y}{\Delta y} = 3$$

**31.** $f(x, y) = \sqrt{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x + y} - \sqrt{x + y}\right)\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}{\Delta x\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\left(\sqrt{x + y + \Delta y} - \sqrt{x + y}\right)\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}{\Delta y\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

**33.** $g(x, y) = 4 - x^2 - y^2$

$g_x(x, y) = -2x$

At $(1, 1)$: $g_x(1, 1) = -2$

$g_y(x, y) = -2y$

At $(1, 1)$: $g_y(1, 1) = -2$

**35.** $z = e^{-x} \cos y$

$$\frac{\partial z}{\partial x} = -e^{-x} \cos y$$

At $(0, 0)$: $\dfrac{\partial z}{\partial x} = -1$

$$\frac{\partial z}{\partial y} = -e^{-x} \sin y$$

At $(0, 0)$: $\dfrac{\partial z}{\partial y} = 0$

**37.** $f(x, y) = \arctan \dfrac{y}{x}$

$$f_x(x, y) = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

At $(2, -2)$: $f_x(2, -2) = \dfrac{1}{4}$

$$f_y(x, y) = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

At $(2, -2)$: $f_y(2, -2) = \dfrac{1}{4}$

**39.** $f(x, y) = \dfrac{xy}{x - y}$

$$f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

At $(2, -2)$: $f_x(2, -2) = -\dfrac{1}{4}$

$$f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

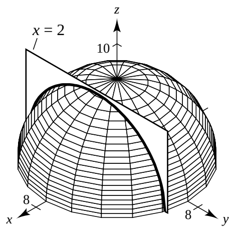
At $(2, -2)$: $f_y(2, -2) = \dfrac{1}{4}$

**41.** $z = \sqrt{49 - x^2 - y^2},\ x = 2,$
$(2, 3, 6)$

Intersecting curve: $z = \sqrt{45 - y^2}$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{45 - y^2}}$$

At $(2, 3, 6)$: $\dfrac{\partial z}{\partial y} = \dfrac{-3}{\sqrt{45 - 9}} = -\dfrac{1}{2}$

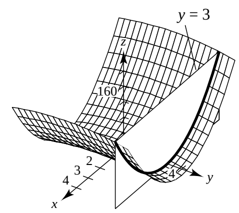



**43.** $z = 9x^2 - y^2,\ y = 3,\ (1, 3, 0)$

Intersecting curve: $z = 9x^2 - 9$

$$\frac{\partial z}{\partial x} = 18x$$

At $(1, 3, 0)$: $\dfrac{\partial z}{\partial x} = 18(1) = 18$




**45.** $f_x(x, y) = 2x + 4y - 4,\ f_y(x, y) = 4x + 2y + 16$

$f_x = f_y = 0$: $2x + 4y = 4$

$4x + 2y = -16$

Solving for $x$ and $y$,

$x = -6$ and $y = 4$.

**47.** $f_x(x, y) = -\dfrac{1}{x^2} + y,\ f_y(x, y) = -\dfrac{1}{y^2} + x$

$f_x = f_y = 0$: $-\dfrac{1}{x^2} + y = 0$ and $-\dfrac{1}{y^2} + x = 0$

$y = \dfrac{1}{x^2}$ and $x = \dfrac{1}{y^2}$

$y = y^4 \implies y = 1 = x$

Points: $(1, 1)$

**49.** (a) The graph is that of $f_y$.

(b) The graph is that of $f_x$.

**51.** $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

**53.** $F(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$

$$= \frac{1}{2}\ln(x^2 + y^2 + z^2)$$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

**55.** $H(x, y, z) = \sin(x + 2y + 3z)$

$H_x(x, y, z) = \cos(x + 2y + 3z)$

$H_y(x, y, z) = 2\cos(x + 2y + 3z)$

$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

**57.** $z = x^2 - 2xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -2$$

$$\frac{\partial z}{\partial y} = -2x + 6y$$

$$\frac{\partial^2 z}{\partial y^2} = 6$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

**59.** $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

**61.** $z = e^x \tan y$

$$\frac{\partial z}{\partial x} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$$

$$\frac{\partial z}{\partial y} = e^x \sec^2 y$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$$

**63.** $z = \arctan \dfrac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

**65.** $z = x \sec y$

$$\frac{\partial z}{\partial x} = \sec y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$$

$$\frac{\partial z}{\partial y} = x \sec y \tan y$$

$$\frac{\partial^2 z}{\partial y^2} = x \sec y(\sec^2 y + \tan^2 y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$$

Therefore, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

There are no points for which $z_x = 0 = z_y$, because

$$\frac{\partial z}{\partial x} = \sec y \neq 0.$$

**67.** $z = \ln\left(\dfrac{x}{x^2 + y^2}\right) = \ln x - \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{(x^2 + y^2)^2}$$

There are no points for which $z_x = z_y = 0$.

**69.** $f(x, y, z) = xyz$

$f_x(x, y, z) = yz$

$f_y(x, y, z) = xz$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = z$

$f_{yx}(x, y, z) = z$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

**71.** $f(x, y, z) = e^{-x} \sin yz$

$f_x(x, y, z) = -e^{-x} \sin yz$

$f_y(x, y, z) = ze^{-x} \cos yz$

$f_{yy}(x, y, z) = -z^2e^{-x} \sin yz$

$f_{xy}(x, y, z) = -ze^{-x} \cos yz$

$f_{yx}(x, y, z) = -ze^{-x} \cos yz$

$f_{yyx}(x, y, z) = z^2e^{-x} \sin yz$

$f_{xyy}(x, y, z) = z^2e^{-x} \sin yz$

$f_{yxy}(x, y, z) = z^2e^{-x} \sin yz$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx}$.

**73.** $z = 5xy$

$$\frac{\partial z}{\partial x} = 5y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial z}{\partial y} = 5x$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0 + 0 = 0.$

**75.** $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$

**77.** $z = \sin(x - ct)$

$$\frac{\partial z}{\partial t} = -c\cos(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$$

$$\frac{\partial z}{\partial x} = \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$$

Therefore, $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}.$

**79.** $z = e^{-t} \cos \dfrac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \cos \frac{x}{c}$$

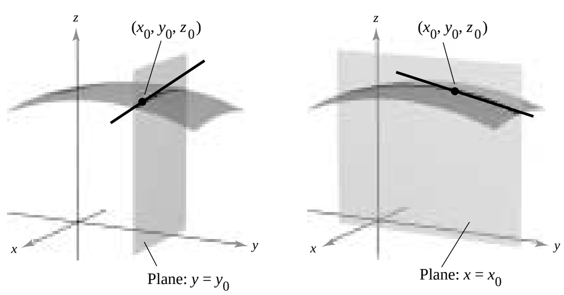
$$\frac{\partial z}{\partial x} = -\frac{1}{c} e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \cos \frac{x}{c}$$

Therefore, $\dfrac{\partial z}{\partial t} = c^2 \dfrac{\partial^2 z}{\partial x^2}.$

**81.** See the definition on page 859.

**83.**

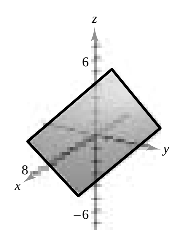



$\dfrac{\partial f}{\partial x}$ denotes the slope of the surface in the $x$-direction.

$\dfrac{\partial f}{\partial y}$ denotes the slope of the surface in the $y$-direction.

**85.** The plane $z = x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} > 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$




**87.** (a) $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left.\frac{\partial C}{\partial x}\right]_{(80,\,20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left.\frac{\partial C}{\partial y}\right]_{(80,\,20)} = 16\sqrt{4} + 205 = 237$$

(b) The fireplace-insert stove results in the cost increasing at a faster rate because $\dfrac{\partial C}{\partial y} > \dfrac{\partial C}{\partial x}.$

**89.** An increase in either price will cause a decrease in demand.

**91.** $T = 500 - 0.6x^2 - 1.5y^2$

$$\frac{\partial T}{\partial x} = -1.2x, \frac{\partial T}{\partial x}(2, 3) = -2.4°/\text{m}$$

$$\frac{\partial T}{\partial y} = -3y = \frac{\partial T}{\partial y}(2, 3) = -9°/\text{m}$$

**93.**

$$PV = mRT$$

$$T = \frac{PV}{mR} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{mR}$$

$$P = \frac{mRT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{mRT}{V^2}$$

$$V = \frac{mRT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{mR}{P}$$

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = \left(\frac{V}{mR}\right)\left(-\frac{mRT}{V^2}\right)\left(\frac{mR}{P}\right)$$

$$= -\frac{mRT}{VP} = -\frac{mRT}{mRT} = -1$$

**95.** (a) $\dfrac{\partial z}{\partial x} = -1.83$

$\dfrac{\partial z}{\partial x} = -1.09$

(b) As the consumption of skim milk $(x)$ increases, the consumption of whole milk $(z)$ decreases.

Similarly, as the consumption of reduced-fat milk $(y)$ increases, the consumption of whole milk $(z)$ decreases.

**97.** $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(x, y) = \dfrac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0/[(\Delta x)^2] - 0}{\Delta x} = 0$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0/[(\Delta y)^2] - 0}{\Delta y} = 0$

(c) $f_{xy}(0, 0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)\Big|_{(0,0)} = \lim\limits_{\Delta y \to 0} \dfrac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\Delta y(-(\Delta y)^4)}{((\Delta y)^2)^2(\Delta y)} = \lim\limits_{\Delta y \to 0} (-1) = -1$

$f_{yx}(0, 0) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)\Big|_{(0,0)} = \lim\limits_{\Delta x \to 0} \dfrac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x((\Delta x)^4)}{((\Delta x)^2)^2(\Delta x)} = \lim\limits_{\Delta x \to 0} 1 = 1$

(d) $f_{yx}$ or $f_{xy}$ or both are not continuous at $(0, 0)$.

**99.** True

**101.** True

## Section 12.4 Differentials

**1.** $z = 3x^2y^3$

$dz = 6xy^3\,dx + 9x^2y^2\,dy$

**3.** $z = \dfrac{-1}{x^2 + y^2}$

$$dz = \frac{2x}{(x^2 + y^2)^2}\,dx + \frac{2y}{(x^2 + y^2)^2}\,dy$$

$$= \frac{2}{(x^2 + y^2)^2}(x\,dx + y\,dy)$$

**5.** $z = x \cos y - y \cos x$

$dz = (\cos y + y \sin x)\,dx + (-x \sin y - \cos x)\,dy = (\cos y + y \sin x)\,dx - (x \sin y + \cos x)\,dy$

**7.** $z = e^x \sin y$

$dz = (e^x \sin y)\,dx + (e^x \cos y)\,dy$

**9.** $w = 2z^3 y \sin x$

$dw = 2z^3 y \cos x\,dx + 2z^3 \sin x\,dy + 6z^2 y \sin x\,dz$

**11.** (a) $f(1, 2) = 4$

$f(1.05, 2.1) = 3.4875$

$\Delta z = f(1.05, 2.1) - f(1, 2) = -0.5125$

(b) $dz = -2x\,dx - 2y\,dy$

$= -2(0.05) - 4(0.1) = -0.5$

**13.** (a) $f(1, 2) = \sin 2$

$f(1.05, 2.1) = 1.05 \sin 2.1$

$\Delta z = f(1.05, 2.1) - f(1, 2) \approx -0.00293$

(b) $dz = \sin y\,dx + x \cos y\,dy$

$= (\sin 2)(0.05) + (\cos 2)(0.1) \approx 0.00385$

**15.** (a) $f(1, 2) = -5$

$f(1.05, 2.1) = -5.25$

$\Delta z = -0.25$

(b) $dz = 3\,dx - 4\,dy$

$= 3(0.05) - 4(0.1) \approx -0.25$

**17.** Let $z = \sqrt{x^2 + y^2}$, $x = 5$, $y = 3$, $dx = 0.05$, $dy = 0.1$. Then: $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

**19.** Let $z = (1 - x^2)/y^2$, $x = 3$, $y = 6$, $dx = 0.05$, $dy = -0.05$. Then: $dz = -\dfrac{2x}{y^2}\,dx + \dfrac{-2(1 - x^2)}{y^3}\,dy$

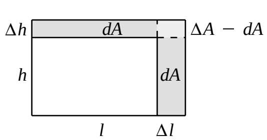
$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

**21.** See the definition on page 869.

**23.** The tangent plane to the surface $z = f(x, y)$ at the point $P$ is a linear approximation of $z$.

**25.** $A = lh$

$dA = l\,dh + h\,dl$




**27.** $V = \dfrac{\pi r^2 h}{3}$

$r = 3$

$h = 6$

$dV = \dfrac{2\pi rh}{3}\,dr + \dfrac{\pi r^2}{3}\,dh = \dfrac{\pi r}{3}(2h\,dr + r\,dh)$

| $\Delta r$ | $\Delta h$ | $dV$ | $\Delta V$ | $\Delta V - dV$ |
|---|---|---|---|---|
| 0.1 | 0.1 | 4.7124 | 4.8391 | 0.1267 |
| 0.1 | −0.1 | 2.8274 | 2.8264 | −0.0010 |
| 0.001 | 0.002 | 0.0565 | 0.0566 | 0.0001 |
| −0.0001 | 0.0002 | −0.0019 | −0.0019 | 0.0000 |

**29.** (a) $dz = -1.83\,dx - 1.09\,dy$

(b) $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy$

$$= -1.83(\pm 0.25) + (-1.09)(\pm 0.25)$$

$$= \pm 0.73$$

Maximum propagated error: $\pm 0.73$

Relative error: $\dfrac{dz}{z} = \dfrac{\pm 0.73}{(-1.83)(7.2) - 1.09(8.5) + 28.7} = \dfrac{\pm 0.73}{6.259} \approx \pm 0.1166 = 11.67\%$

**31.** $V = \pi r^2 h = dV = (2\pi rh)\,dr + (\pi r^2)\,dh$

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$

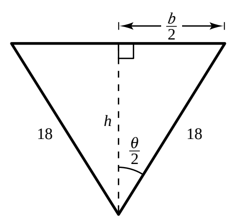
$$= 2(0.04) + (0.02) = 0.10 = 10\%$$

**33.** $A = \frac{1}{2}ab \sin C$

$$dA = \tfrac{1}{2}[(b \sin C)\,da + (a \sin C)\,db + (ab \cos C)\,dC]$$

$$= \tfrac{1}{2}\left[4(\sin 45°)\left(\pm\tfrac{1}{16}\right) + 3(\sin 45°)\left(\pm\tfrac{1}{16}\right) + 12(\cos 45°)(\pm 0.02)\right] \approx \pm 0.24 \text{ in.}^2$$

**35.** (a) $V = \dfrac{1}{2}bhl$




$$= \left(18 \sin \frac{\theta}{2}\right)\left(18 \cos \frac{\theta}{2}\right)(16)(12)$$

$$= 31{,}104 \sin \theta \text{ in.}^3$$

$$= 18 \sin \theta \text{ ft}^3$$

$V$ is maximum when $\sin \theta = 1$ or $\theta = \pi/2$.

(b) $V = \dfrac{s^2}{2}(\sin \theta)l$

$$dV = s(\sin \theta)l\,ds + \frac{s^2}{2}l(\cos \theta)\,d\theta + \frac{s^2}{2}(\sin \theta)\,dl$$

$$= 18\left(\sin \frac{\pi}{2}\right)(16)(12)\left(\frac{1}{2}\right) + \frac{18^2}{2}(16)(12)\left(\cos \frac{\pi}{2}\right)\left(\frac{\pi}{90}\right) + \frac{18^2}{2}\left(\sin \frac{\pi}{2}\right)\left(\frac{1}{2}\right)$$

$$= 1809 \text{ in}^3 \approx 1.047 \text{ ft}^3$$

**37.** $P = \dfrac{E^2}{R}$

$$dP = \frac{2E}{R}\,dE - \frac{E^2}{R^2}\,dR$$

$$\frac{dP}{P} = 2\frac{dE}{E} - \frac{dR}{R} = 2(0.02) - (-0.03) = 0.07 = 7\%$$

**39.** $L = 0.00021\left(\ln \dfrac{2h}{r} - 0.75\right)$

$$dL = 0.00021\left[\frac{dh}{h} - \frac{dr}{r}\right] = 0.00021\left[\frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2}\right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \approx 8.096 \times 10^{-4} \pm dL = 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro–henrys}$$

**41.** $z = f(x, y) = x^2 - 2x + y$

$$\begin{aligned}
\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\
&= (x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)) - (x^2 - 2x + y)\\
&= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y)\\
&= (2x - 2)\,\Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y)\\
&= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = \Delta x \text{ and } \epsilon_2 = 0.
\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

**43.** $z = f(x, y) = x^2 y$

$$\begin{aligned}
\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\
&= (x^2 + 2x(\Delta x) + (\Delta x)^2)(y + \Delta y) - x^2 y\\
&= 2xy(\Delta x) + y(\Delta x)^2 + x^2\Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2\,\Delta y\\
&= 2xy(\Delta x) + x^2\Delta y + (y\Delta x)\,\Delta x + [2x\Delta x + (\Delta x)^2]\,\Delta y\\
&= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = y(\Delta x) \text{ and } \epsilon_2 = 2x\Delta x + (\Delta x)^2.
\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

**45.** $f(x, y) = \begin{cases} \dfrac{3x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(0, 0) = \displaystyle\lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$

$f_y(0, 0) = \displaystyle\lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$

Thus, the partial derivatives exist at $(0, 0)$.

(b) Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \to 0} \frac{3x}{x^2 + 1} = 0$

Along the curve $y = x^2$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$

$f$ is not continuous at $(0, 0)$. Therefore, $f$ is not differentiable at $(0, 0)$. (See Theroem 12.5)

**47.** Essay. For example, we can use the equation $F = ma$:

$$dF = \frac{\partial F}{\partial m}\,dm + \frac{\partial F}{\partial a}\,da = a\,dm + m\,da.$$

## Section 12.5 Chain Rules for Functions of Several Variables

**1.** $w = x^2 + y^2$

$x = e^t$

$y = e^{-t}$

$$\frac{dw}{dt} = 2xe^t + 2y(-e^{-t}) = 2(e^{2t} - e^{-2t})$$

**3.** $w = x \sec y$

$x = e^t$

$y = \pi - t$

$$\begin{aligned}\frac{dw}{dt} &= (\sec y)(e^t) + (x \sec y \tan y)(-1)\\ &= e^t \sec(\pi - t)[1 - \tan(\pi - t)]\\ &= -e^t(\sec t + \sec t \tan t)\end{aligned}$$

**5.** $w = xy,\ x = 2 \sin t,\ y = \cos t$

(a) $\dfrac{dw}{dt} = 2y \cos t + x(-\sin t) = 2y \cos t - x \sin t$

$= 2(\cos^2 t - \sin^2 t) = 2 \cos 2t$

(b) $w = 2 \sin t \cos t = \sin 2t,\ \dfrac{dw}{dt} = 2 \cos 2t$

**7.** $w = x^2 + y^2 + z^2$

$x = e^t \cos t$

$y = e^t \sin t$

$z = e^t$

(a) $\dfrac{dw}{dt} = 2x(-e^t \sin t + e^t \cos t) + 2y(e^t \cos t + e^t \sin t) + 2ze^t = 4e^{2t}$

(b) $w = 2e^{2t},\ \dfrac{dw}{dt} = 4e^{2t}$

**9.** $w = xy + xz + yz,\ x = t - 1,\ y = t^2 - 1,\ z = t$

(a) $\dfrac{dw}{dt} = (y + z) = (x + z)(2t) + (x + y)$

$= (t^2 - 1 + t) + (t - 1 + 1)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)$

(b) $w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$

$$\frac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)$$

**11.** Distance $= f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2}$

$$f'(t) = \frac{1}{2}[(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2]^{-1/2}$$

$$[[2(10 \cos 2t - 7 \cos t)(-20 \sin 2t + 7 \sin t)] + [2(6 \sin 2t - 4 \sin t)(12 \cos 2t - 4 \cos t)]]$$

$$f'\left(\frac{\pi}{2}\right) = \frac{1}{2}[(-10)^2 + 4^2]^{-1/2}[[2(-10)(7)] + (2(-4)(-12)]$$

$$= \frac{1}{2}(116)^{-1/2}(-44) = \frac{22}{2\sqrt{29}} = \frac{-11\sqrt{29}}{20} \approx -2.04$$

**13.** $w = \arctan(2xy),\ x = \cos t,\ y = \sin t,\ t = 0$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2y}{1 + (4x^2y^2)}(-\sin t) + \frac{2x}{1 + (4x^2y^2)}(\cos t)$$

$$= \frac{2\sin t}{1 + 4\cos^2 t\sin^2 t}(-\sin t) + \frac{2\cos t}{1 + 4\cos^2 t\sin^2 t}(\cos t)$$

$$= \frac{2\cos^2 t - 2\sin^2 t}{1 + 4\cos^2 t\sin^2 t}$$

$$\frac{d^2w}{dt^2} = \frac{(1 + 4\cos^2 t\sin^2 t)(-8\cos t\sin t) - (2\cos^2 t - 2\sin^2 t)(8\cos^3 t\sin t - 8\sin^3 t\cos t)}{(1 + 4\cos^2 t\sin^2 t)^2}$$

$$= \frac{-8\cos t\sin t(1 + 2\sin^4 t + 2\cos^4 t)}{(1 + 4\cos^2 t\sin^2 t)^2}$$

At $t = 0$, $\dfrac{d^2w}{dt^2} = 0$.

**15.** $w = x^2 + y^2$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2x + 2y = 2(x + y) = 4s$$

$$\frac{\partial w}{\partial t} = 2x + 2y(-1) = 2(x - y) = 4t$$

When $s = 2$ and $t = -1$,

$$\frac{\partial w}{\partial s} = 8 \text{ and } \frac{\partial w}{\partial t} = -4.$$

**17.** $w = x^2 - y^2$

$x = s\cos t$

$y = s\sin t$

$$\frac{\partial w}{\partial s} = 2x\cos t - 2y\sin t$$

$$= 2s\cos^2 t - 2s\sin^2 t = 2s\cos 2t$$

$$\frac{\partial w}{\partial t} = 2x(-s\sin t) - 2y(s\cos t) = -2s^2\sin 2t$$

When $s = 3$ and $t = \dfrac{\pi}{4}$, $\dfrac{\partial w}{\partial s} = 0$ and $\dfrac{\partial w}{\partial t} = -18$.

**19.** $w = x^2 - 2xy + y^2,\ x = r + \theta,\ y = r - \theta$

(a) $$\frac{\partial w}{\partial r} = (2x - 2y)(1) + (-2x + 2y)(1) = 0$$

$$\frac{\partial w}{\partial \theta} = (2x - 2y)(1) + (-2x + 2y)(-1)$$

$$= 4x - 4y = 4(x - y)$$

$$= 4[(r + \theta) - (r - \theta)] = 8\theta$$

(b) $$w = (r + \theta)^2 - 2(r + \theta)(r - \theta) + (r - \theta)^2$$

$$= (r^2 + 2r\theta + \theta^2) - 2(r^2 - \theta^2) + (r^2 - 2r\theta + \theta^2)$$

$$= 4\theta^2$$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 8\theta$$

**21.** $w = \arctan\dfrac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

(a) $\dfrac{\partial w}{\partial r} = \dfrac{-y}{x^2+y^2}\cos\theta + \dfrac{x}{x^2+y^2}\sin\theta = \dfrac{-r\sin\theta\cos\theta}{r^2} + \dfrac{r\cos\theta\sin\theta}{r^2} = 0$

$\dfrac{\partial w}{\partial\theta} = \dfrac{-y}{x^2+y^2}(-r\sin\theta) + \dfrac{x}{x^2+y^2}(r\cos\theta) = \dfrac{-(r\sin\theta)(-r\sin\theta)}{r^2} + \dfrac{(r\cos\theta)(r\cos\theta)}{r^2} = 1$

(b) $w = \arctan\dfrac{r\sin\theta}{r\cos\theta} = \arctan(\tan\theta) = \theta$

$\dfrac{\partial w}{\partial r} = 0$

$\dfrac{\partial w}{\partial\theta} = 1$

**23.** $w = xyz,\ x = s+t, y = s-t,\ z = st^2$

$$\begin{aligned}\frac{\partial w}{\partial s} &= yz(1) + xz(1) + xy(t^2)\\ &= (s-t)st^2 + (s+t)st^2 + (s+t)(s-t)t^2\\ &= 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2-t^2)\\ \frac{\partial w}{\partial t} &= yz(1) + xz(-1) + xy(2st)\\ &= (s-t)st^2 - (s+t)st^2 + (s+t)(s-t)(2st)\\ &= -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3 = 2st(s^2-2t^2)\end{aligned}$$

**25.** $w = ze^{x/y}, x = s-t, y = s+t, z = st$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{z}{y}e^{x/y}(1) + -\frac{zx}{y^2}e^{x/y}(1) + e^{x/y}(t)\\ &= e^{(s-t/s+t)}\left[\frac{st}{s+t} - \frac{(s-t)st}{(s+t)^2} + t\right]\\ &= e^{(s-t/s+t)}\left[\frac{st(s+t) - s^2t + st^2 + t(s+t)^2}{(s+t)^2}\right]\\ &= e^{(s-t/s+t)}\frac{t(s^2+4st+t^2)}{(s+t)^2}\\ \frac{\partial w}{\partial t} &= \frac{z}{y}e^{x/y}(-1) + -\frac{zx}{y^2}e^{x/y}(1) + e^{x/y}(s)\\ &= e^{(s-t/s+t)}\left[-\frac{st}{s+t} - \frac{st(s-t)}{(s+t)^2} + s\right]\\ &= e^{(s-t/s+t)}\left[\frac{-st(s+t) - st(s-t) + s(s+t)^2}{(s+t)^2}\right]\\ &= e^{(s-t/s+t)}\frac{s(s^2+t^2)}{(s+t)^2}\end{aligned}$$

**27.** $x^2 - 3xy + y^2 - 2x + y - 5 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{2x-3y-2}{-3x+2y+1} = \frac{3y-2x+2}{2y-3x+1}$$

**29.** $\ln\sqrt{x^2+y^2} + xy = 4$

$\dfrac{1}{2}\ln(x^2+y^2) + xy - 4 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{\dfrac{x}{x^2+y^2}+y}{\dfrac{y}{x^2+y^2}+x} = -\frac{x+x^2y+y^3}{y+xy^2+x^3}$$

**31.** $F(x,y,z) = x^2+y^2+z^2-25$

$F_x = 2x$

$F_y = 2y$

$F_z = 2z$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{x}{z}$

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{y}{z}$

**33.** $F(x,y,z) = \tan(x+y) + \tan(y+z) - 1$

$F_x = \sec^2(x+y)$

$F_y = \sec^2(x+y) + \sec^2(y+z)$

$F_z = \sec^2(y+z)$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\sec^2(x+y)}{\sec^2(y+z)}$

$$\begin{aligned}\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{\sec^2(x+y)+\sec^2(y+z)}{\sec^2(y+z)}\\ &= -\left(\frac{\sec^2(x+y)}{\sec^2(y+z)} + 1\right)\end{aligned}$$

**35.** $x^2 + 2yz + z^2 - 1 = 0$

(i) $2x + 2y\dfrac{\partial z}{\partial x} + 2z\dfrac{\partial z}{\partial x} = 0$ implies $\dfrac{\partial z}{\partial x} = -\dfrac{x}{y+z}$.

(ii) $2y\dfrac{\partial z}{\partial y} + 2z + 2z\dfrac{\partial z}{\partial y} = 0$ implies $\dfrac{\partial z}{\partial y} = -\dfrac{z}{y+z}$.

**37.** $e^{xz} + xy = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

**39.** $F(x, y, z, w) = xyz + xzw - yzw + w^2 - 5$

$$F_x = yz + zw$$
$$F_y = xz - zw$$
$$F_z = xy + xw - yw$$
$$F_w = xz - yz + 2w$$
$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{z(y + w)}{xz - yz + 2w}$$
$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{z(x - w)}{xz - yz + 2w}$$
$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{xy + xw - yw}{xz - yz + 2w}$$

**41.** $F(x, y, z, w) = \cos xy + \sin yz + wz - 20$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{y \sin xy}{z}$$
$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$
$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = -\frac{y \cos zy + w}{z}$$

**43.** $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{xy}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

$$xf_x(x, y) + yf_y(x, y) = x\left(\frac{y^3}{(x^2 + y^2)^{3/2}}\right) + y\left(\frac{x^3}{(x^2 + y^2)^{3/2}}\right)$$
$$= \frac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y)$$

**45.** $f(x, y) = e^{x/y}$

$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$

Degree: 0

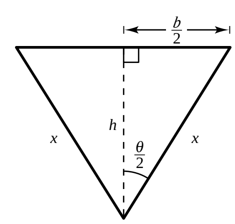
$$xf_x(x, y) + yf_y(x, y) = x\left(\frac{1}{y}e^{x/y}\right) + y\left(-\frac{x}{y^2}e^{x/y}\right) = 0$$

**47.** $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$ (Page 876)

**49.** $w = f(x, y)$ is the explicit form of a function of two variables, as in $z = x^2 + y^2$. The implicit form is of the form $F(x, y, z) = 0$, as in $z - x^2 - y^2 = 0$.

**51.** $A = \dfrac{1}{2}bh = \left(x \sin\dfrac{\theta}{2}\right)\left(x \cos\dfrac{\theta}{2}\right) = \dfrac{x^2}{2}\sin\theta$

$$\frac{dA}{dt} = x \sin\theta\frac{dx}{dt} + \frac{x^2}{2}\cos\theta\frac{d\theta}{dt}$$
$$= 6\left(\sin\frac{\pi}{4}\right)\left(\frac{1}{2}\right) + \frac{6^2}{2}\left(\cos\frac{\pi}{4}\right)\left(\frac{\pi}{90}\right) = \frac{3\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{10}\ \text{m}^2/\text{hr}$$



**53.** (a) $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \frac{1}{3}\pi[2(12)(36)(6) + (12)^2(-4)] = 1536\pi \text{ in.}^3/\text{min}$$

(b) $S = \pi r\sqrt{r^2 + h^2} + \pi r^2$ (Surface area includes base.)

$$\frac{dS}{dt} = \pi\left[\left(\sqrt{r^2+h^2} + \frac{r^2}{\sqrt{r^2+h^2}} + 2r\right)\frac{dr}{dt} + \frac{rh}{\sqrt{r^2+h^2}}\frac{dh}{dt}\right]$$

$$= \pi\left[\left(\sqrt{12^2+36^2} + \frac{144}{\sqrt{12^2+36^2}} + 2(12)\right)(6) + \frac{36(12)}{\sqrt{12^2+36^2}}(-4)\right]$$

$$= \pi\left[\left(12\sqrt{10} + \frac{12}{\sqrt{10}}\right)(6) + 144 + \frac{36}{\sqrt{10}}(-4)\right]$$

$$= \frac{648\pi}{\sqrt{10}} + 144\pi \text{ in.}^2/\text{min} = \frac{36\pi}{5}\left(20 + 9\sqrt{10}\right) \text{ in.}^2/\text{min}$$

**55.** $I = \frac{1}{2}m(r_1^2 + r_2^2)$

$$\frac{dI}{dt} = \frac{1}{2}m\left[2r_1\frac{dr_1}{dt} + 2r_2\frac{dr_2}{dt}\right] = m[(6)(2) + (8)(2)] = 28m \text{ cm}^2/\text{sec}$$

**57.** (a)

$$\tan\phi = \frac{2}{x}$$

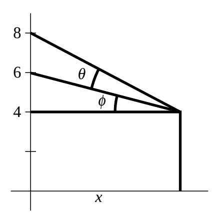
$$\tan(\theta + \phi) = \frac{4}{x}$$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{4}{x}$$

$$\frac{\tan\theta + (2/x)}{1 - (2/x)\tan\theta} = \frac{4}{x}$$

$$x\tan\theta + 2 = 4 - \frac{8}{x}\tan\theta$$

$$x^2\tan\theta - 2x + 8\tan\theta = 0$$




(b) $F(x, \theta) = (x^2 + 8)\tan\theta - 2x = 0$

$$\frac{d\theta}{dx} = -\frac{F_x}{F_\theta} = -\frac{2x\tan\theta - 2}{\sec^2\theta(x^2+8)} = \frac{2\cos^2\theta - 2x\sin\theta\cos\theta}{x^2+8}$$

(c) $\frac{d\theta}{dx} = 0 \implies 2\cos^2\theta = 2x\sin\theta\cos\theta \implies \cos\theta = x\sin\theta \implies \tan\theta = \frac{1}{x}$

Thus, $x^2\left(\frac{1}{x}\right) - 2x + 8\left(\frac{1}{x}\right) = 0 \implies \frac{8}{x} = x \implies x = 2\sqrt{2}$ ft.

**59.**

$$w = f(x, y)$$

$$x = u - v$$

$$y = v - u$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

**61.** $w = f(x, y),\ x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

(a)

$$r\cos\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\cos^2\theta + \frac{\partial w}{\partial y}r\sin\theta\cos\theta$$

$$-\sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\sin^2\theta) - \frac{\partial w}{\partial x}r\sin\theta\cos\theta$$

$$r\cos\theta\frac{\partial w}{\partial r} - \sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\cos^2\theta + r\sin^2\theta)$$

$$r\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}(r\cos\theta) - \frac{\partial w}{\partial \theta}\sin\theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$r\sin\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\sin\theta\cos\theta + \frac{\partial w}{\partial y}r\sin^2\theta$$

$$\cos\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta\cos\theta) + \frac{\partial w}{\partial y}(r\cos^2\theta)$$

$$r\sin\theta\frac{\partial w}{\partial r} + \cos\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial y}(r\sin^2\theta + r\cos^2\theta)$$

$$r\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}r\sin\theta + \frac{\partial w}{\partial \theta}\cos\theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r}$$

(b) $$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta -$$

$$2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

**63.** Given $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) = r\left[\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta\right]$$

Therefore, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -r\left[-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta\right]$$

Therefore, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

## Section 12.6 Directional Derivatives and Gradients

**1.** $f(x, y) = 3x - 4xy + 5y$

$$\mathbf{v} = \frac{1}{2}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j}\right)$$

$$\nabla f(x, y) = (3 - 4y)\mathbf{i} + (-4x + 5)\mathbf{j}$$

$$\nabla f(1, 2) = -5\mathbf{i} + \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \frac{1}{2}\left(-5 + \sqrt{3}\right)$$

**3.** $f(x, y) = xy$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\nabla f(2, 3) = 3\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(2, 3) = \nabla f(2, 3) \cdot \mathbf{u} = \frac{5\sqrt{2}}{2}$$

**5.** $g(x, y) = \sqrt{x^2 + y^2}$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$$

$$\nabla g(3, 4) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} g(3, 4) = \nabla g(3, 4) \cdot \mathbf{u} = -\frac{7}{25}$$

**7.** $h(x, y) = e^x \sin y$

$$\mathbf{v} = -\mathbf{i}$$

$$\nabla h = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$$

$$h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$$

$$D_{\mathbf{u}} h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$$

**9.** $f(x, y, z) = xy + yz + xz$

$$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

$$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$$

$$D_{\mathbf{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2\sqrt{6}}{3}$$

**11.** $h(x, y, z) = x \arctan yz$

$$\mathbf{v} = \langle 1, 2, -1 \rangle$$

$$\nabla h(x, y, z) = \arctan yz\,\mathbf{i} + \frac{xz}{1 + (yz)^2}\mathbf{j} + \frac{xy}{1 + (yz)^2}\mathbf{k}$$

$$\nabla h(4, 1, 1) = \frac{\pi}{4}\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$D_{\mathbf{u}} h(4, 1, 1) = \nabla h(4, 1, 1) \cdot \mathbf{u} = \frac{\pi + 8}{4\sqrt{6}} = \frac{(\pi + 8)\sqrt{6}}{24}$$

**13.** $f(x, y) = x^2 + y^2$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y = \sqrt{2}(x + y)$$

**15.** $f(x, y) = \sin(2x - y)$

$$\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla f = 2\cos(2x - y)\mathbf{i} - \cos(2x - y)\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \cos(2x - y) + \frac{\sqrt{3}}{2}\cos(2x - y)$$

$$= \left(\frac{2 + \sqrt{3}}{2}\right)\cos(2x - y)$$

**17.** $f(x, y) = x^2 + 4y^2$

$$\mathbf{v} = -2\mathbf{i} - 2\mathbf{j}$$

$$\nabla f = 2x\,\mathbf{i} + 8y\,\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

$$D_{\mathbf{u}} f = -\frac{2}{\sqrt{2}}x - \frac{8}{\sqrt{2}}y = -\sqrt{2}(x + 4y)$$

At $P = (3, 1)$, $D_{\mathbf{u}} f = -7\sqrt{2}$.

**19.** $h(x, y, z) = \ln(x + y + z)$

$$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$D_{\mathbf{u}} h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$$

**21.** $f(x, y) = 3x - 5y^2 + 10$

$$\nabla f(x, y) = 3\mathbf{i} - 10y\mathbf{j}$$

$$\nabla f(2, 1) = 3\mathbf{i} - 10\mathbf{j}$$

**23.** $z = \cos(x^2 + y^2)$

$$\nabla z(x, y) = -2x\sin(x^2 + y^2)\mathbf{i} - 2y\sin(x^2 + y^2)\mathbf{j}$$

$$\nabla z(3, -4) = -6\sin 25\mathbf{i} + 8\sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$$

**25.** $w = 3x^2y - 5yz + z^2$

$$\nabla w(x, y, z) = 6xy\mathbf{i} + (3x^2 - 5z)\mathbf{j} + (2z - 5y)\mathbf{k}$$

$$\nabla w(1, 1, -2) = 6\mathbf{i} + 13\mathbf{j} - 9\mathbf{k}$$

**27.** $\overrightarrow{PQ} = 2\mathbf{i} + 4\mathbf{j}$, $\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} = \frac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j},\ \nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$$

$$D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u} = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

**29.** $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$$\nabla f(x, y) = -e^{-x}\cos y\mathbf{i} - e^{-x}\sin y\mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

**31.** $h(x, y) = x\tan y$

$$\nabla h(x, y) = \tan y\mathbf{i} + x\sec^2 y\mathbf{j}$$

$$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$$

$$\left\|\nabla h\left(2, \frac{\pi}{4}\right)\right\| = \sqrt{17}$$

**33.** $g(x, y) = \ln\sqrt[3]{x^2 + y^2} = \frac{1}{3}\ln(x^2 + y^2)$

$$\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$$

$$\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$$

$$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$$

**35.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\|\nabla f(1, 4, 2)\| = 1$$

**37.** $f(x, y, z) = xe^{yz}$

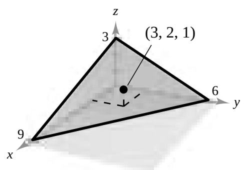
$$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$$

$$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$$

$$\|\nabla f(2, 0, -4)\| = \sqrt{65}$$

**For Exercises 39–45, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta$.**

**39.** $f(x, y) = 3 - \dfrac{x}{3} - \dfrac{y}{2}$




**41.** (a) $D_{4\pi/3} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$

$$= \frac{2 + 3\sqrt{3}}{12}$$

(b) $D_{-\pi/6} f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

$$= \frac{3 - 2\sqrt{3}}{12}$$

**43.** (a) $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$

$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

$D_\mathbf{u} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$

(b) $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{10}$

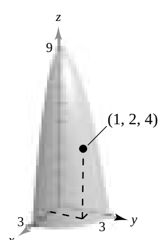
$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

$D_\mathbf{u} f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$

**45.** $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

**For Exercises 47 and 49, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x\cos\theta - 2y\sin\theta = -2(x\cos\theta + y\sin\theta)$.**

**47.** $f(x, y) = 9 - x^2 - y^2$




**49.** $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

**51.** (a) In the direction of the vector $-4\mathbf{i} + \mathbf{j}$.

(b) $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

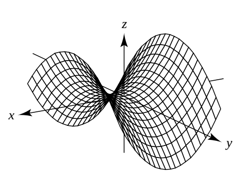
$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$

(Same direction as in part (a).)

(c) $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$, the direction opposite that of the gradient.

**53.** $f(x, y) = x^2 - y^2$, $(4, -3, 7)$

(a)

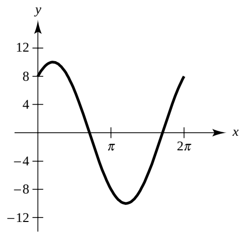



**—CONTINUED–**

**53. —CONTINUED—**

(b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x \cos\theta - 2y \sin\theta$

$D_{\mathbf{u}} f(4, -3) = 8\cos\theta + 6\sin\theta$

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(c) Zeros: $\theta \approx 2.21, 5.36$

These are the angles $\theta$ for which $D_{\mathbf{u}} f(4, 3)$ equals zero.

(d) $g(\theta) = D_{\mathbf{u}} f(4, -3) = 8\cos\theta + 6\sin\theta$

$g'(\theta) = -8\sin\theta + 6\cos\theta$

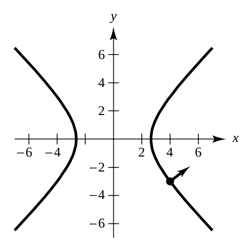
Critical numbers: $\theta \approx 0.64, 3.79$

These are the angles for which $D_{\mathbf{u}} f(4, -3)$ is a maximum (0.64) and minimum (3.79).

(e) $\|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(3)\mathbf{j}\| = \sqrt{64 + 36} = 10$, the maximum value of $D_{\mathbf{u}} f(4, -3)$, at $\theta = 0.64$.

(f) $f(x, y) = x^2 - y^2 = 7$

$\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$ is perpendicular to the level curve at $(4, -3)$.

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**55.** $f(x, y) = x^2 + y^2$

$c = 25,\ P = (3, 4)$

$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$

$x^2 + y^2 = 25$

$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$

**57.** $f(x, y) = \dfrac{x}{x^2 + y^2}$

$c = \dfrac{1}{2},\ P = (1, 1)$

$\nabla f(x, y) = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}\mathbf{i} - \dfrac{2xy}{(x^2 + y^2)^2}\mathbf{j}$

$\dfrac{x}{x^2 + y^2} = \dfrac{1}{2}$

$x^2 + y^2 - 2x = 0$

$\nabla f(1, 1) = -\dfrac{1}{2}\mathbf{j}$

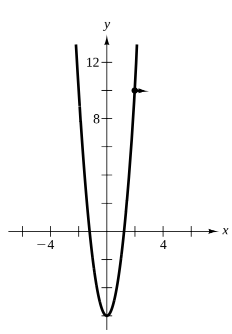
**59.** $4x^2 - y = 6$

$f(x, y) = 4x^2 - y$

$\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$

$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$

$$\frac{\nabla f(2, 10)}{\|\nabla f(2, 10)\|} = \frac{1}{\sqrt{257}}(16\mathbf{i} - \mathbf{j}) = \frac{\sqrt{257}}{257}(16\mathbf{i} - \mathbf{j})$$



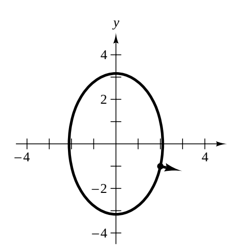

**61.** $9x^2 + 4y^2 = 40$

$f(x, y) = 9x^2 + 4y^2$

$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$

$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$

$$\frac{\nabla f(2, -1)}{\|\nabla f(2, -1)\|} = \frac{1}{\sqrt{85}}(9\mathbf{i} - 2\mathbf{j}) = \frac{\sqrt{85}}{85}(9\mathbf{i} - 2\mathbf{j})$$



**63.** $T = \dfrac{x}{x^2 + y^2}$

$$\nabla T = \frac{y^2 - x^2}{(x^2 + y^2)^2}\mathbf{i} - \frac{2xy}{(x^2 + y^2)^2}\mathbf{j}$$

$$\nabla T(3, 4) = \frac{7}{625}\mathbf{i} - \frac{24}{625}\mathbf{j} = \frac{1}{625}(7\mathbf{i} - 24\mathbf{j})$$

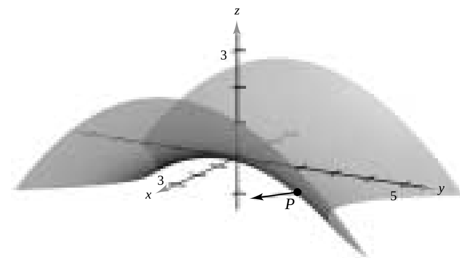
**65.** See the definition, page 885.

**67.** Let $f(x, y)$ be a function of two variables and $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ a unit vector.

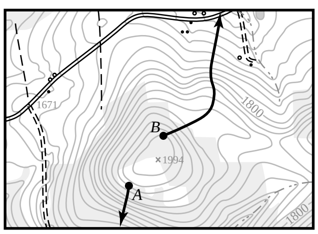
(a) If $\theta = 0°$, then $D_{\mathbf{u}} f = \dfrac{\partial f}{\partial x}$.

(b) If $\theta = 90°$, then $D_{\mathbf{u}} f = \dfrac{\partial f}{\partial y}$.

**69.**




**71.**




**73.** $T(x, y) = 400 - 2x^2 - y^2$, $P = (10, 10)$

$$\frac{dx}{dt} = -4x \qquad \frac{dy}{dt} = -2y$$

$$x(t) = C_1e^{-4t} \qquad y(t) = C_2e^{-2t}$$

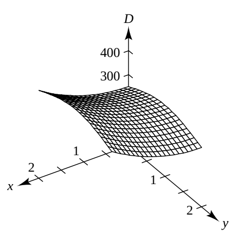
$$10 = x(0) = C_1 \qquad 10 = y(0) = C_2$$

$$x(t) = 10e^{-4t} \qquad y(t) = 10e^{-2t}$$

$$x = \frac{y^2}{10} \qquad y^2(t) = 100e^{-4t}$$

$$y^2 = 10x$$

**75.** (a)




(b) The graph of $-D = -250 - 30x^2 - 50\sin(\pi y/2)$ would model the ocean floor.

(c) $D(1, 0.5) = 250 + 30(1) + 50\sin\dfrac{\pi}{4} \approx 315.4$ ft

(d) $\dfrac{\partial D}{\partial x} = 60x$ and $\dfrac{\partial D}{\partial x}(1, 0.5) = 60$

(e) $\dfrac{\partial D}{\partial y} = 25\pi\cos\dfrac{\pi y}{2}$ and $\dfrac{\partial D}{\partial y}(1, 0.5) = 25\pi\cos\dfrac{\pi}{4} \approx 55.5$

(f) $\nabla D = 60x\mathbf{i} + 25\pi\cos\left(\dfrac{\pi y}{2}\right)\mathbf{j}$

$\nabla D(1, 0.5) = 60\mathbf{i} + 55.5\mathbf{j}$

**77.** True

**79.** True

**81.** Let $f(x, y, z) = e^x\cos y + \dfrac{z^2}{2} + C$. Then $\nabla f(x, y, z) = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j} + z\mathbf{k}$.

## Section 12.7 Tangent Planes and Normal Lines

**1.** $F(x, y, z) = 3x - 5y + 3z - 15 = 0$

$$3x - 5y + 3z = 15 \text{ Plane}$$

**3.** $F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$

$$4x^2 + 9y^2 = 4z^2 \text{ Elliptic cone}$$

**5.** $F(x, y, z) = x + y + z - 4$

$$\nabla F = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

**7.** $F(x, y, z) = \sqrt{x^2 + y^2} - z$

$$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$$

$$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right) = \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) = \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

**9.** $F(x, y, z) = x^2y^4 - z$

$$\nabla F(x, y, z) = 2xy^4\mathbf{i} + 4x^2y^3\mathbf{j} - \mathbf{k}$$

$$\nabla F(1, 2, 16) = 32\mathbf{i} + 32\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{2049}}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k}) = \frac{\sqrt{2049}}{2049}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k})$$

**11.** $F(x, y, z) = \ln\left(\frac{x}{y - z}\right) = \ln x - \ln(y - z)$

$$\nabla F(x, y, z) = \frac{1}{x}\mathbf{i} - \frac{1}{y - z}\mathbf{j} + \frac{1}{y - z}\mathbf{k}$$

$$\nabla F(1, 4, 3) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

**13.** $F(x, y, z) = -x\sin y + z - 4$

$$\nabla F(x, y, z) = -\sin y\mathbf{i} - x\cos y\mathbf{j} + \mathbf{k}$$

$$\nabla F\left(6, \frac{\pi}{6}, 7\right) = -\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{113}}\left(-\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}\right) = \frac{1}{\sqrt{113}}\left(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k}\right) = \frac{\sqrt{113}}{113}\left(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k}\right)$$

**15.** $f(x, y) = 25 - x^2 - y^2$, $(3, 1, 15)$

$$F(x, y, z) = 25 - x^2 - y^2 - z$$

$F_x(x, y, z) = -2x$ $\quad F_y(x, y, z) = -2y$ $\quad F_z(x, y, z) = -1$

$F_x(3, 1, 15) = -6$ $\quad F_y(3, 1, 15) = -2$ $\quad F_z(3, 1, 15) = -1$

$$-6(x - 3) - 2(y - 1) - (z - 15) = 0$$

$$0 = 6x + 2y + z - 35$$

$$6x + 2y + z = 35$$

**17.** $f(x, y) = \sqrt{x^2 + y^2}$, $(3, 4, 5)$

$F(x, y, z) = \sqrt{x^2 + y^2} - z$

$$F_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}} \qquad F_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2}} \qquad F_z(x, y, z) = -1$$

$$F_x(3, 4, 5) = \frac{3}{5} \qquad F_y(3, 4, 5) = \frac{4}{5} \qquad F_z(3, 4, 5) = -1$$

$$\frac{3}{5}(x - 3) + \frac{4}{5}(y - 4) - (z - 5) = 0$$

$$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$$

$$3x + 4y - 5z = 0$$

**19.** $g(x, y) = x^2 - y^2$, $(5, 4, 9)$

$G(x, y, z) = x^2 - y^2 - z$

$$G_x(x, y, z) = 2x \qquad G_y(x, y, z) = -2y \qquad G_z(x, y, z) = -1$$

$$G_x(5, 4, 9) = 10 \qquad G_y(5, 4, 9) = -8 \qquad G_z(5, 4, 9) = -1$$

$$10(x - 5) - 8(y - 4) - (z - 9) = 0$$

$$10x - 8y - z = 9$$

**21.** $z = e^x(\sin y + 1)$, $\left(0, \frac{\pi}{2}, 2\right)$

$F(x, y, z) = e^x(\sin y + 1) - z$

$$F_x(x, y, z) = e^x(\sin y + 1) \qquad F_y(x, y, z) = e^x \cos y \qquad F_z(x, y, z) = -1$$

$$F_x\left(0, \frac{\pi}{2}, 2\right) = 2 \qquad F_y\left(0, \frac{\pi}{2}, 2\right) = 0 \qquad F_z\left(0, \frac{\pi}{2}, 2\right) = -1$$

$$2x - z = -2$$

**23.** $h(x, y) = \ln \sqrt{x^2 + y^2}$, $(3, 4, \ln 5)$

$H(x, y, z) = \ln \sqrt{x^2 + y^2} - z = \frac{1}{2}\ln(x^2 + y^2) - z$

$$H_x(x, y, z) = \frac{x}{x^2 + y^2} \qquad H_y(x, y, z) = \frac{y}{x^2 + y^2} \qquad H_z(x, y, z) = -1$$

$$H_x(3, 4, \ln 5) = \frac{3}{25} \qquad H_y(3, 4, \ln 5) = \frac{4}{25} \qquad H_z(3, 4, \ln 5) = -1$$

$$\frac{3}{25}(x - 3) + \frac{4}{25}(y - 4) - (z - \ln 5) = 0$$

$$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$$

$$3x + 4y - 25z = 25(1 - \ln 5)$$

**25.** $x^2 + 4y^2 + z^2 = 36$, $(2, -2, 4)$

$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$

$$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 8y \qquad F_z(x, y, z) = 2z$$

$$F_x(2, -2, 4) = 4 \qquad F_y(2, -2, 4) = -16 \qquad F_z(2, -2, 4) = 8$$

$$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$$

$$(x - 2) - 4(y + 2) + 2(z - 4) = 0$$

$$x - 4y + 2z = 18$$

**27.** $xy^2 + 3x - z^2 = 4, \ (2, 1, -2)$

$F(x, y, z) = xy^2 + 3x - z^2 - 4$

$F_x(x, y, z) = y^2 + 3 \qquad F_y(x, y, z) = 2xy \qquad F_z(x, y, z) = -2z$

$F_x(2, 1, -2) = 4 \qquad F_y(2, 1, -2) = 4 \qquad F_Z(2, 1, -2) = 4$

$4(x - 2) + 4(y - 1) + 4(z + 2) = 0$

$x + y + z = 1$

**29.** $x^2 + y^2 + z = 9, \ (1, 2, 4)$

$F(x, y, z) = x^2 + y^2 + z - 9$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = 1$

$F_x(1, 2, 4) = 2 \qquad F_y(1, 2, 4) = 4 \qquad F_z(1, 2, 4) = 1$

Direction numbers: 2, 4, 1

Plane: $2(x - 1) + 4(y - 2) + (z - 4) = 0, \ 2x + 4y + z = 14$

Line: $\dfrac{x - 1}{2} = \dfrac{y - 2}{4} = \dfrac{z - 4}{1}$

**31.** $xy - z = 0, \ (-2, -3, 6)$

$F(x, y, z) = xy - z$

$F_x(x, y, z) = y \qquad F_y(x, y, z) = x \qquad F_z(x, y, z) = -1$

$F_x(-2, -3, 6) = -3 \qquad F_y(-2, -3, 6) = -2 \qquad F_z(-2, -3, 6) = -1$

Direction numbers: 3, 2, 1

Plane: $3(x + 2) + 2(y + 3) + (z - 6) = 0, \ 3x + 2y + z = -6$

Line: $\dfrac{x + 2}{3} = \dfrac{y + 3}{2} = \dfrac{z - 6}{1}$

**33.** $z = \arctan \dfrac{y}{x}, \ \left(1, 1, \dfrac{\pi}{4}\right)$

$F(x, y, z) = \arctan \dfrac{y}{x} - z$

$F_x(x, y, z) = \dfrac{-y}{x^2 + y^2} \qquad F_y(x, y, z) = \dfrac{x}{x^2 + y^2} \qquad F_z(x, y, z) = -1$

$F_x\left(1, 1, \dfrac{\pi}{4}\right) = -\dfrac{1}{2} \qquad F_y\left(1, 1, \dfrac{\pi}{4}\right) = \dfrac{1}{2} \qquad F_z\left(1, 1, \dfrac{\pi}{4}\right) = -1$

Direction numbers: $1, -1, 2$

Plane: $(x - 1) - (y - 1) + 2\left(z - \dfrac{\pi}{4}\right) = 0, \ x - y + 2z = \dfrac{\pi}{2}$

Line: $\dfrac{x - 1}{1} = \dfrac{y - 1}{-1} = \dfrac{z - (\pi/4)}{2}$

**35.** $z = f(x, y) = \dfrac{4xy}{(x^2+1)(y^2+1)}, \; -2 \le x \le z, \; 0 \le y \le 3$

(a) Let $F(x, y, z) = \dfrac{4xy}{(x^2+1)(y^2+1)} - z$

$$\nabla F(x, y, z) = \frac{4y}{y^2+1}\left(\frac{x^2+1-2x^2}{(x^2+1)^2}\right)\mathbf{i} + \frac{4x}{x^2+1}\left(\frac{y^2+1-2y^2}{(y^2+1)^2}\right)\mathbf{j} - \mathbf{k}$$

$$= \frac{4y(1-x^2)}{(y^2+1)(x^2+1)^2}\mathbf{i} + \frac{4x(1-y^2)}{(x^2+1)(y^2+1)^2}\mathbf{j} - \mathbf{k}$$

$\nabla F(1, 1, 1) = -\mathbf{k}.$

Direction numbers: $0, 0, -1.$

Line: $x = 1, \; y = 1, \; z = 1 - t$

Tangent plane: $0(x-1) + 0(y-1) - 1(z-1) = 0 \implies z = 1$

(b) $\nabla F\left(-1, 2, -\dfrac{4}{5}\right) = 0\mathbf{i} + \dfrac{-4(-3)}{(2)(5)^2}\mathbf{j} - \mathbf{k} = \dfrac{6}{25}\mathbf{j} - \mathbf{k}$

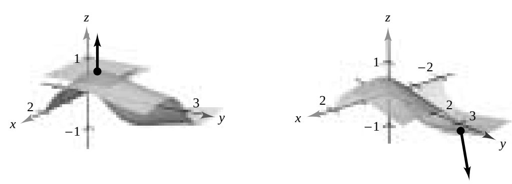
Line: $x = -1, \; y = 2 + \dfrac{6}{25}t, \; z = -\dfrac{4}{5} - t$

Plane: $0(x+1) + \dfrac{6}{25}(y-2) - 1\left(z + \dfrac{4}{5}\right) = 0$

$$6y - 12 - 25z - 20 = 0$$

$$6y - 25z - 32 = 0$$

(c)



(d) At $(1, 1, 1)$, the tangent plane is parallel to the $xy$-plane, implying that the surface is level there. At $\left(-1, 2, -\frac{4}{5}\right)$, the function does not change in the $x$-direction.

**37.** $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

(Theorem 12.13)

**39.** $F(x, y, z) = x^2 + y^2 - 5 \qquad G(x, y, z) = x - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} \qquad \nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$

$\nabla F(2, 1, 2) = 4\mathbf{i} + 2\mathbf{j} \qquad \nabla G(2, 1, 2) = \mathbf{i} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Direction numbers: $1, -2, 1, \; \dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z-2}{1}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{4}{\sqrt{20}\sqrt{2}} = \dfrac{2}{\sqrt{10}} = \dfrac{\sqrt{10}}{5}$; not orthogonal

**41.** $F(x, y, z) = x^2 + z^2 - 25 \qquad G(x, y, z) = y^2 + z^2 - 25$

$\nabla F = 2x\mathbf{i} + 2z\mathbf{k} \qquad \nabla G = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k} \qquad \nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$

**—CONTINUED—**

**41. —CONTINUED—**

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

Direction numbers: $4, 4, -3$, $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-4}{-3}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{64}{(10)(10)} = \dfrac{16}{25}$; not orthogonal

**43.** $F(x, y, z) = x^2 + y^2 + z^2 - 6$ $\qquad G(x, y, z) = x - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $\qquad \nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

$\nabla F(2, 1, 1) = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ $\qquad \nabla G(2, 1, 1) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 2 \\ 1 & -1 & -1 \end{vmatrix} = 6\mathbf{j} - 6\mathbf{k} = 6(\mathbf{j} - \mathbf{k})$

Direction numbers: $0, 1, -1$, $x = 2$, $\dfrac{y-1}{1} = \dfrac{z-1}{-1}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = 0$; orthogonal

**45.** $f(x, y) = 6 - x^2 - \dfrac{y^2}{4}$, $g(x, y) = 2x + y$

(a) $F(x, y, z) = z + x^2 + \dfrac{y^2}{4} - 6$

$\nabla F(x, y, z) = 2x\mathbf{i} + \dfrac{1}{2}y\mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 4) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$G(x, y, z) = z - 2x - y$

$\nabla G(x, y, z) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\nabla G(1, 2, 4) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

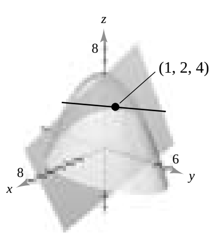
The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j}$$

Using direction numbers $1, -2, 0$, you get $x = 1 + t$, $y = 2 - 2t$, $z = 4$.

$$\cos\theta = \frac{\nabla F \cdot \nabla G}{\|\nabla F\|\,\|\nabla G\|} = \frac{-4 - 1 + 1}{\sqrt{6}\,\sqrt{6}} = \frac{-4}{6} \implies \theta \approx 48.2^\circ$$

(b)




**47.** $F(x, y, z) = 3x^2 + 2y^2 - z - 15$, $(2, 2, 5)$

$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$

$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

$\cos\theta = \dfrac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \dfrac{1}{\sqrt{209}}$

$\theta = \arccos\left(\dfrac{1}{\sqrt{209}}\right) \approx 86.03^\circ$

**49.** $F(x, y, z) = x^2 - y^2 + z$, $(1, 2, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

$\cos\theta = \dfrac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \dfrac{1}{\sqrt{21}}$

$\theta = \arccos\dfrac{1}{\sqrt{21}} \approx 77.40^\circ$

**51.** $F(x, y, z) = 3 - x^2 - y^2 + 6y - z$

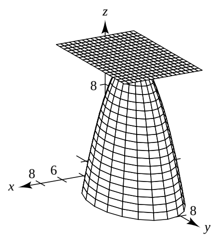
$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$

$-2x = 0,\ x = 0$

$-2y + 6 = 0,\ y = 3$

$z = 3 - 0^2 - 3^2 + 6(3) = 12$

$(0, 3, 12)$ (vertex of paraboloid)




**53.** $T(x, y, z) = 400 - 2x^2 - y^2 - 4z^2,\ (4, 3, 10)$

$\dfrac{dx}{dt} = -4kx \qquad \dfrac{dy}{dt} = -2ky \qquad \dfrac{dz}{dt} = -8kz$

$x(t) = C_1e^{-4kt} \qquad y(t) = C_2e^{-2kt} \qquad z(t) = C_3e^{-8kt}$

$x(0) = C_1 = 4 \qquad y(0) = C_2 = 3 \qquad z(0) = C_3 = 10$

$x = 4e^{-4kt} \qquad y = 3e^{-2kt} \qquad z = 10e^{-8kt}$

**55.** $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$

$F_x(x, y, z) = \dfrac{2x}{a^2}$

$F_y(x, y, z) = \dfrac{2y}{b^2}$

$F_z(x, y, z) = \dfrac{2z}{c^2}$

Plane: $\dfrac{2x_0}{a^2}(x - x_0) + \dfrac{2y_0}{b^2}(y - y_0) + \dfrac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

**57.** $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$

$F_x(x, y, z) = 2a^2x$

$F_y(x, y, z) = 2b^2y$

$F_z(x, y, z) = -2z$

Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$

$$a^2x_0x + b^2y_0y - z_0z = a^2x_0^2 + b^2y_0^2 - z_0^2 = 0$$

Hence, the plane passes through the origin.

**59.** $f(x, y) = e^{x-y}$

$f_x(x, y) = e^{x-y}, \qquad f_y(x, y) = -e^{x-y}$

$f_{xx}(x, y) = e^{x-y}, \qquad f_{yy}(x, y) = e^{x-y}, \qquad f_{xy}(x, y) = -e^{x-y}$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0,0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2$

$= 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$

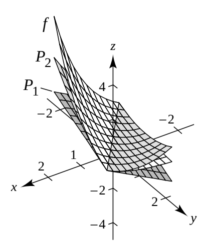
(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$. This is the second–degree Taylor polynomial for $e^{-y}$.

If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$. This is the second–degree Taylor polynomial for $e^x$.

(d)

| $x$ | $y$ | $f(x, y)$ | $P_1(x, y)$ | $P_2(x, y)$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0.9048 | 0.9000 | 0.9050 |
| 0.2 | 0.1 | 1.1052 | 1.1000 | 1.1050 |
| 0.2 | 0.5 | 0.7408 | 0.7000 | 0.7450 |
| 1 | 0.5 | 1.6487 | 1.5000 | 1.6250 |

(e)




**61.** Given $w = F(x, y, z)$ where $F$ is differentiable at

$(x_0, y_0, z_0)$ and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$,

the level surface of $F$ at $(x_0, y_0, z_0)$ is of the form $F(x, y, z) = C$ for some constant $C$. Let

$G(x, y, z) = F(x, y, z) - C = 0.$

Then $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ where $\nabla G(x_0, y_0, z_0)$ is normal to $F(x_0, y_0, z_0) - C = 0$.

Therefore, $\nabla F(x_0, y_0z_0)$ is normal to $F(x_0, y_0, z_0) = C$.

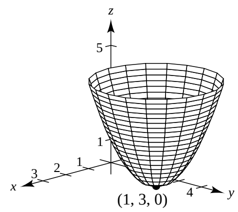
## Section 12.8 Extrema of Functions of Two Variables

**1.** $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum: $(1, 3, 0)$

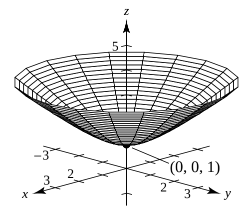
$g_x = 2(x - 1) = 0 \implies x = 1$

$g_y = 2(y - 3) = 0 \implies y = 3$




**3.** $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

Relative minimum: $(0, 0, 1)$




**Check:** $f_x = \dfrac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \implies x = 0$

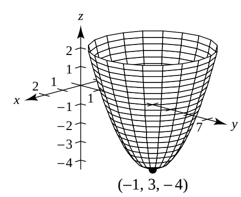
$f_y = \dfrac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \implies y = 0$

$f_{xx} = \dfrac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}},\ f_{yy} = \dfrac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}},\ f_{xy} = \dfrac{-xy}{(x^2 + y^2 + 1)^{3/2}}$

At the critical point $(0, 0)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(0, 0, 1)$ is a relative minimum.

**5.** $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

Relative minimum: $(-1, 3, -4)$




**Check:** $f_x = 2x + 2 = 0 \implies x = -1$

$f_y = 2y - 6 = 0 \implies y = 3$

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0$

At the critical point $(-1, 3)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 3, -4)$ is a relative minimum.

**7.** $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$\left.\begin{array}{l} f_x = 4x + 2y + 2 = 0 \\ f_y = 2x + 2y = 0 \end{array}\right\}$ Solving simultaneously yields $x = -1$ and $y = 1$.

$f_{xx} = 4,\ f_{yy} = 2,\ f_{xy} = 2$

At the critical point $(-1, 1)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 1, -4)$ is a relative minimum.

**9.** $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$\left.\begin{array}{l} f_x = -10x + 4y + 16 = 0 \\ f_y = 4x - 2y = 0 \end{array}\right\}$ Solving simultaneously yields $x = 8$ and $y = 16$.

$f_{xx} = -10,\ f_{yy} = -2,\ f_{xy} = 4$

At the critical point $(8, 16)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(8, 16, 74)$ is a relative maximum.

**11.** $f(x, y) = 2x^2 + 3y^2 - 4x - 12y + 13$

$f_x = 4x - 4 = 4(x - 1) = 0$ when $x = 1$.

$f_y = 6y - 12 = 6(y - 2) = 0$ when $y = 2$.

$f_{xx} = 4,\ f_{yy} = 6,\ f_{xy} = 0$

At the critical point $(1, 2)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 2, -1)$ is a relative minimum.

**13.** $f(x, y) = 2\sqrt{x^2 + y^2} + 3$

$\left.\begin{array}{l} f_x = \dfrac{2x}{\sqrt{x^2 + y^2}} = 0 \\ f_y = \dfrac{2y}{\sqrt{x^2 + y^2}} = 0 \end{array}\right\}\ x = 0, y = 0$

Since $f(x, y) \geq 3$ for all $(x, y)$, $(0, 0, 3)$ is relative minimum.

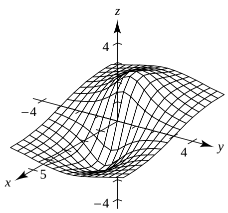
**15.** $g(x, y) = 4 - |x| - |y|$

$(0, 0)$ is the only critical point. Since $g(x, y) \leq 4$ for all $(x, y)$, $(0, 0, 4)$ is relative maximum.

**17.** $z = \dfrac{-4x}{x^2 + y^2 + 1}$

Relative minimum: $(1, 0, -2)$

Relative maximum: $(-1, 0, 2)$

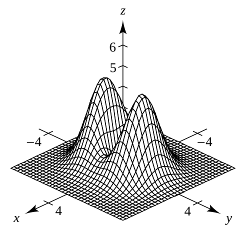



**19.** $z = (x^2 + 4y^2)e^{1-x^2-y^2}$

Relative minimum: $(0, 0, 0)$

Relative maxima: $(0, \pm 1, 4)$

Saddle points: $(\pm 1, 0, 1)$




**21.** $h(x, y) = x^2 - y^2 - 2x - 4y - 4$

$h_x = 2x - 2 = 2(x - 1) = 0$ when $x = 1$.

$h_y = -2y - 4 = -2(y + 2) = 0$ when $y = -2$.

$h_{xx} = 2,\ h_{yy} = -2,\ h_{xy} = 0$

At the critical point $(1, -2)$, $h_{xx}\,h_{yy} - (h_{xy})^2 < 0$. Therefore, $(1, -2, -1)$ is a saddle point.

**23.** $h(x, y) = x^2 - 3xy - y^2$

$$\left.\begin{aligned} h_x &= 2x - 3y = 0 \\ h_y &= -3x - 2y = 0 \end{aligned}\right\} \quad \text{Solving simultaneously yields } x = 0 \text{ and } y = 0.$$

$h_{xx} = 2,\ h_{yy} = -2,\ h_{xy} = -3$

At the critical point $(0, 0)$, $h_{xx}\,h_{yy} - (h_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

**25.** $f(x, y) = x^3 - 3xy + y^3$

$$\left.\begin{aligned} f_x &= 3(x^2 - y) = 0 \\ f_y &= 3(-x + y^2) = 0 \end{aligned}\right\} \quad \text{Solving by substitution yields two critical points } (0, 0) \text{ and } (1, 1).$$

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = -3$

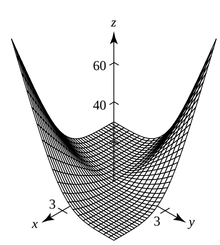
At the critical point $(0, 0)$, $f_{xx}\,f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point. At the critical point $(1, 1)$, $f_{xx} = 6 > 0$ and $f_{xx}\,f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 1, -1)$ is a relative minimum.

**27.** $f(x, y) = e^{-x} \sin y$

$$\left.\begin{aligned} f_x &= -e^{-x}\sin y = 0 \\ f_y &= e^{-x}\cos y = 0 \end{aligned}\right\} \quad \begin{array}{l}\text{Since } e^{-x} > 0 \text{ for all } x \text{ and } \sin y \text{ and } \cos y \text{ are never both zero for a} \\ \text{given value of } y\text{, there are no critical points.}\end{array}$$

**29.** $z = \dfrac{(x - y)^4}{x^2 + y^2} \geq 0$. $z = 0$ if $x = y \neq 0$.

Relative minimum at all points $(x, x)$, $x \neq 0$.



**31.** $f_{xx} f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$

Insufficient information.

**33.** $f_{xx} f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$

$f$ has a saddle point at $(x_0, y_0)$.

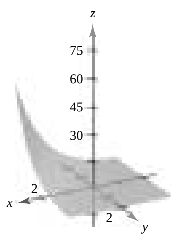
**35.** (a) The function $f$ defined on a region $R$ containing $(x_0, y_0)$ has a relative minimum at $(x_0, y_0)$ if $f(x, y) \geq f(x_0, y_0)$ for all $(x, y)$ in $R$.

(b) The function $f$ defined on a region $R$ containing $(x_0, y_0)$ has a relative maximum at $(x_0, y_0)$ if $f(x, y) \leq f(x_0, y_0)$ for all $(x, y)$ in $R$.

(c) A saddle point is a critical point which is not a relative extremum.

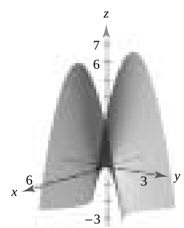
(d) See definition page 906.

**37.**




No extrema

**39.**




Saddle point

**41.** The point $A$ will be a saddle point. The function could be

$$f(x, y) = x^2 - y^2.$$

**43.** $d = f_{xx} f_{yy} - f_{xy}{}^2 = (2)(8) - f_{xy}{}^2 = 16 - f_{xy}{}^2 > 0$

$\implies f_{xy}{}^2 < 16 \implies -4 < f_{xy} < 4$

**45.** $f(x, y) = x^3 + y^3$

$$\left.\begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= 3y^2 = 0 \end{aligned}\right\} \text{Solving yields } x = y = 0$$

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = 0$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(0, 0, 0)$ is a saddle point.

**47.** $f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$

$$\left.\begin{aligned} f_x &= 2(x - 1)(y + 4)^2 = 0 \\ f_y &= 2(x - 1)^2(y + 4) = 0 \end{aligned}\right\} \text{Solving yields the critical points } (1, a) \text{ and } (b, -4).$$

$f_{xx} = 2(y + 4)^2, f_{yy} = 2(x - 1)^2, f_{xy} = 4(x - 1)(y + 4)$

At both $(1, a)$ and $(b, -4)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails.

Absolute minima: $(1, a, 0)$ and $(b, -4, 0)$

**49.** $f(x, y) = x^{2/3} + y^{2/3} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{2}{3\sqrt[3]{x}} \\ f_y &= \frac{2}{3\sqrt[3]{y}} \end{aligned}\right\} f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = -\frac{2}{9x\sqrt[3]{3}}, f_{yy} = -\frac{2}{9y\sqrt[3]{y}}, f_{xy} = 0$$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: 0 at $(0, 0)$

**51.** $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$

$$\left.\begin{aligned} f_x &= 2x = 0 \\ f_y &= 2(y - 3) = 0 \\ f_z &= 2(z + 1) = 0 \end{aligned}\right\} \text{Solving yields the critical point } (0, 3, -1).$$

Absolute minimum: 0 at $(0, 3, -1)$

**53.** $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1$, $0 \le x \le 1$,

$$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4$, $1 \le x \le 2$,

$$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$$

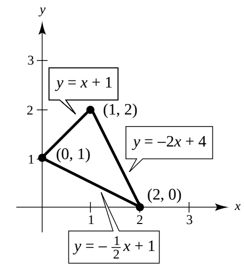
and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1$, $0 \le x \le 2$,

$$f(x, y) = f(x) = 12 - 3x - 2\left(-\frac{1}{2}x + 1\right) = -2x + 10$$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at $(0, 1)$

Absolute minimum: 5 at $(1, 2)$




**55.** $f(x, y) = 3x^2 + 2y^2 - 4y$

$$\left.\begin{aligned} f_x &= 6x = 0 \quad \Longrightarrow x = 0 \\ f_y &= 4y - 4 = 0 \Longrightarrow y = 1 \end{aligned}\right\} \quad f(0, 1) = -2$$

On the line $y = 4$, $-2 \le x \le 2$,

$$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$$

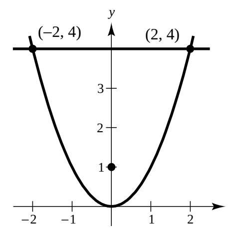
and the maximum is 28, the minimum is 16. On the curve $y = x^2$, $-2 \le x \le 2$,

$$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$$

and the maximum is 28, the minimum is $-\frac{1}{8}$.

Absolute maximum: 28 at $(\pm 2, 4)$

Absolute minimum: $-2$ at $(0, 1)$




**57.** $f(x, y) = x^2 + xy$, $R = \{(x, y): |x| \le 2, |y| \le 1\}$

$$\left.\begin{aligned} f_x &= 2x + y = 0 \\ f_y &= x = 0 \end{aligned}\right\} x = y = 0$$

$f(0, 0) = 0$

Along $y = 1$, $-2 \le x \le 2$, $f = x^2 + x$, $f' = 2x + 1 = 0 \Longrightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2$, $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f(2, 1) = 6$.

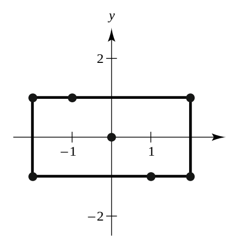
Along $y = -1$, $-2 \le x \le 2$, $f = x^2 - x$, $f' = 2x - 1 = 0 \Longrightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6$, $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$, $f(2, -1) = 2$.

Along $x = 2$, $-1 \le y \le 1$, $f = 4 + 2y \Longrightarrow f' = 2 \ne 0$.

Along $x = -2$, $-1 \le y \le 1$, $f = 4 - 2y \Longrightarrow f' = -2 \ne 0$.

Thus, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$.




**59.** $f(x, y) = x^2 + 2xy + y^2$, $R = \{(x, y): x^2 + y^2 \le 8\}$

$$\left.\begin{aligned} f_x &= 2x + 2y = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\} y = -x$$

$f(x, -x) = x^2 - 2x^2 + x^2 = 0$

On the boundary $x^2 + y^2 = 8$, we have $y^2 = 8 - x^2$ and $y = \pm\sqrt{8 - x^2}$. Thus,

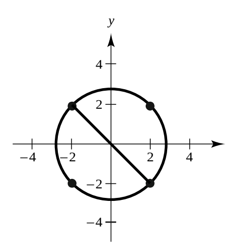
$$f = x^2 \pm 2x\sqrt{8 - x^2} + (8 - x^2) = 8 \pm 2x\sqrt{8 - x^2}$$

$$f' = \pm(8 - x^2)^{-1/2}(-2x) + 2(8 - x^2)^{1/2}) = \pm\frac{16 - 4x^2}{\sqrt{8 - x^2}}.$$

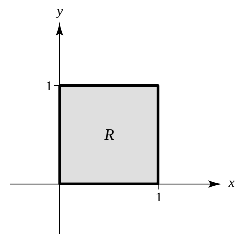
Then, $f' = 0$ implies $16 = 4x^2$ or $x = \pm 2$.

$$f(2, 2) = f(-2, -2) = 16 \quad \text{and} \quad f(2, -2) = f(-2, 2) = 0$$

Thus, the maxima are $f(2, 2) = 16$ and $f(-2, -2) = 16$, and the minima are $f(x, -x) = 0$, $|x| \le 2$.



**61.** $f(x, y) = \dfrac{4xy}{(x^2+1)(y^2+1)}, R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$




$f_x = \dfrac{4(1-x^2)y}{(y^2+1)(x^2+1)} = 0 \implies x = 1 \text{ or } y = 0$

$f_y = \dfrac{4(1-y^2)x}{(x^2+1)(y^2+1)^2} \implies x = 0 \text{ or } y = 1$

For $x = 0$, $y = 0$, also, and $f(0, 0) = 0$.

For $x = 1$, $y = 1$, $f(1, 1) = 1$.

The absolute maximum is $1 = f(1, 1)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)

**63.** False

Let $f(x, y) = |1 - x - y|$.

$(0, 0, 1)$ is a relative maximum, but $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

## Section 12.9 Applications of Extrema of Functions of Two Variables

**1.** A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from the origin to this point is

$$S = x^2 + y^2 + (12 - 2x - 3y)^2$$

$$S_x = 2x + 2(12 - 2x - 3y)(-2)$$

$$S_y = 2y + 2(12 - 2x - 3y)(-3)$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 24$$

$$3x + 5y = 18.$$

Solving simultaneously, we have $x = \frac{12}{7}, y = \frac{18}{7}$ $z = 12 - \frac{24}{7} - \frac{54}{7} = \frac{6}{7}$. Therefore, the distance from the origin to $\left(\frac{12}{7}, \frac{18}{7}, \frac{6}{7}\right)$ is

$$\sqrt{\left(\frac{12}{7}\right)^2 + \left(\frac{18}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{6\sqrt{14}}{7}.$$

**3.** A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 5, 0)$ to a point on the paraboloid is given by

$$S = (x-5)^2 + (y-5)^2 + (x^2+y^2)^2$$

$$S_x = 2(x-5) + 4x(x^2+y^2) = 0$$

$$S_y = 2(y-5) + 4y(x^2+y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$

$$2y^3 + 2x^2y + y - 5 = 0$$

Multiply the first equation by $y$ and the second equation by $x$, and subtract to obtain $x = y$. Then, we have $x = 1$, $y = 1$, $z = 2$ and the distance is

$$\sqrt{(1-5)^2 + (1-5)^2 + (2-0)^2} = 6.$$

**5.** Let $x$, $y$ and $z$ be the numbers. Since $x + y + z = 30$, $z = 30 - x - y$.

$$P = xyz = 30xy - x^2y - xy^2$$

$$\left.\begin{aligned} P_x &= 30y - 2xy - y^2 = y(30 - 2x - y) = 0 \\ P_y &= 30x - x^2 - 2xy = x(30 - x - 2y) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 30 \\ x + 2y &= 30 \end{aligned}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

**7.** Let $x$, $y$, and $z$ be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 30$, we have

$$S = x^2 + y^2 + (30 - x - y)^2$$

$$\left.\begin{aligned} S_x &= 2x + 2(30 - x - y)(-1) = 0 \\ S_y &= 2y + 2(30 - x - y)(-1) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 30 \\ x + 2y &= 30. \end{aligned}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

**9.** Let $x$, $y$, and $z$ be the length, width, and height, respectively. Then the sum of the length and girth is given by $x + (2y + 2z) = 108$ or $x = 108 - 2y - 2z$. The volume is given by

$$V = xyz = 108zy - 2zy^2 - 2yz^2$$

$$V_y = 108z - 4yz - 2z^2 = z(108 - 4y - 2z) = 0$$

$$V_z = 108y - 2y^2 - 4yz = y(108 - 2y - 4z) = 0.$$

Solving the system $4y + 2z = 108$ and $2y + 4z = 108$, we obtain the solution $x = 36$ inches, $y = 18$ inches, and $z = 18$ inches.

**11.** Let $a + b + c = k$. Then

$$V = \frac{4\pi abc}{3} = \frac{4}{3}\pi ab(k - a - b)$$

$$= \frac{4}{3}\pi(kab - a^2b - ab^2)$$

$$\left.\begin{aligned} V_a &= \frac{4\pi}{3}(kb - 2ab - b^2) = 0 \\ V_b &= \frac{4\pi}{3}(ka - a^2 - 2ab) = 0 \end{aligned}\right\} \begin{aligned} kb - 2ab - b^2 &= 0 \\ ka - a^2 - 2ab &= 0. \end{aligned}$$

Solving this system simultaneously yields $a = b$ and substitution yields $b = k/3$. Therefore, the solution is $a = b = c = k/3$.

**13.** Let $x$, $y$, and $z$ be the length, width, and height, respectively and let $V_0$ be the given volume.

Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$\left.\begin{aligned} S_x &= 2\left(y - \frac{V_0}{x^2}\right) = 0 \\ S_y &= 2\left(x - \frac{V_0}{y^2}\right) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0. \end{aligned}$$

Solving simultaneously yields $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

**15.** The distance from $P$ to $Q$ is $\sqrt{x^2 + 4}$. The distance from $Q$ to $R$ is $\sqrt{(y - x)^2 + 1}$. The distance from $R$ to $S$ is $10 - y$.

$$C = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y - x)^2 + 1} + k(10 - y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(\frac{-(y - x)}{\sqrt{(y - x)^2 + 1}}\right) = 0$$

$$C_y = 2k\left(\frac{y - x}{\sqrt{(y - x)^2 + 1}}\right) - k = 0 \implies \frac{y - x}{\sqrt{(y - x)^2 + 1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2 + 4}$$

$$9x^2 = x^2 + 4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y - x) = \sqrt{(y - x)^2 + 1}$$

$$4(y - x)^2 = (y - x)^2 + 1$$

$$(y - x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3} + 3\sqrt{2}}{6}$$

Therefore, $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km and $y = \dfrac{2\sqrt{3} + 3\sqrt{2}}{6} \approx 1.284$ kms.

**17.** Let $h$ be the height of the trough and $r$ the length of the slanted sides. We observe that the area of a trapezoidal cross section is given by

$$A = h\left[\frac{(w - 2r) + [(w - 2r) + 2x]}{2}\right] = (w - 2r + x)h$$

where $x = r\cos\theta$ and $h = r\sin\theta$. Substituting these expressions for $x$ and $h$, we have

$$A(r, \theta) = (w - 2r + r\cos\theta)(r\sin\theta) = wr\sin\theta - 2r^2\sin\theta + r^2\sin\theta\cos\theta$$

Now

$$A_r(r, \theta) = w\sin\theta - 4r\sin\theta + 2r\sin\theta\cos\theta = \sin\theta(w - 4r + 2r\cos\theta) = 0 \implies w = r(4 - 2\cos\theta)$$

$$A_\theta(r, \theta) = wr\cos\theta - 2r^2\cos\theta + r^2\cos 2\theta = 0.$$

Substituting the expression for $w$ from $A_r(r, \theta) = 0$ into the equation $A_\theta(r, \theta) = 0$, we have

$$r^2(4 - 2\cos\theta)\cos\theta - 2r^2\cos\theta + r^2(2\cos^2\theta - 1) = 0$$

$$r^2(2\cos\theta - 1) = 0 \text{ or } \cos\theta = \frac{1}{2}.$$

Therefore, the first partial derivatives are zero when $\theta = \pi/3$ and $r = w/3$. (Ignore the solution $r = \theta = 0$.) Thus, the trapezoid of maximum area occurs when each edge of width $w/3$ is turned up 60° from the horizontal.

**19.** $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$R_{x_1} = -10x_1 - 2x_2 + 42 = 0,\ 5x_1 + x_2 = 21$

$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, x_1 + 8x_2 = 51$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$$R_{x_1x_1} = -10$$

$$R_{x_1x_2} = -2$$

$$R_{x_2x_2} = -16$$

$R_{x_1x_1} < 0$ and $R_{x_1x_1}R_{x_2x_2} - (R_{x_1x_2})^2 > 0$

Thus, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

**21.** $P(x_1, x_2) = 15(x_1 + x_2) - C_1 - C_2$

$$= 15x_1 + 15x_2 - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275)$$

$$= -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$$

$$P_{x_1} = -0.04x_1 + 11 = 0, x_1 = 275$$

$$P_{x_2} = -0.10x_2 + 11 = 0, x_2 = 110$$

$$P_{x_1x_1} = -0.04$$

$$P_{x_1x_2} = 0$$

$$P_{x_2x_2} = -0.10$$

$P_{x_1x_1} < 0$ and $P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$

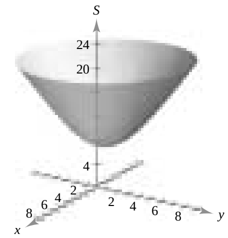
Therefore, profit is maximized when $x_1 = 275$ and $x_2 = 110$.

**23.** (a) $S(x, y) = d_1 + d_2 + d_3$

$$= \sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2}$$

$$= \sqrt{x^2 + y^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2}$$

From the graph we see that the surface has a minimum.




(b) $S_x(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}} + \dfrac{x+2}{\sqrt{(x+2)^2 + (y-2)^2}} + \dfrac{x-4}{\sqrt{(x-4)^2 + (y-2)^2}}$

$S_y(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}} + \dfrac{y-2}{\sqrt{(x+2)^2 + (y-2)^2}} + \dfrac{y-2}{\sqrt{(x-4)^2 + (y-2)^2}}$

(c) $-\nabla S(1, 1) = -S_x(1, 1)\mathbf{i} - S_y(1, 1)\mathbf{j} = -\dfrac{1}{\sqrt{2}}\mathbf{i} - \left(\dfrac{1}{\sqrt{2}} - \dfrac{2}{\sqrt{10}}\right)\mathbf{j}$

$\tan\theta = \dfrac{\left(2/\sqrt{10}\right) - \left(1/\sqrt{2}\right)}{-1/\sqrt{2}} = 1 - \dfrac{2}{\sqrt{5}} \implies \theta \approx 186.027°$

(d) $(x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t) = \left(1 - \dfrac{1}{\sqrt{2}}t, 1 + \left(\dfrac{2}{\sqrt{10}} - \dfrac{1}{\sqrt{2}}\right)t\right)$

$$S\left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right) = \sqrt{2 + \left(\frac{2\sqrt{10}}{5} - 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$
$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} + 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$
$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} - 4\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.344$. Thus, $(x_2, y_2) \approx (0.05, 0.90)$.

(e) $(x_3, y_3) = (x_2 - S_x(x_2, y_2)t, y_2 - S_y(x_2, y_2)t) \approx (0.05 + 0.03t, 0.90 - 0.26t)$

$$S(0.05 + 0.03t, 0.90 - 0.26t) = \sqrt{(0.05 + 0.03t)^2 + (0.90 - 0.26t)^2} + \sqrt{(2.05 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$
$$+ \sqrt{(-3.95 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.78$. Thus $(x_3, y_3) \approx (0.10, 0.44)$.

$(x_4, y_4) = (x_3 - S_x(x_3, y_3)t, y_3 - S_y(x_3, y_3)t) \approx (0.10 - 0.09t, 0.44 - 0.01t)$

$$S(0.10 - 0.09t, 0.45 - 0.01t) = \sqrt{(0.10 - 0.09t)^2 + (0.45 - 0.01t)^2} + \sqrt{(2.10 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$
$$+ \sqrt{(-3.90 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 0.44$. Thus, $(x_4, y_4) \approx (0.06, 0.44)$.

**Note:** The minimum occurs at $(x, y) = (0.0555, 0.3992)$

(f) $-\nabla S(x, y)$ points in the direction that $S$ *decreases* most rapidly. You would use $\nabla S(x, y)$ for maximization problems.

**25.** Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points, too.

**27.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| $-2$ | 0 | 0 | 4 |
| 0 | 1 | 0 | 0 |
| 2 | 3 | 6 | 4 |
| $\sum x_i = 0$ | $\sum y_i = 4$ | $\sum x_iy_i = 6$ | $\sum x_i^2 = 8$ |

$$a = \frac{3(6) - 0(4)}{3(8) - 0^2} = \frac{3}{4},\ b = \frac{1}{3}\left[4 - \frac{3}{4}(0)\right] = \frac{4}{3},$$

$$y = \frac{3}{4}x + \frac{4}{3}$$

(b) $$S = \left(-\frac{3}{2} + \frac{4}{3} - 0\right)^2 + \left(\frac{4}{3} - 1\right)^2 + \left(\frac{3}{2} + \frac{4}{3} - 3\right)^2$$

$$= \frac{1}{6}$$

**29.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 0 | 4 | 0 | 0 |
| 1 | 3 | 3 | 1 |
| 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 4 |
| $\sum x_i = 4$ | $\sum y_i = 8$ | $\sum x_iy_i = 4$ | $\sum x_i^2 = 6$ |

$$a = \frac{4(4) - 4(8)}{4(6) - 4^2} = -2,\ b = \frac{1}{4}[8 + 2(4)] = 4,$$

$$y = -2x + 4$$

(b) $S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$

**31.** $(0, 0), (1, 1), (3, 4), (4, 2), (5, 5)$

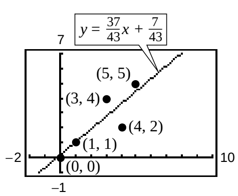
$\sum x_i = 13,\qquad \sum y_i = 12,$

$\sum x_iy_i = 46,\qquad \sum x_i^2 = 51$

$$a = \frac{5(46) - 13(12)}{5(51) - (13)^2} = \frac{74}{86} = \frac{37}{43}$$

$$b = \frac{1}{5}\left[12 - \frac{37}{43}(13)\right] = \frac{7}{43}$$

$$y = \frac{37}{43}x + \frac{7}{43}$$




**33.** $(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$

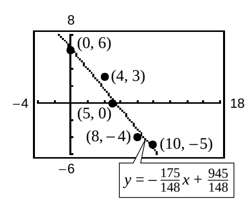
$\sum x_i = 27,\qquad \sum y_i = 0,$

$\sum x_iy_i = -70,\qquad \sum x_i^2 = 205$

$$a = \frac{5(-70) - (27)(0)}{5(205) - (27)^2} = \frac{-350}{296} = -\frac{175}{148}$$

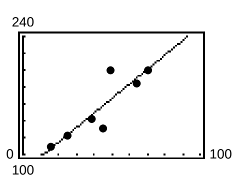
$$b = \frac{1}{5}\left[0 - \left(-\frac{175}{148}\right)(27)\right] = \frac{945}{148}$$

$$y = -\frac{175}{148}x + \frac{945}{148}$$




**35.** (a) $y = 1.7236x + 79.7334$

(b)




(c) For each one-year increase in age, the pressure changes by 1.7236 (slope of line).

**37.** $(1.0, 32), (1.5, 41), (2.0, 48), (2.5, 53)$

$\sum x_i = 7, \sum y_i = 174, \sum x_iy_i = 322, \sum x_i^2 = 13.5$

$a = 14, b = 19, y = 14x + 19$

When $x = 1.6$, $y = 41.4$ bushels per acre.

**39.** $S(a, b, c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} -2x_i^2(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} -2x_i(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial c} = -2\sum_{i=1}^{n}(y_i - ax_i^2 - bx_i - c) = 0$$

$$a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$$

$$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$$

**41.** $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

$\sum x_i = 0$

$\sum y_i = 8$

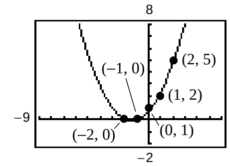
$\sum x_i^2 = 10$

$\sum x_i^3 = 0$

$\sum x_i^4 = 34$

$\sum x_i y_i = 12$

$\sum x_i^2 y_i = 22$




$34a + 10c = 22,\ 10b = 12,\ 10a + 5c = 8$

$a = \frac{3}{7},\ b = \frac{6}{5},\ c = \frac{26}{35},\ y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$

**43.** $(0, 0), (2, 2), (3, 6), (4, 12)$

$\sum x_i = 9$

$\sum y_i = 20$

$\sum x_i^2 = 29$

$\sum x_i^3 = 99$

$\sum x_i^4 = 353$

$\sum x_i y_i = 70$

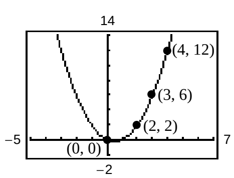
$\sum x_i^2 y_i = 254$

$353a + 99b + 29c = 254$

$99a + 29b + 9c = 70$

$29a + 9b + 4c = 20$

$a = 1,\ b = -1,\ c = 0,\ y = x^2 - x$




**45.** $(0, 0), (2, 15), (4, 30), (6, 50), (8, 65), (10, 70)$

$\sum x_i = 30,$

$\sum y_i = 230,$

$\sum x_i^2 = 220,$

$\sum x_i^3 = 1{,}800,$

$\sum x_i^4 = 15{,}664,$

$\sum x_i y_i = 1{,}670,$

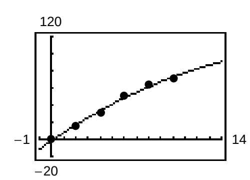
$\sum x_i^2 y_i = 13{,}500$

$15{,}664a + 1{,}800b + 220c = 13{,}500$

$1{,}800a + 220b + 30c = 1{,}670$

$220a + 30b + 6c = 230$

$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$

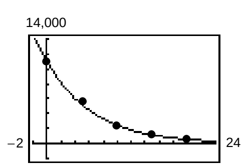



**47.** (a) $\ln P = -0.1499h + 9.3018$

(b) $\ln P = -0.1499h + 9.3018$

$P = e^{-0.1499h + 9.3018} = 10{,}957.7e^{-0.1499h}$

(c)




(d) Same answers.

## Section 12.10 Lagrange Multipliers

**1.** Maximize $f(x, y) = xy$.

Constraint: $x + y = 10$

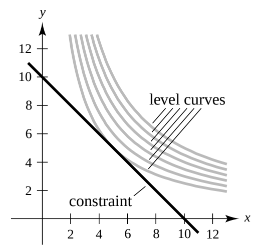
$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$$\left.\begin{aligned} y &= \lambda \\ x &= \lambda \end{aligned}\right\} x = y$$

$$x + y = 10 = \implies x = y = 5$$

$f(5, 5) = 25$




**3.** Minimize $f(x, y) = x^2 + y^2$.

Constraint: $x + y = 4$

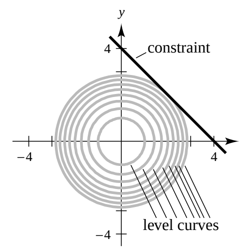
$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \end{aligned}\right\} x = y$$

$$x + y = 4 \implies x = y = 2$$

$f(2, 2) = 8$




**5.** Minimize $f(x, y) = x^2 - y^2$.

Constraint: $x - 2y = -6$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = \lambda\mathbf{i} - 2\lambda\mathbf{j}$

$$2x = \lambda \implies x = \frac{\lambda}{2}$$

$$-2y = -2\lambda \implies y = \lambda$$

$$x - 2y = -6 \implies -\frac{3}{2}\lambda = -6$$

$$\lambda = 4,\ x = 2,\ y = 4$$

$f(2, 4) = -12$

**7.** Maximize $f(x, y) = 2x + 2xy + y$.

Constraint: $2x + y = 100$

$\nabla f = \lambda \nabla g$

$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{aligned} 2 + 2y = 2\lambda &\implies y = \lambda - 1 \\ 2x + 1 = \lambda &\implies x = \frac{\lambda - 1}{2} \end{aligned}\right\} y = 2x$$

$$2x + y = 100 \implies 4x = 100$$

$$x = 25,\ y = 50$$

$f(25, 50) = 2600$

**9.** **Note:** $f(x, y) = \sqrt{6 - x^2 - y^2}$ is maximum when $g(x, y)$ is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.

Constraint: $x + y = 2$

$$\left.\begin{aligned} -2x &= \lambda \\ -2y &= \lambda \end{aligned}\right\} x = y$$

$x + y = 2 \implies x = y = 1$

$f(1, 1) = \sqrt{g(1, 1)} = 2$

**11.** Maximize $f(x, y) = e^{xy}$.

Constraint: $x^2 + y^2 = 8$

$$\left.\begin{aligned} ye^{xy} &= 2x\lambda \\ xe^{xy} &= 2y\lambda \end{aligned}\right\} x = y$$

$x^2 + y^2 = 8 \implies 2x^2 = 8$

$$x = y = 2$$

$f(2, 2) = e^4$

**13.** Maximize or minimize $f(x, y) = x^2 + 3xy + y^2$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} 2x + 3y &= 2x\lambda \\ 3x + 2y &= 2y\lambda \end{aligned}\right\} x^2 = y^2$$

$$x^2 + y^2 = 1 \implies x = \pm\frac{\sqrt{2}}{2}, y = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= 2x + 3y = 0 \\ f_y &= 3x + 2y = 0 \end{aligned}\right\} x = y = 0$$

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 3,\ f_{xx}f_{yy} - (f_{xy})^2 \le 0$

Saddle point: $f(0, 0) = 0$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at

$$\left(\pm\frac{\sqrt{2}}{2},\ \pm\frac{\sqrt{2}}{2}\right)$$

and a minimum of $-\frac{1}{2}$ at

$$\left(\pm\frac{\sqrt{2}}{2},\ \mp\frac{\sqrt{2}}{2}\right).$$

**15.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 6$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 6 \implies x = y = z = 2$

$f(2, 2, 2) = 12$

**17.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 1$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 1 \implies x = y = z = \frac{1}{3}$

$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

**19.** Maximize $f(x, y, z) = xyz$.

Constraints: $x + y + z = 32$

$x - y + z = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\left.\begin{aligned} yz &= \lambda + \mu \\ xz &= \lambda - \mu \\ xy &= \lambda + \mu \end{aligned}\right\} yz = xy \implies x = z$$

$$\left.\begin{aligned} x + y + z &= 32 \\ x - y + z &= 0 \end{aligned}\right\} 2x + 2z = 32 \implies x = z = 8$$

$$y = 16$$

$f(8, 16, 8) = 1024$

**21.** Maximize $f(x, y, z) = xy + yz$.

Constraints: $x + 2y = 6$

$x - 3z = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j}) + \mu(\mathbf{i} - 3\mathbf{k})$

$$\left.\begin{aligned} y &= \lambda + \mu \\ x + z &= 2\lambda \\ y &= -3\mu \end{aligned}\right\} y = \frac{3}{4}\lambda \implies x + z = \frac{8}{3}y$$

$x + 2y = 6 \implies y = 3 - \dfrac{x}{2}$

$x - 3z = 0 \implies z = \dfrac{x}{3}$

$$x + \frac{x}{3} = \frac{8}{3}\left(3 - \frac{x}{2}\right)$$

$$x = 3, y = \frac{3}{2}, z = 1$$

$f\left(3, \dfrac{3}{2}, 1\right) = 6$

**23.** Minimize the square of the distance $f(x, y) = x^2 + y^2$ subject to the constraint $2x + 3y = -1$.

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 3\lambda \end{aligned}\right\} y = \frac{3x}{2}$$

$2x + 3y = -1 \implies x = -\dfrac{2}{13}, y = -\dfrac{3}{13}$

The point on the line is $\left(-\frac{2}{13}, -\frac{3}{13}\right)$ and the desired distance is

$$d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}.$$

**25.** Minimize the square of the distance

$$f(x, y, z) = (x - 2)^2 + (y - 1)^2 + (z - 1)^2$$

subject to the constraint $x + y + z = 1$.

$$\left.\begin{aligned} 2(x - 2) &= \lambda \\ 2(y - 1) &= \lambda \\ 2(z - 1) &= \lambda \end{aligned}\right\} y = z \text{ and } y = x - 1$$

$x + y + z = 1 \implies x + 2(x - 1) = 1$

$$x = 1, y = z = 0$$

The point on the plane is $(1, 0, 0)$ and the desired distance is

$$d = \sqrt{(1 - 2)^2 + (0 - 1)^2 + (0 - 1)^2} = \sqrt{3}.$$

**27.** Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

$$\left.\begin{aligned} 0 &= 2x\lambda + 2\mu \\ 0 &= 2y\lambda + \mu \\ 1 &= 2z\lambda - \mu \end{aligned}\right\} x = 2y$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \implies z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y - 2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for $y$ we have the point

$$\left(\frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3}\right).$$

**29.** Optimization problems that have restrictions or contstraints on the values that can be used to produce the optimal solution are called contrained optimization problems.

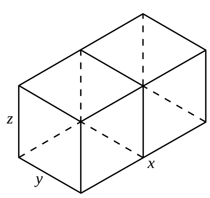
**31.** Maximize $V(x, y, z) = xyz$ subject to the constraint $x + 2y + 2z = 108$.

$$\left.\begin{aligned} yz &= \lambda \\ xz &= 2\lambda \\ xy &= 2\lambda \end{aligned}\right\} y = z \text{ and } x = 2y$$

$$x + 2y + 2z = 108 \implies 6y = 108, y = 18$$

$$x = 36, y = z = 18$$

Volume is maximum when the dimensions are $36 \times 18 \times 18$ inches



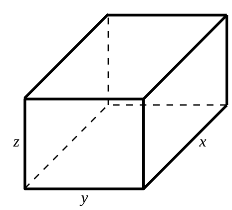

**33.** Minimize $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$ subject to the constraint $xyz = 480$.

$$\left.\begin{aligned} 8y + 6z &= yz\lambda \\ 8x + 6z &= xz\lambda \\ 6x + 6y &= xy\lambda \end{aligned}\right\} x = y, 4y = 3z$$

$$xyz = 480 \implies \tfrac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \tfrac{4}{3}\sqrt[3]{360}$$

Dimensions: $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ feet




**35.** Maximize $V(x, y, z) = (2x)(2y)(2z) = 8xyz$ subject to the constraint $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

$$\left.\begin{aligned} 8yz &= \frac{2x}{a^2}\lambda \\ 8xz &= \frac{2y}{b^2}\lambda \\ 8xy &= \frac{2z}{c^2}\lambda \end{aligned}\right\} \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \implies \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

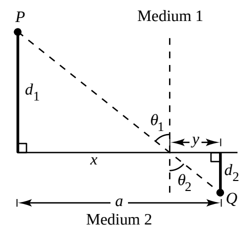
$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

Therefore, the dimensions of the box are $\dfrac{2\sqrt{3}a}{3} \times \dfrac{2\sqrt{3}b}{3} \times \dfrac{2\sqrt{3}c}{3}$.

**37.** Using the formula Time $= \dfrac{\text{Distance}}{\text{Rate}}$, minimize $T(x, y) = \dfrac{\sqrt{d_1^2 + x^2}}{v_1} + \dfrac{\sqrt{d_2^2 + y^2}}{v_2}$ subject to the constraint $x + y = a$.

$$\left.\begin{aligned} \frac{x}{v_1\sqrt{d_2^2 + x^2}} &= \lambda \\ \frac{y}{v_2\sqrt{d_2^2 + y^2}} &= \lambda \end{aligned}\right\} \frac{x}{v_1\sqrt{d_1^2 + x^2}} = \frac{y}{v_2\sqrt{d_2^2 + y^2}}$$

$x + y = a$




Since $\sin\theta_1 = \dfrac{x}{\sqrt{d_1^2 + x^2}}$ and $\sin\theta_2 = \dfrac{y}{\sqrt{d_2^2 + y^2}}$, we have

$$\frac{x/\sqrt{d_1^2 + x^2}}{v_1} = \frac{y/\sqrt{d_2^2 + y^2}}{v_2} \quad \text{or} \quad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

**39.** Maximize $P(p, q, r) = 2pq + 2pr + 2qr$.

Constraint: $p + q + r = 1$

$\nabla P = \lambda \nabla g$

$$\left.\begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned}\right\} \begin{aligned} &\Longrightarrow 3\lambda = 4(p + q + r) = 4(1) \\ &\Longrightarrow \lambda = \tfrac{4}{3} \end{aligned}$$

$p + q + r = 1$

$$\left.\begin{aligned} q + r &= \tfrac{2}{3} \\ p + q + r &= 1 \end{aligned}\right\} \Longrightarrow p = \tfrac{1}{3}, q = \tfrac{1}{3}, r = \tfrac{1}{3}$$

$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{3}$.

**41.** Maximize $P(x, y) = 100x^{0.25}y^{0.75}$

subject to the constraint $48x + 36y = 100{,}000$.

$$25x^{-0.75}y^{0.75} = 48\lambda \Longrightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 36\lambda \Longrightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{48\lambda}{25}\right)\left(\frac{75}{36\lambda}\right)$$

$$\frac{y}{x} = 4$$

$$y = 4x$$

$48x + 36y = 100{,}000 \Longrightarrow 192x = 100{,}000$

$$x = \frac{3125}{6}, y = \frac{6250}{3}$$

Therefore, $P\left(\frac{3125}{6}, \frac{6250}{3}\right) \approx 147{,}314$.

**43.** Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.25}y^{0.75} = 20{,}000$.

$$48 = 25x^{-0.75}y^{0.75}\lambda \Longrightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48}{25\lambda}$$

$$36 = 75x^{0.25}y^{-0.25}\lambda \Longrightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{48}{25\lambda}\right)\left(\frac{75\lambda}{36}\right)$$

$$\frac{y}{x} = 4 \Longrightarrow y = 4x$$

$100x^{0.25}y^{0.75} = 20{,}000 \Longrightarrow x^{0.25}(4x)^{0.75} = 200$

$$x = \frac{200}{4^{0.75}} = \frac{200}{2\sqrt{2}} = 50\sqrt{2}$$

$$y = 4x = 200\sqrt{2}$$

Therefore, $C\left(50\sqrt{2}, 200\sqrt{2}\right) \approx \$13{,}576.45$.

**45.** (a) Maximize $g(\alpha, \beta, \gamma) = \cos\alpha\cos\beta\cos\gamma$ subject to the constraint $\alpha + \beta + \gamma = \pi$.

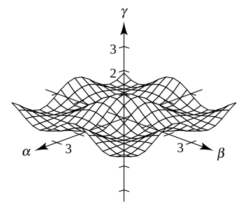
$$\left.\begin{aligned} -\sin\alpha\cos\beta\cos\gamma &= \lambda \\ -\cos\alpha\sin\beta\cos\gamma &= \lambda \\ -\cos\alpha\cos\beta\sin\gamma &= \lambda \end{aligned}\right\} \tan\alpha = \tan\beta = \tan\gamma \implies \alpha = \beta = \gamma$$

$$\alpha + \beta + \gamma = \pi \implies \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{1}{8}$$

(b) $\alpha + \beta + \gamma = \pi \implies \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos\alpha\cos\beta\cos(\pi - (\alpha + \beta)) \\ &= \cos\alpha\cos\beta[\cos\pi\cos(\alpha + \beta) + \sin\pi\sin(\alpha + \beta)] \\ &= -\cos\alpha\cos\beta\cos(\alpha + \beta) \end{aligned}$$




## Review Exercises for Chapter 12

**1.** No, it is not the graph of a function.

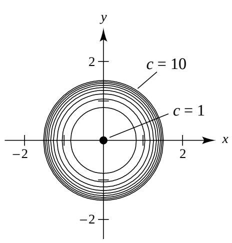
**3.** $f(x, y) = e^{x^2 + y^2}$

The level curves are of the form

$$c = e^{x^2 + y^2}$$

$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.

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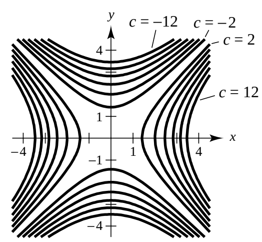
**5.** $f(x, y) = x^2 - y^2$

The level curves are of the form

$$c = x^2 - y^2$$

$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.

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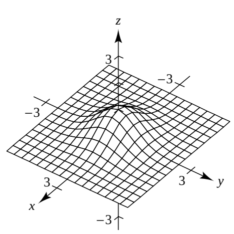
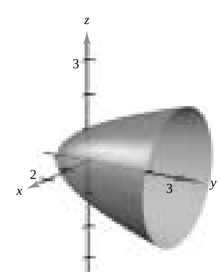
**7.** $f(x, y) = e^{-(x^2 + y^2)}$




**9.** $f(x, y, z) = x^2 - y + z^2 = 1$

$$y = x^2 + z^2 - 1$$

Elliptic paraboloid




**11.** $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$.

**13.** $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{-4x^2y}{x^4 + y^2}$

For $y = x^2$, $\dfrac{-4x^2y}{x^4 + y^2} = \dfrac{-4x^4}{x^4 + x^4} = -2$, for $x \neq 0$

For $y = 0$, $\dfrac{-4x^2y}{x^4 + y^2} = 0$, for $x \neq 0$

Thus, the limit does not exist. Continuous except at $(0, 0)$.

**15.** $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$
$$f_y = -e^x \sin y$$

**17.** $z = xe^y + ye^x$

$$\frac{\partial z}{\partial x} = e^y + ye^x$$
$$\frac{\partial z}{\partial y} = xe^y + e^x$$

**19.** $g(x, y) = \dfrac{xy}{x^2 + y^2}$

$$g_x = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$
$$g_y = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

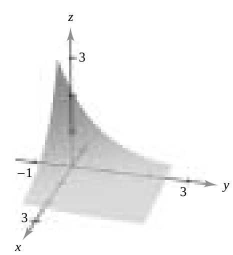
**21.** $f(x, y, z) = z \arctan \dfrac{y}{x}$

$$f_x = \frac{z}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-yz}{x^2 + y^2}$$
$$f_y = \frac{z}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{xz}{x^2 + y^2}$$
$$f_z = \arctan \frac{y}{x}$$

**23.** $u(x, t) = ce^{-n^2 t} \sin(nx)$

$$\frac{\partial u}{\partial x} = cne^{-n^2 t} \cos(nx)$$
$$\frac{\partial u}{\partial t} = -cn^2 e^{-n^2 t} \sin(nx)$$

**25.**




**27.** $f(x, y) = 3x^2 - xy + 2y^3$

$$f_x = 6x - y$$
$$f_y = -x + 6y^2$$
$$f_{xx} = 6$$
$$f_{yy} = 12y$$
$$f_{xy} = -1$$
$$f_{yx} = -1$$

**29.** $h(x, y) = x \sin y + y \cos x$

$$h_x = \sin y - y \sin x$$
$$h_y = x \cos y + \cos x$$
$$h_{xx} = -y \cos x$$
$$h_{yy} = -x \sin y$$
$$h_{xy} = \cos y - \sin x$$
$$h_{yx} = \cos y - \sin x$$

**31.** $z = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x$$
$$\frac{\partial^2 z}{\partial x^2} = 2$$
$$\frac{\partial z}{\partial y} = -2y$$
$$\frac{\partial^2 z}{\partial y^2} = -2$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

**33.** $z = \dfrac{y}{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}$$
$$\frac{\partial^2 z}{\partial x^2} = -2y\left[\frac{-4x^2}{(x^2 + y^2)^3} + \frac{1}{(x^2 + y^2)^2}\right] = 2y\,\frac{3x^2 - y^2}{(x^2 + y^2)^3}$$
$$\frac{\partial z}{\partial y} = \frac{(x^2 + y^2) - 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - 2(x^2 - y^2)(x^2 + y^2)(2y)}{(x^2 + y^2)^4}$$
$$= -2y\,\frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

**35.** $z = x \sin \dfrac{y}{x}$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = \left(\sin\frac{y}{x} - \frac{y}{x}\cos\frac{y}{x}\right)dx + \left(\cos\frac{y}{x}\right)dy$$

**37.** $z^2 = x^2 + y^2$

$2z\,dx = 2x\,dx + 2y\,dy$

$$dz = \frac{x}{z}\,dx + \frac{y}{z}\,dy = \frac{5}{13}\left(\frac{1}{2}\right) + \frac{12}{13}\left(\frac{1}{2}\right) = \frac{17}{26} \approx 0.654 \text{ cm}$$

Percentage error: $\dfrac{dz}{z} = \dfrac{17/26}{13} \approx 0.0503 \approx 5\%$

**39.** $V = \frac{1}{3}\pi r^2 h$

$$dV = \tfrac{2}{3}\pi rh\,dr + \tfrac{1}{3}\pi r^2\,dh = \tfrac{2}{3}\pi(2)(5)\left(\pm\tfrac{1}{8}\right) + \tfrac{1}{3}\pi(2)^2\left(\pm\tfrac{1}{8}\right)$$
$$= \pm\tfrac{5}{6}\pi \pm \tfrac{1}{6}\pi = \pm\pi \text{ in.}^3$$

**41.** $w = \ln(x^2 + y^2),\ x = 2t + 3,\ y = 4 - t$

Chain Rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$$= \frac{2x}{x^2 + y^2}(2) + \frac{2y}{x^2 + y^2}(-1)$$

$$= \frac{2(2t + 3)2}{(2t + 3)^2 + (4 - t)^2} - \frac{2(4 - t)}{(2t + 3)^2 + (4 - t)^2}$$

$$= \frac{10t + 4}{5t^2 + 4t + 25}$$

Substitution: $w = \ln(x^2 + y^2) = \ln\left[(2t + 3)^2 + (4 - t)^2\right]$

$$\frac{dw}{dt} = \frac{2(2t + 3)(2) - 2(4 - t)}{(2t + 3)^2 + (4 - t)^2} = \frac{10t + 4}{5t^2 + 4t + 25}$$

**43.** $u = x^2 + y^2 + z^2,\ x = r\cos t,\ y = r\sin t,\ z = t$

Chain Rule: $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r}$

$$= 2x\cos t + 2y\sin t + 2z(0)$$

$$= 2(r\cos^2 t + r\sin^2 t) = 2r$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= 2x(-r\sin t) + 2y(r\cos t) + 2z$$

$$= 2(-r^2\sin t\cos t + r^2\sin t\cos t) + 2t$$

$$= 2t$$

Substitution: $u(r, t) = r^2\cos^2 t + r^2\sin^2 t + t^2 = r^2 + t^2$

$$\frac{\partial u}{\partial r} = 2r$$

$$\frac{\partial u}{\partial t} = 2t$$

**45.** $x^2y - 2yz - xz - z^2 = 0$

$$2xy - 2y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z - 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2xy + z}{-2y - x - 2z} = \frac{2xy - z}{x + 2y + 2z}$$

$$x^2 - 2y\frac{\partial z}{\partial y} - 2z - x\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 + 2z}{-2y - x - 2z} = \frac{x^2 - 2z}{x + 2y + 2z}$$

**47.** $f(x, y) = x^2y$

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla f(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = 2\sqrt{2} - 2\sqrt{2} = 0$$

**49.** $w = y^2 + xz$

$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$

$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2) \cdot \mathbf{u} = \frac{4}{3} - \frac{4}{3} + \frac{2}{3} = \frac{2}{3}$$

**51.** $z = \dfrac{y}{x^2 + y^2}$

$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$

$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0 \right\rangle$$

$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

**53.** $z = e^{-x}\cos y$

$$\nabla z = -e^{-x}\cos y\mathbf{i} - e^{-x}\sin y\mathbf{j}$$

$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

$$\left\| \nabla z\left(0, \frac{\pi}{4}\right) \right\| = 1$$

**55.** $9x^2 - 4y^2 = 65$

$$f(x, y) = 9x^2 - 4y^2$$

$$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$$

$$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$$

Unit normal: $\dfrac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \dfrac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$

**57.** $F(x, y, z) = x^2y - z = 0$

$$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

Therefore, the equation of the tangent plane is

$4(x - 2) + 4(y - 1) - (z - 4) = 0$ or

$4x + 4y - z = 8$,

and the equation of the normal line is

$$\frac{x - 2}{4} = \frac{y - 1}{4} = \frac{z - 4}{-1}.$$

**59.** $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, -3, 4) = \mathbf{k}$$

Therefore, the equation of the tangent plane is

$z - 4 = 0$ or $z = 4$,

and the equation of the normal line is

$x = 2,\ y = -3,\ z = 4 + t$.

**61.** $F(x, y, z) = x^2 - y^2 - z = 0$

$$G(x, y, z) = 3 - z = 0$$

$$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$$

$$\nabla G = -\mathbf{k}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2(\mathbf{i} + 2\mathbf{j})$$

Therefore, the equation of the tangent line is

$$\frac{x - 2}{1} = \frac{y - 1}{2},\ z = 3.$$

**63.** $f(x, y, z) = x^2 + y^2 + z^2 - 14$

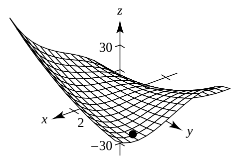
$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ Normal vector to plane.

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

$$\theta = 36.7^\circ$$

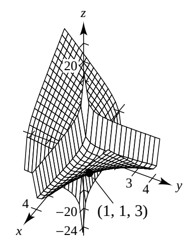
**65.** $f(x, y) = x^3 - 3xy + y^2$




$$f_x = 3x^2 - 3y = 3(x^2 - y) = 0$$
$$f_y = -3x + 2y = 0$$
$$f_{xx} = 6x$$
$$f_{yy} = 2$$
$$f_{xy} = -3$$

From $f_x = 0$, we have $y = x^2$. Substituting this into $f_y = 0$, we have $-3x + 2x^2 = x(2x - 3) = 0$. Thus, $x = 0$ or $\frac{3}{2}$.
At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.
At the critical point $\left(\frac{3}{2}, \frac{9}{4}\right)$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$. Therefore, $\left(\frac{3}{2}, \frac{9}{4}, -\frac{27}{16}\right)$ is a relative minimum.

**67.** $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$




$$f_x = y - \frac{1}{x^2} = 0,\ x^2y = 1$$
$$f_y = x - \frac{1}{y^2} = 0,\ xy^2 = 1$$

Thus, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$$f_{xx} = \frac{2}{x^3}$$
$$f_{xy} = 1$$
$$f_{yy} = \frac{2}{y^3}$$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$. Thus, $(1, 1, 3)$ is a relative minimum.

**69.** The level curves are hyperbolas. There is a critical point at $(0, 0)$, but there are no relative extrema. The gradient is normal to the level curve at any given point at $(x_0, y_0)$.

**71.** $P(x_1, x_2) = R - C_1 - C_2$

$$= [225 - 0.4(x_1 + x_2)](x_1 + x_2) - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$$
$$= -0.45x_1^2 - 0.43x_2^2 - 0.8x_1x_2 + 210x_1 + 210x_2 - 11{,}500$$
$$P_{x_1} = -0.9x_1 - 0.8x_2 + 210 = 0$$
$$0.9x_1 + 0.8x_2 = 210$$
$$P_{x_2} = -0.86x_2 - 0.8x_1 + 210 = 0$$
$$0.8x_1 + 0.86x_2 = 210$$

Solving this system yields $x_1 \approx 94$ and $x_2 \approx 157$.

$$P_{x_1x_1} = -0.9$$
$$P_{x_1x_2} = -0.8$$
$$P_{x_2x_2} = -0.86$$
$$P_{x_1x_1} < 0$$
$$P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$$

Therefore, profit is maximum when $x_1 \approx 94$ and $x_2 \approx 157$.

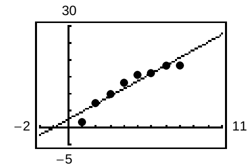
**73.** Maximize $f(x, y) = 4x + xy + 2y$ subject to the constraint $20x + 4y = 2000$.

$$\left.\begin{aligned} 4 + y &= 20\lambda \\ x + 2 &= 4\lambda \end{aligned}\right\} 5x - y = -6$$

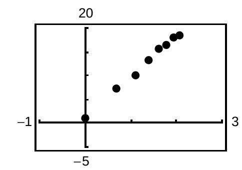
$$20x + 4y = 2000 \implies \begin{array}{r} 5x + y = 500 \\ 5x - y = -6 \\ \hline 10x \quad = 494 \\ x = 49.4 \\ y = 253 \end{array}$$

$f(49.4, 253) = 13{,}201.8$

**75.** (a) $y = 2.29t + 2.34$



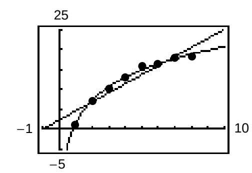

(b)




Yes, the data appears more linear.

(c) $y = 8.37 \ln t + 1.54$

(d)




The logarithmic model is a better fit.

**77.** Optimize $f(x, y, z) = xy + yz + xz$ subject to the constraint $x + y + z = 1$.

$$\left.\begin{aligned} y + z &= \lambda \\ x + z &= \lambda \\ x + y &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 1 \implies x = y = z = \tfrac{1}{3}$$

Maximum: $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

**79.** $PQ = \sqrt{x^2 + 4}$, $QR = \sqrt{y^2 + 1}$, $RS = z$; $x + y + z = 10$

$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + 2$

Constraint: $x + y + z = 10$

$\nabla C = \lambda \nabla g$

$$\frac{3x}{\sqrt{x^2 + 4}}\mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}}\mathbf{j} + \mathbf{k} = \lambda[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$3x = \lambda\sqrt{x^2 + 4}$

$2y = \lambda\sqrt{y^2 + 1}$

$1 = \lambda$

$$9x^2 = x^2 + 4 \implies x^2 = \frac{1}{2}$$

$$4y^2 = y^2 + 1 \implies y^2 = \frac{1}{3}$$

Hence, $x = \dfrac{\sqrt{2}}{2}, y = \dfrac{\sqrt{3}}{3}, z = 10 - \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{3} \approx 8.716$ m.

# CHAPTER 13
# Multiple Integration

# CHAPTER 13
# Multiple Integration

## Section 13.1 Iterated Integrals and Area in the Plane

Solutions to Odd-Numbered Exercises

**1.** $\displaystyle\int_0^x (2x - y)\,dy = \left[2xy - \frac{1}{2}y^2\right]_0^x = \frac{3}{2}x^2$

**3.** $\displaystyle\int_1^{2y} \frac{y}{x}\,dx = \Big[y \ln x\Big]_1^{2y} = y \ln 2y - 0 = y \ln 2y$

**5.** $\displaystyle\int_0^{\sqrt{4-x^2}} x^2 y\,dy = \left[\frac{1}{2}x^2y^2\right]_0^{\sqrt{4-x^2}} = \frac{4x^2 - x^4}{2}$

**7.** $\displaystyle\int_{e^y}^{y} \frac{y \ln x}{x}\,dx = \left[\frac{1}{2}y \ln^2 x\right]_{e^y}^{y} = \frac{1}{2}y[\ln^2 y - \ln^2 e^y] = \frac{y}{2}[(\ln y)^2 - y^2]$

**9.** $\displaystyle\int_0^{x^3} ye^{-y/x}\,dy = \Big[-xye^{-y/x}\Big]_0^{x^3} + x\int_0^{x^3} e^{-y/x}\,dy = -x^4 e^{-x^2} - \Big[x^2e^{-y/x}\Big]_0^{x^3} = x^2(1 - e^{-x^2} - x^2e^{-x^2})$

$u = y,\ du = dy,\ dv = e^{-y/x}\,dy,\ v = -xe^{-y/x}$

**11.** $\displaystyle\int_0^1\int_0^2 (x + y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^2 dx = \int_0^1 (2x + 2)\,dx = \Big[x^2 + 2x\Big]_0^1 = 3$

**13.** $\displaystyle\int_0^1\int_0^x \sqrt{1 - x^2}\,dy\,dx = \int_0^1 \left[y\sqrt{1 - x^2}\right]_0^x dx = \int_0^1 x\sqrt{1 - x^2}\,dx = \left[-\frac{1}{2}\left(\frac{2}{3}\right)(1 - x^2)^{3/2}\right]_0^1 = \frac{1}{3}$

**15.** $\displaystyle\int_1^2\int_0^4 (x^2 - 2y^2 + 1)\,dx\,dy = \int_1^2 \left[\frac{1}{3}x^3 - 2xy^2 + x\right]_0^4 dy$

$$= \int_1^2 \left(\frac{64}{3} - 8y^2 + 4\right) dy = \frac{4}{3}\int_1^2 (19 - 6y^2)\,dy = \left[\frac{4}{3}(19y - 2y^3)\right]_1^2 = \frac{20}{3}$$

**17.** $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} (x + y)\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2 + xy\right]_0^{\sqrt{1-y^2}} dy$

$$= \int_0^1 \left[\frac{1}{2}(1 - y^2) + y\sqrt{1 - y^2}\right] dy = \left[\frac{1}{2}y - \frac{1}{6}y^3 - \frac{1}{2}\left(\frac{2}{3}\right)(1 - y^2)^{3/2}\right]_0^1 = \frac{2}{3}$$

**19.** $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4 - y^2}}\,dx\,dy = \int_0^2 \left[\frac{2x}{\sqrt{4 - y^2}}\right]_0^{\sqrt{4-y^2}} dy = \int_0^2 2\,dy = \Big[2y\Big]_0^2 = 4$

**21.** $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} \theta r\,dr\,d\theta = \int_0^{\pi/2} \left[\theta\frac{r^2}{2}\right]_0^{\sin\theta} d\theta = \int_0^{\pi/2} \frac{1}{2}\theta \sin^2\theta\,d\theta$

$$= \frac{1}{4}\int_0^{\pi/2} (\theta - \theta\cos 2\theta)\,d\theta = \frac{1}{4}\left[\frac{\theta^2}{2} - \left(\frac{1}{4}\cos 2\theta + \frac{\theta}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{\pi^2}{32} + \frac{1}{8}$$

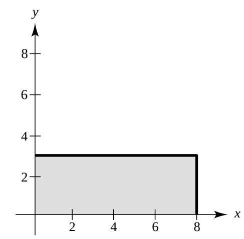
**23.** $\displaystyle\int_1^{\infty}\int_0^{1/x} y\,dy\,dx = \int_1^{\infty}\left[\frac{y^2}{2}\right]_0^{1/x} dx = \frac{1}{2}\int_1^{\infty}\frac{1}{x^2}\,dx = \left[-\frac{1}{2x}\right]_1^{\infty} = 0 + \frac{1}{2} = \frac{1}{2}$

**25.** $\displaystyle\int_1^{\infty}\int_1^{\infty}\frac{1}{xy}\,dx\,dy = \int_1^{\infty}\left[\frac{1}{y}\ln x\right]_1^{\infty} dy = \int_1^{\infty}\left[\frac{1}{y}(\infty) - \frac{1}{y}(0)\right] dy$

Diverges

**27.** $\displaystyle A = \int_0^8\int_0^3 dy\,dx = \int_0^8\Big[y\Big]_0^3 dx = \int_0^8 3\,dx = \Big[3x\Big]_0^8 = 24$

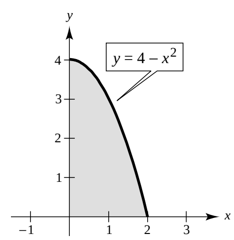
$\displaystyle A = \int_0^3\int_0^8 dx\,dy = \int_0^3\Big[x\Big]_0^8 dy = \int_0^3 8\,dy = \Big[8y\Big]_0^3 = 24$



**29.** $\displaystyle A = \int_0^2\int_0^{4-x^2} dy\,dx = \int_0^2\Big[y\Big]_0^{4-x^2} dx$

$\displaystyle = \int_0^2 (4 - x^2)\,dx$

$\displaystyle = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}$




$\displaystyle A = \int_0^4\int_0^{\sqrt{4-y}} dx\,dy$

$\displaystyle = \int_0^4\Big[x\Big]_0^{\sqrt{4-y}} dy = \int_0^4 \sqrt{4-y}\,dy = -\int_0^4 (4-y)^{1/2}(-1)\,dy = \left[-\frac{2}{3}(4-y)^{3/2}\right]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$

**31.** $\displaystyle A = \int_{-2}^1\int_{x+2}^{4-x^2} dy\,dx$

$\displaystyle = \int_{-2}^1\Big[y\Big]_{x+2}^{4-x^2} dx$

$\displaystyle = \int_{-2}^1 (4 - x^2 - x - 2)\,dx$

$\displaystyle = \int_{-2}^1 (2 - x - x^2)\,dx$

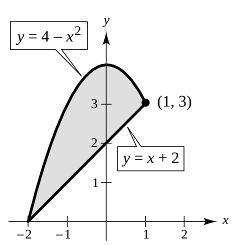
$\displaystyle = \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^1 = \frac{9}{2}$

$\displaystyle A = \int_0^3\int_{-\sqrt{4-y}}^{y-2} dx\,dy + 2\int_3^4\int_0^{\sqrt{4-y}} dx\,dy$

$\displaystyle = \int_0^3\Big[x\Big]_{-\sqrt{4-y}}^{y-2} dy + 2\int_3^4\Big[x\Big]_0^{\sqrt{4-y}} dy$

$\displaystyle = \int_0^3\left(y - 2 + \sqrt{4-y}\right) dy + 2\int_3^4 \sqrt{4-y}\,dy$

$\displaystyle = \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^3 - \left[\frac{4}{3}(4-y)^{3/2}\right]_3^4 = \frac{9}{2}$



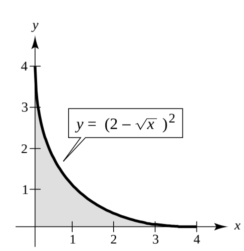

**33.** $\displaystyle\int_0^4\int_0^{(2-\sqrt{x})^2} dy\,dx = \int_0^4\Big[y\Big]_0^{(2-\sqrt{x})^2} dx$

$\displaystyle = \int_0^4\left(4 - 4\sqrt{x} + x\right) dx$

$\displaystyle = \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2}\right]_0^4 = \frac{8}{3}$

$\displaystyle\int_0^4\int_0^{(2-\sqrt{y})^2} dx\,dy = \frac{8}{3}$

Integration steps are similar to those above.



**35.** $A = \int_0^3 \int_0^{2x/3} dy\,dx + \int_3^5 \int_0^{5-x} dy\,dx$

$= \int_0^3 \left[y\right]_0^{2x/3} dx + \int_3^5 \left[y\right]_0^{5-x} dx$

$= \int_0^3 \frac{2x}{3}\,dx + \int_3^5 (5 - x)\,dx$

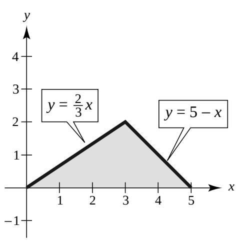
$= \left[\frac{1}{3}x^2\right]_0^3 + \left[5x - \frac{1}{2}x^2\right]_3^5 = 5$

$A = \int_0^2 \int_{3y/2}^{5-y} dx\,dy$

$= \int_0^2 \left[x\right]_{3y/2}^{5-y} dy$

$= \int_0^2 \left(5 - y - \frac{3y}{2}\right) dy$

$= \int_2^2 \left(5 - \frac{5y}{2}\right) dy = \left[5y - \frac{5}{4}y^2\right]_0^2 = 5$




**37.** $\frac{A}{4} = \int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} dy\,dx = \int_0^a \left[y\right]_0^{(b/a)\sqrt{a^2-x^2}} dx$

$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = ab\int_0^{\pi/2} \cos^2\theta\,d\theta$

$(x = a\sin\theta,\ dx = a\cos\theta\,d\theta)$

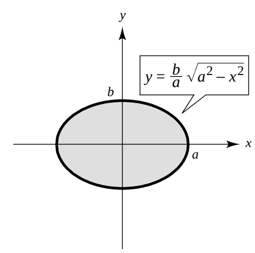
$= \frac{ab}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta = \left[\frac{ab}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2}$

$= \frac{\pi ab}{4}$

Therefore, $A = \pi ab$.

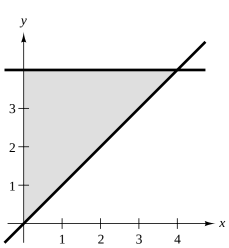
$\frac{A}{4} = \int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} dx\,dy = \frac{\pi ab}{4}$

Therefore, $A = \pi ab$. Integration steps are similar to those above.



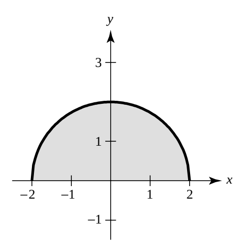

**39.** $\int_0^4 \int_0^y f(x, y)\,dx\,dy,\ 0 \le x \le y,\ 0 \le y \le 4$

$= \int_0^4 \int_x^4 f(x, y)\,dy\,dx$



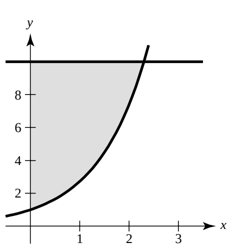
**41.** $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} f(x, y)dy\,dx, 0 \le y \le \sqrt{4 - x^2}, -2 \le x \le 2$

$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\,dy$



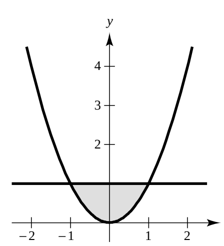
**43.** $\int_1^{10} \int_0^{\ln y} f(x, y)dx\,dy, 0 \le x \le \ln y, 1 \le y \le 10$

$= \int_0^{\ln 10} \int_{e^x}^{10} f(x, y)dy\,dx$

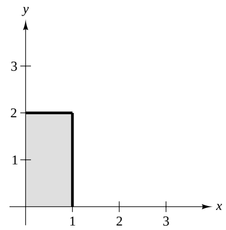


**45.** $\int_{-1}^1 \int_{x^2}^1 f(x, y)\,dy\,dx, x^2 \le y \le 1, 1 \le x \le 1$

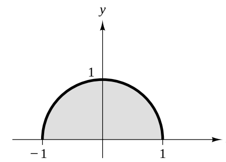
$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx\,dy$



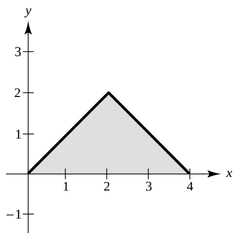
**47.** $\displaystyle\int_0^1\int_0^2 dy\,dx = \int_0^2\int_0^1 dx\,dy = 2$



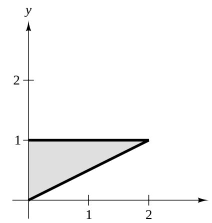
**49.** $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = \int_{-1}^1\int_0^{\sqrt{1-x^2}} dy\,dx = \frac{\pi}{2}$



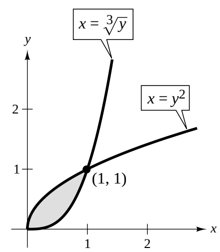
**51.** $\displaystyle\int_0^2\int_0^x dy\,dx + \int_2^4\int_0^{4-x} dy\,dx = \int_0^2\int_y^{4-y} dx\,dy = 4$



**53.** $\displaystyle\int_0^2\int_{x/2}^1 dy\,dx = \int_0^1\int_0^{2y} dx\,dy = 1$

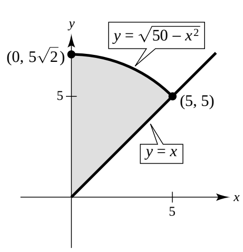


**55.** $\displaystyle\int_0^1\int_{y^2}^{\sqrt[3]{y}} dx\,dy = \int_0^1\int_{x^3}^{\sqrt{x}} dy\,dx = \frac{5}{12}$




**57.** The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

$$\int_0^5\int_x^{\sqrt{50-x^2}} x^2y^2\,dy\,dx = \int_0^5\left[\frac{1}{3}x^2(50-x^2)^{3/2} - \frac{1}{3}x^5\right]dx$$
$$= \frac{15625}{24}\pi$$
$$\int_0^5\int_0^y x^2y^2\,dx\,dy + \int_5^{5\sqrt{2}}\int_0^{\sqrt{50-y^2}} x^2y^2\,dx\,dy = \int_0^5 \frac{1}{3}y^5\,dy + \int_5^{5\sqrt{2}} \frac{1}{3}(50-y^2)^{3/2}y^2\,dy = \frac{15625}{18} + \left(\frac{15625}{18}\pi - \frac{15625}{18}\right)$$
$$= \frac{15625}{24}\pi$$



**59.** $\int_0^2 \int_x^2 x\sqrt{1+y^3}\,dy\,dx = \int_0^2 \int_0^y x\sqrt{1+y^3}\,dx\,dy = \int_0^2 \left[\sqrt{1+y^3}\cdot\frac{x^2}{2}\right]_0^y dy$

$= \frac{1}{2}\int_0^2 \sqrt{1+y^3}\,y^2\,dy = \left[\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}(1+y^3)^{3/2}\right]_0^2 = \frac{1}{9}(27) - \frac{1}{9}(1) = \frac{26}{9}$

**61.** $\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \int_0^1 \left[y\sin(x^2)\right]_0^x dx$

$= \int_0^1 x\sin(x^2)\,dx = \left[-\frac{1}{2}\cos(x^2)\right]_0^1 = -\frac{1}{2}\cos 1 + \frac{1}{2}(1) = \frac{1}{2}(1-\cos 1) \approx 0.2298$

**63.** $\int_0^2 \int_{x^2}^{2x} (x^3 + 3y^2)\,dy\,dx = \frac{1664}{105} \approx 15.848$

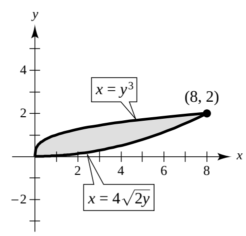
**65.** $\int_0^4 \int_0^y \frac{2}{(x+1)(y+1)}\,dx\,dy = (\ln 5)^2 \approx 2.590$

**67.** (a) $x = y^3 \Leftrightarrow y = x^{1/3}$

$x = 4\sqrt{2y} \Leftrightarrow x^2 = 32y \Leftrightarrow y = \frac{x^2}{32}$

(b) $\int_0^8 \int_{x^2/32}^{x^{1/3}} (x^2y - xy^2)\,dy\,dx$

(c) Both integrals equal $67520/693 \approx 97.43$




**69.** $\int_0^2 \int_0^{4-x^2} e^{xy}\,dy\,dx \approx 20.5648$

**71.** $\int_0^{2\pi} \int_0^{1+\cos\theta} 6r^2\cos\theta\,dr\,d\theta = \frac{15\pi}{2}$

**73.** An iterated integral is a double integral of a function of two variables. First integrate with respect to one variable while holding the other variable constant. Then integrate with respect to the second variable.

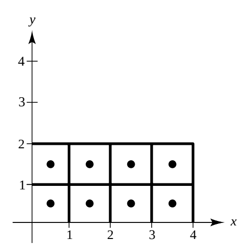
**75.** The region is a rectangle.

**77.** True

## Section 13.2 Double Integrals and Volume

**For Exercise 1–3, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are**

$\left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{3}{2},\frac{1}{2}\right), \left(\frac{5}{2},\frac{1}{2}\right), \left(\frac{7}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{3}{2}\right), \left(\frac{3}{2},\frac{3}{2}\right), \left(\frac{5}{2},\frac{3}{2}\right), \left(\frac{7}{2},\frac{3}{2}\right).$




**1.** $f(x, y) = x + y$

$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = 1 + 2 + 3 + 4 + 2 + 3 + 4 + 5 = 24$

$\int_0^4 \int_0^2 (x+y)\,dy\,dx = \int_0^4 \left[xy + \frac{y^2}{2}\right]_0^2 dx = \int_0^4 (2x+2)\,dx = \left[x^2 + 2x\right]_0^4 = 24$

**3.** $f(x, y) = x^2 + y^2$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{2}{4} + \frac{10}{4} + \frac{26}{4} + \frac{50}{4} + \frac{10}{4} + \frac{18}{4} + \frac{34}{4} + \frac{58}{4} = 52$$

$$\int_0^4 \int_0^2 (x^2 + y^2)\,dy\,dx = \int_0^4 \left[x^2y + \frac{y^3}{3}\right]_0^2 dx = \int_0^4 \left(2x^2 + \frac{8}{3}\right) dx = \left[\frac{2x^3}{3} + \frac{8x}{3}\right]_0^4 = \frac{160}{3}$$

**5.** $$\int_0^4 \int_0^4 f(x, y)dy\,dx \approx (32 + 31 + 28 + 23) + (31 + 30 + 27 + 22) + (28 + 27 + 24 + 19) + (23 + 22 + 19 + 14)$$

$$= 400$$

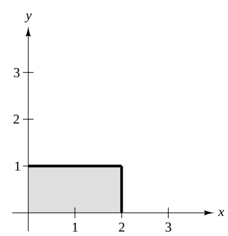
Using the corner of the $i$th square furthest from the origin, you obtain 272.

**7.** $$\int_0^2 \int_0^1 (1 + 2x + 2y)\,dy\,dx = \int_0^2 \left[y + 2xy + y^2\right]_0^1 dx$$

$$= \int_0^2 (2 + 2x)\,dx$$

$$= \left[2x + x^2\right]_0^2$$

$$= 8$$

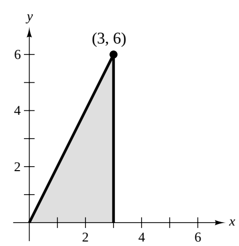



**9.** $$\int_0^6 \int_{y/2}^3 (x + y)\,dx\,dy = \int_0^6 \left[\frac{1}{2}x^2 + xy\right]_{y/2}^3 dy$$

$$= \int_0^6 \left(\frac{9}{2} + 3y - \frac{5}{8}y^2\right) dy$$

$$= \left[\frac{9}{2}y + \frac{3}{2}y^2 - \frac{5}{24}y^3\right]_0^6$$

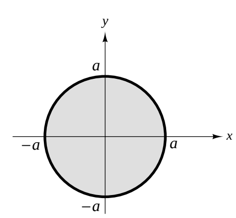
$$= 36$$




**11.** $$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y)\,dy\,dx = \int_{-a}^{a} \left[xy + \frac{1}{2}y^2\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} 2x\sqrt{a^2 - x^2}\,dx$$

$$= \left[-\frac{2}{3}(a^2 - x^2)^{3/2}\right]_{-a}^{a} = 0$$

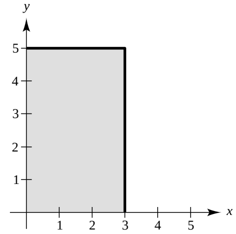



**13.** $$\int_0^5 \int_0^3 xy\,dx\,dy = \int_0^3 \int_0^5 xy\,dy\,dx$$

$$= \int_0^3 \left[\frac{1}{2}xy^2\right]_0^5 dx$$

$$= \frac{25}{2}\int_0^3 x\,dx$$

$$= \left[\frac{25}{4}x^2\right]_0^3 = \frac{225}{4}$$



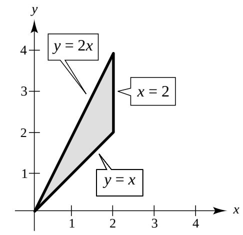
**15.** $\displaystyle\int_0^2\int_{y/2}^y \frac{y}{x^2+y^2}\,dx\,dy + \int_2^4\int_{y/2}^2 \frac{y}{x^2+y^2}\,dx\,dy = \int_0^2\int_x^{2x} \frac{y}{x^2+y^2}\,dy\,dx$

$$= \frac{1}{2}\int_0^2 \Big[\ln(x^2+y^2)\Big]_x^{2x}\,dx$$

$$= \frac{1}{2}\int_0^2 (\ln 5x^2 - \ln 2x^2)\,dx$$

$$= \frac{1}{2}\ln\frac{5}{2}\int_0^2 dx$$

$$= \left[\frac{1}{2}\left(\ln\frac{5}{2}\right)x\right]_0^2 = \ln\frac{5}{2}$$

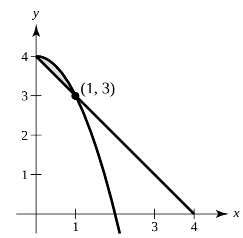



**17.** $\displaystyle\int_3^4\int_{4-y}^{\sqrt{4-y}} -2y\ln x\,dx\,dy = \int_0^1\int_{4-x}^{4-x^2} -2y\ln x\,dy\,dx$

$$= -\int_0^1 \Big[\ln x \cdot y^2\Big]_{4-x}^{4-x^2}\,dx$$

$$= -\int_0^1 [\ln x[(4-x^2)^2 - (4-x)^2]]\,dx$$

$$= \frac{26}{25}$$

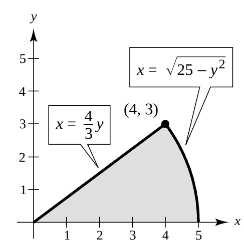



**19.** $\displaystyle\int_0^4\int_0^{3x/4} x\,dy\,dx + \int_4^5\int_0^{\sqrt{25-x^2}} x\,dy\,dx = \int_0^3\int_{4y/3}^{\sqrt{25-y^2}} x\,dx\,dy$

$$= \int_0^3 \left[\frac{1}{2}x^2\right]_{4y/3}^{\sqrt{25-y^2}} dy$$

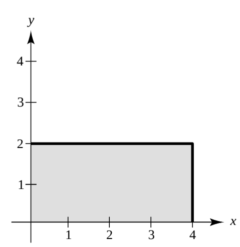
$$= \frac{25}{18}\int_0^3 (9-y^2)\,dy$$

$$= \left[\frac{25}{18}\left(9y - \frac{1}{3}y^3\right)\right]_0^3 = 25$$




**21.** $\displaystyle\int_0^4\int_0^2 \frac{y}{2}\,dy\,dx = \int_0^4 \left[\frac{y^2}{4}\right]_0^2 dx$

$$= \int_0^4 dx = 4$$

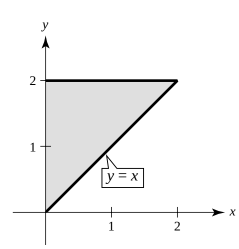


**23.** $\displaystyle\int_0^2\int_0^y (4-x-y)dx\,dy = \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_0^y dy$

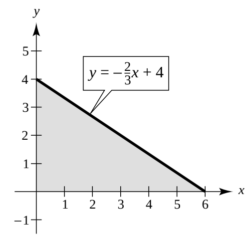
$$= \int_0^2 \left(4y - \frac{y^2}{2} - y^2\right)dy$$

$$= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3}\right]_0^2$$

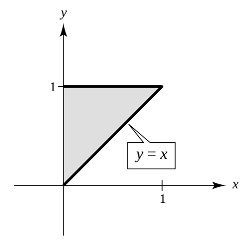
$$= 8 - \frac{8}{6} - \frac{8}{3} = 4$$



**25.** $$\int_0^6 \int_0^{(-2/3)x+4} \left(\frac{12 - 2x - 3y}{4}\right) dy\,dx = \int_0^6 \left[\frac{1}{4}\left(12y - 2xy - \frac{3}{2}y^2\right)\right]_0^{(-2/3)x+4} dx$$
$$= \int_0^6 \left(\frac{1}{6}x^2 - 2x + 6\right) dx$$
$$= \left[\frac{1}{18}x^3 - x^2 + 6x\right]_0^6$$
$$= 12$$



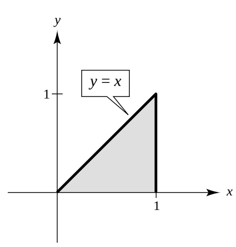

**27.** $$\int_0^1 \int_0^y (1 - xy)\,dx\,dy = \int_0^1 \left[x - \frac{x^2y}{2}\right]_0^y dy$$
$$= \int_0^1 \left(y - \frac{y^3}{2}\right) dy$$
$$= \left[\frac{y^2}{2} - \frac{y^4}{8}\right]_0^1$$
$$= \frac{3}{8}$$



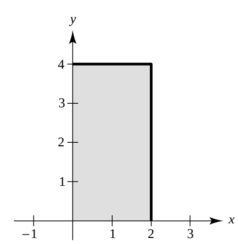

**29.** $$\int_0^\infty \int_0^\infty \frac{1}{(x+1)^2(y+1)^2}\,dy\,dx = \int_0^\infty \left[-\frac{1}{(x+1)^2(y+1)}\right]_0^\infty dx = \int_0^\infty \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{(x+1)}\right]_0^\infty = 1$$

**31.** $$4\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\,dy\,dx = 8\pi$$

**33.** $$V = \int_0^1 \int_0^x xy\,dy\,dx$$
$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_0^x dx = \frac{1}{2}\int_0^1 x^3\,dx$$
$$= \left[\frac{1}{8}x^4\right]_0^1 = \frac{1}{8}$$

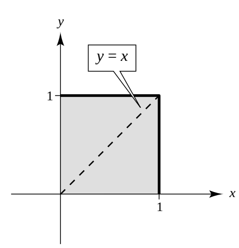



**35.** $$V = \int_0^2 \int_0^4 x^2\,dy\,dx$$
$$= \int_0^2 \left[x^2y\right]_0^4 dx = \int_0^2 4x^2\,dx$$
$$= \left[\frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$$



**37.** Divide the solid into two equal parts.

$$V = 2\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$$
$$= 2\int_0^1 \left[y\sqrt{1 - x^2}\right]_0^x dx$$
$$= 2\int_0^1 x\sqrt{1 - x^2}\,dx$$
$$= \left[-\frac{2}{3}(1 - x^2)^{3/2}\right]_0^1 = \frac{2}{3}$$

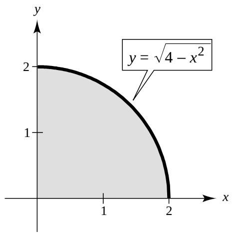


**39.** $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y)\, dy\, dx$

$= \int_0^2 \left[xy + \frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}} dx$

$= \int_0^2 \left(x\sqrt{4 - x^2} + 2 - \frac{1}{2}x^2\right) dx$

$= \left[-\frac{1}{3}(4 - x^2)^{3/2} + 2x - \frac{1}{6}x^3\right]_0^2 = \frac{16}{3}$



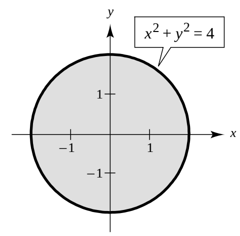

**41.** $V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx$

$= 4\int_0^2 \left[x^2\sqrt{4 - x^2} + \frac{1}{3}(4 - x^2)^{3/2}\right] dx, \quad x = 2\sin\theta$

$= 4\int_0^{\pi/2} \left(16\cos^2\theta - \frac{32}{3}\cos^4\theta\right) d\theta$

$= 4\left[16\left(\frac{\pi}{4}\right) - \frac{32}{3}\left(\frac{3\pi}{16}\right)\right]$

$= 8\pi$




**43.** $V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx = 8\pi$

**45.** $V = \int_0^2 \int_0^{-0.5x+1} \frac{2}{1 + x^2 + y^2}\, dy\, dx \approx 1.2315$

**47.** $f$ is a continuous function such that $0 \le f(x, y) \le 1$ over a region $R$ of area 1. Let $f(m, n) =$ the minimum value of $f$ over $R$ and $f(M, N) =$ the maximum value of $f$ over $R$. Then

$$f(m, n)\int\int_R dA \le \int\int_R f(x, y)\, dA \le f(M, N)\int\int_R dA.$$

Since $\int\int_R dA = 1$ and $0 \le f(m, n) \le f(M, N) \le 1$, we have $0 \le f(m, n)(1) \le \int\int_R f(x, y)\, dA \le f(M, N)(1) \le 1$.

Therefore, $0 \le \int\int_R f(x, y)\, dA \le 1$.

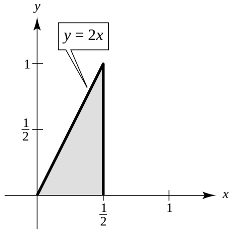
**49.** $\int_0^1 \int_{y/2}^{1/2} e^{-x^2}\, dx\, dy = \int_0^{1/2} \int_0^{2x} e^{-x^2}\, dy\, dx$

$= \int_0^{1/2} 2xe^{-x^2}\, dx$

$= \left[-e^{-x^2}\right]_0^{1/2}$

$= -e^{-1/4} + 1$

$= 1 - e^{-1/4} \approx 0.221$

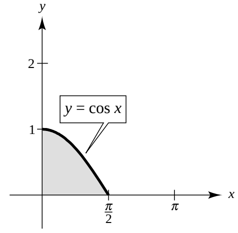



**51.** $\int_0^1 \int_0^{\arccos y} \sin x\sqrt{1 + \sin^2 x}\, dx\, dy$

$= \int_0^{\pi/2} \int_0^{\cos x} \sin x\sqrt{1 + \sin^2 x}\, dy\, dx$

$= \int_0^{\pi/2} (1 + \sin^2 x)^{1/2} \sin x \cos x\, dx$

$= \left[\frac{1}{2} \cdot \frac{2}{3}(1 + \sin^2 x)^{3/2}\right]_0^{\pi/2} = \frac{1}{3}\left[2\sqrt{2} - 1\right]$



**53.** Average $= \dfrac{1}{8}\displaystyle\int_0^4\int_0^2 x\,dy\,dx = \dfrac{1}{8}\int_0^4 2x\,dx = \left[\dfrac{x^2}{8}\right]_0^4 = 2$

**55.** Average $= \dfrac{1}{4}\displaystyle\int_0^2\int_0^2 (x^2 + y^2)\,dx\,dy$

$$= \frac{1}{4}\int_0^2 \left[\frac{x^3}{3} + xy^2\right]_0^2 dy = \frac{1}{4}\int_0^2 \left(\frac{8}{3} + 2y^2\right) dy$$

$$= \left[\frac{1}{4}\left(\frac{8}{3}y + \frac{2}{3}y^3\right)\right]_0^2 = \frac{8}{3}$$

**57.** See the definition on page 946.

**59.** The value of $\displaystyle\int_R\int f(x, y)\,dA$ would be $kB$.

**61.** Average $= \dfrac{1}{1250}\displaystyle\int_{300}^{325}\int_{200}^{250} 100x^{0.6}y^{0.4}\,dx\,dy$

$$= \frac{1}{1250}\int_{300}^{325}\left[(100y^{0.4})\frac{x^{1.6}}{1.6}\right]_{200}^{250} dy = \frac{128{,}844.1}{1250}\int_{300}^{325} y^{0.4}\,dy = 103.0753\left[\frac{y^{1.4}}{1.4}\right]_{300}^{325} \approx 25{,}645.24$$

**63.** $f(x, y) \ge 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^5\int_0^2 \frac{1}{10}\,dy\,dx = \int_0^5 \frac{1}{5}\,dx = 1$$

$$P(0 \le x \le 2, 1 \le y \le 2) = \int_0^2\int_1^2 \frac{1}{10}\,dy\,dx = \int_0^2 \frac{1}{10}\,dx = \frac{1}{5}.$$

**65.** $f(x, y) \ge 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^3\int_3^6 \frac{1}{27}(9 - x - y)\,dy\,dx$$

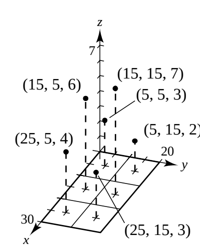
$$= \int_0^3 \frac{1}{27}\left[9y - xy - \frac{y^2}{2}\right]_3^6 dx = \int_0^3 \left(\frac{1}{2} - \frac{1}{9}x\right) dx = \left[\frac{x}{2} - \frac{x^2}{18}\right]_0^3 = 1$$

$$P(0 \le x \le 1, 4 \le y \le 6) = \int_0^1\int_4^6 \frac{1}{27}(9 - x - y)\,dy\,dx = \int_0^1 \frac{2}{27}(4 - x)\,dx = \frac{7}{27}.$$

**67.** Divide the base into six squares, and assume the height at the center of each square is the height of the entire square.

Thus,

$$V \approx (4 + 3 + 6 + 7 + 3 + 2)(100) = 2500m^3.$$




**69.** $\displaystyle\int_0^1\int_0^2 \sin\sqrt{x + y}\,dy\,dx \qquad m = 4, n = 8$

(a) 1.78435

(b) 1.7879

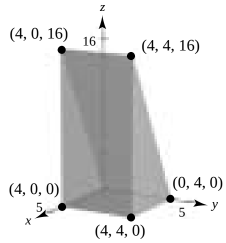
**71.** $\displaystyle\int_4^6\int_0^2 y\cos\sqrt{x}\,dx\,dy \qquad m = 4, n = 8$

(a) 11.0571

(b) 11.0414

**73.** $V \approx 125$

Matches d.



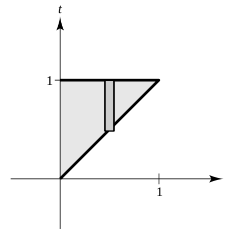

**75.** False

$$V = 8\int_0^1\int_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\,dx\,dy$$

**77.** Average $= \displaystyle\int_0^1 f(x)\,dx = \int_0^1\int_1^x e^{t^2}\,dt\,dx = -\int_0^1\int_x^1 e^{t^2}\,dt\,dx$

$$= -\int_0^1\int_0^t e^{t^2}\,dx\,dt = -\int_0^1 te^{t^2}\,dt$$

$$= \left[-\frac{1}{2}e^{t^2}\right]_0^1 = -\frac{1}{2}(e - 1) = \frac{1}{2}(1 - e)$$



## Section 13.3 Change of Variables: Polar Coordinates

**1.** Rectangular coordinates

**3.** Polar coordinates

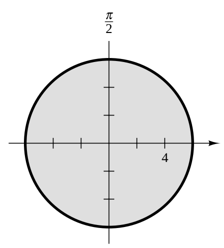
**5.** $R = \{(r, \theta)\colon\ 0 \le r \le 8, 0 \le \theta \le \pi\}$

**7.** $R = \{(r, \theta)\colon\ 0 \le r \le 3 + 3\sin\theta, 0 \le \theta \le 2\pi\}$ Cardioid

**9.** $\displaystyle\int_0^{2\pi}\int_0^6 3r^2\sin\theta\,dr\,d\theta = \int_0^{2\pi}\left[r^3\sin\theta\right]_0^6 d\theta$

$$= \int_0^{2\pi} 216\sin\theta\,d\theta$$

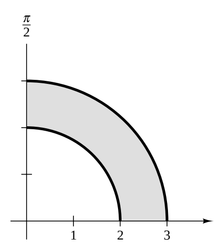
$$= \left[-216\cos\theta\right]_0^{2\pi} = 0$$



**11.** $\displaystyle\int_0^{\pi/2}\int_2^3 \sqrt{9 - r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{1}{3}(9 - r^2)^{3/2}\right]_2^3 d\theta$

$$= \left[\frac{5\sqrt{5}}{3}\theta\right]_0^{\pi/2}$$

$$= \frac{5\sqrt{5}\pi}{6}$$

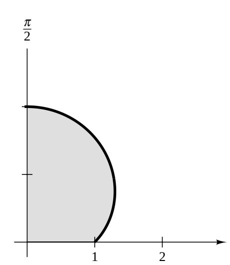


**13.** $\displaystyle\int_0^{\pi/2}\int_0^{1+\sin\theta} \theta r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{\theta r^2}{2}\right]_0^{1+\sin\theta} d\theta$

$$= \int_0^{\pi/2}\frac{1}{2}\theta(1 + \sin\theta)^2\,d\theta$$

$$= \left[\frac{1}{8}\theta^2 + \sin\theta - \theta\cos\theta + \frac{1}{2}\theta\left(-\frac{1}{2}\cos\theta\cdot\sin\theta + \frac{1}{2}\theta\right) + \frac{1}{8}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{3}{32}\pi^2 + \frac{9}{8}$$



**15.** $\displaystyle\int_0^a\int_0^{\sqrt{a^2-y^2}} y\,dx\,dy = \int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\sin\theta\,d\theta = \left[\frac{a^3}{3}(-\cos\theta)\right]_0^{\pi/2} = \frac{a^3}{3}$

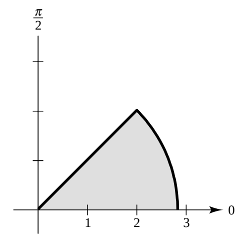
**17.** $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \int_0^{\pi/2}\int_0^3 r^4\,dr\,d\theta = \frac{243}{5}\int_0^{\pi/2} d\theta = \frac{243\pi}{10}$

**19.** $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} xy\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta = 4\int_0^{\pi/2}\cos^5\theta\sin\,d\theta = \left[-\frac{4\cos^6\theta}{6}\right]_0^{\pi/2} = \frac{2}{3}$

**21.** $\displaystyle\int_0^2\int_0^x \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta$

$$= \int_0^{\pi/4}\frac{16\sqrt{2}}{3}\,d\theta$$

$$= \frac{4\sqrt{2}\pi}{3}$$



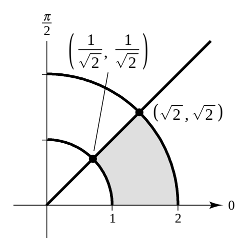

**23.** $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx = \int_0^{\pi/2}\int_0^2 (r\cos\theta + r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 (\cos\theta+\sin\theta)r^2\,dr\,d\theta$

$$= \frac{8}{3}\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta = \left[\frac{8}{3}(\sin\theta-\cos\theta)\right]_0^{\pi/2} = \frac{16}{3}$$

**25.** $\displaystyle\int_0^{1/\sqrt{2}}\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan\frac{y}{x}\,dx\,dy + \int_{1/\sqrt{2}}^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \arctan\frac{y}{x}\,dx\,dy$

$$= \int_0^{\pi/4}\int_1^2 \theta r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\frac{3}{2}\theta\,d\theta = \left[\frac{3\theta^2}{4}\right]_0^{\pi/4} = \frac{3\pi^2}{64}$$




**27.** $\displaystyle V = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^1 r^3\sin 2\theta\,dr\,d\theta = \frac{1}{8}\int_0^{\pi/2}\sin 2\theta\,d\theta = \left[-\frac{1}{16}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}$$

**29.** $\displaystyle V = \int_0^{2\pi}\int_0^5 r^2\,dr\,d\theta = \frac{250\pi}{3}$

**31.** $\displaystyle V = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{\pi/2}\left[-\frac{1}{3}\left(\sqrt{16-r^2}\right)^3\right]_0^{4\cos\theta} d\theta = -\frac{2}{3}\int_0^{\pi/2}(64\sin^3\theta - 64)\,d\theta$

$$= \frac{128}{3}\int_0^{\pi/2}[1-\sin\theta(1-\cos^2\theta)]\,d\theta = \frac{128}{3}\left[\theta+\cos\theta-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{64}{9}(3\pi-4)$$

**33.** $V = \int_0^{2\pi}\int_a^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_a^4 d\theta = \frac{1}{3}\left(\sqrt{16 - a^2}\right)^3(2\pi)$

One-half the volume of the hemisphere is $(64\pi)/3$.

$$\begin{aligned}
\frac{2\pi}{3}(16 - a^2)^{3/2} &= \frac{64\pi}{3}\\
(16 - a^2)^{3/2} &= 32\\
16 - a^2 &= 32^{2/3}\\
a^2 &= 16 - 32^{2/3} = 16 - 8\sqrt[3]{2}\\
a &= \sqrt{4\left(4 - 2\sqrt[3]{2}\right)} = 2\sqrt{4 - 2\sqrt[3]{2}} \approx 2.4332
\end{aligned}$$

**35.** Total Volume $= V = \int_0^{2\pi}\int_0^4 25e^{-r^2/4}\, r\, dr\, d\theta$

$$\begin{aligned}
&= \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^4 d\theta\\
&= \int_0^{2\pi} -50(e^{-4} - 1)\, d\theta\\
&= (1 - e^{-4})\,100\pi \approx 308.40524
\end{aligned}$$

Let $c$ be the radius of the hole that is removed.

$$\begin{aligned}
\frac{1}{10}V &= \int_0^{2\pi}\int_0^c 25e^{-r^2/4}\, r\, dr\, d\theta = \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^c d\theta\\
&= \int_0^{2\pi} -50(e^{-c^2/4} - 1)\, d\theta \Longrightarrow 30.84052 = 100\pi(1 - e^{-c^2/4})\\
&\Longrightarrow e^{-c^2/4} = 0.90183\\
&-\frac{c^2}{4} = -0.10333\\
&c^2 = 0.41331\\
&c = 0.6429\\
&\Longrightarrow \text{diameter} = 2c = 1.2858
\end{aligned}$$

**37.** $A = \int_0^{\pi}\int_0^{6\cos\theta} r\, dr\, d\theta = \int_0^{\pi} 18\cos^2\theta\, d\theta = 9\int_0^{\pi}(1 + \cos 2\theta)\, d\theta = \left[9\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi} = 9\pi$

**39.** $\int_0^{2\pi}\int_0^{1+\cos\theta} r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)\, d\theta$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \frac{3\pi}{2}$$

**41.** $3\int_0^{\pi/3}\int_0^{2\sin 3\theta} r\, dr\, d\theta = \frac{3}{2}\int_0^{\pi/3} 4\sin^2 3\theta\, d\theta = 3\int_0^{\pi/3}(1 - \cos 6\theta)\, d\theta = 3\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \pi$

**43.** Let $R$ be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$, and the lines $\theta = a$ and $\theta = b$.

When using polar coordinates to evaluate a double integral over $R$, $R$ can be partitioned into small polar sectors.

**45.** $r$-simple regions have fixed bounds for $\theta$.

$\theta$-simple regions have fixed bounds for $r$.

**47.** You would need to insert a factor of $r$ because of the $r\,dr\,d\theta$ nature of polar coordinate integrals. The plane regions would be sectors of circles.

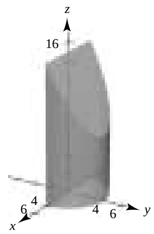
**49.** $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^5 r\sqrt{1+r^3}\sin\sqrt{\theta}\,dr\,d\theta \approx 56.051$

$\left[\textbf{Note:}\text{ This integral equals }\left(\displaystyle\int_{\pi/4}^{\pi/2}\sin\sqrt{\theta}\,d\theta\right)\left(\displaystyle\int_0^5 r\sqrt{1+r^3}\,dr\right)\right]$

**51.** Volume $=$ base $\times$ height

$\approx 8\pi \times 12 \approx 300$

Answer (c)




**53.** False

Let $f(r, \theta) = r - 1$ where $R$ is the circular sector $0 \le r \le 6$ and $0 \le \theta \le \pi$. Then,

$$\int_R\int (r-1)\,dA > 0 \quad \text{but} \quad r - 1 \not> 0 \text{ for all } r.$$

**55.** (a) $\displaystyle I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dA = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2/2}\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left[-e^{-r^2/2}\right]_0^{\infty} d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

(b) Therefore, $I = \sqrt{2\pi}$.

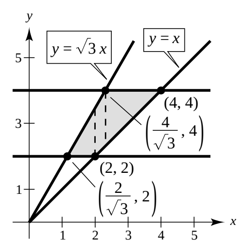
**57.** $\displaystyle\int_{-7}^{7}\int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}} 4000e^{-0.01(x^2+y^2)}\,dy\,dx = \int_0^{2\pi}\int_0^7 4000e^{-0.01r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left[-200{,}000e^{-0.01r^2}\right]_0^7 d\theta$

$= 2\pi(-200{,}000)(e^{-0.49} - 1) = 400{,}000\pi(1 - e^{-0.49}) \approx 486{,}788$

**59.** (a) $\displaystyle\int_2^4\int_{y/\sqrt{3}}^{y} f\,dx\,dy$

(b) $\displaystyle\int_{2/\sqrt{3}}^{2}\int_2^{\sqrt{3}x} f\,dy\,dx + \int_2^{4/\sqrt{3}}\int_x^{\sqrt{3}x} f\,dy\,dx + \int_{4/\sqrt{3}}^{4}\int_x^4 f\,dy\,dx$

(c) $\displaystyle\int_{\pi/4}^{\pi/3}\int_{2\csc\theta}^{4\csc\theta} fr\,dr\,d\theta$




**61.** $\displaystyle A = \frac{\Delta\theta r_2^2}{2} - \frac{\Delta\theta r_1^2}{2} = \Delta\theta\left(\frac{r_1 + r_2}{2}\right)(r_2 - r_1) = r\Delta r\Delta\theta$

## Section 13.4 Center of Mass and Moments of Inertia

**1.** $\displaystyle m = \int_0^4\int_0^3 xy\,dy\,dx = \int_0^4\left[\frac{xy^2}{2}\right]_0^3 dx = \int_0^4 \frac{9}{2}x\,dx = \left[\frac{9x^2}{4}\right]_0^4 = 36$

**3.** $\displaystyle m = \int_0^{\pi/2}\int_0^2 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 \cos\theta\sin\theta\cdot r^3\,dr\,d\theta$

$\displaystyle = \int_0^{\pi/2} 4\cos\theta\sin\theta\,d\theta$

$\displaystyle = \left[4\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 2$

**5.** (a) $m = \int_0^a \int_0^b k\,dy\,dx = kab$

$M_x = \int_0^a \int_0^b ky\,dy\,dx = \dfrac{kab^2}{2}$

$M_y = \int_0^a \int_0^b kx\,dy\,dx = \dfrac{ka^2b}{2}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^2b/2}{kab} = \dfrac{a}{2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{kab^2/2}{kab} = \dfrac{b}{2}$

$(\bar{x}, \bar{y}) = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

(b) $m = \int_0^a \int_0^b ky\,dy\,dx = \dfrac{kab^2}{2}$

$M_x = \int_0^a \int_0^b ky^2\,dy\,dx = \dfrac{kab^3}{3}$

$M_y = \int_0^a \int_0^b kxy\,dy\,dx = \dfrac{ka^2b^2}{4}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^2b^2/4}{kab^2/2} = \dfrac{a}{2}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{kab^3/3}{kab^2/2} = \dfrac{2}{3}b$

$(\bar{x}, \bar{y}) = \left(\dfrac{a}{2}, \dfrac{2}{3}b\right)$

(c) $m = \int_0^a \int_0^b kx\,dy\,dx = k\int_0^a xb\,dx = \dfrac{1}{2}ka^2b$

$M_x = \int_0^a \int_0^b kxy\,dy\,dx = \dfrac{ka^2b^2}{4}$

$M_y = \int_0^a \int_0^b kx^2\,dy\,dx = \dfrac{ka^3b}{3}$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{ka^3b/3}{ka^2b/2} = \dfrac{2}{3}a$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{ka^2b^2/4}{ka^2b/2} = \dfrac{b}{2}$

$(\bar{x}, \bar{y}) = \left(\dfrac{2}{3}a, \dfrac{b}{2}\right)$

**7.** (a) $m = \dfrac{k}{2}bh$

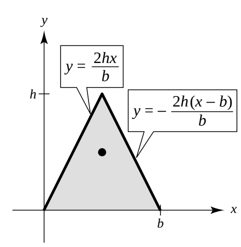
$\bar{x} = \dfrac{b}{2}$ by symmetry

$M_x = \int_0^{b/2} \int_0^{2hx/b} ky\,dy\,dx + \int_{b/2}^{b} \int_0^{-2h(x-b)/b} ky\,dy\,dx$

$= \dfrac{kbh^2}{12} + \dfrac{kbh^2}{12} = \dfrac{kbh^2}{6}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{kbh^2/6}{kbh/2} = \dfrac{h}{3}$

$(\bar{x}, \bar{y}) = \left(\dfrac{b}{2}, \dfrac{h}{3}\right)$

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**7. —CONTINUED—**

(b) $m = \int_0^{b/2}\int_0^{2hx/b} ky\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} ky\,dy\,dx = \frac{kbh^2}{6}$

$M_x = \int_0^{b/2}\int_0^{2hx/b} ky^2\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} ky^2\,dy\,dx = \frac{kbh^3}{12}$

$M_y = \int_0^{b/2}\int_0^{2hx/b} kxy\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kxy\,dy\,dx = \frac{kb^2h^2}{12}$

$\bar{x} = \frac{M_y}{m} = \frac{kb^2h^2/12}{kbh^2/6} = \frac{b}{2}$

$\bar{y} = \frac{M_x}{m} = \frac{kbh^3/12}{kbh^2/6} = \frac{h}{2}$

(c) $m = \int_0^{b/2}\int_0^{2hx/b} kx\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kx\,dy\,dx$

$= \frac{1}{12}kb^2h + \frac{1}{6}kb^2h = \frac{1}{4}kb^2h$

$M_x = \int_0^{b/2}\int_0^{2hx/b} kxy\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kxy\,dy\,dx$

$= \frac{1}{32}kh^2b^2 + \frac{5}{96}kh^2b^2 = \frac{1}{12}kh^2b^2$

$M_y = \int_0^{b/2}\int_0^{2hx/b} kx^2\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kx^2\,dy\,dx$

$= \frac{1}{32}kb^3h + \frac{11}{96}kb^3h = \frac{7}{48}kb^3h$

$\bar{x} = \frac{M_y}{m} = \frac{7kb^3h/48}{kb^2h/4} = \frac{7}{12}b$

$\bar{y} = \frac{M_x}{m} = \frac{kh^2b^2/12}{kb^2h/4} = \frac{h}{3}$

**9.** (a) The $x$-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{b}{2}\right)$

(b) The $x$-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{2b}{3}\right)$

(c) $m = \int_5^{a+5}\int_0^{b} kx\,dy\,dx = \frac{1}{2}k(a+5)^2b - \frac{25}{2}kb$

$M_x = \int_5^{a+5}\int_0^{b} kxy\,dy\,dx = \frac{1}{4}k(a+5)^2b^2 - \frac{25}{4}kb^2$

$M_y = \int_5^{a+5}\int_0^{b} kx^2\,dy\,dx = \frac{1}{3}k(a+5)^3\,b - \frac{125}{3}kb$

$\bar{x} = \frac{M_y}{m} = \frac{2(a^2 + 15a + 75)}{3(a + 10)}$

$\bar{y} = \frac{M_x}{m} = \frac{b}{2}$

**11.** (a) $\bar{x} = 0$ by symmetry

$m = \frac{\pi a^2 k}{2}$

$M_x = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} yk\,dy\,dx = \frac{2a^3k}{3}$

$\bar{y} = \frac{M_x}{m} = \frac{2a^3k}{3} \cdot \frac{2}{\pi a^2 k} = \frac{4a}{3\pi}$

(b) $m = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} k(a-y)\,y\,dy\,dx = \frac{a^4k}{24}(16 - 3\pi)$

$M_x = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} k(a-y)\,y^2\,dy\,dx = \frac{a^5k}{120}(15\pi - 32)$

$M_y = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} kx(a-y)\,y\,dy\,dx = 0$

$\bar{x} = \frac{M_y}{m} = 0$

$\bar{y} = \frac{M_x}{m} = \frac{a}{5}\left[\frac{15\pi - 32}{16 - 3\pi}\right]$

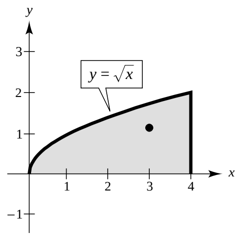
**13.** $m = \int_0^4 \int_0^{\sqrt{x}} kxy\, dy\, dx = \frac{32k}{3}$

$M_x = \int_0^4 \int_0^{\sqrt{x}} kxy^2\, dy\, dx = \frac{256k}{21}$

$M_y = \int_0^4 \int_0^{\sqrt{x}} kx^2 y\, dy\, dx = 32k$

$\bar{x} = \frac{M_y}{m} = \frac{32k}{1} \cdot \frac{3}{32k} = 3$

$\bar{y} = \frac{M_x}{m} = \frac{256k}{21} \cdot \frac{3}{32k} = \frac{8}{7}$

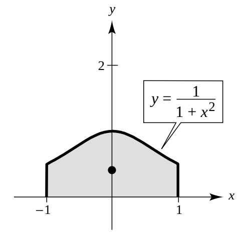



**15.** $\bar{x} = 0$ by symmetry

$m = \int_{-1}^1 \int_0^{1/(1+x^2)} k\, dy\, dx = \frac{k\pi}{2}$

$M_x = \int_{-1}^1 \int_0^{1/(1+x^2)} ky\, dy\, dx = \frac{k}{8}(2 + \pi)$

$\bar{y} = \frac{M_x}{m} = \frac{k}{8}(2 + \pi) \cdot \frac{2}{k\pi} = \frac{2 + \pi}{4\pi}$

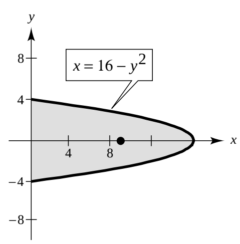



**17.** $\bar{y} = 0$ by symmetry

$m = \int_{-4}^4 \int_0^{16-y^2} kx\, dx\, dy = \frac{8192k}{15}$

$M_y = \int_{-4}^4 \int_0^{16-y^2} kx^2\, dx\, dy = \frac{524{,}288k}{105}$

$\bar{x} = \frac{M_y}{m} = \frac{524{,}288k}{105} \cdot \frac{15}{8192k} = \frac{64}{7}$

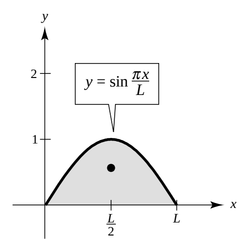



**19.** $\bar{x} = \frac{L}{2}$ by symmetry

$m = \int_0^L \int_0^{\sin \pi x/L} ky\, dy\, dx = \frac{kL}{4}$

$M_x = \int_0^L \int_0^{\sin \pi x/L} ky^2\, dy\, dx = \frac{4kL}{9\pi}$

$\bar{y} = \frac{M_x}{m} = \frac{4kL}{9\pi} \cdot \frac{4}{kL} = \frac{16}{9\pi}$



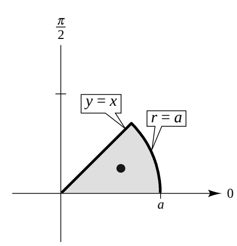

**21.** $m = \frac{\pi a^2 k}{8}$

$M_x = \iint_R ky\, dA = \int_0^{\pi/4} \int_0^a kr^2 \sin\theta\, dr\, d\theta = \frac{ka^3(2 - \sqrt{2})}{6}$

$M_y = \iint_R kx\, dA = \int_0^{\pi/4} \int_0^a kr^2 \cos\theta\, dr\, d\theta = \frac{ka^3\sqrt{2}}{6}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^3\sqrt{2}}{6} \cdot \frac{8}{\pi a^2 k} = \frac{4a\sqrt{2}}{3\pi}$

$\bar{y} = \frac{M_x}{m} = \frac{ka^3(2 - \sqrt{2})}{6} \cdot \frac{8}{\pi a^2 k} = \frac{4a(2 - \sqrt{2})}{3\pi}$



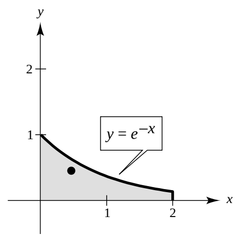
**23.** $m = \int_0^2 \int_0^{e^{-x}} ky\,dy\,dx = \frac{k}{4}(1 - e^{-4})$

$M_x = \int_0^2 \int_0^{e^{-x}} ky^2\,dy\,dx = \frac{k}{9}(1 - e^{-6})$

$M_y = \int_0^2 \int_0^{e^{-x}} kxy\,dy\,dx = \frac{k(1 - 5e^{-4})}{8}$

$\bar{x} = \frac{M_y}{m} = \frac{k(e^4 - 5)}{8e^4} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{e^4 - 5}{2(e^4 - 1)} \approx 0.46$

$\bar{y} = \frac{M_x}{m} = \frac{k(e^6 - 1)}{9e^6} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{4}{9}\left[\frac{e^6 - 1}{e^6 - e^2}\right] \approx 0.45$



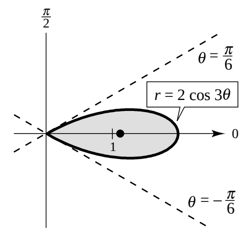

**25.** $\bar{y} = 0$ by symmetry

$m = \iint_R k\,dA = \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr\,dr\,d\theta = \frac{k\pi}{3}$

$M_y = \iint_R kx\,dA$

$= \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr^2 \cos\theta\,dr\,d\theta \approx 1.17k$

$\bar{x} = \frac{M_y}{m} \approx 1.17k\left(\frac{3}{\pi k}\right) \approx 1.12$




**27.** $m = bh$

$I_x = \int_0^b \int_0^h y^2\,dy\,dx = \frac{bh^3}{3}$

$I_y = \int_0^b \int_0^h x^2 dy\,dx = \frac{b^3h}{3}$

$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{b^2}{3}} = \frac{b}{\sqrt{3}} = \frac{\sqrt{3}}{3}b$

$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{h^2}{3}} = \frac{h}{\sqrt{3}} = \frac{\sqrt{3}}{3}h$

**29.** $m = \pi a^2$

$I_x = \iint_R y^2\,dA = \int_0^{2\pi} \int_0^a r^3 \sin^2\theta\,dr\,d\theta = \frac{a^4\pi}{4}$

$I_y = \iint_R x^2\,dA = \int_0^{2\pi} \int_0^a r^3 \cos^2\theta\,dr\,d\theta = \frac{a^4\pi}{4}$

$I_0 = I_x + I_y = \frac{a^4\pi}{4} + \frac{a^4\pi}{4} = \frac{a^4\pi}{2}$

$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{4} \cdot \frac{1}{\pi a^2}} = \frac{a}{2}$

**31.** $m = \frac{\pi a^2}{4}$

$I_x = \iint_R y^2\,dA = \int_0^{\pi/2} \int_0^a r^3 \sin^2\theta\,dr\,d\theta = \frac{\pi a^4}{16}$

$I_y = \iint_R x^2\,dA = \int_0^{\pi/2} \int_0^a r^3 \cos^2\theta\,dr\,d\theta = \frac{\pi a^4}{16}$

$I_0 = I_x + I_y = \frac{\pi a^4}{16} + \frac{\pi a^4}{16} = \frac{\pi a^4}{8}$

$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\pi a^4}{16} \cdot \frac{4}{\pi a^2}} = \frac{a}{2}$

**33.** $\rho = ky$

$m = k\int_0^a \int_0^b y\,dy\,dx = \frac{kab^2}{2}$

$I_x = k\int_0^a \int_0^b y^3\,dy\,dx = \frac{kab^4}{4}$

$I_y = k\int_0^a \int_0^b x^2 y\,ydy\,dx = \frac{ka^3b^2}{6}$

$I_0 = I_x + I_y = \frac{3kab^4 + 2kb^2a^3}{12}$

$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{ka^3b^2/6}{kab^2/2}} = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3}a$

$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{kab^4/4}{kab^2/2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}} = \frac{\sqrt{2}}{2}b$

**35.** $\rho = kx$

$$m = k\int_0^2\int_0^{4-x^2} x\,dy\,dx = 4k$$

$$I_x = k\int_0^2\int_0^{4-x^2} xy^2\,dy\,dx = \frac{32k}{3}$$

$$I_y = k\int_0^2\int_0^{4-x^2} x^3\,dy\,dx = \frac{16k}{3}$$

$$I_0 = I_x + I_y = 16k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16k/3}{4k}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32k/3}{4k}} = \sqrt{\frac{8}{3}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

**37.** $\rho = kxy$

$$m = \int_0^4\int_0^{\sqrt{x}} kxy\,dy\,dx = \frac{32k}{3}$$

$$I_x = \int_0^4\int_0^{\sqrt{x}} kxy^3\,dy\,dx = 16k$$

$$I_y = \int_0^4\int_0^{\sqrt{x}} kx^3y\,dy\,dx = \frac{512k}{5}$$

$$I_0 = I_x + I_y = \frac{592k}{5}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k}{5}\cdot\frac{3}{32k}} = \sqrt{\frac{48}{5}} = \frac{4\sqrt{15}}{5}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16k}{1}\cdot\frac{3}{32k}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

**39.** $\rho = kx$

$$m = \int_0^1\int_{x^2}^{\sqrt{x}} kx\,dy\,dx = \frac{3k}{20}$$

$$I_x = \int_0^1\int_{x^2}^{\sqrt{x}} kxy^2\,dy\,dx = \frac{3k}{56}$$

$$I_y = \int_0^1\int_{x^2}^{\sqrt{x}} kx^3\,dy\,dx = \frac{k}{18}$$

$$I_0 = I_x + I_y = \frac{55k}{504}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k}{18}\cdot\frac{20}{3k}} = \frac{\sqrt{30}}{9}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{3k}{56}\cdot\frac{20}{3k}} = \frac{\sqrt{70}}{14}$$

**41.** $$I = 2k\int_{-b}^{b}\int_0^{\sqrt{b^2-x^2}} (x-a)^2\,dy\,dx = 2k\int_{-b}^{b}(x-a)^2\sqrt{b^2-x^2}\,dx$$

$$= 2k\left[\int_{-b}^{b} x^2\sqrt{b^2-x^2}\,dx - 2a\int_{-b}^{b} x\sqrt{b^2-x^2}\,dx + a^2\int_{-b}^{b}\sqrt{b^2-x^2}\,dx\right]$$

$$= 2k\left[\frac{\pi b^4}{8} + 0 + \frac{\pi a^2 b^2}{2}\right] = \frac{k\pi b^2}{4}(b^2 + 4a^2)$$

**43.** $$I = \int_0^4\int_0^{\sqrt{x}} kx(x-6)^2\,dy\,dx = \int_0^4 kx\sqrt{x}(x^2 - 12x + 36)\,dx = k\left[\frac{2}{9}x^{9/2} - \frac{24}{7}x^{7/2} + \frac{72}{5}x^{5/2}\right]_0^4 = \frac{42{,}752k}{315}$$

**45.** $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)(y-a)^2\,dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)^3\,dy\,dx = \int_0^a \left[-\frac{k}{4}(a-y)^4\right]_0^{\sqrt{a^2-x^2}} dx$

$$= -\frac{k}{4}\int_0^a \left[a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= -\frac{k}{4}\int_0^a \left[a^4 - 4a^3\sqrt{a^2-x^2} + 6a^2(a^2-x^2) - 4a(a^2-x^2)\sqrt{a^2-x^2} + (a^4 - 2a^2x^2 + x^4) - a^4\right] dx$$

$$= -\frac{k}{4}\int_0^a \left[7a^4 - 8a^2x^2 + x^4 - 8a^3\sqrt{a^2-x^2} + 4ax^2\sqrt{a^2-x^2}\right] dx$$

$$= -\frac{k}{4}\left[7a^4x - \frac{8a^2}{3}x^3 + \frac{x^5}{5} - 4a^3\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right) + \frac{a}{2}\left(x(2x^2-a^2)\sqrt{a^2-x^2} + a^4\arcsin\frac{x}{a}\right)\right]_0^a$$

$$= -\frac{k}{4}\left(7a^5 - \frac{8}{3}a^5 + \frac{1}{5}a^5 - 2a^5\pi + \frac{1}{4}a^5\pi\right) = a^5k\left(\frac{7\pi}{16} - \frac{17}{15}\right)$$

**47.** $\rho(x, y) = ky$. $\bar{y}$ will increase

**49.** $\rho(x, y) = kxy$.

Both $\bar{x}$ and $\bar{y}$ will increase

**51.** Let $\rho(x, y)$ be a continuous density function on the planar lamina $R$.

The movements of mass with respect to the $x$- and $y$-axes are

$$M_x = \iint_R y\,\rho(x, y)dA \text{ and } M_y = \iint_R x\,\rho(x, y)\,dA.$$

If $m$ is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

**53.** See the definition on page 968

**55.** $\bar{y} = \frac{L}{2}, A = bL, h = \frac{L}{2}$

$$I_{\bar{y}} = \int_0^b \int_0^L \left(y - \frac{L}{2}\right)^2 dy\,dx$$

$$= \int_0^b \left[\frac{[y - (L/2)]^3}{3}\right]_0^L dx = \frac{L^3b}{12}$$

$$y_a = \bar{y} - \frac{I_{\bar{y}}}{hA} = \frac{L}{2} - \frac{L^3b/12}{(L/2)(bL)} = \frac{L}{3}$$

**57.** $\bar{y} = \frac{2L}{3}. A = \frac{bL}{2}, h = \frac{L}{3}$

$$I_{\bar{y}} = 2\int_0^{b/2} \int_{2Lx/b}^L \left(y - \frac{2L}{3}\right)^2 dy\,dx$$

$$= \frac{2}{3}\int_0^{b/2} \left[\left(y - \frac{2L}{3}\right)^3\right]_{2Lx/b}^L dx$$

$$= \frac{2}{3}\int_0^{b/2} \left[\frac{L}{27} - \left(\frac{2Lx}{b} - \frac{2L}{3}\right)^3\right] dx$$

$$= \frac{2}{3}\left[\frac{L^3x}{27} - \frac{b}{8L}\left(\frac{2Lx}{b} - \frac{2L}{3}\right)^4\right]_0^{b/2} = \frac{L^3b}{36}$$

$$y_a = \frac{2L}{3} - \frac{L^3b/36}{L^2b/6} = \frac{L}{2}$$

# Section 13.5 Surface Area

**1.** $f(x, y) = 2x + 2y$

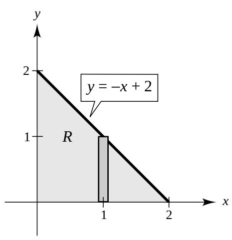
$R =$ triangle with vertices $(0, 0), (2, 0), (0, 2)$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_0^2 \int_0^{2-x} 3\, dy\, dx = 3 \int_0^2 (2 - x)\, dx$$

$$= \left[3\left(2x - \frac{x^2}{2}\right)\right]_0^2 = 6$$



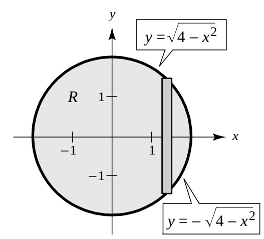

**3.** $f(x, y) = 8 + 2x + 2y$

$R = \{(x, y): x^2 + y^2 \le 4\}$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3\, dy\, dx = \int_0^{2\pi} \int_0^2 3r\, dr\, d\theta = 12\pi$$




**5.** $f(x, y) = 9 - x^2$

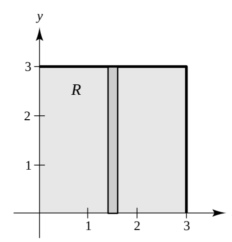
$R =$ square with vertices, $(0, 0), (3, 0), (0, 3), (3, 3)$

$f_x = -2x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4x^2}\, dy\, dx = \int_0^3 3\sqrt{1 + 4x^2}\, dx$$

$$= \left[\frac{3}{4}\left(2x\sqrt{1 + 4x^2} + \ln\left|2x + \sqrt{1 + 4x^2}\right|\right)\right]_0^3 = \frac{3}{4}\left(6\sqrt{37} + \ln\left|6 + \sqrt{37}\right|\right)$$




**7.** $f(x, y) = 2 + x^{3/2}$

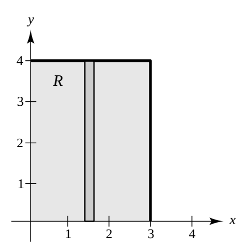
$R =$ rectangle with vertices $(0, 0), (0, 4), (3, 4), (3, 0)$

$f_x = \dfrac{3}{2}x^{1/2},\ f_y = 0$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\frac{9}{4}\right)x} = \frac{\sqrt{4 + 9x}}{2}$$

$$S = \int_0^3 \int_0^4 \frac{\sqrt{4 + 9x}}{2}\, dy\, dx = \int_0^3 4\left(\frac{\sqrt{4 + 9x}}{2}\right) dx$$

$$= \left[\frac{4}{27}(4 + 9x)^{3/2}\right]_0^3 = \frac{4}{27}\left(31\sqrt{31} - 8\right)$$



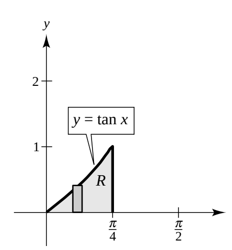

**9.** $f(x, y) = \ln|\sec x|$

$$R = \left\{(x, y): 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \tan x\right\}$$

$f_x = \tan x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \tan^2 x} = \sec x$

$$S = \int_0^{\pi/4} \int_0^{\tan x} \sec x\, dy\, dx = \int_0^{\pi/4} \sec x \tan x\, dx = \left[\sec x\right]_0^{\pi/4} = \sqrt{2} - 1$$



**11.** $f(x, y) = \sqrt{x^2 + y^2}$

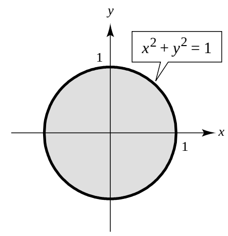
$R = \{(x, y): 0 \le f(x, y) \le 1\}$

$0 \le \sqrt{x^2 + y^2} \le 1,\ x^2 + y^2 \le 1$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}},\ f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

$$S = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2}\, dy\, dx = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\, dr\, d\theta = \sqrt{2}\pi$$



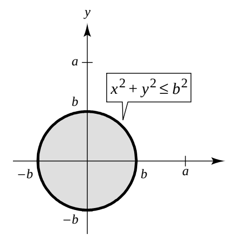

**13.** $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$R = \{(x, y): x^2 + y^2 \le b^2, b < a\}$

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}},\ f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

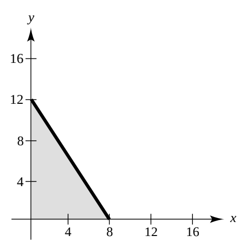
$$S = \int_{-b}^{b}\int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx = \int_0^{2\pi}\int_0^b \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a\left(a - \sqrt{a^2 - b^2}\right)$$




**15.** $z = 24 - 3x - 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_0^8\int_0^{-(3/2)x+12} \sqrt{14}\, dy\, dx = 48\sqrt{14}$$

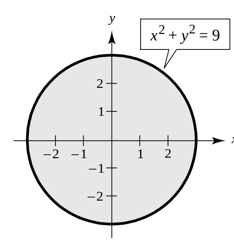



**17.** $z = \sqrt{25 - x^2 - y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}}$$

$$S = 2\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25 - (x^2 + y^2)}}\, dy\, dx$$

$$= 2\int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25 - r^2}}\, r\, dr\, d\theta = 20\pi$$

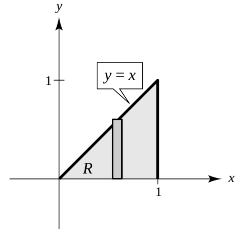



**19.** $f(x, y) = 2y + x^2$

$R$ = triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4x^2}$

$$S = \int_0^1\int_0^x \sqrt{5 + 4x^2}\, dy\, dx = \frac{1}{12}\left(27 - 5\sqrt{5}\right)$$



**21.** $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y):\ 0 \le f(x, y)\}$

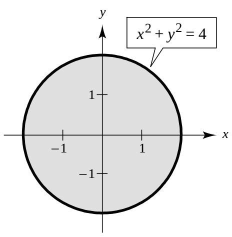
$0 \le 4 - x^2 - y^2,\ x^2 + y^2 \le 4$

$f_x = -2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{\left(17\sqrt{17} - 1\right)\pi}{6}$$




**23.** $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y):\ 0 \le x \le 1,\ 0 \le y \le 1\}$

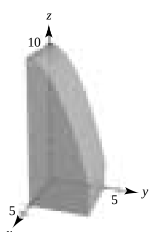
$f_x = -2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^1\int_0^1 \sqrt{(1 + 4x^2) + 4y^2}\, dy\, dx \approx 1.8616$$

**25.** Surface area $> (4) \cdot (6) = 24.$

Matches (e)




**27.** $f(x, y) = e^x$

$R = \{(x, y):\ 0 \le x \le 1,\ 0 \le y \le 1\}$

$f_x = e^x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{2x}}$

$$S = \int_0^1\int_0^1 \sqrt{1 + e^{2x}}\, dy\, dx$$

$$= \int_0^1 \sqrt{1 + e^{2x}} \approx 2.0035$$

**29.** $f(x, y) = x^3 - 3xy + y^3$

$R =$ square with vertices $(1, 1), (-1, 1), (-1, -1), (1, -1)$

$f_x = 3x^2 - 3y = 3(x^2 - y), f_y = -3x + 3y^2 = 3(y^2 - x)$

$$S = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2}\, dy\, dx$$

**31.** $f(x, y) = e^{-x}\sin y$

$f_x = -e^{-x}\sin y, f_y = e^{-x}\cos y$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + e^{-2x}\sin^2 y + e^{-2x}\cos^2 y}$$

$$= \sqrt{1 + e^{-2x}}$$

$$S = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}}\, dy\, dx$$

**33.** $f(x, y) = e^{xy}$

$R = \{(x, y):\ 0 \le x \le 4,\ 0 \le y \le 10\}$

$f_x = ye^{xy},\ f_y = xe^{xy}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2e^{2xy} + x^2e^{2xy}} = \sqrt{1 + e^{2xy}(x^2 + y^2)}$

$$S = \int_0^4\int_0^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)}\, dy\, dx$$

**35.** See the definition on page 972.

**37.** $f(x, y) = \sqrt{1 - x^2}; f_x = \dfrac{-x}{\sqrt{1^2 - x^2}}, f_y = 0$

$$S = \iint_R \sqrt{1 + f_x^{\,2} + f_y^{\,2}}\, dA$$

$$= 16\int_0^1 \int_0^x \frac{1}{\sqrt{1 - x^2}}\, dy\, dx$$

$$= 16\int_0^1 \frac{x}{\sqrt{1 - x^2}}\, dx = \Big[-16(1 - x^2)^{1/2}\Big]_0^1 = 16$$

**39.** (a) $V = \displaystyle\int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \left(20 + \frac{xy}{100} - \frac{x + y}{5}\right) dy\, dx$

$$= \int_0^{50} \left[20\sqrt{50^2 - x^2} + \frac{x}{200}(50^2 - x^2) - \frac{x}{5}\sqrt{50^2 - x^2} - \frac{50^2 - x^2}{10}\right] dy$$

$$= \left[10\left(x\sqrt{50 - x^2} + 50^2 \arcsin\frac{x}{50}\right) + \frac{25}{4}x^2 - \frac{x^4}{800} + \frac{1}{15}(50^2 - x^2)^{3/2} - 250x + \frac{x^3}{30}\right]_0^{50}$$

$$\approx 30{,}415.74 \text{ ft}^3$$

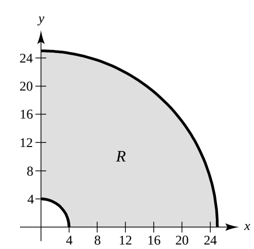
(b) $z = 20 + \dfrac{xy}{100}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{y^2}{100^2} + \frac{x^2}{100^2}} = \frac{\sqrt{100^2 + x^2 + y^2}}{100}$$

$$S = \frac{1}{100}\int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \sqrt{100^2 + x^2 + y^2}\, dy\, dx$$

$$= \frac{1}{100}\int_0^{\pi/2} \int_0^{50} \sqrt{100^2 + r^2}\, r\, dr\, d\theta \approx 2081.53 \text{ ft}^2$$

**41.** (a) $V = \displaystyle\iint_R f(x, y)$




$$= 8\iint_R \sqrt{625 - x^2 - y^2}\, dA \qquad \text{where } R \text{ is the region in the first quadrant}$$

$$= 8\int_0^{\pi/2} \int_4^{25} \sqrt{625 - r^2}\, r\, dr\, d\theta$$

$$= -4\int_0^{\pi/2} \left[\frac{2}{3}(625 - r^2)^{3/2}\right]_4^{25} d\theta$$

$$= -\frac{8}{3}\left[0 - 609\sqrt{609}\right] \cdot \frac{\pi}{2}$$

$$= 812\pi\sqrt{609} \text{ cm}^3$$

(b) $A = \displaystyle\iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = 8\iint_R \sqrt{1 + \frac{x^2}{625 - x^2 - y^2} + \frac{y^2}{625 - x^2 - y^2}}\, dA$

$$= 8\iint_R \frac{25}{\sqrt{625 - x^2 - y^2}}\, dA = 8\int_0^{\pi/2} \int_4^{25} \frac{25}{\sqrt{625 - r^2}}\, r\, dr\, d\theta$$

$$= \lim_{b \to 25^-} \left[-200\sqrt{625 - r^2}\right]_4^b \cdot \frac{\pi}{2} = 100\pi\sqrt{609} \text{ cm}^2$$

## Section 13.6 Triple Integrals and Applications

**1.** $$\int_0^3\int_0^2\int_0^1 (x+y+z)\,dx\,dy\,dx = \int_0^3\int_0^2\left[\frac{1}{2}x^2+xy+xz\right]_0^1 dy\,dx$$
$$= \int_0^3\int_0^2\left(\frac{1}{2}+y+z\right)dy\,dz = \int_0^3\left[\frac{1}{2}y+\frac{1}{2}y^2+yz\right]_0^2 dz = \left[3z+z^2\right]_0^3 = 18$$

**3.** $$\int_0^1\int_0^x\int_0^{xy} x\,dz\,dy\,dx = \int_0^1\int_0^x\left[xz\right]_0^{xy} dy\,dx$$
$$= \int_0^1\int_0^x x^2y\,dy\,dx = \int_0^1\left[\frac{x^2y^2}{2}\right]_0^x dx = \int_0^1 \frac{x^4}{2}\,dx = \left[\frac{x^5}{10}\right]_0^1 = \frac{1}{10}$$

**5.** $$\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\,dy\,dx\,dz = \int_1^4\int_0^1\left[(2ze^{-x^2})y\right]_0^x dx\,dz = \int_1^4\int_0^1 2zxe^{-x^2}\,dx\,dz$$
$$= \int_1^4\left[-ze^{-x^2}\right]_0^1 dz = \int_1^4 z(1-e^{-1})\,dz = \left[(1-e^{-1})\frac{z^2}{2}\right]_1^4 = \frac{15}{2}\left(1-\frac{1}{e}\right)$$

**7.** $$\int_0^4\int_0^{\pi/2}\int_0^{1-x} x\cos y\,dz\,dy\,dx = \int_0^4\int_0^{\pi/2}\left[(x\cos y)z\right]_0^{1-x} dy\,dx = \int_0^4\int_0^{\pi/2} x(1-x)\cos y\,dy\,dx$$
$$= \int_0^4\left[x(1-x)\sin y\right]_0^{\pi/2} dx = \int_0^4 x(1-x)dx = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^4 = 8-\frac{64}{3} = \frac{-40}{3}$$

**9.** $$\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x^2} x\,dz\,dy\,dx = \int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^3\,dy\,dx = \frac{128}{15}$$

**11.** $$\int_0^2\int_0^{\sqrt{4-x^2}}\int_1^4 \frac{x^2\sin y}{z}\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}}\left[x^2\sin y\ln|z|\right]_1^4 dy\,dx$$
$$= \int_0^2\left[x^2\ln 4(-\cos y)\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 x^2\ln 4\left[1-\cos\sqrt{4-x^2}\right]dx \approx 2.44167$$

**13.** $$\int_0^4\int_0^{4-x}\int_0^{4-x-y} dz\,dy\,dx$$

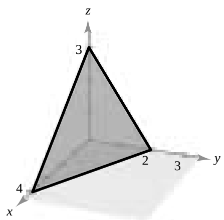
**15.** $$\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} dz\,dy\,dx$$

**17.** $$\int_{-2}^2\int_0^{4-y^2}\int_0^x dz\,dx\,dy = \int_{-2}^2\int_0^{4-y^2} x\,dx\,dy$$
$$= \frac{1}{2}\int_{-2}^2 (4-y^2)^2\,dy = \int_0^2 (16-8y^2+y^4)\,dy = \left[16y-\frac{8}{3}y^3+\frac{1}{5}y^5\right]_0^2 = \frac{256}{15}$$

**19.** $$8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$
$$= 4\int_0^a\left[y\sqrt{a^2-x^2-y^2}+(a^2-x^2)\arcsin\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$
$$= 4\left(\frac{\pi}{2}\right)\int_0^a (a^2-x^2)\,dx = \left[2\pi\left(a^2x-\frac{1}{3}x^3\right)\right]_0^a = \frac{4}{3}\pi a^3$$

**21.** $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^{4-x^2} dz\,dy\,dx = \int_0^2 (4-x^2)^2\,dx = \int_0^2 (16-8x^2+x^4)\,dx = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{256}{15}$

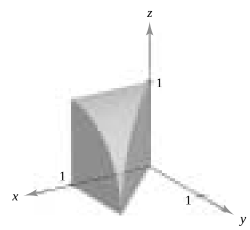
**23.** Plane: $3x + 6y + 4z = 12$




$$\int_0^3\int_0^{(12-4z)/3}\int_0^{(12-4z-3x)/6} dy\,dx\,dz$$

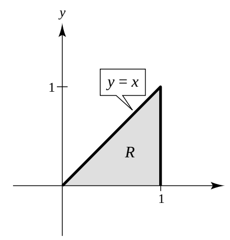
**25.** Top cylinder: $y^2 + z^2 = 1$

Side plane: $x = y$




$$\int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} dz\,dy\,dx$$

**27.** $Q = \{(x, y, z): 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 3\}$




$$\iiint_Q xyz\,dV = \int_0^3\int_0^1\int_y^1 xyz\,dx\,dy\,dz = \int_0^3\int_0^1\int_0^x xyz\,dy\,dx\,dz$$

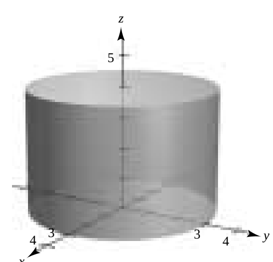
$$= \int_0^1\int_0^3\int_y^1 xyz\,dx\,dz\,dy$$

$$= \int_0^1\int_0^3\int_0^x xyz\,dy\,dz\,dx$$

$$= \int_0^1\int_y^1\int_0^3 xyz\,dz\,dx\,dy$$

$$= \int_0^1\int_0^x\int_0^3 xyz\,dz\,dy\,dx\left(=\frac{9}{16}\right)$$

**29.** $Q = \{(x, y, z): x^2 + y^2 \le 9, 0 \le z \le 4\}$




$$\iiint_Q xyz\,dV = \int_0^4\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\,dy\,dx\,dz$$

$$= \int_0^4\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\,dx\,dy\,dz$$

$$= \int_{-3}^3\int_0^4\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\,dx\,dz\,dy$$

$$= \int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^4 xyz\,dz\,dx\,dy$$

$$= \int_{-3}^3\int_0^4\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\,dy\,dz\,dx$$

$$= \int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^4 xyz\,dz\,dy\,dx\ \ (=0)$$

**31.** $m = k\int_0^6\int_0^{4-(2x/3)}\int_0^{2-(y/2)-(x/3)} dz\,dy\,dx$

$= 8k$

$M_{yz} = k\int_0^6\int_0^{4-(2x/3)}\int_0^{2-(y/2)-(x/3)} x\,dz\,dy\,dx$

$= 12k$

$\bar{x} = \dfrac{M_{yz}}{m} = \dfrac{12k}{8k} = \dfrac{3}{2}$

**33.** $m = k\int_0^4\int_0^4\int_0^{4-x} x\,dz\,dy\,dx = k\int_0^4\int_0^4 x(4-x)\,dy\,dx$

$= 4k\int_0^4 (4x - x^2)\,dx = \dfrac{128k}{3}$

$M_{xy} = k\int_0^4\int_0^4\int_0^{4-x} xz\,dz\,dy\,dx = k\int_0^4\int_0^4 x\dfrac{(4-x)^2}{2}\,dy\,dx$

$= 2k\int_0^4 (16x - 8x^2 + x^3)\,dx = \dfrac{128k}{3}$

$\bar{z} = \dfrac{M_{xy}}{m} = 1$

**35.** $m = k\int_0^b\int_0^b\int_0^b xy\,dz\,dy\,dx = \dfrac{kb^5}{4}$

$M_{yz} = k\int_0^b\int_0^b\int_0^b x^2y\,dz\,dy\,dx = \dfrac{kb^6}{6}$

$M_{xz} = k\int_0^b\int_0^b\int_0^b xy^2\,dz\,dy\,dx = \dfrac{kb^6}{6}$

$M_{xy} = k\int_0^b\int_0^b\int_0^b xyz\,dz\,dy\,dx = \dfrac{kb^6}{8}$

$\bar{x} = \dfrac{M_{yz}}{m} = \dfrac{kb^6/6}{kb^5/4} = \dfrac{2b}{3}$

$\bar{y} = \dfrac{M_{xz}}{m} = \dfrac{kb^6/6}{kb^5/4} = \dfrac{2b}{3}$

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{kb^6/8}{kb^5/4} = \dfrac{b}{2}$

**37.** $\bar{x}$ will be greater than 2, whereas $\bar{y}$ and $\bar{z}$ will be unchanged.

**39.** $\bar{y}$ will be greater than 0, whereas $\bar{x}$ and $\bar{z}$ will be unchanged.

**41.** $m = \dfrac{1}{3}k\pi r^2h$

$\bar{x} = \bar{y} = 0$ by symmetry

$M_{xy} = 4k\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_{h\sqrt{x^2+y^2}/r}^{h} z\,dz\,dy\,dx$

$= \dfrac{3kh^2}{r^2}\int_0^r\int_0^{\sqrt{r^2-x^2}} (r^2 - x^2 - y^2)\,dy\,dx$

$= \dfrac{4kh^2}{3r^2}\int_0^r (r^2 - x^2)^{3/2}\,dx$

$= \dfrac{k\pi r^2h^2}{4}$

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{k\pi r^2h^2/4}{k\pi r^2h/3} = \dfrac{3h}{4}$

**43.** $m = \dfrac{128k\pi}{3}$

$\bar{x} = \bar{y} = 0$ by symmetry

$z = \sqrt{4^2 - x^2 - y^2}$

$$M_{xy} = 4k\int_0^4 \int_0^{\sqrt{4^2-x^2}} \int_0^{\sqrt{4^2-x^2-y^2}} z\,dz\,dy\,dx$$

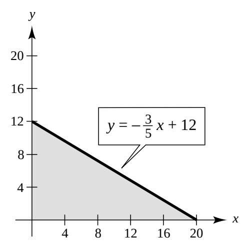
$$= 2k\int_0^4 \int_0^{\sqrt{4^2-x^2}} (4^2 - x^2 - y^2)\,dy\,dx = 2k\int_0^4 \left[16y - x^2y - \frac{1}{3}y^3\right]_0^{\sqrt{4^2-x^2}} dx = \frac{4k}{3}\int_0^4 (4^2 - x^2)^{3/2}\,dx$$

$$= \frac{1024k}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta \qquad (\text{let } x = 4\sin\theta)$$

$$= 64\pi k \qquad \text{by Wallis's Formula}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{64k\pi}{1}\cdot\frac{3}{128k\pi} = \frac{3}{2}$$

**45.** $f(x, y) = \dfrac{5}{12}y$




$$m = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} dz\,dy\,dx = 200k$$

$$M_{yz} = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} x\,dz\,dy\,dx = 1000k$$

$$M_{xz} = k\int_0^{20}\int_0^{-(3/5)+x+12}\int_0^{(5/12)y} y\,dz\,dy\,dx = 1200k$$

$$M_{xy} = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} z\,dz\,dy\,dx = 250k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1000k}{200k} = 5$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1200k}{200k} = 6$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{250k}{200k} = \frac{5}{4}$$

**47.** (a) $I_x = k\displaystyle\int_0^a\int_0^a\int_0^a (y^2 + z^2)\,dx\,dy\,dz = ka\int_0^a\int_0^a (y^2 + z^2)\,dy\,dz$

$$= ka\int_0^a \left[\frac{1}{3}y^3 + z^2y\right]_0^a dz = ka\int_0^a \left(\frac{1}{3}a^3 + az^2\right)dz = \left[ka\left(\frac{1}{3}a^3z + \frac{1}{3}az^3\right)\right]_0^a = \frac{2ka^5}{3}$$

$I_x = I_y = I_z = \dfrac{2ka^5}{3}$ by symmetry

(b) $I_x = k\displaystyle\int_0^a\int_0^a\int_0^a (y^2 + z^2)xyz\,dx\,dy\,dz = \frac{ka^2}{2}\int_0^a\int_0^a (y^3z + yz^3)\,dy\,dz$

$$= \frac{ka^2}{2}\int_0^a \left[\frac{y^4z}{4} + \frac{y^2z^3}{2}\right]_0^a dz = \frac{ka^4}{8}\int_0^a (a^2z + 2z^3)\,dz = \left[\frac{ka^4}{8}\left(\frac{a^2z^2}{2} + \frac{2z^4}{4}\right)\right]_0^a = \frac{ka^8}{8}$$

$I_x = I_y = I_z = \dfrac{ka^8}{8}$ by symmetry

**49.** (a) $I_x = k\int_0^4\int_0^4\int_0^{4-x} (y^2 + z^2)\, dz\, dy\, dx = k\int_0^4\int_0^4 \left[y^2(4 - x) + \frac{1}{3}(4 - x)^3\right] dy\, dx$

$$= k\int_0^4 \left[\frac{y^3}{3}(4 - x) + \frac{y}{3}(4 - x)^3\right]_0^4 dx = k\int_0^4 \left[\frac{64}{3}(4 - x) + \frac{4}{3}(4 - x)^3\right] dx$$

$$= k\left[-\frac{32}{3}(4 - x)^2 - \frac{1}{3}(4 - x)^4\right]_0^4 = 256k$$

$$I_y = k\int_0^4\int_0^4\int_0^{4-x} (x^2 + z^2)\, dz\, dy\, dx = k\int_0^4\int_0^4 \left[x^2(4 - x) + \frac{1}{3}(4 - x)^3\right] dy\, dx$$

$$= 4k\int_0^4 \left[4x^2 - x^3 + \frac{1}{3}(4 - x)^3\right] dx = 4k\left[\frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4 - x)^4\right]_0^4 = \frac{512k}{3}$$

$$I_z = k\int_0^4\int_0^4\int_0^{4-x} (x^2 + y^2)\, dz\, dy\, dx = k\int_0^4\int_0^4 (x^2 + y^2)(4 - x)\, dy\, dx$$

$$= k\int_0^4 \left[\left(x^2y + \frac{y^3}{3}\right)(4 - x)\right]_0^4 dx = k\int_0^4\left(4x^2 + \frac{64}{3}\right)(4 - x)\, dx = 256k$$

(b) $I_x = k\int_0^4\int_0^4\int_0^{4-x} y(y^2 + z^2)\, dz\, dy\, dx = k\int_0^4\int_0^4 \left[y^3(4 - x) + \frac{1}{3}y(4 - x)^3\right] dy\, dx$

$$= k\int_0^4 \left[\frac{y^4}{4}(4 - x) + \frac{y^2}{6}(4 - x)^3\right]_0^4 dx = k\int_0^4 \left[64(4 - x) + \frac{8}{3}(4 - x)^3\right] dx$$

$$= k\left[-32(4 - x)^2 - \frac{2}{3}(4 - x)^4\right]_0^4 = \frac{2048k}{3}$$

$$I_y = k\int_0^4\int_0^4\int_0^{4-x} y(x^2 + z^2)\, dz\, dy\, dx = k\int_0^4\int_0^4 \left[x^2y(4 - x) + \frac{1}{3}y(4 - x)^3\right] dy\, dx$$

$$= 8k\int_0^4\left[4x^2 - x^3 + \frac{1}{3}(4 - x)^3\right] dx = 8k\left[\frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4 - x)^4\right]_0^4 = \frac{1024k}{3}$$

$$I_z = k\int_0^4\int_0^4\int_0^{4-x} y(x^2 + y^2)\, dz\, dy\, dx = k\int_0^4\int_0^4 (x^2y + y^3)(4 - x)\, dx$$

$$= k\int_0^4 \left[\left(\frac{x^2y^2}{2} + \frac{y^4}{4}\right)(4 - x)\right]_0^4 dx = k\int_0^4 (8x^2 + 64)(4 - x)\, dx$$

$$= 8k\int_0^4 (32 - 8x + 4x^2 - x^3)\, dx = \left[8k\left(32x - 4x^2 + \frac{4}{3}x^3 - \frac{1}{4}x^4\right)\right]_0^4 = \frac{2048k}{3}$$

**51.** $I_{xy} = k\int_{-L/2}^{L/2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2\, dz\, dx\, dy = k\int_{-L/2}^{L/2}\int_{-a}^{a} \frac{2}{3}(a^2 - x^2)\sqrt{a^2 - x^2}\, dx\, dy$

$$= \frac{2}{3}\int_{-L/2}^{L/2} k\left[\frac{a^2}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right) - \frac{1}{8}\left(x(2x^2 - a^2)\sqrt{x^2 - a^2} + a^4 \arcsin\frac{x}{a}\right)\right]_{-a}^{a} dy$$

$$= \frac{2k}{3}\int_{-L/2}^{L/2} 2\left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right) dy = \frac{a^4\pi Lk}{4}$$

Since $m = \pi a^2Lk$, $I_{xy} = ma^2/4$.

**—CONTINUED—**

**51. —CONTINUED—**

$$I_{xz} = k\int_{-L/2}^{L/2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\,dz\,dx\,dy = 2k\int_{-L/2}^{L/2}\int_{-a}^{a} y^2\sqrt{a^2-x^2}\,dx\,dy$$

$$= 2k\int_{-L/2}^{L/2}\left[\frac{y^2}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right)\right]_{-a}^{a}dy = k\pi a^2\int_{-L/2}^{L/2} y^2\,dy = \frac{2k\pi a^2}{3}\left(\frac{L^3}{8}\right) = \frac{1}{12}mL^2$$

$$I_{yz} = k\int_{-L/2}^{L/2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,dz\,dx\,dy = 2k\int_{-L/2}^{L/2}\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx\,dy$$

$$= 2k\int_{-L/2}^{L/2}\frac{1}{8}\left[x(2x^2-a^2)\sqrt{a^2-x^2}+a^4\arcsin\frac{x}{a}\right]_{-a}^{a}dy = \frac{ka^4\pi}{4}\int_{-L/2}^{L/2}dy = \frac{ka^4\pi L}{4} = \frac{ma^2}{4}$$

$$I_x = I_{xy} + I_{xz} = \frac{ma^2}{4} + \frac{mL^2}{12} = \frac{m}{12}(3a^2 + L^2)$$

$$I_y = I_{xy} + I_{yz} = \frac{ma^2}{4} + \frac{ma^2}{4} = \frac{ma^2}{2}$$

$$I_z = I_{xz} + I_{yz} = \frac{mL^2}{12} + \frac{ma^2}{4} = \frac{m}{12}(3a^2 + L^2)$$

**53.** $\displaystyle\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x}(x^2+y^2)\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

**55.** See the definition, page 978.

See Theorem 13.4, page 979.

**57.** (a) The annular solid on the right has the greater density.

(b) The annular solid on the right has the greater movement of inertia.

(c) The solid on the left will reach the bottom first. The solid on the right has a greater resistance to rotational motion.

## Section 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

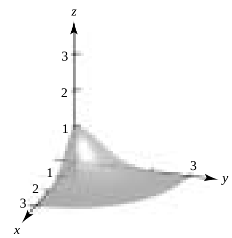
**1.** $$\int_0^4\int_0^{\pi/2}\int_0^2 r\cos\theta\,dr\,d\theta\,dz = \int_0^4\int_0^{\pi/2}\left[\frac{r^2}{2}\cos\theta\right]_0^2 d\theta\,dz$$
$$= \int_0^4\int_0^{\pi/2} 2\cos\theta\,d\theta\,dz = \int_0^4\left[2\sin\theta\right]_0^{\pi/2}dz = \int_0^4 2dz = 8$$

**3.** $$\int_0^{\pi/2}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos^2\theta} r(4-r^2)\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\left(2r^2-\frac{r^4}{4}\right)\sin\theta\right]_0^{2\cos^2\theta}d\theta$$
$$= \int_0^{\pi/2}[8\cos^4\theta - 4\cos^8\theta]\sin\theta\,d\theta = \left[-\frac{8\cos^5\theta}{5}+\frac{4\cos^9\theta}{9}\right]_0^{\pi/2} = \frac{52}{45}$$

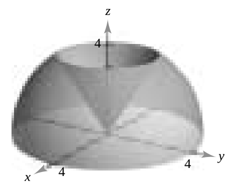
**5.** $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi\,d\theta = -\frac{1}{12}\int_0^{2\pi}\left[\cos^4\phi\right]_0^{\pi/4}d\theta = \frac{\pi}{8}$$

**7.** $$\int_0^4\int_0^z\int_0^{\pi/2} re^r\,d\theta\,dr\,dz = \pi(e^4+3)$$

**9.** $$\int_0^{\pi/2}\int_0^3\int_0^{e^{-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3 re^{-r^2}\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^3 d\theta$$
$$= \int_0^{\pi/2}\frac{1}{2}(1-e^{-9})d\theta$$
$$= \frac{\pi}{4}(1-e^{-9})$$




**11.** $$\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^4 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{64}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\sin\phi\,d\phi\,d\theta$$
$$= \frac{64}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_{\pi/6}^{\pi/2}d\theta$$
$$= \frac{32\sqrt{3}}{3}\int_0^{2\pi}d\theta$$
$$= \frac{64\sqrt{3}\pi}{3}$$




**13.** (a) $$\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cos\theta\,dz\,dr\,d\theta = 0$$

(b) $$\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{4\sec\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\arctan(1/2)}^{\pi/2}\int_0^{\cot\phi\csc\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta = 0$$

**15.** (a) $$\int_0^{2\pi}\int_0^a\int_a^{a+\sqrt{a^2-r^2}} r^2\cos\theta\,dz\,dr\,d\theta = 0$$

(b) $$\int_0^{\pi/4}\int_0^{2\pi}\int_{a\sec\phi}^{2a\cos\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\theta\,d\phi = 0$$

**17.** $$V = 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$
$$= \frac{4}{3}a^3\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta = \frac{4}{3}a^3\left[\theta + \frac{1}{3}\cos\theta(\sin^2\theta+2)\right]_0^{\pi/2} = \frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right) = \frac{2a^3}{9}(3\pi-4)$$

**19.** $$V = 2\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$
$$= 2\int_0^{\pi}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$
$$= 2\int_0^{\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos\theta}d\theta$$
$$= \frac{2a^3}{3}\int_0^{\pi}(1-\sin^3\theta)\,d\theta$$
$$= \frac{2a^3}{3}\left[\theta+\cos\theta-\frac{\cos^3\theta}{3}\right]_0^{\pi}$$
$$= \frac{2a^3}{9}(3\pi-4)$$

**21.** $$m = \int_0^{2\pi}\int_0^2\int_0^{9-r\cos\theta-2r\sin\theta}(kr)r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 kr^2(9-r\cos\theta-2r\sin\theta)dr\,d\theta$$
$$= \int_0^{2\pi}k\left[3r^3-\frac{r^4}{4}\cos\theta-\frac{r^4}{2}\sin\theta\right]_0^2 d\theta$$
$$= \int_0^{2\pi}k[24-4\cos\theta-8\sin\theta]d\theta$$
$$= k\Big[24\theta-4\sin\theta+8\cos\theta\Big]_0^{2\pi}$$
$$= k[48\pi+8-8] = 48k\pi$$

**23.** $z = h - \dfrac{h}{r_0}\sqrt{x^2 + y^2} = \dfrac{h}{r_0}(r_0 - r)$

$$V = 4\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r\, dz\, dr\, d\theta$$

$$= \frac{4h}{r_0}\int_0^{\pi/2}\int_0^{r_0}(r_0 r - r^2)\, dr\, d\theta$$

$$= \frac{4h}{r_0}\int_0^{\pi/2}\frac{r_0^3}{6}\, d\theta$$

$$= \frac{4h}{r_0}\left(\frac{r_0^3}{6}\right)\left(\frac{\pi}{2}\right) = \frac{1}{3}\pi r_0^2 h$$

**25.** $\rho = k\sqrt{x^2 + y^2} = kr$

$\bar{x} = \bar{y} = 0$ by symmetry

$$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2\, dz\, dr\, d\theta$$

$$= \frac{1}{6}k\pi r_0^3 h$$

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2 z\, dz\, dr\, d\theta$$

$$= \frac{1}{30}k\pi r_0^3 h^2$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r_0^3 h^2/30}{k\pi r_0^3 h/6} = \frac{h}{5}$$

**27.** $I_z = 4k\displaystyle\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^3\, dz\, dr\, d\theta$

$$= \frac{4kh}{r_0}\int_0^{\pi/2}\int_0^{r_0}(r_0 r^3 - r^4)\, dr\, d\theta$$

$$= \frac{4kh}{r_0}\left(\frac{r_0^5}{20}\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{10}k\pi r_0^4 h$$

Since the mass of the core is $m = kV = k\left(\frac{1}{3}\pi r_0^2 h\right)$ from Exercise 23, we have $k = 3m/\pi r_0^2 h$. Thus,

$$I_z = \frac{1}{10}k\pi r_0^4 h$$

$$= \frac{1}{10}\left(\frac{3m}{\pi r_0^2 h}\right)\pi r_0^4 h$$

$$= \frac{3}{10}m r_0^2$$

**29.** $m = k(\pi b^2 h - \pi a^2 h) = k\pi h(b^2 - a^2)$

$$I_z = 4k\int_0^{\pi/2}\int_a^b\int_0^h r^3\, dz\, dr\, d\theta$$

$$= 4kh\int_0^{\pi/2}\int_a^b r^3\, dr\, d\theta$$

$$= kh\int_0^{\pi/2}(b^4 - a^4)\, d\theta$$

$$= \frac{k\pi(b^4 - a^4)h}{2}$$

$$= \frac{k\pi(b^2 - a^2)(b^2 + a^2)h}{2}$$

$$= \frac{1}{2}m(a^2 + b^2)$$

**31.** $V = \displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{4\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 16\pi^2$

**33.** $m = 8k\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\sin\phi\, d\rho\, d\theta\, d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\, d\theta\, d\phi$$

$$= k\pi a^4\int_0^{\pi/2}\sin\phi\, d\phi$$

$$= \Big[k\pi a^4(-\cos\phi)\Big]_0^{\pi/2}$$

$$= k\pi a^4$$

**35.** $m = \frac{2}{3}k\pi r^3$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^r \rho^3 \cos\phi \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \frac{1}{2}kr^4\int_0^{\pi/2}\int_0^{\pi/2} \sin 2\phi \, d\theta\, d\phi$$

$$= \frac{kr^4\pi}{4}\int_0^{\pi/2} \sin 2\phi \, d\phi$$

$$= \left[-\frac{1}{8}k\pi r^4 \cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi r^4$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^4/4}{2k\pi r^3/3} = \frac{3r}{8}$$

**37.** $I_z = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi} \rho^4 \sin^3\phi \, d\rho\, d\theta\, d\phi$

$$= \frac{4}{5}k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2} \cos^5\phi \sin^3\phi \, d\theta\, d\phi$$

$$= \frac{2}{5}k\pi\int_{\pi/4}^{\pi/2} \cos^5\phi(1 - \cos^2\phi)\sin\phi \, d\phi$$

$$= \left[\frac{2}{5}k\pi\left(-\frac{1}{6}\cos^6\phi + \frac{1}{8}\cos^8\phi\right)\right]_{\pi/4}^{\pi/2}$$

$$= \frac{k\pi}{192}$$

**39.** $x = r\cos\theta \qquad x^2 + y^2 = r^2$

$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$

$z = z \qquad z = z$

**41.** $\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$

**43.** (a) $r = r_0$: right circular cylinder about $z$-axis

$\theta = \theta_0$: plane parallel to $z$-axis

$z = z_0$: plane parallel to $xy$-plane

(b) $\rho = \rho_0$: sphere of radius $\rho_0$

$\theta = \theta_0$: plane parallel to $z$-axis

$\phi = \phi_0$: cone

**45.** $16\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\int_0^{\sqrt{a^2-x^2-y^2-z^2}} dw\, dz\, dy\, dx$

$$= 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2 - x^2 - y^2 - z^2}\, dz\, dy\, dx$$

$$= 16\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} \sqrt{(a^2 - r^2) - z^2}\, dz(r\, dr\, d\theta)$$

$$= 16\int_0^{\pi/2}\int_0^a \frac{1}{2}\left[z\sqrt{(a^2 - r^2) - z^2} + (a^2 - r^2)\arcsin\frac{z}{\sqrt{a^2 - r^2}}\right]_0^{\sqrt{a^2-r^2}} r\, dr\, d\theta$$

$$= 8\int_0^{\pi/2}\int_0^a \frac{\pi}{2}(a^2 - r^2) r\, dr\, d\theta$$

$$= 4\pi\int_0^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a d\theta$$

$$= a^4\pi\int_0^{\pi/2} d\theta = \frac{a^4\pi^2}{2}$$

## Section 13.8 Change of Variables: Jacobians

**1.** $x = -\frac{1}{2}(u - v)$

$y = \frac{1}{2}(u + v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

**3.** $x = u - v^2$

$y = u + v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (1)(-2v) = 1 + 2v$$

**5.** $x = u\cos\theta - v\sin\theta$

$y = u\sin\theta + v\cos\theta$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \cos^2\theta + \sin^2\theta = 1$$

**7.** $x = e^u \sin v$

$y = e^u \cos v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (e^u \sin v)(-e^u \sin v) - (e^u \cos v)(e^u \cos v) = -e^{2u}$$

**9.** $x = 3u + 2v$

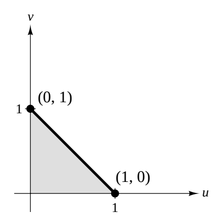
$y = 3v$

$v = \frac{y}{3}$

$u = \frac{x - 2v}{3} = \frac{x - 2(y/3)}{3}$

$= \frac{x}{3} - \frac{2y}{9}$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 0)$ | $(0, 0)$ |
| $(3, 0)$ | $(1, 0)$ |
| $(2, 3)$ | $(0, 1)$ |




**11.** $x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$\iint_R 4(x^2 + y^2)\,dA = \int_{-1}^{1}\int_{-1}^{1} 4\left[\frac{1}{4}(u + v)^2 + \frac{1}{4}(u - v)^2\right]\left(\frac{1}{2}\right)dv\,du$$

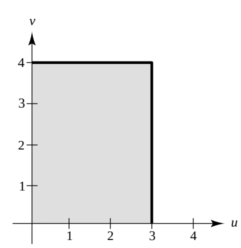
$$= \int_{-1}^{1}\int_{-1}^{1}(u^2 + v^2)\,dv\,du = \int_{-1}^{1} 2\left(u^2 + \frac{1}{3}\right)du = \left[2\left(\frac{u^3}{3} + \frac{u}{3}\right)\right]_{-1}^{1} = \frac{8}{3}$$

**13.** $x = u + v$

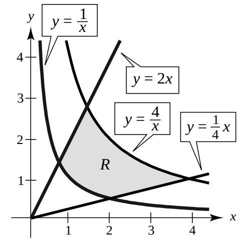
$y = u$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(0) - (1)(1) = -1$$

$$\iint_R y(x - y)\,dA = \int_0^3\int_0^4 uv(1)\,dv\,du = \int_0^3 8u\,du = 36$$



**15.** $\displaystyle\iint_R e^{-xy/2}\,dA$

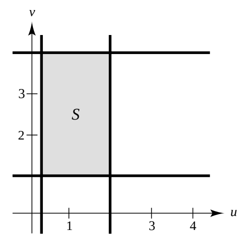



$R$: $y = \dfrac{x}{4}, y = 2x, y = \dfrac{1}{x}, y = \dfrac{4}{x}$

$x = \sqrt{v/u}, y = \sqrt{uv} \implies u = \dfrac{y}{x},\; v = xy$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{2}\dfrac{v^{1/2}}{u^{3/2}} & \dfrac{1}{2}\dfrac{1}{u^{1/2}v^{1/2}} \\ \dfrac{1}{2}\dfrac{v^{1/2}}{u^{1/2}} & \dfrac{1}{2}\dfrac{u^{1/2}}{v^{1/2}} \end{vmatrix} = -\frac{1}{4}\left(\frac{1}{u} + \frac{1}{u}\right) = -\frac{1}{2u}$$

Transformed Region:




$y = \dfrac{1}{x} \implies yx = 1 \implies v = 1$

$y = \dfrac{4}{x} \implies yx = 4 \implies v = 4$

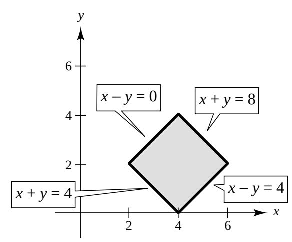
$y = 2x \implies \dfrac{y}{x} = 2 \implies u = 2$

$y = \dfrac{x}{4} \implies \dfrac{y}{x} = \dfrac{1}{4} \implies u = \dfrac{1}{4}$

$$\iint_R e^{-xy/2}\,dA = \int_{1/4}^{2}\int_{1}^{4} e^{-v/2}\left(\frac{1}{2u}\right) dv\,du = -\int_{1/4}^{2}\left[\frac{e^{-v/2}}{u}\right]_1^4 du = -\int_{1/4}^{2}(e^{-2} - e^{-1/2})\frac{1}{u}\,du$$

$$= -\Big[(e^{-2} - e^{-1/2})\ln u\Big]_{1/4}^{2} = -(e^{-2} - e^{-1/2})\left(\ln 2 - \ln\frac{1}{4}\right) = (e^{-1/2} - e^{-2})\ln 8 \approx 0.9798$$

**17.** $u = x + y = 4, \quad v = x - y = 0$




$u = x + y = 8, \quad v = x - y = 4$

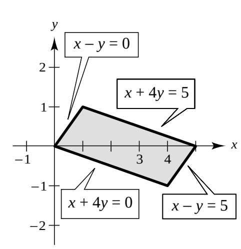
$x = \dfrac{1}{2}(u + v) \qquad y = \dfrac{1}{2}(u - v)$

$\dfrac{\partial(x, y)}{\partial(u, v)} = -\dfrac{1}{2}$

$$\iint_R (x + y)e^{x-y}\,dA = \int_4^8\int_0^4 ue^v\left(\frac{1}{2}\right) dv\,du$$

$$= \frac{1}{2}\int_4^8 u(e^4 - 1)\,du = \left[\frac{1}{4}u^2(e^4 - 1)\right]_4^8 = 12(e^4 - 1)$$

**19.** $u = x + 4y = 0, \quad v = x - y = 0$




$u = x + 4y = 5, \quad v = x - y = 5$

$x = \dfrac{1}{5}(u + 4v), \qquad y = \dfrac{1}{5}(u - v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{5}\right)\left(-\frac{1}{5}\right) - \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = -\frac{1}{5}$$

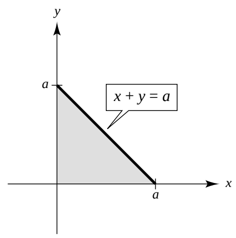
$$\iint_R \sqrt{(x - y)(x + 4y)}\,dA = \int_0^5\int_0^5 \sqrt{uv}\left(\frac{1}{5}\right) du\,dv$$

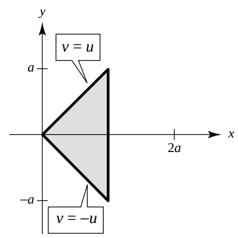
$$= \int_0^5 \left[\frac{1}{5}\left(\frac{2}{3}\right)u^{3/2}\sqrt{v}\right]_0^5 dv = \left[\frac{2\sqrt{5}}{3}\left(\frac{2}{3}\right)v^{3/2}\right]_0^5 = \frac{100}{9}$$

**21.** $u = x + y, v = x - y, x = \dfrac{1}{2}(u + v), y = \dfrac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$$

$$\int_R\int \sqrt{x + y}\, dA = \int_0^a\int_{-u}^u \sqrt{u}\left(\frac{1}{2}\right) dv\, du = \int_0^a u\sqrt{u}\, du = \left[\frac{2}{5}u^{5/2}\right]_0^a = \frac{2}{5}a^{5/2}$$



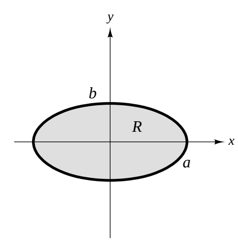



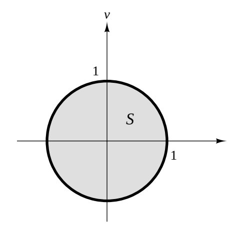

**23.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, x = au, y = bv$

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1$$

$$u^2 + v^2 = 1$$

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\qquad$ $u^2 + v^2 = 1$




(b) $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}$

$$= (a)(b) - (0)(0) = ab$$

(c) $A = \displaystyle\int_S\int ab\, dS$

$$= ab(\pi(1)^2) = \pi ab$$

**25.** Jacobian $= \dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}$

**27.** $x = u(1 - v),\ y = uv(1 - w),\ z = uvw$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 - v & -u & 0 \\ v(1 - w) & u(1 - w) & -uv \\ vw & uw & uv \end{vmatrix} = (1 - v)[u^2v(1 - w) + u^2vw] + u[uv^2(1 - w) + uv^2w]$$

$$= (1 - v)(u^2v) + u(uv^2)$$

$$= u^2v$$

**29.** $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\ \cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi[-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta] - \rho\sin\phi[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta]$$

$$= \cos\phi[-\rho^2\sin\phi\cos\phi(\sin^2\theta + \cos^2\theta)] - \rho\sin\phi[\rho\sin^2\phi(\cos^2\theta + \sin^2\theta)]$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin^3\phi$$

$$= -\rho^2\sin\phi(\cos^2\phi + \sin^2\phi)$$

$$= -\rho^2\sin\phi$$

# Review Exercises for Chapter 13

**1.** $\displaystyle\int_1^{x^2} x \ln y \, dy = \Big[xy(-1 + \ln y)\Big]_1^{x^2} = x^3(-1 + \ln x^2) + x = x - x^3 + x^3 \ln x^2$

**3.** $\displaystyle\int_0^1\int_0^{1+x} (3x + 2y)\, dy\, dx = \int_0^1 \Big[3xy + y^2\Big]_0^{1+x} dx = \int_0^1 (4x^2 + 5x + 1)\, dx = \Big[\frac{4}{3}x^3 + \frac{5}{2}x^2 + x\Big]_0^1 = \frac{29}{6}$

**5.** $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} 4x\, dy\, dx = \int_0^3 4x\sqrt{9 - x^2}\, dx = \Big[-\frac{4}{3}(9 - x^2)^{3/2}\Big]_0^3 = 36$

**7.** $\displaystyle\int_0^3\int_0^{(3-x)/3} dy\, dx = \int_0^1\int_0^{3-3y} dx\, dy$

$$A = \int_0^1\int_0^{3-3y} dx\, dy = \int_0^1 (3 - 3y)\, dy = \Big[3y - \frac{3}{2}y^2\Big]_0^1 = \frac{3}{2}$$

**9.** $\displaystyle\int_{-5}^3\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy\, dx = \int_{-5}^{-4}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\, dy + \int_{-4}^4\int_{-\sqrt{25-y^2}}^{3} dx\, dy + \int_4^5\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\, dy$

$$A = 2\int_{-5}^3\int_0^{\sqrt{25-x^2}} dy\, dx = 2\int_{-5}^3 \sqrt{25 - x^2}\, dx = \Big[x\sqrt{25 - x^2} + 25 \arcsin\frac{x}{5}\Big]_{-5}^3 = \frac{25\pi}{2} + 12 + 25 \arcsin\frac{3}{5} \approx 67.36$$

**11.** $\displaystyle A = 4\int_0^1\int_0^{x\sqrt{1-x^2}} dy\, dx = 4\int_0^1 x\sqrt{1 - x^2}\, dx = \Big[-\frac{4}{3}(1 - x^2)^{3/2}\Big]_0^1 = \frac{4}{3}$

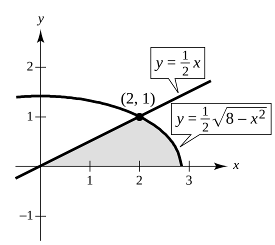
$$A = 4\int_0^{1/2}\int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx\, dy$$

**13.** $\displaystyle A = \int_2^5\int_{x-3}^{\sqrt{x-1}} dy\, dx + 2\int_1^2\int_0^{\sqrt{x-1}} dy\, dx = \int_{-1}^2\int_{y^2+1}^{y+3} dx\, dy = \frac{9}{2}$

**15.** Both integrations are over the common region $R$ shown in the figure. Analytically,

$$\int_0^1\int_{2y}^{2\sqrt{2-y^2}} (x + y)\, dx\, dy = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$

$$\int_0^2\int_0^{x/2} (x + y)\, dy\, dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}/2} (x + y)\, dy\, dx = \frac{5}{3} + \left(\frac{4}{3}\sqrt{2} - \frac{1}{3}\right) = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$




**17.** $\displaystyle V = \int_0^4\int_0^{x^2+4} (x^2 - y + 4)\, dy\, dx$

$$= \int_0^4 \Big[x^2y - \frac{1}{2}y^2 + 4y\Big]_0^{x^2+4} dx$$

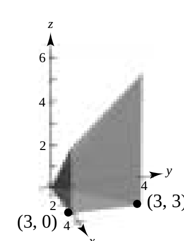
$$= \int_0^4 \left(\frac{1}{2}x^4 + 4x^2 + 8\right) dx$$

$$= \Big[\frac{1}{10}x^5 + \frac{4}{3}x^3 + 8x\Big]_0^4 = \frac{3296}{15}$$

**19.** Volume $\approx$ (base)(height)

$$\approx \frac{9}{2}(3) = \frac{27}{2}$$

Matches (c)



**21.** $\displaystyle\int_0^{\infty}\int_0^{\infty} kxye^{-(x+y)}\,dy\,dx = \int_0^{\infty}\Big[-kxe^{-(x+y)}(y+1)\Big]_0^{\infty}\,dx = \int_0^{\infty} kxe^{-x}\,dx = \Big[-k(x+1)e^{-x}\Big]_0^{\infty} = k$

Therefore, $k = 1$.

$$P = \int_0^1\int_0^1 xye^{-(x+y)}\,dy\,dx \approx 0.070$$

**23.** True

**25.** True

**27.** $\displaystyle\int_0^h\int_0^x \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/4}\int_0^{h\sec\theta} r^2\,dr\,d\theta$

$$= \frac{h^3}{3}\int_0^{\pi/4}\sec^3\theta\,d\theta = \frac{h^3}{6}\Big[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4} = \frac{h^3}{6}\Big[\sqrt{2}+\ln\big(\sqrt{2}+1\big)\Big]$$

**29.** $\displaystyle V = 4\int_0^h\int_0^{\pi/2}\int_1^{\sqrt{1+z^2}} r\,dr\,d\theta\,dz$

$$= 2\int_0^h\int_0^{\pi/2}(1+z^2-1)\,d\theta\,dz$$

$$= \pi\int_0^h z^2\,dz$$

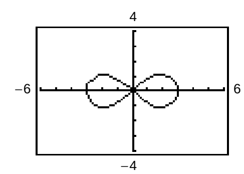
$$= \left[\pi\left(\frac{1}{3}z^3\right)\right]_0^h = \frac{\pi h^3}{3}$$

**31.** (a) $(x^2+y^2)^2 = 9(x^2-y^2)$

$$(r^2)^2 = 9(r^2\cos^2\theta - r^2\sin^2\theta)$$

$$r^2 = 9(\cos^2\theta-\sin^2\theta) = 9\cos 2\theta$$

$$r = 3\sqrt{\cos 2\theta}$$




(b) $\displaystyle A = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta = 9$

(c) $\displaystyle V = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}}\sqrt{9-r^2}\,r\,dr\,d\theta \approx 20.392$

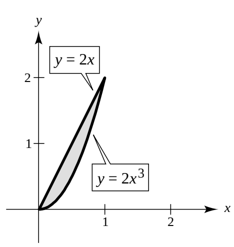
**33.** (a) $\displaystyle m = k\int_0^1\int_{2x^3}^{2x} xy\,dy\,dx = \frac{k}{4}$

$$M_x = k\int_0^1\int_{2x^3}^{2x} xy^2\,dy\,dx = \frac{16k}{55}$$

$$M_y = k\int_0^1\int_{2x^3}^{2x} x^2y\,dy\,dx = \frac{8k}{45}$$

$$\bar{x} = \frac{M_y}{m} = \frac{32}{45}$$

$$\bar{y} = \frac{M_x}{m} = \frac{64}{55}$$




(b) $\displaystyle m = k\int_0^1\int_{2x^3}^{2x}(x^2+y^2)\,dy\,dx = \frac{17k}{30}$

$$M_x = k\int_0^1\int_{2x^3}^{2x} y(x^2+y^2)\,dy\,dx = \frac{392k}{585}$$

$$M_y = k\int_0^1\int_{2x^3}^{2x} x(x^2+y^2)\,dy\,dx = \frac{156k}{385}$$

$$\bar{x} = \frac{M_y}{m} = \frac{936}{1309}$$

$$\bar{y} = \frac{M_x}{m} = \frac{784}{663}$$

**35.** $I_x = \int\int_R y^2\,\rho(x, y)dA = \int_0^a\int_0^b kxy^2\,dy\,dx = \frac{1}{6}kb^3a^2$

$I_y = \int\int_R x^2\,\rho(x, y)dA = \int_0^a\int_0^b kx^3\,dy\,dx = \frac{1}{4}kba^4$

$I_0 = I_x + I_y = \frac{1}{6}kb^3a^2 + \frac{1}{4}kba^4 = \frac{ka^2b}{12}(2b^2 + 3a^2)$

$m = \int\int_R \rho(x, y)dA = \int_0^a\int_0^b kx\,dy\,dx = \frac{1}{2}kba^2$

$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{(1/4)kba^4}{(1/2)kba^2}} = \sqrt{\frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$

$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{(1/6)kb^3a^2}{(1/2)kba^2}} = \sqrt{\frac{b^2}{3}} = \frac{b\sqrt{3}}{3}$

**37.** $S = \int\int_R \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$

$= 4\int_0^4\int_0^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$

$= 4\int_0^{\pi/2}\int_0^4 \sqrt{1 + 4r^2}\,r\,dr\,d\theta$

$= \left[\frac{1}{3}(65^{3/2} - 1)\theta\right]_0^{\pi/2} = \frac{\pi}{6}\left(65\sqrt{65} - 1\right)$

**39.** $f(x, y) = 9 - y^2$

$f_x = 0,\ f_y = -2y$

$S = \int\int_R \sqrt{1 + f_x^2 + f_y^2}\,dA$

$= \int_0^3\int_{-y}^{y} \sqrt{1 + 4y^2}\,dx\,dy$

$= \int_0^3\left[\sqrt{1 + 4y^2}\,x\right]_{-y}^{y} dy$

$= \int_0^3 2\sqrt{1 + 4y^2}\,dy = \frac{1}{4}\frac{2}{3}(1 + 4y^2)^{3/2}\Big]_0^3 = \frac{1}{6}[(37)^{3/2} - 1]$

**41.** $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2}^{9} \sqrt{x^2 + y^2}\,dz\,dy\,dx = \int_0^{2\pi}\int_0^3\int_{r^2}^{9} r^2\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^3 (9r^2 - r^4)\,dr\,d\theta = \int_0^{2\pi}\left[3r^3 - \frac{r^5}{5}\right]_0^3 d\theta = \frac{162}{5}\int_0^{2\pi} d\theta = \frac{324\pi}{5}$

**43.** $\int_0^a\int_0^b\int_0^c (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^a\int_0^b\left(\frac{1}{3}c^3 + cy^2 + cz^2\right)dy\,dz$

$= \int_0^a\left(\frac{1}{3}bc^3 + \frac{1}{3}b^3c + bcz^2\right)dz = \frac{1}{3}abc^3 + \frac{1}{3}ab^3c + \frac{1}{3}a^3bc = \frac{1}{3}abc(a^2 + b^2 + c^2)$

**45.** $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^3\,dz\,dr\,d\theta = \frac{8\pi}{15}$

**47.** $V = 4\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$= 4\int_0^{\pi/2}\int_0^{2\cos\theta} r\sqrt{4-r^2}\,dr\,d\theta$

$= -\int_0^{\pi/2}\left[\frac{4}{3}(4-r^2)^{3/2}\right]_0^{2\cos\theta} d\theta$

$= \frac{32}{3}\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta$

$= \frac{32}{3}\left[\theta + \cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{32}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$

**49.** $m = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$= \frac{4}{3}k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\theta\,d\phi = \frac{2}{3}k\pi\int_{\pi/4}^{\pi/2}\cos^3\phi\sin\phi\,d\phi = \left[-\frac{2}{3}k\pi\left(\frac{1}{4}\cos^4\phi\right)\right]_{\pi/4}^{\pi/2} = \frac{k\pi}{24}$

$M_{xy} = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi}\rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi$

$= k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\cos^5\phi\sin\phi\,d\theta\,d\phi = \frac{1}{2}k\pi\int_{\pi/4}^{\pi/2}\cos^5\phi\sin\phi\,d\phi = \left[-\frac{1}{12}k\pi\cos^6\phi\right]_{\pi/4}^{\pi/2} = \frac{k\pi}{96}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi/96}{k\pi/24} = \frac{1}{4}$

$\bar{x} = \bar{y} = 0$ by symmetry

**51.** $m = k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{k\pi a^3}{6}$

$M_{xy} = k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{k\pi a^4}{16}$

$\bar{x} = \bar{y} = \bar{z} = \frac{M_{xy}}{m} = \frac{k\pi a^4}{16}\left(\frac{6}{k\pi a^3}\right) = \frac{3a}{8}$

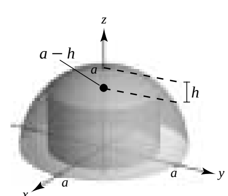
**53.** $I_z = 4k\int_0^{\pi/2}\int_3^4\int_0^{16-r^2} r^3\,dz\,dr\,d\theta$

$= 4k\int_0^{\pi/2}\int_3^4(16r^3 - r^5)\,dr\,d\theta = \frac{833\pi k}{3}$

**55.** $z = f(x,y) = \sqrt{a^2-x^2-y^2}$

$= \sqrt{a^2-r^2}$

$0 \le r \le \sqrt{2ah-h^2}$

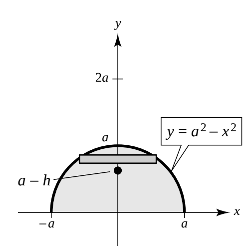



(a) Disc Method

$V = \pi\int_{a-h}^a(a^2-y^2)dy$

$= \pi\left[a^2y - \frac{y^3}{3}\right]_{a-h}^a = \pi\left[\left(a^3-\frac{a^3}{3}\right) - \left(a^2(a-h) - \frac{(a-h)^3}{3}\right)\right]$

$= \pi\left[a^3 - \frac{a^3}{3} - a^3 + a^2h + \frac{a^3}{3} - a^2h + ah^2 - \frac{h^3}{3}\right] = \pi\left[ah^2 - \frac{h^3}{3}\right] = \frac{1}{3}\pi h^2[3a-h]$




Equivalently, use spherical coordinates

$V = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^a\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

**—CONTINUED—**

**55. —CONTINUED—**

(b) $M_{xy} = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^{a} (\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{4}h^2\pi(2a-h)^2$

$\bar{z} = \frac{M_{xy}}{V} = \frac{\frac{1}{4}h^2\pi(2a-h)^2}{\frac{1}{3}h^2\pi(3a-h)} = \frac{3}{4}\frac{(2a-h)^2}{3a-h}$

centroid: $\left(0, 0, \frac{3(2a-h)^2}{4(3a-h)}\right)$

(c) If $h = a$, $\bar{z} = \frac{3(a)^2}{4(2a)} = \frac{3}{8}a$

centroid of hemisphere: $\left(0, 0, \frac{3}{8}a\right)$

(d) $\lim_{h\to 0} \bar{z} = \lim_{h\to 0}\frac{3(2a-h)^2}{4(3a-h)} = \frac{3(4a^2)}{12a} = a$

(e) $x^2 + y^2 = \rho^2\sin^2\phi$

$I_z = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^{a} (\rho^2\sin^2\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$= \frac{h^3}{30}(20a^2 - 15ah + 3h^2)\pi$

(f) If $h = a$, $I_z = \frac{a^3\pi}{30}(20a^2 - 15a^2 + 3a^2) = \frac{4}{15}a^5\pi$

**57.** $\int_0^{2\pi}\int_0^{\pi}\int_0^{6\sin\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

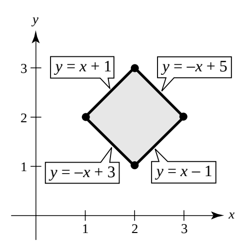
Since $\rho = 6\sin\phi$ represents (in the $yz$-plane) a circle of radius 3 centered at $(0, 3, 0)$, the integral represents the volume of the torus formed by revolving $(0 < \theta < 2\pi)$ this circle about the $z$-axis.

**59.** $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

$= 1(-3) - 2(3) = -9$

**61.** $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}$

$x = \frac{1}{2}(u+v), y = \frac{1}{2}(u-v) \implies u = x + y, v = x - y$




| Boundaries in $xy$-plane | Boundaries in $uv$-plane |
|---|---|
| $x + y = 3$ | $u = 3$ |
| $x + y = 5$ | $u = 5$ |
| $x - y = -1$ | $v = -1$ |
| $x - y = 1$ | $v = 1$ |

$\int\int_R \ln(x+y)dA = \int_3^5\int_{-1}^1 \ln\left(\frac{1}{2}(u+v) + \frac{1}{2}(u-v)\right)\left(\frac{1}{2}\right) dv\, du = \int_3^5\int_{-1}^1 \frac{1}{2}\ln u\, dv\, du = \int_3^5 \ln u\, du = \Big[u\ln u - u\Big]_3^5$

$= (5\ln 5 - 5) - (3\ln u - 3) = 5\ln 5 - 3\ln 3 - 2 \approx 2.751$

## Problem Solving for Chapter 13

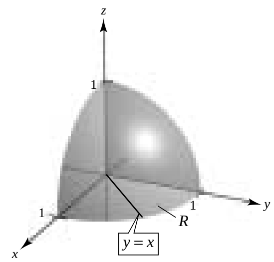
**1.** (a) $V = 16\int\int_R \sqrt{1-x^2}\,dA$

$$= 16\int_0^{\pi/4}\int_0^1 \sqrt{1-r^2\cos^2\theta}\,r\,dr\,d\theta$$

$$= -\frac{16}{3}\int_0^{\pi/4}\frac{1}{\cos^2\theta}\left[(1-\cos^2\theta)^{3/2}-1\right]d\theta$$

$$= -\frac{16}{3}\Big[\sec\theta+\cos\theta-\tan\theta\Big]_0^{\pi/4}$$

$$= 8\left(2-\sqrt{2}\right) \approx 4.6863$$




(b) Programs will vary.

**3.** (a) $\displaystyle\int\frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a}+c$. Let $a^2 = 2-u^2$, $u = v$.

Then $\displaystyle\int\frac{1}{(2-u^2)+v^2}\,dv = \frac{1}{\sqrt{2-u^2}}\arctan\frac{v}{\sqrt{2-u^2}}+C.$

(b) $$I_1 = \int_0^{\sqrt{2}/2}\left[\frac{2}{\sqrt{2-u^2}}\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-u}^{u}du$$

$$= \int_0^{\sqrt{2}/2}\frac{2}{\sqrt{2-u^2}}\left(\arctan\frac{u}{\sqrt{2-u^2}}-\arctan\frac{-u}{\sqrt{2-u^2}}\right)du$$

$$= \int_0^{\sqrt{2}/2}\frac{4}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}\,du$$

Let $u = \sqrt{2}\sin\theta$, $du = \sqrt{2}\cos\theta\,d\theta$, $2-u^2 = 2-2\sin^2\theta = 2\cos^2\theta$.

$$I_1 = 4\int_0^{\pi/6}\frac{1}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)\cdot\sqrt{2}\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/6}\arctan(\tan\theta)d\theta = \frac{4\theta^2}{2}\Big]_0^{\pi/6} = 2\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{18}$$

(c) $$I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}}\left[\frac{2}{\sqrt{2-u^2}}\arctan\frac{v}{\sqrt{2-u^2}}\right]_{u-\sqrt{2}}^{-u+\sqrt{2}}du$$

$$= \int_{\sqrt{2}/2}^{\sqrt{2}}\frac{2}{\sqrt{2-u^2}}\left[\arctan\left(\frac{-u+\sqrt{2}}{\sqrt{2-u^2}}\right)-\arctan\left(\frac{u-\sqrt{2}}{\sqrt{2-u^2}}\right)\right]du$$

$$= \int_{\sqrt{2}/2}^{\sqrt{2}}\frac{4}{\sqrt{2-u^2}}\arctan\left(\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\right)du$$

Let $u = \sqrt{2}\sin\theta$.

$$I_2 = 4\int_{\pi/6}^{\pi/2}\frac{1}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)\cdot\sqrt{2}\cos\theta\,d\theta$$

$$= 4\int_{\pi/6}^{\pi/2}\arctan\left(\frac{1-\sin\theta}{\cos\theta}\right)d\theta$$

**—CONTINUED—**

**3. —CONTINUED—**

(d) $$\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right) = \sqrt{\frac{1-\cos((\pi/2)-\theta)}{1+\cos((\pi/2)-\theta)}} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{(1+\sin\theta)(1-\sin\theta)}} = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} = \frac{1-\sin\theta}{\cos\theta}$$

(e) $$I_2 = 4\int_{\pi/6}^{\pi/2} \arctan\left(\frac{1-\sin\theta}{\cos\theta}\right)d\theta = 4\int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right)d\theta$$

$$= 4\int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\frac{\pi}{2}-\theta\right)d\theta = 2\int_{\pi/6}^{\pi/2}\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$= 2\left[\frac{\pi}{2}\theta - \frac{\theta^2}{2}\right]_{\pi/6}^{\pi/2} = 2\left[\left(\frac{\pi^2}{4}-\frac{\pi^2}{8}\right) - \left(\frac{\pi^2}{12}-\frac{\pi^2}{72}\right)\right]$$

$$= 2\left[\frac{18-9-6+1}{72}\pi^2\right] = \frac{4}{36}\pi^2 = \frac{\pi^2}{9}$$

(f) $$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + \cdots \qquad |xy| < 1$$

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1\int_0^1 [1 + (xy) + (xy)^2 + \cdots]\,dx\,dy$$

$$= \int_0^1\int_0^1 \sum_{K=0}^{\infty}(xy)^K\,dx\,dy = \sum_{K=0}^{\infty}\int_0^1 \frac{x^{K+1}y^K}{K+1}\Big|_0^1\,dy$$

$$= \sum_{K=0}^{\infty}\int_0^1 \frac{y^K}{K+1}\,dy = \sum_{K=0}^{\infty}\frac{y^{K+1}}{(K+1)^2}\Big|_0^1$$

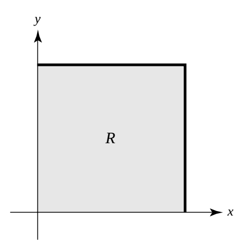
$$= \sum_{K=0}^{\infty}\frac{1}{(K+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

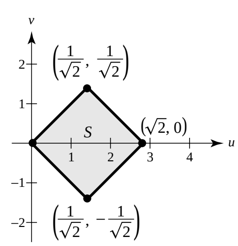
(g) $$u = \frac{x+y}{\sqrt{2}},\ v = \frac{y-x}{\sqrt{2}}$$

$$u - v = \frac{2x}{\sqrt{2}} \implies x = \frac{u-v}{\sqrt{2}}$$

$$u + v = \frac{2y}{\sqrt{2}} \implies y = \frac{u+v}{\sqrt{2}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1$$




| <u>R</u> | | <u>S</u> |
|---|---|---|
| $(0, 0)$ | $\leftrightarrow$ | $(0, 0)$ |
| $(1, 0)$ | $\leftrightarrow$ | $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ |
| $(0, 1)$ | $\leftrightarrow$ | $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ |
| $(1, 1)$ | $\leftrightarrow$ | $(\sqrt{2}, 0)$ |

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{1}{1-\dfrac{u^2}{2}+\dfrac{v^2}{2}}\,dv\,du + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{1}{1-\dfrac{u^2}{2}+\dfrac{v^2}{2}}\,dv\,du$$

$$= I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

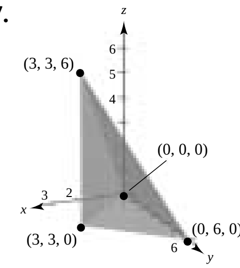
**5.** Boundary in $xy$-plane — Boundary in $uv$-plane

| Boundary in $xy$-plane | Boundary in $uv$-plane |
|---|---|
| $y = \sqrt{x}$ | $u = 1$ |
| $y = \sqrt{2x}$ | $u = 2$ |
| $y = \frac{1}{3}x^2$ | $v = 3$ |
| $y = \frac{1}{4}x^2$ | $v = 4$ |

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3}\left(\frac{v}{u}\right)^{2/3} & \frac{2}{3}\left(\frac{u}{v}\right)^{1/3} \\ \frac{2}{3}\left(\frac{v}{u}\right)^{1/3} & \frac{1}{3}\left(\frac{u}{v}\right)^{2/3} \end{vmatrix} = -\frac{1}{3}$$

$$A = \int_R\int 1\, dA = \int_S\int 1 \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA = \frac{1}{3}$$

**7.**




$$V = \int_0^3\int_0^{2x}\int_x^{6-x} dy\, dz\, dx = 18$$

**9.** From Exercise 55, Section 13.3,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Thus, $\displaystyle\int_0^{\infty} e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}}{2}$ and $\displaystyle\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

$$\int_0^{\infty} x^2 e^{-x^2}\, dx = \left[-\frac{1}{2}xe^{-x^2}\right]_0^{\infty} + \frac{1}{2}\int_0^{\infty} e^{-x^2}\, dx = \frac{1}{2}\,\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

**11.** $f(x, y) = \begin{cases} ke^{-(x+y)/a} & x \ge 0, y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = \int_0^{\infty}\int_0^{\infty} ke^{-(x+y)/a}\, dx\, dy$$

$$= k\int_0^{\infty} e^{-x/a}\, dx \cdot \int_0^{\infty} e^{-y/a}\, dy$$

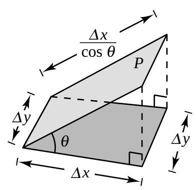
These two integrals are equal to

$$\int_0^{\infty} e^{-x/a}\, dx = \lim_{b\to\infty}\left[(-a)e^{-x/a}\right]_0^b = a.$$

Hence, assuming $a, k > 0$, you obtain

$$1 = ka^2 \quad \text{or} \quad a = \frac{1}{\sqrt{k}}.$$

**13.** $A = l \cdot w = \left(\dfrac{\Delta x}{\cos\theta}\right)\Delta y = \sec\theta\,\Delta x\,\Delta y$




Area in $xy$-plane: $\Delta x\, \Delta y$

# CHAPTER 14
# Vector Analysis

# CHAPTER 14
# Vector Analysis

## Section 14.1 Vector Fields

Solutions to Odd-Numbered Exercises

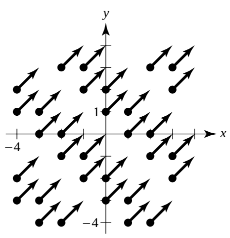
**1.** All vectors are parallel to $y$-axis.
Matches (c)

**3.** All vectors point outward.
Matches (b)

**5.** Vectors are parallel to $x$-axis for $y = n\pi$.
Matches (a)

**7.** $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

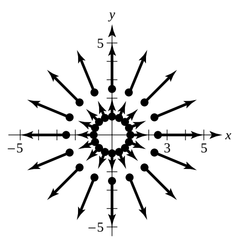
$\|\mathbf{F}\| = \sqrt{2}$




**9.** $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

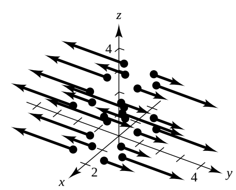
$\|\mathbf{F}\| = \sqrt{x^2 + y^2} = c$

$x^2 + y^2 = c^2$




**11.** $\mathbf{F}(x, y, z) = 3y\mathbf{j}$

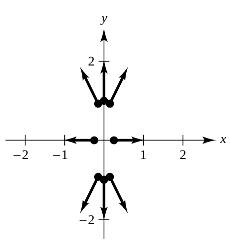
$\|\mathbf{F}\| = 3|y| = c$




**13.** $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$

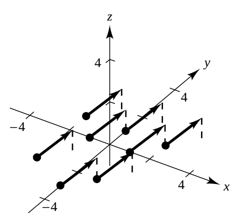
$\|\mathbf{F}\| = \sqrt{16x^2 + y^2} = c$

$$\frac{x^2}{c^2/16} + \frac{y^2}{c^2} = 1$$

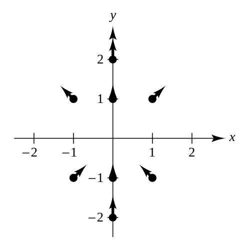



**15.** $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

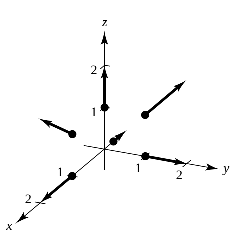
$\|\mathbf{F}\| = \sqrt{3}$




**17.**




**19.**




**21.** $f(x, y) = 5x^2 + 3xy + 10y^2$

$f_x(x, y) = 10x + 3y$

$f_y(x, y) = 3x + 20y$

$\mathbf{F}(x, y) = (10x + 3y)\mathbf{i} + (3x + 20y)\mathbf{j}$

**23.** $f(x, y, z) = z - ye^{x^2}$

$f_x(x, y, z) = -2xye^{x^2}$

$f_y(x, y, z) = -e^{x^2}$

$f_z = 1$

$\mathbf{F}(x, y, z) = -2xye^{x^2}\mathbf{i} - e^{x^2}\mathbf{j} + \mathbf{k}$

**25.** $g(x, y, z) = xy \ln(x + y)$

$$g_x(x, y, z) = y \ln(x + y) + \frac{xy}{x + y}$$

$$g_y(x, y, z) = x \ln(x + y) + \frac{xy}{x + y}$$

$$g_z(x, y, z) = 0$$

$$\mathbf{G}(x, y, z) = \left[\frac{xy}{x + y} + y \ln(x + y)\right]\mathbf{i} + \left[\frac{xy}{x + y} + x \ln(x + y)\right]\mathbf{j}$$

**27.** $\mathbf{F}(x, y) = 12xy\mathbf{i} + 6(x^2 + y)\mathbf{j}$

$M = 12xy$ and $N = 6(x^2 + y)$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = 12x = \frac{\partial M}{\partial y} \implies \mathbf{F} \text{ is conservative.}$$

**29.** $\mathbf{F}(x, y) = \sin y\mathbf{i} + x \cos y\mathbf{j}$

$M = \sin y$ and $N = x \cos y$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = \cos y = \frac{\partial M}{\partial y} \implies \mathbf{F} \text{ is conservative.}$$

**31.** $M = 15y^3, N = -5xy^2$

$$\frac{\partial N}{\partial x} = -5y^2 \neq \frac{\partial M}{\partial y} = 45y^2 \implies \text{Not conservative}$$

**33.** $M = \dfrac{2}{y}e^{2x/y}, N = \dfrac{-2x}{y^2}e^{2x/y}$

$$\frac{\partial N}{\partial x} = \frac{-2(y + 2x)}{y^3}e^{2x/y} = \frac{\partial M}{\partial y} \implies \text{Conservative}$$

**35.** $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

$$\frac{\partial}{\partial y}[2xy] = 2x$$

$$\frac{\partial}{\partial x}[x^2] = 2x$$

Conservative

$$f_x(x, y) = 2xy$$

$$f_y(x, y) = x^2$$

$$f(x, y) = x^2y + K$$

**37.** $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

$$\frac{\partial}{\partial y}[2xye^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

$$\frac{\partial}{\partial x}[x^2e^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

Conservative

$$f_x(x, y) = 2xye^{x^2y}$$

$$f_y(x, y) = x^2e^{x^2y}$$

$$f(x, y) = e^{x^2y} + K$$

**39.** $\mathbf{F}(x, y) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial}{\partial x}\left[\frac{y}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

Conservative

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y) = \frac{y}{x^2 + y^2}$$

$$f(x, y) = \frac{1}{2}\ln(x^2 + y^2) + K$$

**41.** $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} + \sin y\mathbf{j})$

$$\frac{\partial}{\partial y}[e^x \cos y] = -e^x \sin y$$

$$\frac{\partial}{\partial x}[e^x \sin y] = e^x \sin y$$

Not conservative

**43.** $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \; (1, 2, 1)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\textbf{curl F}\,(1, 2, 1) = 2\mathbf{j} - \mathbf{k}$$

**45.** $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} - e^x \cos y\,\mathbf{j}, \; (0, 0, 3)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} = -2e^x \cos y\mathbf{k}$$

$$\textbf{curl F}\,(0, 0, 3) = -2\mathbf{k}$$

**47.** $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan\left(\frac{x}{y}\right) & \frac{1}{2}\ln(x^2 + y^2) & 1 \end{vmatrix} = \left[\frac{x}{x^2 + y^2} - \frac{(-x/y^2)}{1 + (x/y)^2}\right]\mathbf{k} = \frac{2x}{x^2 + y^2}\mathbf{k}$$

**49.** $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x - y) & \sin(y - z) & \sin(z - x) \end{vmatrix} = \cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$$

**51.** $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} - x\cos y\,\mathbf{j} + \mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & -x\cos y & 1 \end{vmatrix} = -2\cos y\,\mathbf{k} \neq \mathbf{0}$$

Not conservative

**53.** $\mathbf{F}(x, y, z) = e^z(y\,\mathbf{i} + x\,\mathbf{j} + xy\,\mathbf{k})$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = ye^z$$
$$f_y(x, y, z) = xe^z$$
$$f_z(x, y, z) = xye^z$$
$$f(x, y, z) = xye^z + K$$

**55.** $\mathbf{F}(x, y, z) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j} + (2z - 1)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{y} & -\frac{x}{y^2} & 2z - 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{1}{y}$$
$$f_y(x, y, z) = -\frac{x}{y^2}$$
$$f_z(x, y, z) = 2z - 1$$
$$f(x, y, z) = \int \frac{1}{y}\,dx = \frac{x}{y} + g(y, z) + K_1$$
$$f(x, y, z) = \int -\frac{x}{y^2}\,dy = \frac{x}{y} + h(x, z) + K_2$$
$$f(x, y, z) = \int (2z - 1)\,dz$$
$$= z^2 - z + p(x, y) + K_3$$
$$f(x, y, z) = \frac{x}{y} + z^2 - z + K$$

**57.** $\mathbf{F}(x, y) = 6x^2\mathbf{i} - xy^2\mathbf{j}$

$$\text{div}\,\mathbf{F}(x, y) = \frac{\partial}{\partial x}[6x^2] + \frac{\partial}{\partial y}[-xy^2]$$
$$= 12x - 2xy$$

**59.** $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + z^2\mathbf{k}$

$$\text{div}\,\mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}[z^2] = \cos x - \sin y + 2z$$

**61.** $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$\text{div}\,\mathbf{F}(x, y, z) = yz + 1 + 1 = yz + 2$

$\text{div}\,\mathbf{F}(1, 2, 1) = 4$

**63.** $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} - e^x \cos y\,\mathbf{j}$

$\text{div}\,\mathbf{F}(x, y, z) = e^x \sin y + e^x \sin y$

$\text{div}\,\mathbf{F}(0, 0, 3) = 0$

**65.** See the definition, page 1008. Examples include velocity fields, gravitational fields and magnetic fields.

**67.** See the definition on page 1014.

**69.** $\mathbf{F}(x, y, z) = \mathbf{i} + 2x\,\mathbf{j} + 3y\,\mathbf{k}$

$\mathbf{G}(x, y, z) = x\,\mathbf{i} - y\,\mathbf{j} + z\,\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix} = (2xz + 3y^2)\mathbf{i} - (z - 3xy)\mathbf{j} + (-y - 2x^2)\mathbf{k}$$

$$\textbf{curl}\,(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz + 3y^2 & 3xy - z & -y - 2x^2 \end{vmatrix} = (-1 + 1)\mathbf{i} - (-4x - 2x)\mathbf{j} + (3y - 6y)\mathbf{k} = 6x\mathbf{j} - 3y\mathbf{k}$$

**71.** $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\textbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\,\mathbf{j} - xz\,\mathbf{k}$$

$$\textbf{curl}(\textbf{curl}\,\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = z\,\mathbf{j} + y\,\mathbf{k}$$

**73.** $\mathbf{F}(x, y, z) = \mathbf{i} + 2x\,\mathbf{j} + 3y\,\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\,\mathbf{j} + z\,\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix}$$

$$= (2xz + 3y^2)\mathbf{i} - (z - 3xy)\mathbf{j} + (-y - 2x^2)\mathbf{k}$$

$\text{div}(\mathbf{F} \times \mathbf{G}) = 2z + 3x$

**75.** $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$

$$\textbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$\text{div}(\textbf{curl}\,\mathbf{F}) = x - x = 0$

**77.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = Q\mathbf{i} + R\,\mathbf{j} + S\mathbf{k}$ where $M, N, P, Q, R,$ and $S$ have continuous partial derivatives.

$\mathbf{F} + \mathbf{G} = (M + Q)\mathbf{i} + (N + R)\mathbf{j} + (P + S)\mathbf{k}$

$$\textbf{curl}(\mathbf{F} + \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M + Q & N + R & P + S \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(P + S) - \frac{\partial}{\partial z}(N + R)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(P + S) - \frac{\partial}{\partial z}(M + Q)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(N + R) - \frac{\partial}{\partial y}(M + Q)\right]\mathbf{k}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} + \left(\frac{\partial S}{\partial y} - \frac{\partial R}{\partial z}\right)\mathbf{i} - \left(\frac{\partial S}{\partial x} - \frac{\partial Q}{\partial z}\right)\mathbf{j} + \left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y}\right)\mathbf{k}$$

$$= \textbf{curl}\,\mathbf{F} + \textbf{curl}\,\mathbf{G}$$

**79.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\begin{aligned}
\operatorname{div}(\mathbf{F} + \mathbf{G}) = \frac{\partial}{\partial x}(M + R) + \frac{\partial}{\partial y}(N + S) + \frac{\partial}{\partial z}(P + T) &= \frac{\partial M}{\partial x} + \frac{\partial R}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial S}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial T}{\partial z} \\
&= \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right] + \left[\frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z}\right] \\
&= \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}
\end{aligned}$$

**81.** $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\begin{aligned}
\nabla \times [\nabla f + (\nabla \times \mathbf{F})] &= \mathbf{curl}(\nabla f + (\nabla \times \mathbf{F})) \\
&= \mathbf{curl}(\nabla f) + \mathbf{curl}(\nabla \times \mathbf{F}) && \text{(Exercise 77)} \\
&= \mathbf{curl}(\nabla \times \mathbf{F}) && \text{(Exercise 78)} \\
&= \nabla \times (\nabla \times \mathbf{F})
\end{aligned}$$

**83.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, then $f\mathbf{F} = fM\mathbf{i} + fN\mathbf{j} + fP\mathbf{k}$.

$$\begin{aligned}
\operatorname{div}(f\mathbf{F}) = \frac{\partial}{\partial x}(fM) + \frac{\partial}{\partial y}(fN) + \frac{\partial}{\partial z}(fP) &= f\frac{\partial M}{\partial x} + M\frac{\partial f}{\partial x} + f\frac{\partial N}{\partial y} + N\frac{\partial f}{\partial y} + f\frac{\partial P}{\partial z} + P\frac{\partial f}{\partial z} \\
&= f\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial N}{\partial z}\right) + \left(\frac{\partial f}{\partial x}M + \frac{\partial f}{\partial y}N + \frac{\partial f}{\partial z}P\right) \\
&= f\operatorname{div}\mathbf{F} + \nabla f \cdot \mathbf{F}
\end{aligned}$$

**In Exercises 85 and 87, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.**

**85.** $\ln f = \dfrac{1}{2}\ln(x^2 + y^2 + z^2)$

$$\nabla(\ln f) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{F}}{f^2}$$

**87.** $f^n = \left(\sqrt{x^2 + y^2 + z^2}\right)^n$

$$\begin{aligned}
\nabla f^n &= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} \\
&\quad + n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} \\
&= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nf^{n-2}\mathbf{F}
\end{aligned}$$

**89.** The winds are stronger over Phoenix. Although the winds over both cities are northeasterly, they are more towards the east over Atlanta.

## Section 14.2 Line Integrals

**1.**
$$x^2 + y^2 = 9$$
$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$
$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t = \frac{x^2}{9}$$
$$\sin^2 t = \frac{y^2}{9}$$
$$x = 3\cos t$$
$$y = 3\sin t$$
$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$$
$$0 \le t \le 2\pi$$

**3.** $$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \le t \le 6 \\ (9-t)\mathbf{i} + 3\mathbf{j}, & 6 \le t \le 9 \\ (12-t)\mathbf{j}, & 9 \le t \le 12 \end{cases}$$

**5.** $$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \sqrt{t}\,\mathbf{j}, & 0 \le t \le 1 \\ (2-t)\mathbf{i} + (2-t)\mathbf{j}, & 1 \le t \le 2 \end{cases}$$

**7.** $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j},\ 0 \le t \le 2;\ \mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$

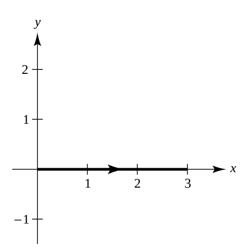
$$\int_C (x-y)\,ds = \int_0^2 (4t-3t)\sqrt{(4)^2+(3)^2}\,dt = \int_0^2 5t\,dt = \left[\frac{5t^2}{2}\right]_0^2 = 10$$

**9.** $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 8t\mathbf{k},\ 0 \le t \le \dfrac{\pi}{2};\ \mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + 8\mathbf{k}$

$$\int_C (x^2+y^2+z^2)\,ds = \int_0^{\pi/2} (\sin^2 t + \cos^2 t + 64t^2)\sqrt{(\cos t)^2 + (-\sin t)^2 + 64}\,dt$$
$$= \int_0^{\pi/2} \sqrt{65}(1+64t^2)\,dt = \left[\sqrt{65}\left(t + \frac{64t^3}{3}\right)\right]_0^{\pi/2} = \sqrt{65}\left(\frac{\pi}{2} + \frac{8\pi^3}{3}\right) = \frac{\sqrt{65}\pi}{6}(3 + 16\pi^2)$$

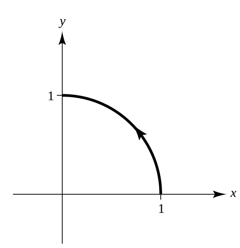
**11.** $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 3$

$$\int_C (x^2+y^2)\,ds = \int_0^3 [t^2 + 0^2]\sqrt{1+0}\,dt$$
$$= \int_0^3 t^2\,dt$$
$$= \left[\frac{1}{3}t^3\right]_0^3 = 9$$

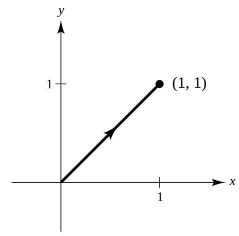



**13.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$$\int_C (x^2+y^2)\,ds = \int_0^{\pi/2} [\cos^2 t + \sin^2 t]\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt$$
$$= \int_0^{\pi/2} dt = \frac{\pi}{2}$$



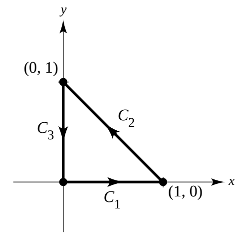
**15.** $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$




$$\int_C \left(x + 4\sqrt{y}\right) ds = \int_0^1 \left(t + 4\sqrt{t}\right)\sqrt{1+1}\, dt$$

$$= \left[\sqrt{2}\left(\frac{t^2}{2} + \frac{8}{3}t^{3/2}\right)\right]_0^1 = \frac{19\sqrt{2}}{6}$$

**17.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ (2-t)\mathbf{i} + (t-1)\mathbf{j}, & 1 \le t \le 2 \\ (3-t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$




$$\int_{C_1} \left(x + 4\sqrt{y}\right) ds = \int_0^1 t\, dt = \frac{1}{2}$$

$$\int_{C_2} \left(x + 4\sqrt{y}\right) ds = \int_1^2 \left[(2-t) + 4\sqrt{t-1}\right]\sqrt{1+1}\, dt$$

$$= \sqrt{2}\left[2t - \frac{t^2}{2} + \frac{8}{3}(t-1)^{3/2}\right]_1^2 = \frac{19\sqrt{2}}{6}$$

$$\int_{C_3} \left(x + 4\sqrt{y}\right) ds = \int_2^3 4\sqrt{3-t}\, dt = \left[-\frac{8}{3}(3-t)^{3/2}\right]_2^3 = \frac{8}{3}$$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \frac{1}{2} + \frac{19\sqrt{2}}{6} + \frac{8}{3} = \frac{19 + 19\sqrt{2}}{6} = \frac{19\left(1+\sqrt{2}\right)}{6}$$

**19.** $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 4\pi$$

$$\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2)^2} = \sqrt{13}$$

$$\text{Mass} = \int_C \rho(x, y, z)\, ds = \int_0^{4\pi} \frac{1}{2}\left[(3\cos t)^2 + (3\sin t)^2 + (2t)^2\right]\sqrt{13}\, dt$$

$$= \frac{\sqrt{13}}{2}\int_0^{4\pi} (9 + 4t^2)\, dt = \left[\frac{\sqrt{13}}{2}\left(9t + \frac{4t^3}{3}\right)\right]_0^{4\pi}$$

$$= \frac{2\sqrt{13}\pi}{3}(27 + 64\pi^2) \approx 4973.8$$

**21.** $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$C$: $\mathbf{r}(t) = 4t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{F}(t) = 4t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}'(t) = 4\mathbf{i} + \mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (16t^2 + t)\, dt$$

$$= \left[\frac{16}{3}t^3 + \frac{1}{2}t^2\right]_0^1 = \frac{35}{6}$$

**23.** $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{i}$

$C$: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{F}(t) = 6\cos t\,\mathbf{i} + 8\sin t\,\mathbf{j}$

$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-12\sin t\cos t + 16\sin t\cos t)\, dt$$

$$= \left[2\sin^2 t\right]_0^{\pi/2} = 2$$

**25.** $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + (x - z)\mathbf{j} + xyz\mathbf{k}$

$C$: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k},\ 0 \le t \le 1$

$\mathbf{F}(t) = t^4\mathbf{i} + (t - 2)\mathbf{j} + 2t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[t^4 + 2t(t - 2)\right] dt$$

$$= \left[\frac{t^5}{5} + \frac{2t^3}{3} - 2t^2\right]_0^1 = -\frac{17}{15}$$

**27.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k},\ 1 \le t \le 3$

$\mathbf{F}(t) = t^2 \ln t\mathbf{i} + 6t^2\mathbf{j} + t^2 \ln^2 t\mathbf{k}$

$d\mathbf{r} = \left(\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}\right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 \left[t^2 \ln t + 12t^3 + t(\ln t)^2\right] dt$$

$$\approx 249.49$$

**29.** $\mathbf{F}(x, y) = -x\mathbf{i} - 2y\mathbf{j}$

$C$: $y = x^3$ from $(0, 0)$ to $(2, 8)$

$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = -t\mathbf{i} - 2t^3\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -t - 6t^5$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (-t - 6t^5)\, dt = \left[-\frac{1}{2}t^2 - t^6\right]_0^2 = -66$$

**31.** $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

$C$: counterclockwise around the triangle whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ \mathbf{i} + (t - 1)\mathbf{j}, & 1 \le t \le 2 \\ (3 - t)\mathbf{i} + (3 - t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$$

On $C_1$: $\mathbf{F}(t) = 2t\mathbf{i},\ \mathbf{r}'(t) = \mathbf{i}$

$$\text{Work} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2t\, dt = 1$$

On $C_2$: $\mathbf{F}(t) = 2\mathbf{i} + (t - 1)\mathbf{j},\ \mathbf{r}'(t) = \mathbf{j}$

$$\text{Work} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (t - 1)\, dt = \frac{1}{2}$$

On $C_3$: $\mathbf{F}(t) = 2(3 - t)\mathbf{i} + (3 - t)\mathbf{j},\ \mathbf{r}'(t) = -\mathbf{i} - \mathbf{j}$

$$\text{Work} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 \left[-2(3 - t) - (3 - t)\right] dt = -\frac{3}{2}$$

$$\text{Total work} = \int_C \mathbf{F} \cdot d\mathbf{r} = 1 + \frac{1}{2} - \frac{3}{2} = 0$$

**33.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

$C$: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} - 5t\mathbf{k}$

$\mathbf{F} \cdot \mathbf{r}' = -5t$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -5t\, dt = -10\pi^2$$

**35.** $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j} + \frac{10}{2\pi}t\mathbf{k},\ 0 \le t \le 2\pi$

$\mathbf{F} = 150\mathbf{k}$

$d\mathbf{r} = \left(3\cos t\mathbf{i} - 3\sin t\mathbf{j} + \frac{10}{2\pi}\mathbf{k}\right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1500}{2\pi}\, dt = \left[\frac{1500}{2\pi}t\right]_0^{2\pi} = 1500 \text{ ft} \cdot \text{lb}$$

**37.** $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = 2t\mathbf{i} + (t - 1)\mathbf{j},\ 1 \le t \le 3$

$\mathbf{r}_1'(t) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = 4t^2\mathbf{i} + 2t(t - 1)\mathbf{j}$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (8t^2 + 2t(t - 1))\, dt = \frac{236}{3}$$

Both paths join (2, 0) and (6, 2). The integrals are negatives of each other because the orientations are different.

(b) $\mathbf{r}_2(t) = 2(3 - t)\mathbf{i} + (2 - t)\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}_2'(t) = -2\mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = 4(3 - t)^2\mathbf{i} + 2(3 - t)(2 - t)\mathbf{j}$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 [-8(3 - t)^2 - 2(3 - t)(2 - t)]\, dt = -\frac{236}{3}$$

**39.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

$C$: $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -2t + 2t = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

**41.** $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \frac{y}{2}\right)\mathbf{j}$

$C$: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (t^3 - 2t^2)\mathbf{i} + \left(t - \frac{t^2}{2}\right)\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = (t^3 - 2t^2) + 2t\left(t - \frac{t^2}{2}\right) = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

**43.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x$ or $x = \frac{y}{5},\ 0 \le y \le 10$

$$\int_C (x + 3y^2)\, dy = \int_0^{10} \left(\frac{y}{5} + 3y^2\right) dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

**45.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies x = \frac{y}{5},\ 0 \le y \le 10,\ dx = \frac{1}{5}\, dy$

$$\int_C xy\, dx + y\, dy = \int_0^{10} \left(\frac{y^2}{25} + y\right) dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \quad \text{OR}$$

$y = 5x,\ dy = 5\, dx,\ 0 \le x \le 2$

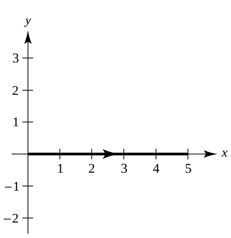
$$\int_C xy\, dx + y\, dy = \int_0^2 (5x^2 + 25x)\, dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

**47.** $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 5$

$x(t) = t, \quad y(t) = 0$

$dx = dt, \quad dy = 0$

$$\int_C (2x - y)\, dx + (x + 3y)\, dy = \int_0^5 2t\, dt = 25$$



**49.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \le t \le 6 \end{cases}$

$C_1$: $x(t) = t,\ y(t) = 0,$

$dx = dt,\ dy = 0$

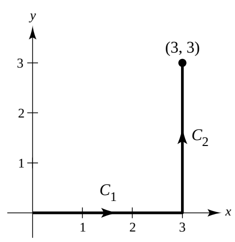
$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 2t\,dt = 9$$

$C_2$: $x(t) = 3,\ y(t) = t - 3$

$dx = 0,\ dy = dt$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^6 [3 + 3(t-3)]\,dt = \left[\frac{3t^2}{2} - 6t\right]_3^6 = \frac{45}{2}$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = 9 + \frac{45}{2} = \frac{63}{2}$$




**51.** $x(t) = t,\ y(t) = 1 - t^2,\ 0 \le t \le 1,\ dx = dt,\ dy = -2t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^1 [(2t - 1 + t^2) + (t + 3 - 3t^2)(-2t)]\,dt$$

$$= \int_0^1 (6t^3 - t^2 - 4t - 1)\,dt = \left[\frac{3t^4}{2} - \frac{t^3}{3} - 2t^2 - t\right]_0^1 = -\frac{11}{6}$$

**53.** $x(t) = t,\ y(t) = 2t^2,\ 0 \le t \le 2$

$dx = dt,\ dy = 4t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^2 (2t - 2t^2)\,dt + (t + 6t^2)4t\,dt$$

$$= \int_0^2 (24t^3 + 2t^2 + 2t)\,dt = \left[6t^4 + \frac{2}{3}t^3 + t^2\right]_0^2 = \frac{316}{3}$$

**55.** $f(x, y) = h$

$C$: line from $(0, 0)$ to $(3, 4)$

$\mathbf{r} = 3t\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{r}'(t)\| = 5$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^1 5h\,dt = 5h$$

**57.** $f(x, y) = xy$

$C$: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

$\|\mathbf{r}'(t)\| = 1$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} \cos t \sin t\,dt$$

$$= \left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{1}{2}$$

**59.** $f(x, y) = h$

$C$: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j},\ 0 \le t \le 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\begin{aligned}\int_C f(x, y)\,ds &= \int_0^1 h\sqrt{1 + 4(1 - t)^2}\,dt \\ &= -\frac{h}{4}\left[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1 \\ &= \frac{h}{4}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \approx 1.4789h\end{aligned}$$

**61.** $f(x, y) = xy$

$C$: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

You could parameterize the curve $C$ as in Exercises 59 and 60. Alternatively, let $x = \cos t$, then:

$$y = 1 - \cos^2 t = \sin^2 t$$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^2 t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + 2\sin t\cos t\,\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4\sin^2 t\cos^2 t} = \sin t\sqrt{1 + 4\cos^2 t}$$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} \cos t\sin^2 t\left(\sin t\sqrt{1 + 4\cos^2 t}\right)dt = \int_0^{\pi/2} \sin^2 t\left[(1 + 4\cos^2 t)^{1/2}\sin t\cos t\right]dt$$

Let $u = \sin^2 t$ and $dv = (1 + 4\cos^2 t)^{1/2}\sin t\cos t$, then $du = 2\sin t\cos t\,dt$ and $v = -\frac{1}{12}(1 + 4\cos^2 t)^{3/2}$.

$$\begin{aligned}\int_C f(x, y)\,ds &= \left[-\frac{1}{12}\sin^2 t(1 + 4\cos^2 t)^{3/2}\right]_0^{\pi/2} + \frac{1}{6}\int_0^{\pi/2}(1 + 4\cos^2 t)^{3/2}\sin t\cos t\,dt \\ &= \left[-\frac{1}{12}\sin^2 t(1 + 4\cos^2 t)^{3/2} - \frac{1}{120}(1 + 4\cos^2 t)^{5/2}\right]_0^{\pi/2} \\ &= \left(-\frac{1}{12} - \frac{1}{120}\right) + \frac{1}{120}(5)^{5/2} = \frac{1}{120}\left(25\sqrt{5} - 11\right) \approx 0.3742\end{aligned}$$

**63.** (a) $f(x, y) = 1 + y^2$

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le 2\pi$$

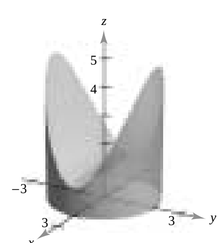
$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 2$$

$$\begin{aligned}S &= \int_C f(x, y)\,ds = \int_0^{2\pi}(1 + 4\sin^2 t)(2)\,dt \\ &= \left[2t + 4(t - \sin t\cos t)\right]_0^{2\pi} = 12\pi \approx 37.70\text{ cm}^2\end{aligned}$$

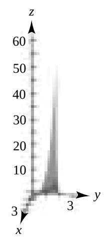
(b) $0.2(12\pi) = \dfrac{12\pi}{5} \approx 7.54\text{ cm}^3$

(c)

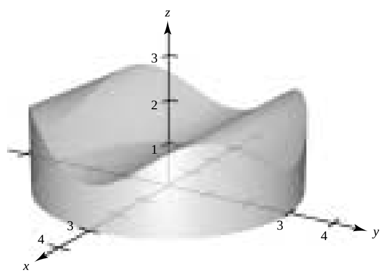


**65.** $S \approx 25$

Matches b




**67.** (a) Graph of: $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + (1 + \sin^2 2t)\mathbf{k}\ \ 0 \le t \le 2\pi$




(b) Consider the portion of the surface in the first quadrant. The curve $z = 1 + \sin^2 2t$ is over the curve $\mathbf{r}_1(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j},\ 0 \le t \le \pi/2$. Hence, the total lateral surface area is

$$4\int_C f(x, y)\,ds = 4\int_0^{\pi/2} (1 + \sin^2 2t)3\,dt = 12\left(\frac{3\pi}{4}\right) = 9\pi \text{ sq. cm}$$

(c) The cross sections parallel to the $xz$-plane are rectangles of height $1 + 4(y/3)^2(1 - y^2/9)$ and base $2\sqrt{9 - y^2}$. Hence,

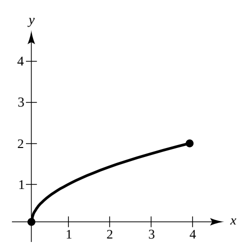
$$\text{Volume} = 2\int_0^3 2\sqrt{9 - y^2}\left(1 + 4\frac{y^2}{9}\left(1 - \frac{y^2}{9}\right)\right)dy \approx 42.412 \text{ cm}^3$$

**69.** See the definition of Line Integral, page 1020.

See Theorem 14.4.

**71.** The greater the height of the surface over the curve, the greater the lateral surface area. Hence,

$$z_3 < z_1 < z_2 < z_4.$$




**73.** False

$$\int_C xy\,ds = \sqrt{2}\int_0^1 t^2\,dt$$

**75.** False, the orientations are different.

## Section 14.3 Conservative Vector Fields and Independence of Path

**1.** $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^4)\,dt = \frac{11}{15}$$

(b) $\mathbf{r}_2(\theta) = \sin\theta\mathbf{i} + \sin^2\theta\mathbf{j},\ 0 \le \theta \le \frac{\pi}{2}$

$\mathbf{r}_2'(\theta) = \cos\theta\mathbf{i} + 2\sin\theta\cos\theta\mathbf{j}$

$\mathbf{F}(t) = \sin^2\theta\mathbf{i} + \sin^3\theta\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\sin^2\theta\cos\theta + 2\sin^4\theta\cos\theta)\,d\theta$$

$$= \left[\frac{\sin^3\theta}{3} + \frac{2\sin^5\theta}{5}\right]_0^{\pi/2} = \frac{11}{15}$$

**3.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec\theta\mathbf{i} + \tan\theta\mathbf{j},\ 0 \le \theta \le \frac{\pi}{3}$

$\mathbf{r}_1'(\theta) = \sec\theta\tan\theta\mathbf{i} + \sec^2\theta\mathbf{j}$

$\mathbf{F}(\theta) = \tan\theta\mathbf{i} - \sec\theta\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/3} (\sec\theta\tan^2\theta - \sec^3\theta)\,d\theta = \int_0^{\pi/3} [\sec\theta(\sec^2\theta - 1) - \sec^3\theta]\,d\theta$$

$$= -\int_0^{\pi/3} \sec\theta\,d\theta = \Big[-\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = -\ln\left(2 + \sqrt{3}\right) \approx -1.317$$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\,\mathbf{i} + \sqrt{t}\,\mathbf{j},\quad 0 \le t \le 3$

$$\mathbf{r}_2'(t) = \frac{1}{2\sqrt{t+1}}\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$$

$\mathbf{F}(t) = \sqrt{t}\,\mathbf{i} - \sqrt{t+1}\,\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[\frac{\sqrt{t}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t}}\right] dt = -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{t}\sqrt{t+1}}\,dt = -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{t^2 + t + (1/4) - (1/4)}}\,dt$$

$$= -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{[t + (1/2)]^2 - (1/4)}}\,dt = \left[-\frac{1}{2}\ln\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t}\right|\right]_0^3$$

$$= -\frac{1}{2}\left[\ln\left(\frac{7}{2} + 2\sqrt{3}\right) - \ln\left(\frac{1}{2}\right)\right] = -\frac{1}{2}\ln\left(7 + 4\sqrt{3}\right) \approx -1.317$$

**5.** $\mathbf{F}(x, y) = e^x\sin y\mathbf{i} + e^x\cos y\mathbf{j}$

$$\frac{\partial N}{\partial x} = e^x\cos y \qquad \frac{\partial M}{\partial y} = e^x\cos y$$

Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

**7.** $\mathbf{F}(x, y) = \dfrac{1}{y}\mathbf{i} + \dfrac{x}{y^2}\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{1}{y^2} \qquad \frac{\partial M}{\partial y} = -\frac{1}{y^2}$$

Since $\dfrac{\partial N}{\partial x} \ne \dfrac{\partial M}{\partial y}$, $\mathbf{F}$ is not conservative.

**9.** $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

$\textbf{curl}\,\mathbf{F} = \mathbf{0} \implies \mathbf{F}$ is conservative.

**11.** $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = 2t^3\mathbf{i} + t^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^3\, dt = 1$$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = 2t^4\mathbf{i} + t^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^4\, dt = 1$$

**13.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = t\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2\, dt = -\frac{1}{3}$$

(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_3'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t^3\, dt = -\frac{1}{2}$$

**15.** $\displaystyle\int_C y^2\, dx + 2xy\, dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative. The potential function is $f(x, y) = xy^2 + k$. Therefore, we can use the Fundamental Theorem of Line Integrals.

(a) $\displaystyle\int_C y^2\, dx + 2xy\, dy = \Big[x^2y\Big]_{(0,0)}^{(4,4)} = 64$

(b) $\displaystyle\int_C y^2\, dx + 2xy\, dy = \Big[x^2y\Big]_{(-1,0)}^{(1,0)} = 0$

(c) and (d) Since $C$ is a closed curve, $\displaystyle\int_C y^2\, dx + 2xy\, dy = 0$.

**17.** $\displaystyle\int_C 2xy\, dx + (x^2 + y^2)\, dy$

Since $\partial M/\partial y = \partial N/\partial x = 2x$,

$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ is conservative.

The potential function is $f(x, y) = x^2y + \dfrac{y^3}{3} + k$.

(a) $\displaystyle\int_C 2xy\, dx + (x^2 + y^2)\, dy = \left[x^2y + \frac{y^3}{3}\right]_{(5,0)}^{(0,4)} = \frac{64}{3}$

(b) $\displaystyle\int_C 2xy\, dx + (x^2 + y^2)\, dy = \left[x^2y + \frac{y^3}{3}\right]_{(2,0)}^{(0,4)} = \frac{64}{3}$

**19.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Since **curl F** $= \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xyz + k$.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xyz\Big]_{(0,2,0)}^{(4,2,4)} = 32$$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 2$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xyz\Big]_{(0,0,0)}^{(4,2,4)} = 32$$

**21.** $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (2t^2 + t)\mathbf{i} + (t^2 - 1)\mathbf{j} + (2t^2 - 4)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3 + 2t^2 - t)\, dt = \frac{2}{3}$$

**—CONTINUED—**

**21. —CONTINUED—**

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t-1)^2\mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 4(2t-1)\mathbf{k}$

$\mathbf{F}(t) = 3t\mathbf{i} + [t^2 - (2t-1)^2]\mathbf{j} + [2t - 4(2t-1)^2]\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [3t + t^2 - (2t-1)^2 + 8t(2t-1) - 16(2t-1)^3]\,dt$$

$$= \int_0^1 [17t^2 - 5t - (2t-1)^2 - 16(2t-1)^3]\,dt = \left[\frac{17t^3}{3} - \frac{5t^2}{2} - \frac{(2t-1)^3}{6} - 2(2t-1)^4\right]_0^1 = \frac{17}{6}$$

**23.** $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xye^z + k$.

(a) $\mathbf{r}_1(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3\mathbf{k},\ 0 \le t \le \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xye^z\Big]_{(4,0,3)}^{(-4,0,3)} = 0$$

(b) $\mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k},\ 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xye^z\Big]_{(4,0,3)}^{(-4,0,3)} = 0$$

**25.** $\displaystyle\int_C (y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{r} = \Big[xy\Big]_{(0,0)}^{(3,8)} = 24$

**27.** $\displaystyle\int_C \cos x \sin y\,dx + \sin x \cos y\,dy = \Big[\sin x \sin y\Big]_{(0,-\pi)}^{(3\pi/2,\,\pi/2)} = -1$

**29.** $\displaystyle\int_C e^x \sin y\,dx + e^x \cos y\,dy = \Big[e^x \sin y\Big]_{(0,0)}^{(2\pi,0)} = 0$

**31.** $\displaystyle\int_C (y + 2z)\,dx + (x - 3z)\,dy + (2x - 3y)\,dz$

$\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xy - 3yz + 2xz$.

(a) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,1,1)} = 0 - 0 = 0$

(b) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(0,0,1)} + \Big[xy - 3yz + 2xz\Big]_{(0,0,1)}^{(1,1,1)} = 0 + 0 = 0$

(c) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,0,0)} + \Big[xy - 3yz + 2xz\Big]_{(1,0,0)}^{(1,1,0)} + \Big[xy - 3yz + 2xz\Big]_{(1,1,0)}^{(1,1,1)} = 0 + 1 + (-1) = 0$

**33.** $\displaystyle\int_C -\sin x\,dx + z\,dy + y\,dz = \Big[\cos x + yz\Big]_{(0,0,0)}^{(\pi/2,3,4)} = 12 - 1 = 11$

**35.** $\mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j}$ is conservative.

Work $= \Big[3x^3y^2 - y\Big]_{(0,0)}^{(5,9)} = 30{,}366$

**37.** $\mathbf{r}(t) = 2\cos 2\pi t\mathbf{i} + 2\sin 2\pi t\mathbf{j}$

$\mathbf{r}'(t) = -4\pi\sin 2\pi t\mathbf{i} + 4\pi\cos 2\pi t\mathbf{j}$

$\mathbf{a}(t) = -8\pi^2\cos 2\pi t\mathbf{i} - 8\pi^2\sin 2\pi t\mathbf{j}$

$\mathbf{F}(t) = m \cdot \mathbf{a}(t) = \dfrac{1}{32}\mathbf{a}(t) = -\dfrac{\pi^2}{4}(\cos 2\pi t\mathbf{i} + \sin 2\pi t\mathbf{j})$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -\frac{\pi^2}{4}(\cos 2\pi t\mathbf{i} + \sin 2\pi t\mathbf{j}) \cdot 4\pi(-\sin 2\pi t\mathbf{i} + \cos 2\pi t\mathbf{j})\,dt = -\pi^3\int_C 0\,dt = 0$$

**39.** Since the sum of the potential and kinetic energies remains constant from point to point, if the kinetic energy is decreasing at a rate of 10 units per minute, then the potential energy is increasing at a rate of 10 units per minute.

**41.** No. The force field is conservative.

**43.** See Theorem 14.5, page 1033.

**45.** (a) The direct path along the line segment joining $(-4, 0)$ to $(3, 4)$ requires less work than the path going from $(-4, 0)$ to $(-4, 4)$ and then to $(3, 4)$.

(b) The closed curve given by the line segments joining $(-4, 0)$, $(-4, 4)$, $(3, 4)$, and $(-4, 0)$ satisfies $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

**47.** False, it would be true if **F** were conservative.

**49.** True

**51.** Let

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial y}\mathbf{i} - \frac{\partial f}{\partial x}\mathbf{j}.$$

Then $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial f}{\partial x}\right) = -\dfrac{\partial^2 f}{\partial x^2}$. Since

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ we have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

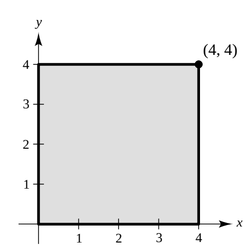
Thus, **F** is conservative. Therefore, by Theorem 14.7, we have

$$\int_C \left(\frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy\right) = \int_C (M\,dx + N\,dy) = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every closed curve in the plane.

## Section 14.4 Green's Theorem

**1.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \le t \le 8 \\ (12-t)\mathbf{i} + 4\mathbf{j}, & 8 \le t \le 12 \\ (16-t)\mathbf{j}, & 12 \le t \le 16 \end{cases}$



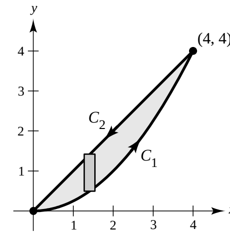

$$\int_C y^2\,dx + x^2\,dy = \int_0^4 [0\,dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16\,dt]$$
$$+ \int_8^{12} [16(-dt) + (12-t)^2(0)] + \int_{12}^{16} [(16-t)^2(0) + 0(-dt)]$$
$$= 0 + 64 - 64 + 0 = 0$$

By Green's Theorem, $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_0^4 (2x - 2y)\,dy\,dx = \int_0^4 (8x - 16)\,dx = 0.$

**3.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \le t \le 4 \\ (8-t)\mathbf{i} + (8-t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$

$$\int_C y^2\,dx + x^2\,dy = \int_0^4 \left[\frac{t^4}{16}(dt) + t^2\left(\frac{t}{2}\,dt\right)\right] + \int_4^8 [(8-t)^2(-dt) + (8-t)^2(-dt)]$$

$$= \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2}\right] dt + \int_4^8 -2(8-t)^2\,dt = \frac{224}{5} - \frac{128}{3} = \frac{32}{15}$$




By Green's Theorem,

$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_{x^2/4}^{x} (2x - 2y)\,dy\,dx = \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16}\right) dx = \frac{32}{15}.$$

**5.** $C$: $x^2 + y^2 = 4$

Let $x = 2\cos t$ and $y = 2\sin t$, $0 \le t \le 2\pi$.

$$\int_C xe^y\,dx + e^x\,dy = \int_0^{2\pi} [2\cos te^{2\sin t}(-2\sin t) + e^{2\cos t}(2\cos t)]\,dt \approx 19.99$$

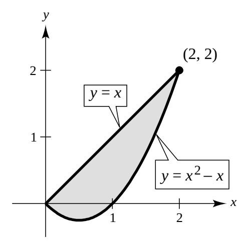
$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (e^x - xe^y)\,dy\,dx = \int_{-2}^{2}\left[2\sqrt{4-x^2}\,e^x - xe^{\sqrt{4-x^2}} + xe^{-\sqrt{4-x^2}}\right] dx \approx 19.99$$

**In Exercises 7 and 9, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$.**

**7.** $\displaystyle\int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2\int_{x^2-x}^{x} dy\,dx$

$$= \int_0^2 (2x - x^2)\,dx$$

$$= \frac{4}{3}$$



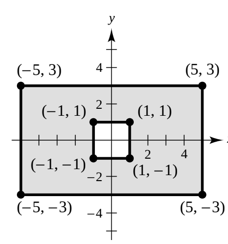

**9.** From the accompanying figure, we see that $R$ is the shaded region. Thus, Green's Theorem yields

$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R\int 1\,dA$$

$$= \text{Area of } R$$

$$= 6(10) - 2(2)$$

$$= 56.$$




**11.** Since the curves $y = 0$ and $y = 4 - x^2$ intersect at $(-2, 0)$ and $(2, 0)$, Green's Theorem yields

$$\int_C 2xy\,dx + (x + y)\,dy = \int_R\int (1 - 2x)\,dA = \int_{-2}^{2}\int_0^{4-x^2} (1 - 2x)\,dy\,dx$$

$$= \int_{-2}^{2}\left[y - 2xy\right]_0^{4-x^2} dx$$

$$= \int_{-2}^{2} (4 - 8x - x^2 + 2x^3)\,dx$$

$$= \left[4x - 4x^2 - \frac{x^3}{3} + \frac{x^4}{2}\right]_{-2}^{2}$$

$$= -\frac{8}{3} - \frac{8}{3} + 16 = \frac{32}{3}.$$

**13.** Since $R$ is the interior of the circle $x^2 + y^2 = a^2$, Green's Theorem yields

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy = \iint_R (2y + 2y)\,dA$$
$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y\,dy\,dx = 4\int_{-a}^{a} 0\,dx = 0.$$

**15.** Since $\dfrac{\partial M}{\partial y} = \dfrac{2x}{x^2 + y^2} = \dfrac{\partial N}{\partial x}$,

we have path independence and

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

**17.** By Green's Theorem,

$$\int_C \sin x \cos y\,dx + (xy + \cos x \sin y)\,dy = \iint_R [(y - \sin x \sin y) - (-\sin x \sin y)]\,dA$$
$$= \int_0^1 \int_x^{\sqrt{x}} y\,dy\,dx = \frac{1}{2}\int_0^1 (x - x^2)\,dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{12}.$$

**19.** By Green's Theorem,

$$\int_C xy\,dx + (x + y)\,dy = \iint_R (1 - x)\,dA$$
$$= \int_0^{2\pi}\int_1^3 (1 - r\cos\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(4 - \frac{26}{3}\cos\theta\right) d\theta = 8\pi.$$

**21.** $\mathbf{F}(x, y) = xy\,\mathbf{i} + (x + y)\,\mathbf{j}$

$C$: $x^2 + y^2 = 4$

$$\text{Work} = \int_C xy\,dx + (x + y)\,dy = \iint_R (1 - x)\,dA = \int_0^{2\pi}\int_0^2 (1 - r\cos\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(2 - \frac{8}{3}\cos\theta\right) d\theta = 4\pi$$

**23.** $\mathbf{F}(x, y) = (x^{3/2} - 3y)\,\mathbf{i} + (6x + 5\sqrt{y})\,\mathbf{j}$

$C$: boundary of the triangle with vertices (0, 0), (5, 0), (0, 5)

$$\text{Work} = \int_C (x^{3/2} - 3y)\,dx + \left(6x + 5\sqrt{y}\right) dy = \iint_R 9\,dA = 9\left(\tfrac{1}{2}\right)(5)(5) = \tfrac{225}{2}$$

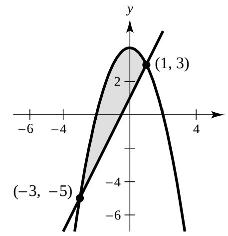
**25.** $C$: let $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$. By Theorem 14.9, we have

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} [a\cos t(a\cos t) - a\sin t(-a\sin t)]\,dt = \frac{1}{2}\int_0^{2\pi} a^2\,dt = \left[\frac{a^2}{2}t\right]_0^{2\pi} = \pi a^2.$$

**27.** From the accompanying figure we see that

$C_1$: $y = 2x + 1,\ dy = 2\,dx$

$C_2$: $y = 4 - x^2,\ dy = -2x\,dx.$




Thus, by Theorem 14.9, we have

$$A = \frac{1}{2}\int_{-3}^{1}[x(2) - (2x + 1)]\,dx + \frac{1}{2}\int_{1}^{-3}[x(-2x) - (4 - x^2)]\,dx$$

$$= \frac{1}{2}\int_{-3}^{1}(-1)\,dx + \frac{1}{2}\int_{1}^{-3}(-x^2 - 4)\,dx$$

$$= \frac{1}{2}\int_{-3}^{1}(-1)\,dx + \frac{1}{2}\int_{-3}^{1}(x^2 + 4)\,dx = \frac{1}{2}\int_{-3}^{1}(3 + x^2)\,dx = \frac{1}{2}\left[3x + \frac{x^3}{3}\right]_{-3}^{1} = \frac{32}{3}.$$

**29.** See Theorem 14.8, page 1042.

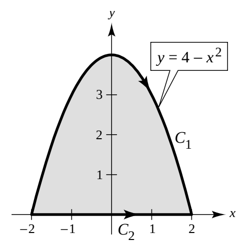
**31.** Answers will vary.

$\mathbf{F}_1(x, y) = y\mathbf{i} + x\mathbf{j}$

$\mathbf{F}_2(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$

$\mathbf{F}_3(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

**33.** $A = \displaystyle\int_{-2}^{2}(4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$




$$\bar{x} = \frac{1}{2A}\int_{C_1} x^2\,dy + \frac{1}{2A}\int_{C_2} x^2\,dy$$

For $C_1$, $dy = -2x\,dx$ and for $C_2$, $dy = 0$. Thus,

$$\bar{x} = \frac{1}{2(32/3)}\int_{2}^{-2} x^2(-2x\,dx) = \left[\frac{3}{64}\left(-\frac{x^4}{2}\right)\right]_{2}^{-2} = 0.$$

To calculate $\bar{y}$, note that $y = 0$ along $C_2$. Thus,

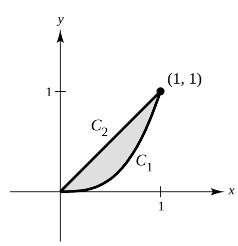
$$\bar{y} = \frac{-1}{2(32/3)}\int_{2}^{-2}(4 - x^2)^2\,dx = \frac{3}{64}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx = \frac{3}{64}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{8}{5}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$$

**35.** Since $A = \displaystyle\int_{0}^{1}(x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4}$, we have $\dfrac{1}{2A} = 2$. On $C_1$ we have $y = x^3$, $dy = 3x^2\,dx$ and on $C_2$ we have $y = x$, $dy = dx$. Thus,

$$\bar{x} = 2\int_{C} x^2\,dy = 2\int_{C_1} x^2(3x^2\,dx) + 2\int_{C_2} x^2\,dx$$

$$= 6\int_{0}^{1} x^4\,dx + 2\int_{1}^{0} x^2\,dx = \frac{6}{5} - \frac{2}{3} = \frac{8}{15}$$




$$\bar{y} = -2\int_{C} y^2\,dx$$

$$= -2\int_{0}^{1} x^6\,dx - 2\int_{1}^{0} x^2\,dx = -\frac{2}{7} + \frac{2}{3} = \frac{8}{21}.$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{15}, \frac{8}{21}\right)$$

**37.** $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} a^2(1 - \cos\theta)^2\, d\theta$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{a^2}{2}(3\pi) = \frac{3\pi a^2}{2}$$

**39.** In this case the inner loop has domain $\dfrac{2\pi}{3} \le \theta \le \dfrac{4\pi}{3}$. Thus,

$$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(1 + 4\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 4\cos\theta + 2\cos 2\theta)\, d\theta = \frac{1}{2}\Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}.$$

**41.** $I = \displaystyle\int_C \frac{y\,dx - x\,dy}{x^2 + y^2}$

(a) Let $\mathbf{F} = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$.

$\mathbf{F}$ is conservative since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$.

$\mathbf{F}$ is defined and has continuous first partials everywhere except at the origin. If $C$ is a circle (a closed path) that does not contain the origin, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = 0.$$

(b) Let $\mathbf{r} = a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j},\ 0 \le t \le 2\pi$ be a circle $C_1$ oriented clockwise inside $C$ (see figure). Introduce line segments $C_2$ and $C_3$ as illustrated in Example 6 of this section in the text. For the region inside C and outside $C_1$, Green's Theorem applies. Note that since $C_2$ and $C_3$ have opposite orientations, the line integrals over them cancel. Thus, $C_4 = C_1 + C_2 + C + C_3$ and

$$\int_{C_4}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{F}\cdot d\mathbf{r} = 0.$$

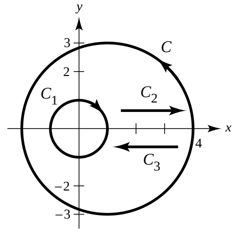
But,

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\left[\frac{(-a\sin t)(-a\sin t)}{a^2\cos^2 t + a^2\sin^2 t} + \frac{(-a\cos t)(-a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\right]dt$$

$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = \Big[t\Big]_0^{2\pi} = 2\pi.$$

Finally, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = -2\pi.$

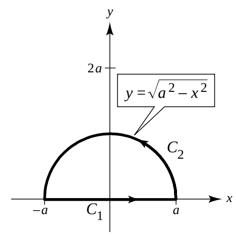
**Note:** If $C$ were orientated clockwise, then the answer would have been $2\pi$.




**43.** Pentagon: $(0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)$

$A = \frac{1}{2}[(0 - 0) + (4 - 0) + (12 - 2) + (1 + 4) + (0 - 0)] = \frac{19}{2}$

**45.** $\displaystyle\int_C y^n\,dx + x^n\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$




For the line integral, use the two paths

$C_1$: $\mathbf{r}_1(x) = x\mathbf{i}$, $-a \le x \le a$

$C_2$: $\mathbf{r}_2(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\,\mathbf{j}$, $x = a$ to $x = -a$

$$\int_{C_1} y^n\,dx + x^n\,dy = 0$$

$$\int_{C_2} y^n\,dx + x^n\,dy = \int_a^{-a} \left[(a^2 - x^2)^{n/2} + x^n \frac{-x}{\sqrt{a^2 - x^2}}\right] dx$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-a}^{a}\int_0^{\sqrt{a^2 - x^2}} \left[nx^{n-1} - ny^{n-1}\right] dy\,dx$$

(a) For $n = 1, 3, 5, 7$, both integrals give 0.

(b) For $n$ even, you obtain

$n = 2: -\frac{4}{3}a^3$ $\quad n = 4: -\frac{16}{15}a^5$ $\quad n = 6: -\frac{32}{35}a^7$ $\quad n = 8: -\frac{256}{315}a^9$

(c) If $n$ is odd and $0 < a < 1$, then the integral equals 0.

**47.** $\displaystyle\int_C (fD_{\mathbf{N}}g - gD_{\mathbf{N}}f)\,ds = \int_C fD_{\mathbf{N}}g\,ds - \int_C gD_{\mathbf{N}}f\,ds$

$$= \iint_R (f\nabla^2 g + \nabla f \cdot \nabla g)\,dA - \iint_R (g\nabla^2 f + \nabla g \cdot \nabla f)\,dA = \iint_R (f\nabla^2 g - g\nabla^2 f)\,dA$$

**49.** $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 0 \implies \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (0)\,dA = 0$$

## Section 14.5 Parametric Surfaces

**1.** $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$

$z = xy$

Matches c.

**3.** $\mathbf{r}(u, v) = 2\cos v\cos u\,\mathbf{i} + 2\cos v\sin u\,\mathbf{j} + 2\sin v\,\mathbf{k}$

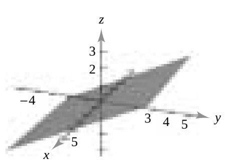
$x^2 + y^2 + z^2 = 4$

Matches b.

**5.** $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \dfrac{v}{2}\mathbf{k}$

$y - 2z = 0$

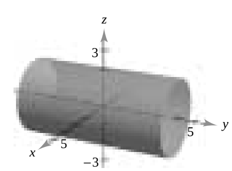
Plane




**7.** $\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\mathbf{j} + 2\sin u\,\mathbf{k}$

$x^2 + z^2 = 4$

Cylinder

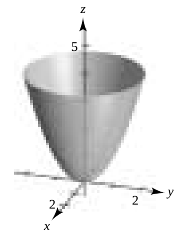


**For Exercises 9 and 11,**

$$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi.$$

**Eliminating the parameter yields**

$$z = x^2 + y^2,\ 0 \le z \le 4.$$




**9.** $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} - u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$z = -(x^2 + y^2)$

The paraboloid is reflected (inverted) through the $xy$-plane.

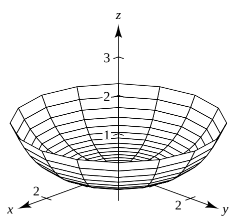
**11.** $\mathbf{s}(u, v) = u\cos v\mathbf{i} - u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 3,\ 0 \le v \le 2\pi$

The height of the paraboloid is increased from 4 to 9.

**13.** $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + u^4\mathbf{k},$

$0 \le u \le 1,\ 0 \le v \le 2\pi$

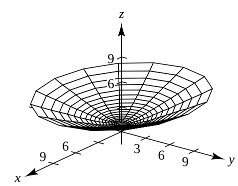
$z = \dfrac{(x^2 + y^2)^2}{16}$




**15.** $\mathbf{r}(u, v) = 2\sinh u\cos v\mathbf{i} + \sinh u\sin v\mathbf{j} + \cosh u\mathbf{k},$

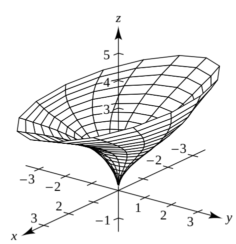
$0 \le u \le 2,\ 0 \le v \le 2\pi$

$\dfrac{z^2}{1} - \dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$




**17.** $\mathbf{r}(u, v) = (u - \sin u)\cos v\mathbf{i} + (1 - \cos u)\sin v\mathbf{j} + u\mathbf{k},$

$0 \le u \le \pi,\ 0 \le v \le 2\pi$




**19.** $z = y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$

**21.** $x^2 + y^2 = 16$

$\mathbf{r}(u, v) = 4\cos u\mathbf{i} + 4\sin u\mathbf{j} + v\mathbf{k}$

**23.** $z = x^2$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + u^2\mathbf{k}$

**25.** $z = 4$ inside $x^2 + y^2 = 9.$

$\mathbf{r}(u, v) = v\cos u\mathbf{i} + v\sin u\mathbf{j} + 4\mathbf{k},\ 0 \le v \le 3$

**27.** Function: $y = \dfrac{x}{2},\ 0 \le x \le 6$

Axis of revolution: $x$-axis

$x = u,\ y = \dfrac{u}{2}\cos v,\ z = \dfrac{u}{2}\sin v$

$0 \le u \le 6,\ 0 \le v \le 2\pi$

**29.** Function: $x = \sin z,\ 0 \le z \le \pi$

Axis of revolution: $z$-axis

$x = \sin u\cos v,\ y = \sin u\sin v,\ z = u$

$0 \le u \le \pi,\ 0 \le v \le 2\pi$

**31.** $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k},\ (1, -1, 1)$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j},\ \mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

At $(1, -1, 1)$, $u = 0$ and $v = 1$.

$\mathbf{r}_u(0, 1) = \mathbf{i} + \mathbf{j},\ \mathbf{r}_v(0, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\mathbf{N} = \mathbf{r}_u(0, 1) \times \mathbf{r}_v(0, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

Tangent plane: $(x - 1) - (y + 1) - 2(z - 1) = 0$

$$x - y - 2z = 0$$

(The original plane!)

**33.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 3u \sin v\mathbf{j} + u^2\mathbf{k},\ (0, 6, 4)$

$\mathbf{r}_u(u, v) = 2 \cos v\mathbf{i} + 3 \sin v\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -2u \sin v\mathbf{i} + 3u \cos v\mathbf{j}$

At $(0, 6, 4)$, $u = 2$ and $v = \pi/2$.

$$\mathbf{r}_u\left(2, \frac{\pi}{2}\right) = 3\mathbf{j} + 4\mathbf{k},\ \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = -4\mathbf{i}$$

$$\mathbf{N} = \mathbf{r}_u\left(2, \frac{\pi}{2}\right) \times \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -4 & 0 & 0 \end{vmatrix} = -16\mathbf{j} + 12\mathbf{k}$$

Direction numbers: $0, 4, -3$

Tangent plane: $4(y - 6) - 3(z - 4) = 0$

$$4y - 3z = 12$$

**35.** $\mathbf{r}(u, v) = 2u\mathbf{i} - \dfrac{v}{2}\mathbf{j} + \dfrac{v}{2}\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 1$

$\mathbf{r}_u(u, v) = 2\mathbf{i},\ \mathbf{r}_v(u, v) = -\dfrac{1}{2}\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\mathbf{j} - \mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2}$

$$A = \int_0^1 \int_0^2 \sqrt{2}\, du\, dv = 2\sqrt{2}$$

**37.** $\mathbf{r}(u, v) = a \cos u\mathbf{i} + a \sin u\mathbf{j} + v\mathbf{k},\ 0 \le u \le 2\pi,\ 0 \le v \le b$

$\mathbf{r}_u(u, v) = -a \sin u\mathbf{i} + a \cos u\mathbf{j}$

$\mathbf{r}_v(u, v) = \mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a \cos u\mathbf{i} + a \sin u\mathbf{j}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a$

$$A = \int_0^b \int_0^{2\pi} a\, du\, dv = 2\pi ab$$

**39.** $\mathbf{r}(u, v) = au \cos v\mathbf{i} + au \sin v\mathbf{j} + u\mathbf{k},\ 0 \le u \le b,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a \cos v\mathbf{i} + a \sin v\mathbf{j} + \mathbf{k}$

$\mathbf{r}_v(u, v) = -au \sin v\mathbf{i} + au \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos v & a \sin v & 1 \\ -au \sin v & au \cos v & 0 \end{vmatrix} = -au \cos v\mathbf{i} - au \sin v\mathbf{j} + a^2u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = au\sqrt{1 + a^2}$

$$A = \int_0^{2\pi} \int_0^b a\sqrt{1 + a^2}\,u\, du\, dv = \pi ab^2\sqrt{1 + a^2}$$

**41.** $\mathbf{r}(u, v) = \sqrt{u}\cos v\,\mathbf{i} + \sqrt{u}\sin v\,\mathbf{j} + u\mathbf{k},\ 0 \le u \le 4,\ 0 \le v \le 2\pi$

$$\mathbf{r}_u(u, v) = \frac{\cos v}{2\sqrt{u}}\mathbf{i} + \frac{\sin v}{2\sqrt{u}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -\sqrt{u}\sin v\,\mathbf{i} + \sqrt{u}\cos v\,\mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\cos v}{2\sqrt{u}} & \dfrac{\sin v}{2\sqrt{u}} & 1 \\ -\sqrt{u}\sin v & \sqrt{u}\cos v & 0 \end{vmatrix} = -\sqrt{u}\cos v\,\mathbf{i} - \sqrt{u}\sin v\,\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u + \frac{1}{4}}$$

$$A = \int_0^{2\pi}\int_0^4 \sqrt{u + \frac{1}{4}}\,du\,dv = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 36.177$$

**43.** See the definition, page 1051.

**45.** (a) From $(-10, 10, 0)$

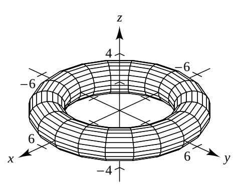
(b) From $(10, 10, 10)$

(c) From $(0, 10, 0)$

(d) From $(10, 0, 0)$

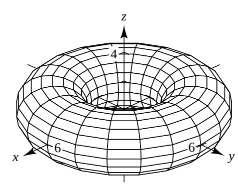
**47.** (a) $\mathbf{r}(u, v) = (4 + \cos v)\cos u\,\mathbf{i} + (4 + \cos v)\sin u\,\mathbf{j} + \sin v\,\mathbf{k},$

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$



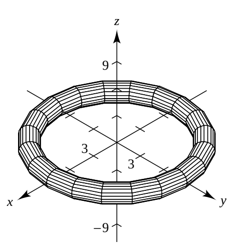

(b) $\mathbf{r}(u, v) = (4 + 2\cos v)\cos u\,\mathbf{i} + (4 + 2\cos v)\sin u\,\mathbf{j} + 2\sin v\,\mathbf{k},$

$0 \le u \le 2\pi, 0 \le v \le 2\pi$



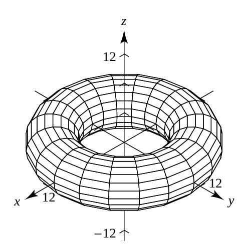

(c) $\mathbf{r}(u, v) = (8 + \cos v)\cos u\,\mathbf{i} + (8 + \cos v)\sin u\,\mathbf{j} + \sin v\,\mathbf{k},$

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$




(d) $\mathbf{r}(u, v) = (8 + 3\cos v)\cos u\,\mathbf{i} + (8 + 3\cos v)\sin u\,\mathbf{j} + 3\sin v\,\mathbf{k},$

$0 \le u \le 2\pi, 0 \le v \le 2\pi$




The radius of the generating circle that is revolved about the $z$-axis is $b$, and its center is $a$ units from the axis of revolution.

**49.** $\mathbf{r}(u, v) = 20 \sin u \cos v\mathbf{i} + 20 \sin u \sin v\mathbf{j} + 20 \cos u\mathbf{k} \;\; 0 \le u \le \pi/3,\;\; 0 \le v \le 2\pi$

$\mathbf{r}_u = 20 \cos u \cos v\mathbf{i} + 20 \cos u \sin v\mathbf{j} - 20 \sin u\mathbf{k}$

$\mathbf{r}_v = -20 \sin u \sin v\mathbf{i} + 20 \sin u \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 \cos u \cos v & 20 \cos u \sin v & -20 \sin u \\ -20 \sin u \sin v & 20 \sin u \cos v & 0 \end{vmatrix}$$

$$= 400 \sin^2 u \cos v\mathbf{i} + 400 \sin^2 u \sin v\mathbf{j} + 400(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v)\mathbf{k}$$

$$= 400[\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \cos u \sin u\mathbf{k}]$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = 400\sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u}$$

$$= 400\sqrt{\sin^4 u + \cos^2 u \sin^2 u}$$

$$= 400\sqrt{\sin^2 u} = 400 \sin u$$

$$S = \iint_S dS = \int_0^{2\pi}\int_0^{\pi/3} 400 \sin u\, du\, dv = \int_0^{2\pi} \Big[-400 \cos u\Big]_0^{\pi/3} dv$$

$$= \int_0^{2\pi} 200\, dv = 400\pi \text{ m}^2$$

**51.** $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + 2v\mathbf{k},\; 0 \le u \le 3,\; 0 \le v \le 2\pi$

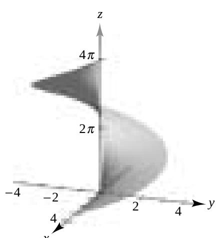
$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j}$

$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v\mathbf{i} - 2 \cos v\mathbf{j} + u\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 + u^2}$$

$$A = \int_0^{2\pi}\int_0^3 \sqrt{4 + u^2}\, du\, dv = \pi\left[3\sqrt{13} + 4 \ln\left(\frac{3 + \sqrt{13}}{2}\right)\right]$$




**53.** Essay

## Section 14.6 Surface Integrals

**1.** $S$: $z = 4 - x,\; 0 \le x \le 4,\; 0 \le y \le 4,\; \dfrac{\partial z}{\partial x} = -1,\; \dfrac{\partial z}{\partial y} = 0$

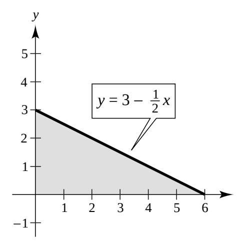
$$\iint_S (x - 2y + z)\, dS = \int_0^4\int_0^4 (x - 2y + 4 - x)\sqrt{1 + (-1)^2 + (0)^2}\, dy\, dx$$

$$= \sqrt{2}\int_0^4\int_0^4 (4 - 2y)\, dy\, dx = 0$$

**3.** $S$: $z = 10$, $x^2 + y^2 \le 1$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\iint_S (x - 2y + z)\, dS = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x - 2y + 10)\sqrt{1 + (0)^2 + (0)^2}\, dy\, dx$$
$$= \int_0^{2\pi}\int_0^1 (r\cos\theta - 2r\sin\theta + 10)r\, dr\, d\theta$$
$$= \int_0^{2\pi}\left(\frac{1}{3}\cos\theta - \frac{2}{3}\sin\theta + 5\right) d\theta$$
$$= \left[\frac{1}{3}\sin\theta + \frac{2}{3}\cos\theta + 5\theta\right]_0^{2\pi} = 10\pi$$

**5.** $S$: $z = 6 - x - 2y$, (first octant) $\dfrac{\partial z}{\partial x} = -1$, $\dfrac{\partial z}{\partial y} = -2$




$$\iint_S xy\, dS = \int_0^6\int_0^{3-(x/2)} xy\sqrt{1 + (-1)^2 + (-2)^2}\, dy\, dx$$
$$= \sqrt{6}\int_0^6 \left[\frac{xy^2}{2}\right]_0^{3-(x/2)} dx$$
$$= \frac{\sqrt{6}}{2}\int_0^6 x\left(9 - 3x + \frac{1}{4}x^2\right) dx$$
$$= \frac{\sqrt{6}}{2}\left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16}\right]_0^6 = \frac{27\sqrt{6}}{2}$$

**7.** $S$: $z = 9 - x^2$, $0 \le x \le 2$, $0 \le y \le x$,

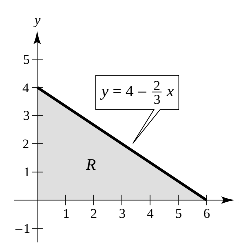
$$\frac{\partial z}{\partial x} = -2x,\ \frac{\partial z}{\partial y} = 0$$
$$\iint_S xy\, dS = \int_0^2\int_y^2 xy\sqrt{1 + 4x^2}\, dx\, dy = \frac{391\sqrt{17} + 1}{240}$$

**9.** $S$: $z = 10 - x^2 - y^2$, $0 \le x \le 2$, $0 \le y \le 2$

$$\iint_S (x^2 - 2xy)\, dS = \int_0^2\int_0^2 (x^2 - 2xy)\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx \approx -11.47$$

**11.** $S$: $2x + 3y + 6z = 12$ (first octant) $\implies z = 2 - \dfrac{1}{3}x - \dfrac{1}{2}y$

$\rho(x, y, z) = x^2 + y^2$




$$m = \iint_R (x^2 + y^2)\sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{2}\right)^2}\, dA$$
$$= \frac{7}{6}\int_0^6\int_0^{4-(2x/3)} (x^2 + y^2)\, dy\, dx$$
$$= \frac{7}{6}\int_0^6 \left[x^2\left(4 - \frac{2}{3}x\right) + \frac{1}{3}\left(4 - \frac{2}{3}x\right)^3\right] dx = \frac{7}{6}\left[\frac{4}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{8}\left(4 - \frac{2}{3}x\right)^4\right]_0^6 = \frac{364}{3}$$

**13.** $S$: $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \dfrac{v}{2}\mathbf{k},\ 0 \le u \le 1,\ 0 \le v \le 2$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \left\| -\frac{1}{2}\mathbf{j} + \mathbf{k} \right\| = \frac{\sqrt{5}}{2}$$

$$\iint_S (y + 5)\, dS = \int_0^2 \int_0^1 (v + 5)\frac{\sqrt{5}}{2}\, du\, dv = 6\sqrt{5}$$

**15.** $S$: $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k},\ 0 \le u \le \dfrac{\pi}{2},\ 0 \le v \le 2$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2\cos u\mathbf{i} + 2\sin u\mathbf{j}\| = 2$

$$\iint_S xy\, dS = \int_0^2 \int_0^{\pi/2} 8\cos u \sin u\, du\, dv = 8$$

**17.** $f(x, y, z) = x^2 + y^2 + z^2$

$S$: $z = x + 2,\ x^2 + y^2 \le 1$

$$\begin{aligned}
\iint_S f(x, y, z)\, dS &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [x^2 + y^2 + (x + 2)^2]\sqrt{1 + (1)^2 + (0)^2}\, dy\, dx \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 [r^2 + (r\cos\theta + 2)^2] r\, dr\, d\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 [r^2 + r^2\cos^2\theta + 4r\cos\theta + 4] r\, dr\, d\theta \\
&= \sqrt{2}\int_0^{2\pi}\left[\frac{r^4}{4} + \frac{r^4}{4}\cos^2\theta + \frac{4r^3}{3}\cos\theta + 2r^2\right]_0^1 d\theta \\
&= \sqrt{2}\int_0^{2\pi}\left[\frac{9}{4} + \left(\frac{1}{4}\right)\frac{1 + \cos 2\theta}{2} + \frac{4}{3}\cos\theta\right] d\theta \\
&= \sqrt{2}\left[\frac{9}{4}\theta + \frac{1}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + \frac{4}{3}\sin\theta\right]_0^{2\pi} = \sqrt{2}\left[\frac{18\pi}{4} + \frac{\pi}{4}\right] = \frac{19\sqrt{2}\pi}{4}
\end{aligned}$$

**19.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S$: $z = \sqrt{x^2 + y^2},\ x^2 + y^2 \le 4$

$$\begin{aligned}
\iint_S f(x, y, z)\, dS &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2}\sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx \\
&= \sqrt{2}\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\sqrt{\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2}}\, dy\, dx \\
&= 2\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx \\
&= 2\int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta = 2\int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^2 d\theta = \left[\frac{16}{3}\theta\right]_0^{2\pi} = \frac{32\pi}{3}
\end{aligned}$$

**21.** $f(x, y, z) = x^2 + y^2 + z^2$

$S$: $x^2 + y^2 = 9$, $0 \le x \le 3$, $0 \le y \le 3$, $0 \le z \le 9$

Project the solid onto the $yz$-plane; $x = \sqrt{9 - y^2}$, $0 \le y \le 3$, $0 \le z \le 9$.

$$\iint_S f(x, y, z)\,dS = \int_0^3\int_0^9 [(9 - y^2) + y^2 + z^2]\sqrt{1 + \left(\frac{y}{\sqrt{9 - y^2}}\right)^2 + (0)^2}\,dz\,dy$$

$$= \int_0^3\int_0^9 (9 + z^2)\frac{3}{\sqrt{9 - y^2}}\,dz\,dy = \int_0^3\left[\frac{3}{\sqrt{9 - y^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^9 dy$$

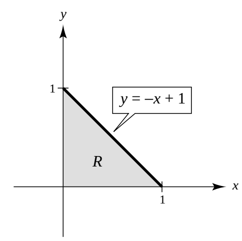
$$= 324\int_0^3 \frac{3}{\sqrt{9 - y^2}}\,dy = \left[972 \arcsin\left(\frac{y}{3}\right)\right]_0^3 = 972\left(\frac{\pi}{2} - 0\right) = 486\pi$$

**23.** $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

$S$: $x + y + z = 1$ (first octant)

$G(x, y, z) = x + y + z - 1$

$\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$




$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_R \mathbf{F}\cdot\nabla G\,dA = \int_0^1\int_0^{1-x} (3z - 4 + y)\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} [3(1 - x - y) - 4 + y]\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} (-1 - 3x - 2y)\,dy\,dx$$

$$= \int_0^1 \Big[-y - 3xy - y^2\Big]_0^{1-x} dx$$

$$= -\int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2]\,dx$$

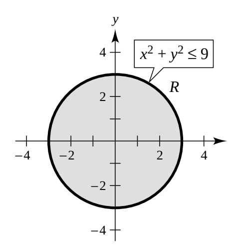
$$= -\int_0^1 (2 - 2x^2)\,dx = -\frac{4}{3}$$

**25.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S$: $z = 9 - x^2 - y^2$, $0 \le z$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$




$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_R \mathbf{F}\cdot\nabla G\,dA = \iint_R (2x^2 + 2y^2 + z)\,dA$$

$$= \iint_R [2x^2 + 2y^2 + (9 - x^2 - y^2)]\,dA$$

$$= \iint_R (x^2 + y^2 + 9)\,dA$$

$$= \int_0^{2\pi}\int_0^3 (r^2 + 9)r\,dr\,d\theta$$

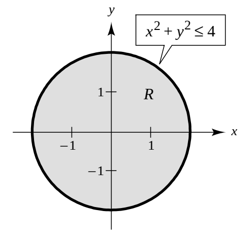
$$= \int_0^{2\pi}\left[\frac{r^4}{4} + \frac{9r^2}{2}\right]_0^3 d\theta = \frac{243\pi}{2}$$

**27.** $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$S$: $z = x^2 + y^2,\ x^2 + y^2 \le 4$

$G(x, y, z) = -x^2 - y^2 + z$

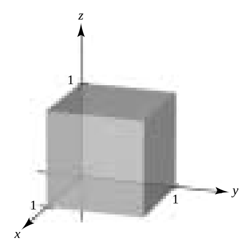
$\nabla G(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$




$$\begin{aligned}\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\!\!\int \mathbf{F} \cdot \nabla G\, dA &= \int_R\!\!\int (-8x + 6y + 5)\, dA \\ &= \int_0^{2\pi}\!\!\int_0^2 [-8r\cos\theta + 6r\sin\theta + 5]r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[-\frac{8}{3}r^3\cos\theta + 2r^3\sin\theta + \frac{5}{2}r^2\right]_0^2 d\theta \\ &= \int_0^{2\pi} \left[-\frac{64}{3}\cos\theta + 16\sin\theta + 10\right] d\theta \\ &= \left[-\frac{64}{3}\sin\theta - 16\cos\theta + 10\theta\right]_0^{2\pi} = 20\pi\end{aligned}$$

**29.** $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$

$S$: unit cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$



$S_1$: The top of the cube

$\mathbf{N} = \mathbf{k},\ z = 1$

$$\int_{S_1}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 y(1)\, dy\, dx = \frac{1}{2}$$

$S_2$: The bottom of the cube

$\mathbf{N} = -\mathbf{k},\ z = 0$

$$\int_{S_2}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 -y(0)\, dy\, dx = 0$$

$S_3$: The front of the cube

$\mathbf{N} = \mathbf{i},\ x = 1$

$$\int_{S_3}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 4(1)y\, dy\, dz = 2$$

$S_4$: The back of the cube

$\mathbf{N} = -\mathbf{i},\ x = 0$

$$\int_{S_4}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 -4(0)y\, dy\, dx = 0$$

$S_5$: The right side of the cube

$\mathbf{N} = \mathbf{j},\ y = 1$

$$\int_{S_5}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 z^2\, dz\, dx = \frac{1}{3}$$

$S_6$: The left side of the cube

$\mathbf{N} = -\mathbf{j},\ y = 0$

$$\int_{S_6}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 -z^2\, dz\, dx = -\frac{1}{3}$$

Therefore,

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \frac{1}{2} + 0 + 2 + 0 + \frac{1}{3} - \frac{1}{3} = \frac{5}{2}.$$

**31.** The surface integral of $f$ over a surface $S$, where $S$ is given by $z = g(x, y)$, is defined as

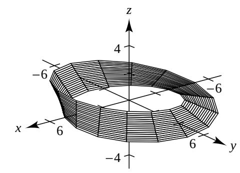
$$\int_S\!\!\int f(x, y, z)\, dS = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i. \text{ (page 1061)}$$

See Theorem 14.10, page 1061.

**33.** See the definition, page 1067.

See Theorem 14.11, page 1067.

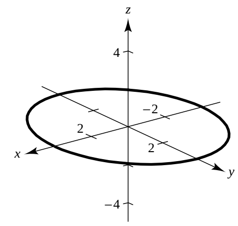
**35.** (a)




(b) If a normal vector at a point $P$ on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c) $\mathbf{r}(u, 0) = 4\cos(2u)\mathbf{i} + 4\sin(2u)\mathbf{j}$

This is a circle.




(d) (construction)

(e) You obtain a strip with a double twist and twice as long as the original Möbius strip.

**37.** $z = \sqrt{x^2 + y^2},\ 0 \le z \le a$

$$m = \iint_S k\, dS = k\iint_R \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dA = k\iint_R \sqrt{2}\, dA = \sqrt{2}\, k\pi a^2$$

$$I_z = \iint_S k(x^2 + y^2)\, dS = \iint_R k(x^2 + y^2)\sqrt{2}\, dA$$

$$= \sqrt{2}k\int_0^{2\pi}\int_0^a r^3\, dr\, d\theta = \frac{\sqrt{2}ka^4}{4}(2\pi)$$

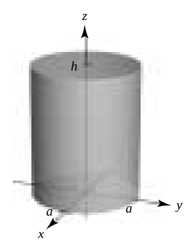
$$= \frac{\sqrt{2}k\pi a^4}{2} = \frac{a^2}{2}\left(\sqrt{2}k\pi a^2\right) = \frac{a^2 m}{2}$$

**39.** $x^2 + y^2 = a^2,\ 0 \le z \le h$

$\rho(x, y, z) = 1$

$y = \pm\sqrt{a^2 - x^2}$

Project the solid onto the $xz$-plane.




$$I_z = 4\iint_S (x^2 + y^2)(1)\, dS$$

$$= 4\int_0^h\int_0^a \left[x^2 + (a^2 - x^2)\right]\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2 + (0)^2}\, dx\, dz$$

$$= 4a^3\int_0^h\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\, dx\, dz$$

$$= 4a^3\int_0^h \left[\arcsin\frac{x}{a}\right]_0^a dz = 4a^3\left(\frac{\pi}{2}\right)(h) = 2\pi a^3 h$$

**41.** $S\colon\ z = 16 - x^2 - y^2,\ z \ge 0$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

$$\iint_S \rho\mathbf{F}\cdot\mathbf{N}\, dS = \iint_R \rho\mathbf{F}\cdot\left(-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}\right) dA = \iint_R 0.5\rho z\mathbf{k}\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\, dA$$

$$= \iint_R 0.5\rho z\, dA = \iint_R 0.5\rho(16 - x^2 - y^2)\, dA$$

$$= 0.5\rho\int_0^{2\pi}\int_0^4 (16 - r^2)r\, dr\, d\theta = 0.5\rho\int_0^{2\pi} 64\, d\theta = 64\pi\rho$$

## Section 14.7 Divergence Theorem

1. **Surface Integral:** There are six surfaces to the cube, each with $dS = \sqrt{1}\,dA$.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F}\cdot\mathbf{N} = -z^2, \quad \int_{S_1}\!\!\int 0\,dA = 0$$

$$z = a, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F}\cdot\mathbf{N} = z^2, \quad \int_{S_2}\!\!\int a^2\,dA = \int_0^a\!\!\int_0^a a^2\,dx\,dy = a^4$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F}\cdot\mathbf{N} = -2x, \quad \int_{S_3}\!\!\int 0\,dA = 0$$

$$x = a, \quad \mathbf{N} = \mathbf{i}, \quad \mathbf{F}\cdot\mathbf{N} = 2x, \quad \int_{S_4}\!\!\int 2a\,dy\,dz = \int_0^a\!\!\int_0^a 2a\,dy\,dz = 2a^3$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F}\cdot\mathbf{N} = 2y, \quad \int_{S_5}\!\!\int 0\,dA = 0$$

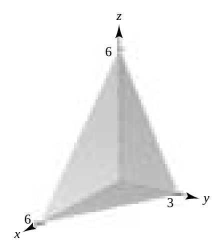
$$y = a, \quad \mathbf{N} = \mathbf{j}, \quad \mathbf{F}\cdot\mathbf{N} = -2y, \quad \int_{S_6}\!\!\int -2a\,dA = \int_0^a\!\!\int_0^a -2a\,dz\,dx = -2a^3$$

Therefore, $\displaystyle\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = a^4 + 2a^3 - 2a^3 = a^4.$

**Divergence Theorem:** Since div $\mathbf{F} = 2z$, the Divergence Theorem yields

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \int_0^a\!\!\int_0^a\!\!\int_0^a 2z\,dz\,dy\,dx = \int_0^a\!\!\int_0^a a^2\,dy\,dx = a^4.$$

3. **Surface Integral:** There are four surfaces to this solid.




$z = 0,\ \mathbf{N} = -\mathbf{k},\ \mathbf{F}\cdot\mathbf{N} = -z$

$$\int_{S_1}\!\!\int 0\,dS = 0$$

$y = 0,\ \mathbf{N} = -\mathbf{j},\ \mathbf{F}\cdot\mathbf{N} = 2y - z,\ dS = dA = dx\,dz$

$$\int_{S_2}\!\!\int -z\,dS = \int_0^6\!\!\int_0^{6-z} -z\,dx\,dz = \int_0^6 (z^2 - 6z)\,dz = -36$$

$x = 0,\ \mathbf{N} = -\mathbf{i},\ \mathbf{F}\cdot\mathbf{N} = y - 2x,\ dS = dA = dz\,dy$

$$\int_{S_3}\!\!\int y\,dS = \int_0^3\!\!\int_0^{6-2y} y\,dz\,dy = \int_0^3 (6y - 2y^2)\,dy = 9$$

$x + 2y + z = 6,\ \mathbf{N} = \dfrac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}},\ \mathbf{F}\cdot\mathbf{N} = \dfrac{2x - 5y + 3z}{\sqrt{6}},\ dS = \sqrt{6}\,dA$

$$\int_{S_4}\!\!\int (2x - 5y + 3z)\,dz\,dy = \int_0^3\!\!\int_0^{6-2y} (18 - x - 11y)\,dx\,dy = \int_0^3 (90 - 90y + 20y^2)\,dy = 45$$

Therefore, $\displaystyle\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = 0 - 36 + 9 + 45 = 18.$

**Divergence Theorem:** Since div $\mathbf{F} = 1$, we have

$$\iiint_Q dV = (\text{Volume of solid}) = \frac{1}{3}(\text{Area of base}) \times (\text{Height}) = \frac{1}{3}(9)(6) = 18.$$

**5.** Since div $\mathbf{F} = 2x + 2y + 2z$, we have

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \int_0^a\int_0^a\int_0^a (2x + 2y + 2z)\,dz\,dy\,dx$$

$$= \int_0^a\int_0^a (2ax + 2ay + a^2)\,dy\,dx = \int_0^a (2a^2x + 2a^3)\,dx = \Big[a^2x^2 + 2a^3x\Big]_0^a = 3a^4.$$

**7.** Since div $\mathbf{F} = 2x - 2x + 2xyz = 2xyz$

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q 2xyz\,dV = \int_0^a\int_0^{2\pi}\int_0^{\pi/2} 2(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$

$$= \int_0^a\int_0^{2\pi}\int_0^{\pi/2} 2\rho^5(\sin\theta\cos\theta)(\sin^3\phi\cos\phi)\,d\phi\,d\theta\,d\rho$$

$$= \int_0^a\int_0^{2\pi} \frac{1}{2}\rho^5\sin\theta\cos\theta\,d\theta\,d\rho = \int_0^a\left[\left(\frac{\rho^5}{2}\right)\frac{\sin^2\theta}{2}\right]_0^{2\pi} d\rho = 0.$$

**9.** Since div $\mathbf{F} = 3$, we have

$$\iiint_Q 3\,dV = 3(\text{Volume of sphere}) = 3\left[\frac{4}{3}\pi(2)^3\right] = 32\pi.$$

**11.** Since div $\mathbf{F} = 1 + 2y - 1 = 2y$, we have

$$\iiint_Q 2y\,dV = \int_0^4\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 2y\,dx\,dy\,dz = \int_0^4\int_{-3}^3 4y\sqrt{9-y^2}\,dy\,dz = \int_0^4\left[-\frac{4}{3}(9-y^2)^{3/2}\right]_{-3}^3 dz = 0.$$

**13.** Since div $\mathbf{F} = 3x^2 + x^2 + 0 = 4x^2$, we have

$$\iiint_Q 4x^2\,dV = \int_0^6\int_0^4\int_0^{4-y} 4x^2\,dz\,dy\,dx = \int_0^6\int_0^4 4x^2(4-y)\,dy\,dx = \int_0^6 32x^2\,dx = 2304.$$

**15.** $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4y\mathbf{j} + xz\mathbf{k}$

div $\mathbf{F} = y + 4 + x$

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q (y + x + 4)\,dV$$

$$= \int_0^3\int_0^\pi\int_0^{2\pi} (\rho\sin\phi\sin\theta + \rho\sin\phi\cos\theta + 4)\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$$

$$= \int_0^3\int_0^\pi\int_0^{2\pi} [\rho^3\sin^2\phi\sin\theta + \rho^3\sin^2\phi\cos\theta + 4\rho^2\sin\phi]\,d\theta\,d\phi\,d\rho$$

$$= \int_0^3\int_0^\pi\Big[-\rho^3\sin^2\phi\cos\theta + \rho^3\sin^2\phi\sin\theta + 4\rho^2\sin\phi\cdot\theta\Big]_0^{2\pi} d\phi\,d\rho$$

$$= \int_0^3\int_0^\pi 8\pi\rho^2\sin\phi\,d\phi\,d\rho$$

$$= \int_0^3\Big[-8\pi\rho^2\cos\phi\Big]_0^\pi d\rho$$

$$= \int_0^3 16\pi\rho^2\,d\rho = \left[\frac{16\pi\rho^3}{3}\right]_0^3 = 144\pi.$$

**17.** Using the Divergence Theorem, we have

$$\int_S\int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}\,(\mathbf{curl F})\, dV$$

$$\mathbf{curl\ F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy + z^2 & 2x^2 + 6yz & 2xz \end{vmatrix} = -6y\mathbf{i} - (2z - 2z)\mathbf{j} + (4x - 4x)\mathbf{k} = -6y\mathbf{i}$$

$$\operatorname{div}\,(\mathbf{curl\ F}) = 0.$$

Therefore, $\displaystyle\iiint_Q \operatorname{div}\,(\mathbf{curl\ F})\, dV = 0.$

**19.** See Theorem 14.12, page 1073.

**21.** Using the triple integral to find volume, we need $\mathbf{F}$ so that

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 1.$$

Hence, we could have $\mathbf{F} = x\mathbf{i}$, $\mathbf{F} = y\mathbf{j}$, or $\mathbf{F} = z\mathbf{k}$.

For $dA = dy\, dz$ consider $\mathbf{F} = x\mathbf{i}$, $x = f(y, z)$, then $\mathbf{N} = \dfrac{\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_y^2 + f_z^2}}$ and $dS = \sqrt{1 + f_y^2 + f_z^2}\, dy\, dz.$

For $dA = dz\, dx$ consider $\mathbf{F} = y\mathbf{j}$, $y = f(x, z)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + \mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_x^2 + f_z^2}}$ and $dS = \sqrt{1 + f_x^2 + f_z^2}\, dz\, dx.$

For $dA = dx\, dy$ consider $\mathbf{F} = z\mathbf{k}$, $z = f(x, y)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$ and $dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$

Correspondingly, we then have $\displaystyle V = \int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_S\int x\, dy\, dz = \int_S\int y\, dz\, dx = \int_S\int z\, dx\, dy.$

**23.** Using the Divergence Theorem, we have $\displaystyle\int_S\int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div}\,(\mathbf{curl\ F})\, dV.$ Let

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

$$\mathbf{curl\ F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\operatorname{div}\,(\mathbf{curl\ F}) = \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0.$$

Therefore, $\displaystyle\int_S\int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q 0\, dV = 0.$

**25.** If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div} \mathbf{F} = 3$.

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div} \mathbf{F}\, dV = \iiint_Q 3\, dV = 3V.$$

**27.** $\displaystyle\int_S\int f D_{\mathbf{N}} g\, dS = \int_S\int f\nabla g \cdot \mathbf{N}\, dS$

$$= \iiint_Q \operatorname{div}\,(f\nabla g)\, dV = \iiint_Q (f \operatorname{div} \nabla g + \nabla f \cdot \nabla g)\, dV = \iiint_Q (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$$

## Section 14.8 Stokes's Theorem

**1.** $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + xyz\mathbf{j} + e^z\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y - z & xyz & e^z \end{vmatrix} = -xy\mathbf{i} - \mathbf{j} + (yz - 2)\mathbf{k}$$

**3.** $\mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z & -4x^2 & \arctan x \end{vmatrix} = \left(2 - \frac{1}{1 + x^2}\right)\mathbf{j} - 8x\mathbf{k}$$

**5.** $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$

$$\begin{aligned}\textbf{curl F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{x^2+y^2} & e^{y^2+z^2} & xyz \end{vmatrix} \\ &= (xz - 2ze^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k} \\ &= z(x - 2e^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}\end{aligned}$$

**7.** In this case, $M = -y + z,\ N = x - z,\ P = x - y$ and $C$ is the circle $x^2 + y^2 = 1,\ z = 0,\ dz = 0.$

**Line Integral:** $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y\,dx + x\,dy$

Letting $x = \cos t,\ y = \sin t,$ we have $dx = -\sin t\,dt,\ dy = \cos t\,dt$ and

$$\int_C -y\,dx + x\,dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t)dt = 2\pi.$$

**Double Integral:** Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1.$

Then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

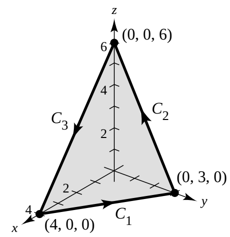
Since

$$z^2 = 1 - x^2 - y^2,\ z_x = \frac{-2x}{2z} = \frac{-x}{z}, \text{ and } z_y = \frac{-y}{z},\ dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}dA = \frac{1}{z}dA.$$

Now, since $\textbf{curl F} = 2\mathbf{k}$, we have

$$\int_S\int (\textbf{curl F}) \cdot \mathbf{N}\,dS = \int_R\int 2z\left(\frac{1}{z}\right) dA = \int_R\int 2\,dA = 2(\text{Area of circle of radius } 1) = 2\pi.$$

**9.** **Line Integral:** From the accompanying figure we see that for




$C_1$: $z = 0$, $dz = 0$

$C_2$: $x = 0$, $dx = 0$

$C_3$: $y = 0$, $dy = 0$.

Hence, $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C xyz\,dx + y\,dy + z\,dz$$

$$= \int_{C_1} y\,dy + \int_{C_2} y\,dy + z\,dz + \int_{C_3} z\,dz$$

$$= \int_0^3 y\,dy + \int_3^0 y\,dy + \int_0^6 z\,dz + \int_6^0 z\,dz = 0.$$

**Double Integral:** $\textbf{curl F} = xy\mathbf{j} - xz\mathbf{k}$

Considering $F(x, y, z) = 3x + 4y + 2z - 12$, then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{29}} \text{ and } dS = \sqrt{29}\,dA.$$

Thus,

$$\int_S\int (\textbf{curl F})\cdot\mathbf{N}\,dS = \int_R\int (4xy - 2xz)\,dy\,dx$$

$$= \int_0^4\int_0^{(-3x+12)/4}\left[4xy - 2x\left(6 - 2y - \frac{3}{2}x\right)\right]dy\,dx$$

$$= \int_0^4\int_0^{(12-3x)/4}(8xy + 3x^2 - 12x)\,dy\,dx$$

$$= \int_0^4 0\,dx = 0.$$

**11.** Let $A = (0, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{j}$. Thus,

$$\mathbf{N} = \frac{\mathbf{U}\times\mathbf{V}}{\|\mathbf{U}\times\mathbf{V}\|} = \frac{-2\mathbf{i} + 2\mathbf{k}}{2\sqrt{2}} = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}.$$

Surface $S$ has direction numbers $-1, 0, 1$, with equation $z - x = 0$ and $dS = \sqrt{2}\,dA$. Since $\textbf{curl F} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, we have

$$\int_S\int (\textbf{curl F})\cdot\mathbf{N}\,dS = \int_R\int \frac{1}{\sqrt{2}}\left(\sqrt{2}\right)dA = \int_R\int dA = (\text{Area of triangle with } a = 1, b = 2) = 1.$$

**13.** $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, $S$: $z = 4 - x^2 - y^2$, $0 \le z$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$$

$$G(x, y, z) = x^2 + y^2 + z - 4$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\int_S\int (\textbf{curl F})\cdot\mathbf{N}\,dS = \int_R\int (4xy + 4yz + 2x)\,dA = \int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4xy + 4y(4 - x^2 - y^2) + 2x\right]dy\,dx$$

$$= \int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4xy + 16y - 4x^2y - 4y^3 + 2x\right]dy\,dx$$

$$= \int_{-2}^2 4x\sqrt{4 - x^2}\,dx = 0$$

**15.** $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y\mathbf{j} + xz\mathbf{k},\ S\colon z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y & xz \end{vmatrix} = z\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{F}\, dS = \int_R\int \frac{yz}{\sqrt{4 - x^2 - y^2}}\, dA = \int_R\int \frac{y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}}\, dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy\, dx = 0$$

**17.** $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\,\mathbf{i} + \arctan\dfrac{x}{y}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -1/2 \ln(x^2 + y^2) & \arctan x/y & 1 \end{vmatrix} = \left[\frac{(1/y)}{1 + (x^2/y^2)} + \frac{y}{x^2 + y^2}\right]\mathbf{k} = \left[\frac{2y}{x^2 + y^2}\right]\mathbf{k}$$

$S\colon z = 9 - 2x - 3y$ over one petal of $r = 2 \sin 2\theta$ in the first octant.

$$G(x, y, z) = 2x + 3y + z - 9$$

$$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\int \frac{2y}{x^2 + y^2}\, dA \\ &= \int_0^{\pi/2}\int_0^{2 \sin 2\theta} \frac{2r \sin\theta}{r^2} r\, dr\, d\theta \\ &= \int_0^{\pi/2}\int_0^{4 \sin\theta \cos\theta} 2 \sin\theta\, dr\, d\theta \\ &= \int_0^{\pi/2} 8 \sin^2\theta \cos\theta\, d\theta = \left[\frac{8 \sin^3\theta}{3}\right]_0^{\pi/2} = \frac{8}{3} \end{aligned}$$

**19.** From Exercise 10, we have $\mathbf{N} = \dfrac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$ and $dS = \sqrt{1 + 4x^2}\, dA$. Since $\mathbf{curl\ F} = xy\mathbf{j} - xz\mathbf{k}$, we have

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int xz\, dA = \int_0^a\int_0^a x^3\, dy\, dx = \int_0^a ax^3\, dx = \left[\frac{ax^4}{4}\right]_0^a = \frac{a^5}{4}.$$

**21.** $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

Letting $\mathbf{N} = \mathbf{k}$, we have $\displaystyle\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = 0.$

**23.** See Theorem 14.13, page 1081.

**25.** (a) $\int_C f\nabla g \cdot d\mathbf{r} = \int_S\int \mathbf{curl}[f\nabla g]\cdot \mathbf{N}\, dS$ (Stoke's Theorem)

$$f\nabla g = f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}$$

$$\mathbf{curl}\,(f\nabla g) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f(\partial g/\partial x) & f(\partial g/\partial y) & f(\partial g/\partial z) \end{vmatrix}$$

$$= \left[\left[f\left(\frac{\partial^2 g}{\partial y\partial z}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z\partial y}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\right]\mathbf{i}$$

$$- \left[\left[f\left(\frac{\partial^2 g}{\partial x\partial z}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z\partial x}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{j}$$

$$+ \left[\left[f\left(\frac{\partial^2 g}{\partial x\partial y}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial y\partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{k}$$

$$= \left[\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\mathbf{i} - \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{j} + \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right) - \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \end{vmatrix} = \nabla f \times \nabla g$$

Therefore, $\int_C f\nabla g \cdot d\mathbf{r} = \int_S\int \mathbf{curl}[f\nabla g]\cdot\mathbf{N}\,dS = \int_S\int [\nabla f \times \nabla g]\cdot \mathbf{N}\,dS.$

(b) $\int_C (f\nabla f)\cdot d\mathbf{r} = \int_S\int (\nabla f \times \nabla f)\cdot\mathbf{N}\,dS$ (using part a.)

$= 0$ since $\nabla f \times \nabla f = 0.$

(c) $\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = \int_C (f\nabla g)\cdot d\mathbf{r} + \int_C (g\nabla f)\cdot d\mathbf{r}$

$= \int_S\int (\nabla f \times \nabla g)\cdot\mathbf{N}\,dS + \int_S\int (\nabla g \times \nabla f)\cdot \mathbf{N}\,dS$ (using part a.)

$= \int_S\int (\nabla f \times \nabla g)\cdot\mathbf{N}\,dS + \int_S\int -(\nabla f \times \nabla g)\cdot\mathbf{N}\,dS = 0$

**27.** Let $\mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\frac{1}{2}\int_C (\mathbf{C}\times\mathbf{r})\cdot d\mathbf{r} = \frac{1}{2}\int_S\int \mathbf{curl}\,(\mathbf{C}\times\mathbf{r})\cdot\mathbf{N}\,dS = \frac{1}{2}\int_S\int 2\mathbf{C}\cdot\mathbf{N}\,dS = \int_S\int \mathbf{C}\cdot\mathbf{N}\,dS$$

since

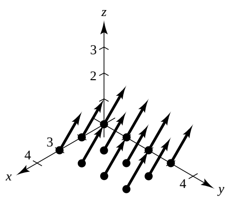
$$\mathbf{C}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$$

and

$$\mathbf{curl}(\mathbf{C}\times\mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{C}.$$

## Review Exercises for Chapter 14

**1.** $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$



**3.** $f(x, y, z) = 8x^2 + xy + z^2$

$\mathbf{F}(x, y, z) = (16x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$

**5.** Since $\partial M/\partial y = -1/y^2 \neq \partial N/\partial x$, $\mathbf{F}$ is not conservative.

**7.** Since $\partial M/\partial y = 12xy = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = 6xy^2 - 3x^2$ and $N = \partial U/\partial y = 6x^2y + 3y^2 - 7$, partial integration yields $U = 3x^2y^2 - x^3 + h(y)$ and $U = 3x^2y^2 + y^3 - 7y + g(x)$ which suggests $h(y) = y^3 - 7y$, $g(x) = -x^3$, and $U(x, y) = 3x^2y^2 - x^3 + y^3 - 7y + C$.

**9.** Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 1 \neq \frac{\partial P}{\partial x}.$$

$\mathbf{F}$ is not conservative.

**11.** Since

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2z} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{-1}{yz^2} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{x}{y^2z^2} = \frac{\partial P}{\partial y},$$

$\mathbf{F}$ is conservative. From

$$M = \frac{\partial U}{\partial x} = \frac{1}{yz}, \quad N = \frac{\partial U}{\partial y} = \frac{-x}{y^2z}, \quad P = \frac{\partial U}{\partial z} = \frac{-x}{yz^2}$$

we obtain

$$U = \frac{x}{yz} + f(y, z), \quad U = \frac{x}{yz} + g(x, z), \quad U = \frac{x}{yz} + h(x, y) \implies f(x, y, z) = \frac{x}{yz} + K$$

**13.** Since $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = 2x + 2y + 2z$

(b) $\textbf{curl F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

**15.** Since $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = -y\sin x - x\cos y + xy$

(b) $\textbf{curl F} = xz\mathbf{i} - yz\mathbf{j} + (\cos x - \sin y + \sin y - \cos x)\mathbf{k} = xz\mathbf{i} - yz\mathbf{j}$

**17.** Since $\mathbf{F} = \arcsin x\mathbf{i} + xy^2\mathbf{j} + yz^2\mathbf{k}$:

(a) $\operatorname{div}\mathbf{F} = \dfrac{1}{\sqrt{1-x^2}} + 2xy + 2yz$

(b) $\mathbf{curl\,F} = z^2\mathbf{i} + y^2\mathbf{k}$

**19.** Since $\mathbf{F} = \ln(x^2+y^2)\mathbf{i} + \ln(x^2+y^2)\mathbf{j} + z\mathbf{k}$:

(a) $\operatorname{div}\mathbf{F} = \dfrac{2x}{x^2+y^2} + \dfrac{2y}{x^2+y^2} + 1$

$$= \frac{2x+2y}{x^2+y^2} + 1$$

(b) $\mathbf{curl\,F} = \dfrac{2x-2y}{x^2+y^2}\mathbf{k}$

**21.** (a) Let $x = t, y = t, -1 \le t \le 2$, then $ds = \sqrt{2}\,dt$.

$$\int_C (x^2+y^2)\,ds = \int_{-1}^{2} 2t^2\sqrt{2}\,dt = \left[2\sqrt{2}\left(\frac{t^3}{3}\right)\right]_{-1}^{2} = 6\sqrt{2}$$

(b) Let $x = 4\cos t, y = 4\sin t, 0 \le t \le 2\pi$, then $ds = 4\,dt$.

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} 16(4\,dt) = 128\pi$$

**23.** $x = \cos t + t\sin t, y = \sin t - t\cos t, 0 \le t \le 2\pi, \dfrac{dx}{dt} = t\cos t, \dfrac{dy}{dt} = t\sin t$

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} [(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2]\sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = \int_0^{2\pi} [t^3 + t]\,dt$$

$$= 2\pi^2(1+2\pi^2)$$

**25.** (a) Let $x = 2t, y = -3t, 0 \le t \le 1$

$$\int_C (2x-y)\,dx + (x+3y)\,dy = \int_0^1 [7t(2) + (-7t)(-3)]\,dt = \int_0^1 35t\,dt = \frac{35}{2}$$

(b) $x = 3\cos t, y = 3\sin t, dx = -3\sin t\,dt, dy = 3\cos t\,dt, 0 \le t \le 2\pi$

$$\int_C (2x-y)\,dx + (x+3y)\,dy = \int_0^{2\pi} (9 + 9\sin t\cos t)\,dt = 18\pi$$

**27.** $\displaystyle\int_C (2x+y)\,ds, \mathbf{r}(t) = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}, 0 \le t \le \frac{\pi}{2}$

$x'(t) = -3a \cdot \cos^2 t\sin t$

$y'(t) = 3a \cdot \sin^2 t\cos t$

$$\int_C (2x+y)\,ds = \int_0^{\pi/2} (2(a\cdot\cos^3 t) + a\cdot\sin^3 t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = \frac{9a^2}{5}$$

**29.** $f(x, y) = 5 + \sin(x+y)$

$C$: $y = 3x$ from $(0, 0)$ to $(2, 6)$

$\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}, 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{10}$

Lateral surface area:

$$\int_{C_2} f(x, y)\,ds = \int_0^2 [5 + \sin(t+3t)]\sqrt{10}\,dt = \sqrt{10}\int_0^2 (5 + \sin 4t)\,dt = \frac{\sqrt{10}}{4}(41 - \cos 8) \approx 32.528$$

**31.** $d\mathbf{r} = (2t\mathbf{i} + 3t^2\mathbf{j})\,dt$

$\mathbf{F} = t^5\mathbf{i} + t^4\mathbf{j},\ 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^6\,dt = \frac{5}{7}$$

**33.** $d\mathbf{r} = [(-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k}]\,dt$

$\mathbf{F} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} t\,dt = 2\pi^2$$

**35.** Let $x = t,\ y = -t,\ z = 2t^2,\ -2 \le t \le 2,\ d\mathbf{r} = [\mathbf{i} - \mathbf{j} + 4t\mathbf{k}]\,dt.$

$\mathbf{F} = (-t - 2t^2)\mathbf{i} + (2t^2 - t)\mathbf{j} + (2t)\mathbf{k}$

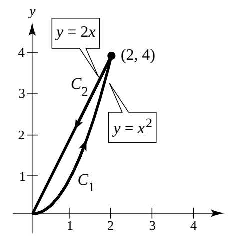
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^{2} 4t^2\,dt = \left[\frac{4t^3}{3}\right]_{-2}^{2} = \frac{64}{3}$$

**37.** For $y = x^2$, $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

For $y = 2x$, $\mathbf{r}_2(t) = (2 - t)\mathbf{i} + (4 - 2t)\mathbf{j},\ 0 \le t \le 2$

$$\int_C xy\,dx + (x^2 + y^2)\,dy = \int_{C_1} xy\,dx + (x^2 + y^2)\,dy + \int_{C_2} xy\,dx + (x^2 + y^2)\,dy$$

$$= \frac{100}{3} + (-32) = \frac{4}{3}$$




**39.** $\mathbf{F} = x\mathbf{i} - \sqrt{y}\,\mathbf{j}$ is conservative.

$$\text{Work} = \left[\frac{1}{2}x^2 - \frac{2}{3}y^{3/2}\right]_{(0,\,0)}^{(4,\,8)} = \frac{1}{2}(16) - \left(\frac{2}{3}\right)8^{3/2} = \frac{8}{3}\left(3 - 4\sqrt{2}\right)$$

**41.** $$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz = \Big[x^2yz\Big]_{(0,\,0,\,0)}^{(1,\,4,\,3)} = 12$$

**43.** (a) $$\int_C y^2\,dx + 2xy\,dy = \int_0^1 [(1 + t)^2(3) + 2(1 + 3t)(1 + t)]\,dt$$

$$= \int_0^1 3(t^2 + 2t + 1) + 2(3t^2 + 4t + 1)]\,dt$$

$$= \int_0^1 (9t^2 + 14t + 5)\,dt$$

$$= \Big[3t^3 + 7t^2 + 5t\Big]_0^1 = 15$$

(b) $$\int_C y^2\,dx + 2xy\,dy = \int_1^4 \left[t(1) + 2(t)\left(\sqrt{t}\right)\frac{1}{2\sqrt{t}}\right]dt$$

$$= \int_1^4 (t + t)\,dt$$

$$= \Big[t^2\Big]_1^4 = 15$$

(c) $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\,\mathbf{j} = \nabla f$ where $f(x, y) = xy^2$.

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4(2)^2 - 1(1)^2 = 15$$

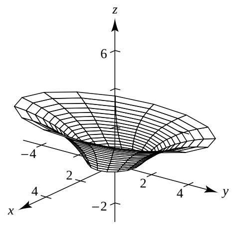
**45.** $$\int_C y\,dx + 2x\,dy = \int_0^2\int_0^2 (2 - 1)\,dy\,dx = \int_0^2 2\,dx = 4$$

**47.** $$\int_C xy^2\,dx + x^2y\,dy = \iint_R (2xy - 2xy)\,dA = 0$$

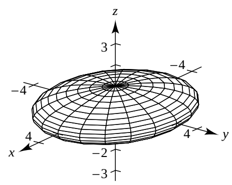
**49.** $\displaystyle\int_C xy\,dx + x^2\,dy = \int_0^1\int_{x^2}^{x} x\,dy\,dx = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{12}$

**51.** $\mathbf{r}(u, v) = \sec u \cos v\,\mathbf{i} + (1 + 2\tan u)\sin v\,\mathbf{j} + 2u\,\mathbf{k}$

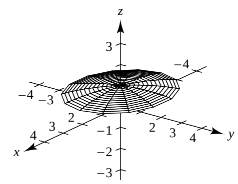
$0 \le u \le \dfrac{\pi}{3},\quad 0 \le v \le 2\pi$



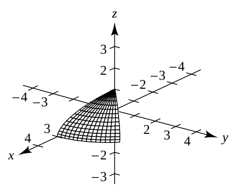

**53.** (a)



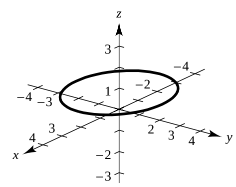

(b)




(c)




(d)




The space curve is a circle:

$$\mathbf{r}\left(u, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}\cos u\,\mathbf{i} + \frac{3\sqrt{2}}{2}\sin u\,\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

(e) $\mathbf{r}_u = -3\cos v \sin u\,\mathbf{i} + 3\cos v\cos u\,\mathbf{j}$

$\mathbf{r}_v = -3\sin v\cos u\,\mathbf{i} - 3\sin v\sin u\,\mathbf{j} + \cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\cos v\sin u & 3\cos v\cos u & 0 \\ -3\sin v\cos u & -3\sin v\sin u & \cos v \end{vmatrix}$$

$$= (3\cos^2 v\cos u)\mathbf{i} + (3\cos^2 v\sin u)\mathbf{j} + (9\cos v\sin v\sin^2 u + 9\cos v\sin v\cos^2 u)\mathbf{k}$$

$$= (3\cos^2 v\cos u)\mathbf{i} + (3\cos^2 v\sin u)\mathbf{j} + (9\cos v\sin v)\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9\cos^4 v\cos^2 u + 9\cos^4 v\sin^2 u + 81\cos^2 v\sin^2 v}$$

$$= \sqrt{9\cos^4 v + 81\cos^2 v\sin^2 v}$$

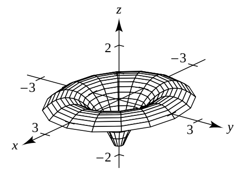
Using a Symbolic integration utility,

$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv \approx 14.44$$

(f) Similarly,

$$\int_0^{\pi/4}\int_0^{\pi/2} \|\mathbf{r}_u \times \mathbf{r}_v\|\,dv\,du \approx 4.27$$

**55.** $S: \mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + (u-1)(2-u)\mathbf{k}, \quad 0 \le u \le 2,\ 0 \le v \le 2\pi$




$$\mathbf{r}_u(u, v) = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + (3 - 2u)\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 3 - 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (2u-3)u\cos v\,\mathbf{i} + (2u-3)u\sin v\,\mathbf{j} + u\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{(2u-3)^2 + 1}$$

$$\iint_S (x + y)\,dS = \int_0^{2\pi}\int_0^2 (u\cos v + u\sin v)\,u\sqrt{(2u-3)^2+1}\,du\,dv$$

$$= \int_0^2\int_0^{2\pi} (\cos v + \sin v)u^2\sqrt{(2u-3)^2+1}\,dv\,du = 0$$

**57.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

$Q$: solid region bounded by the coordinates planes and the plane $2x + 3y + 4z = 12$

**Surface Integral**: There are four surfaces for this solid.

$$z = 0 \qquad \mathbf{N} = -\mathbf{k}, \qquad \mathbf{F}\cdot\mathbf{N} = -z, \qquad \iint_{S_1} 0\,dS = 0$$

$$y = 0, \qquad \mathbf{N} = -\mathbf{j}, \qquad \mathbf{F}\cdot\mathbf{N} = -xy, \qquad \iint_{S_2} 0\,dS = 0$$

$$x = 0, \qquad \mathbf{N} = -\mathbf{i}, \qquad \mathbf{F}\cdot\mathbf{N} = -x^2, \qquad \iint_{S_3} 0\,dS = 0$$

$$2x + 3y + 4z = 12,\ \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}},\ dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)}dA = \frac{\sqrt{29}}{4}\,dA$$

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{N}\,dS = \frac{1}{4}\iint_R (2x^2 + 3xy + 4z)\,dA$$

$$= \frac{1}{4}\int_0^6\int_0^{4-(2x/3)} (2x^2 + 3xy + 12 - 2x - 3y)\,dy\,dx$$

$$= \frac{1}{4}\int_0^6\left[2x^2\left(\frac{12-2x}{3}\right) + \frac{3x}{2}\left(\frac{12-2x}{3}\right)^2 + 12\left(\frac{12-2x}{3}\right) - 2x\left(\frac{12-2x}{3}\right) - \frac{3}{2}\left(\frac{12-2x}{3}\right)^2\right]dx$$

$$= \frac{1}{6}\int_0^6 (-x^3 + x^2 + 24x + 36)\,dx = \frac{1}{6}\left[-\frac{x^4}{4} + \frac{x^3}{3} + 12x^2 + 36x\right]_0^6 = 66$$

**Divergence Theorem:** Since div $\mathbf{F} = 2x + x + 1 = 3x + 1$, Divergence Theorem yields

$$\iiint_Q \operatorname{div}\mathbf{F}\,dV = \int_0^6\int_0^{(12-2x)/3}\int_0^{(12-2x-3y)/4} (3x + 1)\,dz\,dy\,dx$$

$$= \int_0^6\int_0^{(12-2x)/3} (3x + 1)\left(\frac{12 - 2x - 3y}{4}\right)dy\,dx$$

$$= \frac{1}{4}\int_0^6 (3x + 1)\left[12y - 2xy - \frac{3}{2}y^2\right]_0^{(12-2x)/3} dx$$

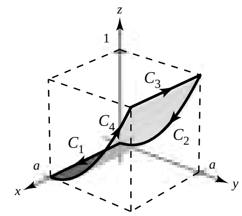
$$= \frac{1}{4}\int_0^6 (3x + 1)\left[4(12 - 2x) - 2x\left(\frac{12-2x}{3}\right) - \frac{3}{2}\left(\frac{12-2x}{3}\right)^2\right]dx$$

$$= \frac{1}{4}\int_0^6 \frac{2}{3}(3x^3 - 35x^2 + 96x + 36)\,dx = \frac{1}{6}\left[\frac{3x^4}{4} - \frac{35x^3}{3} + 48x^2 + 36x\right]_0^6 = 66.$$

**59.** $\mathbf{F}(x, y, z) = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$

$S$: portion of $z = y^2$ over the square in the $xy$-plane with vertices $(0, 0)$, $(a, 0)$, $(a, a)$, $(0, a)$

**Line Integral:** Using the line integral we have:




$C_1$: $y = 0, \quad dy = 0$

$C_2$: $x = 0, \quad dx = 0, \quad z = y^2, \quad dz = 2y\,dy$

$C_3$: $y = a, \quad dy = 0, \quad z = a^2, \quad dz = 0$

$C_4$: $x = a, \quad dx = 0, \quad z = y^2, \quad dz = 2y\,dy$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy + xyz\,dz$$

$$= \int_{C_1} dx + \int_{C_2} 0 + \int_{C_3} (\cos a + a\cos x)\,dx + \int_{C_4} (\sin a - a\sin y)\,dy + ay^3(2y\,dy)$$

$$= \int_0^a dx + \int_a^0 (\cos a + a\cos x)\,dx + \int_0^a (\sin a - a\sin y)\,dy + \int_0^a 2ay^4\,dy$$

$$= a + \Big[x\cos a + a\sin x\Big]_a^0 + \Big[y\sin a + a\cos y\Big]_0^a + \left[2a\frac{y^5}{5}\right]_0^a$$

$$= a - a\cos a - a\sin a + a\sin a + a\cos a - a + \frac{2a^6}{5} = \frac{2a^6}{5}$$

**Double Integral:** Considering $f(x, y, z) = z - y^2$, we have:

$$\mathbf{N} = \frac{\nabla f}{\|\nabla f\|} = \frac{-2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4y^2}},\ dS = \sqrt{1 + 4y^2}\,dA, \text{ and } \mathbf{curl\ F} = xz\mathbf{i} - yz\mathbf{j}.$$

Hence,

$$\iint_S (\mathbf{curl\ F})\cdot\mathbf{N}\,dS = \int_0^a\int_0^a 2y^2z\,dy\,dx = \int_0^a\int_0^a 2y^4\,dy\,dx = \int_0^a \frac{2a^5}{5}\,dx = \frac{2a^6}{5}.$$

## Problem Solving for Chapter 14

**1.** (a) $\nabla T = \dfrac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{i} + z\mathbf{k}]$

$\mathbf{N} = x\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{k}$

$dS = \dfrac{1}{\sqrt{1 - x^2}}\,dy\,dx$

$$\text{Flux} = \iint_S -k\nabla T\cdot N\,dS$$

$$= 25k\iint_R \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right] dA$$

$$= 25k\int_{-1/2}^{1/2}\int_0^1 \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{1 - x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}}\right] dy\,dx$$

$$= 25k\int_{-1/2}^{1/2}\int_0^1 \frac{1}{(1 + y^2)^{3/2}(1 - x^2)^{1/2}}\,dy\,dx$$

$$= 25k\int_0^1 \frac{1}{(1 + y^2)^{3/2}}\,dy \int_{-1/2}^{1/2}\frac{1}{(1 - x^2)^{1/2}}\,dx$$

$$= 25k\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\pi}{3}\right) = 25k\frac{\sqrt{2}\pi}{6}$$

—CONTINUED—

**1. —CONTINUED—**

(b) $\mathbf{r}(u, v) = \langle \cos u, v, \sin u\rangle$

$\mathbf{r}_u = \langle -\sin u, 0, \cos u\rangle, \mathbf{r}_v = \langle 0, 1, 0\rangle$

$\mathbf{r}_u \times \mathbf{r}_v = \langle -\cos u, 0, -\sin u\rangle$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$$

$$= \frac{-25}{(v^2 + 1)^{3/2}}[\cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}]$$

$$\nabla T \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \frac{-25}{(v^2 + 1)^{3/2}}(-\cos^2 u - \sin^2 u) = \frac{25}{(v^2 + 1)^{3/2}}$$

$$\text{Flux} = \int_0^1 \int_{\pi/3}^{2\pi/3} \frac{25k}{(v^2 + 1)^{3/2}}\, du\, dv = 25k\frac{\sqrt{2}\pi}{6}$$

**3.** $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$

$\mathbf{r}'(t) = \langle -3\sin t, 3\cos t, 2\rangle, \|\mathbf{r}'(t)\| = \sqrt{13}$

$$I_x = \int_C (y^2 + z^2)\rho\, ds = \int_0^{2\pi} (9\sin^2 t + 4t^2)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi(32\pi^2 + 27)$$

$$I_y = \int_C (x^2 + z^2)\rho\, ds = \int_0^{2\pi} (9\cos^2 t + 4t^2)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi(32\pi^2 + 27)$$

$$I_z = \int_C (x^2 + y^2)\rho\, ds = \int_0^{2\pi} (9\cos^2 t + 9\sin^2 t)\sqrt{13}\, dt = 18\pi\sqrt{13}$$

**5.** $$\frac{1}{2}\int_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} [a(\theta - \sin\theta)(a\sin\theta)\, d\theta - a(1 - \cos\theta)(a(1 - \cos\theta))\, d\theta]$$

$$= \frac{1}{2}a^2\int_0^{2\pi} [\theta\sin\theta - \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\, d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi} (\theta\sin\theta + 2\cos\theta - 2)\, d\theta$$

$$= -3\pi a^2$$

Hence, the area is $3\pi a^2$.

**7.** (a) $\mathbf{r}(t) = t\mathbf{j}, 0 \le t \le 1$

$\mathbf{r}'(t) = \mathbf{j}$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t\mathbf{i} + \mathbf{j}) \cdot \mathbf{j}\, dt = \int_0^1 dt = 1$$

(b) $\mathbf{r}(t) = (t - t^2)\mathbf{i} + t\mathbf{j}, 0 \le t \le 1$

$\mathbf{r}'(t) = (1 - 2t)\mathbf{i} + \mathbf{j}$

$$W = \mathbf{F} \cdot d\mathbf{r} = \int_0^1 ((2t - t^2)\mathbf{i} + [(t - t^2)^2 + 1]\mathbf{j}) \cdot ((1 - 2t)\mathbf{i} + \mathbf{j})\, dt$$

$$= \int_0^1 [(1 - 2t)(2t - t^2) + (t^4 - 2t^3 + t^2 + 1)]\, dt$$

$$= \int_0^1 (t^4 - 4t^2 + 2t + 1)\, dt = \frac{13}{15}$$

**—CONTINUED—**

**7. —CONTINUED—**

(c) $\mathbf{r}(t) = c(t - t^2)\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}'(t) = c(1 - 2t)\mathbf{i} + \mathbf{j}$

$$\mathbf{F} \cdot d\mathbf{r} = (c(t - t^2) + t)(c(1 - 2t)) + (c^2(t - t^2)^2 + 1)(1)$$
$$= c^2t^4 - 2c^2t^2 + c^2t - 2ct^2 + ct + 1$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{30}c^2 - \frac{1}{6}c + 1$$

$$\frac{dW}{dc} = \frac{1}{15}c - \frac{1}{6} = 0 \implies c = \frac{5}{2}$$

$$\frac{d^2W}{dc^2} = \frac{1}{15} > 0 \quad c = \frac{5}{2} \text{ minimum.}$$

**9.** $\mathbf{v} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$

$$= \langle a_2z - a_3y, -a_1z + a_3x, a_1y - a_2x \rangle$$

$\mathbf{curl}(\mathbf{v} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{v}$

By Stoke's Theorem,

$$\int_C (\mathbf{v} \times \mathbf{r})\, d\mathbf{r} = \int_S\!\!\int \mathbf{curl}(\mathbf{v} \times \mathbf{r}) \cdot \mathbf{N}\, dS$$
$$= \int_S\!\!\int 2\mathbf{v} \cdot N\, dS.$$

**11.** $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = \dfrac{m}{(x^2 + y^2)^{5/2}}[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]$

$$M = \frac{3mxy}{(x^2 + y^2)^{5/2}} = 3mxy(x^2 + y^2)^{-5/2}$$

$$\frac{\partial M}{\partial y} = 3mxy\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2y)\right] + (x^2 + y^2)^{-5/2}(3mx)$$
$$= 3mx(x^2 + y^2)^{-7/2}[-5y^2 + (x^2 + y^2)] = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

$$N = \frac{m(2y^2 - x^2)}{(x^2 + y^2)^{5/2}} = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$$

$$\frac{\partial N}{\partial x} = m(2y^2 - x^2)\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2x)\right] + (x^2 + y^2)^{-5/2}(-2mx)$$
$$= mx(x^2 + y^2)^{-7/2}[(2y^2 - x^2)(-5) + (x^2 + y^2)(-2)]$$
$$= mx(x^2 + y^2)^{-7/2}(3x^2 - 12y^2) = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

Therefore, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ and $\mathbf{F}$ is conservative.

# PART II

## CHAPTER P
## Preparation for Calculus

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# CHAPTER P
# Preparation for Calculus

## Section P.1 Graphs and Models

Solutions to Even-Numbered Exercises

**2.** $y = \sqrt{9 - x^2}$

$x$-intercepts: $(-3, 0), (3, 0)$

$y$-intercept: $(0, 3)$

Matches graph (d)

**4.** $y = x^3 - x$

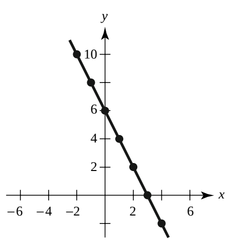
$x$-intercepts: $(0, 0), (-1, 0), (1, 0)$

$y$-intercept: $(0, 0)$

Matches graph (c)

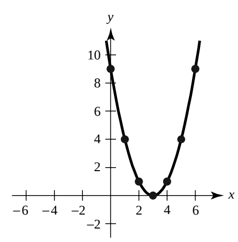
**6.** $y = 6 - 2x$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|
| $y$ | $10$ | $8$ | $6$ | $4$ | $2$ | $0$ | $-2$ |



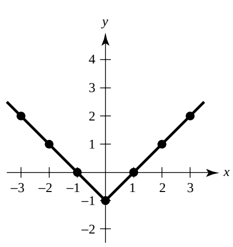

**8.** $y = (x - 3)^2$

| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|---|---|---|---|---|---|---|---|
| $y$ | $9$ | $4$ | $1$ | $0$ | $1$ | $4$ | $9$ |



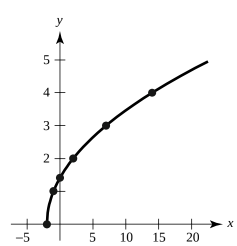

**10.** $y = |x| - 1$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | $2$ | $1$ | $0$ | $-1$ | $0$ | $1$ | $2$ |




**12.** $y = \sqrt{x + 2}$

| $x$ | $-2$ | $-1$ | $0$ | $2$ | $7$ | $14$ |
|---|---|---|---|---|---|---|
| $y$ | $0$ | $1$ | $\sqrt{2}$ | $2$ | $3$ | $4$ |

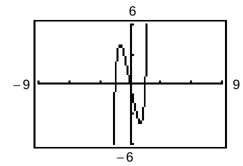


**14.**

| |
|---|
| Xmin = -30 |
| Xmax = 30 |
| Xscl = 5 |
| Ymin = -10 |
| Ymax = 40 |
| Yscl = 5 |

Note that $y = 10$ when $x = 0$ or $x = 10$.

**16.**




(a) $(-0.5, y) = (-0.5, 2.47)$

(b) $(x, -4) = (-1.65, -4)$ and $(x, -4) = (1, -4)$

**18.** $y^2 = x^3 - 4x$

$y$-intercept: $y^2 = 0^3 - 4(0)$

$y = 0; (0, 0)$

$x$-intercepts: $0 = x^3 - 4x$

$0 = x(x - 2)(x + 2)$

$x = 0, \pm 2; (0, 0), (\pm 2, 0)$

**20.** $y = (x - 1)\sqrt{x^2 + 1}$

$y$-intercept: $y = (0 - 1)\sqrt{0^2 + 1}$

$y = -1; (0, -1)$

$x$-intercepts: $0 = (x - 1)\sqrt{x^2 + 1}$

$x = 1; (1, 0)$

**22.** $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$

$y$-intercept: $y = \dfrac{0^2 + 3(0)}{[3(0) + 1]^2}$

$y = 0; (0, 0)$

$x$-intercepts: $0 = \dfrac{x^2 + 3x}{(3x + 1)^2}$

$0 = \dfrac{x(x + 3)}{(3x + 1)^2}$

$x = 0, -3; (0, 0), (-3, 0)$

**24.** $y = 2x - \sqrt{x^2 + 1}$

$y$-intercept: $y = 2(0) - \sqrt{0^2 + 1}$

$y = -1; (0, -1)$

$x$-intercepts: $0 = 2x - \sqrt{x^2 + 1}$

$2x = \sqrt{x^2 + 1}$

$4x^2 = x^2 + 1$

$3x^2 = 1$

$x^2 = \dfrac{1}{3}$

$x = \pm\dfrac{\sqrt{3}}{3}$

$x = \dfrac{\sqrt{3}}{3}; \left(\dfrac{\sqrt{3}}{3}, 0\right)$

**Note:** $x = -\sqrt{3}/3$ is an extraneous solution.

**26.** $y = x^2 - x$

No symmetry with respect to either axis or the origin.

**28.** Symmetric with respect to the origin since

$(-y) = (-x)^3 + (-x)$

$-y = -x^3 - x$

$y = x^3 + x.$

**30.** Symmetric with respect to the $x$-axis since

$x(-y)^2 = xy^2 = -10.$

**32.** Symmetric with respect to the origin since

$(-x)(-y) - \sqrt{4 - (-x)^2} = 0$

$xy - \sqrt{4 - x^2} = 0.$

**34.** $y = \dfrac{x^2}{x^2 + 1}$ is symmetric with respect to the $y$-axis

since $y = \dfrac{(-x)^2}{(-x)^2 + 1} = \dfrac{x^2}{x^2 + 1}$.

**36.** $|y| - x = 3$ is symmetric with respect to the $x$-axis

since $|-y| - x = 3$

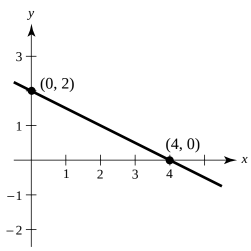
$|y| - x = 3$.

**38.** $y = -\dfrac{x}{2} + 2$

Intercepts:

$(4, 0), (0, 2)$

Symmetry: none

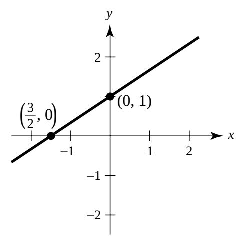



**40.** $y = \frac{2}{3}x + 1$

Intercepts:

$(0, 1), \left(-\frac{3}{2}, 0\right)$

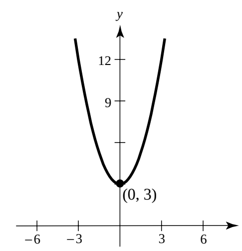
Symmetry: none




**42.** $y = x^2 + 3$

Intercept: $(0, 3)$

Symmetry: $y$-axis

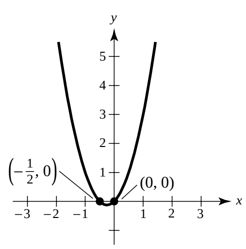



**44.** $y = 2x^2 + x = x(2x + 1)$

Intercepts:

$(0, 0), \left(-\frac{1}{2}, 0\right)$

Symmetry: none

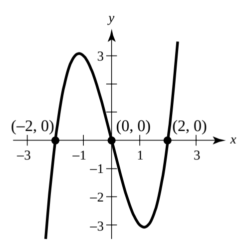



**46.** $y = x^3 - 4x$

Intercepts:

$(0, 0), (2, 0), (-2, 0)$

Symmetry: origin



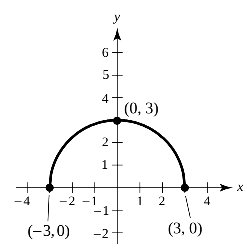

**48.** $y = \sqrt{9 - x^2}$

Intercepts:

$(-3, 0), (3, 0), (0, 3)$

Symmetry: $y$-axis

Domain: $[-3, 3]$

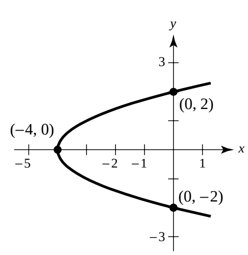



**50.** $x = y^2 - 4$

Intercepts:

$(0, 2), (0, -2), (-4, 0)$

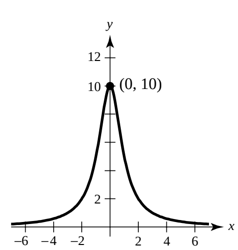
Symmetry: $x$-axis




**52.** $y = \dfrac{10}{x^2 + 1}$

Intercepts: $(0, 10)$

Symmetry: $y$-axis

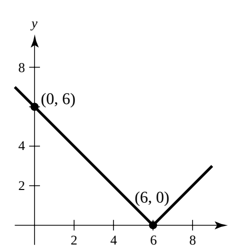



**54.** $y = |6 - x|$

Intercepts:

$(0, 6), (6, 0)$

Symmetry: none



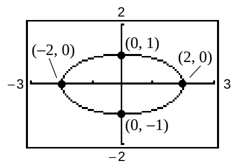
**56.** $x^2 + 4y^2 = 4 \Rightarrow y = \pm\dfrac{\sqrt{4 - x^2}}{2}$

Intercepts:

$(-2, 0), (2, 0), (0, -1), (0, 1)$

Symmetry: origin and both axes

Domain: $[-2, 2]$



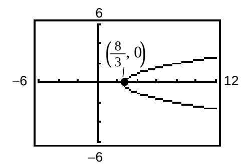

**58.** $3x - 4y^2 = 8$

$$4y^2 = 3x - 8$$

$$y = \pm\sqrt{\tfrac{3}{4}x - 2}$$

Intercept: $\left(\frac{8}{3}, 0\right)$

Symmetry: $x$-axis




**60.** $y = \left(x + \frac{5}{2}\right)(x - 2)\left(x - \frac{3}{2}\right)$ (other answers possible)

**62.** Some possible equations:

$$x = y^2$$

$$x = |y|$$

$$x = y^4 + 1$$

$$x^2 + y^2 = 25$$

**64.** $2x - 3y = 13 \Rightarrow y = \dfrac{2x - 13}{3}$

$5x + 3y = 1 \Rightarrow y = \dfrac{1 - 5x}{3}$

$$\frac{2x - 13}{3} = \frac{1 - 5x}{3}$$

$$2x - 13 = 1 - 5x$$

$$7x = 14$$

$$x = 2$$

The corresponding $y$-value is $y = -3$.

Point of intersection: $(2, -3)$

**66.** $5x - 6y = 9 \Rightarrow y = \dfrac{5x - 9}{6}$

$-7x + 3y = -18 \Rightarrow y = \dfrac{7x - 18}{3}$

$$\frac{5x - 9}{6} = \frac{7x - 18}{3}$$

$$5x - 9 = 14x - 36$$

$$27 = 9x$$

$$x = 3$$

The corresponding $y$-value is $y = 1$.

Point of intersection: $(3, 1)$

**68.** $x = 3 - y^2 \Rightarrow y^2 = 3 - x$

$y = x - 1$

$$3 - x = (x - 1)^2$$

$$3 - x = x^2 - 2x + 1$$

$$0 = x^2 - x - 2 = (x + 1)(x - 2)$$

$$x = -1 \text{ or } x = 2$$

The corresponding $y$-values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

**70.** $x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2$

$2x + y = 10 \Rightarrow y = 10 - 2x$

$$25 - x^2 = (10 - 2x)^2$$

$$25 - x^2 = 100 - 40x + 4x^2$$

$$0 = 5x^2 - 40x + 75 = 5(x - 3)(x - 5)$$

$$x = 3 \text{ or } x = 5$$

The corresponding $y$-values are $y = 4$ and $y = 0$.

Points of intersection: $(3, 4), (5, 0)$

**72.** $y = x^3 - 4x$

$y = -(x + 2)$

$x^3 - 4x = -(x + 2)$

$x^3 - 3x + 2 = 0$

$(x - 1)^2(x + 2) = 0$

$x = 1 \quad \text{or} \quad x = -2$

The corresponding $y$-values are $y = -3$ and $y = 0$.

Points of intersection: $(1, -3), (-2, 0)$

**74.** $y = x^4 - 2x^2 + 1$

$y = 1 - x^2$

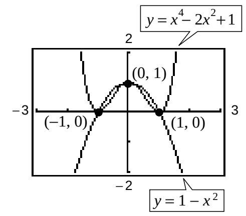
$1 - x^2 = x^4 - 2x^2 + 1$

$0 = x^4 - x^2$

$0 = x^2(x + 1)(x - 1)$

$x = -1, 0, 1$

$(-1, 0), (0, 1), (1, 0)$




**76.** $y = kx + 5$ matches (b).

Use $(1, 7)$: $7 = k(1) + 5 \Longrightarrow k = 2$, thus, $y = 2x + 5$.

$y = x^2 + k$ matches (d).

Use $(1, -9)$: $-9 = (1)^2 + k \Longrightarrow k = -10$, thus, $y = x^2 - 10$.

$y = kx^{3/2}$ matches (a).

Use $(1, 3)$: $3 = k(1)^{3/2} \Longrightarrow k = 3$, thus, $y = 3x^{3/2}$.

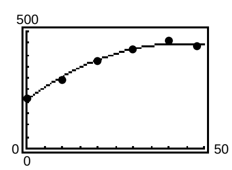
$xy = k$ matches (c).

Use $(1, 36)$: $(1)(36) = k \Longrightarrow k = 36$, thus, $xy = 36$.

**78.** (a) Using a graphing utility, you obtain

$y = -0.1283t^2 + 11.0988t + 207.1116$

(b)




(c) For the year 2004, $t = 54$ and $y \approx 432.3$ acres per farm.

**80.** (a) If $(x, y)$ is on the graph, then so is $(-x, y)$ by $y$-axis symmetry. Since $(-x, y)$ is on the graph, then so is $(-x, -y)$ by $x$-axis symmetry. Hence, the graph is symmetric with respect to the origin. The converse is not true. For example, $y = x^3$ has origin symmetry but is not symmetric with respect to either the $x$-axis or the $y$-axis.

(b) Assume that the graph has $x$-axis and origin symmetry. If $(x, y)$ is on the graph, so is $(x, -y)$ by $x$-axis symmetry. Since $(x, -y)$ is on the graph, then so is $(-x, -(-y)) = (-x, y)$ by origin symmetry. Therefore, the graph is symmetric with respect to the $y$-axis. The argument is similar for $y$-axis and origin symmetry.

**82.** True

**84.** True; the $x$-intercept is

$$\left(-\frac{b}{2a}, 0\right).$$

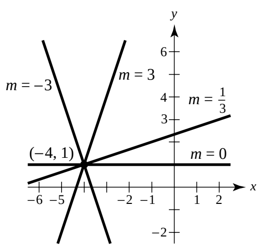
# Section P.2 Linear Models and Rates of Change

**2.** $m = 2$

**4.** $m = -1$

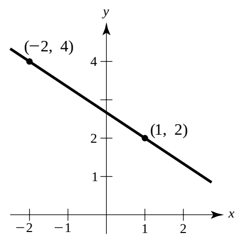
**6.** $m = \frac{40}{3}$

**8.**

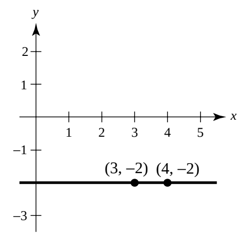



**10.** $m = \dfrac{4 - 2}{-2 - 1}$

$= -\dfrac{2}{3}$

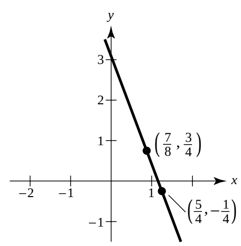



**12.** $m = \dfrac{-2 - (-2)}{4 - 1} = 0$




**14.** $m = \dfrac{(3/4) - (-1/4)}{(7/8) - (5/4)}$

$= \dfrac{1}{-3/8} = -\dfrac{8}{3}$




**16.** Since the slope is undefined, the line is vertical and its equation is $x = -3$. Therefore, three additional points are $(-3, 2)$, $(-3, 3)$, and $(-3, 5)$.

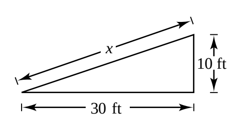
**18.** The equation of this line is

$$y + 2 = 2(x + 2)$$
$$y = 2x + 2.$$

Therefore, three additional points are $(-3, -4)$, $(-1, 0)$, and $(0, 2)$.

**20.** (a) Slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{1}{3}$

(b)




By the Pythagorean Theorem,

$$x^2 = 30^2 + 10^2 = 1000$$
$$x \approx 31.623 \text{ feet.}$$

**22.** (a) $m = 400$ indicates that the revenues increase by 400 in one day.

(b) $m = 100$ indicates that the revenues increase by 100 in one day.

(c) $m = 0$ indicates that the revenues do not change from one day to the next.

**24.** $6x - 5y = 15$

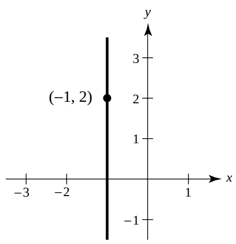
$$y = \frac{6}{5}x - 3$$

Therefore, the slope is $m = \frac{6}{5}$ and the $y$-intercept is $(0, -3)$.

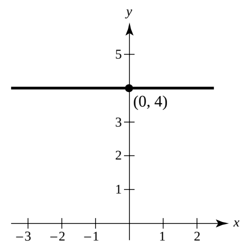
**26.** $y = -1$

The line is horizontal. Therefore, the slope is $m = 0$ and the $y$-intercept is $(0, -1)$.

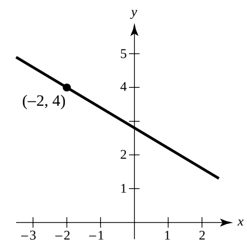
**28.** 
$$x = -1$$
$$x + 1 = 0$$




**30.** 
$$y = 4$$
$$y - 4 = 0$$

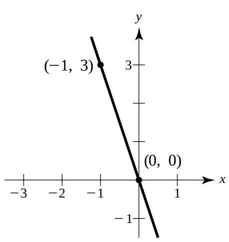



**32.** 
$$y - 4 = -\frac{3}{5}(x + 2)$$
$$5y - 20 = -3x - 6$$
$$3x + 5y - 14 = 0$$



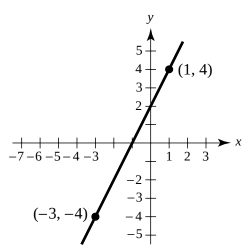

**34.** $m = \dfrac{3 - 0}{-1 - 0} = -3$

$$y - 0 = -3(x - 0)$$
$$y = -3x$$
$$3x + y = 0$$



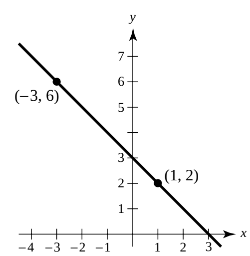

**36.** $m = \dfrac{4 - (-4)}{1 - (-3)} = \dfrac{8}{4} = 2$

$$y - 4 = 2(x - 1)$$
$$y - 4 = 2x - 2$$
$$0 = 2x - y + 2$$



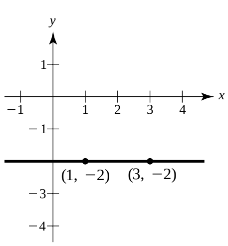

**38.** $m = \dfrac{6 - 2}{-3 - 1} = \dfrac{4}{-4} = -1$

$$y - 2 = -1(x - 1)$$
$$y - 2 = -x + 1$$
$$x + y - 3 = 0$$




**40.** $m = 0$

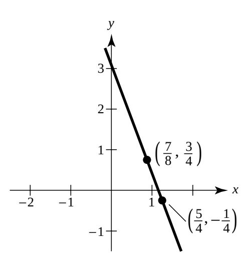
$$y = -2$$
$$y + 2 = 0$$




**42.** $m = \dfrac{(3/4) - (-1/4)}{(7/8) - (5/4)}$

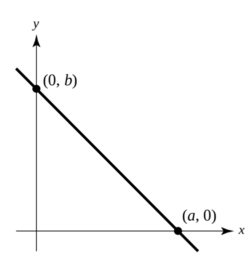
$$= \frac{1}{-3/8} = -\frac{8}{3}$$

$$y + \frac{1}{4} = \frac{-8}{3}\left(x - \frac{5}{4}\right)$$
$$12y + 3 = -32x + 40$$
$$32x + 12y - 37 = 0$$




**44.** $m = -\dfrac{b}{a}$

$$y = \frac{-b}{a}x + b$$
$$\frac{b}{a}x + y = b$$
$$\frac{x}{a} + \frac{y}{b} = 1$$



**46.** $\dfrac{x}{-2/3} + \dfrac{y}{-2} = 1$

$$\frac{-3x}{2} - \frac{y}{2} = 1$$

$$3x + y = -2$$

$$3x + y + 2 = 0$$

**48.** $\dfrac{x}{a} + \dfrac{y}{a} = 1$

$$\frac{-3}{a} + \frac{4}{a} = 1$$

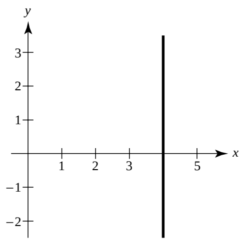
$$\frac{1}{a} = 1$$

$$a = 1 \Longrightarrow x + y = 1$$

$$x + y - 1 = 0$$

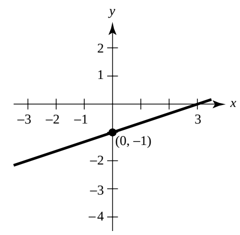
**50.** $x = 4$

$$x - 4 = 0$$



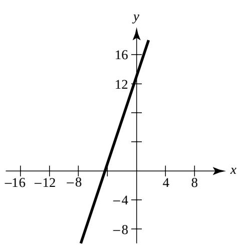
**52.** $y = \frac{1}{3}x - 1$

$$3y - x + 3 = 0$$



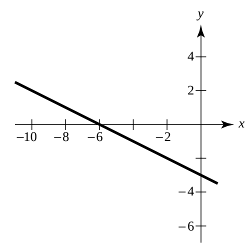

**54.** $y - 1 = 3(x + 4)$

$$y = 3x + 13$$

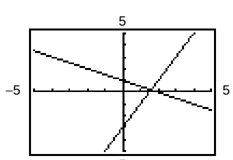


**56.** $x + 2y + 6 = 0$

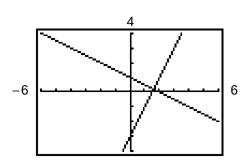
$$y = -\frac{1}{2}x - 3$$



**58.**



The lines do not appear perpendicular.



The lines appear perpendicular.

The lines are perpendicular because their slopes 2 and $-\frac{1}{2}$ are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.

**60.** $x + y = 7$

$$y = -x + 7$$

$$m = -1$$

(a) $$y - 2 = -1(x + 3)$$

$$y - 2 = -x - 3$$

$$x + y + 1 = 0$$

(b) $$y - 2 = 1(x + 3)$$

$$y - 2 = x + 3$$

$$x - y + 5 = 0$$

**62.** $3x + 4y = 7$

$$y = -\tfrac{3}{4}x + \tfrac{7}{4}$$

$$m = -\tfrac{3}{4}$$

(a) $$y - 4 = -\tfrac{3}{4}(x + 6)$$

$$4y - 16 = -3x - 18$$

$$3x + 4y + 2 = 0$$

(b) $$y - 4 = \tfrac{4}{3}(x + 6)$$

$$3y - 12 = 4x + 24$$

$$4x - 3y + 36 = 0$$

**64.** (a) $y = 0$

(b) $x = -1 \Longrightarrow x + 1 = 0$

**66.** The slope is 4.50.

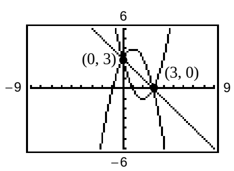
Hence, $V = 4.5(t - 1) + 156$

$= 4.5t + 151.5$

**68.** The slope is $-5600$. Hence, $V = -5600(t - 1) + 245{,}000$

$= -5600t + 250{,}600$

**70.**




You can use the graphing utility to determine that the points of intersection are $(0, 3)$ and $(3, 0)$. Analytically,

$$x^2 - 4x + 3 = -x^2 + 2x + 3$$

$$2x^2 - 6x = 0$$

$$2x(x - 3) = 0$$

$$x = 0 \Longrightarrow y = 3 \Longrightarrow (0, 3)$$

$$x = 3 \Longrightarrow y = 0 \Longrightarrow (3, 0).$$

The slope of the line joining $(0, 3)$ and $(3, 0)$ is $m = (0 - 3)/(3 - 0) = -1$. Hence, an equation of the line is

$$y - 3 = -1(x - 0)$$

$$y = -x + 3.$$

**72.** $m_1 = \dfrac{-6 - 4}{7 - 0} = -\dfrac{10}{7}$

$m_2 = \dfrac{11 - 4}{-5 - 0} = -\dfrac{7}{5}$

$m_1 \neq m_2$

The points are not collinear.

**74.** Equations of medians:

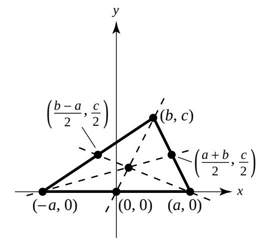
$$y = \frac{c}{b}x$$

$$y = \frac{c}{3a + b}(x + a)$$

$$y = \frac{c}{-3a + b}(x - a)$$

Solving simultaneously, the point of intersection is

$$\left(\frac{b}{3}, \frac{c}{3}\right).$$




**76.** The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(b, \frac{a^2 - b^2}{c}\right)$ is:

$$m_1 = \frac{[(a^2 - b^2)/c] - (c/3)}{b - (b/3)} = \frac{(3a^2 - 3b^2 - c^2)/(3c)}{(2b)/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(0, \frac{-a^2 + b^2 + c^2}{2c}\right)$ is:

$$m_2 = \frac{[(-a^2 + b^2 + c^2)/(2c)] - (c/3)}{0 - (b/3)} = \frac{(-3a^2 + 3b^2 + 3c^2 - 2c^2)/(6c)}{-b/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

$$m_1 = m_2$$

Therefore, the points are collinear.

**78.** $C = 0.34x + 150$. If $x = 137$, $C = 0.34(137) + 150 = \$196.58$

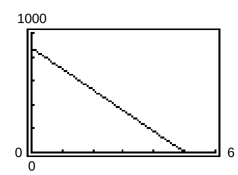
**80.** (a) Depreciation per year:

$\frac{875}{5} = \$175$

$y = 875 - 175x$

where $0 \le x \le 5$.

(b) $y = 875 - 175(2) = \$525$



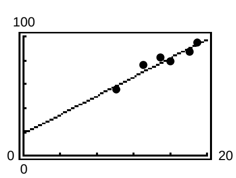

(c) $200 = 875 - 175x$

$175x = 675$

$x \approx 3.86$ years

**82.** (a) $y = 18.91 + 3.97x$ ($x$ = quiz score, $y$ = test score)

(b)




(c) If $x = 17$, $y = 18.91 + 3.97(17) = 86.4$.

(d) The slope shows the average increase in exam score for each unit increase in quiz score.

(e) The points would shift vertically upward 4 units. The new regression line would have a $y$-intercept 4 greater than before: $y = 22.91 + 3.97x$.

**84.** $4x + 3y - 10 = 0 \Longrightarrow d = \dfrac{|4(2) + 3(3) - 10|}{\sqrt{4^2 + 3^2}} = \dfrac{7}{5}$

**86.** $x + 1 = 0 \Longrightarrow d = \dfrac{|1(6) + (0)(2) + 1|}{\sqrt{1^2 + 0^2}} = 7$

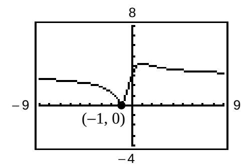
**88.** A point on the line $3x - 4y = 1$ is $(-1, -1)$. The distance from the point $(-1, -1)$ to $3x - 4y - 10 = 0$ is

$$d = \frac{|-3 + 4 - 10|}{5} = \frac{9}{5}.$$

**90.** $y = mx + 4 \Longrightarrow mx + (-1)y + 4 = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m3 + (-1)(1) + 4|}{\sqrt{m^2 + (-1)^2}}$$

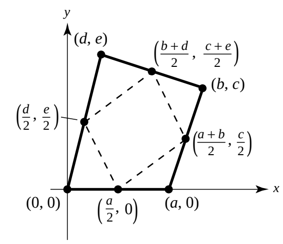
$$= \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$




The distance is 0 when $m = -1$. In this case, the line $y = -x + 4$ contains the point $(3, 1)$.

**92.** For simplicity, let the vertices of the quadrilateral be $(0, 0)$, $(a, 0)$, $(b, c)$, and $(d, e)$, as shown in the figure. The midpoints of the sides are

$$\left(\frac{a}{2}, 0\right), \left(\frac{a + b}{2}, \frac{c}{2}\right), \left(\frac{b + d}{2}, \frac{c + e}{2}\right), \text{ and } \left(\frac{d}{2}, \frac{e}{2}\right).$$




The slope of the opposite sides are equal:

$$\frac{\frac{c}{2} - 0}{\frac{a + b}{2} - \frac{a}{2}} = \frac{\frac{c + e}{2} - \frac{e}{2}}{\frac{b + d}{2} - \frac{d}{2}} = \frac{c}{b}$$

$$\frac{0 - \frac{e}{2}}{\frac{a}{2} - \frac{d}{2}} = \frac{\frac{c}{2} - \frac{c + e}{2}}{\frac{a + b}{2} - \frac{b + d}{2}} = -\frac{e}{a - d}$$

Therefore, the figure is a paralleogram.

**94.** If $m_1 = -1/m_2$, then $m_1 m_2 = -1$. Let $L_3$ be a line with slope $m_3$ that is perpendicular to $L_1$. Then $m_1 m_3 = -1$. Hence, $m_2 = m_3 \Longrightarrow L_2$ and $L_3$ are parallel. Therefore, $L_2$ and $L_1$ are also perpendicular.

**96.** False; if $m_1$ is positive, then $m_2 = -1/m_1$ is negative.

## Section P.3 Functions and Their Graphs

**2.** (a) $f(-2) = \sqrt{-2 + 3} = \sqrt{1} = 1$

(b) $f(6) = \sqrt{6 + 3} = \sqrt{9} = 3$

(c) $f(c) = \sqrt{c + 3}$

(d) $f(x + \Delta x) = \sqrt{x + \Delta x + 3}$

**4.** (a) $g(4) = 4^2(4 - 4) = 0$

(b) $g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2\left(\frac{3}{2} - 4\right) = \frac{9}{4}\left(-\frac{5}{2}\right) = -\frac{45}{8}$

(c) $g(c) = c^2(c - 4) = c^3 - 4c^2$

(d) $g(t + 4) = (t + 4)^2(t + 4 - 4)$

$= (t + 4)^2 t = t^3 + 8t^2 + 16t$

**6.** (a) $f(\pi) = \sin \pi = 0$

(b) $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

(c) $f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

**8.** $\dfrac{f(x) - f(1)}{x - 1} = \dfrac{3x - 1 - (3 - 1)}{x - 1} = \dfrac{3(x - 1)}{x - 1} = 3, x \neq 1$

**10.** $\dfrac{f(x) - f(1)}{x - 1} = \dfrac{x^3 - x - 0}{x - 1} = \dfrac{x(x + 1)(x - 1)}{x - 1} = x(x + 1), x \neq 1$

**12.** $g(x) = x^2 - 5$

Domain: $(-\infty, \infty)$

Range: $[-5, \infty)$

**14.** $h(t) = \cot t$

Domain: all $t \neq k\pi$, $k$ an integer

Range: $(-\infty, \infty)$

**16.** $g(x) = \dfrac{2}{x - 1}$

Domain: $(-\infty, 1), (1, \infty)$

Range: $(-\infty, 0), (0, \infty)$

**18.** $f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2) = (-2)^2 + 2 = 6$

(b) $f(0) = 0^2 + 2 = 2$

(c) $f(1) = 1^2 + 2 = 3$

(d) $f(s^2 + 2) = 2(s^2 + 2)^2 = 2s^4 + 8s^2 + 10$

(Note: $s^2 + 2 > 1$ for all $s$)

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

**20.** $f(x) = \begin{cases} \sqrt{x + 4}, & x \le 5 \\ (x - 5)^2, & x > 5 \end{cases}$

(a) $f(-3) = \sqrt{-3 + 4} = \sqrt{1} = 1$

(b) $f(0) = \sqrt{0 + 4} = 2$

(c) $f(5) = \sqrt{5 + 4} = 3$

(d) $f(10) = (10 - 5)^2 = 25$

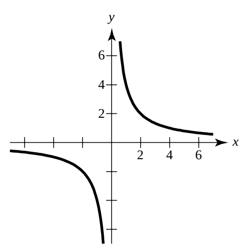
Domain: $[-4, \infty)$

Range: $[0, \infty)$

**22.** $g(x) = \dfrac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$

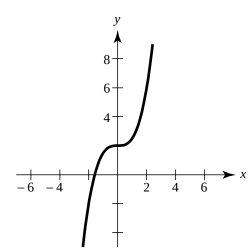
Range: $(-\infty, 0), (0, \infty)$




**24.** $f(x) = \frac{1}{2}x^3 + 2$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$




**26.** $f(x) = x + \sqrt{4 - x^2}$

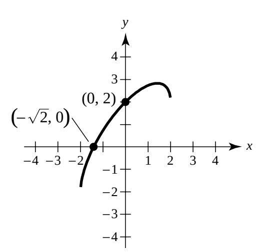
Domain: $[-2, 2]$

Range:

$[-2, 2\sqrt{2}] \approx [-2, 2.83]$

$y$-intercept: $(0, 2)$

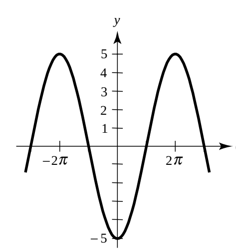
$x$-intercept: $\left(-\sqrt{2}, 0\right)$




**28.** $h(\theta) = -5 \cos \dfrac{\theta}{2}$

Domain: $(-\infty, \infty)$

Range: $[-5, 5]$




**30.** $\sqrt{x^2 - 4} - y = 0 \Rightarrow y = \sqrt{x^2 - 4}$

$y$ is a function of $x$. Vertical lines intersect the graph at most once.

**32.** $x^2 + y^2 = 4$

$y = \pm\sqrt{4 - x^2}$

$y$ is not a function of $x$. Some vertical lines intersect the graph twice.

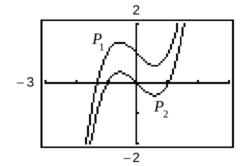
**34.** $x^2 + y = 4 \Rightarrow y = 4 - x^2$

$y$ is a function of $x$ since there is one value of $y$ for each $x$.

**36.** $x^2y - x^2 + 4y = 0 \Rightarrow y = \dfrac{x^2}{x^2 + 4}$

$y$ is a function of $x$ since there is one value of $y$ for each $x$.

**38.** $p_1(x) = x^3 - x + 1$ has one zero. $p_2(x) = x^3 - x$ has three zeros. Every cubic polynomial has at least one zero. Given $p(x) = Ax^3 + Bx^2 + Cx + D$, we have $p \to -\infty$ as $x \to -\infty$ and $p \to \infty$ as $x \to \infty$ if $A > 0$. Furthermore, $p \to \infty$ as $x \to -\infty$ and $p \to -\infty$ as $x \to \infty$ if $A < 0$. Since the graph has no breaks, the graph must cross the $x$-axis at least one time.

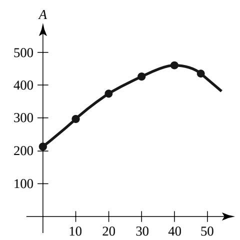



**40.** The function is $f(x) = cx$. Since $(1, 1/4)$ satisfies the equation, $c = 1/4$. Thus, $f(x) = (1/4)x$.

**42.** The function is $h(x) = c\sqrt{|x|}$. Since $(1, 3)$ satisfies the equation, $c = 3$. Thus, $h(x) = 3\sqrt{|x|}$.

**44.** The student travels $\dfrac{2-0}{4-0} = \dfrac{1}{2}$ mi/min during the first 4 minutes. The student is stationary for the following 2 minutes. Finally, the student travels $\dfrac{6-2}{10-6} = 1$ mi/min during the final 4 minutes.

**46.** (a)



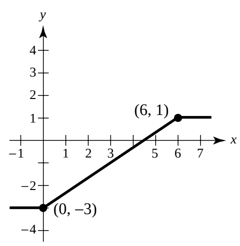
(b) $A(15) \approx 345$ acres/farm

**48.** (a) $g(x) = f(x - 4)$

$g(6) = f(2) = 1$

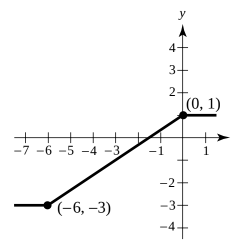
$g(0) = f(-4) = -3$

Shift $f$ right 4 units



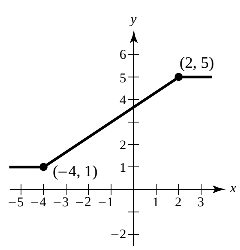

(b) $g(x) = f(x + 2)$

Shift $f$ left 2 units



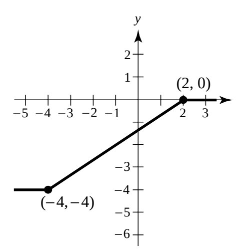

(c) $g(x) = f(x) + 4$

Vertical shift upwards 4 units




(d) $g(x) = f(x) - 1$

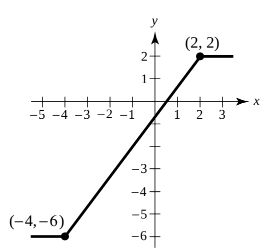
Vertical shift down 1 unit




(e) $g(x) = 2f(x)$

$g(2) = 2f(2) = 2$

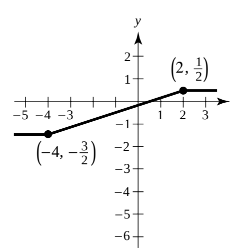
$g(-4) = 2f(-4) = -6$




(f) $g(x) = \frac{1}{2}f(x)$

$g(2) = \frac{1}{2}f(2) = \frac{1}{2}$

$g(-4) = \frac{1}{2}f(-4) = -\frac{3}{2}$




**50.** (a) $h(x) = \sin(x + (\pi/2)) + 1$ is a horizontal shift $\pi/2$ units to the left, followed by a vertical shift 1 unit upwards.

(b) $h(x) = -\sin(x - 1)$ is a horizontal shift 1 unit to the right followed by a reflection about the $x$-axis.

**52.** (a) $f(g(1)) = f(0) = 0$

(b) $g(f(1)) = g(1) = 0$

(c) $g(f(0)) = g(0) = -1$

(d) $f(g(-4)) = f(15) = \sqrt{15}$

(e) $f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$

(f) $g(f(x)) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 - 1 = x - 1 \quad (x \geq 0)$

**54.** $f(x) = x^2 - 1, g(x) = \cos x$

$(f \circ g)(x) = f(g(x)) = f(\cos x) = \cos^2 x - 1$

Domain: $(-\infty, \infty)$

$(g \circ f)(x) = g(x^2 - 1) = \cos(x^2 - 1)$

Domain: $(-\infty, \infty)$

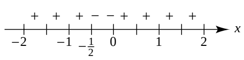
No, $f \circ g \neq g \circ f$.

**56.** $(f \circ g)(x) = f\left(\sqrt{x+2}\right) = \dfrac{1}{\sqrt{x+2}}$

Domain: $(-2, \infty)$

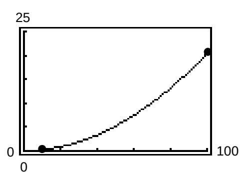
$(g \circ f)(x) = g\left(\dfrac{1}{x}\right) = \sqrt{\dfrac{1}{x} + 2} = \sqrt{\dfrac{1 + 2x}{x}}$

You can find the domain of $g \circ f$ by determining the intervals where $(1 + 2x)$ and $x$ are both positive, or both negative.




Domain: $\left(-\infty, -\frac{1}{2}\right], (0, \infty)$

**58.** (a)




(b) $H(1.6x) = 0.002(1.6x)^2 + 0.005(1.6x) - 0.029$

$= 0.00512x^2 + 0.008x - 0.029$

**60.** $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$

Odd

**62.** $f(-x) = \sin^2(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x$

Even

**64.** (a) If $f$ is even, then $(-4, 9)$ is on the graph.

(b) If $f$ is odd, then $(-4, -9)$ is on the graph.

**66.** $f(-x) = a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + \cdots + a_2(-x)^2 + a_0$

$= a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$

$= f(x)$

Even

**68.** Let $F(x) = f(x)g(x)$ where $f$ is even and $g$ is odd. Then

$F(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -F(x).$

Thus, $F(x)$ is odd.

**70.** (a) Let $F(x) = f(x) \pm g(x)$ where $f$ and $g$ are even. Then, $F(-x) = f(-x) \pm g(-x) = f(x) \pm g(x) = F(x)$. Thus, $F(x)$ is even.

(b) Let $F(x) = f(x) \pm g(x)$ where f and g are odd. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \mp g(x) = -F(x)$. Thus, $F(x)$ is odd.

(c) Let $F(x) = f(x) \pm g(x)$ where f is odd and g is even. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \pm g(x)$. Thus, $F(x)$ is neither odd nor even.

**72.** By equating slopes, $\dfrac{y-2}{0-3} = \dfrac{0-2}{x-3}$

$$y - 2 = \frac{6}{x-3}$$

$$y = \frac{6}{x-3} + 2 = \frac{2x}{x-3},$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{2x}{x-3}\right)^2}.$$

**74.** True

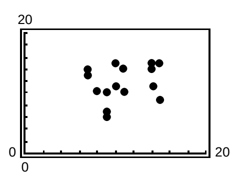
**76.** False; let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. Thus, $3f(x) \neq f(3x)$.

## Section P.4 Fitting Models to Data

**2.** Trigonometric function

**4.** No relationship

**6.** (a)

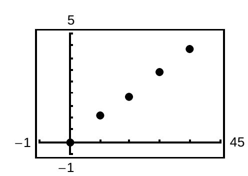



No, the relationship does not appear to be linear.

(b) Quiz scores are dependent on several variables such as study time, class attendance, etc. These variables may change from one quiz to the next.

**8.** (a) $s = 9.7t + 0.4$

(b)

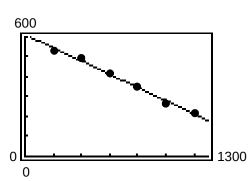



The model fits well.

(c) If $t = 2.5$, $s = 24.65$ meters/second.

**10.** (a) Linear model: $H = -0.3323t + 612.9333$

(b)



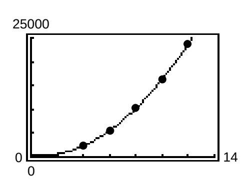

The fit is very good.

(c) When $t = 500$,

$$H = -0.3323(500) + 612.9333 \approx 446.78.$$

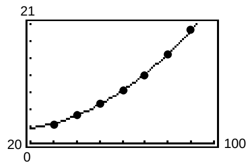
**12.** (a) $S = 180.89x^2 - 205.79x + 272$

(b)



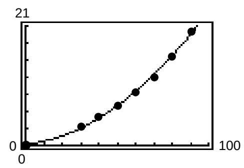

(c) When $x = 2$, $S \approx 583.98$ pounds.

**14.** (a) $t = 0.00271s^2 - 0.0529s + 2.671$

(b) 

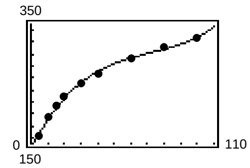

(c) The curve levels off for $s < 20$.

(d) $t = 0.002s^2 + 0.0346s + 0.183$




The model is better for low speeds.

**16.** (a) $T = 2.9856 \times 10^{-4} p^3 - 0.0641 p^2 + 5.2826p + 143.1$

(b) 

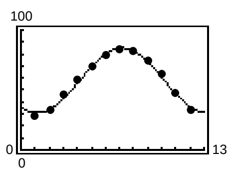

(c) For $T = 300°\text{F}$, $p \approx 68.29$ pounds per square inch.

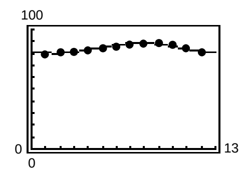
(d) The model is based on data up to 100 pounds per square inch.

**18.** (a) $H(t) = 84.4 + 4.28 \sin\left(\dfrac{\pi t}{6} + 3.86\right)$

One model is

$$C(t) = 58 + 27 \sin\left(\frac{\pi t}{6} + 4.1\right).$$

(b) 


(c) 


(d) The average in Honolulu is 84.4.

The average in Chicago is 58.

(e) The period is 12 months (1 year).

(f) Chicago has greater variability $(27 > 4.28)$.

**20.** Answers will vary.

## Review Exercises for Chapter P

**2.** $y = (x - 1)(x - 3)$

$x = 0 \Longrightarrow y = (0 - 1)(0 - 3) = 3 \Longrightarrow (0, 3)$ $y$-intercept

$y = 0 \Longrightarrow 0 = (x - 1)(x - 3) \Longrightarrow x = 1, 3 \Longrightarrow (1, 0), (3, 0)$ $x$-intercepts

**4.** $xy = 4$

$x = 0$ and $y = 0$ are both impossible. No intercepts.

**6.** Symmetric with respect to $y$-axis since

$y = (-x)^4 - (-x)^2 + 3$

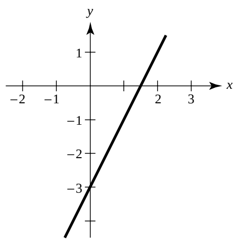
$y = x^4 - x^2 + 3.$

**8.** $4x - 2y = 6$

$$y = 2x - 3$$

Slope: $2$

$y$-intercept: $-3$



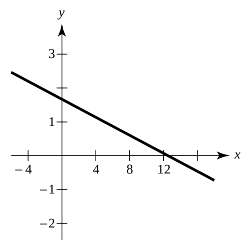
**10.** $0.02x + 0.15y = 0.25$

$$2x + 15y = 25$$

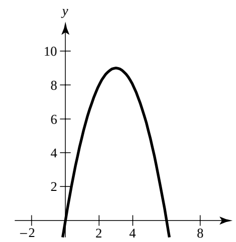
$$y = -\frac{2}{15}x + \frac{5}{3}$$

Slope: $-\frac{2}{15}$

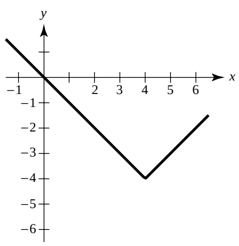
$y$-intercept: $\frac{5}{3}$



**12.** $y = x(6 - x)$



**14.** $y = |x - 4| - 4$



**16.** $y = 8\sqrt[3]{x - 6}$

Xmin = -40
Xmax = 40
Xscl = 10
Ymin = -40
Ymax = 40
Yscl = 10

**18.**

$$y = x + 1$$
$$(x + 1) - x^2 = 7$$
$$0 = x^2 - x + 6$$

No real solution
No points of intersection
The graphs of $y = x + 1$ and $y = x^2 + 7$ do not intersect.

**20.** $y = kx^3$

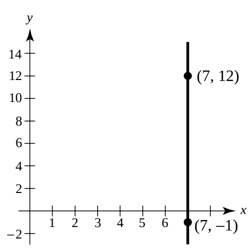
(a) $4 = k(1)^3 \Longrightarrow k = 4$ and $y = 4x^3$

(b) $1 = k(-2)^3 \Longrightarrow k = -\frac{1}{8}$ and $y = -\frac{1}{8}x^3$

(c) $0 = k(0)^3 \Longrightarrow$ any $k$ will do!

(d) $-1 = k(-1)^3 \Longrightarrow k = 1 \Longrightarrow y = x^3$

**22.**

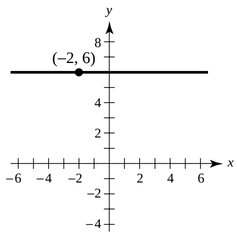



The line is vertical and has no slope.

**24.**
$$\frac{3 - (-1)}{-3 - t} = \frac{3 - 6}{-3 - 8}$$
$$\frac{4}{-3 - t} = \frac{-3}{-11}$$
$$-44 = 9 + 3t$$
$$-53 = 3t$$
$$t = -\frac{53}{3}$$

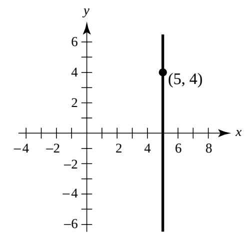
**26.** $y - 6 = 0(x - (-2))$

$y = 6$ Horizontal line




**28.** $m$ is undefined. Line is vertical.

$x = 5$




**30.** (a) $y - 3 = -\frac{2}{3}(x - 1)$

$3y - 9 = -2x + 2$

$2x + 3y - 11 = 0$

(b) Slope of perpendicular line is 1.

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(c) $m = \frac{4 - 3}{2 - 1} = 1$

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(d) $y = 3$

$y - 3 = 0$

**32.** (a) $C = 9.25t + 13.50t + 36{,}500$

$= 22.75t + 36{,}500$

(b) $R = 30t$

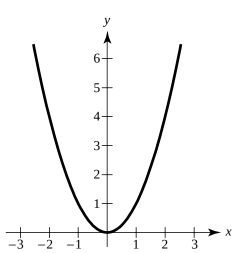
(c) $30t = 22.75t + 36{,}500$

$7.25t = 36{,}000$

$t \approx 5034.48$ hours to break even.

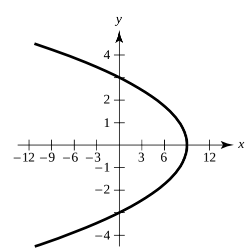
**34.** $x^2 - y = 0$

Function of $x$ since there is one value for $y$ for each $x$.



**36.** $x = 9 - y^2$

Not a function of $x$ since there are two values of $y$ for some $x$.



**38.** (a) $f(-4) = (-4)^2 + 2 = 18$ (because $-4 < 0$)

(b) $f(0) = |0 - 2| = 2$

(c) $f(1) = |1 - 2| = 1$

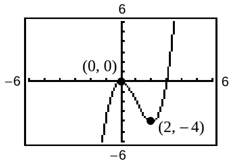
**40.** $f(x) = 1 - x^2$ and $g(x) = 2x + 1$

(a) $f(x) - g(x) = (1 - x^2) - (2x + 1) = -x^2 - 2x$

(b) $f(x)g(x) = (1 - x^2)(2x + 1) = -2x^3 - x^2 + 2x + 1$

(c) $g(f(x)) = g(1 - x^2) = 2(1 - x^2) + 1 = 3 - 2x^2$

**42.** $f(x) = x^3 - 3x^2$



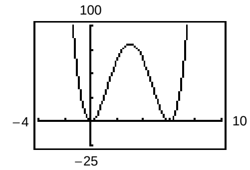

(a) The graph of $g$ is obtained from $f$ by a vertical shift down 1 unit, followed by a reflection in the $x$-axis:

$$g(x) = -[f(x) - 1]$$
$$= -x^3 + 3x^2 + 1$$

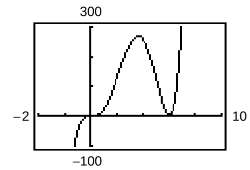
(b) The graph of $g$ is obtained from $f$ by a vertical shift upwards of 1 and a horizontal shift of 2 to the right.

$$g(x) = f(x - 2) + 1$$
$$= (x - 2)^3 - 3(x - 2)^2 + 1$$

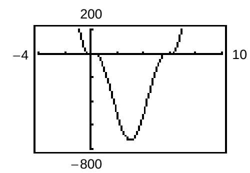
**44.** (a) $f(x) = x^2(x - 6)^2$




(b) $g(x) = x^3(x - 6)^2$



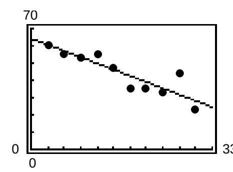

(c) $h(x) = x^3(x - 6)^3$




**46.** For company (a) the profit rose rapidly for the first year, and then leveled off. For the second company (b), the profit dropped, and then rose again later.

**48.** (a) $y = -1.204x + 64.2667$

(b)




(c) The data point $(27, 44)$ is probably an error. Without this point, the new model is

$$y = -1.4344x + 66.4387.$$

## Problem Solving for Chapter P

**2.** Let $y = mx + 1$ be a tangent line to the circle from the point $(0, 1)$. Then

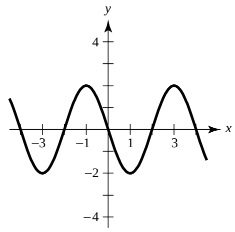
$$x^2 + (y + 1)^2 = 1$$
$$x^2 + (mx + 1 + 1)^2 = 1$$
$$(m^2 + 1)x^2 + 4mx + 3 = 0$$

Setting the discriminant $b^2 - 4ac$ equal to zero,

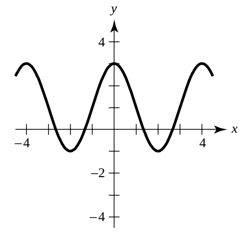
$$16m^2 - 4(m^2 + 1)(3) = 0$$
$$16m^2 - 12m^2 = 12$$
$$4m^2 = 12$$
$$m = \pm\sqrt{3}$$

Tangent lines: $y = \sqrt{3}x + 1$ and $y = -\sqrt{3}x + 1$.

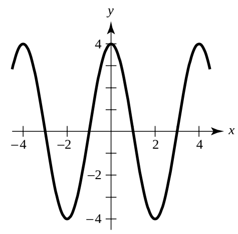
**4.** (a) $f(x + 1)$



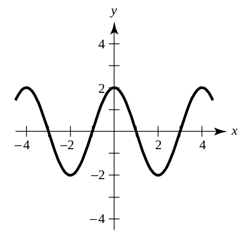

(b) $f(x) + 1$



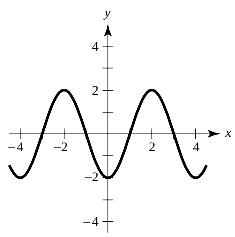

(c) $2f(x)$



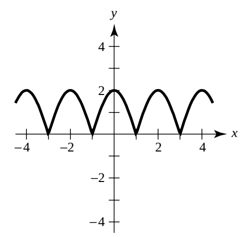

(d) $f(-x)$



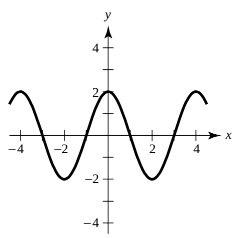

(e) $-f(x)$




(f) $|f(x)|$




(g) $f(|x|)$

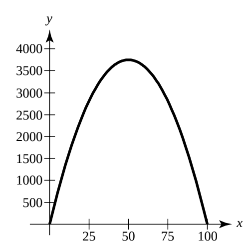



**6.** (a) $4y + 3x = 300 \implies y = \dfrac{300 - 3x}{4}$

$$A(x) = x(2y) = x\left(\frac{300 - 3x}{2}\right) = \frac{-3x^2 + 300x}{2}$$

Domain: $0 < x < 100$

(b)




Maximum of 3750 ft$^2$ at $x = 50$ ft, $y = 37.5$ ft.

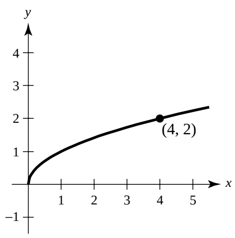
(c)
$$\begin{aligned} A(x) &= -\frac{3}{2}(x^2 - 100x) \\ &= -\frac{3}{2}(x^2 - 100x + 2500) + 3750 \\ &= -\frac{3}{2}(x - 50)^2 + 3750 \end{aligned}$$

$A(50) = 3750$ square feet is the maximum area, where $x = 50$ ft and $y = 37.5$ ft.

**8.** Let $d$ be the distance from the starting point to the beach.

$$\begin{aligned}\text{Average velocity} &= \frac{\text{distance}}{\text{time}}\\ &= \frac{2d}{\dfrac{d}{120}+\dfrac{d}{60}}\\ &= \frac{2}{\dfrac{1}{120}+\dfrac{1}{60}}\\ &= 80 \text{ km/hr}\end{aligned}$$

**10.**




(a) Slope $= \dfrac{3-2}{9-4} = \dfrac{1}{5}$. Slope of tangent line is greater than $\dfrac{1}{5}$.

(b) Slope $= \dfrac{2-1}{4-1} = \dfrac{1}{3}$. Slope of tangent line is less than $\dfrac{1}{3}$.

(c) Slope $= \dfrac{2.1-2}{4.1-4} = \dfrac{10}{41}$. Slope of tangent line is greater than $\dfrac{10}{41}$.

(d) $$\begin{aligned}\text{Slope} &= \frac{f(4+h)-f(4)}{(4+h)-4}\\ &= \frac{\sqrt{4+h}-2}{h}\end{aligned}$$

(e) $$\begin{aligned}\frac{\sqrt{4+h}-2}{h} &= \frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\\ &= \frac{(4+h)-4}{h\left(\sqrt{4+h}+2\right)}\\ &= \frac{1}{\sqrt{4+h}+2},\ h \neq 0\end{aligned}$$

As $h$ gets closer to 0, the slope gets closer to $\dfrac{1}{4}$. The slope is $\dfrac{1}{4}$ at the point $(4, 2)$.

**12.** (a) $$\frac{I}{\sqrt{x^2+y^2}} = \frac{kI}{\sqrt{(x-4)^2+y^2}}$$

$$(x-4)^2 + y^2 = k^2(x^2+y^2)$$

$$(k^2-1)x^2 + (k^2-1)y^2 + 8x = 16$$

If $k = 1$, then $x = 2$ is a vertical line. So, assume $k^2 - 1 \neq 0$. Then

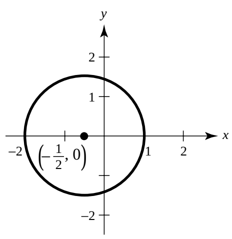
$$x^2 + y^2 + \frac{8x}{k^2-1} = \frac{16}{k^2-1}$$

$$\left(x + \frac{4}{k^2-1}\right)^2 + y^2 = \frac{16}{k^2-1} + \frac{16}{(k^2-1)^2}$$

$$\left(x + \frac{4}{k^2-1}\right)^2 + y^2 = \left(\frac{4k}{k^2-1}\right)^2, \text{ Circle}$$

(b) If $k = 3$, $\left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$

(c) For large $k$, the center of the circle is near $(0, 0)$, and the radius becomes smaller.




**14.** $f(x) = y = \dfrac{1}{1-x}$

(a) Domain: all $x \neq 1$

Range: all $y \neq 0$

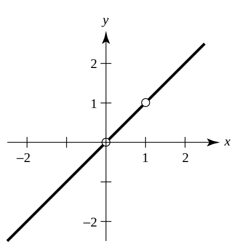
(b) $$f(f(x)) = f\left(\frac{1}{1-x}\right) = \frac{1}{1 - \left(\frac{1}{1-x}\right)} = \frac{1}{\frac{1-x-1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$$

Domain: all $x \neq 0, 1$

(c) $$f(f(f(x))) = f\left(\frac{x-1}{x}\right) = \frac{1}{1 - \left(\frac{x-1}{x}\right)} = \frac{1}{\frac{1}{x}} = x$$

Domain: all $x \neq 0, 1$

(d) The graph is not a line. It has holes at $(0, 0)$ and $(1, 1)$.



# CHAPTER 1
## Limits and Their Properties

# C H A P T E R 1
# Limits and Their Properties

## Section 1.1 A Preview of Calculus

Solutions to Even-Numbered Exercises

**2.** Calculus: velocity is not constant

Distance $\approx$ (20 ft/sec)(15 seconds) $=$ 300 feet

**4.** Precalculus: rate of change $=$ slope $=$ 0.08

**6.** Precalculus: Area $= \pi\left(\sqrt{2}\right)^2$

$= 2\pi$

**8.** Precalculus: Volume $= \pi(3)^2 6 = 54\pi$

**10.** (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

## Section 1.2 Finding Limits Graphically and Numerically

**2.**

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.2564 | 0.2506 | 0.2501 | 0.2499 | 0.2494 | 0.2439 |

$\lim_{x\to 2} \frac{x-2}{x^2-4} \approx 0.25$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

**4.**

| $x$ | $-3.1$ | $-3.01$ | $-3.001$ | $-2.999$ | $-2.99$ | $-2.9$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.2485$ | $-0.2498$ | $-0.2500$ | $-0.2500$ | $-0.2502$ | $-0.2516$ |

$\lim_{x\to -3} \frac{\sqrt{1-x}-2}{x+3} \approx -0.25$ $\left(\text{Actual limit is } -\frac{1}{4}.\right)$

**6.**

| $x$ | 3.9 | 3.99 | 3.999 | 4.001 | 4.01 | 4.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.0408 | 0.0401 | 0.0400 | 0.0400 | 0.0399 | 0.0392 |

$\lim_{x\to 4} \frac{[x/(x+1)]-(4/5)}{x-4} \approx 0.04$ $\left(\text{Actual limit is } \frac{1}{25}.\right)$

**8.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.0500 | 0.0050 | 0.0005 | $-0.0005$ | $-0.0050$ | $-0.0500$ |

$\lim_{x\to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

**10.** $\lim_{x\to 1} (x^2 + 2) = 3$

**12.** $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^2 + 2) = 3$

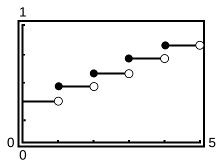
**14.** $\lim_{x\to 3} \dfrac{1}{x-3}$ does not exist since the function increases and decreases without bound as $x$ approaches 3.

**16.** $\lim_{x\to 0} \sec x = 1$

**18.** $\lim_{x\to 1} \sin(\pi x) = 0$

**20.** $C(t) = 0.35 - 0.12[\![-(t-1)]\!]$

(a)




(b)

| $t$ | 3 | 3.3 | 3.4 | 3.5 | 3.6 | 3.7 | 4 |
|---|---|---|---|---|---|---|---|
| $C(t)$ | 0.59 | 0.71 | 0.71 | 0.71 | 0.71 | 0.71 | 0.71 |

$\lim_{t\to 3.5} C(t) = 0.71$

(c)

| $t$ | 3 | 2.5 | 2.9 | 3 | 3.1 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $C(t)$ | 0.47 | 0.59 | 0.59 | 0.59 | 0.71 | 0.71 | 0.71 |

$\lim_{t\to 3.5} C(t)$ does not exist. The values of $C$ jump from 0.59 to 0.71 at $t = 3$.

**22.** You need to find $\delta$ such that $0 < |x-2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$\begin{aligned} -0.2 &< x^2 - 4 < 0.2 \\ 4 - 0.2 &< \quad x^2 \quad < 4 + 0.2 \\ 3.8 &< \quad x^2 \quad < 4.2 \\ \sqrt{3.8} &< \quad x \quad < \sqrt{4.2} \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2 \end{aligned}$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$-\left(\sqrt{4.2} - 2\right) < x - 2 < \sqrt{4.2} - 2$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < \epsilon = 0.2.$$

**24.** $\lim_{x\to 4} \left(4 - \dfrac{x}{2}\right) = 2$

$$\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$$
$$\left|2 - \frac{x}{2}\right| < 0.01$$
$$\left|-\frac{1}{2}(x - 4)\right| < 0.01$$
$$0 < |x - 4| < 0.02 = \delta$$

Hence, if $0 < |x - 4| < \delta = 0.02$, you have

$$\left|-\frac{1}{2}(x - 4)\right| < 0.01$$
$$\left|2 - \frac{x}{2}\right| < 0.01$$
$$\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$$
$$|f(x) - L| < 0.01$$

**26.** $\lim_{x\to 5}(x^2+4)=29$

$$|(x^2+4)-29|<0.01$$
$$|x^2-25|<0.01$$
$$|(x+5)(x-5)|<0.01$$
$$|x-5|<\frac{0.01}{|x+5|}$$

If we assume $4<x<6$, then $\delta=0.01/11\approx 0.0009$.

Hence, if $0<|x-5|<\delta=\dfrac{0.01}{11}$, you have

$$|x-5|<\frac{0.01}{11}<\frac{1}{|x+5|}(0.01)$$
$$|x-5||x+5|<0.01$$
$$|x^2-25|<0.01$$
$$|(x^2+4)-29|<0.01$$
$$|f(x)-L|<0.01$$

**28.** $\lim_{x\to -3}(2x+5)=-1$

Given $\epsilon>0$:

$$|(2x+5)-(-1)|<\epsilon$$
$$|2x+6|<\epsilon$$
$$2|x+3|<\epsilon$$
$$|x+3|<\frac{\epsilon}{2}=\delta$$

Hence, let $\delta=\epsilon/2$.

Hence, if $0<|x+3|<\delta=\dfrac{\epsilon}{2}$, you have

$$|x+3|<\frac{\epsilon}{2}$$
$$|2x+6|<\epsilon$$
$$|(2x+5)-(-1)|<\epsilon$$
$$|f(x)-L|<\epsilon$$

**30.** $\lim_{x\to 1}\left(\frac{2}{3}x+9\right)=\frac{2}{3}(1)+9=\frac{29}{3}$

Given $\epsilon>0$:

$$\left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right|<\epsilon$$
$$\left|\tfrac{2}{3}x-\tfrac{2}{3}\right|<\epsilon$$
$$\tfrac{2}{3}|x-1|<\epsilon$$
$$|x-1|<\tfrac{3}{2}\epsilon$$

Hence, let $\delta=(3/2)\epsilon$.

Hence, if $0<|x-1|<\delta=\frac{3}{2}\epsilon$, you have

$$|x-1|<\tfrac{3}{2}\epsilon$$
$$\left|\tfrac{2}{3}x-\tfrac{2}{3}\right|<\epsilon$$
$$\left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right|<\epsilon$$
$$|f(x)-L|<\epsilon$$

**32.** $\lim_{x\to 2}(-1)=-1$

Given $\epsilon>0$: $|-1-(-1)|<\epsilon$

$$0<\epsilon$$

Hence, any $\delta>0$ will work.

Hence, for any $\delta>0$, you have

$$|(-1)-(-1)|<\epsilon$$
$$|f(x)-L|<\epsilon$$

**34.** $\lim_{x\to 4}\sqrt{x}=\sqrt{4}=2$

Given $\epsilon>0$: $\quad\left|\sqrt{x}-2\right|<\epsilon$

$$\left|\sqrt{x}-2\right|\left|\sqrt{x}+2\right|<\epsilon\left|\sqrt{x}+2\right|$$
$$|x-4|<\epsilon\left|\sqrt{x}+2\right|$$

Assuming $1<x<9$, you can choose $\delta=3\epsilon$. Then,

$$0<|x-4|<\delta=3\epsilon\implies|x-4|<\epsilon\left|\sqrt{x}+2\right|$$
$$\implies\left|\sqrt{x}-2\right|<\epsilon.$$

**36.** $\lim_{x\to 3}|x-3|=0$

Given $\epsilon>0$:

$$|(x-3)-0|<\epsilon$$
$$|x-3|<\epsilon=\delta$$

Hence, let $\delta=\epsilon$.

Hence for $0<|x-3|<\delta=\epsilon$, you have

$$|x-3|<\epsilon$$
$$||x-3|-0|<\epsilon$$
$$|f(x)-L|<\epsilon$$

**38.** $\lim_{x\to-3}(x^2+3x)=0$

Given $\epsilon > 0$:

$$|(x^2+3x)-0| < \epsilon$$
$$|x(x+3)| < \epsilon$$
$$|x+3| < \frac{\epsilon}{|x|}$$

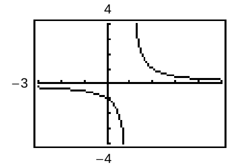
If we assume $-4 < x < -2$, then $\delta = \epsilon/4$.

Hence for $0 < |x-(-3)| < \delta = \dfrac{\epsilon}{4}$, you have

$$|x+3| < \frac{1}{4}\epsilon < \frac{1}{|x|}\epsilon$$
$$|x(x+3)| < \epsilon$$
$$|x^2+3x-0| < \epsilon$$
$$|f(x)-L| < \epsilon$$

**40.** $f(x)=\dfrac{x-3}{x^2-4x+3}$

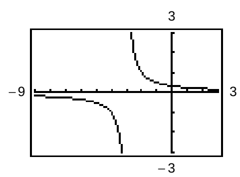
$\lim_{x\to3} f(x) = \dfrac{1}{2}$




The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$.

**42.** $f(x)=\dfrac{x-3}{x^2-9}$

$\lim_{x\to3} f(x) = \dfrac{1}{6}$




The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{6}\right)$.

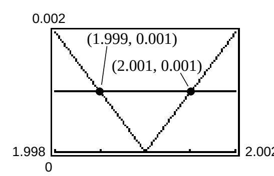
**44.** (a) No. The fact that $f(2)=4$ has no bearing on the existence of the limit of $f(x)$ as $x$ approaches 2.

(b) No. The fact that $\lim_{x\to2} f(x) = 4$ has no bearing on the value of $f$ at 2.

**46.** Let $p(x)$ be the atmospheric pressure in a plane at altitude $x$ (in feet).

$$\lim_{x\to0^+} p(x) = 14.7 \text{ lb/in}^2$$

**48.**




Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x-2| < 0.001, \left|\frac{x^2-4}{x-2}-4\right| < 0.001.$$

**50.** True

**52.** False; let

$$f(x)=\begin{cases} x^2-4x, & x\neq 4 \\ 10, & x=4 \end{cases}.$$

$\lim_{x\to4} f(x) = \lim_{x\to4}(x^2-4x)=0$ and $f(4)=10\neq 0$

**54.** $\lim_{x\to4}\dfrac{x^2-x-12}{x-4}=7$

| $n$ | $4+[0.1]^n$ | $f(4+[0.1]^n)$ |
|---|---|---|
| 1 | 4.1 | 7.1 |
| 2 | 4.01 | 7.01 |
| 3 | 4.001 | 7.001 |
| 4 | 4.0001 | 7.0001 |

| $n$ | $4-[0.1]^n$ | $f(4-[0.1]^n)$ |
|---|---|---|
| 1 | 3.9 | 6.9 |
| 2 | 3.99 | 6.99 |
| 3 | 3.999 | 6.999 |
| 4 | 3.9999 | 6.9999 |

**56.** $f(x) = mx + b, m \neq 0$. Let $\epsilon > 0$ be given. Take $\delta = \dfrac{\epsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\epsilon}{|m|}$, then

$$|m||x - c| < \epsilon$$
$$|mx - mc| < \epsilon$$
$$|(mx + b) - (mc + b)| < \epsilon$$

which shows that $\lim\limits_{x \to c} (mx + b) = mc + b$.

**58.** $\lim\limits_{xc} g(x) = L, L > 0$. Let $\epsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $|g(x) - L| < \epsilon = \frac{1}{2}L$.

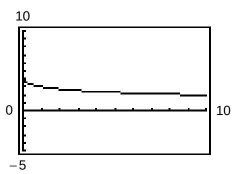
That is,

$$-\tfrac{1}{2}L < g(x) - L < \tfrac{1}{2}L$$
$$\tfrac{1}{2}L < \quad g(x) \quad < \tfrac{3}{2}L$$

Hence for $x$ in the interval $(c - \delta, c + \delta), x \neq c$,

$$g(x) > \tfrac{1}{2}L > 0.$$

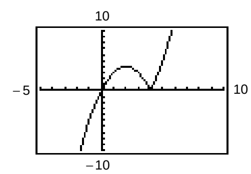
## Section 1.3 Evaluating Limits Analytically

**2.** 


$g(x) = \dfrac{12\left(\sqrt{x} - 3\right)}{x - 9}$

(a) $\lim\limits_{x \to 4} g(x) = 2.4$

(b) $\lim\limits_{x \to 0} g(x) = 4$

**4.** 


$f(t) = t|t - 4|$

(a) $\lim\limits_{t \to 4} f(t) = 0$

(b) $\lim\limits_{t \to -1} f(t) = -5$

**6.** $\lim\limits_{x \to -2} x^3 = (-2)^3 = -8$

**8.** $\lim\limits_{x \to -3} (3x + 2) = 3(-3) + 2 = -7$

**10.** $\lim\limits_{x \to 1} (-x^2 + 1) = -(1)^2 + 1 = 0$

**12.** $\lim\limits_{x \to 1} (3x^3 - 2x^2 + 4) = 3(1)^3 - 2(1)^2 + 4 = 5$

**14.** $\lim\limits_{x \to -3} \dfrac{2}{x + 2} = \dfrac{2}{-3 + 2} = -2$

**16.** $\lim\limits_{x \to 3} \dfrac{2x - 3}{x + 5} = \dfrac{2(3) - 3}{3 + 5} = \dfrac{3}{8}$

**18.** $\lim\limits_{x \to 3} \dfrac{\sqrt{x + 1}}{x - 4} = \dfrac{\sqrt{3 + 1}}{3 - 4} = -2$

**20.** $\lim\limits_{x \to 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = 2$

**22.** $\lim\limits_{x \to 0} (2x - 1)^3 = [2(0) - 1]^3 = -1$

**24.** (a) $\lim\limits_{x \to -3} f(x) = (-3) + 7 = 4$

(b) $\lim\limits_{x \to 4} g(x) = 4^2 = 16$

(c) $\lim\limits_{x \to -3} g(f(x)) = g(4) = 16$

**26.** (a) $\lim\limits_{x \to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim\limits_{x \to 21} g(x) = \sqrt[3]{21 + 6} = 3$

(c) $\lim\limits_{x \to 4} g(f(x)) = g(21) = 3$

**28.** $\lim\limits_{x \to \pi} \tan x = \tan \pi = 0$

**30.** $\lim\limits_{x \to 1} \sin \dfrac{\pi x}{2} = \sin \dfrac{\pi}{2} = 1$

**32.** $\lim\limits_{x \to \pi} \cos 3x = \cos 3\pi = -1$

**34.** $\lim\limits_{x \to 5\pi/3} \cos x = \cos \dfrac{5\pi}{3} = \dfrac{1}{2}$

**36.** $\lim\limits_{x \to 7} \sec\left(\dfrac{\pi x}{6}\right) = \sec \dfrac{7\pi}{6} = \dfrac{-2\sqrt{3}}{3}$

**38.** (a) $\lim_{x\to c}[4f(x)] = 4\lim_{x\to c} f(x) = 4\left(\frac{3}{2}\right) = 6$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \frac{3}{2} + \frac{1}{2} = 2$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{3/2}{1/2} = 3$

**40.** (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c}\frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \frac{27}{18} = \frac{3}{2}$

(c) $\lim_{x\to c}[f(x)]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}[f(x)]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

**42.** $f(x) = x - 3$ and $h(x) = \frac{x^2 - 3x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to -2} h(x) = \lim_{x\to -2} f(x) = -5$

(b) $\lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = -3$

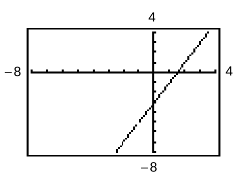
**44.** $g(x) = \frac{1}{x - 1}$ and $f(x) = \frac{x}{x^2 - x}$ agree except at $x = 0$.

(a) $\lim_{x\to 1} f(x)$ does not exist.

(b) $\lim_{x\to 0} f(x) = -1$

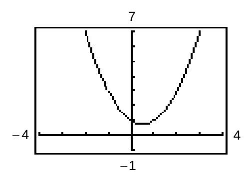
**46.** $f(x) = \frac{2x^2 - x - 3}{x + 1}$ and $g(x) = 2x - 3$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -5$




**48.** $f(x) = \frac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = 3$




**50.** $\lim_{x\to 2}\frac{2 - x}{x^2 - 4} = \lim_{x\to 2}\frac{-(x - 2)}{(x - 2)(x + 2)}$

$= \lim_{x\to 2}\frac{-1}{x + 2} = -\frac{1}{4}$

**52.** $\lim_{x\to 4}\frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x\to 4}\frac{(x - 4)(x - 1)}{(x - 4)(x + 2)}$

$= \lim_{x\to 4}\frac{(x - 1)}{(x + 2)} = \frac{3}{6} = \frac{1}{2}$

**54.** $\lim_{x\to 0}\frac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2 + x} - \sqrt{2}}{x}\cdot\frac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}}$

$= \lim_{x\to 0}\frac{2 + x - 2}{\left(\sqrt{2 + x} + \sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2 + x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

**56.** $\lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3} = \lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3}\cdot\frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x\to 3}\frac{x - 3}{(x - 3)\left[\sqrt{x + 1} + 2\right]} = \lim_{x\to 3}\frac{1}{\sqrt{x + 1} + 2} = \frac{1}{4}$

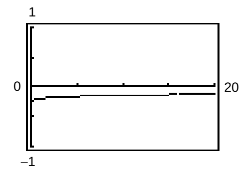
**58.** $\lim_{x\to 0}\frac{\frac{1}{x + 4} - \frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{4 - (x + 4)}{4(x + 4)}}{x} = \lim_{x\to 0}\frac{-1}{4(x + 4)} = -\frac{1}{16}$

**60.** $\lim_{\Delta x\to 0}\frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x) = 2x$

**62.** $\displaystyle \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

**64.** $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$



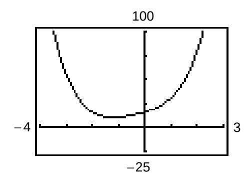

| $x$ | 15.9 | 15.99 | 15.999 | 16 | 16.001 | 16.01 | 16.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-.1252$ | $-.125$ | $-.125$ | ? | $-.125$ | $-.125$ | $-.1248$ |

It appears that the limit is $-0.125$.

Analytically, $\displaystyle \lim_{x \to 16} \frac{4 - \sqrt{x}}{x - 16} = \lim_{x \to 16} \frac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)}$

$$= \lim_{x \to 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.$$

**66.** $\displaystyle \lim_{x \to 2} \frac{x^5 - 32}{x - 2} = 80$




| $x$ | 1.9 | 1.99 | 1.999 | 1.9999 | 2.0 | 2.0001 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 72.39 | 79.20 | 79.92 | 79.99 | ? | 80.01 | 80.08 | 80.80 | 88.41 |

Analytically, $\displaystyle \lim_{x \to 2} \frac{x^5 - 32}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2}$

$$= \lim_{x \to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$$

(*Hint:* Use long division to factor $x^5 - 32$.)

**68.** $\displaystyle \lim_{x \to 0} \frac{3(1 - \cos x)}{x} = \lim_{x \to 0} \left[3\left(\frac{1 - \cos x}{x}\right)\right] = (3)(0) = 0$

**70.** $\displaystyle \lim_{\theta \to 0} \frac{\cos\theta \tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

**72.** $\displaystyle \lim_{x \to 0} \frac{\tan^2 x}{x} = \lim_{x \to 0} \frac{\sin^2 x}{x\cos^2 x} = \lim_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x}\right]$

$$= (1)(0) = 0$$

**74.** $\displaystyle \lim_{\phi \to \pi} \phi \sec\phi = \pi(-1) = -\pi$

**76.** $\displaystyle \lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x} = \lim_{x \to \pi/4} \frac{\cos x - \sin x}{\sin x \cos x - \cos^2 x}$

$$= \lim_{x \to \pi/4} \frac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)}$$

$$= \lim_{x \to \pi/4} \frac{-1}{\cos x}$$

$$= \lim_{x \to \pi/4} (-\sec x)$$

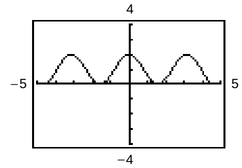
$$= -\sqrt{2}$$

**78.** $\displaystyle \lim_{x \to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \to 0} \left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right] = 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

**80.** $f(h) = (1 + \cos 2h)$

| $h$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(h)$ | 1.98 | 1.9998 | 2 | ? | 2 | 1.9998 | 1.98 |

Analytically, $\lim\limits_{h\to 0} (1 + \cos 2h) = 1 + \cos(0) = 1 + 1 = 2.$

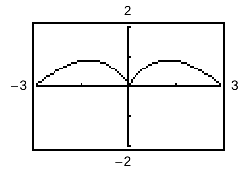



The limit appear to equal 2.

**82.** $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.215 | 0.0464 | 0.01 | ? | 0.01 | 0.0464 | 0.215 |

Analytically, $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}} = \lim\limits_{x\to 0} \sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$




The limit appear to equal 0.

**84.** $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

**86.** $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h}$$

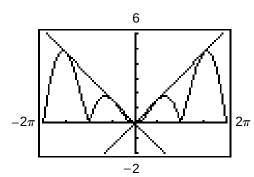
$$= \lim_{h\to 0} \frac{h(2x + h - 4)}{h} = \lim_{h\to 0} (2x + h - 4) = 2x - 4$$

**88.** $$\lim_{x\to a} [b - |x - a|] \le \lim_{x\to a} f(x) \le \lim_{x\to a} [b + |x - a|]$$

$$b \le \lim_{x\to a} f(x) \le b$$

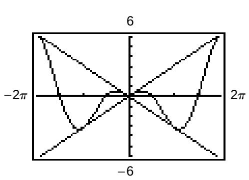
Therefore, $\lim\limits_{x\to a} f(x) = b.$

**90.** $f(x) = |x \sin x|$



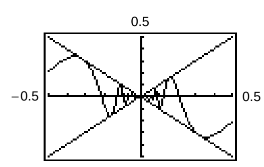

$\lim\limits_{x\to 0} |x \sin x| = 0$

**92.** $f(x) = |x| \cos x$




$\lim\limits_{x\to 0} |x| \cos x = 0$

**94.** $h(x) = x \cos \dfrac{1}{x}$

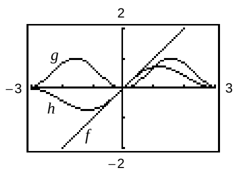



$\lim\limits_{x\to 0} \left(x \cos \dfrac{1}{x}\right) = 0$

**96.** $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

**98.** If a function $f$ is squeezed between two functions $h$ and $g$, $h(x) \le f(x) \le g(x)$, and $h$ and $g$ have the same limit $L$ as $x \to c$, then $\lim\limits_{x\to c} f(x)$ exists and equals $L$.

**100.** $f(x) = x$, $g(x) = \sin^2 x$, $h(x) = \dfrac{\sin^2 x}{x}$




When you are "close to" 0 the magnitude of $g$ is "smaller" than the magnitude of $f$ and the magnitude of $g$ is approaching zero "faster" than the magnitude of $f$.
Thus, $|g|/|f| \approx 0$ when $x$ is "close to" 0

**102.** $s(t) = -16t^2 + 1000 = 0$ when $t = \sqrt{\dfrac{1000}{16}} = \dfrac{5\sqrt{10}}{2}$ seconds

$$\lim_{t\to 5\sqrt{10}/2} \frac{s\left(\frac{5\sqrt{10}}{2}\right) - s(t)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t\to 5\sqrt{10}/2} \frac{0 - (-16t^2 + 1000)}{\frac{5\sqrt{10}}{2} - t}$$

$$= \lim_{t\to 5\sqrt{10}/2} \frac{16\left(t^2 - \frac{125}{2}\right)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t\to 5\sqrt{10}/2} \frac{16\left(t + \frac{5\sqrt{10}}{2}\right)\left(t - \frac{5\sqrt{10}}{2}\right)}{-\left(t - \frac{5\sqrt{10}}{2}\right)}$$

$$= \lim_{t\to 5\sqrt{10}/2} -16\left(t + \frac{5\sqrt{10}}{2}\right) = -80\sqrt{10} \text{ ft/sec} \approx -253 \text{ ft/sec}$$

**104.** $-4.9t^2 + 150 = 0$ when $t = \sqrt{\dfrac{150}{4.9}} = \sqrt{\dfrac{1500}{49}} \approx 5.53$ seconds.

The velocity at time $t = a$ is

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t} = \lim_{t\to a} \frac{(-4.9a^2 + 150) - (-4.9t^2 + 150)}{a - t} = \lim_{t\to a} \frac{-4.9(a - t)(a + t)}{a - t}$$

$$= \lim_{t\to a} -4.9(a + t) = -2a(4.9) = -9.8a \text{ m/sec.}$$

Hence, if $a = \sqrt{1500/49}$, the velocity is $-9.8\sqrt{1500/49} \approx -54.2$ m/sec.

**106.** Suppose, on the contrary, that $\lim\limits_{x\to c} g(x)$ exists. Then, since $\lim\limits_{x\to c} f(x)$ exists, so would $\lim\limits_{x\to c} [f(x) + g(x)]$, which is a contradiction. Hence, $\lim\limits_{x\to c} g(x)$ does not exist.

**108.** Given $f(x) = x^n$, $n$ is a positive integer, then

$$\lim_{x\to c} x^n = \lim_{x\to c} (xx^{n-1}) = \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right]$$

$$= c\left[\lim_{x\to c} (xx^{n-2})\right] = c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right]$$

$$= c(c)\lim_{x\to c} (xx^{n-3}) = \cdots = c^n.$$

**110.** Given $\lim\limits_{x\to c} f(x) = 0$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \epsilon$ for $|x - c| < \delta$. Therefore, $\lim\limits_{x\to c} |f(x)| = 0$.

**112.** (a) If $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} [-|f(x)|] = 0$.

$$-|f(x)| \le f(x) \le |f(x)|$$

$$\lim_{x\to c} [-|f(x)|] \le \lim_{x\to c} f(x) \le \lim_{x\to c} |f(x)|$$

$$0 \le \lim_{x\to c} f(x) \le 0$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. Since $||f(x)| - |L|| \le |f(x) - L| < \epsilon$ for $|x - c| < \delta$, then $\lim_{x\to c} |f(x)| = |L|$.

**114.** True. $\lim_{x\to 0} x^3 = 0^3 = 0$

**116.** False. Let $f(x) = \begin{cases} x & x \ne 1 \\ 3 & x = 1 \end{cases}$, $c = 1$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \ne 1$.

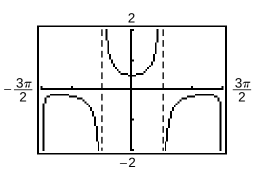
**118.** False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all $x \ne 0$. But $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

**120.**
$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$$
$$= \lim_{x\to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$
$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x}$$
$$= \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{x\to 0} \frac{\sin x}{1 + \cos x}\right]$$
$$= (1)(0) = 0$$

**122.** $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of $f$ is all $x \ne 0$, $\pi/2 + n\pi$.

(b)




The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}$$
$$= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

Hence, $\displaystyle\lim_{x\to 0} \frac{\sec x - 1}{x^2} = \lim_{x\to 0} \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$
$$= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

**124.** The calculator was set in degree mode, instead of radian mode.

## Section 1.4 Continuity and One-Sided Limits

**2.** (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

The function is continuous at $x = -2$.

**4.** (a) $\lim_{x\to -2^+} f(x) = 2$

(b) $\lim_{x\to -2^-} f(x) = 2$

(c) $\lim_{x\to -2} f(x) = 2$

The function is NOT continuous at $x = -2$.

**6.** (a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to -1^-} f(x) = 2$

(c) $\lim_{x\to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

**8.** $\lim_{x\to 2^+} \dfrac{2-x}{x^2-4} = \lim_{x\to 2^+} -\dfrac{1}{x+2} = -\dfrac{1}{4}$

**10.** 
$$\begin{aligned}\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}\\ &= \lim_{x\to 4^-} \frac{x-4}{(x-4)\left(\sqrt{x}+2\right)}\\ &= \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4}\end{aligned}$$

**12.** $\lim_{x\to 2^+} \dfrac{|x-2|}{x-2} = \lim_{x\to 2^+} \dfrac{x-2}{x-2} = 1$

**14.**
$$\begin{aligned}\lim_{\Delta x\to 0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} &= \lim_{\Delta x\to 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}\\ &= \lim_{\Delta x\to 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}\\ &= \lim_{\Delta x\to 0^+} (2x + \Delta x + 1)\\ &= 2x + 0 + 1 = 2x + 1\end{aligned}$$

**16.** $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-x^2 + 4x - 2) = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 4x + 6) = 2$

$\lim_{x\to 2} f(x) = 2$

**18.** $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x) = 0$

**20.** $\lim_{x\to \pi/2} \sec x$ does not exist since

$\lim_{x\to (\pi/2)^+} \sec x$ and $\lim_{x\to (\pi/2)^-} \sec x$ do not exist.

**22.** $\lim_{x\to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

**24.** $\lim_{x\to 1} \left(1 - \left[\!\left[-\dfrac{x}{2}\right]\!\right]\right) = 1 - (-1) = 2$

**26.** $f(x) = \dfrac{x^2-1}{x+1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

**28.** $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x-1, & x > 1 \end{cases}$ has discontinuity at $x = 1$ since $f(1) = 2 \neq \lim_{x\to 1} f(x) = 1$.

**30.** $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

**32.** $g(2)$ is not defined. $g$ is continuous on $[-1, 2)$.

**34.** $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real $x$.

**36.** $f(x) = \cos \dfrac{\pi x}{2}$ is continuous for all real $x$.

**38.** $f(x) = \dfrac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$ do not exist.

**40.** $f(x) = \dfrac{x - 3}{x^2 - 9}$ has a nonremovable discontinuity at $x = -3$ since $\lim_{x\to -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{1}{x + 3} = \frac{1}{6}.$$

**42.** $f(x) = \dfrac{x - 1}{(x + 2)(x - 1)}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x + 2} = \frac{1}{3}.$$

**44.** $f(x) = \dfrac{|x - 3|}{x - 3}$

has a nonremovable discontinuity at $x = 3$ since $\lim_{x\to 3} f(x)$ does not exist.

**46.** $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

**1.** $f(1) = 1^2 = 1$

**2.** $\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (-2x + 3) = 1 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} x^2 = 1 \end{aligned}\right\} \lim_{x\to 1} f(x) = 1$

**3.** $f(1) = \lim_{x\to 1} f(x)$

$f$ is continuous at $x = 1$, therefore, $f$ is continuous for all real $x$.

**48.** $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

**1.** $f(2) = -2(2) = -4$

**2.** $\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (-2x) = -4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (x^2 - 4x + 1) = -3 \end{aligned}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, $f$ has a nonremovable discontinuity at $x = 2$.

**50.** $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases} = \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$ has **possible** discontinuities at $x = 1$, $x = 5$.

**1.** $f(1) = \csc \dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc \dfrac{5\pi}{6} = 2$

**2.** $\lim_{x\to 1} f(x) = 2$ $\qquad \lim_{x\to 5} f(x) = 2$

**3.** $f(1) = \lim_{x\to 1} f(x)$ $\qquad f(5) = \lim_{x\to 5} f(x)$

$f$ is continuous at $x = 1$ and $x = 5$, therefore, $f$ is continuous for all real $x$.

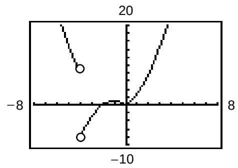
**52.** $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, $k$ is an integer.

**54.** $f(x) = 3 - [\![x]\!]$ has nonremovable discontinuities at each integer $k$.

**56.** $\lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0^-} f(x) = 0$

$f$ is not continuous at $x = -4$




**58.** $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} \dfrac{4 \sin x}{x} = 4$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} (a - 2x) = a$

Let $a = 4$.

**60.** $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim\limits_{x \to a} (x + a) = 2a$

Find $a$ such that $2a = 8 \implies a = 4$.

**62.** $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

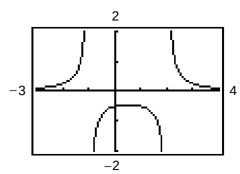
Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.
Because $f \circ g$ is not defined for $x < 1$, it is better to say that $f \circ g$ is discontinuous from the right at $x = 1$.

**64.** $f(g(x)) = \sin x^2$

Continuous for all real $x$

**66.** $h(x) = \dfrac{1}{(x + 1)(x - 2)}$

Nonremovable discontinuity at $x = -1$ and $x = 2$.

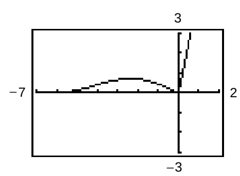



**68.** $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (5x) = 0$




Therefore, $\lim\limits_{x \to 0} f(x) = 0 = f(0)$ and $f$ is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)

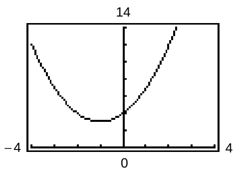
**70.** $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty]$

**72.** $f(x) = \dfrac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

**74.** $f(x) = \dfrac{x^3 - 8}{x - 2}$




The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that $f$ has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

**76.** $f(x) = x^3 + 3x - 2$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of $c$ between 0 and 1.

**78.** $f(x) = \dfrac{-4}{x} + \tan\dfrac{\pi x}{8}$ is continuous on $[1, 3]$.

$f(1) = -4 + \tan\dfrac{\pi}{8} < 0$ and $f(3) = -\dfrac{4}{3} + \tan\dfrac{3\pi}{8} > 0.$

By the Intermediate Value Theorem, $f(1) = 0$ for at least one value of $c$ between 1 and 3.

**80.** $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

**82.** $h(\theta) = 1 + \theta - 3\tan\theta$

$h$ is continuous on $[0, 1]$.

$h(0) = 1 > 0$ and $h(1) \approx -2.67 < 0.$

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value $\theta$ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

**84.** $f(x) = x^2 - 6x + 8$

$f$ is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

Thus, $f(2) = 0$.

**86.** $f(x) = \dfrac{x^2 + x}{x - 1}$

$f$ is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\dfrac{5}{2}\right) = \dfrac{35}{6}$ and $f(4) = \dfrac{20}{3}$

$$\frac{35}{6} < 6 < \frac{20}{3}$$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

$c = 3$ ($x = 2$ is not in the interval.)

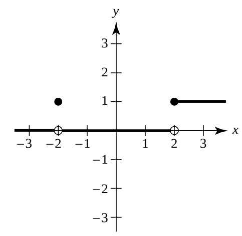
Thus, $f(3) = 6$.

**88.** A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$. Answers will vary.

(a) $f(x) = \dfrac{|x-2|}{x-2}$

(b) $f(x) = \dfrac{\sin(x+2)}{x+2}$

(c) $f(x) = \begin{cases} 1, & \text{if } x \geq 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$



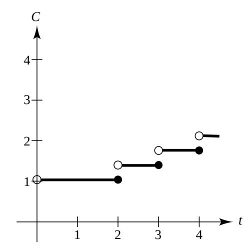

**90.** If $f$ and $g$ are continuous for all real $x$, then so is $f + g$ (Theorem 1.11, part 2). However, $f/g$ might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then $f$ and $g$ are continuous for all real $x$, but $f/g$ is not continuous at $x = \pm 1$.

**92.** $C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 + 0.36[\![t-1]\!], & t > 2,\ t \text{ is not an integer} \\ 1.04 + 0.36(t-2), & t > 2,\ t \text{ is an integer} \end{cases}$

Nonremovable discontinuity at each integer greater than 2.

You can also write $C$ as

$$C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 - 0.36[\![2-t]\!], & t > 2 \end{cases}.$$




**94.** Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

Since $f(0) < 0$ and $f(10) > 0$, then there must be a value $t$ in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time $t$, where $0 \leq t \leq 10$, the position functions for the run up and the run down are equal.

**96.** Suppose there exists $x_1$ in $[a, b]$ such that $f(x_1) > 0$ and there exists $x_2$ in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of $x$ in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, $f$ would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all $x$ in $[a, b]$ or $f(x) < 0$ for all $x$ in $[a, b]$.

**98.** If $x = 0$, then $f(0) = 0$ and $\lim_{x\to 0} f(x) = 0$. Hence, $f$ is continuous at $x = 0$.

If $x \neq 0$, then $\lim_{t\to x} f(t) = 0$ for $x$ rational, whereas $\lim_{t\to x} f(t) = \lim_{t\to x} kt = kx \neq 0$ for $x$ irrational. Hence, $f$ is not continuous for all $x \neq 0$.

**100.** True

**1.** $f(c) = L$ is defined.

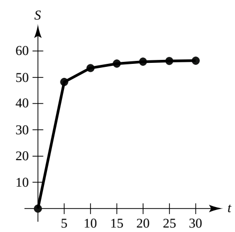
**2.** $\lim_{x\to c} f(x) = L$ exists.

**3.** $f(c) = \lim_{x\to c} f(x)$

All of the conditions for continuity are met.

**102.** False; a rational function can be written as $P(x)/Q(x)$ where $P$ and $Q$ are polynomials of degree $m$ and $n$, respectively. It can have, at most, $n$ discontinuities.

**104.** (a)




(b) There appears to be a limiting speed and a possible cause is air resistance.

**106.** Let $y$ be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists $x$ between 0 and $x_0$ such that $\tan x = y$. The argument is similar if $y < 0$.

**108.** **1.** $f(c)$ is defined.

**2.** $\lim_{x \to c} f(x) = \lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$ exists.

$[\text{Let } x = c + \Delta x. \text{ As } x \to c, \Delta x \to 0]$

**3.** $\lim_{x \to c} f(x) = f(c)$.

Therefore, $f$ is continuous at $x = c$.

**110.** Define $f(x) = f_2(x) - f_1(x)$. Since $f_1$ and $f_2$ are continuous on $[a, b]$, so is $f$.

$$f(a) = f_2(a) - f_1(a) > 0 \quad \text{and} \quad f(b) = f_2(b) - f_1(b) < 0.$$

By the Intermediate Value Theorem, there exists $c$ in $[a, b]$ such that $f(c) = 0$.

$$f(c) = f_2(c) - f_1(c) = 0 \implies f_1(c) = f_2(c)$$

## Section 1.5 Infinite Limits

2. $\displaystyle \lim_{x \to -2^+} \frac{1}{x+2} = \infty$

$\displaystyle \lim_{x \to -2^-} \frac{1}{x+2} = -\infty$

**4.** $\displaystyle \lim_{x \to -2^+} \sec \frac{\pi x}{4} = \infty$

$\displaystyle \lim_{x \to -2^-} \sec \frac{\pi x}{4} = -\infty$

**6.** $f(x) = \dfrac{x}{x^2 - 9}$

| $x$ | $-3.5$ | $-3.1$ | $-3.01$ | $-3.001$ | $-2.999$ | $-2.99$ | $-2.9$ | $-2.5$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1.077$ | $-5.082$ | $-50.08$ | $-500.1$ | 499.9 | 49.92 | 4.915 | 0.9091 |

$\displaystyle \lim_{x \to -3^-} f(x) = -\infty$

$\displaystyle \lim_{x \to -3^+} f(x) = \infty$

**8.** $f(x) = \sec \dfrac{\pi x}{6}$

| $x$ | $-3.5$ | $-3.1$ | $-3.01$ | $-3.001$ | $-2.999$ | $-2.99$ | $-2.9$ | $-2.5$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-3.864$ | $-19.11$ | $-191.0$ | $-1910$ | $1910$ | $191.0$ | $19.11$ | $3.864$ |

$\lim_{x \to -3^-} f(x) = -\infty$

$\lim_{x \to -3^+} f(x) = \infty$

**10.** $\lim_{x \to 2^+} \dfrac{4}{(x-2)^3} = \infty$

$\lim_{x \to 2^-} \dfrac{4}{(x-2)^3} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

**12.** $\lim_{x \to 0^-} \dfrac{2+x}{x^2(1-x)} = \lim_{x \to 0^+} \dfrac{2+x}{x^2(1-x)} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

$\lim_{x \to 1^-} \dfrac{2+x}{x^2(1-x)} = \infty$

$\lim_{x \to 1^+} \dfrac{2+x}{x^2(1-x)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

**14.** No vertical asymptote since the denominator is never zero.

**16.** $\lim_{s \to -5^-} h(s) = -\infty$ and $\lim_{s \to -5^+} h(s) = \infty$.

Therefore, $s = -5$ is a vertical asymptote.

$\lim_{s \to 5^-} h(s) = -\infty$ and $\lim_{s \to 5^+} h(s) = \infty$.

Therefore, $s = 5$ is a vertical asymptote.

**18.** $f(x) = \sec \pi x = \dfrac{1}{\cos \pi x}$ has vertical asymptotes at

$$x = \frac{2n+1}{2},\ n \text{ any integer.}$$

**20.** $g(x) = \dfrac{(1/2)x^3 - x^2 - 4x}{3x^2 - 6x - 24} = \dfrac{1}{6}\dfrac{x(x^2 - 2x - 8)}{x^2 - 2x - 8}$

$= \dfrac{1}{6}x,$

$x \neq -2, 4$

No vertical asymptotes. The graph has holes at $x = -2$ and $x = 4$.

**22.** $f(x) = \dfrac{4(x^2 + x - 6)}{x(x^3 - 2x^2 - 9x + 18)} = \dfrac{4(x+3)(x-2)}{x(x-2)(x^2-9)} = \dfrac{4}{x(x-3)},\ x \neq -3, 2$

Vertical asymptotes at $x = 0$ and $x = 3$. The graph has holes at $x = -3$ and $x = 2$.

**24.** $h(x) = \dfrac{x^2 - 4}{x^3 + 2x^2 + x + 2} = \dfrac{(x+2)(x-2)}{(x+2)(x^2+1)}$

has no vertical asymptote since

$$\lim_{x \to -2} h(x) = \lim_{x \to -2} \frac{x-2}{x^2+1} = -\frac{4}{5}.$$

**26.** $h(t) = \dfrac{t(t-2)}{(t-2)(t+2)(t^2+4)} = \dfrac{t}{(t+2)(t^2+4)},\ t \neq 2$

Vertical asymptote at $t = -2$. The graph has a hole at $t = 2$.

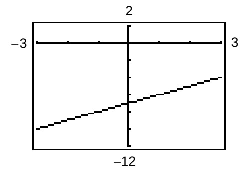
**28.** $g(\theta) = \dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta\cos\theta}$ has vertical asymptotes at

$$\theta = \frac{(2n+1)\pi}{2} = \frac{\pi}{2} + n\pi,\ n \text{ any integer.}$$

There is no vertical asymptote at $\theta = 0$ since

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1.$$

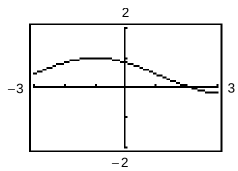
**30.** $\displaystyle\lim_{x\to -1}\frac{x^2-6x-7}{x+1} = \lim_{x\to -1}(x-7) = -8$




Removable discontinuity at $x = -1$

**32.** $\displaystyle\lim_{x\to -1}\frac{\sin(x+1)}{x+1} = 1$

Removable discontinuity at $x = -1$




**34.** $\displaystyle\lim_{x\to 1^+}\frac{2+x}{1-x} = -\infty$

**36.** $\displaystyle\lim_{x\to 4^-}\frac{x^2}{x^2+16} = \frac{1}{2}$

**38.** $\displaystyle\lim_{x\to -(1/2)^+}\frac{6x^2+x-1}{4x^2-4x-3} = \lim_{x\to -(1/2)^+}\frac{3x-1}{2x-3} = \frac{5}{8}$

**40.** $\displaystyle\lim_{x\to 3}\frac{x-2}{x^2} = \frac{1}{9}$

**42.** $\displaystyle\lim_{x\to 0^-}\left(x^2 - \frac{1}{x}\right) = \infty$

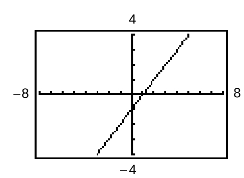
**44.** $\displaystyle\lim_{x\to(\pi/2)^+}\frac{-2}{\cos x} = \infty$

**46.** $\displaystyle\lim_{x\to 0}\frac{(x+2)}{\cot x} = \lim_{x\to 0}\left[(x+2)\tan x\right] = 0$

**48.** $\displaystyle\lim_{x\to(1/2)^-} x^2\tan\pi x = \infty$ and $\displaystyle\lim_{x\to(1/2)^+} x^2\tan\pi x = -\infty$. Therefore, $\displaystyle\lim_{x\to(1/2)} x^2\tan\pi x$ does not exist.

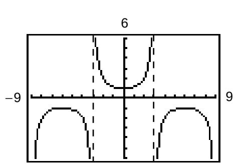
**50.** $f(x) = \dfrac{x^3-1}{x^2+x+1}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x-1) = 0$$




**52.** $f(x) = \sec\dfrac{\pi x}{6}$

$$\lim_{x\to 3^+} f(x) = -\infty$$




**54.** The line $x = c$ is a vertical asymptote if the graph of $f$ approaches $\pm\infty$ as $x$ approaches $c$.

**56.** No. For example, $f(x) = \dfrac{1}{x^2+1}$ has no vertical asymptote.

**58.** $P = \dfrac{k}{V}$

$\displaystyle\lim_{V\to 0^+}\frac{k}{V} = k(\infty) = \infty$ (In this case we know that $k > 0$.)

**60.** (a) $r = 50\pi \sec^2 \dfrac{\pi}{6} = \dfrac{200\pi}{3}$ ft/sec

(b) $r = 50\pi \sec^2 \dfrac{\pi}{3} = 200\pi$ ft/sec

(c) $\lim\limits_{\theta \to (\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

**62.** $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

**64.** (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

| $x$ | 30 | 40 | 50 | 60 |
|---|---|---|---|---|
| $y$ | 150 | 66.667 | 50 | 42.857 |

(c) $\lim\limits_{x \to 25^+} \dfrac{25x}{x - 25} = \infty$

As $x$ gets close to 25 mph, $y$ becomes larger and larger.

**66.** (a) $A = \dfrac{1}{2}bh - \dfrac{1}{2}r^2\,\theta = \dfrac{1}{2}(10)(10 \tan \theta) - \dfrac{1}{2}(10)^2\theta$

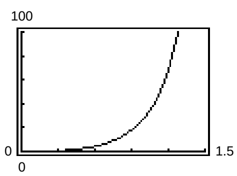
$$= 50 \tan \theta - 50\,\theta$$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(b)

| $\phi$ | 0.3 | 0.6 | 0.9 | 1.2 | 1.5 |
|---|---|---|---|---|---|
| $f(\theta)$ | 0.47 | 4.21 | 18.0 | 68.6 | 630.1 |

(c)




(d) $\lim\limits_{\theta \to \pi/2^-} A = \infty$

**68.** False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The graph of $f$ has a hole at $(1, 2)$, not a vertical asymptote.

**70.** True

**72.** Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$ and $\lim\limits_{x \to 0} \dfrac{1}{x^4} = \infty$, but

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x \to 0} \left(\frac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$$

**74.** Given $\lim\limits_{x \to c} f(x) = \infty$, let $g(x) = 1$. then $\lim\limits_{x \to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

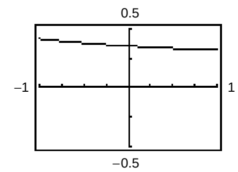
## Review Exercises for Chapter 1

**2.** Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

**4.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.358 | 0.354 | 0.354 | 0.354 | 0.353 | 0.349 |

$\lim_{x\to 0} f(x) \approx 0.2$




**6.** $g(x) = \dfrac{3x}{x-2}$ (a) $\lim_{x\to 2} g(x)$ does not exist. (b) $\lim_{x\to 0} g(x) = 0$

**8.** $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3.$

Let $\epsilon > 0$ be given. We need

$$\left|\sqrt{x} - 3\right| < \epsilon \implies \left|\sqrt{x}+3\right|\left|\sqrt{x}-3\right| < \epsilon\left|\sqrt{x}+3\right|$$
$$|x-9| < \epsilon\left|\sqrt{x}+3\right|$$

Assuming $4 < x < 16$, you can choose $\delta = 5\epsilon$.

Hence, for $0 < |x-9| < \delta = 5\epsilon$, you have

$$|x-9| < 5\epsilon < \left|\sqrt{x}+3\right|\epsilon$$
$$\left|\sqrt{x}-3\right| < \epsilon$$
$$|f(x) - L| < \epsilon$$

**10.** $\lim_{x\to 5} 9 = 9$. Let $\epsilon > 0$ be given. $\delta$ can be any positive number. Hence, for $0 < |x-5| < \delta$, you have

$$|9-9| < \epsilon$$
$$|f(x) - L| < \epsilon$$

**12.** $\lim_{y\to 4} 3|y-1| = 3|4-1| = 9$

**14.** $\lim_{t\to 3} \dfrac{t^2-9}{t-3} = \lim_{t\to 3}(t+3) = 6$

**16.** $$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$
$$= \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4}$$

**18.** $$\lim_{s\to 0} \frac{\left(1/\sqrt{1+s}\right)-1}{s} = \lim_{s\to 0}\left[\frac{\left(1/\sqrt{1+s}\right)-1}{s} \cdot \frac{\left(1/\sqrt{1+s}\right)+1}{\left(1/\sqrt{1+s}\right)+1}\right]$$
$$= \lim_{s\to 0} \frac{[1/(1+s)]-1}{s\left[\left(1/\sqrt{1+s}\right)+1\right]} = \lim_{s\to 0} \frac{-1}{(1+s)\left[\left(1/\sqrt{1+s}\right)+1\right]} = -\frac{1}{2}$$

**20.** $$\lim_{x\to -2} \frac{x^2-4}{x^3+8} = \lim_{x\to -2} \frac{(x+2)(x-2)}{(x+2)(x^2-2x+4)}$$
$$= \lim_{x\to -2} \frac{x-2}{x^2-2x+4}$$
$$= -\frac{4}{12} = -\frac{1}{3}$$

**22.** $\lim_{x\to(\pi/4)} \dfrac{4x}{\tan x} = \dfrac{4(\pi/4)}{1} = \pi$

**24.** $\displaystyle\lim_{\Delta x\to 0}\frac{\cos(\pi+\Delta x)+1}{\Delta x} = \lim_{\Delta x\to 0}\frac{\cos\pi\cos\Delta x - \sin\pi\sin\Delta x + 1}{\Delta x}$

$\displaystyle = \lim_{\Delta x\to 0}\left[-\frac{(\cos\Delta x - 1)}{\Delta x}\right] - \lim_{\Delta x\to 0}\left[\sin\pi\frac{\sin\Delta x}{\Delta x}\right]$

$= -0 - (0)(1) = 0$

**26.** $\lim_{x\to c}[f(x)+2g(x)] = -\frac{3}{4} + 2\left(\frac{2}{3}\right) = \frac{7}{12}$

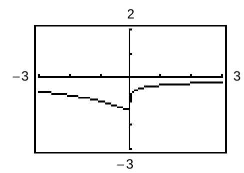
**28.** $f(x) = \dfrac{1-\sqrt[3]{x}}{x-1}$

(a)

| $x$ | 1.1 | 1.01 | 1.001 | 1.0001 |
|---|---|---|---|---|
| $f(x)$ | $-0.3228$ | $-0.3322$ | $-0.3332$ | $-0.3333$ |

$\displaystyle\lim_{x\to 1^+}\frac{1-\sqrt[3]{x}}{x-1} \approx -0.333 \quad \left(\text{Actual limit is } -\tfrac{1}{3}.\right)$

(b)




(c) $\displaystyle\lim_{x\to 1^+}\frac{1-\sqrt[3]{x}}{x-1} = \lim_{x\to 1^+}\frac{1-\sqrt[3]{x}}{x-1}\cdot\frac{1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}{1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}$

$\displaystyle = \lim_{x\to 1^+}\frac{1-x}{(x-1)\left[1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right]}$

$\displaystyle = \lim_{x\to 1^+}\frac{-1}{1+\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}$

$\displaystyle = -\frac{1}{3}$

**30.** $s(t) = 0 \implies -4.9t^2 + 200 = 0 \implies t^2 \approx 40.816 \implies t \approx 6.39$ sec

When $t = 6.39$, the velocity is approximately

$\displaystyle\lim_{t\to a}\frac{s(a)-s(t)}{a-t} = \lim_{t\to a} -4.9(a+t)$

$\displaystyle = \lim_{t\to 6.39} -4.9(6.39+6.39) = -62.6$ m/sec.

**32.** $\lim_{x\to 4}[\![x-1]\!]$ does not exist. The graph jumps from 2 to 3 at $x = 4$.

**34.** $\lim_{x\to 1^+} g(x) = 1 + 1 = 2.$

**36.** $\lim_{s\to -2} f(s) = 2$

**38.** $f(x) = \begin{cases}\dfrac{3x^2-x-2}{x-1}, & x \neq 1\\ 0, & x = 1\end{cases}$

$\displaystyle\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{3x^2-x-2}{x-1}$

$\displaystyle = \lim_{x\to 1}(3x+2) = 5 \neq 0$

Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1)\cup(1,\infty)$

**40.** $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x\to 2^-} (5 - x) = 3$

$\lim_{x\to 2^+} (2x - 3) = 1$

Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

**42.** $f(x) = \sqrt{\dfrac{x+1}{x}} = \sqrt{1 + \dfrac{1}{x}}$

$\lim_{x\to 0^+} \sqrt{1 + \dfrac{1}{x}} = \infty$

Domain: $(-\infty, -1], (0, \infty)$

Nonremovable discontinuity at $x = 0$
Continuous on $(-\infty, -1] \cup (0, \infty)$

**44.** $f(x) = \dfrac{x+1}{2x+2}$

$\lim_{x\to -1} \dfrac{x+1}{2(x+1)} = \dfrac{1}{2}$

Removable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

**46.** $f(x) = \tan 2x$

Nonremovable discontinuities when

$$x = \frac{(2n+1)\pi}{4}$$

Continuous on

$$\left(\frac{(2n-1)\pi}{4}, \frac{(2n+1)\pi}{4}\right)$$

for all integers $n$.

**48.** $\lim_{x\to 1^+} (x + 1) = 2$

$\lim_{x\to 3^-} (x + 1) = 4$

Find $b$ and $c$ so that $\lim_{x\to 1^-} (x^2 + bx + c) = 2$ and $\lim_{x\to 3^+} (x^2 + bx + c) = 4$.

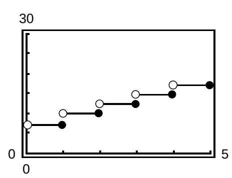
Consequently we get $1 + b + c = 2$ and $9 + 3b + c = 4$.

Solving simultaneously, $b = -3$ and $c = 4$.

**50.** $C = 9.80 + 2.50[-[\![-x]\!] - 1],\ x > 0$

$= 9.80 - 2.50[[\![-x]\!] + 1]$

$C$ has a nonremovable discontinuity at each integer.




**52.** $f(x) = \sqrt{(x-1)x}$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

(b) $\lim_{x\to 0^-} f(x) = 0$

(c) $\lim_{x\to 1^+} f(x) = 0$

**54.** $h(x) = \dfrac{4x}{4 - x^2}$

Vertical asymptotes at $x = 2$ and $x = -2$

**56.** $f(x) = \csc \pi x$

Vertical asymptote at every integer $k$

**58.** $\lim_{x\to (1/2)^+} \dfrac{x}{2x - 1} = \infty$

**60.** $\lim_{x\to -1^-} \dfrac{x+1}{x^4 - 1} = \lim_{x\to -1^-} \dfrac{1}{(x^2+1)(x-1)} = -\dfrac{1}{4}$

**62.** $\lim_{x\to -1^+} \dfrac{x^2 - 2x + 1}{x + 1} = \infty$

**64.** $\lim_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2 - 4}} = -\infty$

**66.** $\lim_{x\to 0^+} \dfrac{\sec x}{x} = \infty$

**68.** $\lim_{x\to 0^-} \dfrac{\cos^2 x}{x} = -\infty$

**70.** $f(x) = \dfrac{\tan 2x}{x}$

(a)

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 2.0271 | 2.0003 | 2.0000 | 2.0000 | 2.0003 | 2.0271 |

$$\lim_{x\to 0} \frac{\tan 2x}{x} = 2$$

(b) Yes, define

$$f(x) = \begin{cases} \dfrac{\tan 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Now $f(x)$ is continuous at $x = 0$.

## Problem Solving for Chapter 1

**2.** (a) Area $\triangle PAO = \frac{1}{2}bh = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area $\triangle PBO = \frac{1}{2}bh = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{x^2}{2}$

(b) $a(x) = \dfrac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO} = \dfrac{x^2/2}{x/2} = x$

| $x$ | 4 | 2 | 1 | 0.1 | 0.01 |
|---|---|---|---|---|---|
| Area $\triangle PAO$ | 2 | 1 | 1/2 | 1/20 | 1/200 |
| Area $\triangle PBO$ | 8 | 2 | 1/2 | 1/200 | 1/20,000 |
| $a(x)$ | 4 | 2 | 1 | 1/10 | 1/100 |

(c) $\displaystyle\lim_{x\to 0^+} a(x) = \lim_{x\to 0^+} x = 0$

**4.** (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\dfrac{3}{4}$ Tangent line: $y - 4 = -\dfrac{3}{4}(x-3)$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

(c) Let $Q = (x, y) = \left(x, \sqrt{25 - x^2}\right)$

$$m_x = \frac{\sqrt{25-x^2} - 4}{x - 3}$$

(d)
$$\begin{aligned}
\lim_{x\to 3} m_x &= \lim_{x\to 3} \frac{\sqrt{25-x^2}-4}{x-3} \cdot \frac{\sqrt{25-x^2}+4}{\sqrt{25-x^2}+4} \\
&= \lim_{x\to 3} \frac{25 - x^2 - 16}{(x-3)\left(\sqrt{25-x^2}+4\right)} \\
&= \lim_{x\to 3} \frac{(3-x)(3+x)}{(x-3)\left(\sqrt{25-x^2}+4\right)} \\
&= \lim_{x\to 3} \frac{-(3+x)}{\sqrt{25-x^2}+4} = \frac{-6}{4+4} = -\frac{3}{4}
\end{aligned}$$

This is the slope of the tangent line at $P$.

**6.**
$$\begin{aligned}
\frac{\sqrt{a+bx} - \sqrt{3}}{x} &= \frac{\sqrt{a+bx} - \sqrt{3}}{x} \cdot \frac{\sqrt{a+bx} + \sqrt{3}}{\sqrt{a+bx} + \sqrt{3}} \\
&= \frac{(a+bx) - 3}{x\left(\sqrt{a+bx} + \sqrt{3}\right)}
\end{aligned}$$

Letting $a = 3$ simplifies the numerator.

Thus,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{3+bx} - \sqrt{3}}{x} &= \lim_{x\to 0} \frac{bx}{x\left(\sqrt{3+bx} + \sqrt{3}\right)} \\
&= \lim_{x\to 0} \frac{b}{\sqrt{3+bx} + \sqrt{3}}.
\end{aligned}$$

Setting $\dfrac{b}{\sqrt{3} + \sqrt{3}} = \sqrt{3}$, you obtain $b = 6$.

Thus, $a = 3$ and $b = 6$.

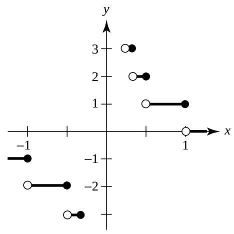
**8.** $\displaystyle\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (a^2 - 2) = a^2 - 2$

$\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{ax}{\tan x} = a \left(\text{because } \lim_{x\to 0} \frac{\tan x}{x} = 1\right)$

Thus,

$$\begin{aligned}
a^2 - 2 &= a \\
a^2 - a - 2 &= 0 \\
(a-2)(a+1) &= 0 \\
a &= -1, 2
\end{aligned}$$

**10.**




(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = -\infty$

$\lim_{x\to 0^+} f(x) = \infty$

(c) $f$ is continuous for all real numbers except

$x = 0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

**12.** (a) $$v^2 = \frac{192{,}000}{r} + v_0^2 - 48$$

$$\frac{192{,}000}{r} = v^2 - v_0^2 + 48$$

$$r = \frac{192{,}000}{v - v_0^2 + 48}$$

$$\lim_{v\to 0} r = \frac{192{,}000}{48 - v_0^2}$$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ feet/sec.

(b) $$v^2 = \frac{1920}{r} + v_0^2 - 2.17$$

$$\frac{1920}{r} = v^2 - v_0^2 + 2.17$$

$$r = \frac{1920}{v^2 - v_0^2 + 2.17}$$

$$\lim_{v\to 0} r = \frac{1920}{2.17 - v_0^2}$$

Let $v_0 = \sqrt{2.17}$ mi/sec $\quad (\approx 1.47$ mi/sec$)$.

(c) $$r = \frac{10{,}600}{v^2 - v_0^2 + 6.99}$$

$$\lim_{v\to 0} r = \frac{10{,}600}{6.99 - v_0^2}$$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Since this is smaller than the escape velocity for earth, the mass is less.

**14.** Let $a \neq 0$ and let $\epsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta$, then $|f(x) - L| < \epsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \epsilon.$$

As a counterexample, let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$.

Then $\lim_{x\to 0} f(x) = 1 = L$,

but $\lim_{x\to 0} f(ax) = \lim_{x\to 0} f(0) = 2$.

# CHAPTER 2
# Differentiation

# CHAPTER 2
# Differentiation

## Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Even-Numbered Exercises

**2.** (a) $m = \dfrac{1}{4}$

(b) $m = 1$

**4.** (a) $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{5 - 2}{3} = 1$

$\dfrac{f(4) - f(3)}{4 - 3} \approx \dfrac{5 - 4.75}{1} = 0.25$

Thus, $\dfrac{f(4) - f(1)}{4 - 1} > \dfrac{f(4) - f(3)}{4 - 3}$

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

**6.** $g(x) = \dfrac{3}{2}x + 1$ is a line. Slope $= \dfrac{3}{2}$

**8.** Slope at $(2, 1) = \displaystyle\lim_{\Delta x \to 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{5 - (2 + \Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5 - 4 - 4(\Delta x) - (\Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (-4 - \Delta x) = -4$$

**10.** Slope at $(-2, 7) = \displaystyle\lim_{\Delta t \to 0} \frac{h(-2 + \Delta t) - h(-2)}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{(-2 + \Delta t)^2 + 3 - 7}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{4 - 4(\Delta t) + (\Delta t)^2 - 4}{\Delta t}$$

$$= \lim_{\Delta t \to 0} (-4 + \Delta t) = -4$$

**12.** $g(x) = -5$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-5 - (-5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

**14.** $f(x) = 3x + 2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3 = 3$$

**16.** $f(x) = 9 - \dfrac{1}{2}x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

**18.** $f(x) = 1 - x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (-2x - \Delta x) = -2x$$

**20.** $f(x) = x^3 + x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x$$

**22.** $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x + \Delta x)^2} - \dfrac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x + \Delta x)^2 x^2}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

**24.** $f(x) = \dfrac{4}{\sqrt{x}}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{4}{\sqrt{x + \Delta x}} - \dfrac{4}{\sqrt{x}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x\sqrt{x}\sqrt{x + \Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{4x - 4(x + \Delta x)}{\Delta x\sqrt{x}\sqrt{x + \Delta x}\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-4}{\sqrt{x}\sqrt{x + \Delta x}\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \frac{-4}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = \frac{-2}{x\sqrt{x}}$$

**26.** (a) $f(x) = x^2 + 2x + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 2$$

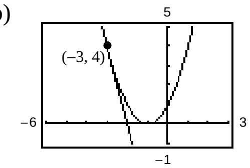
At $(-3, 4)$, the slope of the tangent line is $m = 2(-3) + 2 = -4$.

The equation of the tangent line is

$$y - 4 = -4(x + 3)$$

$$y = -4x - 8.$$

(b)




**28.** (a) $f(x) = x^3 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + 1] - (x^3 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2$$

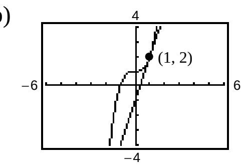
At $(1, 2)$, the slope of the tangent line is $m = 3(1)^2 = 3$.

The equation of the tangent line is

$$y - 2 = 3(x - 1)$$

$$y = 3x - 1.$$

(b)




**30.** (a) $f(x) = \sqrt{x - 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x - 1} - \sqrt{x - 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x - 1) - (x - 1)}{\Delta x\left(\sqrt{x + \Delta x - 1} + \sqrt{x - 1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}}$$

At $(5, 2)$, the slope of the tangent line is

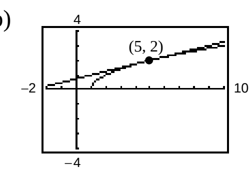
$$m = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{4}$$

The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x - 5)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

(b)



**32.** (a) $f(x) = \dfrac{1}{x+1}$

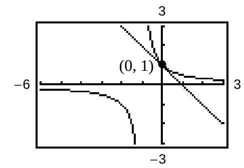
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x+\Delta x+1} - \dfrac{1}{x+1}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x+1) - (x+\Delta x+1)}{\Delta x(x+\Delta x+1)(x+1)}$$
$$= \lim_{\Delta x \to 0} -\frac{1}{(x+\Delta x+1)(x+1)}$$
$$= -\frac{1}{(x+1)^2}$$

At $(0, 1)$, the slope of the tangent line is

$$m = \frac{-1}{(0+1)^2} = -1.$$

The equation of the tangent line is $y = -x + 1$.

(b)




**34.** Using the limit definition of derivative, $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$
$$x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, at the points $(1, 3)$ and $(-1, 1)$ the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$$y - 3 = 3(x - 1) \text{ and } y - 1 = 3(x + 1)$$
$$y = 3x \qquad\qquad y = 3x + 4$$

**36.** Using the limit definition of derivative, $f'(x) = \dfrac{-1}{2(x-1)^{3/2}}$.

Since the slope of the given line is $-\dfrac{1}{2}$, we have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$
$$1 = (x-1)^{3/2}$$
$$1 = x - 1 \Rightarrow x = 2$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$
$$y = -\frac{1}{2}x + 2$$

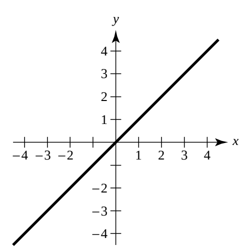
**38.** $h(-1) = 4$ because the tangent line passes through $(-1, 4)$

$$h'(-1) = \frac{6-4}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$$

**40.** $f(x) = x^2 \implies f'(x) = 2x$ (d)

**42.** $f'$ does not exist at $x = 0$. Matches (c)

**44.**




Answers will vary.

Sample answer: $y = x$

**46.** (a) Yes. $\lim_{\Delta x \to 0} \dfrac{f(x + 2\Delta x) - f(x)}{2Dx} = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

(b) No. The numerator does not approach zero.

(c) Yes. $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x) - f(x - \Delta x) + f(x)}{2\Delta x}$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{2\Delta x} + \frac{f(x - \Delta x) - f(x)}{2(-\Delta x)}\right]$$

$$= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x)$$

(d) Yes. $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

**48.** Let $(x_0, y_0)$ be a point of tangency on the graph of $f$. By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and $(x_0, y_0)$ equals the derivative of $f$ at $x_0$:

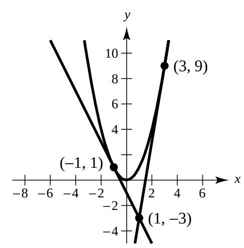
$$\frac{-3 - y_0}{1 - x_0} = 2x_0$$

$$-3 - y_0 = (1 - x_0)2x_0$$

$$-3 - x_0^2 = 2x_0 - 2x_0^2$$

$$x_0^2 - 2x_0 - 3 = 0$$

$$(x_0 - 3)(x_0 + 1) = 0 \implies x_0 = 3, -1$$




Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and $-2$. The equations of the tangent lines are

$$y + 3 = 6(x - 1) \qquad y + 3 = -2(x - 1)$$

$$y = 6x - 9 \qquad y = -2x - 1$$

**50.** (a) $f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

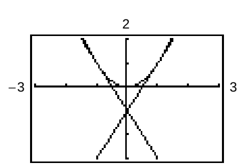
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $x = -1$, $f'(-1) = -2$ and the tangent line is

$y - 1 = -2(x + 1)$ or $y = -2x - 1$.

At $x = 0$, $f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $f'(1) = 2$ and the tangent line is $y = 2x - 1$.




For this function, the slopes of the tangent lines are always distinct for different values of $x$.

(b) $g'(x) = \lim_{\Delta x \to 0} \dfrac{g(x + \Delta x) - g(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2$$

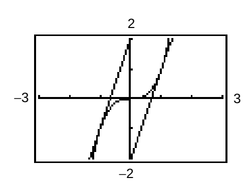
At $x = -1$, $g'(-1) = 3$ and the tangent line is

$y + 1 = 3(x + 1)$ or $y = 3x + 2$.

At $x = 0$, $g'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $g'(1) = 3$ and the tangent line is

$y - 1 = 3(x - 1)$ or $y = 3x - 2$.

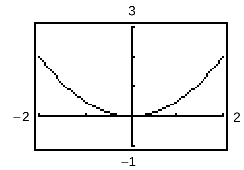



For this function, the slopes of the tangent lines are sometimes the same.

**52.** $f(x) = \frac{1}{2}x^2$

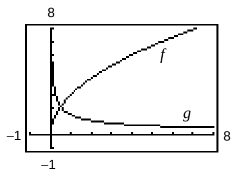
By the limit definition of the derivative we have $f'(x) = x$.

| $x$ | $-2$ | $-1.5$ | $-1$ | $-0.5$ | $0$ | $0.5$ | $1$ | $1.5$ | $2$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 1.125 | 0.5 | 0.125 | 0 | 0.125 | 0.5 | 1.125 | 2 |
| $f'(x)$ | $-2$ | $-1.5$ | $-1$ | $-0.5$ | 0 | 0.5 | 1 | 1.5 | 2 |




**54.** $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= \left(3\sqrt{x + 0.01} - 3\sqrt{3}\right)100$

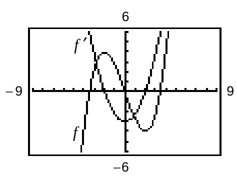



The graph of $g(x)$ is approximately the graph of $f'(x)$.

**56.** $f(2) = \dfrac{1}{4}(2^3) = 2,\ f(2.1) = 2.31525$

$f'(2) \approx \dfrac{2.31525 - 2}{2.1 - 2} = 3.1525$ [Exact: $f'(2) = 3$]

**58.** $f(x) = \dfrac{x^3}{4} - 3x$ and $f'(x) = \dfrac{3}{4}x^2 - 3$




**60.** $f(x) = x + \dfrac{1}{x}$

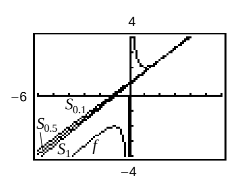
$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2) = \frac{(2 + \Delta x) + \dfrac{1}{2 + \Delta x} - \dfrac{5}{2}}{\Delta x}(x - 2) + \frac{5}{2}$$

$$= \frac{2(2 + \Delta x)^2 + 2 - 5(2 + \Delta x)}{2(2 + \Delta x)\,\Delta x}(x - 2) + \frac{5}{2} = \frac{(2\Delta x + 3)}{2(2 + \Delta x)}(x - 2) + \frac{5}{2}$$

(a) $\Delta x = 1$: $S_{\Delta x} = \dfrac{5}{6}(x - 2) + \dfrac{5}{2} = \dfrac{5}{6}x + \dfrac{5}{6}$

$\Delta x = 0.5$: $S_{\Delta x} = \dfrac{4}{5}(x - 2) + \dfrac{5}{2} = \dfrac{4}{5}x + \dfrac{9}{10}$

$\Delta x = 0.1$: $S_{\Delta x} = \dfrac{16}{21}(x - 2) + \dfrac{5}{2} = \dfrac{16}{21}x + \dfrac{41}{42}$




(b) As $\Delta x \to 0$, the line approaches the tangent line to $f$ at $\left(2, \frac{5}{2}\right)$.

**62.** $g(x) = x(x - 1) = x^2 - x,\ c = 1$

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - x - 0}{x - 1} = \lim_{x\to 1} \frac{x(x - 1)}{x - 1}$$

$$= \lim_{x\to 1} x = 1$$

**64.** $f(x) = x^3 + 2x,\ c = 1$

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x^2 + x + 3)}{x - 1} = \lim_{x\to 1} (x^2 + x + 3) = 5$$

**66.** $f(x) = \frac{1}{x},\ c = 3$

$$f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to 3} \frac{(1/x) - (1/3)}{x - 3} = \lim_{x\to 3} \frac{3 - x}{3x} \cdot \frac{1}{x - 3} = \lim_{x\to 3} \left(-\frac{1}{3x}\right) = -\frac{1}{9}$$

**68.** $g(x) = (x + 3)^{1/3},\ c = -3$

$$g'(-3) = \lim_{x\to -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x\to -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x\to -3} \frac{1}{(x + 3)^{2/3}}$$

Does not exist.

**70.** $f(x) = |x - 4|,\ c = 4$

$$f'(4) = \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4} \frac{|x - 4| - 0}{x - 4} = \lim_{x\to 4} \frac{|x - 4|}{x - 4}$$

Does not exist.

**72.** $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph.)

**74.** $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

**76.** $f(x)$ is differentiable everywhere except at $x = 0$. (Sharp turn in the graph)

**78.** $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

**80.** $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

**82.** $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x\to 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x\to 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty. \text{ (Vertical tangent)}$$

The limit from the right does not exist since $f$ is undefined for $x > 1$. Therefore, $f$ is not differentiable at $x = 1$.

**84.** $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x - 1}{x - 1} = \lim_{x\to 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x\to 1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore, $f$ is not differentiable at $x = 1$.

**86.** Note that $f$ is continuous at $x = 2$. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

The derivative from the left is

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{\left(\frac{1}{2}x + 1\right) - 2}{x - 2} = \lim_{x\to 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

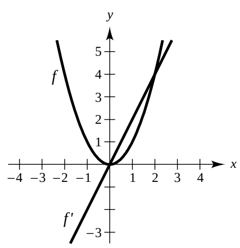
The derivative from the right is

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2}$$

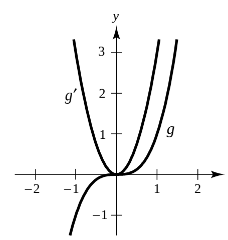
$$= \lim_{x\to 2^+} \frac{2x - 4}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2(x - 2)}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}.$$

The one-sided limits are equal. Therefore, $f$ is differentiable at $x = 2$. $\left(f'(2) = \frac{1}{2}\right)$

**88.** (a) $f(x) = x^2$ and $f'(x) = 2x$




(b) $g(x) = x^3$ and $g'(x) = 3x^2$




(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then $f'(x) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x\to 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x}$$

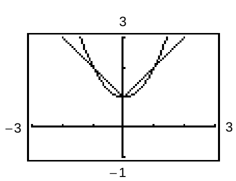
$$= \lim_{\Delta x\to 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3$$

Hence, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof, since you must verify the conjecture for all integer values of $n$, $n \geq 2$.

**90.** False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

**92.** True—see Theorem 2.1

**94.**




As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. $y_2$ is not differentiable at $(0, 1)$.

## Section 2.2 Basic Differentiation Rules and Rates of Change

**2.** (a) $y = x^{-1/2}$
$y' = -\frac{1}{2}x^{-3/2}$
$y'(1) = -\frac{1}{2}$

(b) $y = x^{-1}$
$y' = -x^{-2}$
$y'(1) = -1$

(c) $y = x^{-3/2}$
$y' = -\frac{3}{2}x^{-5/2}$
$y'(1) = -\frac{3}{2}$

(d) $y = x^{-2}$
$y' = -2x^{-3}$
$y'(1) = -2$

**4.** $f(x) = -2$
$f'(x) = 0$

**6.** $y = x^8$
$y' = 8x^7$

**8.** $y = \dfrac{1}{x^8} = x^{-8}$
$y' = 8x^{-9} = \dfrac{-8}{x^9}$

**10.** $y = \sqrt[4]{x} = x^{1/4}$
$y' = \dfrac{1}{4}x^{-3/4} = \dfrac{1}{4x^{3/4}}$

**12.** $g(x) = 3x - 1$
$g'(x) = 3$

**14.** $y = t^2 + 2t - 3$
$y' = 2t + 2$

**16.** $y = 8 - x^3$
$y' = -3x^2$

**18.** $f(x) = 2x^3 - x^2 + 3x$
$f'(x) = 6x^2 - 2x + 3$

**20.** $g(t) = \pi \cos t$
$g'(t) = -\pi \sin t$

**22.** $y = 5 + \sin x$
$y' = \cos x$

**24.** $y = \dfrac{5}{(2x)^3} + 2\cos x = \dfrac{5}{8}x^{-3} + 2\cos x$
$y' = \dfrac{5}{8}(-3)x^{-4} - 2\sin x = \dfrac{-15}{8x^4} - 2\sin x$

| | *Function* | *Rewrite* | *Derivative* | *Simplify* |
|---|---|---|---|---|
| **26.** | $y = \dfrac{2}{3x^2}$ | $y = \dfrac{2}{3}x^{-2}$ | $y' = -\dfrac{4}{3}x^{-3}$ | $y' = -\dfrac{4}{3x^3}$ |
| **28.** | $y = \dfrac{\pi}{(3x)^2}$ | $y = \dfrac{\pi}{9}x^{-2}$ | $y' = -\dfrac{2\pi}{9}x^{-3}$ | $y' = -\dfrac{2\pi}{9x^3}$ |
| **30.** | $y = \dfrac{4}{x^{-3}}$ | $y = 4x^3$ | $y' = 12x^2$ | $y' = 12x^2$ |

**32.** $f(t) = 3 - \dfrac{3}{5t}, \left(\dfrac{3}{5}, 2\right)$
$f'(t) = \dfrac{3}{5t^2}$
$f'\left(\dfrac{3}{5}\right) = \dfrac{5}{3}$

**34.** $y = 3x^3 - 6, (2, 18)$
$y' = 9x^2$
$y'(2) = 36$

**36.** $f(x) = 3(5 - x)^2, \ (5, 0)$
$= 3x^2 - 30x + 75$
$f'(x) = 6x - 30$
$f'(5) = 0$

**38.** $g(t) = 2 + 3\cos t, \ (\pi, -1)$
$g'(t) = -3\sin t$
$g'(\pi) = 0$

**40.** $f(x) = x^2 - 3x - 3x^{-2}$
$f'(x) = 2x - 3 + 6x^{-3}$
$= 2x - 3 + \dfrac{6}{x^3}$

**42.** $f(x) = x + x^{-2}$
$f'(x) = 1 - 2x^{-3}$
$= 1 - \dfrac{2}{x^3}$

**44.** $h(x) = \dfrac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$
$h'(x) = 2 - \dfrac{1}{x^2} = \dfrac{2x^2 - 1}{x^2}$

**46.** $y = 3x(6x - 5x^2) = 18x^2 - 15x^3$
$y' = 36x - 45x^2$

**48.** $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$
$f'(x) = \dfrac{1}{3}x^{-2/3} + \dfrac{1}{5}x^{-4/5} = \dfrac{1}{3x^{2/3}} + \dfrac{1}{5x^{4/5}}$

**50.** $f(t) = t^{2/3} - t^{1/3} + 4$
$f'(t) = \dfrac{2}{3}t^{-1/3} - \dfrac{1}{3}t^{-2/3} = \dfrac{2}{3t^{1/3}} - \dfrac{1}{3t^{2/3}}$

**52.** $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x = 2x^{-1/3} + 3\cos x$

$f'(x) = \dfrac{-2}{3}x^{-4/3} - 3\sin x = \dfrac{-2}{3x^{4/3}} - 3\sin x$

**54.** (a) $y = x^3 + x$

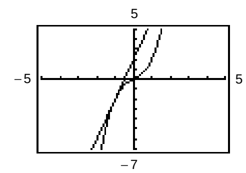
$y' = 3x^2 + 1$

At $(-1, -2)$: $y' = 3(-1)^2 + 1 = 4$.

Tangent line: $y + 2 = 4(x + 1)$

$4x - y + 2 = 0$

(b)




**56.** (a) $y = (x^2 + 2x)(x + 1)$

$= x^3 + 3x^2 + 2x$

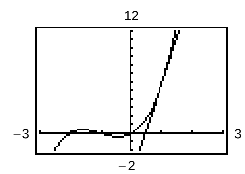
$y' = 3x^2 + 6x + 2$

At $(1, 6)$: $y' = 3(1)^2 + 6(1) + 2 = 11$.

Tangent line: $y - 6 = 11(x - 1)$

$0 = 11x - y - 5$

(b)




**58.** $y = x^3 + x$

$y' = 3x^2 + 1 > 0$ for all $x$.

Therefore, there are no horizontal tangents.

**60.** $y = x^2 + 1$

$y' = 2x = 0 \implies x = 0$

At $x = 0$, $y = 1$.

Horizontal tangent: $(0, 1)$

**62.** $y = \sqrt{3}x + 2\cos x,\ 0 \le x < 2\pi$

$y' = \sqrt{3} - 2\sin x = 0$

$\sin x = \dfrac{\sqrt{3}}{2} \implies x = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

At $x = \dfrac{\pi}{3}$, $y = \dfrac{\sqrt{3}\pi + 3}{3}$.

At $x = \dfrac{2\pi}{3}$, $y = \dfrac{2\sqrt{3}\pi - 3}{3}$.

Horizontal tangents: $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}\pi + 3}{3}\right), \left(\dfrac{2\pi}{3}, \dfrac{2\sqrt{3}\pi - 3}{3}\right)$

**64.** $k - x^2 = -4x + 7$ Equate functions

$-2x = -4$ Equate derivatives

Hence, $x = 2$ and $k - 4 = -8 + 7 \implies k = 3$

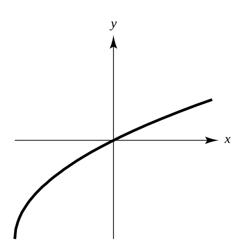
**66.** $k\sqrt{x} = x + 4$ Equate functions

$\dfrac{k}{2\sqrt{x}} = 1$ Equate derivatives

Hence, $k = 2\sqrt{x}$ and

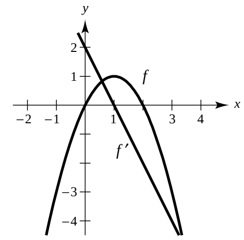
$\left(2\sqrt{x}\right)\sqrt{x} = x + 4 \implies 2x = x + 4 \implies x = 4 \implies k = 4$

**68.** The graph of a function $f$ such that $f' > 0$ for all $x$ and the rate of change the function is decreasing (i.e. $f'' < 0$) would, in general, look like the graph at the right.



**70.** $g(x) = -5f(x) \Longrightarrow g'(x) = -5f'(x)$

**72.**




If $f$ is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

**74.** $m_1$ is the slope of the line tangent to $y = x$. $m_2$ is the slope of the line tangent to $y = 1/x$. Since

$$y = x \implies y' = 1 \implies m_1 = 1 \text{ and } y = \frac{1}{x} \implies y' = \frac{-1}{x^2} \implies m_2 = \frac{-1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \implies x^2 = 1 \implies x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular at the points intersection.

**76.** $f(x) = \dfrac{2}{x}$, $(5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2 y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

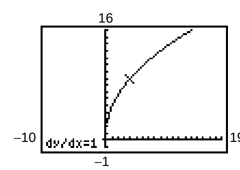
The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of $f$. The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $$y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$$

$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

**78.** $f'(4) = 1$

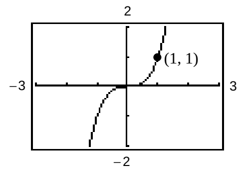


**80.** (a) Nearby point: $(1.0073138, 1.0221024)$

Secant line: $y - 1 = \dfrac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$

$$y = 3.022(x - 1) + 1$$

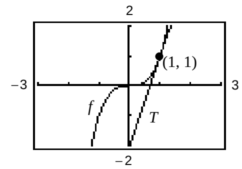
(Answers will vary.)




(b) $f'(x) = 3x^2$

$T(x) = 3(x - 1) + 1 = 3x - 2$

(c) The accuracy worsens at you move away from $(1, 1)$.




(d)

| $\Delta x$ | $-3$ | $-2$ | $-1$ | $-0.5$ | $-0.1$ | 0 | 0.1 | 0.5 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-8$ | $-1$ | 0 | 0.125 | 0.729 | 1 | 1.331 | 3.375 | 8 | 27 | 64 |
| $T(x)$ | $-8$ | $-5$ | $-2$ | $-0.5$ | 0.7 | 1 | 1.3 | 2.5 | 4 | 7 | 10 |

The accuracy decreases more rapidly than in Exercise 59 because $y = x^3$ is less "linear" than $y = x^{3/2}$.

**82.** True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

**84.** True. If $y = x/\pi = (1/\pi) \cdot x$, then $dy/dx = (1/\pi)(1) = 1/\pi$.

**86.** False. If $f(x) = \dfrac{1}{x^n} = x^{-n}$, then $f'(x) = -nx^{-n-1} = \dfrac{-n}{x^{n+1}}$

**88.** $f(t) = t^2 - 3, [2, 2.1]$

$f'(t) = 2t$

Instantaneous rate of change:

$$(2, 1) \implies f'(2) = 2(2) = 4$$

$$(2.1, 1.41) \implies f'(2.1) = 4.2$$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

**90.** $f(x) = \sin x, \left[0, \dfrac{\pi}{6}\right]$

$f'(x) = \cos x$

Instantaneous rate of change:

$$(0, 0) \implies f'(0) = 1$$

$$\left(\frac{\pi}{6}, \frac{1}{2}\right) \implies f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

**92.**

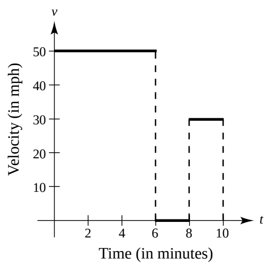
$$\begin{aligned} s(t) &= -16t^2 - 22t + 220 \\ v(t) &= -32t - 22 \\ v(3) &= -118 \text{ ft/sec} \\ s(t) &= -16t^2 - 22t + 220 \\ &= 112 \text{ (height after falling 108 ft)} \\ -16t^2 - 22t + 108 &= 0 \\ -2(t - 2)(8t + 27) &= 0 \\ t &= 2 \\ v(2) &= -32(2) - 22 \\ &= -86 \text{ ft/sec} \end{aligned}$$

**94.** $s(t) = -4.9t^2 + v_0t + s_0$

$= -4.9t^2 + s_0 = 0$ when $t = 6.8$.

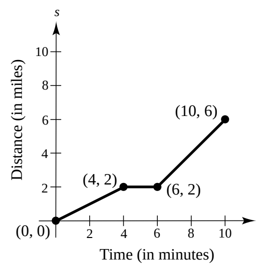
$s_0 = 4.9t^2 = 4.9(6.8)^2 = 226.6 \text{ m}$

**96.**




(The velocity has been converted to miles per hour)

**98.** This graph corresponds with Exercise 75.




**100.** $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$.

Average velocity: $$\frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} = \frac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c)]}{2\Delta t}$$

$$= \frac{-(1/2)a(t_0^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a(t_0^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t}$$

$$= \frac{-2at_0\Delta t}{2\Delta t}$$

$$= -at_0$$

$$= s'(t_0) \quad \text{Instantaneous velocity at } t = t_0$$

**102.** $V = s^3, \frac{dV}{ds} = 3s^2$

When $s = 4$ cm, $\frac{dV}{ds} = 48$ cm$^2$.

**104.** $C = (\text{gallons of fuel used})(\text{cost per gallon})$

$$= \left(\frac{15{,}000}{x}\right)(1.25) = \frac{18{,}750}{x}$$

$$\frac{dC}{dx} = -\frac{18{,}750}{x^2}$$

| $x$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|
| $C$ | 1875 | 1250 | 537.5 | 750 | 625 | 535.71 | 468.75 |
| $\frac{dC}{dx}$ | $-187.5$ | $-83.333$ | $-46.875$ | $-30$ | $-20.833$ | $-15.306$ | $-11.719$ |

The driver who gets 15 miles per gallon would benefit more from a 1 mile per gallon increase in fuel efficiency. The rate of change is larger when $x = 15$.

**106.** $\frac{dT}{dt} = K(T - T_a)$

**108.** $y = \dfrac{1}{x},\ x > 0$

$y' = -\dfrac{1}{x^2}$

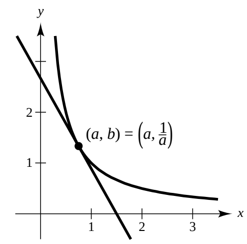
At $(a, b)$, the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}.$$

The $x$-intercept is $(2a, 0)$.

The $y$-intercept is $\left(0, \dfrac{2}{a}\right)$.

The area of the triangle is $A = \dfrac{1}{2}bh = \dfrac{1}{2}(2a)\left(\dfrac{2}{a}\right) = 2$.




**110.** $y = x^2$

$y' = 2x$

(a) Tangent lines through $(0, a)$:

$$\begin{aligned} y - a &= 2x(x - 0) \\ x^2 - a &= 2x^2 \\ -a &= x^2 \\ \pm\sqrt{-a} &= x \end{aligned}$$

The points of tangency are $\left(\pm\sqrt{-a}, -a\right)$. At $\left(\sqrt{-a}, -a\right)$ the slope is $y'\left(\sqrt{-a}\right) = 2\sqrt{-a}$. At $\left(-\sqrt{-a}, -a\right)$ the slope is $y'\left(-\sqrt{-a}\right) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}\left(x - \sqrt{-a}\right)$ and $y + a = -2\sqrt{-a}\left(x + \sqrt{-a}\right)$

$y = 2\sqrt{-a}\,x + a \qquad\qquad y = -2\sqrt{-a}\,x + a$

**Restriction:** $a$ must be negative.

(b) Tangent lines through $(a, 0)$:

$$\begin{aligned} y - 0 &= 2x(x - a) \\ x^2 &= 2x^2 - 2ax \\ 0 &= x^2 - 2ax = x(x - 2a) \end{aligned}$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$ the slope is $y'(0) = 0$. At $(2a, 4a^2)$ the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$y = 0 \qquad\qquad y = 4ax - 4a^2$

**Restriction:** None, $a$ can be any real number.

**112.** $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, $n$ an integer.

$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

You can verify this by graphing $f_1$ and $f_2$ and observing the locations of the sharp turns.

## Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

**2.** $f(x) = (6x + 5)(x^3 - 2)$

$$f'(x) = (6x + 5)(3x^2) + (x^3 - 2)(6)$$
$$= 18x^3 + 15x^2 + 6x^3 - 12$$
$$= 24x^3 + 15x^2 - 12$$

**4.** $g(s) = \sqrt{s}(4 - s^2) = s^{1/2}(4 - s^2)$

$$g'(s) = s^{1/2}(-2s) + (4 - s^2)\frac{1}{2}s^{-1/2} = -2s^{3/2} + \frac{4 - s^2}{2s^{1/2}}$$
$$= \frac{4 - 5s^2}{2s^{1/2}}$$

**6.** $g(x) = \sqrt{x}\sin x$

$$g'(x) = \sqrt{x}\cos x + \sin x\left(\frac{1}{2\sqrt{x}}\right) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$$

**8.** $g(t) = \dfrac{t^2 + 2}{2t - 7}$

$$g'(t) = \frac{(2t - 7)(2t) - (t^2 + 2)(2)}{(2t - 7)^2} = \frac{2t^2 - 14t - 4}{(2t - 7)^2}$$

**10.** $h(s) = \dfrac{s}{\sqrt{s} - 1}$

$$h'(s) = \frac{(\sqrt{s} - 1)(1) - s\left(\frac{1}{2}s^{-1/2}\right)}{\left(\sqrt{s} - 1\right)^2}$$
$$= \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{\left(\sqrt{s} - 1\right)^2} = \frac{\sqrt{s} - 2}{2\left(\sqrt{s} - 1\right)^2}$$

**12.** $f(t) = \dfrac{\cos t}{t^3}$

$$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t\sin t + 3\cos t}{t^4}$$

**14.** $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$f'(x) = (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2)$$
$$= 3x^2(x - 1)^2 + 2(x - 1)^2(x^2 + x + 1)$$
$$= (x - 1)^2(5x^2 + 2x + 2)$$
$$f'(1) = 0$$

**16.** $f(x) = \dfrac{x + 1}{x - 1}$

$$f'(x) = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}$$
$$= \frac{x - 1 - x - 1}{(x - 1)^2}$$
$$= -\frac{2}{(x - 1)^2}$$
$$f'(2) = -\frac{2}{(2 - 1)^2} = -2$$

**18.** $f(x) = \dfrac{\sin x}{x}$

$$f'(x) = \frac{(x)(\cos x) - (\sin x)(1)}{x^2}$$
$$= \frac{x\cos x - \sin x}{x^2}$$
$$f'\left(\frac{\pi}{6}\right) = \frac{(\pi/6)\left(\sqrt{3}/2\right) - (1/2)}{\pi^2/36}$$
$$= \frac{3\sqrt{3}\pi - 18}{\pi^2}$$
$$= \frac{3\left(\sqrt{3}\pi - 6\right)}{\pi^2}$$

| | *Function* | *Rewrite* | *Derivative* | *Simplify* |
|---|---|---|---|---|
| **20.** | $y = \dfrac{5x^2 - 3}{4}$ | $y = \dfrac{5}{4}x^2 - \dfrac{3}{4}$ | $y' = \dfrac{10}{4}x$ | $y' = \dfrac{5x}{2}$ |

| *Function* | *Rewrite* | *Derivative* | *Simplify* |
|---|---|---|---|
| **22.** $y = \dfrac{4}{5x^2}$ | $y = \dfrac{4}{5}x^{-2}$ | $y' = -\dfrac{8}{5}x^{-3}$ | $y' = -\dfrac{8}{5x^3}$ |
| **24.** $y = \dfrac{3x^2 - 5}{7}$ | $y = \dfrac{3}{7}x^2 - \dfrac{5}{7}$ | $y' = \dfrac{6x}{7}$ | $y' = \dfrac{6}{7}x$ |

**26.** $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}$$

**28.** $f(x) = x^4\left[1 - \dfrac{2}{x + 1}\right] = x^4\left[\dfrac{x - 1}{x + 1}\right]$

$$f'(x) = x^4\left[\frac{(x + 1) - (x - 1)}{(x + 1)^2}\right] + \left[\frac{x - 1}{x + 1}\right](4x^3)$$

$$= 2x^3\left[\frac{2x^2 + x - 2}{(x + 1)^2}\right]$$

**30.** $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right) = x^{1/3}(x^{1/2} + 3)$

$$f'(x) = x^{1/3}\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 3)\left(\frac{1}{3}x^{-2/3}\right)$$

$$= \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

**Alternate solution:**

$$f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$$

$$= x^{5/6} + 3x^{1/3}$$

$$f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

**32.** $h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$

$$h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

**34.** $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right) = 2x - \dfrac{x^2}{x + 1}$

$$g'(x) = 2 - \frac{(x + 1)2x - x^2(1)}{(x + 1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x + 1)^2} = \frac{x^2 + 2x + 2}{(x + 1)^2}$$

**36.** $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$

$$f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1)$$

$$= (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1)$$

$$= 2x^5 + x^4 + 3x^3 + x - 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x$$

$$= 6x^5 + 4x^3 - 3x^2 - 1$$

**38.** $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$

$$f'(x) = \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$$

$$= \frac{-4xc^2}{(c^2 + x^2)^2}$$

**40.** $f(\theta) = (\theta + 1)\cos\theta$

$$f'(\theta) = (\theta + 1)(-\sin\theta) + (\cos\theta)(1)$$

$$= \cos\theta - (\theta + 1)\sin\theta$$

**42.** $f(x) = \dfrac{\sin x}{x}$

$f'(x) = \dfrac{x \cos x - \sin x}{x^2}$

**44.** $y = x + \cot x$

$y' = 1 - \csc^2 x = -\cot^2 x$

**46.** $h(s) = \dfrac{1}{s} - 10 \csc s$

$h'(s) = -\dfrac{1}{s^2} + 10 \csc s \cot s$

**48.** $y = \dfrac{\sec x}{x}$

$y' = \dfrac{x \sec x \tan x - \sec x}{x^2}$

$= \dfrac{\sec x(x \tan x - 1)}{x^2}$

**50.** $y = x \sin x + \cos x$

$y' = x \cos x + \sin x - \sin x = x \cos x$

**52.** $f(x) = \sin x \cos x$

$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$

$= \cos 2x$

**54.** $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

$h'(\theta) = 5\theta \sec \theta \tan \theta + 5 \sec \theta + \theta \sec^2 \theta + \tan \theta$

**56.** $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

$f'(x) = 2\dfrac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2}$ (form of answer may vary)

**58.** $f(\theta) = \dfrac{\sin \theta}{1 - \cos \theta}$

$f'(\theta) = \dfrac{1}{\cos \theta - 1} = \dfrac{\cos \theta - 1}{(1 - \cos \theta)^2}$

(form of answer may vary)

**60.** $f(x) = \tan x \ \cot x = 1$

$f'(x) = 0$

$f'(1) = 0$

**62.** $f(x) = \sin x(\sin x + \cos x)$

$f'(x) = \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x$

$= \sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x$

$= \sin 2x + \cos 2x$

$f'\left(\dfrac{\pi}{4}\right) = \sin \dfrac{\pi}{2} + \cos \dfrac{\pi}{2} = 1$

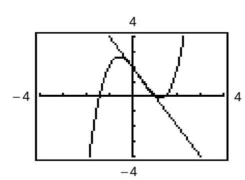
**64.** (a) $f(x) = (x - 1)(x^2 - 2)$, $(0, 2)$

$f'(x) = (x - 1)(2x) + (x^2 - 2)(1) = 3x^2 - 2x - 2$

$f'(0) = -2 =$ slope at $(0, 2)$.

Tangent line: $y - 2 = -2x \Longrightarrow y = -2x + 2$

(b)



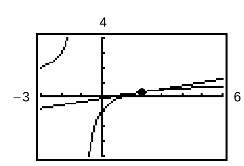

**66.** (a) $f(x) = \dfrac{x - 1}{x + 1}$, $\left(2, \dfrac{1}{3}\right)$

$f'(x) = \dfrac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2}$

$f'(2) = \dfrac{2}{9} =$ slope at $\left(2, \dfrac{1}{3}\right)$.

Tangent line: $y - \dfrac{1}{3} = \dfrac{2}{9}(x - 2) \Longrightarrow y = \dfrac{2}{9}x - \dfrac{1}{9}$

(b)



**68.** (a) $f(x) = \sec x, \quad \left(\dfrac{\pi}{3}, 2\right)$

$f'(x) = \sec x \tan x$

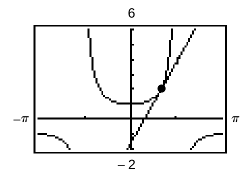
$f'\left(\dfrac{\pi}{3}\right) = 2\sqrt{3} = \text{slope at } \left(\dfrac{\pi}{3}, 2\right).$

Tangent line:

$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$$

(b)




**70.** $f(x) = \dfrac{x^2}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2)(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$$

$f'(x) = 0$ when $x = 0$.

Horizontal tangent is at $(0, 0)$.

**72.** $f'(x) = \dfrac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \dfrac{x \cos x - \sin x}{x^2}$

$$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

$$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$$

$f$ and $g$ differ by a constant.

**74.** $f(x) = \dfrac{\cos x}{x^n} = x^{-n} \cos x$

$$\begin{aligned} f'(x) &= -x^{-n} \sin x - nx^{-n-1} \cos x \\ &= -x^{-n-1}(x \sin x + n \cos x) \\ &= -\frac{x \sin x + n \cos x}{x^{n+1}} \end{aligned}$$

When $n = 1$: $f'(x) = -\dfrac{x \sin x + \cos x}{x^2}$.

When $n = 2$: $f'(x) = -\dfrac{x \sin x + 2 \cos x}{x^3}$.

When $n = 3$: $f'(x) = -\dfrac{x \sin x + 3 \cos x}{x^4}$.

When $n = 4$: $f'(x) = -\dfrac{x \sin x + 4 \cos x}{x^5}$.

For general $n$, $f'(x) = -\dfrac{x \sin x + n \cos x}{x^{n+1}}$.

**76.** $V = \pi r^2 h = \pi(t + 2)\left(\dfrac{1}{2}\sqrt{t}\right)$

$$= \frac{1}{2}(t^{3/2} + 2t^{1/2})\pi$$

$$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t + 2}{4t^{1/2}}\pi \text{ cubic inches/sec}$$

**78.** $P = \dfrac{k}{V}$

$$\frac{dP}{dV} = -\frac{k}{V^2}$$

**80.** $f(x) = \sec x$

$g(x) = \csc x, \quad [0, 2\pi)$

$f'(x) = g'(x)$

$$\sec x \tan x = -\csc x \cot x \implies \frac{\sec x \tan x}{\csc x \cot x} = -1 \implies \frac{\dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x}}{\dfrac{1}{\sin x} \cdot \dfrac{\cos x}{\sin x}} = -1 \implies$$

$$\frac{\sin^3 x}{\cos^3 x} = -1 \implies \tan^3 x = -1 \implies \tan x = -1$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

**82.** (a) $n(t) = -9.6643t^2 + 90.7414t + 77.5029$

$v(t) = -276.4643t^2 + 2987.6929t + 1809.9714$

(b) $A = \dfrac{v(t)}{n(t)} \approx \dfrac{-276.46t^2 + 2987.69t + 1809.97}{-9.66t^2 + 90.74t + 77.50}$

$A$ represents the average retail value (in millions of dollars) per 1000 motor homes.

(c) $A'(t) \approx \dfrac{40.46(x^2 - 2.09x + 17.83)}{(x^2 - 9.39x - 8.02)^2}$

**84.** $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3}$

$f''(x) = \dfrac{192}{x^4}$

**86.** $f(x) = \dfrac{x^2 + 2x - 1}{x} = x + 2 - \dfrac{1}{x}$

$f'(x) = 1 + \dfrac{1}{x^2}$

$f''(x) = -\dfrac{2}{x^3}$

**88.** $f(x) = \sec x$

$f'(x) = \sec x \tan x$

$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$

$= \sec x(\sec^2 x + \tan^2 x)$

**90.** $f''(x) = 2 - 2x^{-1}$

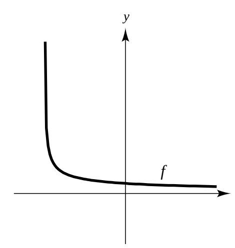
$f'''(x) = 2x^{-2} = \dfrac{2}{x^2}$

**92.** $f^{(4)}(x) = 2x + 1$

$f^{(5)}(x) = 2$

$f^{(6)}(x) = 0$

**94.** The graph of a differentiable function $f$ such that $f > 0$ and $f' < 0$ for all real numbers $x$ would in general look like the graph below.




**96.** $f(x) = 4 - h(x)$

$f'(x) = -h'(x)$

$f'(2) = -h'(2) = -4$

**98.** $f(x) = g(x)h(x)$

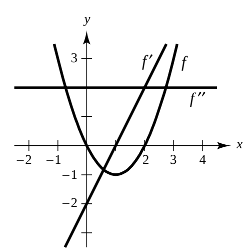
$f'(x) = g(x)h'(x) + h(x)g'(x)$

$f'(2) = g(2)h'(2) + h(2)g'(2)$

$= (3)(4) + (-1)(-2)$

$= 14$

**100.**




It appears that $f$ is quadratic; so $f'$ would be linear and $f''$ would be constant.

**102.** $s(t) = -8.25t^2 + 66t$

$v(t) = -16.50t + 66$

$a(t) = -16.50$

| $t$(sec) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $s(t)$ (ft) | 0 | 57.75 | 99 | 123.75 | 132 |
| $v(t) = s'(t)$ (ft/sec) | 66 | 49.5 | 33 | 16.5 | 0 |
| $a(t) = v'(t)$ (ft/sec²) | −16.5 | −16.5 | −16.5 | −16.5 | −16.5 |

Average velocity on:

$[0, 1]$ is $\dfrac{57.75 - 0}{1 - 0} = 57.75$.

$[1, 2]$ is $\dfrac{99 - 57.75}{2 - 1} = 41.25$.

$[2, 3]$ is $\dfrac{123.75 - 99}{3 - 2} = 24.75$.

$[3, 4]$ is $\dfrac{132 - 123.75}{4 - 3} = 8.25$.

**104.** (a) $f(x) = x^n$

$f^n(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

(b) $f(x) = \dfrac{1}{x}$

$$f^{(n)}(x) = \frac{(-1)^n(n)(n-1)(n-2)\cdots(2)(1)}{x^{n+1}} = \frac{(-1)^n n!}{x^{n+1}}$$

**Note:** $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ (read "$n$ factorial.")

**106.** $[xf(x)]' = xf'(x) + f(x)$

$[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$

$[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$

In general, $[xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$.

**108.** $f(x) = \sin x \qquad f\left(\dfrac{\pi}{2}\right) = 1$

$f'(x) = \cos x \qquad f'\left(\dfrac{\pi}{2}\right) = 0$

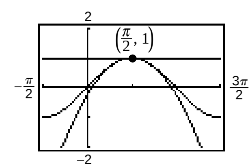
$f''(x) = -\sin x \qquad f''\left(\dfrac{\pi}{2}\right) = -1$

(a) $P_1(x) = f'(a)(x-a) + f(a) = 0\left(x - \dfrac{\pi}{2}\right) + 1 = 1$

$$P_2(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a) = \frac{1}{2}(-1)\left(x - \frac{\pi}{2}\right)^2 + 1$$

$$= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

(b)




(c) $P_2$ is a better approximation than $P_1$.

(d) The accuracy worsens as you move farther away from $x = a = \dfrac{\pi}{2}$.

**110.** True. $y$ is a fourth-degree polynomial.

$\dfrac{d^n y}{dx^n} = 0$ when $n > 4$.

**112.** True

**114.** True. If $v(t) = c$ then

$a(t) = v'(t) = 0$.

**116.** (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$

$= fg'' - f''g$ True

(b) $(fg)'' = (fg' + f'g)'$

$= fg'' + f'g' + f'g' + f''g$

$= fg'' + 2f'g' + f''g$

$\neq fg'' + f''g$ False

## Section 2.4 The Chain Rule

| $y = f(g(x))$ | $u = g(x)$ | $y = f(u)$ |
|---|---|---|
| **2.** $y = \dfrac{1}{\sqrt{x+1}}$ | $u = x + 1$ | $y = u^{-1/2}$ |
| **4.** $y = 3\tan(\pi x^2)$ | $u = \pi x^2$ | $y = 3\tan u$ |
| **6.** $y = \cos\dfrac{3x}{2}$ | $u = \dfrac{3x}{2}$ | $y = \cos u$ |

**8.** $y = (2x^3 + 1)^2$

$y' = 2(2x^3 + 1)(6x^2) = 12x^2(2x^3 + 1)$

**10.** $y = 3(4 - x^2)^5$

$y' = 15(4 - x^2)(-2x) = -30x(4 - x^2)$

**12.** $f(t) = (9t + 2)^{2/3}$

$$f'(t) = \frac{2}{3}(9t + 2)^{-1/3}(9) = \frac{6}{\sqrt[3]{9t + 2}}$$

**14.** $g(x) = \sqrt{5 - 3x} = (5 - 3x)^{1/2}$

$$g'(x) = \frac{1}{2}(5 - 3x)^{-1/2}(-3) = \frac{-3}{2\sqrt{5 - 3x}}$$

**16.** $g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = |x - 1|$

$$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$$

**18.** $f(x) = -3(2 - 9x)^{1/4}$

$$f'(x) = -\frac{3}{4}(2 - 9x)^{-3/4}(-9) = \frac{27}{4(2 - 9x)^{3/4}}$$

**20.** $s(t) = (t^2 + 3t - 1)^{-1}$

$$s'(t) = -1(t^2 + 3t - 1)^{-2}(2t + 3)$$

$$= \frac{-(2t + 3)}{(t^2 + 3t - 1)^2}$$

**22.** $y = -5(t + 3)^{-3}$

$$y' = 15(t + 3)^{-4} = \frac{15}{(t + 3)^4}$$

**24.** $g(t) = (t^2 - 2)^{-1/2}$

$$g'(t) = -\frac{1}{2}(t^2 - 2)^{-3/2}(2t) = -\frac{t}{(t^2 - 2)^{3/2}}$$

**26.** $f(x) = x(3x - 9)^3$

$$f'(x) = x[3(3x - 9)^2(3)] + (3x - 9)^3(1)$$

$$= (3x - 9)^2[9x + 3x - 9]$$

$$= 27(x - 3)^2(4x - 3)$$

**28.** $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

$$y' = \frac{1}{2}x^2\left(\frac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2}$$

$$= \frac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2}$$

$$= \frac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}}$$

**30.** $y = \dfrac{x}{\sqrt{x^4 + 4}}$

$$y' = \frac{(x^4 + 4)^{1/2}(1) - x\frac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4}$$

$$= \frac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} = \frac{4 - x^4}{(x^4 + 4)^{3/2}}$$

**32.** $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

$$h'(t) = 2\left(\frac{t^2}{t^3 + 2}\right)\left(\frac{(t^3 + 2)(2t) - t^2(3t^2)}{(t^3 + 2)^2}\right)$$

$$= \frac{2t^2(4t - t^4)}{(t^3 + 2)^3} = \frac{2t^3(4 - t^3)}{(t^3 + 2)^3}$$

**34.** $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

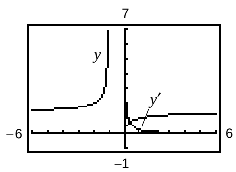
$$g'(x) = 3\left(\frac{3x^2 - 2}{2x + 3}\right)^2\left(\frac{(2x + 3)(6x) - (3x^2 - 2)(2)}{(2x + 3)^2}\right)$$

$$= \frac{3(3x^2 - 2)^2(6x^2 + 18x + 4)}{(2x + 3)^4} = \frac{6(3x^2 - 2)^2(3x^2 + 9x + 2)}{(2x + 3)^4}$$

**36.** $y = \sqrt{\dfrac{2x}{x + 1}}$

$$y' = \frac{1}{\sqrt{2x}(x + 1)^{3/2}}$$

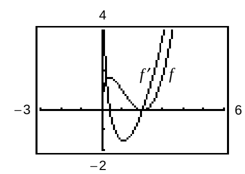
$y'$ has no zeros.




**38.** $f(x) = \sqrt{x}(2 - x)^2$

$$f'(x) = \frac{(x - 2)(5x - 2)}{2\sqrt{x}}$$

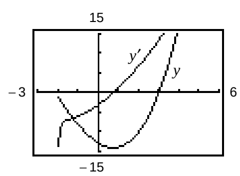
The zeros of $f'$ correspond to the points on the graph of $f$ where the tangent lines are horizontal.




**40.** $y = (t^2 - 9)\sqrt{t + 2}$

$$y' = \frac{5t^2 + 8t - 9}{2\sqrt{t + 2}}$$

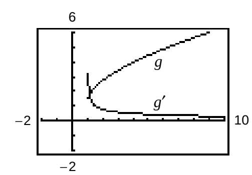
The zero of $y'$ corresponds to the point on the graph of $y$ where the tangent line is horizontal.




**42.** $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

$$g'(x) = \frac{1}{2\sqrt{x - 1}} + \frac{1}{2\sqrt{x + 1}}$$

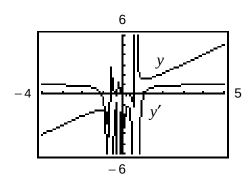
$g'$ has no zeros.




**44.** $y = x^2 \tan \dfrac{1}{x}$

$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$

The zeros of $y'$ correspond to the points on the graph of $y$ where the tangent lines are horizontal.




**46.** (a) $y = \sin 3x$

$y' = 3 \cos 3x$

$y'(0) = 3$

3 cycles in $[0, 2\pi]$

(b) $y = \sin\left(\dfrac{x}{2}\right)$

$y' = \left(\dfrac{1}{2}\right)\cos\left(\dfrac{x}{2}\right)$

$y'(0) = \dfrac{1}{2}$

Half cycle in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is $a$.

**48.** $y = \sin \pi x$

$$\frac{dy}{dx} = \pi \cos \pi x$$

**50.** $h(x) = \sec(x^2)$

$$h'(x) = 2x \sec(x^2) \tan(x^2)$$

**52.** $y = \cos (1 - 2x)^2 = \cos ((1 - 2x)^2)$

$$y' = -\sin (1 - 2x)^2(2(1 - 2x)(-2)) = 4(1 - 2x) \sin(1 - 2x)^2$$

**54.** $g(\theta) = \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)$

$$g'(\theta) = \sec\left(\frac{1}{2}\theta\right) \sec^2\left(\frac{1}{2}\theta\right)\frac{1}{2} + \tan\left(\frac{1}{2}\theta\right) \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)\frac{1}{2}$$

$$= \frac{1}{2} \sec\left(\frac{1}{2}\theta\right)\left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right)\right]$$

**56.** $g(v) = \dfrac{\cos v}{\csc v} = \cos v \cdot \sin v$

$$g'(v) = \cos v(\cos v) + \sin v(-\sin v) = \cos^2 v - \sin^2 v = \cos 2v$$

**58.** $y = 2 \tan^3 x$

$$y' = 6 \tan^2 x \cdot \sec^2 x$$

**60.** $g(t) = 5 \cos^2 \pi t = 5(\cos \pi t)^2$

$$g'(t) = 10 \cos \pi t(-\sin \pi t)(\pi)$$

$$= -10\pi(\sin \pi t)(\cos \pi t) = -5\pi \sin 2\pi t$$

**62.** $h(t) = 2 \cot^2(\pi t + 2)$

$$h'(t) = 4 \cot(\pi t + 2)(-\csc^2(\pi t + 2)(\pi))$$

$$= -4\pi \cot(\pi t + 2) \csc^2(\pi t + 2)$$

**64.** $y = 3x - 5 \cos(\pi x)^2$

$$= 3x - 5 \cos(\pi^2 x^2)$$

$$\frac{dy}{dx} = 3 + 5 \sin(\pi^2 x^2)(2\pi^2 x)$$

$$= 3 + 10\pi^2 x \sin(\pi x)^2$$

**66.** $y = \sin x^{1/3} + (\sin x)^{1/3}$

$$y' = \cos x^{1/3}\left(\frac{1}{3}x^{-2/3}\right) + \frac{1}{3}(\sin x)^{-2/3} \cos x$$

$$= \frac{1}{3}\left[\frac{\cos x^{1/3}}{x^{2/3}} + \frac{\cos x}{(\sin x)^{2/3}}\right]$$

**68.** $y = (3x^3 + 4x)^{1/5}, \quad (2, 2)$

$$y' = \frac{1}{5}(3x^3 + 4x)^{-4/5}(9x^2 + 4)$$

$$= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}}$$

$$y'(2) = \frac{1}{2}$$

**70.** $f(x) = \dfrac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}, \quad \left(4, \dfrac{1}{16}\right)$

$$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$

$$f'(4) = -\frac{5}{32}$$

**72.** $f(x) = \dfrac{x + 1}{2x - 3}, \quad (2, 3)$

$$f'(x) = \frac{(2x - 3)(1) - (x + 1)(2)}{(2x - 3)^2} = \frac{-5}{(2x - 3)^2}$$

$$f'(2) = -5$$

**74.** $y = \dfrac{1}{x} + \sqrt{\cos x}, \quad \left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$y'(\pi/2)$ is undefined.

**76.** (a) $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}, \quad (2, 2)$

$$f'(x) = \frac{1}{3}x\left[\frac{1}{2}(x^2 + 5)^{-1/2}(2x)\right] + \frac{1}{3}(x^2 + 5)^{1/2}$$

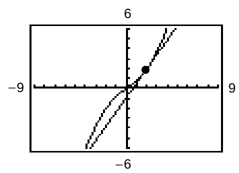
$$= \frac{x^2}{3\sqrt{x^2 + 5}} + \frac{1}{3}\sqrt{x^2 + 5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x - 2) \Longrightarrow 13x - 9y - 8 = 0$$

(b)




**78.** (a) $f(x) = \tan^2 x, \quad \left(\frac{\pi}{4}, 1\right)$

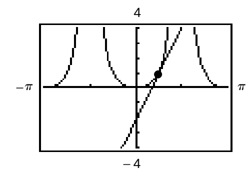
$$f'(x) = 2\tan x \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$$

Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Longrightarrow 4x - y + (1 - \pi) = 0$$

(b)




**80.** $f(x) = (x - 2)^{-1}$

$$f'(x) = -(x - 2)^{-2} = \frac{-1}{(x - 2)^2}$$

$$f''(x) = 2(x - 2)^{-3} = \frac{2}{(x - 2)^3}$$

**82.** $f(x) = \sec^2 \pi x$

$$f'(x) = 2\sec \pi x(\pi \sec \pi x \tan \pi x)$$

$$= 2\pi \sec^2 \pi x \tan \pi x$$

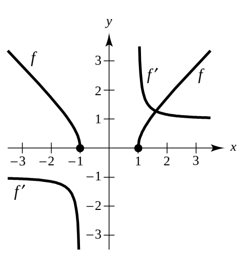
$$f''(x) = 2\pi \sec^2 \pi x(\sec^2 \pi x)(\pi) + 2\pi \tan \pi x(2\pi \sec^2 \pi x \tan \pi x)$$

$$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$

$$= 2\pi^2 \sec^2 \pi x(\sec^2 \pi x + 2\tan^2 \pi x)$$

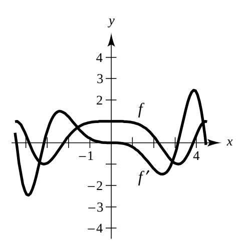
$$= 2\pi^2 \sec^2 \pi x(3\sec^2 \pi x - 2)$$

**84.**




$f$ is decreasing on $(-\infty, -1)$ so $f'$ must be negative there. $f$ is increasing on $(1, \infty)$ so $f'$ must be postive there.

**86.**




The zeros of $f'$ correspond to the points where the graph of $f$ has horizontal tangents.

**88.** $g(x) = f(x^2)$

$$g'(x) = f'(x^2)(2x) \Longrightarrow g'(x) = 2x\,f'(x^2)$$

**90.** (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Longrightarrow g'(x) = 0$

$g'(x) = 2\sin x \cos x + 2\cos x(-\sin x) = 0$

(b) $\tan^2 x + 1 = \sec^2 x$

$g(x) + 1 = f(x)$

Taking derivatives of both sides,

$g'(x) = f'(x)$.

Equivalently, $f'(x) = 2\sec x \cdot \sec x \cdot \tan x$ and $g'(x) = 2\tan x \cdot \sec^2 x$, which are the same.

**92.** $y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$

$v = y' = \frac{1}{3}[-12\sin 12t] - \frac{1}{4}[12\cos 12t]$

$= -4\sin 12t - 3\cos 12t$

When $t = \pi/8$, $y = 0.25$ feet and $v = 4$ feet per second.

**94.** $y = A\cos \omega t$

(a) Amplitude: $A = \dfrac{3.5}{2} = 1.75$

$y = 1.75\cos \omega t$

Period: $10 \Longrightarrow \omega = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

$y = 1.75\cos \dfrac{\pi t}{5}$

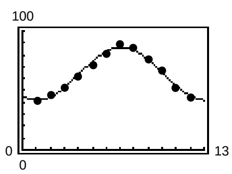
(b) $v = y' = 1.75\left[-\dfrac{\pi}{5}\sin \dfrac{\pi t}{5}\right]$

$= -0.35\,\pi \sin \dfrac{\pi t}{5}$

**96.** (a) Using a graphing utility, or by trial and error, you obtain a model of the form

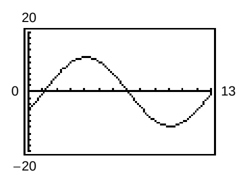
$T(t) = 64.18 - 22.15\sin\left(\dfrac{\pi t}{6} + 1\right)$

(b)




(c) $T'(t) = -22.15\cos\left(\dfrac{\pi t}{6} + 1\right)\left(\dfrac{\pi}{6}\right)$

$= -11.60\cos\left(\dfrac{\pi t}{6} + 1\right)$




(d) The temperature changes most rapidly when $t \approx 4.1$ (April) and $t \approx 10.1$ (October). The temperature changes most slowly ($T'(t) = 0$) when $t \approx 1.1$ (January) and $t \approx 7.1$ (July).

**98.** (a) $g(x) = f(x) - 2 \Longrightarrow g'(x) = f'(x)$

(b) $h(x) = 2f(x) \Longrightarrow h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \Longrightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know $f'(-3x)$.

$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$

$r'(-1) = -3f'(3) = (-3)(-4) = 12$

(d) $s(x) = f(x + 2) \Longrightarrow s'(x) = f'(x + 2)$

Hence, you need to know $f'(x + 2)$.

$s'(-2) = f'(0) = -\frac{1}{3}$, etc.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | $4$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-1$ | $-2$ | $-4$ |
| $g'(x)$ | $4$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $-1$ | $-2$ | $-4$ |
| $h'(x)$ | $8$ | $\frac{4}{3}$ | $-\frac{2}{3}$ | $-2$ | $-4$ | $-8$ |
| $r'(x)$ | | $12$ | $1$ | | | |
| $s'(x)$ | $-\frac{1}{3}$ | $-1$ | $-2$ | $-4$ | | |

**100.** $f(x+p) = f(x)$ for all $x$.

(a) Yes, $f'(x+p) = f'(x)$, which shows that $f'$ is periodic as well.

(b) Yes, let $g(x) = f(2x)$, so $g'(x) = 2f'(2x)$. Since $f'$ is periodic, so is $g'$.

**102.** If $f(-x) = -f(x)$, then

$$\frac{d}{dx}[f(-x)] = \frac{d}{x}[-f(x)]$$
$$f'(-x)(-1) = -f'(x)$$
$$f'(-x) = f'(x).$$

Thus, $f'(x)$ is even.

**104.** $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}\left[\sqrt{u^2}\right] = \frac{1}{2}(u^2)^{-1/2}(2uu') = \frac{uu'}{\sqrt{u^2}} = u'\frac{u}{|u|}, u \neq 0$$

**106.** $f(x) = |x^2 - 4|$

$$f'(x) = 2x\left(\frac{x^2-4}{|x^2-4|}\right), x \neq \pm 2$$

**108.** $f(x) = |\sin x|$

$$f'(x) = \cos x\left(\frac{\sin x}{|\sin x|}\right), x \neq k\pi$$

**110.** (a) $f(x) = \sec(2x)$

$$f'(x) = 2(\sec 2x)(\tan 2x)$$
$$f''(x) = 2[2(\sec 2x)(\tan 2x)]\tan 2x + 2(\sec 2x)(\sec^2 2x)(2)$$
$$= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x]$$

$$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

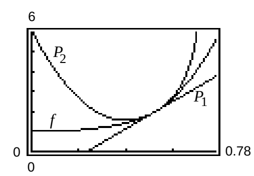
$$f'\left(\frac{\pi}{6}\right) = 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$$

$$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$$

$$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$P_2(x) = \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$
$$= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

(b)




(c) $P_2$ is a better approximation than $P_1$.

(d) The accuracy worsens as you move away from $x = \pi/6$.

**112.** False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2\cos 2x)$.

**114.** False. First apply the Product Rule.

## Section 2.5 Implicit Differentiations

**2.** $$x^2 - y^2 = 16$$
$$2x - 2yy' = 0$$
$$y' = \frac{x}{y}$$

**4.** $$x^3 + y^3 = 8$$
$$3x^2 + 3y^2y' = 0$$
$$y' = -\frac{x^2}{y^2}$$

**6.** $$x^2y + y^2x = -2$$
$$x^2y' + 2xy + y^2 + 2yxy' = 0$$
$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$
$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

**8.** $$(xy)^{1/2} - x + 2y = 0$$
$$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$$
$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} - 1 + 2y' = 0$$
$$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}\,y' = 0$$
$$y' = \frac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$$

**10.** $$2 \sin x \cos y = 1$$
$$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$$
$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$
$$= \cot x \cot y$$

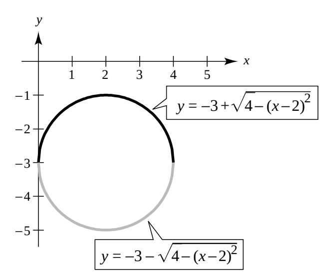
**12.** $$(\sin \pi x + \cos \pi y)^2 = 2$$
$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$
$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$
$$y' = \frac{\cos \pi x}{\sin \pi y}$$

**14.** $$\cot y = x - y$$
$$(-\csc^2 y)y' = 1 - y'$$
$$y' = \frac{1}{1 - \csc^2 y}$$
$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

**16.** $$x = \sec\frac{1}{y}$$
$$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{\mathrm{y}}$$
$$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$$

**18.** (a) $$(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$$
$$(x - 2)^2 + (y + 3)^2 = 4 \text{ (Circle)}$$
$$(y + 3)^2 = 4 - (x - 2)^2$$
$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)




(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}[4 - (x - 2)^2]^{-1/2}(-2)(x - 2)$$
$$= \frac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2}}$$
$$= \frac{-(x - 2)}{\pm\sqrt{4 - (x - 2)^2}}$$
$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$
$$= \frac{-(x - 2)}{y + 3}$$

(d) Implicitly:

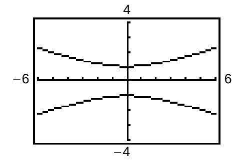
$$2x + 2yy' - 4 + 6y' = 0$$
$$(2y + 6)y' = -2(x - 2)$$
$$y' = \frac{-(x - 2)}{y + 3}$$

**20.** (a) $9y^2 = x^2 + 9$

$$y^2 = \frac{x^2}{9} + 1 = \frac{x^2 + 9}{9}$$

$$y = \frac{\pm\sqrt{x^2 + 9}}{3}$$

(b)




(c) Explicitly: $\dfrac{dy}{dx} = \dfrac{\pm\frac{1}{2}(x^2 + 9)^{-1/2}(2x)}{3} = \dfrac{\pm x}{3\sqrt{x^2 + 9}} = \dfrac{\pm x}{3(\pm 3y)} = \dfrac{x}{9y}$

(d) Implicitly: $9y^2 - x^2 = 9$

$$18yy' - 2x = 0$$

$$18yy' = 2x$$

$$y' = \frac{2x}{18y} = \frac{x}{9y}$$

**22.** $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At $(1, 1)$: $y' = \dfrac{2}{3}$.

**24.** $(x + y)^3 = x^3 + y^3$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At $(-1, 1)$: $y' = -1$.

**26.** $x^3 + y^3 = 4xy + 1$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$(3y^2 - 4x)y' = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(3y^2 - 4x)}$$

At $(2, 1)$, $y' = \dfrac{4 - 12}{3 - 8} = \dfrac{8}{5}$

**28.** $x\cos y = 1$

$$x[-y'\sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x\sin y}$$

$$= \frac{1}{x}\cot y = \frac{\cot y}{x}$$

At $\left(2, \dfrac{\pi}{3}\right)$: $y' = \dfrac{1}{2\sqrt{3}}$.

**30.** $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At $(2, 2)$: $y' = 2$.

**32.** $x^3 + y^3 - 6xy = 0$

$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $\left(\dfrac{4}{3}, \dfrac{8}{3}\right)$: $y' = \dfrac{(16/3) - (16/9)}{(64/9) - (8/3)} = \dfrac{32}{40} = \dfrac{4}{5}$.

**34.**

$$\cos y = x$$
$$-\sin y \cdot y' = 1$$
$$y' = \frac{-1}{\sin y}, \; 0 < y < \pi$$
$$\sin^2 y + \cos^2 y = 1$$
$$\sin^2 y = 1 - \cos^2 y$$
$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$
$$y' = \frac{-1}{\sqrt{1 - x^2}}, \; -1 < x < 1$$

**36.**

$$x^2y^2 - 2x = 3$$
$$2x^2yy' + 2xy^2 - 2 = 0$$
$$x^2yy' + xy^2 - 1 = 0$$
$$y' = \frac{1 - xy^2}{x^2y}$$
$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$
$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$
$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$$
$$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$$
$$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$$
$$y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}$$

**38.**

$$1 - xy = x - y$$
$$y - xy = x - 1$$
$$y = \frac{x - 1}{1 - x} = -1$$
$$y' = 0$$
$$y'' = 0$$

**40.**

$$y^2 = 4x$$
$$2yy' = 4$$
$$y' = \frac{2}{y}$$
$$y'' = -2y^{-2}y' = \left[\frac{-2}{y^2}\right] \cdot \frac{2}{y} = \frac{-4}{y^3}$$

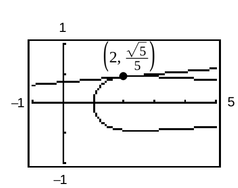
**42.**

$$y^2 = \frac{x - 1}{x^2 + 1}$$
$$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$$
$$= \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$$
$$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$$

At $\left(2, \frac{\sqrt{5}}{5}\right)$: $y' = \frac{1 + 4 - 4}{\left[\left(2\sqrt{5}\right)/5\right](4 + 1)^2} = \frac{1}{10\sqrt{5}}$.

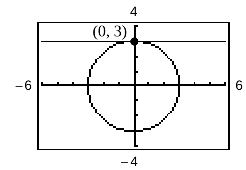
Tangent line: $y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$

$$10\sqrt{5}y - 10 = x - 2$$
$$x - 10\sqrt{5}y + 8 = 0$$



**44.** $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

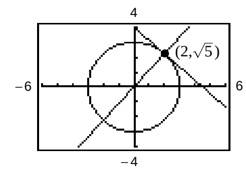



At $(0, 3)$:

Tangent line: $y = 3$

Normal line: $x = 0$.

At $(2, \sqrt{5})$:




Tangent line: $y - \sqrt{5} = \dfrac{-2}{\sqrt{5}}(x - 2) \Rightarrow 2x + \sqrt{5}y - 9 = 0$

Normal line: $y - \sqrt{5} = \dfrac{\sqrt{5}}{2}(x - 2) \Rightarrow \sqrt{5}x - 2y = 0$.

**46.** $y^2 = 4x$

$2yy' = 4$

$$y' = \frac{2}{y} = 1 \text{ at } (1, 2)$$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

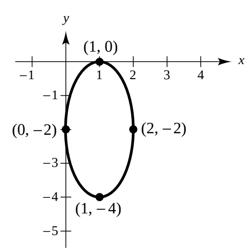
$$\begin{aligned}(x - 1)^2 + [(3 - x) - 2]^2 &= 16\\ 2(x - 1)^2 &= 16\\ x &= 1 \pm 2\sqrt{2}.\end{aligned}$$

Centers of the circles: $\left(1 + 2\sqrt{2}, 2 - 2\sqrt{2}\right)$ and $\left(1 - 2\sqrt{2}, 2 + 2\sqrt{2}\right)$

Equations: $\left(x - 1 - 2\sqrt{2}\right)^2 + \left(y - 2 + 2\sqrt{2}\right)^2 = 16$

$\left(x - 1 + 2\sqrt{2}\right)^2 + \left(y - 2 - 2\sqrt{2}\right)^2 = 16$

**48.** $4x^2 + y^2 - 8x + 4y + 4 = 0$




$8x + 2yy' - 8 + 4y' = 0$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$

Horizontal tangents occur when $x = 1$:

$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$

$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$

Horizontal tangents: $(1, 0)$, $(1, -4)$.

Vertical tangents occur when $y = -2$:

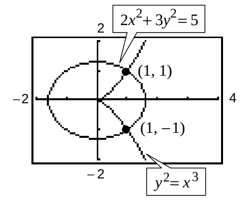
$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$

$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$

Vertical tangents: $(0, -2)$, $(2, -2)$.

**50.** Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$$2x^2 + 3x^3 = 5 \quad \text{and} \quad 3x^3 + 2x^2 - 5 = 0$$




Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$

$\underline{y^2 = x^3}$:　　　$\underline{2x^2 + 3y^2 = 5}$:

$$2yy' = 3x^2 \qquad 4x + 6yy' = 0$$

$$y' = \frac{3x^2}{2y} \qquad y' = -\frac{2x}{3y}$$

At $(1, 1)$, the slopes are:

$$y' = \frac{3}{2} \qquad y' = -\frac{2}{3}.$$

At $(1, -1)$, the slopes are:

$$y' = -\frac{3}{2} \qquad y' = \frac{2}{3}.$$

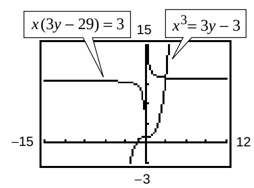
Tangents are perpendicular.

**52.** Rewriting each equation and differentiating,

$$x^3 = 3(y - 1) \qquad x(3y - 29) = 3$$

$$y = \frac{x^3}{3} + 1 \qquad y = \frac{1}{3}\left(\frac{3}{x} + 29\right)$$

$$y' = x^2 \qquad y' = -\frac{1}{x^2}.$$

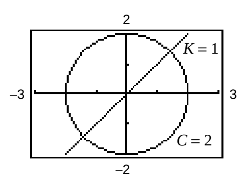


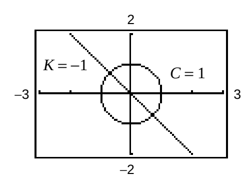

For each value of $x$, the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

**54.** $x^2 + y^2 = C^2 \qquad y = Kx$

$$2x + 2yy' = 0 \qquad y' = K$$

$$y' = -\frac{x}{y}$$




At the point of intersection $(x, y)$ the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

**56.** $x^2 - 3xy^2 + y^3 = 10$

(a) $2x - 3y^2 - 6xyy' + 3y^2y' = 0$

$$(-6xy + 3y^2)y' = 3y^2 - 2x$$

$$y' = \frac{3y^2 - 2x}{3y^2 - 6xy}$$

(b) $2x\dfrac{dx}{dt} - 3y^2\dfrac{dx}{dt} - 6xy\dfrac{dy}{dt} + 3y^2\dfrac{dy}{dt} = 0$

$$(2x - 3y^2)\frac{dx}{dt} = (6xy - 3y^2)\frac{dy}{dt}$$

**58.** (a) $4 \sin x \cos y = 1$

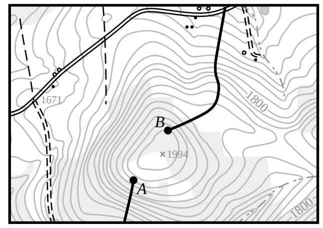
$$4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

(b) $4 \sin x(-\sin y)\dfrac{dy}{dt} + 4 \cos x \dfrac{dx}{dt} \cos y = 0$

$$\cos x \cos y \frac{dx}{dt} = \sin x \sin y \frac{dy}{dt}$$

**60.** Given an implicit equation, first differentiate both sides with respect to $x$. Collect all terms involving $y'$ on the left, and all other terms to the right. Factor out $y'$ on the left side. Finally, divide both sides by the left-hand factor that does not contain $y'$.

**62.** 


Use starting point B.

**64.** $$\sqrt{x} + \sqrt{y} = \sqrt{c}$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at $(x_0, y_0)$:

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

$x$-intercept: $\left(x_0 + \sqrt{x_0}\sqrt{y_0}, 0\right)$

$y$-intercept: $\left(0, y_0 + \sqrt{x_0}\sqrt{y_0}\right)$

Sum of intercepts:

$$\left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

## Section 2.6 Related Rates

**2.** $y = 2(x^2 - 3x)$

$$\frac{dy}{dt} = (4x - 6)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4x - 6}\frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = [4(3) - 6](2) = 12$

(b) When $x = 1$ and $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = \frac{1}{4(1) - 6}(5) = -\frac{5}{2}$

**4.** $$x^2 + y^2 = 25$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right)\frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}.$$

**6.** $y = \dfrac{1}{1 + x^2}$

$\dfrac{dx}{dt} = 2$

$\dfrac{dy}{dt} = \left[\dfrac{-2x}{(1 + x^2)^2}\right]\dfrac{dx}{dt}$

(a) When $x = -2$,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25} \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 0 \text{ cm/sec.}$$

(c) When $x = 2$,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = \frac{-8}{25} \text{ cm/sec.}$$

**8.** $y = \sin x$

$\dfrac{dx}{dt} = 2$

$\dfrac{dy}{dt} = \cos x \dfrac{dx}{dt}$

(a) When $x = \pi/6$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{6}\right)(2) = \sqrt{3} \text{ cm/sec.}$$

(b) When $x = \pi/4$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{4}\right)(2) = \sqrt{2} \text{ cm/sec.}$$

(c) When $x = \pi/3$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{3}\right)(2) = 1 \text{ cm/sec.}$$

**10.** (a) $\dfrac{dx}{dt}$ negative $\Longrightarrow \dfrac{dy}{dt}$ negative

(b) $\dfrac{dy}{dt}$ positive $\Longrightarrow \dfrac{dx}{dt}$ positive

**12.** Answers will vary. See page 145.

**14.** $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$

$\dfrac{dx}{dt} = 2$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2\sin x \cos x)\frac{dx}{dt} = \frac{x + \sin x \cos x}{\sqrt{x^2 + \sin^2 x}}\,\frac{dx}{dt} = \frac{2 + 2\sin x \cos x}{\sqrt{x^2 + \sin^2 x}}$$

**16.** $A = \pi r^2$

$\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$

If $dr/dt$ is constant, $dA/dt$ is not constant.

$\dfrac{dA}{dt}$ depends on $r$ and $\dfrac{dr}{dt}$.

**18.** $V = \dfrac{4}{3}\pi r^3$

$\dfrac{dr}{dt} = 2$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

(a) When $r = 6$, $\dfrac{dV}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{min}$.

When $r = 24$, $\dfrac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi \text{ in}^3/\text{min}$.

(b) If $dr/dt$ is constant, $dV/dt$ is proportional to $r^2$.

**20.** $V = x^3$

$\dfrac{dx}{dt} = 3$

$\dfrac{dV}{dt} = 3x^2 \dfrac{dx}{dt}$

(a) When $x = 1$,

$$\frac{dV}{dt} = 3(1)^2(3) = 9 \text{ cm}^3/\text{sec.}$$

(b) When $x = 10$,

$$\frac{dV}{dt} = 3(10)^2(3) = 900 \text{ cm}^3/\text{sec.}$$

**22.** $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$

$\frac{dr}{dt} = 2$

$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

(a) When $r = 6$,

$$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min}.$$

(b) When $r = 24$,

$$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min}.$$

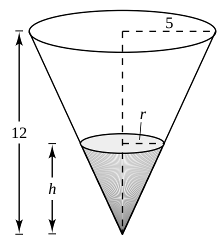
**24.** $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144}h^3 = \frac{25\pi}{3(144)}h^3$

$\left(\text{By similar triangles, } \frac{r}{5} = \frac{h}{12} \implies r = \frac{5}{12}h.\right)$

$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{25\pi}{144}h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right)\frac{dV}{dt}$

When $h = 8$, $\frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi}$ ft/min.

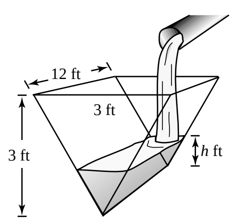



**26.** $V = \frac{1}{2}bh(12) = 6bh = 6h^2$ (since $b = h$)

(a) $\frac{dV}{dt} = 12h\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{1}{12h}\frac{dV}{dt}$

When $h = 1$ and $\frac{dV}{dt} = 2$, $\frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6}$ ft/min

(b) If $\frac{dh}{dt} = \frac{3}{8}$ and $h = 2$, then $\frac{dV}{dt} = 12(2)\left(\frac{3}{8}\right) = 9 \text{ ft}^3/\text{min}$.



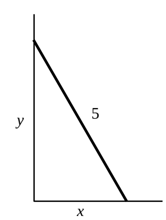

**28.** $x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$$\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{0.15y}{x} \text{ since } \frac{dy}{dt} = 0.15.$$

When $x = 2.5$,

$$y = \sqrt{18.75}, \frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5}0.15 \approx -0.26 \text{ m/sec}$$




**30.** Let $L$ be the length of the rope.

(a) $L^2 = 144 + x^2$

$2L\frac{dL}{dt} = 2x\frac{dx}{dt}$

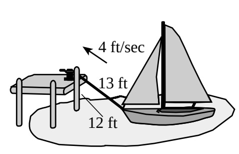
$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x}$ since $\frac{dL}{dt} = -4$ ft/sec.

When $L = 13$,

$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$

$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4$ ft/sec.

Speed of the boat increases as it approaches the dock.




(b) If $\frac{dx}{dt} = -4$, and $L = 13$,

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt}$$

$$= \frac{5}{13}(-4)$$

$$= \frac{-20}{13} \text{ ft/sec}$$

As $L \to 0$, $\frac{dL}{dt}$ increases.

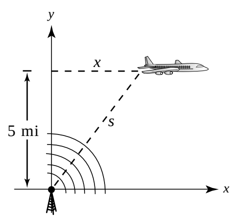
**32.** $x^2 + y^2 = s^2$

$$2x\frac{dx}{dt} + 0 = 2s\frac{ds}{dt} \qquad \left(\text{since } \frac{dy}{dt} = 0\right)$$

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(-240) = \frac{-480}{\sqrt{3}} = -160\sqrt{3} \approx -277.13 \text{ mph.}$$

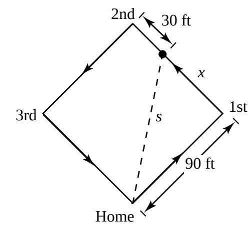



**34.** $s^2 = 90^2 + x^2$

$$x = 60$$

$$\frac{dx}{dt} = 28$$

$$\frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$




When $x = 60$,

$$s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$$

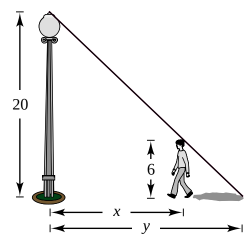
$$\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(28) = \frac{56}{\sqrt{13}} \approx 15.53 \text{ ft/sec.}$$

**36.** (a) $$\frac{20}{6} = \frac{y}{y - x}$$

$$20y - 20x = 6y$$

$$14y = 20x$$

$$y = \frac{10}{7}x$$




$$\frac{dx}{dt} = -5$$

$$\frac{dy}{dt} = \frac{10}{7}\frac{dx}{dt} = \frac{10}{7}(-5) = \frac{-50}{7} \text{ ft/sec}$$

(b) $$\frac{d(y - x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{-50}{7} - (-5) = \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$$

**38.** $x(t) = \frac{3}{5}\sin \pi t,\ x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}$, $y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ m.

Lowest point: $\left(0, \frac{4}{5}\right)$

(c) When $x = \frac{3}{10}$, $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and

$$\frac{3}{10} = \frac{3}{5}\sin \pi t \Longrightarrow \sin \pi t = \frac{1}{2} \Longrightarrow t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5}\pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longrightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $$\frac{dy}{dt} = \frac{-3/10}{\sqrt{15/4}} \cdot \frac{3}{5}\pi \cos\left(\frac{\pi}{6}\right)$$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}.$$

Speed $= \left|\frac{-9\sqrt{5}\pi}{125}\right| \approx 0.5058$ m/sec

**40.** $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$\frac{dR_1}{dt} = 1$

$\frac{dR_2}{dt} = 1.5$

$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{R_1^{\,2}} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^{\,2}} \cdot \frac{dR_2}{dt}$

When $R_1 = 50$ and $R_2 = 75$,

$R = 30$

$$\frac{dR}{dt} = (30)^2\left[\frac{1}{(50)^2}(1) + \frac{1}{(75)^2}(1.5)\right]$$

$$= 0.6 \text{ ohms/sec.}$$

**42.** $rg \tan\theta = v^2$

$32r \tan\theta = v^2$, $r$ is a constant.

$$32r \sec^2\theta \frac{d\theta}{dt} = 2v\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v}\sec^2\theta\ \frac{d\theta}{dt}$$

Likewise, $\frac{d\theta}{dt} = \frac{v}{16r}\cos^2\theta\ \frac{dv}{dt}$.

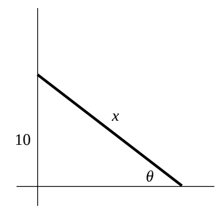
**44.** $\sin\theta = \frac{10}{x}$

$\frac{dx}{dt} = (-1)\text{ft/sec}$

$$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2}\frac{dx}{dt}(\sec\theta)$$

$$= \frac{-10}{25^2}(-1)\frac{25}{\sqrt{25^2 - 10^2}} = \frac{10}{25}\frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}} = \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$

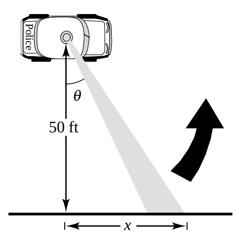



**46.** $\tan\theta = \frac{x}{50}$

$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = 50\sec^2\theta\left(\frac{d\theta}{dt}\right)$$




(a) When $\theta = 30°$, $\frac{dx}{dt} = \frac{200\pi}{3}$ ft/sec.

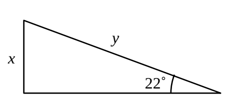
(b) When $\theta = 60°$, $\frac{dx}{dt} = 200\pi$ ft/sec.

(c) When $\theta = 70°$, $\frac{dx}{dt} \approx 427.43\pi$ ft/sec.

**48.** $\sin 22° = \frac{x}{y}$

$$0 = -\frac{x}{y^2} \cdot \frac{dy}{dt} + \frac{1}{y} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y} \cdot \frac{dy}{dt} = (\sin 22°)(240) \approx 89.9056 \text{ mi/hr}$$




**50.** (a) $dy/dt = 3(dx/dt)$ means that $y$ changes three times as fast as $x$ changes.

(b) $y$ changes slowly when $x \approx 0$ or $x \approx L$. $y$ changes more rapidly when $x$ is near the middle of the interval.

**52.** $L^2 = 144 + x^2$; acceleration of the boat $= \dfrac{d^2x}{dt^2}$.

First derivative: $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$

$$L\frac{dL}{dt} = x\frac{dx}{dt}$$

Second derivative: $L\dfrac{d^2L}{dt^2} + \dfrac{dL}{dt} \cdot \dfrac{dL}{dt} = x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} \cdot \dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right)\left[L\frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right]$$

When $L = 13$, $x = 5$, $\dfrac{dx}{dt} = -10.4$, and $\dfrac{dL}{dt} = -4$ (see Exercise 30). Since $\dfrac{dL}{dt}$ is constant, $\dfrac{d^2L}{dt^2} = 0$.

$$\frac{d^2x}{dt^2} = \frac{1}{5}[13(0) + (-4)^2 - (-10.4)^2]$$

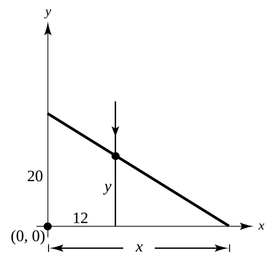
$$= \frac{1}{5}[16 - 108.16] = \frac{1}{5}[-92.16] = -18.432 \text{ ft/sec}^2$$

**54.** $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$




By similar triangles, $\dfrac{20}{x} = \dfrac{y}{x - 12}$

$$20x - 240 = xy.$$

When $y = 15.1$, $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

$$x = \frac{240}{4.9}.$$

$$20x - 240 = xy$$

$$20\frac{dx}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y}\frac{dy}{dt}$$

At $t = 1$, $\dfrac{dx}{dt} = \dfrac{240/4.9}{20 - 15.1}(-9.8) \approx -97.96$ m/sec.

# Review Exercises for Chapter 2

**2.** $f(x) = \dfrac{x+1}{x-1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{x+\Delta x+1}{x+\Delta x-1} - \dfrac{x+1}{x-1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x+1)(x-1) - (x+\Delta x-1)(x+1)}{\Delta x(x+\Delta x-1)(x-1)}$$

$$= \lim_{\Delta x \to 0} \frac{(x^2 + x\Delta x + x - x - \Delta x - 1) - (x^2 + x\Delta x - x + x + \Delta x - 1)}{\Delta x(x+\Delta x-1)(x-1)}$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x(x+\Delta x-1)(x-1)} = \lim_{\Delta x \to 0} \frac{-2}{(x+\Delta x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

**4.** $f(x) = \dfrac{2}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{2}{x+\Delta x} - \dfrac{2}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x - (2x + 2\Delta x)}{\Delta x(x+\Delta x)x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x(x+\Delta x)x}$$

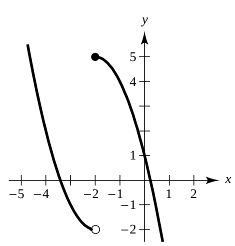
$$= \lim_{\Delta x \to 0} \frac{-2}{(x+\Delta x)x} = \frac{-2}{x^2}$$

**6.** $f$ is differentiable for all $x \neq -3$.

**8.** $f(x) = \begin{cases} x^2 + 4x + 2, & \text{if } x < -2 \\ 1 - 4x - x^2, & \text{if } x \geq -2 \end{cases}$

(a) Nonremovable discontinuity at $x = -2$.

(b) Not differentiable at $x = -2$ because the function is discontinuous there.




**10.** Using the limit defintion, you obtain $h'(x) = \dfrac{3}{8} - 4x$.

At $x = -2$, $h'(-2) = \dfrac{3}{8} - 4(-2) = \dfrac{67}{8}$.

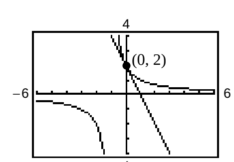
**12.** (a) Using the limit definition, $f'(x) = \dfrac{-2}{(x+1)^2}$.

At $x = 0$, $f'(0) = -2$. The tangent line is

$$y - 2 = -2(x - 0)$$

$$y = -2x + 2$$

(b)



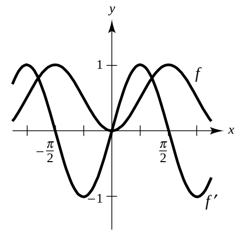
**14.** $f'(2) = \lim_{x\to2} \dfrac{f(x) - f(2)}{x - 2}$

$= \lim_{x\to2} \dfrac{\dfrac{1}{x+1} - \dfrac{1}{3}}{x - 2}$

$= \lim_{x\to2} \dfrac{3 - x - 1}{(x - 2)(x + 1)3}$

$= \lim_{x\to2} \dfrac{-1}{(x + 1)3} = \dfrac{-1}{9}$

**16.**




**18.** $y = -12$

$y' = 0$

**20.** $g(x) = x^{12}$

$g'(x) = 12x^{11}$

**22.** $f(t) = -8t^5$

$f'(t) = -40t^4$

**24.** $g(s) = 4s^4 - 5s^2$

$g'(s) = 16s^3 - 10s$

**26.** $f(x) = x^{1/2} - x^{-1/2}$

$f'(x) = \dfrac{1}{2}x^{-1/2} + \dfrac{1}{2}x^{-3/2} = \dfrac{x + 1}{2x^{3/2}}$

**28.** $h(x) = \dfrac{2}{9}x^{-2}$

$h'(x) = \dfrac{-4}{9}x^{-3} = \dfrac{-4}{9x^3}$

**30.** $g(\alpha) = 4 \cos \alpha + 6$

$g'(\alpha) = -4 \sin \alpha$

**32.** $g(\alpha) = \dfrac{5 \sin \alpha}{3} - 2\alpha$

$g'(\alpha) = \dfrac{5 \cos \alpha}{3} - 2$

**34.** $s = -16t^2 + s_0$

First ball:

$-16t^2 + 100 = 0$

$t = \sqrt{\dfrac{100}{16}} = \dfrac{10}{4} = 2.5$ seconds to hit ground

Second ball:

$-16t^2 + 75 = 0$

$t^2 = \sqrt{\dfrac{75}{16}} = \dfrac{5\sqrt{3}}{4} \approx 2.165$ seconds to hit ground

Since the second ball was released one second after the first ball, the first ball will hit the ground first. The second ball will hit the ground $3.165 - 2.5 = 0.665$ second later.

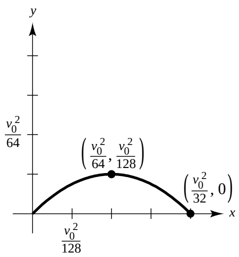
**36.** $s(t) = -16t^2 + 14{,}400 = 0$

$16t^2 = 14{,}400$

$t = 30$ sec

Since 600 mph $= \frac{1}{6}$ mi/sec, in 30 seconds the bomb will move horizontally $\left(\frac{1}{6}\right)(30) = 5$ miles.

**38.**




(a) $y = x - \dfrac{32}{v_0^2}x^2 = x\left(1 - \dfrac{32}{v_0^2}x\right)$

$= 0$ if $x = 0$ or $x = \dfrac{v_0^2}{32}$.

Projectile strikes the ground when $x = v_0^2/32$.

Projectile reaches its maximum height at $x = v_0^2/64$. (one-half the distance)

(b) $y' = 1 - \dfrac{64}{v_0^2}x$

When $x = \dfrac{v_0^2}{64}$, $y' = 1 - \dfrac{64}{v_0^2}\left(\dfrac{v_0^2}{64}\right) = 0$.

(c) $y = x - \dfrac{32}{v_0^2}x^2 = x\left(1 - \dfrac{32}{v_0^2}x\right) = 0$

when $x = 0$ and $x = x_0^2/32$. Therefore, the range is $x = v_0^2/32$. When the initial velocity is doubled the range is

$$x = \frac{(2v_0)^2}{32} = \frac{4v_0^2}{32}$$

or four times the initial range. From part (a), the maximum height occurs when $x = v_0^2/64$. The maximum height is

$$y\left(\frac{v_0^2}{64}\right) = \frac{v_0^2}{64} - \frac{32}{v_0^2}\left(\frac{v_0^2}{64}\right)^2 = \frac{v_0^2}{64} - \frac{v_0^2}{128} = \frac{v_0^2}{128}.$$

If the initial velocity is doubled, the maximum height is

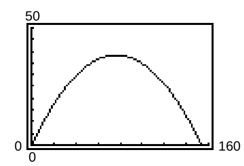
$$y\left[\frac{(2v_0)^2}{64}\right] = \frac{(2v_0)^2}{128} = 4\left(\frac{v_0^2}{128}\right)$$

or four times the original maximum height.

(d) $v_0 = 70$ ft/sec

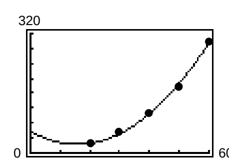
Range: $x = \dfrac{v_0^2}{32} = \dfrac{(70)^2}{32} = 153.125$ ft

Maximum height: $y = \dfrac{v_0^2}{128} = \dfrac{(70)^2}{128} \approx 38.28$ ft

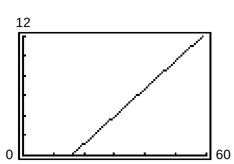



**40.** (a) $y = 0.14x^2 - 4.43x + 58.4$

(b)




(c)




(d) If $x = 65$, $y \approx 362$ feet.

(e) As the speed increases, the stopping distance increases at an increasing rate.

**42.** $g(x) = (x^3 - 3x)(x + 2)$

$$g'(x) = (x^3 - 3x)(1) + (x + 2)(3x^2 - 3)$$
$$= x^3 - 3x + 3x^3 + 6x^2 - 3x - 6$$
$$= 4x^3 + 6x^2 - 6x - 6$$

**44.** $f(t) = t^3 \cos t$

$$f'(t) = t^3(-\sin t) + \cos t(3t^2)$$
$$= -t^3 \sin t + 3t^2 \cos t$$

**46.** $f(x) = \dfrac{x + 1}{x - 1}$

$$f'(x) = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}$$
$$= \frac{-2}{(x - 1)^2}$$

**48.** $f(x) = \dfrac{6x - 5}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(6) - (6x - 5)(2x)}{(x^2 + 1)^2}$$
$$= \frac{2(3 + 5x - 3x^2)}{(x^2 + 1)^2}$$

**50.** $f(x) = 9(3x^2 - 2x)^{-1}$

$$f'(x) = -9(3x^2 - 2x)^{-2}(6x - 2) = \frac{18(1 - 3x)}{(3x^2 - 2x)^2}$$

**52.** $y = \dfrac{\sin x}{x^2}$

$$y' = \frac{(x^2)\cos x - (\sin x)(2x)}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}$$

**54.** $y = 2x - x^2 \tan x$

$$y' = 2 - x^2 \sec^2 x - 2x \tan x$$

**56.** $y = \dfrac{1 + \sin x}{1 - \sin x}$

$$y' = \frac{(1 - \sin x)\cos x - (1 + \sin x)(-\cos x)}{(1 - \sin x)^2}$$
$$= \frac{2 \cos x}{(1 - \sin x)^2}$$

**58.** $v(t) = 36 - t^2,\ 0 \le t \le 6$

$$a(t) = v'(t) = -2t$$
$$v(4) = 36 - 16 = 20 \text{ m/sec}$$
$$a(4) = -8 \text{ m/sec}$$

**60.** $f(x) = 12x^{1/4}$

$$f'(x) = 3x^{-3/4}$$
$$f''(x) = \frac{-9}{4}x^{-7/4} = \frac{-9}{4x^{7/4}}$$

**62.** $h(t) = 4 \sin t - 5 \cos t$

$$h'(t) = 4 \cos t + 5 \sin t$$
$$h''(t) = -4 \sin t + 5 \cos t$$

**64.** $y = \dfrac{(10 - \cos x)}{x}$

$$xy + \cos x = 10$$
$$xy' + y - \sin x = 0$$
$$xy' = \sin x - y$$
$$xy' + y = (\sin x - y) + y = \sin x$$

**66.** $f(x) = (x^2 - 1)^{1/3}$

$$f'(x) = \frac{1}{3}(x^2 - 1)^{-2/3}(2x)$$
$$= \frac{2x}{3(x^2 - 1)^{2/3}}$$

**68.** $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$

$$f'(x) = 5\left(x^2 + \frac{1}{x}\right)^4\left(2x - \frac{1}{x^2}\right)$$

**70.** $h(\theta) = \dfrac{\theta}{(1-\theta)^3}$

$$h'(\theta) = \frac{(1-\theta)^3 - \theta[3(1-\theta)^2(-1)]}{(1-\theta)^6}$$

$$= \frac{(1-\theta)^2(1-\theta+3\theta)}{(1-\theta)^6} = \frac{2\theta+1}{(1-\theta)^4}$$

**72.** $y = 1 - \cos 2x + 2\cos^2 x$

$$y' = 2\sin 2x - 4\cos x \sin x$$

$$= 2[2\sin x\cos x] - 4\sin x\cos x$$

$$= 0$$

**74.** $y = \csc 3x + \cot 3x$

$$y' = -3\csc 3x\cot 3x - 3\csc^2 3x$$

$$= -3\csc 3x(\cot 3x + \csc 3x)$$

**76.** $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

$$y' = \sec^6 x(\sec x\tan x) - \sec^4 x(\sec x\tan x)$$

$$= \sec^5 x\tan x(\sec^2 x - 1)$$

$$= \sec^5 x\tan^3 x$$

**78.** $f(x) = \dfrac{3x}{\sqrt{x^2+1}}$

$$f'(x) = \frac{3(x^2+1)^{1/2} - 3x\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$

$$= \frac{3(x^2+1) - 3x^2}{(x^2+1)^{3/2}}$$

$$= \frac{3}{(x^2+1)^{3/2}}$$

**80.** $y = \dfrac{\cos(x-1)}{x-1}$

$$y' = \frac{-(x-1)\sin(x-1) - \cos(x-1)(1)}{(x-1)^2}$$

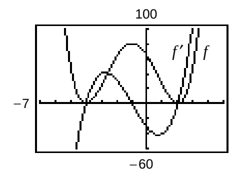
$$= -\frac{1}{(x-1)^2}[(x-1)\sin(x-1) + \cos(x-1)]$$

**82.** $f(x) = [(x-2)(x+4)]^2 = (x^2+2x-8)^2$

$$f'(x) = 4(x^3 + 3x^2 - 6x - 8)$$

$$= 4(x-2)(x+1)(x+4)$$

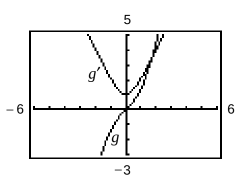
The zeros of $f'$ correspond to the points on the graph of $f$ where the tangent line is horizontal.




**84.** $g(x) = x(x^2+1)^{1/2}$

$$g'(x) = \frac{2x^2+1}{\sqrt{x^2+1}}$$

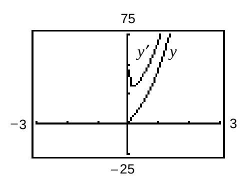
$g'$ does not equal zero for any value of $x$. The graph of $g$ has no horizontal tangent lines.




**86.** $y = \sqrt{3x}(x+2)^3$

$$y' = \frac{3(x+2)^2(7x+2)}{2\sqrt{3x}}$$

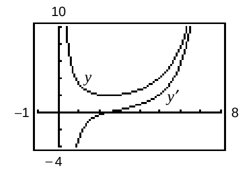
$y'$ does not equal zero for any $x$ in the domain. The graph has no horizontal tangent lines.



**88.** $y = 2\csc^3(\sqrt{x})$

$$y' = -\frac{3}{\sqrt{x}}\csc^3\sqrt{x}\cot\sqrt{x}$$

The zero of $y'$ corresponds to the point on the graph of $y$ where the tangent line is horizontal.




**90.** $y = x^{-1} + \tan x$

$$y' = -x^{-2} + \sec^2 x$$

$$y'' = 2x^{-3} + 2\sec x(\sec x\tan x)$$

$$= \frac{2}{x^3} + 2\sec^2 x\tan x$$

**92.** $y = \sin^2 x$

$$y' = 2\sin x\cos x = \sin 2x$$

$$y'' = 2\cos 2x$$

**94.** $g(x) = \dfrac{6x - 5}{x^2 + 1}$

$$g'(x) = \frac{2(-3x^2 + 5x + 3)}{(x^2 + 1)^2}$$

$$g''(x) = \frac{2(6x^3 - 15x^2 - 18x + 5)}{(x^2 + 1)^3}$$

**96.** $h(x) = x\sqrt{x^2 - 1}$

$$h'(x) = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$h''(x) = \frac{x(2x^2 - 3)}{(x^2 - 1)^{3/2}}$$

**98.** $v = \sqrt{2gh} = \sqrt{2(32)h} = 8\sqrt{h}$

$$\frac{dv}{dh} = \frac{4}{\sqrt{h}}$$

(a) When $h = 9$, $\dfrac{dv}{dh} = \dfrac{4}{3}$ ft/sec.

(b) When $h = 4$, $\dfrac{dv}{dh} = 2$ ft/sec.

**100.**

$$x^2 + 9y^2 - 4x + 3y = 0$$

$$2x + 18yy' - 4 + 3y' = 0$$

$$3(6y + 1)y' = 4 - 2x$$

$$y' = \frac{4 - 2x}{3(6y + 1)}$$

**102.**

$$y^2 = x^3 - x^2y + xy - y^2$$

$$0 = x^3 - x^2y + xy - 2y^2$$

$$0 = 3x^2 - x^2y' - 2xy + xy' + y - 4yy'$$

$$(x^2 - x + 4y)y' = 3x^2 - 2xy + y$$

$$y' = \frac{3x^2 - 2xy + y}{x^2 - x + 4y}$$

**104.**

$$\cos(x + y) = x$$

$$-(1 + y')\sin(x + y) = 1$$

$$-y'\sin(x + y) = 1 + \sin(x + y)$$

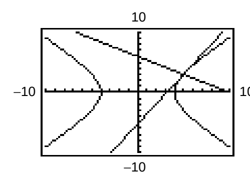
$$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)}$$

$$= -\csc(x + 1) - 1$$

**106.** $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$




At (5, 3): $y' = \dfrac{5}{3}$

Tangent line: $y - 3 = \dfrac{5}{3}(x - 5)$

$$5x - 3y - 16 = 0$$

Normal line: $y - 3 = -\dfrac{3}{5}(x - 5)$

$$3x + 5y - 30 = 0$$

**108.** Surface area $= A = 6x^2$, $x$ length of edge.

$$\frac{dx}{dt} = 5$$

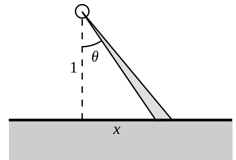
$$\frac{da}{dt} = 12x\frac{dx}{dt} = 12(4.5)(5) = 270 \text{ cm}^2/\text{sec}$$

**110.** $\tan\theta = x$

$$\frac{d\theta}{dt} = 3(2\pi) \text{ rad/min}$$

$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{dx}{dt}$$

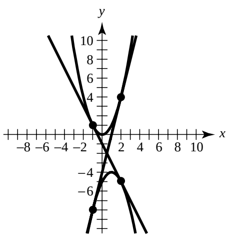
$$\frac{dx}{dt} = (\tan^2\theta + 1)(6\pi) = 6\pi(x^2 + 1)$$




When $x = \frac{1}{2}, \frac{dx}{dt} = 6\pi\left(\frac{1}{4} + 1\right) = \frac{15\pi}{2}$ km/min $= 450\pi$ km/hr.

## Problem Solving for Chapter 2

**2.**




Let $(a, a^2)$ and $(b, -b^2 + 2b - 5)$ be the points of tangency.

For $y = x^2, y' = 2x$ and for $y = -x^2 + 2x - 5, y' = -2x + 2$.

Thus, $2a = -2b + 2 \Rightarrow a + b = 1$, or $a = 1 - b$. Furthermore, the slope of the common tangent line is

$$\frac{a^2 - (-b^2 + 2b - 5)}{a - b} = \frac{(1 - b)^2 + b^2 - 2b + 5}{(1 - b) - b} = -2b + 2$$

$$\Rightarrow \frac{1 - 2b + b^2 + b^2 - 2b + 5}{1 - 2b} = -2b + 2$$

$$\Rightarrow 2b^2 - 4b + 6 = 4b^2 - 6b + 2$$

$$\Rightarrow 2b^2 - 2b - 4 = 0$$

$$\Rightarrow b^2 - b - 2 = 0$$

$$\Rightarrow (b - 2)(b + 1) = 0$$

$$b = 2, -1$$

For $b = 2, a = 1 - b = -1$ and the points of tangency are $(-1, 1), (2, -5)$. The tangent line has slope $-2$:

$$y - 1 = -2(x = 1) \Rightarrow y = -2x - 1$$

For $b = -1, a = 1 - b = 2$ and the points of tangency are $(2, 4)$ and $(-1, -8)$. The tangent line has slope 4:

$$y - 4 = 4(x - 2) \Rightarrow y = 4x - 4.$$

**4.** (a) $y = x^2, y' = 2x$. Slope $= 4$ at $(2, 4)$.

Tangent line: $y - 4 = 4(x - 2)$

$y = 4x - 4$

(b) Slope of normal line: $-\frac{1}{4}$.

Normal line: $y - 4 = -\frac{1}{4}(x - 2)$

$y = -\frac{1}{4}x + \frac{9}{2}$

$y = -\frac{1}{4}x + \frac{9}{2} = x^2 \Rightarrow 4x^2 + x - 18 = 0 \Rightarrow (4x + 9)(x - 2) = 0$

$x = 2, -\frac{9}{4}$. Second intersection point: $\left(-\frac{9}{4}, \frac{81}{16}\right)$

(c) Tangent line: $y = 0$

Normal line: $x = 0$

**—CONTINUED—**

**4. —CONTINUED—**

(d) Let $(a, a^2)$, $a \neq 0$, be a point on the parabola $y = x^2$. Tangent line at $(a, a^2)$ is $y = 2a(x - a) + a^2$. Normal line at $(a, a^2)$ is $y = -\frac{1}{2a}(x - a) + a^2$. To find points of intersection, solve

$$x^2 = -\frac{1}{2a}(x - a) + a^2$$

$$x^2 + \frac{1}{2a}x = a^2 + \frac{1}{2}$$

$$x^2 + \frac{1}{2a}x + \frac{1}{16a^2} = a^2 + \frac{1}{2} + \frac{1}{16a^2}$$

$$\left(x + \frac{1}{4a}\right)^2 = \left(a + \frac{1}{4a}\right)^2$$

$$x + \frac{1}{4a} = \pm\left(a + \frac{1}{4a}\right)$$

$x + \frac{1}{4a} = a + \frac{1}{4a} \Rightarrow x = a$ (Point of tangency)

$$x + \frac{1}{4a} = -\left(a + \frac{1}{4a}\right) \Rightarrow x = -a - \frac{1}{2a} = -\frac{2a^2 + 1}{2a}$$

The normal line intersects a second time at $x = -\frac{2a^2 + 1}{2a}$.

**6.** $f(x) = a + b\cos cx$

$f'(x) = -bc\sin cx$

At $(0, 1)$: $a + b = 1$ Equation 1

At $\left(\frac{\pi}{4}, \frac{3}{2}\right)$: $a + b\cos\left(\frac{c\pi}{4}\right) = \frac{3}{2}$ Equation 2

$-bc\sin\left(\frac{c\pi}{4}\right) = 1$ Equation 3

From Equation 1, $a = 1 - b$. Equation 2 becomes $(1 - b) + b\cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \Rightarrow -b + b\cos\frac{c\pi}{4} = \frac{1}{2}$

From Equation 3, $b = \frac{-1}{c\sin\left(\frac{c\pi}{4}\right)}$. Thus $\frac{1}{c\sin\left(\frac{c\pi}{4}\right)} + \frac{-1}{c\sin\left(\frac{c\pi}{4}\right)}\cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}$

$$1 - \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right)$$

Graphing the equation $g(c) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right) + \cos\left(\frac{c\pi}{4}\right) - 1$, you see that many values of $c$ will work.

One answer: $c = 2$, $b = -\frac{1}{2}$, $a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2} - \frac{1}{2}\cos 2x$

**8.** (a) $b^2y^2 = x^3(a - x);\ a, b > 0$

$$y^2 = \frac{x^3(a - x)}{b^2}$$

Graph $y_1 = \dfrac{\sqrt{x^3(a - x)}}{b}$ and $y_2 = -\dfrac{\sqrt{x^3(a - x)}}{b}$

(b) $a$ determines the $x$-intercept on the right: $(a, 0)$.

$b$ affects the height.

(c) Differentiating implicitly.

$$2b^2yy' = 3x^2(a - x) - x^3 = 3ax^2 - 4x^3$$

$$y' = \frac{(3ax^2 - 4x^3)}{2b^2y} = 0$$

$$\Longrightarrow 3ax^2 = 4x^3$$

$$3a = 4x$$

$$x = \frac{3a}{4}.$$

$$b^2y^2 = \left(\frac{3a}{4}\right)^3\left(a - \frac{3a}{4}\right) = \frac{27a^3}{64}\left(\frac{1}{4}a\right)$$

$$y^2 = \frac{27a^4}{256b^2} \Longrightarrow y = \pm\frac{3\sqrt{3}a^2}{16b}$$

Two points: $\left(\dfrac{3a}{4}, \dfrac{3\sqrt{3}a^2}{16b}\right), \left(\dfrac{3a}{4}, \dfrac{-3\sqrt{3}a^2}{16b}\right)$

**10.** (a) $y = x^{1/3} \Longrightarrow \dfrac{dy}{dt} = \dfrac{1}{3}x^{-2/3}\dfrac{dx}{dt}$

$$1 = \frac{1}{3}(8)^{-2/3}\frac{dx}{dt}$$

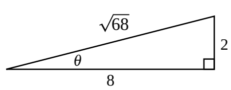
$$\frac{dx}{dt} = 12 \text{ cm/sec}$$

(b) $D = \sqrt{x^2 + y^2} \Longrightarrow \dfrac{dD}{dt} = \dfrac{1}{2}(x^2 + y^2)\left(2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}\right)$

$$= \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{\sqrt{x^2 + y^2}}$$

$$= \frac{8(12) + 2(1)}{\sqrt{64 + 4}} = \frac{98}{\sqrt{68}} = \frac{49}{\sqrt{17}} \text{ cm/sec.}$$

(c) $\tan\theta = \dfrac{y}{x} \Longrightarrow \sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$




From the triangle, $\sec\theta = \dfrac{\sqrt{68}}{8}$. Hence $\dfrac{d\theta}{dt} = \dfrac{8(1) - 2(12)}{64\left(\dfrac{68}{64}\right)} = \dfrac{-16}{68} = \dfrac{-4}{17}$ rad/sec

**12.** $E'(x) = \lim_{\Delta x \to 0} \dfrac{E(x + \Delta x) - E(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{E(x)E(\Delta x) - E(x)}{\Delta x}$$

$$= \lim_{\triangle x \to 0} E(x)\left(\frac{E(\Delta x) - 1}{\Delta x}\right)$$

$$= E(x) \lim_{\Delta x \to 0} \frac{E(\Delta x) - 1}{\Delta x}$$

But, $E'(0) = \lim_{\Delta x \to 0} \dfrac{E(\Delta x) - E(0)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{E(\Delta x) - 1}{\Delta x} = 1.$

Thus, $E'(x) = E(x)E'(0) = E(x)$ exists for all $x$.

For example: $E(x) = e^x$.

**14.** (a) $v(t) = -\dfrac{27}{5}t + 27$ ft/sec

$a(t) = -\dfrac{27}{5}$ ft/sec$^2$

(b) $v(t) = -\dfrac{27}{5}t + 27 = 0 \Longrightarrow \dfrac{27}{5}t = 27 \Longrightarrow t = 5$ seconds

$S(5) = -\dfrac{27}{10}(5)^2 + 27(5) + 6 = 73.5$ feet

(c) The acceleration due to gravity on Earth is greater in magnitude than that on the moon.

# CHAPTER 3
# Applications of Differentiation

# C H A P T E R 3
# Applications of Differentiation

## Section 3.1 Extrema on an Interval

Solutions to Even-Numbered Exercises

**2.** $f(x) = \cos\dfrac{\pi x}{2}$

$$f'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2}$$

$$f'(0) = 0$$

$$f'(2) = 0$$

**4.** $f(x) = -3x\sqrt{x+1}$

$$f'(x) = -3x\left[\frac{1}{2}(x+1)^{-1/2}\right] + \sqrt{x+1}\,(-3)$$

$$= -\frac{3}{2}(x+1)^{-1/2}[x + 2(x+1)]$$

$$= -\frac{3}{2}(x+1)^{-1/2}(3x+2)$$

$$f'\left(-\frac{2}{3}\right) = 0$$

**6.** Using the limit definition of the derivative,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{(4-|x|)-4}{x} = 1$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{(4-|x|)-4}{x-0} = -1$$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

**8.** Critical number: $x = 0$.

$x = 0$: neither

**10.** Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum

**12.** $g(x) = x^2(x^2 - 4) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x = 4x(x^2 - 2)$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

**14.** $f(x) = \dfrac{4x}{x^2+1}$

$$f'(x) = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4(1-x^2)}{(x^2+1)^2}$$

Critical numbers: $x = \pm 1$

**16.** $f(\theta) = 2\sec\theta + \tan\theta, 0 < \theta < 2\pi$

$$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta$$

$$= \sec\theta(2\tan\theta + \sec\theta)$$

$$= \sec\theta\left[2\left(\frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta}\right]$$

$$= \sec^2\theta(2\sin\theta + 1)$$

On $(0, 2\pi)$, critical numbers: $\theta = \dfrac{7\pi}{6}, \theta = \dfrac{11\pi}{6}$

**18.** $f(x) = \dfrac{2x + 5}{3}, [0, 5]$

$f'(x) = \dfrac{2}{3} \Longrightarrow$ No critical numbers

Left endpoint: $\left(0, \dfrac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

**20.** $f(x) = x^2 + 2x - 4, [-1, 1]$

$f'(x) = 2x + 2 = 2(x + 1)$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

**22.** $f(x) = x^3 - 12x, [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

**Note:** $x = -2$ is not in the interval.

**24.** $g(x) = \sqrt[3]{x}, [-1, 1]$

$g'(x) = \dfrac{1}{3x^{2/3}}$

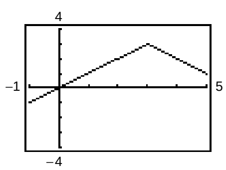
Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

**26.** $y = 3 - |t - 3|, [-1, 5]$

From the graph, you see that $t = 3$ is a critical number.




Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

**28.** $h(t) = \dfrac{t}{t - 2}, [3, 5]$

$h'(t) = \dfrac{-2}{(t - 2)^2}$

Left endpoint: $(3, 3)$ Maximum

Right endpoint: $\left(5, \dfrac{5}{3}\right)$ Minimum

**30.** $g(x) = \sec x, \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\dfrac{\pi}{6}, \dfrac{2}{\sqrt{3}}\right) \approx \left(-\dfrac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\dfrac{\pi}{3}, 2\right)$ Maximum

Critical number: $(0, 1)$ Minimum

**32.** $y = x^2 - 2 - \cos x, [-1, 3]$

$y' = 2x - \sin x$

Left endpoint: $(-1, -1.5403)$

Right endpoint: $(3, 7.99)$ Maximum

Critical number: $(0, -3)$ Minimum

**34.** (a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

**36.** (a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

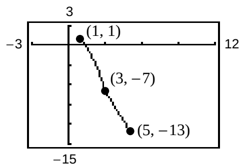
(c) Maximum: $(0, 2)$

(d) Maximum: $(1, \sqrt{3})$

**38.** $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$

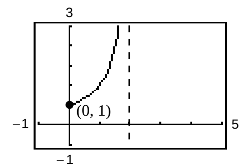
Left endpoint: $(1, 1)$ Maximum

Right endpoint: $(5, -13)$ Minimum

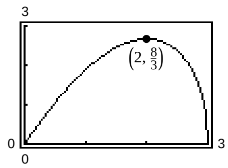



**40.** $f(x) = \dfrac{2}{2 - x}, [0, 2)$

Left endpoint: $(0, 1)$ Minimum




**42.** (a)




Maximum: $\left(2, \dfrac{8}{3}\right)$

Minimum: $(0, 0), (3, 0)$

(b) $f(x) = \dfrac{4}{3}x\sqrt{3 - x}, [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right]$$

$$= \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)[-x + 2(3 - x)]$$

$$= \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Critical number: $x = 2$

$f(0) = 0$ Minimum

$f(3) = 0$ Minimum

$f(2) = \dfrac{8}{3}$

Maximum: $\left(2, \dfrac{8}{3}\right)$

**44.** $f(x) = \dfrac{1}{x^2 + 1}, \left[\dfrac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

Setting $f''' = 0$, we have $x = 0, \pm 1$.

$|f''(1)| = \dfrac{1}{2}$ is the maximum value.

**46.** $f(x) = \dfrac{1}{x^2 + 1}, [-1, 1]$

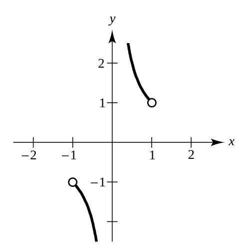
$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4} \quad \text{(See Exercise 44.)}$$

$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

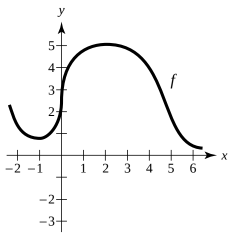
$$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$$

$|f^{(4)}(0)| = 24$ is the maximum value.

**48.** Let $f(x) = 1/x$. $f$ is continuous on $(0, 1)$ but does not have a maximum. $f$ is also continuous on $(-1, 0)$ but does not have a minimum. This can occur if one of the endpoints is an infinite discontinuity.



**50.**




**52.** (a) No

(b) Yes

**54.** (a) No

(b) Yes

**56.** $x = \dfrac{v^2 \sin 2\theta}{32}, \dfrac{\pi}{4} \leq \theta \leq \dfrac{3\pi}{4}$

$\dfrac{d\theta}{dt}$ is constant.

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} \text{ (by the Chain Rule)}$$

$$= \frac{v^2 \cos 2\theta}{16}\frac{d\theta}{dt}$$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for $dx/dt$ and $\theta = \pi/2$ indicates a maximum for $dx/dt$. This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. Thus, the lawn farthest from the spinkler gets the most water.

**58.** $C = 2x + \dfrac{300{,}000}{x}, 1 \leq x \leq 300$

$C(1) = 300{,}002$

$C(300) = 1600$

$C' = 2 - \dfrac{300{,}000}{x^2} = 0$

$2x^2 = 300{,}000$

$x^2 = 150{,}000$

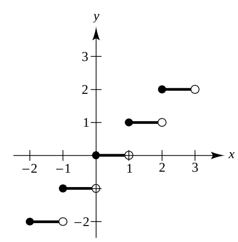
$x = 100\sqrt{15} \approx 387 > 300$ (outside of interval)

$C$ is minimized when $x = 300$ units.

Yes, if $1 \leq x \leq 400$, then $x = 387$ would minimize $C$.

**60.** $f(x) = [\![x]\!]$

The derivative of $f$ is undefined at every integer and is zero at any noninteger real number. All real numbers are critical numbers.




**62.** True. This is stated in the Extreme Value Theorem.

**64.** False. Let $f(x) = x^2$. $x = 0$ is a critical number of $f$.

$g(x) = f(x - k)$

$= (x - k)^2$

$x = k$ is a critical number of $g$.

## Section 3.2 Rolle's Theorem and the Mean Value Theorem

**2.** Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ since $f$ is not continuous at $x = 2\pi$.

**4.** $f(x) = x(x - 3)$

$x$-intercepts: $(0, 0), (3, 0)$

$f'(x) - 2x - 3 = 0$ at $x = \dfrac{3}{2}$.

**6.** $f(x) = -3x\sqrt{x + 1}$

$x$-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x + 1)^{-1/2} - 3(x + 1)^{1/2} = -3(x + 1)^{-1/2}\left(\frac{x}{2} + (x + 1)\right)$$

$$f'(x) = -3(x + 1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}.$$

**8.** $f(x) = x^2 - 5x + 4,\ [1, 4]$

$f(1) = f(4) = 0$

$f$ is continuous on $[1, 4]$. $f$ is differentiable on $(1, 4)$.

Rolle's Theorem applies.

$$f'(x) = 2x - 5$$

$$2x - 5 = 0 \implies x = \frac{5}{2}$$

$c$ value: $\frac{5}{2}$

**10.** $f(x) = (x - 3)(x + 1)^2, [-1, 3]$

$f(-1) = f(3) = 0$

$f$ is continuous on $[-1, 3]$. $f$ is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$\begin{aligned} f'(x) &= (x - 3)(2)(x + 1) + (x + 1)^2 \\ &= (x + 1)[2x - 6 + x + 1] \\ &= (x + 1)(3x - 5) \end{aligned}$$

$c$ value: $\frac{5}{3}$

**12.** $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

$f$ is continuous on $[0, 6]$. $f$ is not differentiable on $(0, 6)$ since $f'(3)$ does not exist. Rolle's Theorem does not apply.

**14.** $f(x) = \frac{x^2 - 1}{x},\ [-1, 1]$

$f(-1) = f(1) = 0$

$f$ is not continuous on $[-1, 1]$ since $f(0)$ does not exist. Rolle's Theorem does not apply.

**16.** $f(x) = \cos x,\ [0, 2\pi]$

$f(0) = f(2\pi) = 1$

$f$ is continuous on $[0, 2\pi]$. $f$ is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = -\sin x$$

$c$ value: $\pi$

**18.** $f(x) = \cos 2x,\ \left[-\frac{\pi}{12}, \frac{\pi}{6}\right]$

$$f\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f\left(-\frac{\pi}{12}\right) \neq f\left(\frac{\pi}{0}\right)$$

Rolle's Theorem does not apply.

**20.** $f(x) = \sec x,\ \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$f$ is continuous on $[-\pi/4, \pi/4]$. $f$ is differentiable on $(-\pi/4, \pi/4)$. Rolle's Theorem applies.

$$\begin{aligned} f'(x) &= \sec x \tan x \\ \sec x \tan x &= 0 \\ x &= 0 \end{aligned}$$

$c$ value: $0$

**22.** $f(x) = x - x^{1/3},\ [0, 1]$

$f(0) = f(1) = 0$

$f$ is continuous on $[0, 1]$. $f$ is differentiable on $(0, 1)$. (**Note:** $f$ is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

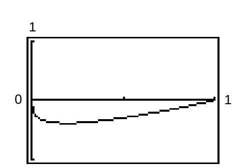
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

$c$ value: $\frac{\sqrt{3}}{9} \approx 0.1925$



**24.** $f(x) = \dfrac{x}{2} - \sin\dfrac{\pi x}{6},\ [-1, 0]$

$f(-1) = f(0) = 0$

$f$ is continuous on $[-1, 0]$. $f$ is differentiable on $(-1, 0)$. Rolle's Theorem applies.

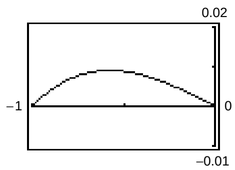
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi} \quad [\text{Value needed in } (-1, 0).]$$

$$\approx -0.5756 \text{ radian}$$

$c$ value: $-0.5756$




**26.** $C(x) = 10\left(\dfrac{1}{x} + \dfrac{x}{x+3}\right)$

(a) $C(3) = C(6) = \dfrac{25}{3}$

(b) $C'(x) = 10\left(-\dfrac{1}{x^2} + \dfrac{3}{(x+3)^2}\right) = 0$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

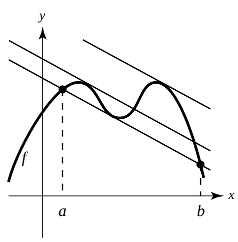
$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

In the interval $(3, 6)$: $c = \dfrac{3 + 3\sqrt{3}}{2} \approx 4.098.$

**28.**




**30.** $f(x) = |x - 3|,\ [0, 6]$

$f$ is not differentiable at $x = 3$.

**32.** $f(x) = x(x^2 - x - 2)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = -1$$

$$f'(x) = 3x^2 - 2x - 2 = -1$$

$$(3x + 1)(x - 1) = 0$$

$$c = -\frac{1}{3}$$

**34.** $f(x) = (x + 1)/x$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - (1/2)} = \frac{(3/2) - 3}{3/2} = -1$$

$$f'(x) = \frac{-1}{x^2} = -1$$

$$x^2 = 1$$

$$c = 1$$

**36.** $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

$$f'(x) = 3x^2 = 1$$

$$x = \pm\frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \dfrac{\sqrt{3}}{3}$.

**38.** $f(x) = 2\sin x + \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = 2\cos x + 2\cos 2x = 0$$

$$2[\cos x + 2\cos^2 x - 1] = 0$$

$$2(2\cos x - 1)(\cos x + 1) = 0$$

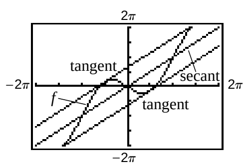
$$\cos x = \frac{1}{2}$$

$$\cos x = -1$$

$$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

In the interval $(0, \pi)$: $c = \dfrac{\pi}{3}$.

**40.** $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

(a) 


(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$\cos x = 0$

$$c = \pm\frac{\pi}{2}, \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\dfrac{\pi}{2} - 2\right) = 1\left(x - \dfrac{\pi}{2}\right)$

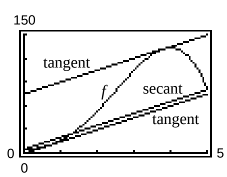
$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

**42.** $f(x) = -x^4 + 4x^3 + 8x^2 + 5$, $(0, 5)$, $(5, 80)$

$$m = \frac{80 - 5}{5 - 0} = 15$$

(a) 


(b) Secant line: $y - 5 = 15(x - 0)$

$$0 = 15x - y + 5$$

$$f'(x) = -4x^3 + 12x^2 + 16x$$

$$\frac{f(5) - f(1)}{5 - 1} = 15$$

$$-4c^3 + 12c^2 + 16c = 15$$

$$0 = 4c^3 - 12c^2 - 16c + 15$$

$$c \approx 0.67 \text{ or } c \approx 3.79$$

(c) First tangent line: $y - f(c) = m(x - c)$

$$y - 9.59 = 15(x - 0.67)$$

$$0 = 15x - y - 0.46$$

Second tangent line: $y - f(c) = m(x - c)$

$$y - 131.35 = 15(x - 3.79)$$

$$0 = 15x - y + 74.5$$

**44.** $S(t) = 200\left(5 - \dfrac{9}{2+t}\right)$

(a) $\dfrac{S(12) - S(0)}{12 - 0} = \dfrac{200[5 - (9/14)] - 200[5 - (9/2)]}{12} = \dfrac{450}{7}$

(b) $$S'(t) = 200\left(\frac{9}{(2+t)^2}\right) = \frac{450}{7}$$

$$\frac{1}{(2+t)^2} = \frac{1}{28}$$

$$2 + t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

**46.** $f(a) = f(b)$ and $f'(c) = 0$ where $c$ is in the interval $(a, b)$.

(a) $g(x) = f(x) + k$

$g(a) = g(b) = f(a) + k$

$g'(x) = f'(x) \Longrightarrow g'(c) = 0$

Interval: $[a, b]$

Critical number of $g$: $c$

(b) $g(x) = f(x - k)$

$g(a + k) = g(b + k) = f(a)$

$g'(x) = f'(x - k)$

$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of $g$: $c + k$

(c) $g(x) = f(kx)$

$g\left(\dfrac{a}{k}\right) = g\left(\dfrac{b}{k}\right) = f(a)$

$g'(x) = kf'(kx)$

$g'\left(\dfrac{c}{k}\right) = kf'(c) = 0$

Interval: $\left[\dfrac{a}{k}, \dfrac{b}{k}\right]$

Critical number of $g$: $\dfrac{c}{k}$

**48.** Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

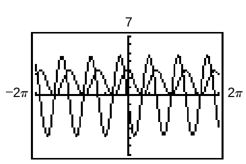
$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/hr}.$$

By the Mean Value Theorem, there exists a time to, $0 < t_0 < 5$, such that $T'(t_0) = -222$.

**50.** $f(x) = 3\cos^2\left(\dfrac{\pi x}{2}\right), \quad f'(x) = 6\cos\left(\dfrac{\pi x}{2}\right)\left(-\sin\left(\dfrac{\pi x}{2}\right)\right)\left(\dfrac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a)




(b) $f$ and $f'$ are both continuous on the entire real line.

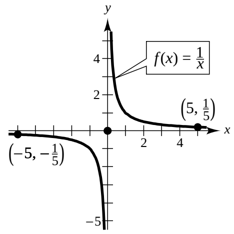
(c) Since $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Since $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

(d) $\lim\limits_{x \to 3^-} f'(x) = 0$

$\lim\limits_{x \to 3^+} f'(x) = 0$

**52.** $f$ is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$




**54.** False. $f$ must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

**56.** True

**58.** Suppose $f(x)$ is not constant on $(a, b)$. Then there exists $x_1$ and $x_2$ in $(a, b)$ such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all $x$ in $(a, b)$.

**60.** Suppose $f(x)$ has two fixed points $c_1$ and $c_2$. Then, by the Mean Value Theorem, there exists $c$ such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all $x$.

**62.** Let $f(x) = \cos x$. $f$ is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists $c$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c||b - a|$$

$$|\cos b - \cos a| \leq |b - a| \text{ since } |-\sin c| \leq 1.$$

## Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

**2.** $y = -(x + 1)^2$

Increasing on: $(-\infty, -1)$

Decreasing on: $(-1, \infty)$

**4.** $f(x) = x^4 - 2x^2$

Increasing on: $(-1, 0), (1, \infty)$

Decreasing on: $(-\infty, -1), (0, 1)$

**6.** $y = \dfrac{x^2}{x + 1}$

$y' = \dfrac{x(x + 2)}{(x + 1)^2}$

Critical numbers: $x = 0, -2$ Discontinuity: $x = -1$

| Test intervals: | $-\infty < x < -2$ | $-2 < x < -1$ | $-1 < x < 0$ | $0 < x < \infty$ |
|---|---|---|---|---|
| Sign of $f'(x)$: | $y' > 0$ | $y' < 0$ | $y' < 0$ | $y' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

Increasing on $(-\infty, -2), (0, \infty)$

Decreasing on $(-2, -1), (-1, 0)$

**8.** $h(x) = 27x - x^3$

$h'(x) = 27 - 3x^2 = 3(3 - x)(3 + x)$

$h'(x) = 0$

Critical numbers: $x = \pm 3$

| Test intervals: | $-\infty < x < -3$ | $-3 < x < 3$ | $3 < x < \infty$ |
|---|---|---|---|
| Sign of $h'(x)$: | $h' < 0$ | $h' > 0$ | $h' < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

Increasing on $(-3, 3)$

Decreasing on $(-\infty, -3), (3, \infty)$

**10.** $y = x + \dfrac{4}{x}$

$y' = \dfrac{(x - 2)(x + 2)}{x^2}$

Critical numbers: $x = \pm 2$ Discontinuity: 0

| Test intervals: | $-\infty < x < -2$ | $-2 < x < 0$ | $0 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|---|
| Sign of $y'$: | $y' > 0$ | $y' < 0$ | $y' < 0$ | $y' > 0$ |
| Conclusion: | Increasing | Decreasing | Decreasing | Increasing |

Increasing: $(-\infty, -2), (2, \infty)$

Decreasing: $(-2, 0), (0, 2)$

**12.** $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

| Test intervals: | $-\infty < x < -4$ | $-4 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(-4, \infty)$

Decreasing on: $(-\infty, -4)$

Relative minimum: $(-4, -6)$

**14.** $f(x) = -(x^2 + 8x + 12)$

$f'(x) = -2x - 8 = 0$

Critical number: $x = -4$

| Test intervals: | $-\infty < x < -4$ | $-4 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ |
| Conclusion: | Increasing | Decreasing |

Increasing on: $(-\infty, -4)$

Decreasing on: $(-4, \infty)$

Relative maximum: $(-4, 4)$

**16.** $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

| Test intervals: | $-\infty < x < 0$ | $0 < x < 4$ | $4 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on $(-\infty, 0)$, $(4, \infty)$

Decreasing on $(0, 4)$

Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

**18.** $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

| Test intervals: | $-\infty < x < -2$ | $-2 < x < 0$ | $0 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $(-\infty, -2)$, $(0, \infty)$

Decreasing on: $(-2, 0)$

Relative maximum: $(-2, 0)$

Relative minimum: $(0, -4)$

**20.** $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

| Test intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

Relative minimum: $(2, -44)$

**22.** $f(x) = x^{2/3} - 4$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

Critical number: $x = 0$

| Test intervals: | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(0, \infty)$

Decreasing on: $(-\infty, 0)$

Relative minimum: $(0, -4)$

**24.** $f(x) = (x - 1)^{1/3}$

$f'(x) = \frac{1}{3(x-1)^{2/3}}$

Critical number: $x = 1$

| Test intervals: | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' > 0$ |
| Conclusion: | Increasing | Increasing |

Increasing on: $(-\infty, \infty)$

No relative extrema

**26.** $f(x) = |x + 3| - 1$

$f'(x) = \frac{x+3}{|x+3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$

Critical number: $x = -3$

| Test intervals: | $-\infty < x < -3$ | $-3 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ |
| Conclusion: | Decreasing | Increasing |

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

Relative minimum: $(-3, -1)$

**28.** $f(x) = \frac{x}{x+1}$

$f'(x) = \frac{(x+1)(1) - (x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$

Discontinuity: $x = -1$

| Test intervals: | $-\infty < x < -1$ | $-1 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' > 0$ |
| Conclusion: | Increasing | Increasing |

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema

**30.** $f(x) = \dfrac{x+3}{x^2} = \dfrac{1}{x} + \dfrac{3}{x^2}$

$f'(x) = -\dfrac{1}{x^2} - \dfrac{6}{x^3} = \dfrac{-(x+6)}{x^3}$

Critical number: $x = -6$

Discontinuity: $x = 0$

| Test intervals: | $-\infty < x < -6$ | $-6 < x < 0$ | $0 < x < \infty$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' < 0$ | $f' > 0$ | $f' < 0$ |
| Conclusion: | Decreasing | Increasing | Decreasing |

Increasing on: $(-6, 0)$

Decreasing on: $(-\infty, -6), (0, \infty)$

Relative minimum: $\left(-6, -\dfrac{1}{12}\right)$

**32.** $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

$f'(x) = \dfrac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2} = \dfrac{x^2 - 4x + 10}{(x-2)^2}$

Discontinuity: $x = 2$

| Test intervals: | $-\infty < x < 2$ | $2 < x < \infty$ |
|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' > 0$ |
| Conclusion: | Increasing | Increasing |

Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema

**34.** $f(x) = \sin x \cos x = \dfrac{1}{2}\sin 2x,\ 0 < x < 2\pi$

$f'(x) = \cos 2x = 0$

Critical numbers: $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

| Test intervals: | $0 < x < \dfrac{\pi}{4}$ | $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$ | $\dfrac{3\pi}{4} < x < \dfrac{5\pi}{4}$ | $\dfrac{5\pi}{4} < x < \dfrac{7\pi}{4}$ | $\dfrac{7\pi}{4} < x < 2\pi$ |
|---|---|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{4}\right), \left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right), \left(\dfrac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right), \left(\dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right)$

Relative maxima: $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right), \left(\dfrac{5\pi}{4}, \dfrac{1}{2}\right)$

Relative minima: $\left(\dfrac{3\pi}{4}, -\dfrac{1}{2}\right), \left(\dfrac{7\pi}{4}, -\dfrac{1}{2}\right)$

**36.** $f(x) = \dfrac{\sin x}{1 + \cos^2 x}, \; 0 < x < 2\pi$

$f'(x) = \dfrac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

| Test intervals: | $0 < x < \dfrac{\pi}{2}$ | $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$ | $\dfrac{3\pi}{2} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$: | $f' > 0$ | $f' < 0$ | $f' > 0$ |
| Conclusion: | Increasing | Decreasing | Increasing |

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

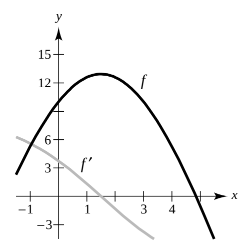
Relative maximum: $\left(\dfrac{\pi}{2}, 1\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -1\right)$

**38.** $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), [0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)




(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

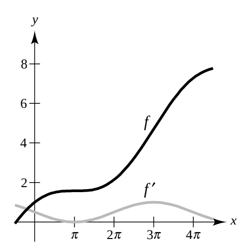
| | |
|---|---|
| $\left(0, \dfrac{3}{2}\right)$ | $\left(\dfrac{3}{2}, 5\right)$ |
| $f'(x) > 0$ | $f'(x) < 0$ |
| Increasing | Decreasing |

$f$ is increasing when $f'$ is positive and decreasing when $f'$ is negative.

**40.** $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}, [0, 4\pi]$

(a) $f'(x) = \dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2}$

(b)




(c) $\dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2} = 0$

$\sin\dfrac{x}{2} = 1$

$\dfrac{x}{2} = \dfrac{\pi}{2}$

Critical number: $x = \pi$

(d) Intervals:

| | |
|---|---|
| $(0, \pi)$ | $(\pi, 4\pi)$ |
| $f'(x) > 0$ | $f'(x) > 0$ |
| Increasing | Increasing |

$f$ is increasing when $f'$ is positive.

**42.** $f(t) = \cos^2 t - \sin^2 t = -2\sin^2 t = g(t),\ -2 < t < 2$

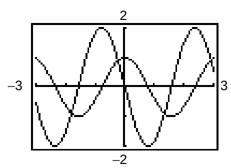
$f'(t) = -4\sin t\cos t = -2\sin 2t$

$f$ symmetric with respect to $y$-axis

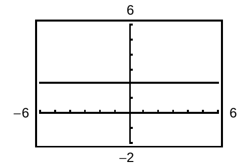
zeros of $f$: $\pm\dfrac{\pi}{4}$

Relative maximum: $(0, 1)$

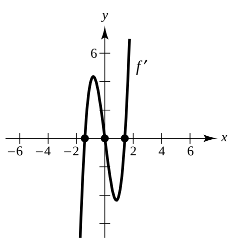
Relative minimum: $\left(-\dfrac{\pi}{2}, -1\right), \left(\dfrac{\pi}{2}, -1\right)$



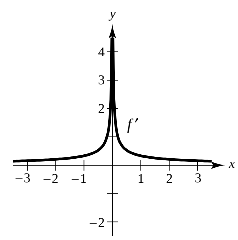

**44.** $f(x)$ is a line of slope $\approx 2 \Longrightarrow f'(x) = 2$.




**46.** $f$ is a 4th degree polynomial $\Longrightarrow f'$ is a cubic polynomial.




**48.** $f$ has positive slope




**In Exercises 50–54, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.**

**50.** $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

**52.** $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(0) = -f'(0) > 0$

**54.** $g(x) = f(x - 10)$

$g'(x) = f'(x - 10)$

$g'(8) = f'(-2) < 0$

**56.** Critical number: $x = 5$

$f'(4) = -2.5 \Longrightarrow f$ is decreasing at $x = 4$.

$f'(6) = 3 \Longrightarrow f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

**58.** $s(t) = 4.9(\sin\theta)t^2$

(a) $v(t) = 9.8(\sin\theta)t$ speed $= |9.8(\sin\theta)t|$

(b) If $\theta = \pi/2$, the speed is maximum,

$v(t) = 9.8t$.

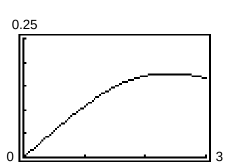
**60.** $C = \dfrac{3t}{27 + t^3},\ t \geq 0$

(a)

| $t$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $C(t)$ | 0 | 0.055 | 0.107 | 0.148 | 0.171 | 0.176 | 0.167 |

The concentration seems greater near $t = 2.5$ hours.

(b)




The concentration is greatest when $t \approx 2.38$ hours.

(c) $C' = \dfrac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$

$= \dfrac{3(27 - 2t^3)}{(27 + t^3)^2}$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

**62.** $P = 2.44x - \dfrac{x^2}{20{,}000} - 5000,\ 0 \le x \le 35{,}000$

$P' = 2.44 - \dfrac{x}{10{,}000} = 0$

$x = 24{,}400$

Increasing when $0 < x < 24{,}400$ hamburgers.

Decreasing when $24{,}400 < x < 35{,}000$ hamburgers.

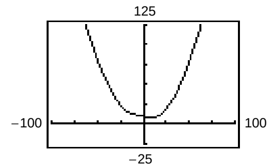
| Test intervals: | $0 < x < 24{,}400$ | $24{,}400 < x < 35{,}000$ |
|---|---|---|
| Sign of $P'$: | $P' > 0$ | $P' < 0$ |

**64.** $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

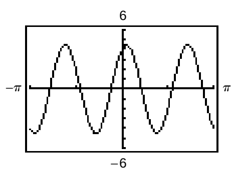
$T = 10°,\ R \approx 8.3666\Omega$

(b)




The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10°$.

**66.** $f(x) = 2\sin(3x) + 4\cos(3x)$




The maximum value is approximately 4.472. You could use calculus by finding $f'(x)$ and then observing that the maximum value of $f$ occurs at a point where $f'(x) = 0$. For instance, $f'(0.154) \approx 0$, and $f(0.154) = 4.472$.

**68.** (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

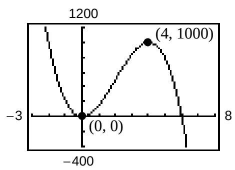
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

| | | |
|---|---|---|
| (0, 0): | $0 = a_0$ | $(f(0) = 0)$ |
| | $0 = a_1$ | $(f'(0) = 0)$ |
| (4, 1000): | $1000 = 64a_3 + 16a_2$ | $(f(4) = 1000)$ |
| | $0 = 48a_3 + 8a_2$ | $(f'(4) = 0)$ |

(c) The solution is $a_0 = a_1 = 0,\ a_2 = \dfrac{375}{2},\ a_3 = \dfrac{-125}{4}$

$$f(x) = \frac{-125}{4}x^3 + \frac{375}{2}x^2.$$

(d)



**70.** (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

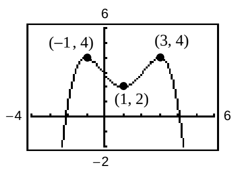
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

| | | |
|---|---|---|
| $(1, 2)$: | $2 = a_4 + a_3 + a_2 + a_1 + a_0$ | $(f(1) = 2)$ |
| | $0 = 4a_4 + 3a_3 + 2a_2 + a_1$ | $(f'(1) = 0)$ |
| $(-1, 4)$: | $4 = a_4 - a_3 + a_2 - a_1 + a_0$ | $(f(-1) = 4)$ |
| | $0 = -4a_4 + 3a_3 - 2a_2 + a_1$ | $(f'(-1) = 0)$ |
| $(3, 4)$: | $4 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$ | $(f(3) = 4)$ |
| | $0 = 108a_4 + 27a_3 + 6a_2 + a_1$ | $(f'(3) = 0)$ |

(c) The solution is $a_0 = \frac{23}{8}$, $a_1 = -\frac{3}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$, $a_4 = -\frac{1}{8}$

$$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}.$$

(d)




**72.** False

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

**74.** True

If $f(x)$ is an $n$th-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

**76.** False.

The function might not be continuous.

**78.** Suppose $f'(x)$ changes from positive to negative at $c$. Then there exists $a$ and $b$ in $I$ such that $f'(x) > 0$ for all $x$ in $(a, c)$ and $f'(x) < 0$ for all $x$ in $(c, b)$. By Theorem 3.5, $f$ is increasing on $(a, c)$ and decreasing on $(c, b)$. Therefore, $f(c)$ is a maximum of $f$ on $(a, b)$ and thus, a relative maximum of $f$.

## Section 3.4 Concavity and the Second Derivative Test

**2.** $y = -x^3 + 3x^2 - 2$, $y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

**4.** $f(x) = \dfrac{x^2 - 1}{2x + 1}$, $y'' = \dfrac{-6}{(2x + 1)^3}$

Concave upward: $\left(-\infty, -\frac{1}{2}\right)$

Concave downward: $\left(-\frac{1}{2}, \infty\right)$

**6.** $y = \dfrac{1}{270}(-3x^5 + 40x^3 + 135x)$, $y'' = \dfrac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2)$, $(0, 2)$

Concave downward: $(-2, 0)$, $(2, \infty)$

**8.** $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

**10.** $y = x + 2\csc x, \quad (-\pi, \pi)$

$y' = 1 - 2\csc x \cot x$

$y'' = -2\csc x(-\csc^2 x) - 2\cot x(-\csc x \cot x)$

$= 2(\csc^3 x + \csc x \cot^2 x)$

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

**12.** $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$f''(x) = 12x - 6 = 0$ when $x = \frac{1}{2}$.

| Test interval | $-\infty < x < \frac{1}{2}$ | $\frac{1}{2} < x < \infty$ |
|---|---|---|
| Sign of $f''(x)$ | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion | Concave downward | Concave upward |

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{2}\right)$

**14.** $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2 = 0$ when $x = 0$.

However, $(0, 3)$ is not a point of inflection since $f''(x) \geq 0$ for all $x$.

Concave upward on $(-\infty, \infty)$

**16.** $f(x) = x^3(x - 4)$

$f'(x) = x^3 + 3x^2(x - 4)$

$= x^2[x + 3(x - 4)] = 4x^2(x - 3)$

$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$

$f''(x) = 12x(x - 2) = 0$ when $x = 0, 2$.

| Test interval | $-\infty < x < 0$ | $0 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Sign of $f''(x)$ | $f''(x) > 0$ | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion | Concave upward | Concave downward | Concave upward |

Points of inflection: $(0, 0), (2, -16)$

**18.** $f(x) = x\sqrt{x + 1}$, Domain: $[-1, \infty)$

$f'(x) = (x)\dfrac{1}{2}(x + 1)^{-1/2} + \sqrt{x + 1} = \dfrac{3x + 2}{2\sqrt{x + 1}}$

$f''(x) = \dfrac{6\sqrt{x + 1} - (3x + 2)(x + 1)^{-1/2}}{4(x + 1)} = \dfrac{3x + 4}{4(x + 1)^{3/2}}$

$f''(x) > 0$ on the entire domain of $f$ (except for $x = -1$, for which $f''(x)$ is undefined).
There are no points of inflection.

Concave upward on $(-1, \infty)$

**20.** $f(x) = \dfrac{x + 1}{\sqrt{x}}$ Domain: $x > 0$

$f'(x) = \dfrac{x - 1}{2x^{3/2}}$

$f''(x) = \dfrac{3 - x}{4x^{5/2}}$

Point of inflection: $\left(3, \dfrac{4}{\sqrt{3}}\right) = \left(3, \dfrac{4\sqrt{3}}{3}\right)$

| Test intervals | $0 < x < 3$ | $3 < x < \infty$ |
|---|---|---|
| Sign of $f''(x)$ | $f'' > 0$ | $f'' < 0$ |
| Conclusion | Concave upward | Concave downward |

**22.** $f(x) = 2\csc\dfrac{3x}{2},\ 0 < x < 2\pi$

$f'(x) = -3\csc\dfrac{3x}{2}\cot\dfrac{3x}{2}$

$f''(x) = \dfrac{9}{2}\left(\csc^3\dfrac{3x}{2} + \csc\dfrac{3x}{2}\cot^2\dfrac{3x}{2}\right) \neq 0$ for any $x$ in the domain of $f$.

Concave upward: $\left(0, \dfrac{2\pi}{3}\right), \left(\dfrac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right)$

No points of inflection

**24.** $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f''(x) = 0$ when $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$.

| Test interval: | $0 < x < \frac{3\pi}{4}$ | $\frac{3\pi}{4} < x < \frac{7\pi}{4}$ | $\frac{7\pi}{4} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f''(x) < 0$ | $f''(x) > 0$ | $f''(x) < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

Points of inflection: $\left(\dfrac{3\pi}{4}, 0\right), \left(\dfrac{7\pi}{4}, 0\right)$

**26.** $f(x) = x + 2\cos x,\ [0, 2\pi]$

$f'(x) = 1 - 2\sin x$

$f''(x) = -2\cos x$

$f''(x) = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

| Test intervals: | $0 < x < \frac{\pi}{2}$ | $\frac{\pi}{2} < x < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < x < \frac{3\pi}{2}$ |
|---|---|---|---|
| Sign of $f''(x)$: | $f'' < 0$ | $f'' > 0$ | $f'' < 0$ |
| Conclusion: | Concave downward | Concave upward | Concave downward |

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

**28.** $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

**30.** $f(x) = -(x - 5)^2$

$f'(x) = -2(x - 5)$

$f''(x) = -2$

Critical number: $x = 5$

$f''(5) < 0$

Therefore, $(5, 0)$ is a relative maximum.

**32.** $f(x) = x^3 - 9x^2 + 27x$

$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$

$f''(x) = 6(x - 3)$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \geq 0$ for all $x$ and, therefore, there are no relative extrema.

**34.** $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

$g'(x) = \dfrac{-(x - 4)(x - 1)(x + 2)}{2}$

$g''(x) = 3 + 3x - \frac{3}{2}x^2$

Critical numbers: $x = -2, 1, 4$

$g''(-2) = -9 < 0$

$(-2, 0)$ is a relative maximum.

$g''(1) = 9/2 > 0$

$(1, -10.125)$ is a relative minimum.

$g''(4) = -9 < 0$

$(4, 0)$ is a relative maximum.

**36.** $f(x) = \sqrt{x^2 + 1}$

$f'(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

Critical number: $x = 0$

$f''(x) = \dfrac{1}{(x^2 + 1)^{3/2}}$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

**38.** $f(x) = \dfrac{x}{x - 1}$

$f'(x) = \dfrac{-1}{(x - 1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

**40.** $f(x) = 2 \sin x + \cos 2x,\ 0 \leq x \leq 2\pi$

$f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x - 4 \sin x \cos x = 2 \cos x(1 - 2 \sin x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

$f''(x) = -2 \sin x - 4 \cos 2x$

$f''\left(\frac{\pi}{6}\right) < 0$

$f''\left(\frac{\pi}{2}\right) > 0$

$f''\left(\frac{5\pi}{6}\right) < 0$

$f''\left(\frac{3\pi}{2}\right) > 0$

Relative maxima: $\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$

Relative minima: $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$

**42.** $f(x) = x^2\sqrt{6 - x^2}$, $[-\sqrt{6}, \sqrt{6}]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0$, $x = \pm 2$.

$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

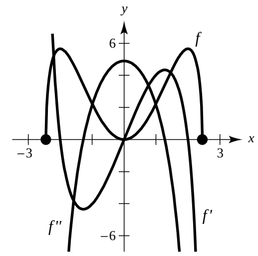
$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow \left(\pm 2, 4\sqrt{2}\right)$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)




The graph of $f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

**44.** $f(x) = \sqrt{2x}\sin x$, $[0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \dfrac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$f''(x) = -\sqrt{2x}\sin x + \dfrac{\cos x}{\sqrt{2x}} + \dfrac{\cos x}{\sqrt{2x}} - \dfrac{\sin x}{2x\sqrt{2x}}$

$= \dfrac{2\cos x}{\sqrt{2x}} - \dfrac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$

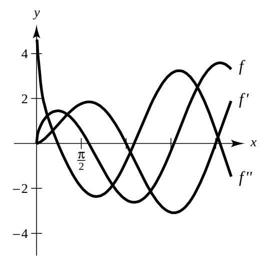
$= \dfrac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$

(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

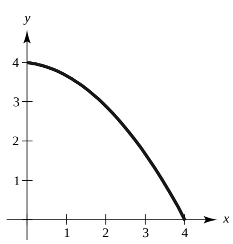
Points of inflection: $(0.75, 0.83)$, $(3.42, -0.72)$

(c)




$f$ is increasing when $f' > 0$ and decreasing when $f' < 0$. $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

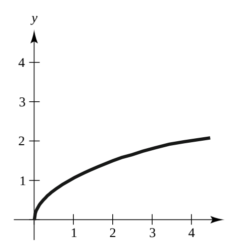
**46.** (a)




$f' < 0$ means $f$ decreasing

$f'$ decreasing means concave downward

(b)




$f' > 0$ means $f$ increasing

$f'$ decreasing means concave downward

**48.** (a) The rate of change of sales is increasing.
$S'' > 0$

(b) The rate of change of sales is decreasing.
$S' > 0$, $S'' < 0$

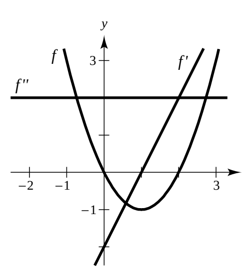
(c) The rate of change of sales is constant.
$S' = C$, $S'' = 0$

(d) Sales are steady.
$S = C$, $S' = 0$, $S'' = 0$

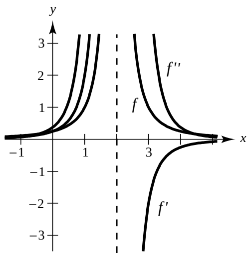
(e) Sales are declining, but at a lower rate.
$S' < 0$, $S'' > 0$

(f) Sales have bottomed out and have started to rise.
$S' > 0$

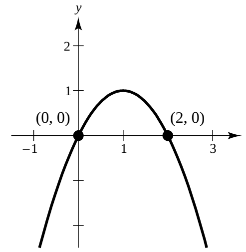
**50.**



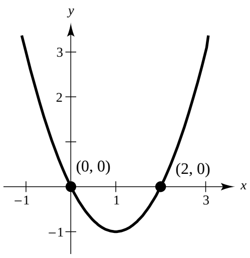
**52.**



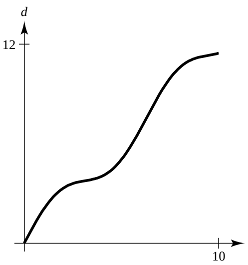

**54.**




**56.**




**58.** (a)




(b) Since the depth $d$ is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of $d$ is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

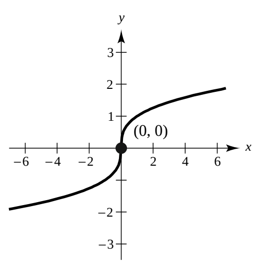
**60.** (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Inflection point: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.




**62.** $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned}\right\} 56a + 12b + 2c = -2 \Longrightarrow 28a + 6b + c = -1$$

$f'(2) = 12a + 4b + c = 0,\ \ f'(4) = 48a + 8b + c = 0,\ \ f''(3) = 18a + 2b = 0$

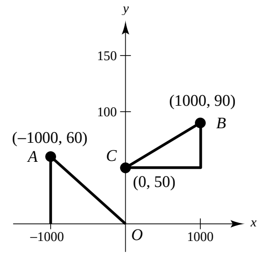
$$\begin{aligned} 28a + 6b + c &= -1 & \qquad 18a + 2b &= 0 \\ 12a + 4b + c &= 0 & 16a + 2b &= -1 \\ 16a + 2b &= -1 & 2a &= 1 \end{aligned}$$

$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

**64.** (a) line $OA$: $y = -0.06x$ slope: $-0.06$

line $CB$: $y = 0.04x + 50$ slope: $0.04$




$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $\quad 60 = (-1000)^3 a + (1000)^2 b - 1000c + d$

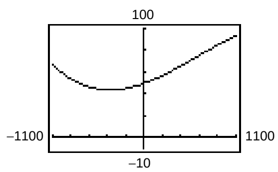
$\quad -0.06 = (1000)^2 3a - 2000b + c$

$(1000, 90)$: $\quad 90 = (1000)^3 a + (1000)^2 b + 1000c + d$

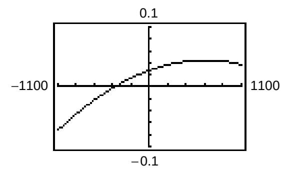
$\quad 0.04 = (1000)^2 3a + 2000b + c$

The solution to this system of 4 equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$




(c)

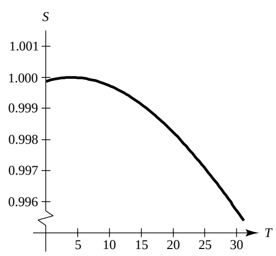



(d) The steepest part of the road is 6% at the point $A$.

**66.** $S = \dfrac{5.755T^3}{10^8} - \dfrac{8.521T^2}{10^6} + \dfrac{0.654T}{10^4} + 0.99987, \quad 0 < T < 25$

(a) The maximum occurs when $T \approx 4°$ and $S \approx 0.999999$.

(b)




(c) $S(20°) \approx 0.9982$

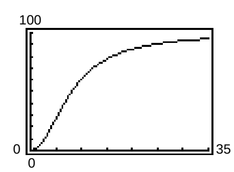
**68.** $C = 2x + \dfrac{300{,}000}{x}$

$C' = 2 - \dfrac{300{,}000}{x^2} = 0$ when $x = 100\sqrt{15} \approx 387$

By the First Derivative Test, $C$ is minimized when $x \approx 387$ units.

**70.** $S = \dfrac{100t^2}{65 + t^2}, \ t > 0$

(a)




(b) $S'(t) = \dfrac{13{,}000t}{(65 + t^2)^2}$

$S''(t) = \dfrac{13{,}000(65 - 3t^2)}{(65 + t^2)^3} = 0 \implies t = 4.65$

$S$ is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As $t$ increases, the speed increases, but at a slower rate.

**72.** $f(x) = 2(\sin x + \cos x), \qquad f(0) = 2$

$f'(x) = 2(\cos x - \sin x), \qquad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x), \qquad f''(0) = -2$

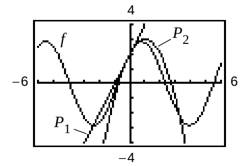
$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

$P_2(x) = 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$

$P_2'(x) = 2 - 2x$

$P_2''(x) = -2$




The values of $f$, $P_1$, $P_2$, and their first derivatives are equal at $x = 0$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

**74.** $f(x) = \dfrac{\sqrt{x}}{x - 1}, \qquad f(2) = \sqrt{2}$

$f'(x) = \dfrac{-(x + 1)}{2\sqrt{x}(x - 1)^2}, \qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2 + 6x - 1}{4x^{3/2}(x - 1)^3}, \qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

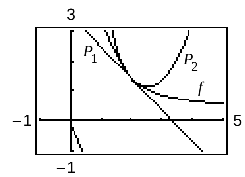
$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x - 2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x - 2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x - 2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x - 2) + \dfrac{23\sqrt{2}}{32}(x - 2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x - 2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$




The values of $f$, $P_1$, $P_2$ and their first derivatives are equal at $x = 2$. The values of the second derivatives of $f$ and $P_2$ are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

**76.** $f(x) = x(x - 6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x - 2)(x - 6) = 0$

$f''(x) = 6x - 24 = 6(x - 4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of $f$.

**78.** $p(x) = ax^3 + bx^2 + cx + d$

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$

$6ax + 2b = 0$

$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$

$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

**80.** False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

**82.** True

$y = \sin(bx)$

Slope: $y' = b\cos(bx)$

$-b \le y' \le b$ (Assume $b > 0$)

**84.** False. For example, let $f(x) = (x - 2)^4$.

## Section 3.5 Limits at Infinity

**2.** $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

**4.** $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

**6.** $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

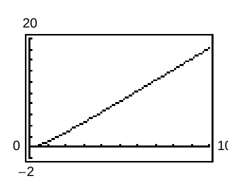
No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e)

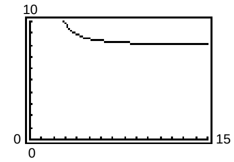
**8.** $f(x) = \dfrac{2x^2}{x + 1}$

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 18.18 | 198.02 | 1998.02 | 19,998 | 199,998 | 1,999,998 |




$\lim_{x\to\infty} f(x) = \infty$ (Limit does not exist.)

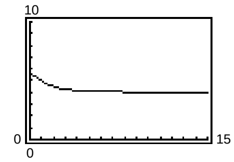
**10.** $f(x) = \dfrac{8x}{\sqrt{x^2 - 3}}$



| $x$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 8.12 | 8.001 | 8 | 8 | 8 | 8 | 8 |

$\lim_{x\to\infty} f(x) = 8$

**12.** $f(x) = 4 + \dfrac{3}{x^2 + 2}$



| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 5 | 4.03 | 4.0003 | 4.0 | 4.0 | 4 | 4 |

$\lim_{x\to\infty} f(x) = 4$

**14.** (a) $h(x) = \dfrac{f(x)}{x} = \dfrac{5x^2 - 3x + 7}{x} = 5x - 3 + \dfrac{7}{x}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^2 - 3x + 7}{x^2} = 5 - \dfrac{3}{x} + \dfrac{7}{x^2}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^2 - 3x + 7}{x^3} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{7}{x^3}$

$\lim_{x\to\infty} h(x) = 0$

**16.** (a) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x - 1} = -\dfrac{2}{3}$

(c) $\lim_{x\to\infty} \dfrac{3 - 2x^2}{3x - 1} = -\infty$ (Limit does not exist)

**18.** (a) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2} + 1} = \dfrac{5}{4}$

(c) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x} + 1} = \infty$ (Limit does not exist)

**20.** $\lim_{x\to\infty} \dfrac{3x^3 + 2}{9x^3 - 2x^2 + 7} = \dfrac{3}{9} = \dfrac{1}{3}$

**22.** $\lim_{x\to\infty} \left(4 + \dfrac{3}{x}\right) = 4 + 0 = 4$

**24.** $\lim_{x\to\infty} \left(\dfrac{1}{2}x - \dfrac{4}{x^2}\right) = -\infty$ (Limit does not exist)

**26.** $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 + 1}} = \lim_{x\to-\infty} \dfrac{1}{\left(\dfrac{\sqrt{x^2 + 1}}{-\sqrt{x^2}}\right)}$ $\left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

$= \lim_{x\to-\infty} \dfrac{-1}{\sqrt{x + (1/x)}} = -1$

**28.** $\lim_{x \to -\infty} \dfrac{-3x + 1}{\sqrt{x^2 + x}} = \lim_{x \to -\infty} \dfrac{-3 + (1/x)}{\dfrac{\sqrt{x^2 + x}}{-\sqrt{x^2}}}$, $\left(\text{for } x < 0 \text{ we have } -\sqrt{x^2} = x\right)$

$$= \lim_{x \to -\infty} \frac{3 - (1/x)}{\sqrt{1 + (1/x)}} = 3$$

**30.** $\lim_{x \to \infty} \dfrac{x - \cos x}{x} = \lim_{x \to \infty} \left(1 - \dfrac{\cos x}{x}\right)$

$$= 1 - 0 = 1$$

**Note:**

$\lim_{x \to \infty} \dfrac{\cos x}{x} = 0$ by the Squeeze Theorem since

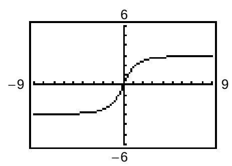
$-\dfrac{1}{x} \le \dfrac{\cos x}{x} \le \dfrac{1}{x}$.

**32.** $\lim_{x \to \infty} \cos\left(\dfrac{1}{x}\right) = \cos 0 = 1$

**34.** $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

$\lim_{x \to \infty} f(x) = 3$

$\lim_{x \to -\infty} f(x) = -3$




Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.

**36.** $\lim_{x \to \infty} x \tan \dfrac{1}{x} = \lim_{t \to 0^+} \dfrac{\tan t}{t} = \lim_{t \to 0^+} \left[\dfrac{\sin t}{t} \cdot \dfrac{1}{\cos t}\right]$

$$= (1)(1) = 1$$

(Let $x = 1/t$.)

**38.** $\lim_{x \to \infty} \left(2x - \sqrt{4x^2 + 1}\right) = \lim_{x \to \infty} \left[\left(2x - \sqrt{4x^2 + 1}\right) \cdot \dfrac{2x + \sqrt{4x^2 + 1}}{2x + \sqrt{4x^2 + 1}}\right] = \lim_{x \to \infty} \dfrac{-1}{2x + \sqrt{4x^2 + 1}} = 0$

**40.** $\lim_{x \to -\infty} \left(3x + \sqrt{9x^2 - x}\right) = \lim_{x \to -\infty} \left[\left(3x + \sqrt{9x^2 - x}\right) \cdot \dfrac{3x - \sqrt{9x^2 - x}}{3x - \sqrt{9x^2 - x}}\right]$

$$= \lim_{x \to -\infty} \frac{x}{3x - \sqrt{9x^2 - x}}$$

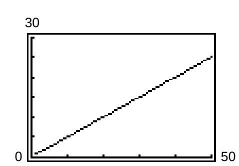
$$= \lim_{x \to -\infty} \frac{1}{3 - \dfrac{\sqrt{9x^2 - x}}{-\sqrt{x^2}}} \quad \left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$$

$$= \lim_{x \to -\infty} \frac{1}{3 + \sqrt{9 - (1/x)}} = \frac{1}{6}$$

**42.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.0 | 5.1 | 50.1 | 500.1 | 5000.1 | 50,000.1 | 500,000.1 |

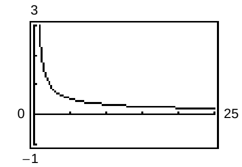
$\lim_{x \to \infty} \dfrac{x^2 - x\sqrt{x^2 - x}}{1} \cdot \dfrac{x^2 + x\sqrt{x^2 - x}}{x^2 + x\sqrt{x^2 - x}} = \lim_{x \to \infty} \dfrac{x^3}{x^2 + x\sqrt{x^2 - x}} = \infty$




Limit does not exist.

**44.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2.000 | 0.348 | 0.101 | 0.032 | 0.010 | 0.003 | 0.001 |




$$\lim_{x\to\infty} \frac{x+1}{x\sqrt{x}} = 0$$

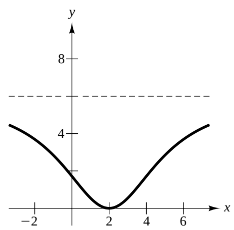
**46.** $x = 2$ is a critical number.

$f'(x) < 0$ for $x < 2$.

$f'(x) > 0$ for $x > 2$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 6$$

For example, let $f(x) = \dfrac{-6}{0.1(x-2)^2 + 1} + 6$.



**48.** (a) The function is even: $\lim_{x\to-\infty} f(x) = 5$

(b) The function is odd: $\lim_{x\to-\infty} f(x) = -5$

**50.** $y = \dfrac{x-3}{x-2}$

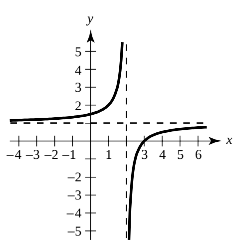
Intercepts: $(3, 0), \left(0, \dfrac{3}{2}\right)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x\to-\infty} \frac{x-3}{x-2} = 1 = \lim_{x\to\infty} \frac{x-3}{x-2}.$$

Discontinuity: $x = 2$ (Vertical asymptote)



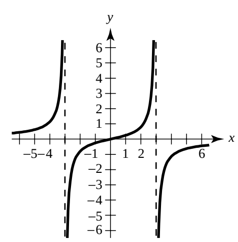
**52.** $y = \dfrac{2x}{9 - x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 3$



**54.** $y = \dfrac{x^2}{x^2 - 9}$

Intercept: $(0, 0)$

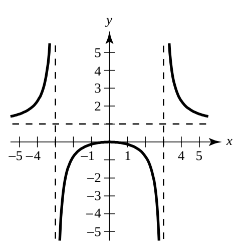
Symmetry: $y$-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x\to-\infty} \frac{x^2}{x^2-9} = 1 = \lim_{x\to\infty} \frac{x^2}{x^2-9}.$$

Discontinuities: $x = \pm 3$ (Vertical asymptotes)

Relative maximum: $(0, 0)$



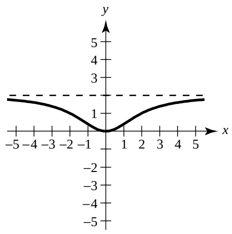
**56.** $y = \dfrac{2x^2}{x^2 + 4}$

Intercept: $(0, 0)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 2$

Relative minimum: $(0, 0)$




**58.** $x^2y = 4$

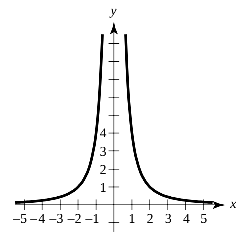
Intercepts: none

Symmetry: $y$-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{4}{x^2} = 0 = \lim_{x \to \infty} \frac{4}{x^2}.$$

Discontinuity: $x = 0$ (Vertical asymptote)




**60.** $y = \dfrac{2x}{1 - x^2}$

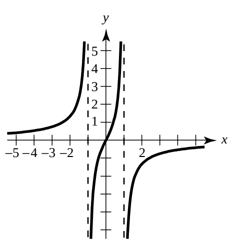
Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x^2} = 0 = \lim_{x \to \infty} \frac{2x}{1 - x^2}.$$

Discontinuities: $x = \pm 1$ (Vertical asymptotes)




**62.** $y = 1 + \dfrac{1}{x}$

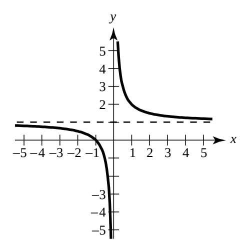
Intercept: $(-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right) = 1 = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



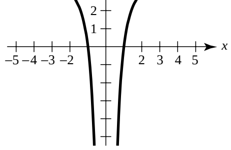

**64.** $y = 4\left(1 - \dfrac{1}{x^2}\right)$

Intercepts: $(\pm 1, 0)$

Symmetry: $y$-axis

Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 0$




**66.** $y = \dfrac{x}{\sqrt{x^2 - 4}}$

Domain: $(-\infty, -2), (2, \infty)$

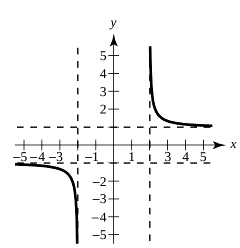
Intercepts: none

Symmetry: origin

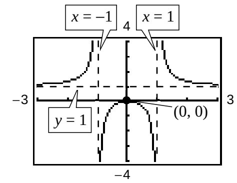
Horizontal asymptotes: $y = \pm 1$ since

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}} = 1, \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - 4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)



**68.** $f(x) = \dfrac{x^2}{x^2 - 1}$




$$f'(x) = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2} = 0 \text{ when } x = 0.$$

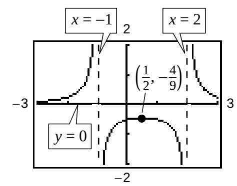
$$f''(x) = \frac{(x^2 - 1)^2(-2) + 2x(2)(x^2 - 1)(2x)}{(x^2 - 1)^4} = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}$$

Since $f''(0) < 0$, then $(0, 0)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of $f$, there are no points of inflection.

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$

**70.** $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x + 1)(x - 2)}$




$$f'(x) = \frac{-(2x - 1)}{(x^2 - x - 2)^2} = 0 \text{ when } x = \frac{1}{2}.$$

$$f''(x) = \frac{(x^2 - x - 2)^2(-2) + (2x - 1)(2)(x^2 - x - 2)(2x - 1)}{(x^2 - x - 2)^4}$$

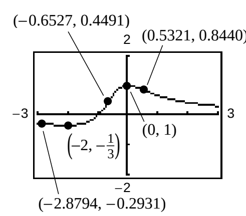
$$= \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}$$

Since $f''\left(\frac{1}{2}\right) < 0$, then $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of $f$, there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 0$

**72.** $f(x) = \dfrac{x + 1}{x^2 + x + 1}$




$$f'(x) = \frac{-x(x + 2)}{(x^2 + x + 1)^2} = 0 \text{ when } x = 0, -2.$$

$$f''(x) = \frac{2(x^3 + 3x^2 - 1)}{(x^2 + x + 1)^3} = 0 \text{ when } x \approx 0.5321, -0.6527, -2.8794.$$

$f''(0) < 0$

Therefore, $(0, 1)$ is a relative maximum.

$f''(-2) > 0$

Therefore,

$$\left(-2, -\frac{1}{3}\right)$$

is a relative minimum.

Points of inflection: $(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and $(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$

**74.** $g(x) = \dfrac{2x}{\sqrt{3x^2 + 1}}$

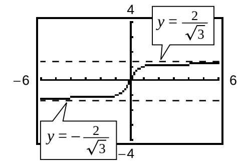
$g'(x) = \dfrac{2}{(3x^2 + 1)^{3/2}}$

$g''(x) = \dfrac{-18x}{(3x^2 + 1)^{5/2}}$

No relative extrema. Point of inflection: $(0, 0)$.

Horizontal asymptotes: $y = \pm\dfrac{2}{\sqrt{3}}$

No vertical asymptotes




**76.** $f(x) = \dfrac{2 \sin 2x}{x}$ Hole at $(0, 4)$

$f'(x) = \dfrac{4x \cos 2x - 2 \sin 2x}{x^2}$

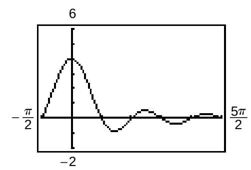
There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

Relative minima: $(\pm 2.25, -0.869)$, $(\pm 5.45, -0.365)$

Relative maxima: $(\pm 3.87, 0.513)$

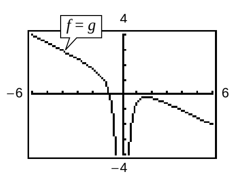
Horizontal asymptote: $y = 0$

No vertical asymptotes




**78.** $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$, $g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

(a)

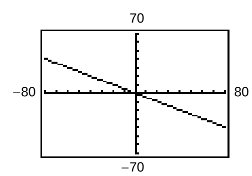



(b) $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$= -\left[\dfrac{x^3}{2x^2} - \dfrac{2x^2}{2x^2} + \dfrac{2}{2x^2}\right]$

$= -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2} = g(x)$

(c)

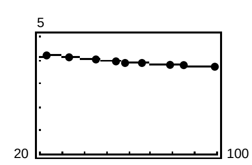



The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

**80.** $\displaystyle\lim_{v_1/v_2 \to \infty} 100\left[1 - \frac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

**82.** $y = \dfrac{3.351t^2 + 42.461t - 543.730}{t^2}$

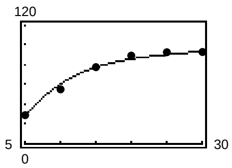
(a)




(b) Yes. $\displaystyle\lim_{t \to \infty} y = 3.351$

**84.** $S = \dfrac{100t^2}{65 + t^2},\ t > 0$

(a)




(b) Yes. $\lim\limits_{t\to\infty} S = \dfrac{100}{1} = 100$

**86.** $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by $x^m$.

**Case 1:** If $n < m$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{\dfrac{a_n}{x^{m-n}} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{0}{b_m} = 0$

**Case 2:** If $m = n$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{a_n}{b_m}.$

**Case 3:** If $n > m$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n x^{n-m} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty.$

**88.** False. Let $y_1 = \sqrt{x+1}$, then $y_1(0) = 1$. Thus, $y_1{}' = 1/\left(2\sqrt{x+1}\right)$ and $y_1{}'(0) = 1/2$. Finally,

$$y_1{}'' = -\frac{1}{4(x+1)^{3/2}} \text{ and } y_1{}''(0) = -\frac{1}{4}.$$

Let $p = ax^2 + bx + 1$, then $p(0) = 1$. Thus, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \ge 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/\left(2\sqrt{x+1}\right), & x > 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/(4(x+1)^{3/2}), & x > 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real $x$, but $f(x)$ *increases* without bound.

## Section 3.6 A Summary of Curve Sketching

**2.** The slope of $f$ approaches $\infty$ as $x \to 0^-$, and approaches $-\infty$ as $x \to 0^+$. Matches (C)

**4.** The slope is positive up to approximately $x = 1.5$. Matches (B)

**6.** (a) $x_0, x_2, x_4$

(b) $x_2, x_3$

(c) $x_1$

(d) $x_1$

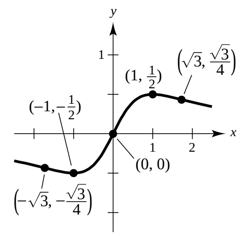
(e) $x_2, x_3$

**8.** $y = \dfrac{x}{x^2 + 1}$

$y' = \dfrac{1 - x^2}{(x^2 + 1)^2} = \dfrac{(1 - x)(x + 1)}{(x^2 + 1)^2} = 0$ when $x = \pm 1$.

$y'' = -\dfrac{2x(3 - x^2)}{(x^2 + 1)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

Horizontal asymptote: $y = 0$




| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -\sqrt{3}$ | | $-$ | $-$ | Decreasing, concave down |
| $x = -\sqrt{3}$ | $-\dfrac{\sqrt{3}}{4}$ | $-$ | 0 | Point of inflection |
| $-\sqrt{3} < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-\dfrac{1}{2}$ | 0 | $+$ | Relative minimum |
| $-1 < x < 0$ | | $+$ | $+$ | Increasing, concave up |
| $x = 0$ | 0 | $+$ | 0 | Point of inflection |
| $0 < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | $\dfrac{1}{2}$ | 0 | $-$ | Relative maximum |
| $1 < x < \sqrt{3}$ | | $-$ | $-$ | Decreasing, concave down |
| $x = \sqrt{3}$ | $\dfrac{\sqrt{3}}{4}$ | $-$ | 0 | Point of inflection |
| $\sqrt{3} < x < \infty$ | | $-$ | $+$ | Decreasing, concave up |

**10.** $y = \dfrac{x^2 + 1}{x^2 - 9}$

$y' = \dfrac{-20x}{(x^2 - 9)^2} = 0 \quad \text{when } x = 0$

$y'' = \dfrac{60(x^2 + 3)}{(x^2 - 9)^3} < 0 \text{ when } x = 0$

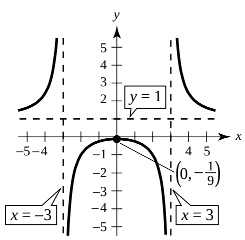
Therefore, $\left(0, -\dfrac{1}{9}\right)$ is a relative maximum.

Intercept: $\left(0, -\dfrac{1}{9}\right)$

Vertical asymptotes: $x = \pm 3$

Horizontal asymptote: $y = 1$

Symmetric about $y$-axis




**12.** $f(x) = \dfrac{x + 2}{x} = 1 + \dfrac{2}{x}$

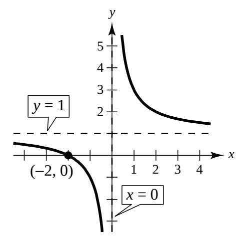
$f'(x) = \dfrac{-2}{x^2} < 0 \text{ when } x \neq 0.$

$f''(x) = \dfrac{4}{x^3} \neq 0$

Intercept: $(-2, 0)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$




**14.** $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x - 4)(x^2 + 4x + 16)}{x^3} = 0 \text{ when } x = 4.$

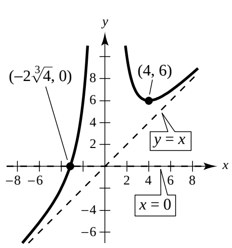
$f''(x) = \dfrac{192}{x^4} > 0 \text{ if } x \neq 0.$

Therefore, $(4, 6)$ is a relative minimum.

Intercept; $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$




**16.** $f(x) = \dfrac{x^3}{x^2 - 4} = x + \dfrac{4x}{x^2 - 4}$

$f'(x) = \dfrac{x^2(x^2 - 12)}{(x^2 - 4)^2} = 0 \qquad \text{when } x = 0, \pm 2\sqrt{3}$

$f''(x) = \dfrac{8x(x^2 + 12)}{(x^2 - 4)^3} = 0 \text{ when } x = 0$

Intercept: $(0, 0)$

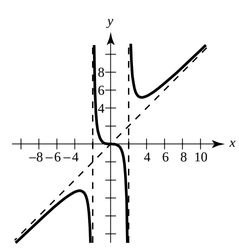
Relative maximum: $\left(-2\sqrt{3}, -3\sqrt{3}\right)$

Relative minimum: $\left(2\sqrt{3}, 3\sqrt{3}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm 2$

Slant asymptote: $y = x$



**18.** $y = \dfrac{2x^2 - 5x + 5}{x - 2} = 2x - 1 + \dfrac{3}{x - 2}$

$y' = 2 - \dfrac{3}{(x - 2)^2} = \dfrac{2x^2 - 8x + 5}{(x - 2)^2} = 0$ when $x = \dfrac{4 \pm \sqrt{6}}{2}$.

$y'' = \dfrac{6}{(x - 2)^3} \neq 0$

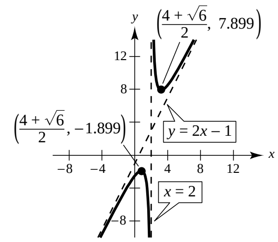
Relative maximum: $\left(\dfrac{4 - \sqrt{6}}{2}, -1.8990\right)$

Relative minimum: $\left(\dfrac{4 + \sqrt{6}}{2}, 7.8990\right)$

Intercept: $(0, -5/2)$

Vertical asymptote: $x = 2$

Slant asymptote: $y = 2x - 1$




**20.** $g(x) = x\sqrt{9 - x}$ Domain: $x \leq 9$

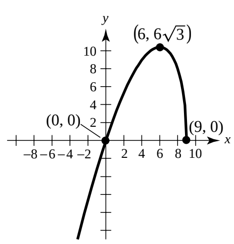
$g'(x) = \dfrac{3(6 - x)}{2\sqrt{9 - x}} = 0$ when $x = 6$

$g''(x) = \dfrac{3(x - 12)}{4(9 - x)^{3/2}} < 0$ when $x = 6$

Relative maximum: $\left(6, 6\sqrt{3}\right)$

Intercepts: $(0, 0), (9, 0)$

Concave downward on $(-\infty, 9)$




**22.** $y = x\sqrt{16 - x^2}$ Domain: $-4 \leq x \leq 4$

$y' = \dfrac{2(8 - x^2)}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$

$y'' = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}} = 0$ when $x = 0$

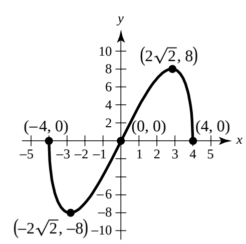
Relative maximum: $\left(2\sqrt{2}, 8\right)$

Relative minimum: $\left(-2\sqrt{2}, -8\right)$

Intercepts: $(0, 0), (\pm 4, 0)$

Symmetric with respect to the origin

Point of inflection: $(0, 0)$




**24.** $y = 3(x - 1)^{2/3} - (x - 1)^2$

$y' = \dfrac{2}{(x - 1)^{1/3}} - 2(x - 1) = \dfrac{2 - 2(x - 1)^{4/3}}{(x - 1)^{1/3}} = 0$ when $x = 0, 2$

$(y'$ undefined for $x = 1)$

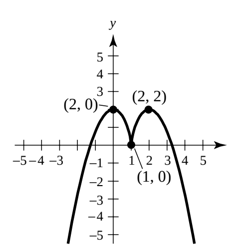
$y'' = \dfrac{-2}{3(x - 1)^{4/3}} - 2 < 0$ for all $x \neq 1$

Concave downward on $(-\infty, 1)$ and $(1, \infty)$

Relative maximum: $(0, 2), (2, 2)$

Relative minimum: $(1, 0)$

Intercepts: $(0, 2), (1, 0), (-1.280, 0), (3.280, 0)$

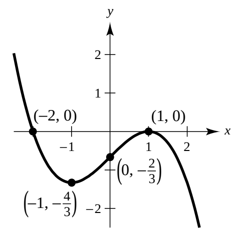


**26.** $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$

$y'' = -2x = 0$ when $x = 0$

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-\frac{4}{3}$ | 0 | $+$ | Relative minimum |
| $-1 < x < 0$ | | $+$ | $+$ | Increasing, concave up |
| $x = 0$ | $-\frac{2}{3}$ | $+$ | 0 | Point of inflection |
| $0 < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | 0 | 0 | $-$ | Relative maximum |
| $1 < x < \infty$ | | $-$ | $-$ | Decreasing, concave down |

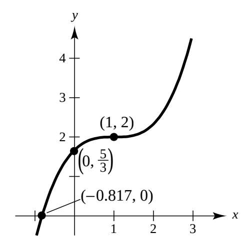



**28.** $f(x) = \frac{1}{3}(x - 1)^3 + 2$

$f'(x) = (x - 1)^2 = 0$ when $x = 1$.

$f''(x) = 2(x - 1) = 0$ when $x = 1$.

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | 2 | 0 | 0 | Point of inflection |
| $1 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |




**30.** $f(x) = (x + 1)(x - 2)(x - 5)$

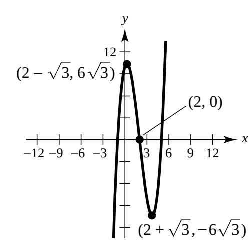
$f'(x) = (x + 1)(x - 2) + (x + 1)(x - 5) + (x - 2)(x - 5)$

$= 3(x^2 - 4x + 1) = 0$ when $x = 2 \pm \sqrt{3}$.

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 2 - \sqrt{3}$ | | $+$ | $-$ | Increasing, concave down |
| $x = 2 - \sqrt{3}$ | $6\sqrt{3}$ | 0 | $-$ | Relative maximum |
| $2 - \sqrt{3} < x < 2$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 2$ | 0 | $-$ | 0 | Point of inflection |
| $2 < x < 2 + \sqrt{3}$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 2 + \sqrt{3}$ | $-6\sqrt{3}$ | 0 | $+$ | Relative minimum |
| $2 + \sqrt{3} < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

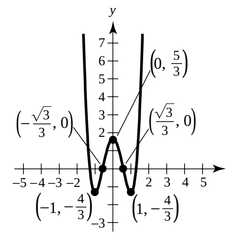
Intercepts: $(0, 10), (-1, 0), (2, 0), (5, 0)$



**32.** $y = 3x^4 - 6x^2 + \frac{5}{3}$

$y' = 12x^3 - 12x = 12x(x^2 - 1) = 0$ when $x = 0, x = \pm 1$.

$y'' = 36x^2 - 12 = 12(3x^2 - 1) = 0$ when $x = \pm\frac{\sqrt{3}}{3}$.

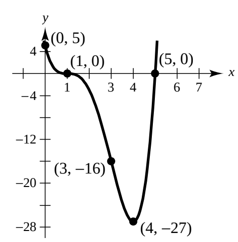



| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < -1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = -1$ | $-4/3$ | 0 | $+$ | Relative minimum |
| $-1 < x < -\frac{\sqrt{3}}{3}$ | | $+$ | $+$ | Increasing, concave up |
| $x = -\frac{\sqrt{3}}{3}$ | 0 | $+$ | 0 | Point of inflection |
| $-\frac{\sqrt{3}}{3} < x < 0$ | | $+$ | $-$ | Increasing, concave down |
| $x = 0$ | $5/3$ | 0 | $-$ | Relative maximum |
| $0 < x < \frac{\sqrt{3}}{3}$ | | $-$ | $-$ | Decreasing, concave down |
| $x = \frac{\sqrt{3}}{3}$ | 0 | $-$ | 0 | Point of inflection |
| $\frac{\sqrt{3}}{3} < x < 1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 1$ | $-4/3$ | 0 | $+$ | Relative minimum |
| $1 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**34.** $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

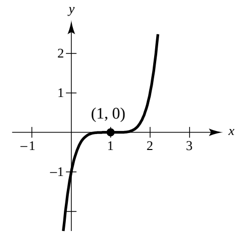
$f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2 = 0$ when $x = 1, x = 4$.

$f''(x) = 12x^2 - 48x + 36 = 12(x - 3)(x - 1) = 0$ when $x = 3, x = 1$.




| | $f(x)$ | $f'(x)$ | $f''(x)$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 1$ | 0 | 0 | 0 | Point of inflection |
| $1 < x < 3$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 3$ | $-16$ | $-$ | 0 | Point of inflection |
| $3 < x < 4$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 4$ | $-27$ | 0 | $+$ | Relative minimum |
| $4 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**36.** $y = (x - 1)^5$

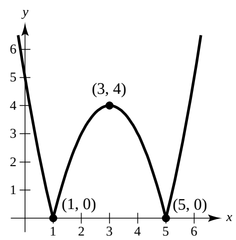



$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 1$ | | $+$ | $-$ | Increasing, concave down |
| $x = 1$ | 0 | 0 | 0 | Point of inflection |
| $1 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**38.** $y = |x^2 - 6x + 5|$



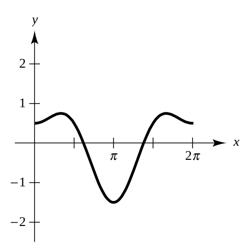

$$y' = \frac{2(x-3)(x^2-6x+5)}{|x^2-6x+5|} = \frac{2(x-3)(x-5)(x-1)}{|(x-5)(x-1)|}$$

$= 0$ when $x = 3$ and undefined when $x = 1$, $x = 5$.

$$y'' = \frac{2(x^2-6x+5)}{|x^2-6x+5|} = \frac{2(x-5)(x-1)}{|(x-5)(x-1)|}$$ undefined when $x = 1$, $x = 5$.

| | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $-\infty < x < 1$ | | $-$ | $+$ | Decreasing, concave up |
| $x = 1$ | 0 | Undefined | Undefined | Relative minimum, point of inflection |
| $1 < x < 3$ | | $+$ | $-$ | Increasing, concave down |
| $x = 3$ | 4 | 0 | $-$ | Relative maximum |
| $3 < x < 5$ | | $-$ | $-$ | Decreasing, concave down |
| $x = 5$ | 0 | Undefined | Undefined | Relative minimum, point of inflection |
| $5 < x < \infty$ | | $+$ | $+$ | Increasing, concave up |

**40.** $y = \cos x - \frac{1}{2}\cos 2x,\ 0 \le x \le 2\pi$



$y' = -\sin x + \sin 2x = -\sin x(1 - 2\cos x) = 0$ when $x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$.

$y'' = -\cos x + 2\cos 2x = -\cos x + 2(2\cos^2 x - 1)$

$= 4\cos^2 x - \cos x - 2 = 0$ when $\cos x = \frac{1 \pm \sqrt{33}}{8} \approx 0.8431, -0.5931$.

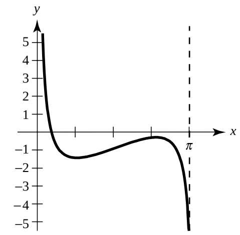
Therefore, $x \approx 0.5678$ or $5.7154$, $x \approx 2.2057$ or $4.0775$.

Relative maxima: $\left(\frac{\pi}{3}, \frac{3}{4}\right), \left(\frac{5\pi}{3}, \frac{3}{4}\right)$

Relative minimum: $\left(\pi, -\frac{3}{2}\right)$

Inflection points: $(0.5678, 0.6323)$, $(2.2057, -0.4449)$, $(5.7154, 0.6323)$, $(4.0775, -0.4449)$

**42.** $y = 2(x - 2) + \cot x, \ 0 < x < \pi$



$y' = 2 - \csc^2 x = 0$ when $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

$y'' = 2\csc^2 x \cot x = 0$ when $x = \dfrac{\pi}{2}$

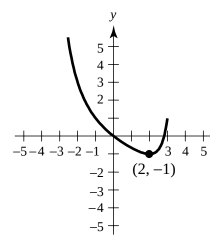
Relative maximum: $\left(\dfrac{3\pi}{4}, \dfrac{3\pi}{2} - 5\right)$

Relative minimum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 3\right)$

Point of inflection: $\left(\dfrac{\pi}{2}, \pi - 4\right)$

Vertical asymptotes: $x = 0, \pi$

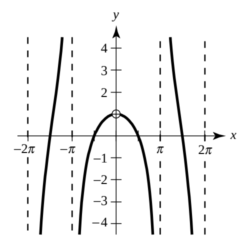
**44.** $y = \sec^2\left(\dfrac{\pi x}{8}\right) - 2\tan\left(\dfrac{\pi x}{8}\right) - 1, \ -3 < x < 3$




$y' = 2\sec^2\left(\dfrac{\pi x}{8}\right)\tan\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) - 2\sec^2\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) = 0 \implies x = 2$

Relative minimum: $(2, -1)$

**46.** $g(x) = x \cot x, \ -2\pi < x < 2\pi$



$g'(x) = \dfrac{\sin x \cos x - x}{\sin^2 x}$

$g'(0)$ does not exist. But $\lim\limits_{x \to 0} x \cot x = \lim\limits_{x \to 0} \dfrac{x}{\tan x} = 1.$

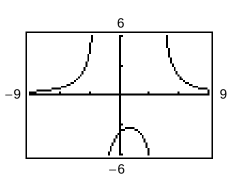
Vertical asymptotes: $x = \pm 2\pi, \pm\pi$

Intercepts: $\left(-\dfrac{3\pi}{2}, 0\right), \left(-\dfrac{\pi}{2}, 0\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{2}, 0\right)$

Symmetric with respect to $y$-axis.

Decreasing on $(0, \pi)$ and $(\pi, 2\pi)$

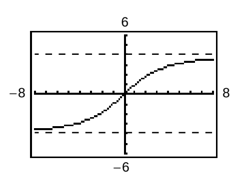
**48.** $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$



$x = -2, 4$ vertical asymptote

$y = 0$ horizontal asymptote

**50.** $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$



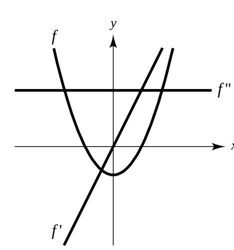
$y = \pm 4$ horizontal asymptotes

$(0, 0)$ point of inflection

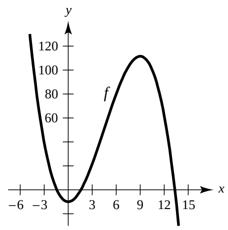
**52.** $f''$ is constant.

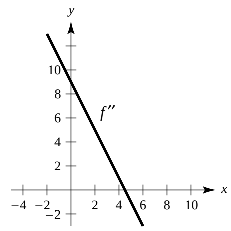
$f'$ is linear.

$f$ is quadratic.



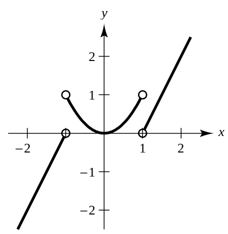
**54.**

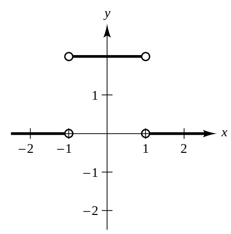



(any vertical translate of $f$ will do)

**56.**



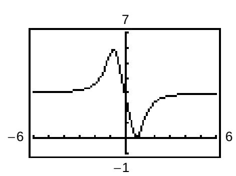

(any vertical translate of the 3 segments of $f$ will do)

**58.** If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

**60.** $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none

Horizontal asymptote: $y = 3$

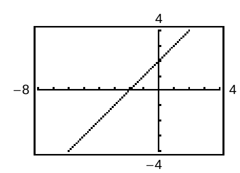



The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

**62.** $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

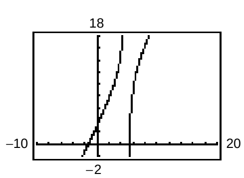
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.




hole at $(1, 3)$

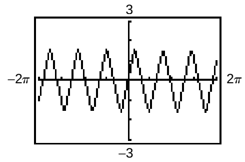
**64.** $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$




The graph appears to approach the slant asymptote $y = 2x + 2$.

**66.** $f(x) = \tan(\sin \pi x)$

(a)




(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$= -\tan(\sin \pi x) = -f(x)$

Symmetry with repect to the origin

(c) Periodic with period 2

(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of $f$ is concave downward.

**68.** Vertical asymptote: $x = -3$

Horizontal asymptote: none

$$y = \frac{x^2}{x + 3}$$

**70.** Vertical asymptote: $x = 0$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x} = \frac{1 - x^2}{x}$$

**72.** $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2),\ a \neq 0$

$f'(x) = a^2x - a = a(ax - 1) = 0$ when $x = \frac{1}{a}$.

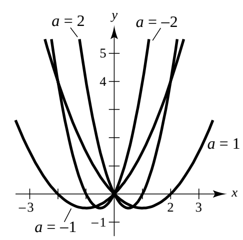
$f''(x) = a^2 > 0$ for all $x$.

(a) Intercepts: $(0, 0), \left(\frac{2}{a}, 0\right)$

Relative minimum: $\left(\frac{1}{a}, -\frac{1}{2}\right)$

Points of inflection: none

(b)




**74.** Tangent line at $P$: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x$-intercept: $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

$y$-intercept: $(0, f(x_0) - x_0 f'(x_0))$

(c) Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0) f'(x_0)$$

$x$-intercept: $(x_0 + f(x_0) f'(x_0), 0)$

(d) Let $x = 0$: $y - y_0 = \frac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

$y$-intercept: $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \frac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\frac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = {y_0}^2 + \left(\frac{f(x_0)}{f'(x_0)}\right) = \frac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC|^2 = \left|\frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$$

(g) $|AB| = |x_0 - (x_0 + f(x_0) f'(x_0))| = |f(x_0) f'(x_0)|$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + {y_0}^2$

$$|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

# Section 3.7 Optimization Problems

**2.** Let $x$ and $y$ be two positive numbers such that $x + y = S$.

$$P = xy = x(S - x) = Sx - x^2$$

$$\frac{dP}{dx} = S - 2x = 0 \text{ when } x = \frac{S}{2}.$$

$$\frac{d^2P}{dx^2} = -2 < 0 \text{ when } x = \frac{S}{2}.$$

$P$ is a maximum when $x = y = S/2$.

**4.** Let $x$ and $y$ be two positive numbers such that $xy = 192$.

$$S = x + 3y = \frac{192}{y} + 3y$$

$$\frac{dS}{dy} = 3 - \frac{192}{y^2} = 0 \text{ when } y = 8.$$

$$\frac{d^2S}{dy^2} = \frac{384}{y^3} > 0 \text{ when } y = 8.$$

$S$ is minimum when $y = 8$ and $x = 24$.

**6.** Let $x$ and $y$ be two positive numbers such that $x + 2y = 100$.

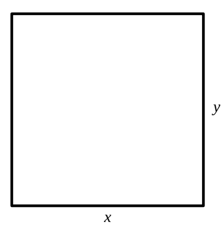
$$P = xy = y(100 - 2y) = 100y - 2y^2$$

$$\frac{dP}{dy} = 100 - 4y = 0 \text{ when } y = 25.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 25.$$

$P$ is a maximum when $x = 50$ and $y = 25$.

**8.** Let $x$ be the length and $y$ the width of the rectangle.




$$2x + 2y = P$$

$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

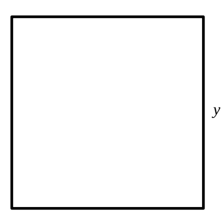
$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

$A$ is maximum when $x = y = P/4$ units. (A square!)

**10.** Let $x$ be the length and $y$ the width of the rectangle.




$$xy = A$$

$$y = \frac{A}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

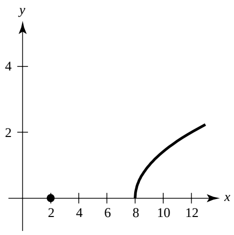
$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

$P$ is minimum when $x = y = \sqrt{A}$ centimeters.

(A square!)

**12.** $f(x) = \sqrt{x - 8}$, $(2, 0)$

From the graph, it is clear that $(8, 0)$ is the closest point on the graph of $f$ to $(2, 0)$.




**14.** $f(x) = (x + 1)^2$, $(5, 3)$

$$\begin{aligned} d &= \sqrt{(x - 5)^2 + [(x + 1)^2 - 3]^2} \\ &= \sqrt{x^2 - 10x + 25 + (x^2 + 2x - 2)^2} \\ &= \sqrt{x^2 - 10x + 25 + x^4 + 4x^3 - 8x + 4} \\ &= \sqrt{x^4 + 4x^3 + x^2 - 18x + 29} \end{aligned}$$

Since $d$ is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$\begin{aligned} g(x) &= x^4 + 4x^3 + x^2 - 18x + 29 \\ g'(x) &= 4x^3 + 12x^2 + 2x - 18 \\ &= 2(x - 1)(2x^2 + 8x + 9) \end{aligned}$$

By the First Derivative Test, $x = 1$ yields a minimum. Hence, $(1, 4)$ is closest to $(5, 3)$.

**16.** $F = \dfrac{v}{22 + 0.02v^2}$

$$\begin{aligned} \frac{dF}{dv} &= \frac{22 - 0.02v^2}{(22 + 0.02v^2)^2} \\ &= 0 \text{ when } v = \sqrt{1100} \approx 33.166. \end{aligned}$$

By the First Derivative Test, the flow rate on the road is maximized when $v \approx 33$ mph.

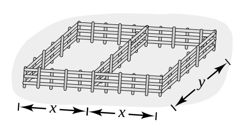
**18.** $4x + 3y = 200$ is the perimeter. (see figure)

$$A = 2xy = 2x\left(\frac{200 - 4x}{3}\right) = \frac{8}{3}(50x - x^2)$$

$$\frac{dA}{dx} = \frac{8}{3}(50 - 2x) = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 25.$$

$A$ is a maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.




**20.** (a)

| Height, $x$ | Length & Width | Volume |
|---|---|---|
| 1 | $24 - 2(1)$ | $1[24 - 2(1)]^2 = 484$ |
| 2 | $24 - 2(2)$ | $2[24 - 2(2)]^2 = 800$ |
| 3 | $24 - 2(3)$ | $3[24 - 2(3)]^2 = 972$ |
| 4 | $24 - 2(4)$ | $4[24 - 2(4)]^2 = 1024$ |
| 5 | $24 - 2(5)$ | $5[24 - 2(5)]^2 = 980$ |
| 6 | $24 - 2(6)$ | $6[24 - 2(6)]^2 = 864$ |

(b) $V = x(24 - 2x)^2$, $0 < x < 12$

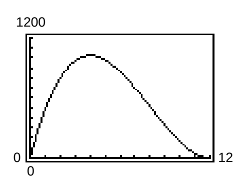
(c) $$\begin{aligned} \frac{dV}{dx} &= 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x) \\ &= 12(12 - x)(4 - x) = 0 \text{ when } x = 12, 4 \text{ (12 is not in the domain).} \end{aligned}$$

$$\frac{d^2V}{dx^2} = 12(2x - 16)$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = 4.$$

When $x = 4$, $V = 1024$ is maximum.

(d)



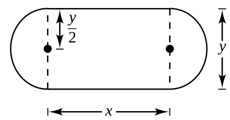

The maximum volume seems to be 1024.

**22.** (a) $P = 2x + 2\pi r$

$= 2x + 2\pi\left(\dfrac{y}{2}\right)$

$= 2x + \pi y = 200$

$\Rightarrow y = \dfrac{200 - 2x}{\pi} = \dfrac{2}{\pi}(100 - x)$



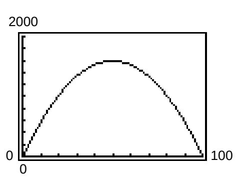

(b)

| Length, $x$ | Width, $y$ | Area, $xy$ |
|---|---|---|
| 10 | $\frac{2}{\pi}(100 - 10)$ | $(10)\frac{2}{\pi}(100 - 10) \approx 573$ |
| 20 | $\frac{2}{\pi}(100 - 20)$ | $(20)\frac{2}{\pi}(100 - 20) \approx 1019$ |
| 30 | $\frac{2}{\pi}(100 - 30)$ | $(30)\frac{2}{\pi}(100 - 30) \approx 1337$ |
| 40 | $\frac{2}{\pi}(100 - 40)$ | $(40)\frac{2}{\pi}(100 - 40) \approx 1528$ |
| 50 | $\frac{2}{\pi}(100 - 50)$ | $(50)\frac{2}{\pi}(100 - 50) \approx 1592$ |
| 60 | $\frac{2}{\pi}(100 - 60)$ | $(60)\frac{2}{\pi}(100 - 60) = 1528$ |

The maximum area of the rectangle is approximately 1592 $m^2$.

(c) $A = xy = x\dfrac{2}{\pi}(100 - x) = \dfrac{2}{\pi}(100x - x^2)$

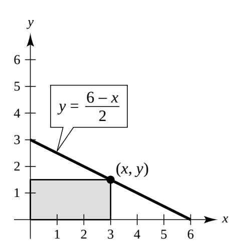
(e)




Maximum area is approximately

1591.55 $m^2$ ($x = 50$ m).

**24.** You can see from the figure that $A = xy$ and $y = \dfrac{6 - x}{2}$.



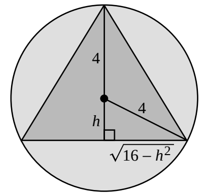

$A = x\left(\dfrac{6 - x}{2}\right) = \dfrac{1}{2}(6x - x^2).$

$\dfrac{dA}{dx} = \dfrac{1}{2}(6 - 2x) = 0$ when $x = 3$.

$\dfrac{d^2A}{dx^2} = -1 < 0$ when $x = 3$.

$A$ is a maximum when $x = 3$ and $y = 3/2$.

**26.** (a) $A = \frac{1}{2}\text{base} \times \text{height}$




$$= \frac{1}{2}\left(2\sqrt{16-h^2}\right)(4+h)$$

$$= \sqrt{16-h^2}(4+h)$$

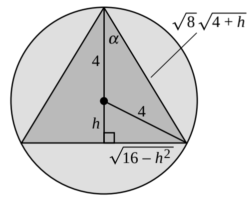
$$\frac{dA}{dh} = \frac{1}{2}(16-h^2)^{-1/2}(-2h)(4+h) + (16-h^2)^{1/2}$$

$$= (16-h^2)^{-1/2}\left[-h(4+h) + (16-h^2)\right]$$

$$= \frac{-2(h^2+2h-8)}{\sqrt{16-h^2}} = \frac{-2(h+4)(h-2)}{\sqrt{16-h^2}}$$

$\frac{dA}{dh} = 0$ when $h = 2$, which is a maximum by the First Derivative Test.
Hence, the sides are $2\sqrt{16-h^2} = 4\sqrt{3}$, an equilateral triangle. Area $= 12\sqrt{3}$ sq. units.

(b) $\cos\alpha = \frac{4+h}{\sqrt{8}\sqrt{4+h}} = \frac{\sqrt{4+h}}{\sqrt{8}}$




$$\tan\alpha = \frac{\sqrt{16-h^2}}{4+h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{16-h^2}\right)(4+h)$$

$$= (4+h)^2\tan\alpha$$

$$= 64\cos^4\alpha\tan\alpha$$

$$A'(\alpha) = 64\left[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$$

$$\Longrightarrow \cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$$

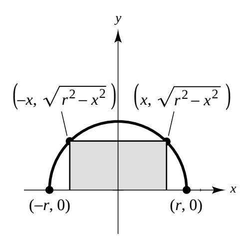
$$1 = 4\cos\alpha\sin\alpha\tan\alpha$$

$$\frac{1}{4} = \sin^2\alpha$$

$$\sin\alpha = \frac{1}{2} \Longrightarrow \alpha = 30^\circ \text{ and } A = 12\sqrt{3}.$$

(c) Equilateral triangle

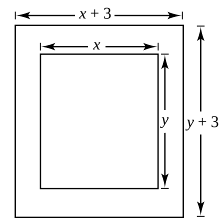
**28.** $A = 2xy = 2x\sqrt{r^2-x^2}$ (see figure)




$$\frac{dA}{dx} = \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, $A$ is maximum when the rectangle has dimensions $\sqrt{2}r$ by $\left(\sqrt{2}r\right)/2$.

**30.** $xy = 36 \implies y = \dfrac{36}{x}$




$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \implies 3x^2 = 108 \implies x = 6, y = 6$$

Dimensions: $9 \times 9$

**32.** $V = \pi r^2 h = V_0$ cubic units or $h = \dfrac{V_0}{\pi r^2}$

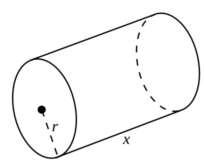
$$S = 2\pi r^2 + 2\pi rh = 2\left(\pi r^2 + \frac{V_0}{r}\right)$$

$$\frac{dS}{dr} = 2\left(2\pi r - \frac{V_0}{r^2}\right) = 0 \text{ when } r = \sqrt[3]{\frac{V_0}{2\pi}} \text{ units.}$$

$$h = \frac{V_0}{\pi\left(\sqrt[3]{V_0/2\pi}\right)^2} = \frac{V_0(2\pi)^{2/3}}{\pi V_0^{2/3}} = \frac{2V_0^{1/3}}{(2\pi)^{1/3}} = 2r$$

By the First Derivative Test, this will yield the minimum surface area.

**34.** $V = \pi r^2 x$




$$x + 2\pi r = 108 \implies x = 108 - 2\pi r \quad \text{(see figure)}$$

$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

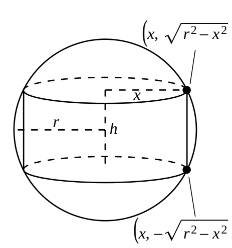
$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.

**36.** $V = \pi x^2 h = \pi x^2\left(2\sqrt{r^2 - x^2}\right) = 2\pi x^2\sqrt{r^2 - x^2}$ (see figure)




$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}(2r^2 - 3x^2)$$

$$= 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \implies x = \frac{\sqrt{6}r}{3}.$$

By the First Derivative Test, the volume is a maximum when

$$x = \frac{\sqrt{6}r}{3} \text{ and } h = \frac{2r}{\sqrt{3}}.$$

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$

**38.** No. The volume will change because the shape of the container changes when squeezed.

**40.** $V = 3000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let $k$ = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi rh) = k\left[8\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{6000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{6000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}} \approx 5.636 \text{ feet and } h \approx 22.545 \text{ feet.}$$

By the Second Derivative Test, we have

$$\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}}.$$

Therefore, these dimensions will produce a minimum cost.

**42.** (a) Let $x$ be the side of the triangle and $y$ the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2,\ 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$, when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(b) Let $x$ be the side of the square and $y$ the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2,\ 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let $x$ be the side of the pentagon and $y$ the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}\left(\sqrt{3}\right)\left(\frac{20 - 5x}{6}\right)^2,\ 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(d) Let $x$ be the side of the hexagon and $r$ the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2,\ 0 \le x \le \frac{10}{3}.$$

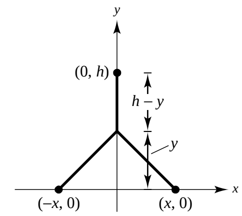
$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

**44.** Let $A$ be the amount of the power line.




$$A = h - y + 2\sqrt{x^2 + y^2}$$

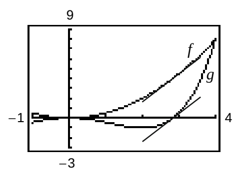
$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.

**46.** $f(x) = \frac{1}{2}x^2 \qquad g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on $[0, 4]$

(a)




(b) $d(x) = f(x) - g(x) = \frac{1}{2}x^2 - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) = x^2 - \frac{1}{16}x^4$

$$d'(x) = 2x - \frac{1}{4}x^3 = 0 \Longrightarrow 8x = x^3$$

$$\Longrightarrow x = 0, 2\sqrt{2} \text{ (in } [0, 4])$$

The maximum distance is $d = 4$ when $x = 2\sqrt{2}$.

(c) $f'(x) = x$, Tangent line at $\left(2\sqrt{2}, 4\right)$ is

$$y - 4 = 2\sqrt{2}\left(x - 2\sqrt{2}\right)$$

$$y = 2\sqrt{2}x - 4.$$

$g'(x) = \frac{1}{4}x^3 - x$, Tangent line at $\left(2\sqrt{2}, 0\right)$ is

$$y - 0 = \left(\tfrac{1}{4}\left(2\sqrt{2}\right)^3 - 2\sqrt{2}\right)\left(x - 2\sqrt{2}\right)$$

$$y = 2\sqrt{2}x - 8.$$

The tangent lines are parallel and 4 vertical units apart.

(d) The tangent lines will be parallel. If $d(x) = f(x) - g(x)$, then $d'(x) = 0 = f'(x) - g'(x)$ implies that $f'(x) = g'(x)$ at the point $x$ where the distance is maximum.

**48.** Let $F$ be the illumination at point $P$ which is $x$ units from source 1.

$$F = \frac{kI_1}{x^2} + \frac{kI_2}{(d - x)^2}$$

$$\frac{dF}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(d - x)^3} = 0 \text{ when } \frac{2kI_1}{x^3} = \frac{2kI_2}{(d - x)^3}.$$

$$\frac{\sqrt[3]{I_1}}{\sqrt[3]{I_2}} = \frac{x}{d - x}$$

$$(d - x)\sqrt[3]{I_1} = x\sqrt[3]{I_2}$$

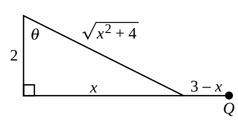
$$d\sqrt[3]{I_1} = x\left(\sqrt[3]{I_1} + \sqrt[3]{I_2}\right)$$

$$x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2F}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(d - x)^4} > 0 \text{ when } x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}.$$

This is the minimum point.

**50.** (a) $T = \dfrac{\sqrt{x^2 + 4}}{2} + \dfrac{(3 - x)}{4}$




(b)

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} - \frac{1}{4} = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{2}$$

$$2x^2 = x^2 + 4$$

$$x^2 = 4$$

$$x = 2$$

$$T(2) = \sqrt{2} + \frac{1}{4} \text{ hours}$$

**—CONTINUED—**

**50. —CONTINUED—**

(c) $$T = \frac{\sqrt{x^2+4}}{v_1} + \frac{(3-x)}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+4}} - \frac{1}{v_2} = 0$$

$$\frac{x}{\sqrt{x^2+4}} = \frac{v_1}{v_2}$$

$$\sin\theta = \frac{v_1}{v_2}$$

$\theta$ depends on $\dfrac{v_1}{v_2}$ only.

(d) $$\text{Cost} = \sqrt{x^2+4}\,C_1 + (3-x)C_2 = \frac{\sqrt{x^2+4}}{(1/C_1)} + \frac{(3-x)}{(1/C_2)}$$

From above, $\sin\theta = \dfrac{1/C_1}{1/C_2} = \dfrac{C_2}{C_1}$

**52.** $$T = \frac{\sqrt{x^2+d_1^2}}{v_1} + \frac{\sqrt{d_2^2+(a-x)^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+d_1^2}} + \frac{x-a}{v_2\sqrt{d_2^2+(a-x)^2}} = 0$$

Since

$$\frac{x}{\sqrt{x^2+d_1^2}} = \sin\theta_1 \text{ and } \frac{x-a}{\sqrt{d_2^2+(a-x)^2}} = -\sin\theta_2$$

we have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Longrightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2+d_1^2)^{3/2}} + \frac{d_2^2}{v_2[d_2^2+(a-x)^2]^{3/2}} > 0$$

this condition yields a minimum time.

**54.** $$C(x) = 2k\sqrt{x^2+4} + k(4-x)$$

$$C'(x) = \frac{2xk}{\sqrt{x^2+4}} - k = 0$$

$$2x = \sqrt{x^2+4}$$

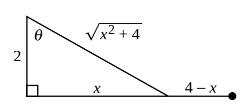
$$4x^2 = x^2+4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$

Or, use Exercise 50(d): $\sin\theta = \dfrac{C_2}{C_1} = \dfrac{1}{2} \implies \theta = 30°$.

Thus, $x = \dfrac{2}{\sqrt{3}}$.




**56.** $$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144-r^2}$$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)(144-r^2)^{-1/2}(-2r) + 2r\sqrt{144-r^2}\right]$$

$$= \frac{1}{3}\pi\left[\frac{288r-3r^3}{\sqrt{144-r^2}}\right] = \pi\left[\frac{r(96-r^2)}{\sqrt{144-r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, $V$ is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone: $S = \pi(4\sqrt{6})\sqrt{(4\sqrt{6})^2 + (4\sqrt{3})^2} = 48\sqrt{6}\pi$

Area of sector: $144\pi - 48\sqrt{6}\pi = \dfrac{1}{2}\theta r^2 = 72\theta$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}(3-\sqrt{6}) \approx 1.153 \text{ radians or } 66°$$

**58.** Let $d$ be the amount deposited in the bank, $i$ be the interest rate paid by the bank, and $P$ be the profit.

$P = (0.12)d - id$

$d = ki^2$ (since $d$ is proportional to $i^2$)

$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$

$\dfrac{dP}{di} = k(0.24i - 3i^2) = 0$ when $i = \dfrac{0.24}{3} = 0.08$.

$\dfrac{d^2P}{di^2} = k(0.24 - 6i) < 0$ when $i = 0.08$ (**Note:** $k > 0$).

The profit is a maximum when $i = 8\%$.

**60.** $P = -\dfrac{1}{10}s^3 + 6s^2 + 400$

(a) $\dfrac{dP}{ds} = -\dfrac{3}{10}s^2 + 12s = -\dfrac{3}{10}s(s - 40) = 0$ when $x = 0, s = 40$.

$\dfrac{d^2P}{ds^2} = -\dfrac{3}{5}s + 12$

$\dfrac{d^2P}{ds^2}(0) > 0 \Longrightarrow s = 0$ yields a minimum.

$\dfrac{d^2P}{ds^2}(40) < 0 \Longrightarrow s = 40$ yields a maximum.

The maximum profit occurs when $s = 40$, which corresponds to \$40,000 ($P = \$3{,}600{,}000$).

(b) $\dfrac{d^2P}{ds^2} = -\dfrac{3}{5}s + 12 = 0$ when $s = 20$.

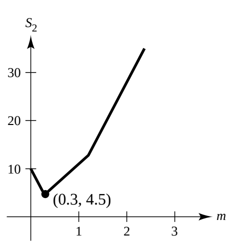
The point of diminishing returns occurs when $s = 20$, which corresonds to \$20,000 being spent on advertising.

**62.** $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

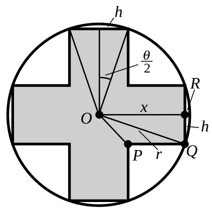
Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7$ mi.



**64.** (a) Label the figure so that $r^2 = x^2 + h^2$.
Then, the area $A$ is 8 times the area of the region given by $OPQR$:




$$A = 8\left[\frac{1}{2}h^2 + (x - h)h\right]$$
$$= 8\left[\frac{1}{2}(r^2 - x^2) + \left(x - \sqrt{r^2 - x^2}\right)\sqrt{r^2 - x^2}\right]$$
$$= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$
$$A'(x) = 8\sqrt{r^2 - x^2} - \frac{8x^2}{\sqrt{r^2 - x^2}} + 8x = 0$$
$$\frac{8x^2}{\sqrt{r^2 - x^2}} = 8x + 8\sqrt{r^2 - x^2}$$
$$x^2 = x\sqrt{r^2 - x^2} + (r^2 - x^2)$$
$$2x^2 - r^2 = x\sqrt{r^2 - x^2}$$
$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2 - x^2)$$
$$5x^4 - 5x^2r^2 + r^4 = 0 \quad \text{Quadratic in } x^2.$$
$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10}\left[5 \pm \sqrt{5}\right].$$

Take positive value.

$$x = r\sqrt{\frac{5 + \sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$

(b) Note that $\sin\dfrac{\theta}{2} = \dfrac{h}{r}$ and $\cos\dfrac{\theta}{2} = \dfrac{x}{r}$. The area $A$ of the cross equals the sum of two large rectangles minus the common square in the middle.

$$A = 2(2x)(2h) - 4h^2 = 8xh - 4h^2$$
$$= 8r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} - 4r^2\sin^2\frac{\theta}{2}$$
$$= 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$
$$A'(\theta) = 4r^2\left(\cos\theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0$$
$$\cos\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\theta$$
$$\tan\theta = 2$$
$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4°$$

(c) Note that $x^2 = \dfrac{r^2}{10}\left(5 + \sqrt{5}\right)$ and $r^2 - x^2 = \dfrac{r^2}{10}\left(5 - \sqrt{5}\right)$.

$$A(x) = 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$
$$= 8\left[\frac{r^2}{10}\left(5 + \sqrt{5}\right)\frac{r^2}{10}\left(5 - \sqrt{5}\right)\right]^{1/2} + 4\frac{r^2}{10}\left(5 + \sqrt{5}\right) - 4r^2$$
$$= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2$$
$$= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2$$
$$= 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2\left(\sqrt{5} - 1\right)$$

Using the angle approach, note that $\tan\theta = 2$, $\sin\theta = \dfrac{2}{\sqrt{5}}$ and $\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1}{2}(1 - \cos\theta) = \dfrac{1}{2}\left(1 - \dfrac{1}{\sqrt{5}}\right)$.

$$\text{Thus, } A(\theta) = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$
$$= 4r^2\left(\frac{2}{\sqrt{5}} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right)$$
$$= \frac{4r^2\left(\sqrt{5} - 1\right)}{2} = 2r^2\left(\sqrt{5} - 1\right)$$

# Section 3.8 Newton's Method

**2.** $f(x) = 2x^2 - 3$

$f'(x) = 4x$

$x_1 = 1$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1 | $-1$ | 4 | $-\frac{1}{4}$ | $\frac{5}{4}$ |
| 2 | $\frac{5}{4} = 1.25$ | 0.125 | 5.0 | 0.025 | 1.225 |

**4.** $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.1000 | 0.1003 | 1.0101 | 0.0993 | 0.0007 |
| 2 | 0.0007 | 0.0007 | 1.0000 | 0.0007 | 0.0000 |

**6.** $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

Approximation of the zero of $f$ is 0.755.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.5000 | $-0.4688$ | 1.3125 | $-0.3571$ | 0.8571 |
| 2 | 0.8571 | 0.3196 | 3.6983 | 0.0864 | 0.7707 |
| 3 | 0.7707 | 0.0426 | 2.7641 | 0.0154 | 0.7553 |
| 4 | 0.7553 | 0.0011 | 2.6272 | 0.0004 | 0.7549 |

**8.** $f(x) = x - 2\sqrt{x + 1}$

$f'(x) = 1 - \dfrac{1}{\sqrt{x + 1}}$

Approximation of the zero of $f$ is 4.8284.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 5 | 0.1010 | 0.5918 | 0.1707 | 4.8293 |
| 2 | 4.8293 | 0.0005 | 0.5858 | .00085 | 4.8284 |

**10.** $f(x) = 1 - 2x^3$

$f'(x) = -6x^2$

Approximation of the zero of $f$ is 0.7937.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1 | $-1$ | $-6$ | 0.1667 | 0.8333 |
| 2 | 0.8333 | $-0.1573$ | $-4.1663$ | 0.0378 | 0.7955 |
| 3 | 0.7955 | $-0.0068$ | $-3.7969$ | 0.0018 | 0.7937 |
| 4 | 0.7937 | 0.0000 | $-3.7798$ | 0.0000 | 0.7937 |

**12.** $f(x) = \frac{1}{2}x^4 - 3x - 3$

$f'(x) = 2x^3 - 3$

Approximation of the zero of $f$ is $-0.8937$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $-1$ | 0.5 | $-5$ | $-0.1$ | $-0.9$ |
| 2 | $-0.9$ | 0.0281 | $-4.458$ | $-0.0063$ | $-0.8937$ |
| 3 | $-0.8937$ | 0.0001 | $-4.4276$ | 0.0000 | $-0.8937$ |

Approximation of the zero of $f$ is 2.0720.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2 | $-1$ | 13 | $-0.0769$ | 2.0769 |
| 2 | 2.0769 | 0.0725 | 14.9175 | 0.0049 | 2.0720 |
| 3 | 2.0720 | $-0.0003$ | 14.7910 | 0.0000 | 2.0720 |

**14.** $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

Approximation of the zero of $f$ is 0.866.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.9000 | 0.1074 | 3.2133 | 0.0334 | 0.8666 |
| 2 | 0.8666 | 0.0034 | 3.0151 | 0.0011 | 0.8655 |
| 3 | 0.8655 | 0.0001 | 3.0087 | 0.0000 | 0.8655 |

**16.** $h(x) = f(x) - g(x) = 3 - x - \dfrac{1}{x^2 + 1}$

$h'(x) = -1 + \dfrac{2x}{(x^2 + 1)^2}$

Point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 2.893$.

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\frac{h(x_n)}{h'(x_n)}$ | $x_n - \frac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.9000 | $-0.0063$ | $-0.9345$ | 0.0067 | 2.8933 |
| 2 | 2.8933 | 0.0000 | $-0.9341$ | 0.0000 | 2.8933 |

**18.** $h(x) = f(x) - g(x) = x^2 - \cos x$

$h'(x) = 2x + \sin x$

One point of intersection of the graphs of $f$ and $g$ occurs when $x \approx 0.824$. Since $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the $y$-axis, the other point of intersection occurs when $x \approx -0.824$.

| $n$ | $x_n$ | $h(x_n)$ | $h'(x_n)$ | $\frac{h(x_n)}{h'(x_n)}$ | $x_n - \frac{h(x_n)}{h'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.8000 | $-0.0567$ | 2.3174 | $-0.0245$ | 0.8245 |
| 2 | 0.8245 | 0.0009 | 2.3832 | 0.0004 | 0.8241 |

**20.** $f(x) = x^n - a = 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{x_i^n - a}{nx_i^{n-1}}$$

$$= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

**22.** $x_{i+1} = \dfrac{x_i^2 + 5}{2x_i}$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 2.0000 | 2.2500 | 2.2361 | 2.2361 |

$\sqrt{5} \approx 2.236$

**24.** $x_{i+1} = \dfrac{2x_i^3 + 15}{3x_i^2}$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 2.5000 | 2.4667 | 2.4662 | 2.4662 |

$\sqrt[3]{15} \approx 2.466$

**26.** $f(x) = \tan x$

$f'(x) = \sec^2 x$

Approximation of the zero: 3.142

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 3.0000 | $-0.1425$ | 1.0203 | $-0.1397$ | 3.1397 |
| 2 | 3.1397 | $-0.0019$ | 1.0000 | $-0.0019$ | 3.1416 |
| 3 | 3.1416 | 0.0000 | 1.0000 | 0.0000 | 3.1416 |

**28.** $y = 4x^3 - 12x^2 + 12x - 3 = f(x)$

$y' = 12x^2 - 24x + 12 = f'(x)$

$x_1 = \dfrac{3}{2}$

$f'(x_2) = 0$; therefore, the method fails.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $\frac{3}{2}$ | $\frac{3}{2}$ | 3 | $\frac{1}{2}$ | 1 |
| 2 | 1 | 1 | 0 | — | — |

**30.** $f(x) = 2\sin x + \cos 2x$

$f'(x) = 2\cos x - 2\sin 2x$

$x_1 = \dfrac{3\pi}{2}$

Fails because $f'(x_1) = 0$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|---|
| 1 | $\frac{3\pi}{2}$ | $-3$ | 0 |

**32.** Newton's Method could fail if $f'(c) \approx 0$, or if the initial value $x_1$ is far from $c$.

**34.** Let $g(x) = f(x) - x = \cot x - x$

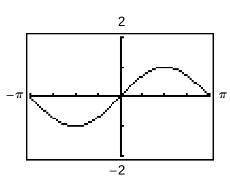
$g'(x) = -\csc^2 x - 1.$

The fixed point is approximately 0.86.

| $n$ | $x_n$ | $g(x_n)$ | $g'(x_n)$ | $\dfrac{g(x_n)}{g'(x_n)}$ | $x_n - \dfrac{g(x_n)}{g'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | $-0.3579$ | $-2.4123$ | 0.1484 | 0.8516 |
| 2 | 0.8516 | 0.0240 | $-2.7668$ | $-0.0087$ | 0.8603 |
| 3 | 0.8603 | 0.0001 | $-2.7403$ | 0.0000 | 0.8603 |

**36.** $f(x) = \sin x$, $f'(x) = \cos x$

(a)



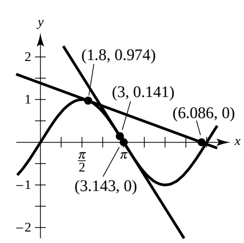

(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)




The $x$-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess $x_1$ is not "close to" the desired zero of the function, the $x$-intercept of the tangent line may approximate another zero of the function.

**38.** (a) $x_{n+1} = x_n(2 - 3x_n)$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 0.3000 | 0.3300 | 0.3333 | 0.3333 |

$\frac{1}{3} \approx 0.333$

(b) $x_{n+1} = x_n(2 - 11x_n)$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 0.1000 | 0.0900 | 0.0909 | 0.0909 |

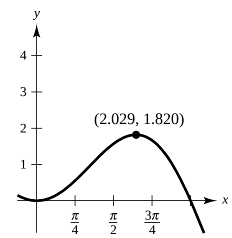
$\frac{1}{11} \approx 0.091$

**40.** $f(x) = x \sin x,\ (0, \pi)$

$f'(x) = x \cos x + \sin x = 0$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$[F'(x) = 2 \cos x - x \sin x]$




| $n$ | $x_n$ | $F(x_n)$ | $F'(x_n)$ | $\dfrac{F(x_n)}{F'(x_n)}$ | $x_n - \dfrac{F(x_n)}{F'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 2.0000 | 0.0770 | $-2.6509$ | $-0.0290$ | 2.0290 |
| 2 | 2.0290 | $-0.0007$ | $-2.7044$ | 0.0002 | 2.0288 |

Approximation to the critical number: 2.029

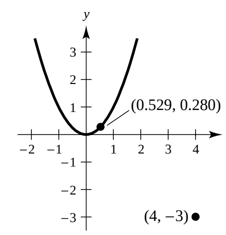
**42.** $y = f(x) = x^2,\ (4, -3)$

$d = \sqrt{(x-4)^2 + (y+3)^2} = \sqrt{(x-4)^2 + (x^2+3)^2} = \sqrt{x^4 + 7x^2 - 8x + 25}$

$d$ is minimum when $D = x^4 + 7x^2 - 8x + 25$ is minimum.

$g(x) = D' = 4x^3 + 14x - 8$

$g'(x) = 12x^2 + 14$




| $n$ | $x_n$ | $g(x_n)$ | $g'(x_n)$ | $\dfrac{g(x_n)}{g'(x_n)}$ | $x_n - \dfrac{g(x_n)}{g'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.5000 | $-0.5000$ | 17.0000 | $-0.0294$ | 0.5294 |
| 2 | 0.5294 | 0.0051 | 17.3632 | 0.0003 | 0.5291 |
| 3 | 0.5291 | $-0.0001$ | 17.3594 | 0.0000 | 0.5291 |

$x \approx 0.529$

Point closest to $(4, -3)$ is approximately $(0.529, 0.280)$.

**44.** Maximize: $C = \dfrac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300.$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 4.5000 | 12.4375 | 915.0000 | 0.0136 | 4.4864 |
| 2 | 4.4864 | 0.0658 | 904.3822 | 0.0001 | 4.4863 |

Since $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

**46.** $170 = 0.808x^3 - 17.974x^2 + 71.248x + 110.843,\ 1 \le x \le 5$

Let $f(x) = 0.808x^3 - 17.974x^2 + 71.248x - 59.157$

$f'(x) = 2.424x^2 - 35.948x + 71.248.$

From the graph, choose $x_1 = 1$ and $x_1 = 3.5$. Apply Newton's Method.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | −5.0750 | 37.7240 | −0.1345 | 1.1345 |
| 2 | 1.1345 | −0.2805 | 33.5849 | −0.0084 | 1.1429 |
| 3 | 1.1429 | 0.0006 | 33.3293 | 0.0000 | 1.1429 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 3.5000 | 4.6725 | −24.8760 | −0.1878 | 3.6878 |
| 2 | 3.6878 | −0.3286 | −28.3550 | 0.0116 | 3.6762 |
| 3 | 3.6762 | −0.0009 | −28.1450 | 0.0000 | 3.6762 |

The zeros occur when $x \approx 1.1429$ and $x \approx 3.6762$. These approximately correspond to engine speeds of 1143 rev/min and 3676 rev/min.

**48.** True

**50.** True

**52.** $f(x) = \sqrt{4 - x^2}\sin(x - 2)$

Domain: $[-2, 2]$

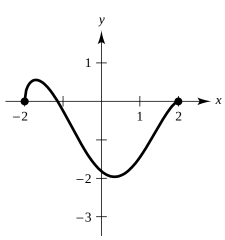
$x = -2$ and $x = 2$ are both zeros.

$f'(x) = \sqrt{4 - x^2}\cos(x - 2) - \dfrac{x}{\sqrt{4 - x^2}}\sin(x - 2)$

Let $x_1 = -1$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | −1.0000 | −0.2444 | −1.7962 | 0.1361 | −1.1361 |
| 2 | −1.1361 | −0.0090 | −1.6498 | 0.0055 | −1.1416 |
| 3 | −1.1416 | 0.0000 | −1.6422 | 0.0000 | −1.1416 |

Zeros: $x = \pm 2,\ x \approx -1.142$



## Section 3.9 Differentials

**2.** $f(x) = \dfrac{6}{x^2} = 6x^{-2}$

$f'(x) = -12x^{-3} = \dfrac{-12}{x^3}$

Tangent line at $\left(2, \dfrac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \dfrac{6}{x^2}$ | 1.6620 | 1.5151 | 1.5 | 1.4851 | 1.3605 |
| $T(x) = -\dfrac{3}{2}x + \dfrac{9}{2}$ | 1.65 | 1.515 | 1.5 | 1.485 | 1.35 |

**4.** $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $\left(2, \sqrt{2}\right)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$

$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \sqrt{x}$ | 1.3784 | 1.4107 | 1.4142 | 1.4177 | 1.4491 |
| $T(x) = \dfrac{x}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}$ | 1.3789 | 1.4107 | 1.4142 | 1.4177 | 1.4496 |

**6.** $f(x) = \csc x$

$f'(x) = -\csc x \cot x$

Tangent line at $(2, \csc 2)$: $y - f(2) = f'(2)(x - 2)$

$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$

$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

| $x$ | 1.9 | 1.99 | 2 | 2.01 | 2.1 |
|---|---|---|---|---|---|
| $f(x) = \csc x$ | 1.0567 | 1.0948 | 1.0998 | 1.1049 | 1.1585 |
| $T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$ | 1.0494 | 1.0947 | 1.0998 | 1.1048 | 1.1501 |

**8.** $y = f(x) = 1 - 2x^2, f'(x) = -4x, x = 0, \Delta x = dx = -0.1$

$$\Delta y = f(x + \Delta x) - f(x) = f(-0.1) - f(0) = [1 - 2(-0.1)^2] - [1 - 2(0)^2] = -0.02$$

$$dy = f'(x)\,dx = f'(0)(-0.1) = (0)(-0.1) = 0$$

**10.** $y = f(x) = 2x + 1, f'(x) = 2, x = 2, \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x) = f(2.01) - f(2) = [2(2.01) + 1] - [2(2) + 1] = 0.02$$

$$dy = f'(x)\,dx = f'(2)(0.01) = 2(0.01) = 0.02$$

**12.** $y = 3x^{2/3}$

$$dy = 2x^{-1/3}dx = \frac{2}{x^{1/3}}dx$$

**14.** $y = \sqrt{9 - x^2}$

$$dy = \frac{1}{2}(9 - x^2)^{-1/2}(-2x)dx = \frac{-x}{\sqrt{9 - x^2}}\,dx$$

**16.** $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$$dy = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)dx = \frac{x - 1}{2x\sqrt{x}}\,dx$$

**18.** $y = x \sin x$

$$dy = (x \cos x + \sin x)\,dx$$

**20.** $y = \dfrac{\sec^2 x}{x^2 + 1}$

$$dy = \left[\frac{(x^2 + 1)2\sec^2 x \tan x - \sec^2 x(2x)}{(x^2 + 1)^2}\right]dx$$

$$= \left[\frac{2\sec^2 x(x^2 \tan x + \tan x - x)}{(x^2 + 1)^2}\right]dx$$

**22.** (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + (-1)(-0.1) = 1.1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + (-1)(0.04) = 0.96$$

**24.** (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + 0(-0.1) = 1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + 0(0.04) = 1$$

**26.** (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + (3)(-0.07) = 7.79$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + (3)(0.1) = 8.3$$

**28.** (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + 5(-0.07) = 7.65$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + 5(0.1) = 8.5$$

**30.** $A = \frac{1}{2}bh,\ b = 36,\ h = 50$

$db = dh = \pm 0.25$

$dA = \frac{1}{2}b\,dh + \frac{1}{2}h\,db$

$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$

$= \pm 10.75$ square centimeters

**32.** $x = 12$ inches

$\Delta x = dx = \pm 0.03$ inch

(a) $V = x^3$

$dV = 3x^2\,dx = 3(12)^2(\pm 0.03)$

$= \pm 12.96$ cubic inches

(b) $S = 6x^2$

$dS = 12x\,dx = 12(12)(\pm 0.03)$

$= \pm 4.32$ square inches

**34.** (a) $C = 56$ centimeters

$\Delta C = dC = \pm 1.2$ centimeters

$$C = 2\pi r \Longrightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{1}{4\pi}C^2$$

$$dA = \frac{1}{2\pi}C\,dC = \frac{1}{2\pi}(56)(\pm 1.2) = \frac{33.6}{\pi}$$

$$\frac{dA}{A} = \frac{33.6/\pi}{[1/(4\pi)](56)^2} \approx 0.042857 = 4.2857\%$$

(b) $$\frac{dA}{A} = \frac{(1/2\pi)C\,dC}{(1/4\pi)C^2} = \frac{2dC}{C} \le 0.03$$

$$\frac{dC}{C} \le \frac{0.03}{2} = 0.015 = 1.5\%$$

**36.** $P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$, $x$ changes from 115 to 120

$dP = (500 - 2x - x + 77)dx = (577 - 3x)\,dx = [577 - 3(115)](120 - 115) = 1160$

Approximate percentage change: $\dfrac{dP}{P}(100) = \dfrac{1160}{43517.50}(100) \approx 2.7\%$

**38.** $V = \frac{4}{3}\pi r^3$, $r = 100$ cm, $dr = 0.2$ cm

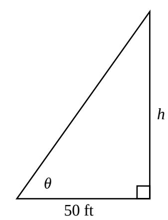
$\Delta V \approx dV = 4\pi r^2\,dr = 4\pi(100)^2(0.2) = 8000\pi \text{ cm}^3$

**40.**
$$E = IR$$
$$R = \frac{E}{I}$$
$$dR = -\frac{E}{I^2}dI$$
$$\frac{dR}{R} = \frac{-(E/I^2)dI}{E/I} = -\frac{dI}{I}$$
$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

**42.** See Exercise 41.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(9.5\cot\theta)(9.5) = 45.125\cot\theta$$
$$dA = -45.125\csc^2\theta\,d\theta$$
$$\left|\frac{dA}{A}\right| = \frac{\csc^2\theta\,d\theta}{\cot\theta} = \frac{d\theta}{\sin\theta\cos\theta}$$
$$= \frac{0.25^\circ}{(\sin 26.75^\circ)(\cos 26.75^\circ)}$$
$$\approx \frac{0.0044}{(\sin 0.4669)(\cos 0.4669)}$$
$$\approx 0.0109 = 1.09\% \text{ (in radians)}$$

**44.** $h = 50\tan\theta$




$\theta = 71.5^\circ = 1.2479$ radians

$$dh = 50\sec^2\theta \cdot d\theta$$
$$\left|\frac{dh}{x}\right| = \left|\frac{50\sec^2(1.2479)}{50\tan(1.2479)}d\theta\right| \le 0.06$$
$$\left|\frac{9.9316}{2.9886}d\theta\right| \le 0.06$$
$$|d\theta| \le 0.018$$

**46.** Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$

$$f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$$
$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

**48.** Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2\,dx$$
$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01) = 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

**50.** Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$
$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

**52.** Propagated error $= f(x + \Delta x) - f(x)$,

relative error $= \left|\dfrac{dy}{y}\right|$, and the percent error $= \left|\dfrac{dy}{y}\right| \times 100$.

**54.** True, $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = a$

**56.** False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

$$dy = f'(x)\,dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

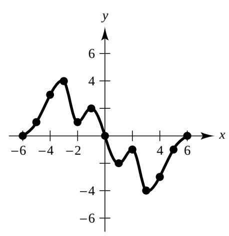
Thus, $dy > \Delta y$ in this example.

## Review Exercises for Chapter 3

**2.** (a) $f(4) = -f(-4) = -3$

(b) $f(-3) = -f(3) = -(-4) = 4$

(c)




At least six critical numbers on $(-6, 6)$.

(d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value $c$ in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

(e) No, $\lim_{x \to 0} f(x)$ exists because $f$ is continuous at $(0, 0)$.

(f) Yes, $f$ is differentiable at $x = 2$.

**4.** $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

$$f'(x) = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}$$
$$= \frac{1}{(x^2 + 1)^{3/2}}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $\left(2, 2/\sqrt{5}\right)$ Maximum

**6.** No. $f$ is not differentiable at $x = 2$.

**8.** No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

**10.**
$$f(x) = \frac{1}{x},\ 1 \le x \le 4$$
$$f'(x) = -\frac{1}{x^2}$$
$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$
$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$
$$c = 2$$

**12.**
$$f(x) = \sqrt{x} - 2x,\ 0 \le x \le 4$$
$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$
$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$
$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$
$$c = 1$$

**14.** $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$\dfrac{f(b) - f(a)}{b - a} = \dfrac{21 - 1}{4 - 0} = 5$

$f'(c) = 4c - 3 = 5$

$c = 2 = \text{Midpoint of } [0, 4]$

**16.** $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

| Interval | $-\infty < x < -1$ | $-1 < x < \infty$ |
|---|---|---|
| Sign of $g'(x)$ | $g'(x) > 0$ | $g'(x) > 0$ |
| Conclusion | Increasing | Increasing |

**18.** $f(x) = \sin x + \cos x, \quad 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \dfrac{\pi}{4}, x = \dfrac{5\pi}{4}$

| Interval | $0 < x < \dfrac{\pi}{4}$ | $\dfrac{\pi}{4} < x < \dfrac{5\pi}{4}$ | $\dfrac{5\pi}{4} < x < 2\pi$ |
|---|---|---|---|
| Sign of $f'(x)$ | $f'(x) > 0$ | $f'(x) < 0$ | $f'(x) > 0$ |
| Conclusion | Increasing | Decreasing | Increasing |

**20.** $g(x) = \dfrac{3}{2}\sin\left(\dfrac{\pi x}{2} - 1\right), \ [0, 4]$

$g'(x) = \dfrac{3}{2}\left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{\pi x}{2} - 1\right)$

$= 0 \text{ when } x = 1 + \dfrac{2}{\pi}, 3 + \dfrac{2}{\pi}$

Relative maximum: $\left(1 + \dfrac{2}{\pi}, \dfrac{3}{2}\right)$

Relative minimum: $\left(3 + \dfrac{2}{\pi}, -\dfrac{3}{2}\right)$

| Test Interval | $0 < x < 1 + \dfrac{2}{\pi}$ | $1 + \dfrac{2}{\pi} < x < 3 + \dfrac{2}{\pi}$ | $3 + \dfrac{2}{\pi} < x < 4$ |
|---|---|---|---|
| Sign of $g'(x)$ | $g'(x) > 0$ | $g'(x) < 0$ | $g'(x) > 0$ |
| Conclusion | Increasing | Decreasing | Increasing |

**22.** (a) $y = A\sin\left(\sqrt{k/m}\,t\right) + B\cos\left(\sqrt{k/m}\,t\right)$

$y' = A\sqrt{k/m}\cos\left(\sqrt{k/m}\,t\right) - B\sqrt{k/m}\sin\left(\sqrt{k/m}\,t\right)$

$= 0 \text{ when } \dfrac{\sin\sqrt{k/m}\,t}{\cos\sqrt{k/m}\,t} = \dfrac{A}{B} \Longrightarrow \tan\left(\sqrt{k/m}\,t\right) = \dfrac{A}{B}.$

Therefore,

$\sin\left(\sqrt{k/m}\,t\right) = \dfrac{A}{\sqrt{A^2 + B^2}}$

$\cos\left(\sqrt{k/m}\,t\right) = \dfrac{B}{\sqrt{A^2 + B^2}}.$

When $v = y' = 0$,

$y = A\left(\dfrac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\dfrac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$

(b) Period: $\dfrac{2\pi}{\sqrt{k/m}}$

Frequency: $\dfrac{1}{2\pi/\sqrt{k/m}} = \dfrac{1}{2\pi}\sqrt{k/m}$

**24.** $f(x) = (x+2)^2(x-4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x = 0$ when $x = 0$.

Point of inflection: $(0, -16)$

| Test Interval | $-\infty < x < 0$ | $0 < x < \infty$ |
|---|---|---|
| Sign of $f''(x)$ | $f''(x) < 0$ | $f''(x) > 0$ |
| Conclusion | Concave downward | Concave upward |

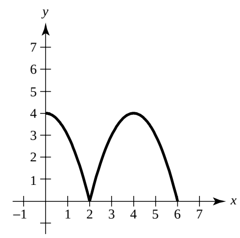
**26.** $h(t) = t - 4\sqrt{t+1}$ Domain: $[-1, \infty)$

$$h'(t) = 1 - \frac{2}{\sqrt{t+1}} = 0 \implies t = 3$$

$$h''(t) = \frac{1}{(t+1)^{3/2}}$$

$h''(3) = \frac{1}{8} > 0$ $(3, -5)$ is a relative minimum.

**28.**




**30.** $C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$

$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$

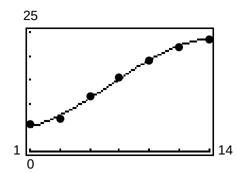
$$\frac{Qs}{x^2} = \frac{r}{2}$$

$$x^2 = \frac{2Qs}{r}$$

$$x = \sqrt{\frac{2Qs}{r}}$$

**32.** (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$

(b)




(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the $t^3$ coefficient term is negative.

**34.** $\lim_{x\to\infty} \frac{2x}{3x^2+5} = \lim_{x\to\infty} \frac{2/x}{3 + 5/x^2} = 0$

**36.** $\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{3}{\sqrt{1 + 4/x^2}} = 3$

**38.** $g(x) = \frac{5x^2}{x^2+2}$

$$\lim_{x\to\infty} \frac{5x^2}{x^2+2} = \lim_{x\to\infty} \frac{5}{1 + (2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

**40.** $f(x) = \frac{3x}{\sqrt{x^2+2}}$

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{x\to\infty} \frac{3x/x}{\sqrt{x^2+2}/\sqrt{x^2}} = \lim_{x\to\infty} \frac{3}{\sqrt{1 + (2/x^2)}} = 3$$

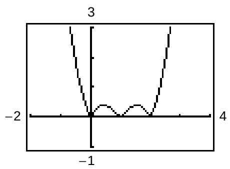
$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{x\to-\infty} \frac{3x/x}{\sqrt{x^2+2}/\left(-\sqrt{x^2}\right)} = \lim_{x\to-\infty} \frac{3}{-\sqrt{1 + (2/x^2)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

**42.** $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0), (1, 0), (2, 0)$

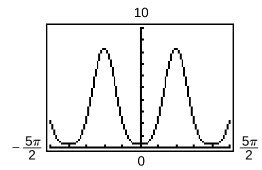
Relative maxima: $(1.577, 0.38), (0.423, 0.38)$




**44.** $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

Relative minima: $(2\pi k, 0.29)$ where $k$ is any integer.

Relative maxima: $((2k-1)\pi, 8.29)$ where $k$ is any integer.




**46.** $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0$ when $x = 0, 3$.

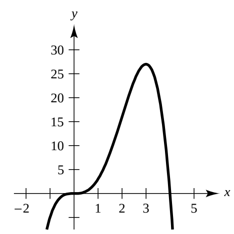
$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0$ when $x = 0, 2$.

$f''(3) < 0$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0), (2, 16)$

Intercepts: $(0, 0), (4, 0)$




**48.** $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = 4x(x^2 - 4) = 0$ when $x = 0, \pm 2$.

$f''(x) = 4(3x^2 - 4) = 0$ when $x = \pm\dfrac{2\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 16)$ is a relative maximum.

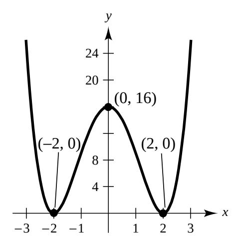
$f''(\pm 2) > 0$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $\left(\pm 2\sqrt{3}/3, 64/9\right)$

Intercepts: $(-2, 0), (0, 16), (2, 0)$

Symmetry with respect to $y$-axis




**50.** $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $\left[-\frac{16,875}{256}, \infty\right)$

$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$

$= (4x - 7)(x + 2)^2 = 0$ when $x = -2, \frac{7}{4}$.

$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$

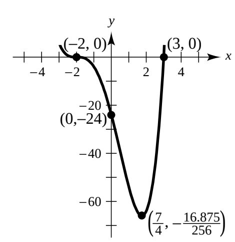
$= 6(2x - 1)(x + 2) = 0$ when $x = -2, \frac{1}{2}$.

$f''\left(\frac{7}{4}\right) > 0$

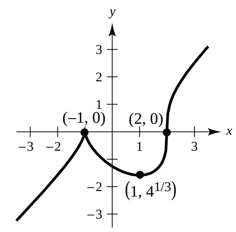
Therefore, $\left(\frac{7}{4}, -\frac{16,875}{256}\right)$ is a relative minimum.

Points of inflection: $(-2, 0), \left(\frac{1}{2}, -\frac{625}{16}\right)$

Intercepts: $(-2, 0), (0, -24), (3, 0)$



**52.** $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$




Graph of Exercise 39 translated 2 units to the right ($x$ replaces by $x - 2$).

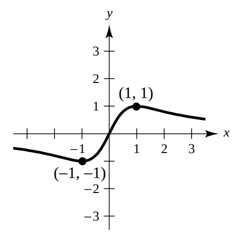
$(-1, 0)$ is a relative maximum.

$\left(1, -\sqrt[3]{4}\right)$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$

**54.** $f(x) = \dfrac{2x}{1 + x^2}$




Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$$f'(x) = \frac{2(1 - x)(1 + x)}{(1 + x^2)^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = \frac{-2x(3 - x^2)}{(1 + x^2)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

$f''(1) < 0$

Therefore, $(1, 1)$ is a relative maximum.

$f''(-1) > 0$

Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $\left(-\sqrt{3}, -\sqrt{3}/2\right), (0, 0), \left(\sqrt{3}, \sqrt{3}/2\right)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$

**56.** $f(x) = \dfrac{x^2}{1 + x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \dfrac{1}{2}\right]$

$$f'(x) = \frac{(1 + x^4)(2x) - x^2(4x^3)}{(1 + x^4)^2} = \frac{2x(1 - x)(1 + x)(1 + x^2)}{(1 + x^4)^2} = 0 \text{ when } x = 0, \pm 1.$$

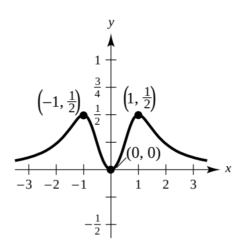
$$f''(x) = \frac{(1 + x^4)^2(2 - 10x^4) - (2x - 2x^5)(2)(1 + x^4)(4x^3)}{(1 + x^4)^4} = \frac{2(1 - 12x^4 + 3x^8)}{(1 + x^4)^3} = 0 \text{ when } x = \pm\sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}.$$

$f''(\pm 1) < 0$

Therefore, $\left(\pm 1, \dfrac{1}{2}\right)$ are relative maxima.

$f''(0) > 0$

Therefore, $(0, 0)$ is a relative minimum.




Points of inflection: $\left(\pm\sqrt[4]{\dfrac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm\sqrt[4]{\dfrac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the $y$-axis

Horizontal asymptote: $y = 0$

**58.** $f(x) = x^2 + \dfrac{1}{x} = \dfrac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = 2x - \dfrac{1}{x^2} = \dfrac{2x^3 - 1}{x^2} = 0$ when $x = \dfrac{1}{\sqrt[3]{2}}$.

$f''(x) = 2 + \dfrac{2}{x^3} = \dfrac{2(x^3 + 1)}{x^3} = 0$ when $x = -1$.

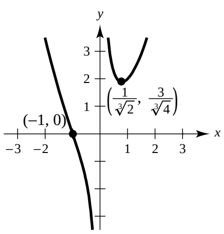
$f''\left(\dfrac{1}{\sqrt[3]{2}}\right) > 0$

Therefore, $\left(\dfrac{1}{\sqrt[3]{2}}, \dfrac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

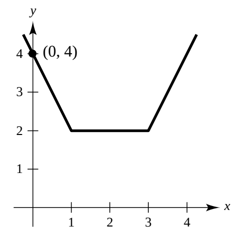



**60.** $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \le 1 \\ 2, & 1 < x \le 3 \\ 2x - 4, & x > 3 \end{cases}$

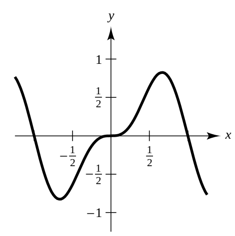
Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$




**62.** $f(x) = \dfrac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$



Domain: $[-1, 1]$; Range: $\left[\dfrac{-3\sqrt{3}}{2\pi}, \dfrac{3\sqrt{3}}{2\pi}\right]$

$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$

Critical Numbers: $x = \pm\dfrac{2}{3}, 0$

$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0$ when $x = 0, \pm 1, \pm 0.420$.

By the First Derivative Test: $\left(-\dfrac{2}{3}, \dfrac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\dfrac{2}{3}, \dfrac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462), (0.420, 0.462), (\pm 1, 0), (0, 0)$

Intercepts: $(-1, 0), (0, 0), (1, 0)$

Symmetric with respect to the origin

**64.** $f(x) = x^n$, $n$ is a positive integer.

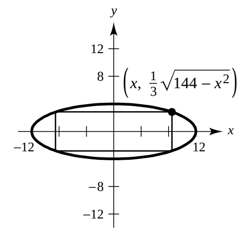
(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when $n$ is even.

(b) $f''(x) = n(n - 1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when $n$ is odd and $n \ge 3$.

**66.** Ellipse: $\dfrac{x^2}{144} + \dfrac{y^2}{16} = 1,\ y = \dfrac{1}{3}\sqrt{144 - x^2}$



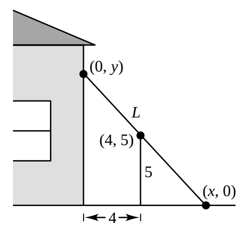

$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\frac{dA}{dx} = \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right]$$

$$= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.$$

The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \dfrac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

**68.** We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,




$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

Let $f(x) = L^2 = x^2 + \left(\dfrac{5x}{x - 4}\right)^2$

$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

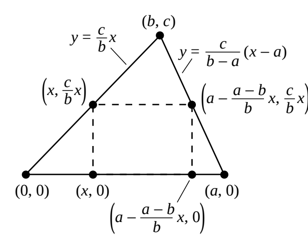
$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4}\sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}}\sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$

**70.** Label triangle with vertices $(0, 0)$, $(a, 0)$, and $(b, c)$. The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The $y$-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the $x$-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b - a}(\bar{x} - a)$$




$$(b - a)x = b(\bar{x} - a)$$

$$\bar{x} = \frac{b - a}{b}x + a = a - \frac{a - b}{b}x.$$

Finally, the lower right vertex is

$$\left(a - \frac{a - b}{b}x, 0\right).$$

Width of rectangle: $a - \dfrac{a - b}{b}x - x$

Height of rectangle: $\dfrac{c}{b}x$ (see figure)

$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a - b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

**72.** You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point $(a, b)$ on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y-b}{0-a} = \frac{b-0}{a-x} \Longrightarrow y = \frac{-bx}{a-x}.$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{-bx}{a-x}\right)^2$.

$$f'(x) = 2x + 2\left(\frac{-bx}{a-x}\right)\left[\frac{-ab}{(a-x)^2}\right]$$

$$\frac{2x[(a-x)^3 + ab^2]}{(a-x)^3} = 0 \text{ when } x = 0, a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$\begin{aligned} L &= \sqrt{\left(a + \sqrt[3]{ab^2}\right)^2 + \left(b + \sqrt[3]{a^2b}\right)^2} \\ &= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2} \\ &= (a^{2/3} + b^{2/3})^{3/2} \text{ meters} \end{aligned}$$

**74.** Using Exercise 73 as a guide we have $L_1 = a \csc\theta$ and $L_2 = b\sec\theta$. Then $dL/d\theta = -a\csc\theta\cot\theta + b\sec\theta\tan\theta = 0$ when

$$\tan\theta = \sqrt[3]{a/b},\ \sec\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}},\ \csc\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a\csc\theta + b\sec\theta = a\frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b\frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

**76.** Total cost $=$ (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\begin{aligned} \frac{dT}{dv} &= \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41{,}250}{50v^2} \\ &= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.} \end{aligned}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

**78.** $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$f'(x) = 3x^2 + 2$

$f$ changes sign in $[-1, 0]$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $-0.5000$ | $-0.1250$ | 2.7500 | $-0.0455$ | $-0.4545$ |
| 2 | $-0.4545$ | $-0.0029$ | 2.6197 | $-0.0011$ | $-0.4534$ |

On the interval $[-1, 0]$: $x \approx -0.453$.

**80.** Find the zeros of $f(x) = \sin \pi x + x - 1$.

$f'(x) = \pi \cos \pi x + 1$

From the graph you can see that $f(x)$ has three real zeros.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.2000 | −0.2122 | 3.5416 | −0.0599 | 0.2599 |
| 2 | 0.2599 | −0.0113 | 3.1513 | −0.0036 | 0.2635 |
| 3 | 0.2635 | 0.0000 | 3.1253 | 0.0000 | 0.2635 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.0000 | 0.0000 | −2.1416 | 0.0000 | 1.0000 |

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.8000 | 0.2122 | 3.5416 | 0.0599 | 1.7401 |
| 2 | 1.7401 | 0.0113 | 3.1513 | 0.0036 | 1.7365 |
| 3 | 1.7365 | 0.0000 | 3.1253 | 0.0000 | 1.7365 |

The three real zeros of $f(x)$ are $x \approx 0.264$, $x = 1$, and $x \approx 1.737$.

**82.** $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}}dx$$

**84.** $p = 75 - \frac{1}{4}x$

$$\Delta p = p(8) - p(7)$$

$$= \left(75 - \frac{8}{4}\right) - \left(75 - \frac{7}{4}\right) = -\frac{1}{4}$$

$$dp = -\frac{1}{4}dx = -\frac{1}{4}(1) = -\frac{1}{4}$$

[$\Delta p = dp$ because $p$ is linear]

## Problem Solving for Chapter 3

**2.** (a)

$$dV = 3x^2 dx = 3x^2\Delta x$$

$$\Delta V = (x + \Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\Delta V - dV = 3x(\Delta x)^2 + (\Delta x)^3 = [\underbrace{3x\Delta x + (\Delta x)^2}_{\varepsilon}]\Delta x$$

$$= \varepsilon\Delta x, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0.$$

(b) Let $\varepsilon = \frac{\Delta y}{\Delta x} - f'(x)$. Then $\varepsilon \to 0$ as $\Delta x \to 0$.

Furthermore, $\Delta y - dy = \Delta y - f'(x)dx = \varepsilon\Delta x$.

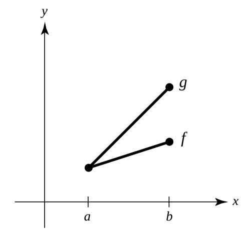
**4.** Let $h(x) = g(x) - f(x)$, which is continuous on $[a, b]$ and differentiable on $(a, b)$. $h(a) = 0$ and $h(b) = g(b) - f(b)$.

By the Mean Value Theorem, there exists $c$ in $(a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{g(b) - f(b)}{b - a}.$$

Since $h'(c) = g'(c) - f'(c) > 0$ and $b - a > 0$,

$$g(b) - f(b) > 0 \implies g(b) > f(b).$$



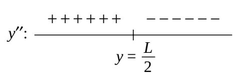

**6.** (a) $f' = 2ax + b, f'' = 2a \neq 0$. No points of inflection.

(b) $f' = 3ax^2 - 2bx + c, f'' = 6ax + 2b = 0 \implies x = \dfrac{-b}{3a}$. One point of inflection.

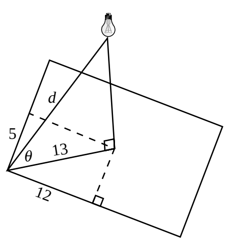
(c) $y' = ky(L - y) = kLy - ky^2$

$y'' = kLy' - 2kyy' = ky'(L - 2y)$

If $y = \dfrac{L}{2}$, then $y'' = 0$ and this is a point of inflection because of the analysis below.




**8.**




$d = \sqrt{13^2 + x^2}$, $\sin\theta = \dfrac{x}{d}$.

Let $A$ be the amount of illumination at one of the corners, as indicated in the figure. Then

$$A = \frac{kI}{(13^2 + x^2)}\sin\theta = \frac{kIx}{(13^2 + x^2)^{3/2}}$$

$$A'(x) = kI\frac{(x^2 + 169)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2 + 169)^{1/2}(2x)}{(169 + x^2)^3} = 0$$

$$\implies (x^2 + 169)^{3/2} = 3x^2(x^2 + 169)^{1/2}$$

$$x^2 + 169 = 3x^2$$

$$2x^2 = 169$$

$$x = \frac{13}{\sqrt{2}} \approx 9.19 \text{ feet}$$

By the First Derivative Test, this is a maximum.

**10.** Let $T$ be the intersection of $PQ$ and $RS$. Let $MN$ be the perpendicular to $SQ$ and $PR$ passing through $T$.

Let $TM = x$ and $TN = b - x$.

$$\frac{SN}{b-x} = \frac{MR}{x} \implies SN = \frac{b-x}{x}MR$$

$$\frac{NQ}{b-x} = \frac{PM}{x} \implies NQ = \frac{b-x}{x}PM$$

$$SQ = \frac{b-x}{x}(MR + PM) = \frac{b-x}{x}d$$

$$A(x) = \text{Area} = \frac{1}{2}dx + \frac{1}{2}\left(\frac{b-x}{x}d\right)(b-x) = \frac{1}{2}d\left[x + \frac{(b-x)^2}{x}\right] = \frac{1}{2}d\left[\frac{2x^2 - 2bx + b^2}{x}\right]$$

$$A'(x) = \frac{1}{2}d\left[\frac{x(4x - 2b) - (2x^2 - 2bx + b^2)}{x^2}\right]$$

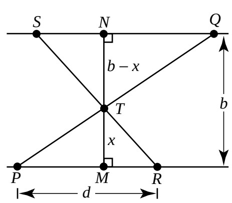
$$A'(x) = 0 \implies 4x^2 - 2xb = 2x^2 - 2bx + b^2$$

$$2x^2 = b^2$$

$$x = \frac{b}{\sqrt{2}}$$

Hence, we have $SQ = \dfrac{b-x}{x}d = \dfrac{b - \left(b/\sqrt{2}\right)}{b/\sqrt{2}}d = \left(\sqrt{2} - 1\right)d.$

Using the Second Derivative Test, this is a minimum. There is no maximum.




**12.** (a) Let $M > 0$ be given. Take $N = \sqrt{M}$. Then whenever $x > N = \sqrt{M}$, you have

$$f(x) = x^2 > M.$$

(b) Let $\varepsilon > 0$ be given. Let $M = \sqrt{\dfrac{1}{\varepsilon}}$. Then whenever $x > M = \sqrt{\dfrac{1}{\varepsilon}}$, you have

$$x^2 > \frac{1}{\varepsilon} \implies \frac{1}{x^2} < \varepsilon \implies \left|\frac{1}{x^2} - 0\right| < \varepsilon.$$

(c) Let $\varepsilon > 0$ be given. There exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.
Let $\delta = \dfrac{1}{N}$. Let $x = \dfrac{1}{y}$.

If $0 < y < \delta = \dfrac{1}{N}$, then $\dfrac{1}{x} < \dfrac{1}{N} \implies x > N$ and

$$|f(x) - L| = \left|f\left(\frac{1}{y}\right) - L\right| < \varepsilon.$$

**14.** Distance $= \sqrt{4^2 + x^2} + \sqrt{(4 - x)^2 + 4^2} = f(x)$

$$f'(x) = \frac{x}{\sqrt{4^2 + x^2}} + \frac{4 - x}{\sqrt{(4 - x)^2 + 4^2}} = 0$$

$$x\sqrt{(4 - x)^2 + 4^2} = (x - 4)\sqrt{4^2 + x^2}$$

$$x^2[16 - 8x + x^2 + 16] = (x^2 - 8x + 16)(16 + x^2)$$

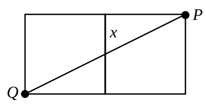
$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 32x^2 - 128x + 256$$

$$128x = 256$$

$$x = 2$$

The bug should head towards the midpoint of the opposite side.

Without Calculus: Imagine opening up the cube:




The shortest distance is the line $PQ$, passing through the midpoint.

**16.** (a) $s = \dfrac{v \dfrac{\text{km}}{\text{hr}}\left(1000 \dfrac{\text{m}}{\text{km}}\right)}{\left(3600 \dfrac{\text{sec}}{\text{hr}}\right)} = \dfrac{5}{18}v$

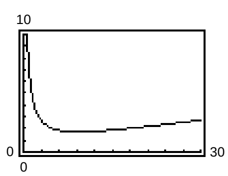
| $v$ | 20 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|
| $s$ | 5.56 | 11.11 | 16.67 | 22.22 | 27.78 |
| $d$ | 5.1 | 13.7 | 27.2 | 44.2 | 66.4 |

$d(t) = 0.071s^2 + 0.389s + 0.727$

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(t)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ more seconds. Hence,

$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

(c)




$$T = \frac{1}{s}(0.071s^2 + 0.389s + 0.727) + \frac{5.5}{s}$$

The minimum is attained when $s \approx 9.365$ m/sec.

(d) $$T(s) = 0.071s + 0.389 + \frac{6.227}{s}$$

$$T'(s) = 0.071 - \frac{6.227}{s^2} \Longrightarrow s^2 = \frac{6.227}{0.071}$$

$$\Longrightarrow s \approx 9.365 \text{ m/sec}$$

$T(9.365) \approx 1.719$ seconds

$$9.365 \text{ m/sec} \cdot \frac{3600}{1000} = 3.37 \text{ km/hr}$$

(e) $d(9.365) = 10.597$ m

**18.** (a)

| $x$ | 0 | 0.5 | 1 | 2 |
|---|---|---|---|---|
| $\sqrt{1 + x}$ | 1 | 1.2247 | 1.4142 | 1.7321 |
| $\frac{1}{2}x + 1$ | 1 | 1.25 | 1.5 | 2 |

(b) Let $f(x) = \sqrt{1 + x}$. Using the Mean Value Theorem on the interval $[0, x]$, there exists $c$, $0 < c < x$, satisfying

$$f'(c) = \frac{1}{2\sqrt{1 + c}} = \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{1 + x} - 1}{x}.$$

Thus $\sqrt{1 + x} = \dfrac{x}{2\sqrt{1 + c}} + 1 < \dfrac{x}{2} + 1$ (because $\sqrt{1 + c} > 1$).

# CHAPTER 4
# Integration

# CHAPTER 4
# Integration

## Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Even-Numbered Exercises

**2.** $\dfrac{d}{dx}\left(x^4 + \dfrac{1}{x} + C\right) = 4x^3 - \dfrac{1}{x^2}$

**4.** $\dfrac{d}{dx}\left(\dfrac{2(x^2+3)}{3\sqrt{x}} + C\right) = \dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2} + 2x^{-1/2} + C\right)$

$= x^{1/2} - x^{-3/2} = \dfrac{x^2 - 1}{x^{3/2}}$

**6.** $\dfrac{dr}{d\theta} = \pi$

$r = \pi\theta + C$

**Check:** $\dfrac{d}{d\theta}[\pi\theta + C] = \pi$

**8.** $\dfrac{dy}{dx} = 2x^{-3}$

$y = \dfrac{2x^{-2}}{-2} + C = \dfrac{-1}{x^2} + c$

**Check:** $\dfrac{d}{dx}\left[\dfrac{-1}{x^2} + C\right] = 2x^{-3}$

| | *Given* | *Rewrite* | *Integrate* | *Simplify* |
|---|---|---|---|---|
| **10.** | $\int \frac{1}{x^2}\,dx$ | $\int x^{-2}\,dx$ | $\frac{x^{-1}}{-1} + C$ | $-\frac{1}{x} + C$ |
| **12.** | $\int x(x^2+3)\,dx$ | $\int (x^3 + 3x)\,dx$ | $\frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C$ | $\frac{1}{4}x^4 + \frac{3}{2}x^2 + C$ |
| **14.** | $\int \frac{1}{(3x^2)}dx$ | $\frac{1}{9}\int x^{-2}dx$ | $\frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C$ | $\frac{-1}{9x} + C$ |

**16.** $\int (5 - x)dx = 5x - \dfrac{x^2}{2} + C$

**Check:** $\dfrac{d}{dx}\left[5x - \dfrac{x^2}{2} + C\right] = 5 - x$

**18.** $\int (4x^3 + 6x^2 - 1)dx = x^4 + 2x^3 - x + C$

**Check:** $\dfrac{d}{dx}[x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$

**20.** $\int (x^3 - 4x + 2)dx = \dfrac{x^4}{4} - 2x^2 + 2x + C$

**Check:** $\dfrac{d}{dx}\left[\dfrac{x^4}{4} - 2x^2 + 2x + C\right] = x^3 - 4x + 2$

**22.** $\int\left(\sqrt{x} + \dfrac{1}{2\sqrt{x}}\right)dx = \int\left(x^{1/2} + \dfrac{1}{2}x^{-1/2}\right)dx = \dfrac{x^{3/2}}{3/2} + \dfrac{1}{2}\left(\dfrac{x^{1/2}}{1/2}\right) + C = \dfrac{2}{3}x^{3/2} + x^{1/2} + C$

**Check:** $\dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \dfrac{1}{2}x^{-1/2} = \sqrt{x} + \dfrac{1}{2\sqrt{x}}$

**24.** $\int\left(\sqrt[4]{x^3}+1\right)dx = \int (x^{3/4}+1)\,dx = \frac{4}{7}x^{7/4}+x+C$

**Check:** $\frac{d}{dx}\left(\frac{4}{7}x^{7/4}+x+C\right) = x^{3/4}+1 = \sqrt[4]{x^3}+1$

**26.** $\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx = \frac{x^{-3}}{-3}+C = -\frac{1}{3x^3}+C$

**Check:** $\frac{d}{dx}\left(-\frac{1}{3x^3}+C\right) = \frac{1}{x^4}$

**28.** $\int \frac{x^2+2x-3}{x^4}dx = \int (x^{-2}+2x^{-3}-3x^{-4})dx$

$$= \frac{x^{-1}}{-1}+\frac{2x^{-2}}{-2}-\frac{3x^{-3}}{-3}+C$$

$$= \frac{-1}{x}-\frac{1}{x^2}+\frac{1}{x^3}+C$$

**Check:** $\frac{d}{dx}\left[-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}+C\right] = x^{-2}+2x^{-3}-3x^{-4}$

$$= \frac{x^2+2x-3}{x^4}$$

**30.** $\int (2t^2-1)^2\,dt = \int (4t^4-4t^2+1)\,dt$

$$= \frac{4}{5}t^5-\frac{4}{3}t^3+t+C$$

**Check:** $\frac{d}{dt}\left(\frac{4}{5}t^5-\frac{4}{3}t^3+t+C\right) = 4t^4-4t^2+1$

$$= (2t^2-1)^2$$

**32.** $\int (1+3t)t^2\,dt = \int (t^2+3t^3)\,dt = \frac{1}{3}t^3+\frac{3}{4}t^4+C$

**Check:** $\frac{d}{dt}\left(\frac{1}{3}t^3+\frac{3}{4}t^4+C\right) = t^2+3t^3 = (1+3t)t^2$

**34.** $\int 3\,dt = 3t+C$

**Check:** $\frac{d}{dt}(3t+C) = 3$

**36.** $\int (t^2-\sin t)\,dt = \frac{1}{3}t^3+\cos t+C$

**Check:** $\frac{d}{dt}\left(\frac{1}{3}t^3+\cos t+C\right) = t^2-\sin t$

**38.** $\int (\theta^2+\sec^2\theta)\,d\theta = \frac{1}{3}\theta^3+\tan\theta+C$

**Check:** $\frac{d}{d\theta}\left(\frac{1}{3}\theta^3+\tan\theta+C\right) = \theta^2+\sec^2\theta$

**40.** $\int \sec y(\tan y-\sec y)\,dy = \int (\sec y\tan y-\sec^2 y)\,dy$

$$= \sec y-\tan y+C$$

**Check:** $\frac{d}{dy}(\sec y-\tan y+C) = \sec y\tan y-\sec^2 y$

$$= \sec y(\tan y-\sec y)$$

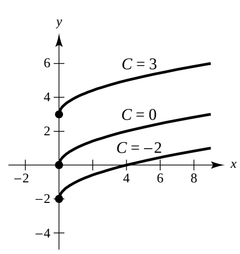
**42.** $\int \frac{\cos x}{1-\cos^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx = \int\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)dx$

$$= \int \csc x\cot x\,dx = -\csc x+C$$

**Check:** $\frac{d}{dx}[-\csc x+C] = \csc x\cot x + \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}$

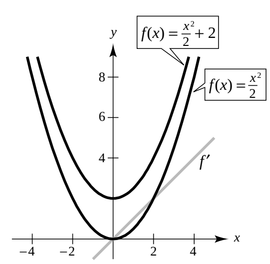
$$= \frac{\cos x}{1-\cos^2 x}$$

**44.** $f(x) = \sqrt{x}$



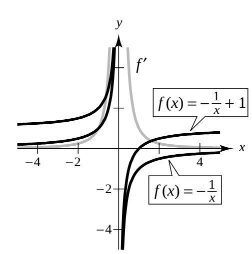

**46.** $f'(x) = x$

$f(x) = \frac{x^2}{2}+C$




**48.** $f'(x) = \frac{1}{x^2}$

$f(x) = -\frac{1}{x}+C$



**50.** $\dfrac{dy}{dx} = 2(x-1) = 2x - 2, \ (3, 2)$

$$y = \int 2(x-1)\,dx = x^2 - 2x + C$$

$$2 = (3)^2 - 2(3) + C \implies C = -1$$

$$y = x^2 - 2x - 1$$

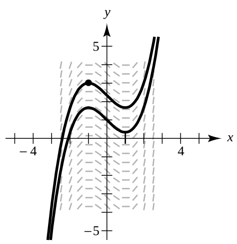
**52.** $\dfrac{dy}{dx} = -\dfrac{1}{x^2} = -x^{-2}, \ (1,3)$

$$y = \int -x^{-2}\,dx = \frac{1}{x} + C$$

$$3 = \frac{1}{1} + C \implies C = 2$$

$$y = \frac{1}{x} + 2, \ x > 0$$

**54.** (a)




(b) $\dfrac{dy}{dx} = x^2 - 1, \ (-1, 3)$

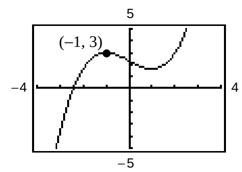
$$y = \frac{x^3}{3} - x + C$$

$$3 = \frac{(-1)^3}{3} - (-1) + C$$

$$3 = -\frac{1}{3} + 1 + C$$

$$C = \frac{7}{3}$$

$$y = \frac{x^3}{3} - x + \frac{7}{3}$$




**56.** $g'(x) = 6x^2, g(0) = -1$

$$g(x) = \int 6x^2\,dx = 2x^3 + C$$

$$g(0) = -1 = 2(0)^3 + C \implies C = -1$$

$$g(x) = 2x^3 - 1$$

**58.** $f'(s) = 6s - 8s^3, \ f(2) = 3$

$$f(s) = \int (6s - 8s^3)\,ds = 3s^2 - 2s^4 + C$$

$$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \implies C = 23$$

$$f(s) = 3s^2 - 2s^4 + 23$$

**60.** $f''(x) = x^2$

$f'(0) = 6$

$f(0) = 3$

$$f'(x) = \int x^2\,dx = \frac{1}{3}x^3 + C_1$$

$$f'(0) = 0 + C_1 = 6 \implies C_1 = 6$$

$$f'(x) = \frac{1}{3}x^3 + 6$$

$$f(x) = \int \left(\frac{1}{3}x^3 + 6\right) dx = \frac{1}{12}x^4 + 6x + C_2$$

$$f(0) = 0 + 0 + C_2 = 3 \implies C_2 = 3$$

$$f(x) = \frac{1}{12}x^4 + 6x + 3$$

**62.** $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$$f'(x) = \int \sin x\,dx = -\cos x + C_1$$

$$f'(0) = -1 + C_1 = 1 \implies C_1 = 2$$

$$f'(x) = -\cos x + 2$$

$$f(x) = \int (-\cos x + 2)\,dx = -\sin x + 2x + C_2$$

$$f(0) = 0 + 0 + C_2 = 6 \implies C_2 = 6$$

$$f(x) = -\sin x + 2x + 6$$

**64.** $\dfrac{dP}{dt} = k\sqrt{t},\ 0 \le t \le 10$

$$P(t) = \int kt^{1/2}\,dt = \frac{2}{3}kt^{3/2} + C$$

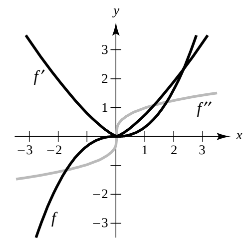
$$P(0) = 0 + C = 500 \implies C = 500$$

$$P(1) = \frac{2}{3}k + 500 = 600 \implies k = 150$$

$$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$$

$$P(7) = 100(7)^{3/2} + 500 \approx 2352 \text{ bacteria}$$

**66.** Since $f''$ is negative on $(-\infty, 0)$, $f'$ is decreasing on $(-\infty, 0)$. Since $f''$ is positive on $(0, \infty)$, $f'$ is increasing on $(0, \infty)$. $f'$ has a relative minimum at $(0, 0)$. Since $f'$ is positive on $(-\infty, \infty)$, $f$ is increasing on $(-\infty, \infty)$.




**68.** $f''(t) = a(t) = -32 \text{ ft/sec}^2$

$$f'(0) = v_0$$

$$f(0) = s_0$$

$$f'(t) = v(t) = \int -32\,dt = -32t + C_1$$

$$f'(0) = 0 + C_1 = v_0 \implies C_1 = v_0$$

$$f'(t) = -32t + v_0$$

$$f(t) = s(t) = \int (-32t + v_0)\,dt = -16t^2 + v_0t + C_2$$

$$f(0) = 0 + 0 + C_2 = s_0 \implies C_2 = s_0$$

$$f(t) = -16t^2 + v_0t + s_0$$

**70.** $v_0 = 16$ ft/sec

$s_0 = 64$ ft

(a) $$s(t) = -16t^2 + 16t + 64 = 0$$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $$v(t) = s'(t) = -32t + 16$$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

**72.** From Exercise 71, $f(t) = -4.9t^2 + 1600$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1600$$

$$4.9t^2 = 1600$$

$$t^2 = \frac{1600}{4.9} \implies t \approx \sqrt{326.53} \approx 18.1 \text{ sec}$$

**74.** From Exercise 71, $f(t) = -4.9t^2 + v_0t + 2$. If

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this $t$ value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9\,v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9\,v_0^2 + 9.8\,v_0^2 = (9.8)^2\,198$$

$$4.9\,v_0^2 = (9.8)^2\,198$$

$$v_0^2 = 3880.8 \implies v_0 \approx 62.3 \text{ m/sec.}$$

**76.** $\int v\,dv = -GM\int \frac{1}{y^2}\,dy$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}{v_0}^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}{v_0}^2 - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}{v_0}^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + {v_0}^2 - \frac{2GM}{R}$$

$$v^2 = {v_0}^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

**78.** $x(t) = (t-1)(t-3)^2 \qquad 0 \le t \le 5$

$\quad = t^3 - 7t^2 + 15t - 9$

(a) $v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$

$a(t) = v'(t) = 6t - 14$

(b) $v(t) > 0$ when $0 < t < \frac{5}{3}$ and $3 < t < 5$.

(c) $a(t) = 6t - 14 = 0$ when $t = \frac{7}{3}$.

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

**80.** (a) $a(t) = \cos t$

$$v(t) = \int a(t)\,dt = \int \cos t\,dt = \sin t + C_1 = \sin t \ \ (\text{since } v_0 = 0)$$

$$f(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \implies C_2 = 4$$

$$f(t) = -\cos t + 4$$

(b) $v(t) = 0 = \sin t$ for $t = k\pi$, $k = 0, 1, 2, \ldots$

**82.** $v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$

$30 \text{ mph} = 44 \text{ ft/sec}$

$15 \text{ mph} = 22 \text{ ft/sec}$

$a(t) = -a$

$v(t) = -at + 66$

$s(t) = -\frac{a}{2}t^2 + 66t$ (Let $s(0) = 0$.)

$v(t) = 0$ after car moves 132 ft.

$-at + 66 = 0$ when $t = \frac{66}{a}$.

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$= 132$ when $a = \frac{33}{2} = 16.5$.

$a(t) = -16.5$

$v(t) = -16.5t + 66$

$s(t) = -8.25t^2 + 66t$

(a) $-16.5t + 66 = 44$

$$t = \frac{22}{16.5} \approx 1.333$$

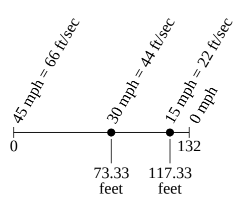
$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

(b) $-16.5t + 66 = 22$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$

(c)




It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

**84.** No, car 2 will be ahead of car 1. If $v_1(t)$ and $v_2(t)$ are the respective velocities, then $\int_0^{30} |v_2(t)|\, dt > \int_0^{30} |v_1(t)|\, dt$.

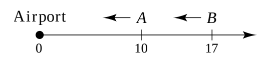
**86.** (a) $v = 0.6139t^3 - 5.525t^2 + 0.0492t + 65.9881$

(b) $s(t) = \int v(t)dt = \dfrac{0.6139t^4}{4} - \dfrac{5.525t^3}{3} + \dfrac{0.0492t^2}{2} + 65.9881t$

(**Note:** Assume $s(0) = 0$ is initial position)

$s(6) \approx 196.1$ feet

**88.** Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.




$v_A(t) = k_A t - 150$ $\qquad v_B = k_B t - 250$

$s_A(t) = \frac{1}{2}k_A t^2 - 150t + 10$ $\qquad s_B = \frac{1}{2}k_B t^2 - 250t + 17$

(a) When aircraft A lands at time $t_A$ you have

$$v_A(t_A) = k_A t_A - 150 = -100 \implies k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}.$$

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625 \implies S_A(t) = S_1(t) = \frac{625}{2}t^2 - 150t + 10$$

Similarly, when aircraft B lands at time $t_B$ you have

$$v_B(t_B) = k_B t_B - 250 = -115 \implies k_B = \frac{135}{t_B}$$

$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

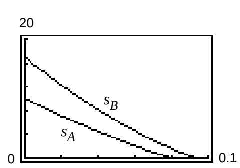
$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

$$t_B = \frac{34}{365}.$$

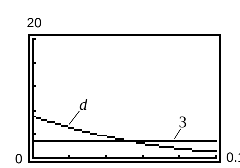
$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49{,}275}{34} \implies S_B(t) = S_2(t) = \frac{49{,}275}{68}t^2 - 250t + 17$$

(b)




(c) $d = s_B(t) - s_A(t)$

Yes, $d < 3$ for $t > 0.0505$.




**90.** True

**92.** True

**94.** False. $f$ has an infinite number of antiderivatives, each differing by a constant.

**96.** $$\frac{d}{dx}\left[[s(x)]^2 + [c(x)]^2\right] = 2s(x)s'(x) + 2c(x)c'(x)$$
$$= 2s(x)c(x) - 2c(x)s(x)$$
$$= 0$$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant $k$. Since,

$s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore,

$[s(x)]^2 + [c(x)]^2 = 1$.

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

## Section 4.2 Area

**2.** $\displaystyle\sum_{k=3}^{6} k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$

**4.** $\displaystyle\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

**6.** $\displaystyle\sum_{i=1}^{4} [(i-1)^2 + (i+1)^3] = (0+8) + (1+27) + (4+64) + (9+125) = 238$

**8.** $\displaystyle\sum_{i=1}^{15} \frac{5}{1+i}$

**10.** $\displaystyle\sum_{j=1}^{4} \left[1 - \left(\frac{j}{4}\right)^2\right]$

**12.** $\displaystyle\frac{2}{n}\sum_{i=1}^{n} \left[1 - \left(\frac{2i}{n} - 1\right)^2\right]$

**14.** $\displaystyle\frac{1}{n}\sum_{i=0}^{n-1} \sqrt{1 - \left(\frac{i}{n}\right)^2}$

**16.** $$\sum_{i=1}^{15} (2i - 3) = 2\sum_{i=1}^{15} i - 3(15)$$
$$= 2\left[\frac{15(16)}{2}\right] - 45 = 195$$

**18.** $$\sum_{i=1}^{10} (i^2 - 1) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1$$
$$= \left[\frac{10(11)(21)}{6}\right] - 10 = 375$$

**20.** $$\sum_{i=1}^{10} i(i^2 + 1) = \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i$$
$$= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2}\right] = 3080$$

**22.** sum seq($x$ $\boxed{\wedge}$ 3 − 2$x$, $x$, 1, 15, 1) = 14,160 (*TI-82*)

$$\sum_{i=1}^{15} (i^3 - 2i) = \frac{(15)^2(15+1)^2}{4} - 2\,\frac{15(15+1)}{2}$$
$$= \frac{(15)^2(16)^2}{4} - 15(16) = 14{,}160$$

**24.** $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

**26.** $$S = \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2}\right] = \frac{55}{6}$$
$$s = \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}\right] = \frac{9}{2} = 4.5$$

**28.** $$S(8) = \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{1} + 2\right)\frac{1}{4}$$
$$+ \left(\sqrt{\frac{5}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{2} + 2\right)\frac{1}{4}$$
$$= \frac{1}{4}\left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 6.038$$
$$s(8) = (0 + 2)\frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} \approx 5.685$$

**30.** $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$$

$$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$$

**32.** $\lim_{n\to\infty}\left[\left(\frac{64}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right] = \frac{64}{6}\lim_{n\to\infty}\left[\frac{2n^3 + 3n^2 + n}{n^3}\right] = \frac{64}{6}(2) = \frac{64}{3}$

**34.** $\lim_{n\to\infty}\left[\left(\frac{1}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{1}{2}\lim_{n\to\infty}\left[\frac{n^2 + n}{n^2}\right] = \frac{1}{2}(1) = \frac{1}{2}$

**36.** $\sum_{j=1}^{n}\frac{4j+3}{n^2} = \frac{1}{n^2}\sum_{j=1}^{n}(4j+3) = \frac{1}{n^2}\left[\frac{4n(n+1)}{2} + 3n\right] = \frac{2n+5}{n} = S(n)$

$$S(10) = \frac{25}{10} = 2.5$$

$$S(100) = 2.05$$

$$S(1000) = 2.005$$

$$S(10{,}000) = 2.0005$$

**38.** $\sum_{i=1}^{n}\frac{4i^2(i-1)}{n^4} = \frac{4}{n^4}\sum_{i=1}^{n}(i^3 - i^2) = \frac{4}{n^4}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$

$$= \frac{4}{n^3}\left[\frac{n^3 + 2n^2 + n}{4} - \frac{2n^2 + 3n + 1}{6}\right]$$

$$= \frac{1}{3n^3}[3n^3 + 6n^2 + 3n - 4n^2 - 6n - 2]$$

$$= \frac{1}{3n^3}[3n^3 + 2n^2 - 3n - 2] = S(n)$$

$$S(10) = 1.056$$

$$S(100) = 1.006566$$

$$S(1000) = 1.00066567$$

$$S(10{,}000) = 1.000066657$$

**40.** $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{4}{2}\left(1 + \frac{1}{n}\right) = 2$

**42.** $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n^3}\sum_{i=1}^{n}(n + 2i)^2$

$$= \lim_{n\to\infty}\frac{2}{n^3}\left[\sum_{i=1}^{n} n^2 + 4n\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\frac{2}{n^3}\left[n^3 + (4n)\left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6}\right]$$

$$= 2\lim_{n\to\infty}\left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2}\right]$$

$$= 2\left(1 + 2 + \frac{4}{3}\right) = \frac{26}{3}$$

**44.** $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n+2i)^3$

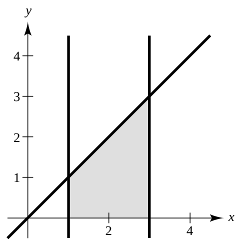
$$= 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n^3+6n^2i+12ni^2+8i^3)$$

$$= 2\lim_{n\to\infty}\frac{1}{n^4}\left[n^4+6n^2\left(\frac{n(n+1)}{2}\right)+12n\left(\frac{n(n+1)(2n+1)}{6}\right)+8\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= 2\lim_{n\to\infty}\left(1+3+\frac{3}{n}+4+\frac{6}{n}+\frac{2}{n^2}+2+\frac{4}{n}+\frac{2}{n^2}\right)$$

$$= 2\lim_{n\to\infty}\left(10+\frac{13}{n}+\frac{4}{n^2}\right) = 20$$

**46.** (a)




(b) $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1+\frac{2}{n} < 1+\frac{4}{n} < \cdots < 1+\frac{2n}{n} = 3$$

$$1 < 1+1\left(\frac{2}{n}\right) < 1+2\left(\frac{2}{n}\right) < \cdots < 1+(n-1)\left(\frac{2}{n}\right) < 1+n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left[1+(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1+(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1+i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1+i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

| $x$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| $s(n)$ | 3.6 | 3.8 | 3.96 | 3.98 |
| $S(n)$ | 4.4 | 4.2 | 4.04 | 4.02 |

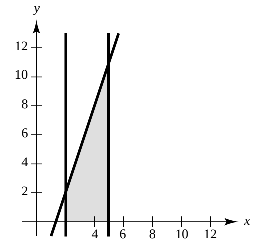
(f) $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\left[1+(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n+\frac{2}{n}\left(\frac{n(n+1)}{2}-n\right)\right]$

$$= \lim_{n\to\infty}\left[2+\frac{2n+2}{n}-\frac{4}{n}\right] = \lim_{n\to\infty}\left[4-\frac{2}{n}\right] = 4$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1+i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[n+\left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[2+\frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4+\frac{2}{n}\right] = 4$$

**48.** $y = 3x - 4$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}\right)$



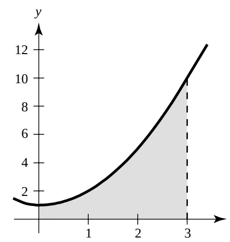

$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^{n} i - 12$$

$$= 6 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$

**50.** $y = x^2 + 1$ on $[0, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}\right)$



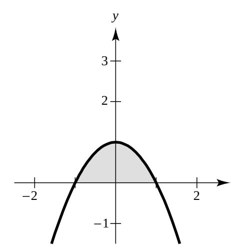

$$S(n) = \sum_{l=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{l=1}^{n}\left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right)$$

$$= \frac{27}{n^3}\sum_{l=1}^{n} l^2 + \frac{3}{n}\sum_{l=1}^{n} 1$$

$$= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2}\frac{2n^2 + 3n + 1}{n^2} + 3$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{9}{2}(2) + 3 = 12$$

**52.** $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$



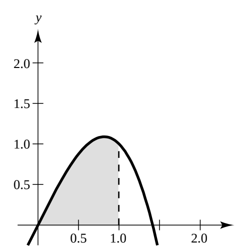

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\frac{1}{2}\text{Area} = \lim_{n\to\infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

**54.** $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$




Since $y$ both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$

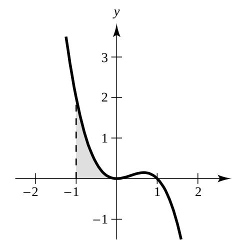
**56.** $y = x^2 - x^3$ on $[-1, 0]$. $\left(\textbf{Note: } \Delta x = \dfrac{0 - (-1)}{n} = \dfrac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\left(\frac{1}{n}\right)\right] = 2 - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$



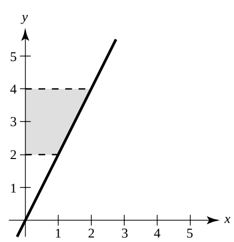

**58.** $g(y) = \dfrac{1}{2}y,\ 2 \le y \le 4.$ $\left(\textbf{Note: } \Delta y = \dfrac{4 - 2}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)$$

$$= \frac{2}{n}\left[n + \frac{1}{n}\,\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + 1 = 3$$




**60.** $f(y) = 4y - y^2,\ 1 \le y \le 2.$ $\left(\textbf{Note: } \Delta y = \dfrac{2 - 1}{n} = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2\right]$$

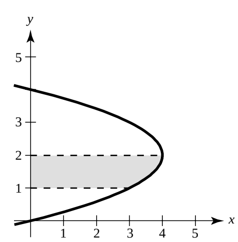
$$= \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\left[3n + \frac{2}{n}\,\frac{n(n+1)}{2} - \frac{1}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$




**62.** $h(y) = y^3 + 1,\ 1 \le y \le 2$ $\left(\textbf{Note: } \Delta y = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

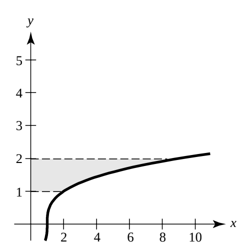
$$= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^3 + 1\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right)$$

$$= \frac{1}{n}\left[2n + \frac{1}{n^3}\,\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\,\frac{n(n+1)}{2}\right]$$

$$= 2 + \frac{(n+1)^2}{n^2 4} + \frac{1}{2}\,\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



**64.** $f(x) = x^2 + 4x,\ 0 \le x \le 4,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = 1,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{2},\ c_3 = \frac{5}{2},\ c_4 = \frac{7}{2}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}[c_i^2 + 4c_i](1)$$

$$= \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right]$$

$$= 53$$

**66.** $f(x) = \sin x,\ 0 \le x \le \dfrac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{8},\ c_1 = \frac{\pi}{16},\ c_2 = \frac{3\pi}{16},\ c_3 = \frac{5\pi}{16},\ c_4 = \frac{7\pi}{16}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\sin c_i)\left(\frac{\pi}{8}\right)$$

$$= \frac{\pi}{8}\left(\sin\frac{\pi}{16} + \sin\frac{3\pi}{16} + \sin\frac{5\pi}{16} + \sin\frac{7\pi}{16}\right) \approx 1.006$$

**68.** $f(x) = \dfrac{8}{x^2 + 1}$ on $[2, 6]$.

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| Approximate area | 2.3397 | 2.3755 | 2.3824 | 2.3848 | 2.3860 |

**70.** $f(x) = \cos\sqrt{x}$ on $[0, 2]$.

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| Approximate area | 1.1041 | 1.1053 | 1.1055 | 1.1056 | 1.1056 |

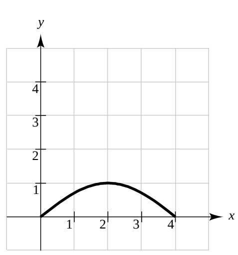
**72.** See the Definition of Area. Page 259.

**74.** $f(x) = \sqrt[3]{x},\ 0 \le x \le 8$

| $n$ | 10 | 20 | 50 | 100 | 200 |
|---|---|---|---|---|---|
| $s(n)$ | 10.998 | 11.519 | 11.816 | 11.910 | 11.956 |
| $S(n)$ | 12.598 | 12.319 | 12.136 | 12.070 | 12.036 |
| $M(n)$ | 12.040 | 12.016 | 12.005 | 12.002 | 12.001 |

(**Note:** exact answer is 12.)

**76.**

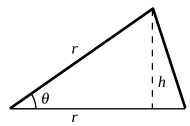



a. $A \approx 3$ square units

**78.** True. (Theorem 4.3)

**80.** (a) $\theta = \dfrac{2\pi}{n}$

(b) $\sin\theta = \dfrac{h}{r}$




$$h = r\sin\theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

(c) $A_n = n\left(\dfrac{1}{2}r^2\sin\dfrac{2\pi}{n}\right) = \dfrac{r^2 n}{2}\sin\dfrac{2\pi}{n} = \pi r^2\left(\dfrac{\sin(2\pi/n)}{2\pi/n}\right)$

Let $x = 2\pi/n$. As $n \to \infty$, $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0}\pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

**82.** (a) $\displaystyle\sum_{i=1}^{n} 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^{k} 2i = k(k+1).$$

Then we have
$$\begin{aligned}\sum_{i=1}^{k+1} 2i &= \sum_{i=1}^{k} 2i + 2(k+1)\\ &= k(k+1) + 2(k+1)\\ &= (k+1)(k+2)\end{aligned}$$

Which shows that the formula is true for $n = k + 1$.

(b) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Then we have
$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3\\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3\\ &= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]\\ &= \frac{(k+1)^2}{4}(k+2)^2\end{aligned}$$

which shows that the formula is true for $n = k + 1$.

## Section 4.3 Riemann Sums and Definite Integrals

**2.** $f(x) = \sqrt[3]{x}$, $y = 0$, $x = 0$, $x = 1$, $c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]\\ &= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(3i^3 - 3i^2 + i)\\ &= \lim_{n\to\infty}\frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]\\ &= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}\end{aligned}$$

**4.** $y = x$ on $[-2, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i$$

$$= -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

**6.** $y = 3x^2$ on $[1, 3]$. $\left(\text{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} 3\left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$$

$$= \frac{6}{n}\sum_{i=1}^{n}\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)$$

$$= \frac{6}{n}\left[n + \frac{4}{n}\,\frac{n(n+1)}{2} + \frac{4}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 6 + 12\,\frac{n+1}{n} + 4\,\frac{(n+1)(2n+1)}{n^2}$$

$$\int_{1}^{3} 3x^2\,dx = \lim_{n\to\infty}\left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2}\right]$$

$$= 6 + 12 + 8 = 26$$

**8.** $y = 3x^2 + 2$ on $[-1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-(-1)}{n} = \dfrac{3}{n}; \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(-1 + \frac{3i}{n}\right)^2 + 2\right]$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2\right]$$

$$= \frac{3}{n}\left[3n - \frac{18}{n}\,\frac{n(n+1)}{2} + \frac{27}{n^2}\,\frac{n(n+1)(2n+1)}{6} + 2n\right]$$

$$= 15 - \frac{27(n+1)}{n} + \frac{27}{2}\,\frac{(n+1)(2n+1)}{n^2}$$

$$\int_{-1}^{2}(3x^2 + 2)dx = \lim_{n\to\infty}\left[15 - 27\frac{(n+1)}{n} + \frac{27}{2}\,\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= 15 - 27 + 27 = 15$$

**10.** $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i = \int_0^4 6x(4-x)^2\,dx$

on the interval $[0, 4]$.

**12.** $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\left(\frac{3}{c_i^2}\right)\Delta x_i = \int_1^3 \frac{3}{x^2}\,dx$

on the interval $[1, 3]$.

**14.** $\displaystyle\int_0^2 (4 - 2x)\,dx$

**16.** $\displaystyle\int_0^2 x^2\,dx$

**18.** $\displaystyle\int_{-1}^{1}\frac{1}{x^2+1}\,dx$

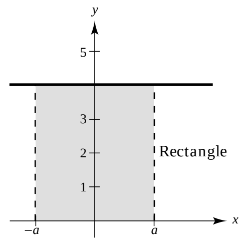
**20.** $\displaystyle\int_0^{\pi/4}\tan x\,dx$

**22.** $\int_0^2 (y-2)^2\,dy$

**24.** Rectangle

$A = bh = 2(4)(a)$

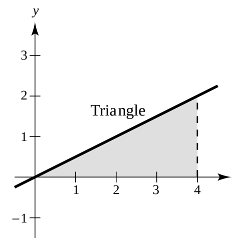
$A = \int_{-a}^{a} 4\,dx = 8a$




**26.** Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$

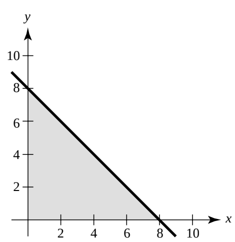
$A = \int_0^4 \frac{x}{2}\,dx = 4$




**28.** Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$

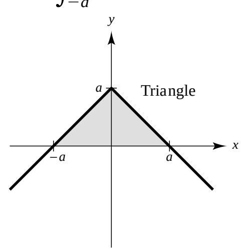
$A = \int_0^8 (8-x)dx = 32$




**30.** Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$

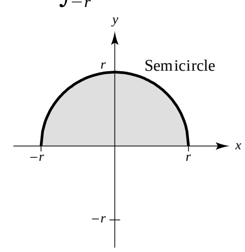
$A = \int_{-a}^{a} (a - |x|)\,dx = a^2$




**32.** Semicircle

$A = \frac{1}{2}\pi r^2$

$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2$




**In Exercises 34–40, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$.**

**34.** $\int_2^2 x^3\,dx = 0$

**36.** $\int_2^4 15\,dx = 15\int_2^4 dx = 15(2) = 30$

**38.** $\int_2^4 (x^3+4)dx = \int_2^4 x^3\,dx + 4\int_2^4 dx = 60 + 4(2) = 68$

**40.** $\int_2^4 (6+2x-x^3)dx = 6\int_2^4 dx + 2\int_2^4 x\,dx - \int_2^4 x^3\,dx$

$= 6(2) + 2(6) - 60 = -36$

**42.** (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

**44.** (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx = 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

**46.** (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx = 4 + 10 = 14$

(b) $\int_{-2}^3 f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ ($f$ even)

(d) $\int_{-5}^5 f(x)\,dx = 0$ ($f$ odd)

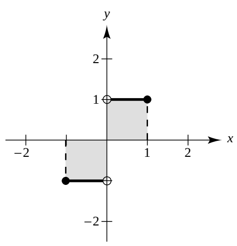
**48.** The right endpoint approximation will be less than the actual area: <

**50.** The average of Exercise 39 and Exercise 40 consists of a trapezoidal approximation, and is greater than the exact area: >

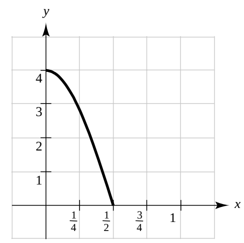
**52.** $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^1 \frac{|x|}{x}\,dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the $x$-axis for $-1 \le x \le 1$. You see that the integral equals 0.

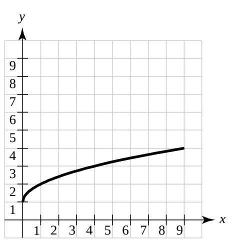



**54.**




b. $A \approx \frac{4}{3}$ square units

**56.**




c. Area $\approx 27$.

**58.** $\int_0^3 \frac{5}{x^2 + 1}\,dx$

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $L(n)$ | 7.9224 | 7.0855 | 6.8062 | 6.6662 | 6.5822 |
| $M(n)$ | 6.2485 | 6.2470 | 7.2460 | 6.2457 | 6.2455 |
| $R(n)$ | 4.5474 | 5.3980 | 5.6812 | 5.8225 | 5.9072 |

**60.** $\int_0^3 x \sin x\,dx$

| $n$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $L(n)$ | 2.8186 | 2.9985 | 3.0434 | 3.0631 | 3.0740 |
| $M(n)$ | 3.1784 | 3.1277 | 3.1185 | 3.1152 | 3.1138 |
| $R(n)$ | 3.1361 | 3.1573 | 3.1493 | 3.1425 | 3.1375 |

**62.** False

$$\int_0^1 x\sqrt{x}\,dx \ne \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{x}\,dx\right)$$

**64.** True

**66.** False

$$\int_{-2}^4 x\,dx = 6$$

**68.** $f(x) = \sin x, [0, 2\pi]$

$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$

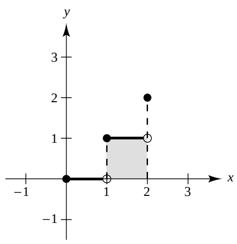
$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$

$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

**70.** To find $\int_0^2 [\![x]\!]\,dx$, use a geometric approach.



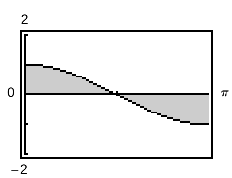

Thus,

$$\int_0^2 [\![x]\!]\,dx = 1(2 - 1) = 1.$$

## Section 4.4 The Fundamental Theorem of Calculus

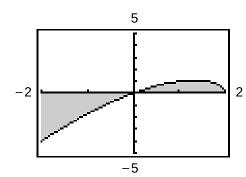
**2.** $f(x) = \cos x$

$$\int_0^{\pi} \cos x\,dx = 0$$




**4.** $f(x) = x\sqrt{2 - x}$

$\int_{-2}^{2} x\sqrt{2 - x}\,dx$ is negative.




**6.** $\int_2^7 3\,dv = \Big[3v\Big]_2^7 = 3(7) - 3(2) = 15$

**8.** $\int_2^5 (-3v + 4)\,dv = \left[-\frac{3}{2}v^2 + 4v\right]_2^5 = \left(-\frac{75}{2} + 20\right) - (-6 + 8) = -\frac{39}{2}$

**10.** $\int_1^3 (3x^2 + 5x - 4)dx = \left[x^3 + \frac{5x^2}{2} - 4x\right]_1^3 = \left(27 + \frac{45}{2} - 12\right) - \left(1 + \frac{5}{2} - 4\right)$

$$= 38$$

**12.** $\int_{-1}^{1} (t^3 - 9t)\,dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2\right]_{-1}^{1} = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = 0$

**14.** $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du = \left[\frac{u^2}{2} + \frac{1}{u}\right]_{-2}^{-1} = \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$

**16.** $\displaystyle\int_{-3}^{3} v^{1/3}\,dv = \left[\frac{3}{4}v^{4/3}\right]_{-3}^{3} = \frac{3}{4}\left[\left(\sqrt[3]{-3}\right)^4\right] - \left(\sqrt[3]{-3}\right)^4\right] = 0$

**18.** $\displaystyle\int_{1}^{8} \sqrt{\frac{2}{x}}\,dx = \sqrt{2}\int_{1}^{8} x^{-1/2}\,dx = \left[\sqrt{2}(2)x^{1/2}\right]_{1}^{8} = \left[2\sqrt{2x}\right]_{1}^{8} = 8 - 2\sqrt{2}$

**20.** $\displaystyle\int_{0}^{2} (2 - t)\sqrt{t}\,dt = \int_{0}^{2} (2t^{1/2} - t^{3/2})\,dt = \left[\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_{0}^{2} = \left[\frac{t\sqrt{t}}{15}(20 - 6t)\right]_{0}^{2} = \frac{2\sqrt{2}}{15}(20 - 12) = \frac{16\sqrt{2}}{15}$

**22.** $\displaystyle\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx = \frac{1}{2}\int_{-8}^{-1} (x^{2/3} - x^{5/3})\,dx$

$$= \frac{1}{2}\left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3}\right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24 - 15x)\right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$$

**24.** $\displaystyle\int_{1}^{4} (3 - 1x - 31)\,dx = \int_{1}^{3} [3 + (x - 3)]\,dx + \int_{3}^{4} [3 - (x - 3)]\,dx$

$$= \int_{1}^{3} x\,dx + \int_{3}^{4} (6 - x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{1}^{3} + \left[6x - \frac{x^2}{2}\right]_{3}^{4}$$

$$= \left(\frac{9}{2} - \frac{1}{2}\right) + \left[(24 - 8) - \left(18 - \frac{9}{2}\right)\right]$$

$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

**26.** $\displaystyle\int_{0}^{4} \left|x^2 - 4x + 3\right|\,dx = \int_{0}^{1} (x^2 - 4x + 3)\,dx - \int_{1}^{3} (x^2 - 4x + 3)\,dx + \int_{3}^{4} (x^2 - 4x + 3)\,dx$ (split up the integral at the zeros $x = 1, 3$)

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{0}^{1} - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{1}^{3} + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{3}^{4}$$

$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)$$

$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

**28.** $\displaystyle\int_{0}^{\pi/4} \frac{1 - \sin^2\theta}{\cos^2\theta}\,d\theta = \int_{0}^{\pi/4} d\theta = \left[\theta\right]_{0}^{\pi/4} = \frac{\pi}{4}$

**30.** $\displaystyle\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\,dx = \left[2x + \cot x\right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1\right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$

**32.** $\displaystyle\int_{-\pi/2}^{\pi/2} (2t + \cos t)\,dt = \left[t^2 + \sin t\right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1\right) - \left(\frac{\pi^2}{4} - 1\right) = 2$

**34.** $\displaystyle P = \frac{2}{\pi}\int_{0}^{\pi/2} \sin\theta\,d\theta = \left[-\frac{2}{\pi}\cos\theta\right]_{0}^{\pi/2} = -\frac{2}{\pi}(0 - 1) = \frac{2}{\pi} \approx 63.7\%$

**36.** $\displaystyle A = \int_{-1}^{1} (1 - x^4)\,dx = \left[x - \frac{1}{5}x^5\right]_{-1}^{1} = \frac{8}{5}$

**38.** $\displaystyle A = \int_{1}^{2} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{1}^{2} = -\frac{1}{2} + 1 = \frac{1}{2}$

**40.** $\displaystyle A = \int_{0}^{\pi} (x + \sin x)\,dx = \left[\frac{x^2}{2} - \cos x\right]_{0}^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$

**42.** Since $y \geq 0$ on $[0.\ 8]$,

$$\text{Area} = \int_0^8 (1 + x^{1/3})\,dx = \left[x + \frac{3}{4}x^{4/3}\right]_0^8 = 8 + \frac{3}{4}(16) = 20$$

**44.** Since $y \geq 0$ on $[0, 3]$,

$$A = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}.$$

**46.** $\displaystyle\int_1^3 \frac{9}{x^3}\,dx = \left[-\frac{9}{2x^2}\right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

**48.** $\displaystyle\int_{-\pi/3}^{\pi/3} \cos x\,dx = \Big[\sin x\Big]_{-\pi/3}^{\pi/3} = \sqrt{3}$

$$f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3}$$

$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

$$c \approx \pm 0.5971$$

**50.** $\displaystyle\frac{1}{3 - 1}\int_1^3 \frac{4(x^2 + 1)}{x^2}\,dx = 2\int_1^3 (1 + x^{-2})dx = 2\left[x - \frac{1}{x}\right]_1^3$

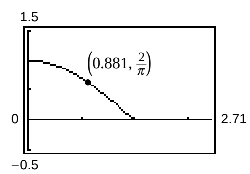
$$= 2\left(3 - \frac{1}{3}\right) = \frac{16}{3}$$

**52.** $\displaystyle\frac{1}{(\pi/2) - 0}\int_0^{\pi/2} \cos x\,dx = \left[\frac{2}{\pi}\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$

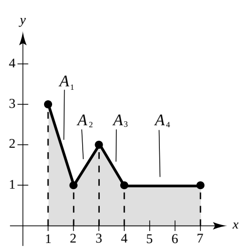



**54.** (a) $\displaystyle\int_1^7 f(x)\,dx = \text{Sum of the areas}$

$$= A_1 + A_2 + A_3 + A_4$$

$$= \frac{1}{2}(3 + 1) + \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 1) + (3)(1)$$

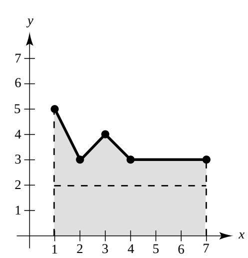
$$= 8$$




(b) $\displaystyle\text{Average value} = \frac{\int_1^7 f(x)\,dx}{7 - 1} = \frac{8}{6} = \frac{4}{3}$

(c) $A = 8 + (6)(2) = 20$

$$\text{Average value} = \frac{20}{6} = \frac{10}{3}$$



**56.** $\int_2^6 f(x)\,dx = (\text{area or region } B) = \int_0^6 f(x)\,dx - \int_0^2 f(x)\,dx$

$= 3.5 - (-1.5) = 5.0$

**58.** $\int_0^2 -2f(x)\,dx = -2\int_0^2 f(x)\,dx = -2(-1.5) = 3.0$

**60.** Average value $= \frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}(3.5) = 0.5833$

**62.** $\frac{1}{R-0}\int_0^R k(R^2 - r^2)\,dr = \frac{k}{R}\left[R^2r - \frac{r^3}{3}\right]_0^R = \frac{2kR^2}{3}$

**64.** $P = 5\left(\sqrt{t} + 30\right)$

(a)

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P$ | 155 | 157.071 | 158.660 | 160 | 161.180 | 162.247 |

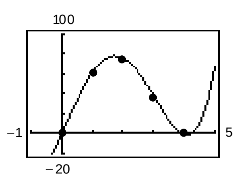
Average profit $\approx \frac{1}{6}(155 + 157.071 + 158.660 + 160 + 161.180 + 162.247) = \frac{954.158}{6} \approx 159.026$

(b) $\frac{1}{6}\int_{0.5}^{6.5} 5\left(\sqrt{t} + 30\right)dt = \frac{1}{6}\left[5\left(\frac{2}{3}t^{3/2} + 30t\right)\right]_{5.0}^{6.5} \approx \frac{954.061}{6} \approx 159.010$

(c) The definite integral yields a better approximation.

**66.** (a) $R = 2.33t^4 - 14.67t^3 + 3.67t^2 + 70.67t$

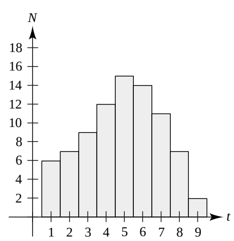
(b)




(c) $\int_0^4 R(t)\,dt = \left[\frac{2.33t^5}{5} - \frac{14.67t^4}{4} + \frac{3.67t^3}{3} + \frac{70.67t^2}{2}\right]_0^4$

$= 181.957$

**68.** (a) histogram

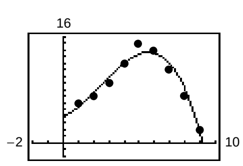



(b) $[6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980$ customers

(c) Using a graphing utility, you obtain

$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052 + 4.10317.$

(d)




(e) $\int_0^9 N(t)\,dt \approx 85.162$

The estimated number of customers is $(85.162)(60) \approx 5110$.

(f) Between 3 P.M. and 7 P.M., the number of customers is approximately $\left(\int_3^7 N(t)\,dt\right)(60) \approx (50.28)(60) \approx 3017.$

Hence, $3017/240 \approx 12.6$ per minute.

**70.** $F(x) = \int_2^x (t^3 + 2t - 2)\,dt = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x$

$= \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4)$

$= \frac{x^4}{4} + x^2 - 2x - 4$

$F(2) = 4 + 4 - 4 - 4 = 0$ $\left[\text{Note: } F(2) = \int_2^2 (t^3 + 2t - 2)\,dt = 0\right]$

$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$

$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$

**72.** $F(x) = \int_2^x \frac{-2}{t^3}\,dt = -\int_2^x 2t^{-3}\,dt = \frac{1}{t^2}\Big]_2^x = \frac{1}{x^2} - \frac{1}{4}$

$F(2) = \frac{1}{4} - \frac{1}{4} = 0$

$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$

$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$

**74.** $F(x) = \int_0^x \sin\theta\,d\theta = -\cos\theta\Big]_0^x = -\cos x + \cos 0 = 1 - \cos x$

$F(2) = 1 - \cos 2 \approx 1.4161$

$F(5) = 1 - \cos 5 \approx 0.7163$

$F(8) = 1 - \cos 8 \approx 1.1455$

**76.** (a) $\int_0^x t(t^2 + 1)\,dt = \int_0^x (t^3 + t)\,dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$

(b) $\frac{d}{dx}\left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right] = x^3 + x = x(x^2 + 1)$

**78.** (a) $\int_4^x \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$

(b) $\frac{d}{dx}\left[\frac{2}{3}x^{3/2} - \frac{16}{3}\right] = x^{1/2} = \sqrt{x}$

**80.** (a) $\int_{\pi/3}^x \sec t \tan t\,dt = \Big[\sec t\Big]_{\pi/3}^x = \sec x - 2$

(b) $\frac{d}{dx}[\sec x - 2] = \sec x \tan x$

**82.** $F(x) = \int_1^x \frac{t^2}{t^2 + 1}\,dt$

$F'(x) = \frac{x^2}{x^2 + 1}$

**84.** $F(x) = \int_1^x \sqrt[4]{t}\,dt$

$F'(x) = \sqrt[4]{x}$

**86.** $F(x) = \int_0^x \sec^3 t\,dt$

$F'(x) = \sec^3 x$

**88.** $F(x) = \int_{-x}^{x} t^3\,dt = \left[\frac{t^4}{4}\right]_{-x}^{x} = 0$

$F'(x) = 0$

**Alternate solution**

$$\begin{aligned} F(x) &= \int_{-x}^{x} t^3\,dt \\ &= \int_{-x}^{0} t^3\,dt + \int_{0}^{x} t^3\,dt \\ &= -\int_{0}^{-x} t^3\,dt + \int_{0}^{x} t^3\,dt \end{aligned}$$

$F'(x) = -(-x)^3(-1) + (x^3) = 0$

**90.** $F(x) = \int_{2}^{x^2} t^{-3}\,dt = \left[\frac{t^{-2}}{-2}\right]_{2}^{x^2} = \left[-\frac{1}{2t^2}\right]_{2}^{x^2} = \frac{-1}{2x^4} + \frac{1}{8} \Longrightarrow F'(x) = 2x^{-5}$

**Alternate solution:** $F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$

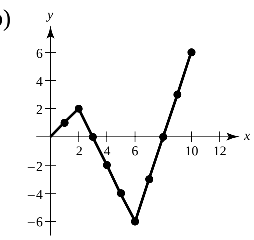
**92.** $F(x) = \int_{0}^{x^2} \sin\theta^2\,d\theta$

$F'(x) = \sin(x^2)^2\,(2x) = 2x\sin x^4$

**94.** (a)

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $g(x)$ | 1 | 2 | 0 | $-2$ | $-4$ | $-6$ | $-3$ | 0 | 3 | 6 |

(b)




(c) Minimum of $g$ at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$ $g$ increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of $g$: $x = 3, x = 8$.

**96.** (a) $g(t) = 4 - \frac{4}{t^2}$

$\lim_{t\to\infty} g(t) = 4$

Horizontal asymptote: $y = 4$

(b) $$\begin{aligned} A(x) &= \int_{1}^{x}\left(4 - \frac{4}{t^2}\right)dt \\ &= \left[4t + \frac{4}{t}\right]_{1}^{x} = 4x + \frac{4}{x} - 8 \\ &= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x} \end{aligned}$$

$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\left(4x + \frac{4}{x} - 8\right) = \infty + 0 - 8 = \infty$

The graph of $A(x)$ does not have a horizontal asymptote.

**98.** True

**100.** Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t)\,dt = \Big[F(t)\Big]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$

$$\begin{aligned} \frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] &= \frac{d}{dx}[F(v(x)) - F(u(x)) \\ &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x). \end{aligned}$$

**102.** $G(x) = \int_0^x \left[ s \int_0^s f(t)\, dt \right] ds$

(a) $G(0) = \int_0^0 \left[ s \int_0^s f(t)\, dt \right] ds = 0$

(b) Let $F(s) = s \int_0^s f(t)\, dt$.

$$G(x) = \int_0^x F(s)\, ds$$

$$G'(x) = F(x) = x \int_0^x f(t)\, dt$$

$$G'(0) = 0 \int_0^0 f(t)\, dt = 0$$

(c) $G''(x) = x \cdot f(x) + \int_0^x f(t)\, dt$

(d) $G''(0) = 0 \cdot f(0) + \int_0^0 f(t)\, dt = 0$

**104.** $x(t) = (t-1)(t-3)^2 = t^3 - 7t^2 + 15t - 9$

$x'(t) = 3t^2 - 14t + 15$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)|\, dt \approx 27.37 \text{ units}$$

## Section 4.5 Integration by Substitution

| $\int f(g(x))g'(x)\, dx$ | $u = g(x)$ | $du = g'(x)\, dx$ |
|---|---|---|
| **2.** $\int x^2\sqrt{x^3+1}\, dx$ | $x^3 + 1$ | $3x^2\, dx$ |
| **4.** $\int \sec 2x \tan 2x\, dx$ | $2x$ | $2\, dx$ |
| **6.** $\int \frac{\cos x}{\sin^2 x}\, dx$ | $\sin x$ | $\cos x\, dx$ |

**8.** $\int (x^2 - 9)^3(2x)\, dx = \frac{(x^2-9)^4}{4} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^2-9)^4}{4} + C\right] = \frac{4(x^2-9)^3}{4}(2x) = (x^2-9)^3(2x)$

**10.** $\int (1-2x^2)^{1/3}(-4x)\, dx = \frac{3}{4}(1-2x^2)^{4/3} + C$

**Check:** $\frac{d}{dx}\left[\frac{3}{4}(1-2x^2)^{4/3} + C\right] = \frac{3}{4} \cdot \frac{4}{3}(1-2x^2)^{1/3}(-4x) = (1-2x^2)^{1/3}(-4x)$

**12.** $\int x^2(x^3+5)^4\, dx = \frac{1}{3}\int (x^3+5)^4(3x^2)\, dx = \frac{1}{3}\frac{(x^3+5)^5}{5} + C = \frac{(x^3+5)^5}{15} + C$

**Check:** $\frac{d}{dx}\left[\frac{(x^3+5)^5}{15} + C\right] = \frac{5(x^3+5)^4(3x^2)}{15} = (x^3+5)^4x^2$

**14.** $\int x(4x^2+3)^3\, dx = \frac{1}{8}\int (4x^2+3)^3(8x)\, dx = \frac{1}{8}\left[\frac{(4x^2+3)^4}{4}\right] + C = \frac{(4x^2+3)^4}{32} + C$

**Check:** $\frac{d}{dx}\left[\frac{(4x^2+3)^4}{32} + C\right] = \frac{4(4x^2+3)^3(8x)}{32} = x(4x^2+3)^3$

**16.** $\displaystyle\int t^3\sqrt{t^4+5}\,dt = \frac{1}{4}\int (t^4+5)^{1/2}(4t^3)\,dt = \frac{1}{4}\frac{(t^4+5)^{3/2}}{3/2} + C = \frac{1}{6}(t^4+5)^{3/2} + C$

**Check:** $\displaystyle\frac{d}{dt}\left[\frac{1}{6}(t^4+5)^{3/2} + C\right] = \frac{1}{6}\cdot\frac{3}{2}(t^4+5)^{1/2}(4t^3) = (t^4+5)^{1/2}(t^3)$

**18.** $\displaystyle\int u^2\sqrt{u^3+2}\,du = \frac{1}{3}\int (u^3+2)^{1/2}(3u^2)\,du = \frac{1}{3}\frac{(u^3+2)^{3/2}}{3/2} + C = \frac{2(u^3+2)^{3/2}}{9} + C$

**Check:** $\displaystyle\frac{d}{du}\left[\frac{2(u^3+2)^{3/2}}{9} + C\right] = \frac{2}{9}\cdot\frac{3}{2}(u^3+2)^{1/2}(3u^2) = (u^3+2)^{1/2}(u^2)$

**20.** $\displaystyle\int \frac{x^3}{(1+x^4)^2}\,dx = \frac{1}{4}\int (1+x^4)^{-2}(4x^3)\,dx = -\frac{1}{4}(1+x^4)^{-1} + C = \frac{-1}{4(1+x^4)} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{-1}{4(1+x^4)} + C\right] = \frac{1}{4}(1+x^4)^{-2}(4x^3) = \frac{x^3}{(1+x^4)^2}$

**22.** $\displaystyle\int \frac{x^2}{(16-x^3)^2}\,dx = -\frac{1}{3}\int (16-x^3)^{-2}(-3x^2)\,dx = -\frac{1}{3}\left[\frac{(16-x^3)^{-1}}{-1}\right] + C = \frac{1}{3(16-x^3)} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{1}{3(16-x^3)} + C\right] = \frac{1}{3}(-1)(16-x^3)^{-2}(3x^2) = \frac{x^2}{(16-x^3)^2}$

**24.** $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int (1+x^4)^{-1/2}(4x^3)\,dx = \frac{1}{4}\frac{(1+x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1+x^4}}{2} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{\sqrt{1+x^4}}{2} + C\right] = \frac{1}{2}\cdot\frac{1}{2}(1+x^4)^{-1/2}(4x^3) = \frac{x^3}{\sqrt{1+x^4}}$

**26.** $\displaystyle\int\left[x^2 + \frac{1}{(3x)^2}\right]dx = \int\left(x^2 + \frac{1}{9}x^{-2}\right)dx = \frac{x^3}{3} + \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4-1}{9x} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\frac{1}{3}x^3 - \frac{1}{9}x^{-1} + C\right] = x^2 + \frac{1}{9}x^{-2} = x^2 + \frac{1}{(3x)^2}$

**28.** $\displaystyle\int \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int x^{-1/2}\,dx = \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \sqrt{x} + C$

**Check:** $\displaystyle\frac{d}{dx}\left[\sqrt{x} + C\right] = \frac{1}{2\sqrt{x}}$

**30.** $\displaystyle\int \frac{t+2t^2}{\sqrt{t}}\,dt = \int (t^{1/2} + 2t^{3/2})\,dt = \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C = \frac{2}{15}t^{3/2}(5+6t) + C$

**Check:** $\displaystyle\frac{d}{dt}\left[\frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C\right] = t^{1/2} + 2t^{3/2} = \frac{t+2t^2}{\sqrt{t}}$

**32.** $\displaystyle\int\left(\frac{t^3}{3} + \frac{1}{4t^2}\right)dt = \int\left(\frac{1}{3}t^3 + \frac{1}{4}t^{-2}\right)dt = \frac{1}{3}\left(\frac{t^4}{4}\right) + \frac{1}{4}\left(\frac{t^{-1}}{-1}\right) + C = \frac{1}{12}t^4 - \frac{1}{4t} + C$

**Check:** $\displaystyle\frac{d}{dt}\left[\frac{1}{12}t^4 - \frac{1}{4t} + C\right] = \frac{1}{3}t^3 + \frac{1}{4t^2}$

**34.** $\int 2\pi y(8 - y^{3/2})\,dy = 2\pi\int (8y - y^{5/2})\,dy = 2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$

**Check:** $\frac{d}{dy}\left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C\right] = \frac{d}{dy}\left[2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C\right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$

**36.** $y = \int \frac{10x^2}{\sqrt{1 + x^3}}\,dx$

$= \frac{10}{3}\int (1 + x^3)^{-1/2}(3x^2)\,dx$

$= \frac{10}{3}\left[\frac{(1 + x^3)^{1/2}}{1/2}\right] + C$

$= \frac{20}{3}\sqrt{1 + x^3} + C$

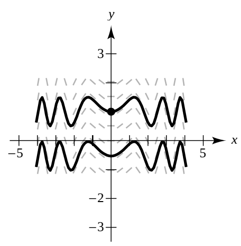
**38.** $y = \int \frac{x - 4}{\sqrt{x^2 - 8x + 1}}\,dx$

$= \frac{1}{2}\int (x^2 - 8x + 1)^{-1/2}(2x - 8)\,dx$

$= \frac{1}{2}\left[\frac{(x^2 - 8x + 1)^{1/2}}{1/2}\right] + C$

$= \sqrt{x^2 - 8x + 1} + C$

**40.** (a)




(b) $\frac{dy}{dx} = x\cos x^2,\ (0, 1)$

$y = \int x\cos x^2\,dx = \frac{1}{2}\int \cos(x^2)2x\,dx$

$= \frac{1}{2}\sin(x^2) + C$

$(0, 1)$: $1 = \frac{1}{2}\sin(0) + C \Rightarrow C = 1$

$y = \frac{1}{2}\sin(x^2) + 1$

**42.** $\int 4x^3\sin x^4\,dx = \int \sin x^4(4x^3)\,dx = -\cos x^4 + C$

**44.** $\int \cos 6x\,dx = \frac{1}{6}\int (\cos 6x)(6)\,dx = \frac{1}{6}\sin 6x + C$

**46.** $\int x\sin x^2\,dx = \frac{1}{2}\int (\sin x^2)(2x)\,dx = -\frac{1}{2}\cos x^2 + C$

**48.** $\int \sec(1 - x)\tan(1 - x)\,dx = -\int [\sec(1 - x)\tan(1 - x)](-1)\,dx = -\sec(1 - x) + C$

**50.** $\int \sqrt{\tan x}\sec^2 x\,dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$

**52.** $\int \frac{\sin x}{\cos^3 x}\,dx = -\int (\cos x)^{-3}(-\sin x)\,dx = -\frac{(\cos x)^{-2}}{-2} + C = \frac{1}{2\cos^2 x} + C = \frac{1}{2}\sec^2 x + C$

**54.** $\int \csc^2\left(\frac{x}{2}\right)dx = 2\int \csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = -2\cot\left(\frac{x}{2}\right) + C$

**56.** $f(x) = \int \pi\sec\pi x\tan\pi x\,dx = \sec\pi x + C$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$f(x) = \sec\pi x - 1.$

**58.** $u = 2x + 1$, $x = \frac{1}{2}(u - 1)$, $dx = \frac{1}{2}\,du$

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}\,du\\ &= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C\\ &= \frac{u^{3/2}}{30}(3u - 5) + C\\ &= \frac{1}{30}(2x+1)^{3/2}[3(2x+1) - 5] + C\\ &= \frac{1}{30}(2x+1)^{3/2}(6x - 2) + C\\ &= \frac{1}{15}(2x+1)^{3/2}(3x - 1) + C\end{aligned}$$

**60.** $u = 2 - x$, $x = 2 - u$, $dx = -du$

$$\begin{aligned}\int (x+1)\sqrt{2-x}\,dx &= -\int (3-u)\sqrt{u}\,du\\ &= -\int (3u^{1/2} - u^{3/2})\,du\\ &= -\left(2u^{3/2} - \frac{2}{5}u^{5/2}\right) + C\\ &= -\frac{2u^{3/2}}{5}(5 - u) + C\\ &= -\frac{2}{5}(2-x)^{3/2}[5 - (2-x)] + C\\ &= -\frac{2}{5}(2-x)^{3/2}(x+3) + C\end{aligned}$$

**62.** Let $u = x + 4$, $x = u - 4$, $du = dx$.

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x+4}}\,dx &= \int \frac{2(u-4)+1}{\sqrt{u}}\,du\\ &= \int (2u^{1/2} - 7u^{-1/2})\,du\\ &= \frac{4}{3}u^{3/2} - 14u^{1/2} + C\\ &= \frac{2}{3}u^{1/2}(2u - 21) + C\\ &= \frac{2}{3}\sqrt{x+4}[2(x+4) - 21] + C\\ &= \frac{2}{3}\sqrt{x+4}(2x - 13) + C\end{aligned}$$

**64.** $u = t - 4$, $t = u + 4$, $dt = du$

$$\begin{aligned}\int t\sqrt[3]{t-4}\,dt &= \int (u+4)u^{1/3}\,du\\ &= \int (u^{4/3} + 4u^{1/3})\,du\\ &= \frac{3}{7}u^{7/3} + 3u^{4/3} + C\\ &= \frac{3u^{4/3}}{7}(u + 7) + C\\ &= \frac{3}{7}(t-4)^{4/3}[(t-4) + 7] + C\\ &= \frac{3}{7}(t-4)^{4/3}(t+3) + C\end{aligned}$$

**66.** Let $u = x^3 + 8$, $du = 3x^2\,dx$.

$$\begin{aligned}\int_{-2}^{4} x^2(x^3+8)^2\,dx &= \frac{1}{3}\int_{-2}^{4}(x^3+8)^2(3x^2)\,dx = \left[\frac{1}{3}\,\frac{(x^3+8)^3}{3}\right]_{-2}^{4}\\ &= \frac{1}{9}[(64+8)^3 - (-8+8)^3] = 41{,}472\end{aligned}$$

**68.** Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_0^1 (1-x^2)^{1/2}(-2x)\,dx = \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

**70.** Let $u = 1 + 2x^2$, $du = 4x\,dx$.

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx = \frac{1}{4}\int_0^2 (1+2x^2)^{-1/2}(4x)\,dx = \left[\frac{1}{2}\sqrt{1+2x^2}\right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

**72.** Let $u = 4 + x^2$, $du = 2x\,dx$.

$$\int_0^2 x\sqrt[3]{4+x^2}\,dx = \frac{1}{2}\int_0^2 (4+x^2)^{1/3}(2x)\,dx = \left[\frac{3}{8}(4+x^2)^{4/3}\right]_0^2 = \frac{3}{8}(8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2}\sqrt[3]{4} \approx 3.619$$

**74.** Let $u = 2x - 1$, $du = 2\,dx$, $x = \frac{1}{2}(u+1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\begin{aligned}\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx &= \int_1^9 \frac{1/2(u+1)}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{4}\int_1^9 (u^{1/2} + u^{-1/2})\,du \\ &= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_1^9 \\ &= \frac{1}{4}\left[\left(\frac{2}{3}(27) + 2(3)\right) - \left(\frac{2}{3} + 2\right)\right] \\ &= \frac{16}{3}\end{aligned}$$

**76.** $$\int_{\pi/3}^{\pi/2} (x + \cos x)\,dx = \left[\frac{x^2}{2} + \sin x\right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1\right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2}\right) = \frac{5\pi^2}{72} + \frac{2-\sqrt{3}}{2}$$

**78.** $u = x + 2$, $x = u - 2$, $dx = du$

When $x = -2$, $u = 0$. When $x = 6$, $u = 8$.

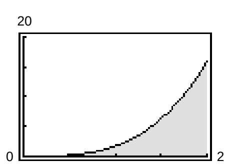
$$\text{Area} = \int_{-2}^6 x^2\sqrt[3]{x+2}\,dx = \int_0^8 (u-2)^2\sqrt[3]{u}\,du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3})\,du = \left[\frac{3}{10}u^{10/3} - \frac{12}{7}u^{7/3} + 3u^{4/3}\right]_0^8 = \frac{4752}{35}$$

**80.** $$A = \int_0^\pi (\sin x + \cos 2x)\,dx = \left[-\cos x + \frac{1}{2}\sin 2x\right]_0^\pi = 2$$

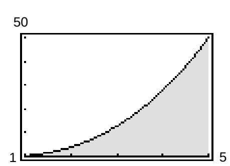
**82.** Let $u = 2x$, $du = 2\,dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx = \frac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x(2)\,dx = \left[-\frac{1}{2}\csc 2x\right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

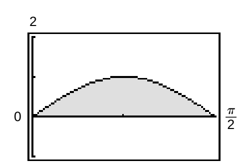
**84.** $\displaystyle\int_0^2 x^3\sqrt{x+2}\,dx \approx 7.581$




**86.** $\displaystyle\int_1^5 x^2\sqrt{x-1}\,dx \approx 67.505$




**88.** $\displaystyle\int_0^{\pi/2} \sin 2x\,dx \approx 1.0$




**90.** $$\int \sin x \cos x\,dx = \int (\sin x)^1(\cos x\,dx) = \frac{\sin^2 x}{2} + C_1$$

$$\int \sin x \cos x\,dx = -\int (\cos x)^1(-\sin x\,dx) = -\frac{\cos^2 x}{2} + C_2$$

$$-\frac{\cos^2 x}{2} + C_2 = -\frac{(1 - \sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2$$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

**92.** $f(x) = \sin^2 x \cos x$ is even.

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx = \int_0^{\pi/2} \sin^2 x(\cos x)\, dx = 2\left[\frac{\sin^3 x}{3}\right]_0^{\pi/2} = \frac{2}{3}$$

**94.** $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$$

**96.** (a) $\int_{-\pi/4}^{\pi/4} \sin x \, dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x \, dx = 2\int_0^{\pi/4} \cos x \, dx = \Big[2 \sin x\Big]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the $y$-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x \, dx = 2\int_0^{\pi/2} \cos x \, dx = \Big[2 \sin x\Big]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$ since $\sin(-x)\cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

**98.** $\int_{-\pi}^{\pi} (\sin 3x + \cos 3x)\, dx = \int_{-\pi}^{\pi} \sin 3x \, dx + \int_{-\pi}^{\pi} \cos 3x \, dx = 0 + 2\int_0^{\pi} \cos 3x \, dx = \left[\frac{2}{3} \sin 3x\right]_0^{\pi} = 0$

**100.** If $u = 5 - x^2$, then $du = -2x\, dx$ and $\int x(5 - x^2)^3 \, dx = -\frac{1}{2}\int (5 - x^2)^3(-2x)\, dx = -\frac{1}{2}\int u^3 \, du$.

**102.** $\frac{dQ}{dt} = k(100 - t)^2$

$Q(t) = \int k(100 - t)^2 \, dt = -\frac{k}{3}(100 - t)^3 + C$

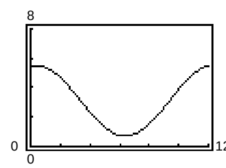
$Q(100) = C = 0$

$Q(t) = -\frac{k}{3}(100 - t)^3$

$Q(0) = -\frac{k}{3}(100)^3 = 2{,}000{,}000 \implies k = -6$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) =$ \$250,000.

**104.** $R = 3.121 + 2.399 \sin(0.524t + 1.377)$

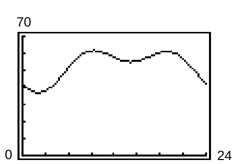
(a) 


Relative minimum: $(6.4, 0.7)$ or June

Relative maximum: $(0.4, 5.5)$ or January

(b) $\int_0^{12} R(t)\, dt \approx 37.47$ inches

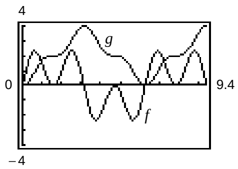
(c) $\frac{1}{3}\int_9^{12} R(t)\, dt \approx \frac{1}{3}(13) = 4.33$ inches

**106.** (a) 


Maximum flow: $R \approx 61.713$ at $t = 9.36$.
[(18.861, 61.178) is a relative maximum.]

(b) Volume $= \int_0^{24} R(t)\, dt \approx 1272$ (5 thousand of gallons)

**108.** (a)

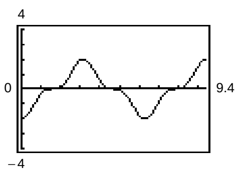



(b) $g$ is nonnegative because the graph of $f$ is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on $g$ that correspond to the extrema of $f$ are points of inflection of $g$.

(d) No, some zeros of $f$, like $x = \pi/2$, do not correspond to an extrema of $g$. The graph of $g$ continues to increase after $x = \pi/2$ because $f$ remains above the $x$-axis.

(e)




The graph of $h$ is that of $g$ shifted 2 units downward.

$$\begin{aligned} g(t) &= \int_0^t f(x)\,dx \\ &= \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^t f(x)\,dx = 2 + h(t). \end{aligned}$$

**110.** False

$$\int x(x^2+1)^2\,dx = \frac{1}{2}\int (x^2+1)(2x)\,dx = \frac{1}{4}(x^2+1)^2 + C$$

**112.** True

$$\int_a^b \sin x\,dx = \Big[-\cos x\Big]_a^b = -\cos b + \cos a = -\cos(b+2\pi) + \cos a = \int_a^{b+2\pi} \sin x\,dx$$

**114.** False

$$\int \sin^2 2x \cos 2x\,dx = \frac{1}{2}\int (\sin 2x)^2(2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^3}{3} + C = \frac{1}{6}\sin^3 2x + C$$

**116.** Because $f$ is odd, $f(-x) = -f(x)$. Then

$$\begin{aligned} \int_{-a}^a f(x)\,dx &= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \\ &= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx. \end{aligned}$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\begin{aligned} \int_{-a}^1 f(x)\,dx &= -\int_0^a f(-u)(-du) + \int_0^a f(x)\,dx \\ &= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0 \end{aligned}$$

## Section 4.6 Numerical Integration

**2.** Exact: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx = \left[\frac{x^3}{6}+x\right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx \approx \frac{1}{8}\left[1 + 2\left(\frac{(1/4)^2}{2}+1\right) + 2\left(\frac{(1/2)^2}{2}+1\right) + 2\left(\frac{(3/4)^2}{2}+1\right) + \left(\frac{1^2}{2}+1\right) = \frac{75}{64} \approx 1.1719\right.$

Simpson's: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx \approx \frac{1}{12}\left[1 + 4\left(\frac{(1/4)^2}{2}+1\right) + 2\left(\frac{(1/2)^2}{2}+1\right) + 4\left(\frac{(3/4)^2}{2}+1\right) + \left(\frac{1^2}{2}+1\right)\right] = \frac{7}{6} \approx 1.1667$

**4.** Exact: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^2 = 0.5000$

Trapezoidal: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{8}\left[1 + 2\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 2\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5090$

Simpson's: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 4\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5004$

**6.** Exact: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

**8.** Exact: $\displaystyle\int_1^3 (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\displaystyle\int_1^3 (4-x^2)\,dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] - 5\right\} = -0.7500$

Simpson's: $\displaystyle\int_1^3 (4-x^2)\,dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 5\right] \approx -0.6667$

**10.** Exact: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[(x^2+1)^{3/2}\right]_0^2 = \frac{1}{3}(5^{3/2}-1) \approx 3.393$

Trapezoidal: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.457$

Simpson's: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.392$

**12.** Trapezoidal: $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

**14.** Trapezoidal: $\displaystyle\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\displaystyle\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

**16.** Trapezoidal: $\displaystyle\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

$\approx 0.271$

Simpson's: $\displaystyle\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

$\approx 0.257$

Graphing utility: 0.256

**18.** Trapezoidal: $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{16}\left[\sqrt{2} + 2\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 2\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Simpson's: $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{24}\left[\sqrt{2} + 4\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 4\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Graphing utility: 1.910

**20.** Trapezoidal: $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

**22.** Trapezoidal: Linear polynomials

Simpson's: Quadratic polynomials

**24.** $f(x) = \dfrac{1}{x+1}$

$f'(x) = \dfrac{-1}{(x+1)^2}$

$f''(x) = \dfrac{2}{(x+1)^3}$

$f'''(x) = \dfrac{-6}{(x+1)^4}$

$f^{(4)}(x) = \dfrac{24}{(x+1)^5}$

(a) Trapezoidal: Error $\le \dfrac{(1-0)^2}{12(4^2)}(2) = \dfrac{1}{96} \approx 0.01$ since

$f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\le \dfrac{(1-0)^5}{180(4^4)}(24) = \dfrac{1}{1920} \approx 0.0005$

since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

**26.** $f''(x) = \dfrac{2}{(1+x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\leq \dfrac{1}{12n^2}(2) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

$f^{(4)}(x) = \dfrac{24}{(1+x)^5}$ in $[0, 1]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\leq \dfrac{1}{180n^4}(24) < 0.00001$, $n^4 > 13{,}333.33$, $n > 10.75$; let $n = 12$. (In Simpson's Rule $n$ must be even.)

**28.** $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\dfrac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{2}{9}$.

Trapezoidal: Error $\leq \dfrac{8}{12n^4}\left(\dfrac{2}{9}\right) < 0.00001$, $n^2 > 14{,}814.81$, $n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\dfrac{56}{81(x+1)^{10/3}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{56}{81}$.

Simpson's: Error $\leq \dfrac{32}{180n^4}\left(\dfrac{56}{81}\right) < 0.00001$, $n^4 > 12{,}290.81$, $n > 10.53$; let $n = 12$. (In Simpson's Rule $n$ must be even.)

**30.** $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2\sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\leq \dfrac{(1-0)^3}{12n^2}(2.2853) < 0.00001$, $n^2 > 19{,}044.17$, $n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\leq \dfrac{(1-0)^5}{180n^4}(28.4285) < 0.00001$, $n^4 > 15{,}793.61$, $n > 11.21$; let $n = 12$.

**32.** The program will vary depending upon the computer or programmable calculator that you use.

**34.** $f(x) = \sqrt{1-x^2}$ on $[0, 1]$.

| $n$ | $L(n)$ | $M(n)$ | $R(n)$ | $T(n)$ | $S(n)$ |
|---|---|---|---|---|---|
| 4 | 0.8739 | 0.7960 | 0.6239 | 0.7489 | 0.7709 |
| 8 | 0.8350 | 0.7892 | 0.7100 | 0.7725 | 0.7803 |
| 10 | 0.8261 | 0.7881 | 0.7261 | 0.7761 | 0.7818 |
| 12 | 0.8200 | 0.7875 | 0.7367 | 0.7783 | 0.7826 |
| 16 | 0.8121 | 0.7867 | 0.7496 | 0.7808 | 0.7836 |
| 20 | 0.8071 | 0.7864 | 0.7571 | 0.7821 | 0.7841 |

**36.** $f(x) = \dfrac{\sin x}{x}$ on $[1, 2]$.

| $n$ | $L(n)$ | $M(n)$ | $R(n)$ | $T(n)$ | $S(n)$ |
|---|---|---|---|---|---|
| 4 | 0.7070 | 0.6597 | 0.6103 | 0.6586 | 0.6593 |
| 8 | 0.6833 | 0.6594 | 0.6350 | 0.6592 | 0.6593 |
| 10 | 0.6786 | 0.6594 | 0.6399 | 0.6592 | 0.6593 |
| 12 | 0.6754 | 0.6594 | 0.6431 | 0.6593 | 0.6593 |
| 16 | 0.6714 | 0.6594 | 0.6472 | 0.6593 | 0.6593 |
| 20 | 0.6690 | 0.6593 | 0.6496 | 0.6593 | 0.6593 |

**38.** Simpson's Rule: $n = 8$

$$8\sqrt{3}\int_0^{\pi/2} \sqrt{1 - \frac{2}{3}\sin^2\theta\, d\theta} \approx \frac{\sqrt{3}\pi}{6}\left[\sqrt{1 - \frac{2}{3}\sin^2 0} + 4\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{2}}\right]$$
$$\approx 17.476$$

**40.** (a) Trapezoidal:

$$\int_0^2 f(x)\,dx \approx \frac{2}{2(8)}[4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x)\,dx \approx \frac{2}{3(8)}[4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844$$

Integrating, $\displaystyle\int_0^2 y\,dx \approx 12.53$

**42.** Simpson's Rule: $n = 6$

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\,dx \approx \frac{4}{3(6)}\left[1 + \frac{4}{1 + (1/6)^2} + \frac{2}{1 + (2/6)^2} + \frac{4}{1 + (3/6)^2} + \frac{2}{1 + (4/6)^2} + \frac{4}{1 + (5/6)^2} + \frac{1}{2}\right]$$
$$\approx 3.14159$$

**44.** Area $\approx \dfrac{120}{2(12)}[75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$

$= 7435$ sq m

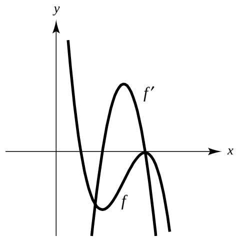
**46.** The quadratic polynomial

$$p(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}y_3$$

passes through the three points.

# Review Exercises for Chapter 4

**2.**




**4.** $u = 3x$

$du = 3\,dx$

$$\int \frac{2}{\sqrt[3]{3x}}\,dx = \frac{2}{3}\int (3x)^{-1/3}(3)\,dx = (3x)^{2/3} + C$$

**6.** $$\int \frac{x^3 - 2x^2 + 1}{x^2}\,dx = \int (x - 2 + x^{-2})\,dx$$

$$= \frac{1}{2}x^2 - 2x - \frac{1}{x} + C$$

**8.** $$\int (5\cos x - 2\sec^2 x)\,dx = 5\sin x - 2\tan x + C$$

**10.** $f''(x) = 6(x - 1)$

$$f'(x) = \int 6(x - 1)\,dx = 3(x - 1)^2 + C_1$$

Since the slope of the tangent line at $(2, 1)$ is 3, it follows that $f'(2) = 3 + C_1 = 3$ when $C_1 = 0$.

$$f'(x) = 3(x - 1)^2$$

$$f(x) = \int 3(x - 1)^2\,dx = (x - 1)^3 + C_2$$

$$f(2) = 1 + C_2 = 1 \text{ when } C_2 = 0.$$

$$f(x) = (x - 1)^3$$

**12.** 45 mph $=$ 66 ft/sec

30 mph $=$ 44 ft/sec

$a(t) = -a$

$v(t) = -at + 66$ since $v(0) = 66$ ft/sec.

$s(t) = -\frac{a}{2}t^2 + 66t$ since $s(0) = 0$.

Solving the system

$$v(t) = -at + 66 = 44$$

$$s(t) = -\frac{a}{2}t^2 + 66t = 264$$

we obtain $t = 24/5$ and $a = 55/12$. We now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. Thus,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mph to rest is

$$475.2 - 264 = 211.2 \text{ ft.}$$

**14.** $a(t) = -9.8$ m/sec$^2$

$v(t) = -9.8t + v_0 = -9.8t + 40$

$s(t) = -4.9t^2 + 40t \quad (s(0) = 0)$

(a) $v(t) = -9.8t + 40 = 0$ when $t = \dfrac{40}{9.8} \approx 4.08$ sec.

(b) $s(4.08) \approx 81.63$ m

(c) $v(t) = -9.8t + 40 = 20$ when $t = \dfrac{20}{9.8} \approx 2.04$ sec.

(d) $s(2.04) \approx 61.2$ m

**16.** $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3, x_5 = 7$

(a) $\frac{1}{5}\sum_{i=1}^{5} x_i = \frac{1}{5}(2 - 1 + 5 + 3 + 7) = \frac{16}{5}$

(b) $\sum_{i=1}^{5} \frac{1}{x_i} = \frac{1}{2} - 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{7} = \frac{37}{210}$

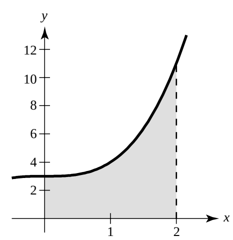
(c) $\sum_{i=1}^{5} (2x_i - x_i^2) = [2(2) - (2)^2] + [2(-1) - (-1)^2] + [2(5) - (5)^2] + [2(3) - (3)^2] + [2(7) - (7)^2] = -56$

(d) $\sum_{i=2}^{5} (x_i - x_{i-1}) = (-1 - 2) + [5 - (-1)] + (3 - 5) + (7 - 3) = 5$

**18.** $y = 9 - \frac{1}{4}x^2, \Delta x = 1, n = 4$

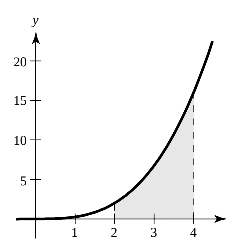
$$S(4) = 1\left[\left(9 - \frac{1}{4}(4)\right) + \left(9 - \frac{1}{4}(9)\right) + \left(9 - \frac{1}{4}(16)\right) + 9 - \frac{1}{4}(25)\right]$$
$$\approx 22.5$$
$$s(4) = 1\left[\left(9 - \frac{1}{4}(9)\right) + \left(9 - \frac{1}{4}(16)\right) + \left(9 - \frac{1}{4}(25)\right) + (9 - 9)\right]$$
$$\approx 14.5$$

**20.** $y = x^2 + 3, \Delta x = \frac{2}{n}$ right endpoints



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 3\right]\left(\frac{2}{n}\right)$$
$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\frac{4i^2}{n^2} + 3\right]$$
$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} + 3n\right]$$
$$= \lim_{n\to\infty} \left[\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} + 6\right] = \frac{8}{3} + 6 = \frac{26}{3}$$

**22.** $y = \frac{1}{4}x^3, \Delta x = \frac{2}{n}$



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{4}\left(2 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$$
$$= \lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{n}\left[8 + \frac{24i}{n} + \frac{24i^2}{n^2} + \frac{8i^3}{n^3}\right]$$
$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}\right]$$
$$= \lim_{n\to\infty} \frac{4}{n}\left[n + \frac{3}{n}\frac{n(n+1)}{2} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\frac{n^2(n+1)^2}{4}\right]$$
$$= 4 + 6 + 4 + 1 = 15$$

**24.** (a) $S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3 + 4) = \frac{5mb^2}{8}$

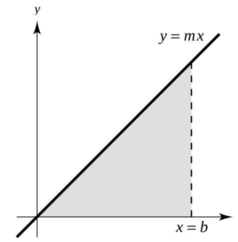
$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3) = \frac{3mb^2}{8}$

(b) $S(n) = \sum_{i=1}^{n} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^{n}\left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2\sum_{i=1}^{n} i = \frac{mb^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$

$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=0}^{n-1} i = \frac{mb^2}{n^2}\left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$

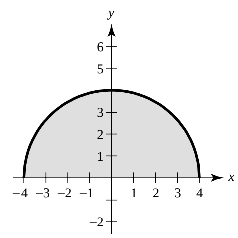
(c) Area $= \lim_{n\to\infty} \frac{mb^2(n+1)}{2n} = \lim_{n\to\infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}$(base)(height)

(d) $\int_0^b mx\,dx = \left[\frac{1}{2}mx^2\right]_0^b = \frac{1}{2}mb^2$




**26.** $\lim_{\|\Delta\|\to\infty} \sum_{i=1}^{n} 3c_i(9 - c_i^2)\,\Delta x_i = \int_1^3 3x(9 - x^2)\,dx$

**28.**




$\int_{-4}^{4} \sqrt{16 - x^2}\,dx = \frac{1}{2}\,\pi(4)^2 = 8\pi$ (semicircle)

**30.** (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_4^4 f(x)\,dx = 0$

(d) $\int_3^6 -10\,f(x)\,dx = -10\int_3^6 f(x)\,dx = -10(-1) = 10$

**32.** $\int_1^3 \frac{12}{x^3}\,dx = \left[\frac{12x^{-2}}{-2}\right]_1^3 = \left[\frac{-6}{x^2}\right]_1^3 = \frac{-6}{9} + 6 = \frac{16}{3}$, (d)

**34.** $\int_{-1}^{1} (t^2 + 2)\,dt = \left[\frac{t^3}{3} + 2t\right]_{-1}^{1} = \frac{14}{3}$

**36.** $\int_{-2}^{2} (x^4 + 2x^2 - 5)\,dx = \left[\frac{x^5}{5} + \frac{2x^3}{3} - 5x\right]_{-2}^{2}$

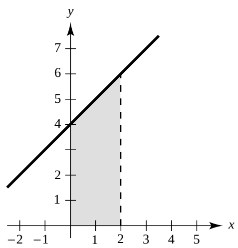
$$= \left(\frac{32}{5} + \frac{16}{3} - 10\right) - \left(-\frac{32}{5} - \frac{16}{3} + 10\right)$$

$$= \frac{52}{15}$$

**38.** $\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx = \int_1^2 (x^{-2} - x^{-3})\,dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8}\right) - \left(-1 + \frac{1}{2}\right) = \frac{1}{8}$

**40.** $\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt = \left[\tan t\right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

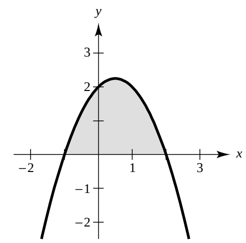
**42.** $\displaystyle\int_0^2 (x+4)\,dx = \left[\frac{x^2}{2} + 4x\right]_0^2 = 10$




**44.** $\displaystyle\int_{-1}^2 (-x^2 + x + 2)\,dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^2$

$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

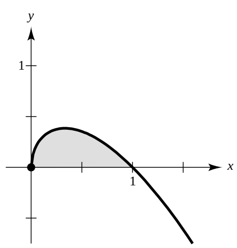
$$= \frac{10}{3} + \frac{7}{6} = \frac{9}{2}$$




**46.** $\displaystyle\int_0^1 \sqrt{x}(1-x)\,dx = (x^{1/2} - x^{3/2})\,dx$

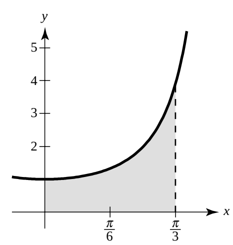
$$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$




**48.** Area $\displaystyle= \int_0^{\pi/3} \sec^2 x\,dx$

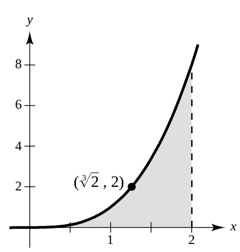
$$= \tan x\Big]_0^{\pi/3} = \sqrt{3}$$




**50.** $\displaystyle\frac{1}{2-0}\int_0^2 x^3\,dx = \left[\frac{x^4}{8}\right]_0^2 = 2$

$x^3 = 2$

$x = \sqrt[3]{2}$




**52.** $F'(x) = \dfrac{1}{x^2}$

**54.** $F'(x) = \csc^2 x$

**56.** $\displaystyle\int\left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + x^{-2})\,dx$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + C$$

**58.** $u = x^3 + 3,\ du = 3x^2\,dx$

$$\int x^2\sqrt{x^3+3}\,dx = \frac{1}{3}\int (x^3+3)^{1/2}\,3x^2\,dx = \frac{2}{9}(x^3+3)^{3/2} + C$$

**60.** $u = x^2 + 6x - 5,\ du = (2x+6)\,dx$

$$\int \frac{x+3}{(x^2+6x-5)^2} = \frac{1}{2}\int \frac{2x+6}{(x^2+6x-5)^2}\,dx = \frac{-1}{2}(x^2+6x-5)^{-1} + C = \frac{-1}{2(x^2+6x-5)} + C$$

**62.** $$\int x \sin 3x^2\,dx = \frac{1}{6}\int (\sin 3x^2)(6x)\,dx = -\frac{1}{6}\cos 3x^2 + C$$

**64.** $$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int (\sin x)^{-1/2}\cos x\,dx = 2(\sin x)^{1/2} + C = 2\sqrt{\sin x} + C$$

**66.** $$\int \sec 2x \tan 2x\,dx = \frac{1}{2}\int (\sec 2x \tan 2x)(2)\,dx = \frac{1}{2}\sec 2x + C$$

**68.** $$\int \cot^4 \alpha \csc^2 \alpha\,d\alpha = -\int (\cot \alpha)^4(-\csc^2 \alpha)\,d\alpha = -\frac{1}{5}\cot^5 \alpha + C$$

**70.** $$\int_0^1 x^2(x^3+1)^3\,dx = \frac{1}{3}\int_0^1 (x^3+1)^3(3x^2)\,dx = \frac{1}{12}\Big[(x^3+1)^4\Big]_0^1 = \frac{1}{12}(16-1) = \frac{5}{4}$$

**72.** $$\int_3^6 \frac{x}{3\sqrt{x^2-8}}\,dx = \frac{1}{6}\int_3^6 (x^2-8)^{-1/2}(2x)\,dx = \left[\frac{1}{3}(x^2-8)^{1/2}\right]_3^6 = \frac{1}{3}\left(2\sqrt{7}-1\right)$$

**74.** $u = x + 1,\ x = u - 1,\ dx = du$

When $x = -1$, $u = 0$. When $x = 0$, $u = 1$.

$$2\pi\int_{-1}^0 x^2\sqrt{x+1}\,dx = 2\pi\int_0^1 (u-1)^2\sqrt{u}\,du$$

$$= 2\pi\int_0^1 (u^{5/2} - 2u^{3/2} + u^{1/2})\,du = 2\pi\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 = \frac{32\pi}{105}$$

**76.** $\displaystyle\int_{-\pi/4}^{\pi/4} \sin 2x\,dx = 0$ since $\sin 2x$ is an odd function.

**78.** $u = 1 - x,\ x = 1 - u,\ dx = -du$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{1155}{32}x^3(1-x)^{3/2}\,dx = \frac{1155}{32}\int_{1-a}^{1-b} -(1-u)^3 u^{3/2}\,du$$

$$= \frac{1155}{32}\int_{1-a}^{1-b} (u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2})\,du = \frac{1155}{32}\left[\frac{2}{11}u^{11/2} - \frac{2}{3}u^{9/2} + \frac{6}{7}u^{7/2} - \frac{2}{5}u^{5/2}\right]_{1-a}^{1-b}$$

$$= \frac{1155}{32}\left[\frac{2u^{5/2}}{1155}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b}$$

(a) $$P_{0,\,0.25} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_1^{0.75} \approx 0.025 = 2.5\%$$

(b) $$P_{0.5,\,1} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{0.5}^{0} \approx 0.736 = 73.6\%$$

**80.** $\displaystyle\int_0^2 1.75 \sin \frac{\pi t}{2}\, dt = -\frac{2}{\pi}\left[1.75 \cos \frac{\pi t}{2}\right]_0^2 = -\frac{2}{\pi}(1.75)(-1 - 1) = \frac{7}{\pi} \approx 2.2282$ liters

Increase is

$$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048 \text{ liters.}$$

**82.** Trapezoidal Rule $(n = 4)$: $\displaystyle\int_0^1 \frac{x^{3/2}}{3 - x^2}\, dx \approx \frac{1}{8}\left[0 + \frac{2(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{2(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2}\right] \approx 0.172$

Simpson's Rule $(n = 4)$: $\displaystyle\int_0^1 \frac{x^{3/2}}{3 - x^2}\, dx \approx \frac{1}{12}\left[0 + \frac{4(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{4(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2}\right] \approx 0.166$

Graphing utility: 0.166

**84.** Trapezoidal Rule $(n = 4)$: $\displaystyle\int_0^\pi \sqrt{1 + \sin^2 x}\, dx \approx 3.820$

Simpson's Rule $(n = 4)$: 3.820

Graphing utility: 3.820

## Problem Solving for Chapter 4

**2.** (a) $F(x) = \displaystyle\int_2^x \sin t^2\, dt$

| $x$ | 0 | 1.0 | 1.5 | 1.9 | 2.0 | 2.1 | 2.5 | 3.0 | 4.0 | 5.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $F(x)$ | $-0.8048$ | $-0.4945$ | $-0.0265$ | 0.0611 | 0 | $-0.0867$ | $-0.3743$ | $-0.0312$ | $-0.0576$ | $-0.2769$ |

(b) $G(x) = \dfrac{1}{x - 2}\displaystyle\int_2^x \sin t^2\, dt$

| $x$ | 1.9 | 1.95 | 1.99 | 2.01 | 2.05 | 2.1 |
|---|---|---|---|---|---|---|
| $G(x)$ | $-0.6106$ | $-0.6873$ | $-0.7436$ | $-0.7697$ | $-0.8174$ | $-0.8671$ |

$\displaystyle\lim_{x\to 2} G(x) \approx -0.75$

(c) $$\begin{aligned} F'(2) &= \lim_{x\to 2} \frac{F(x) - F(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{1}{x - 2}\int_2^x \sin t^2\, dt \\ &= \lim_{x\to 2} G(x) \end{aligned}$$

Since $F'(x) = \sin x^2$, $F'(2) = \sin 4 = \displaystyle\lim_{x\to 2} G(x)$.

(**Note:** $\sin 4 \approx -0.7568$)

**4.** Let $d$ be the distance traversed and $a$ be the uniform acceleration.
We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{1}{2}at^2.$$

$s(t) = d$ when $t = \sqrt{\dfrac{2d}{a}}$.

The highest speed is $v = a\sqrt{\dfrac{2d}{a}} = \sqrt{2ad}$.

The lowest speed is $v = 0$.

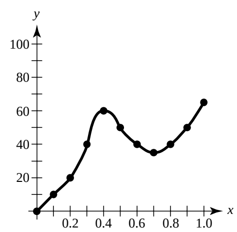
The mean speed is $\dfrac{1}{2}\left(\sqrt{2ad} + 0\right) = \sqrt{\dfrac{ad}{2}}$.

The time necessary to traverse the distance $d$ at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

**6.** (a)




(b) $v$ is increasing (positive acceleration) on $(0, 0.4)$ and $(0.7, 1.0)$.

(c) Average acceleration $= \dfrac{v(0.4) - v(0)}{0.4 - 0} = \dfrac{60 - 0}{0.4} = 150$ mi/hr$^2$

(d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t)\,dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

**8.** $\int_0^x f(t)(x-t)\,dt = \int_0^x xf(t)\,dt - \int_0^x tf(t)\,dt = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$

Thus, $\frac{d}{dx}\int_0^x f(t)(x-t)\,dt = xf(x) + \int_0^x f(t)\,dt - xf(x) = \int_0^x f(t)\,dt$

Differentiating the other integral,

$$\frac{d}{dx}\int_0^x \left(\int_0^x f(v)\,dv\right)dt = \int_0^x f(v)\,dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x-t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv\right)dt + C$$

Letting $x = 0$, we see that $C = 0$.

**10.** Consider $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$. The corresponding Riemann Sum using right-hand endpoints is

$$S(n) = \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right]$$

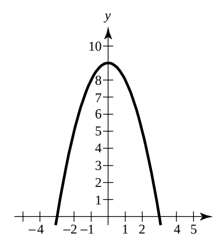
$$= \frac{1}{n^{3/2}}\left[\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right]$$

Thus, $\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}$.

**12.** (a) Area $= \int_{-3}^3 (9 - x^2)\,dx = 2\int_0^3 (9 - x^2)\,dx$

$$= 2\left[9x - \frac{x^3}{3}\right]_0^3$$

$$= 2[27 - 9] = 36$$




(b) Base $= 6$, height $= 9$. Area $= \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$.

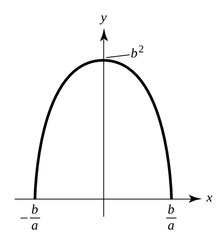
(c) Let the parabola be given by $y = b^2 - a^2x^2$, $a, b > 0$.

Area $= 2\int_0^{b/a} (b^2 - a^2x^2)\,dx$

$$= 2\left[b^2x - a^2\frac{x^3}{3}\right]_0^{b/a}$$

$$= 2\left[b^2\left(\frac{b}{a}\right) - \frac{a^2}{3}\left(\frac{b}{a}\right)^3\right]$$

$$= 2\left[\frac{b^3}{a} - \frac{1}{3}\frac{b^3}{a}\right] = \frac{4}{3}\frac{b^3}{a}$$




Base $= \frac{2b}{a}$, height $= b^2$

Archimedes' Formula: Area $= \frac{2}{3}\left(\frac{2b}{a}\right)(b^2) = \frac{4}{3}\frac{b^3}{a}$

**14.** (a) $(1 + i)^3 = 1 + 3i + 3i^2 + i^3 \implies (1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $3i^2 + 3i + 1 = (i + 1)^3 - i^3$

$$\sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}[(i + 1)^3 - i^3]$$
$$= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [((n + 1)^3 - n^3)]$$
$$= (n + 1)^3 - 1$$

Hence, $(n + 1)^3 = \sum_{i=1}^{n}(3i^2 + 3i + 1) + 1$

(c) $(n + 1)^3 - 1 = \sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}3i^2 + \dfrac{3(n)(n + 1)}{2} + n$

$$\implies \sum_{i=1}^{n}3i^2 = n^3 + 3n^2 + 3n - \frac{3n(n + 1)}{2} - n$$
$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$
$$= \frac{2n^3 + 3n^2 + n}{2}$$
$$= \frac{n(n + 1)(2n + 1)}{2}$$
$$\implies \sum_{i=1}^{n}i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

**16.** (a) $C = 0.1\displaystyle\int_8^{20}\left[12 \sin \frac{\pi(t - 8)}{12}\right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t - 8)}{12}\right]_8^{20} = \frac{-14.4}{\pi}(-1 - 1) \approx \$9.17$

(b) $C = 0.1\displaystyle\int_{10}^{18}\left[12 \sin \frac{\pi(t - 8)}{12} - 6\right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t - 8)}{12} - 0.6t\right]_{10}^{18}$

$$= \left[\frac{-14.4}{\pi}\left(\frac{-\sqrt{3}}{2}\right) - 10.8\right] - \left[\frac{-14.4}{\pi}\left(\frac{\sqrt{3}}{2}\right) - 6\right] \approx \$3.14$$

Savings $\approx 9.17 - 3.14 = \$6.03$.

**18.** (a) Let $A = \displaystyle\int_0^b \frac{f(x)}{f(x) + f(b - x)}\, dx$.

Let $u = b - x$, $du = -dx$.

$$A = \int_b^0 \frac{f(b - u)}{f(b - u) + f(u)}(-du)$$
$$= \int_0^b \frac{f(b - u)}{f(b - u) + f(u)}\, du$$
$$= \int_0^b \frac{f(b - x)}{f(b - x) + f(x)}\, dx$$

Then,

$$2A = \int_0^b \frac{f(x)}{f(x) + f(b - x)}\, dx + \int_0^b \frac{f(b - x)}{f(b - x) + f(x)}\, dx$$
$$= \int_0^b 1\, dx = b.$$

Thus, $A = \dfrac{b}{2}$.

(b) $b = 1 \implies \displaystyle\int_0^1 \frac{\sin x}{\sin(1 - x) + \sin x}\, dx = \frac{1}{2}$

# CHAPTER 5
# Logarithmic, Exponential, and Other Transcendental Functions

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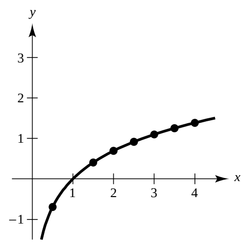
# Logarithmic, Exponential, and Other Transcendental Functions

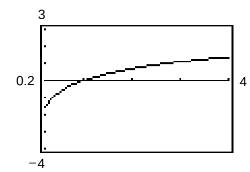
## Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Even-Numbered Exercises

**2.** (a)



(b)



The graphs are identical.

**4.** (a) $\ln 8.3 \approx 2.1163$

(b) $\displaystyle\int_1^{8.3} \frac{1}{t}\,dt \approx 2.1163$

**6.** (a) $\ln 0.6 \approx -0.5108$

(b) $\displaystyle\int_1^{0.6} \frac{1}{t}\,dt \approx -0.5108$

**8.** $f(x) = -\ln x$

Reflection in the $x$-axis

Matches (d)

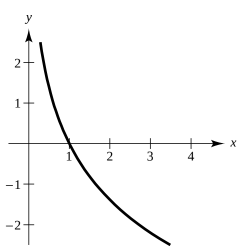
**10.** $f(x) = -\ln(-x)$

Reflection in the $y$-axis and the $x$-axis

Matches (c)

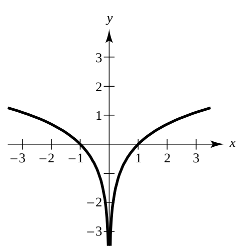
**12.** $f(x) = -2\ln x$

Domain: $x > 0$



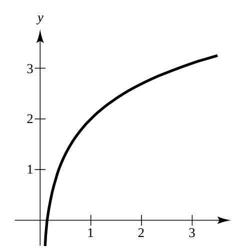
**14.** $f(x) = \ln|x|$

Domain: $x \neq 0$



**16.** $g(x) = 2 + \ln x$

Domain: $x > 0$



**18.** (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2\ln 2 \approx -1.3862$

(b) $\ln 24 = 3\ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2\ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3\ln 2 + 2\ln 3) \approx -4.2765$

**20.** $\ln\sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2}\ln 2$

**22.** $\ln xyz = \ln x + \ln y + \ln z$

**24.** $\ln\sqrt{a-1} = \ln(a-1)^{1/2} = \left(\frac{1}{2}\right)\ln(a-1)$

**26.** $\ln 3e^2 = \ln 3 + 2\ln e = 2 + \ln 3$

**28.** $\ln \dfrac{1}{e} = \ln 1 - \ln e = -1$

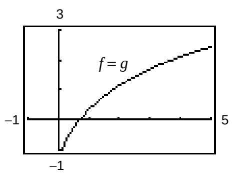
**30.** $3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$

$$= \ln \frac{x^3y^2}{z^4}$$

**32.** $2[\ln x - \ln(x + 1) - \ln(x - 1)] = 2 \ln \dfrac{x}{(x + 1)(x - 1)} = \ln\left(\dfrac{x}{x^2 - 1}\right)^2$

**34.** $\dfrac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)] = \dfrac{3}{2} \ln \dfrac{x^2 + 1}{(x + 1)(x - 1)} = \ln \sqrt{\left(\dfrac{x^2 + 1}{x^2 - 1}\right)^3}$

**36.**




**38.** $\lim\limits_{x \to 6^-} \ln(6 - x) = -\infty$

**40.** $\lim\limits_{x \to 5^+} \ln \dfrac{x}{\sqrt{x - 4}} = \ln 5 \approx 1.6094$

**42.** $y = \ln x^{3/2} = \dfrac{3}{2} \ln x$

$y' = \dfrac{3}{2x}$

At $(1, 0)$, $y' = \frac{3}{2}$.

**44.** $y = \ln x^{1/2} = \dfrac{1}{2} \ln x$

$y' = \dfrac{1}{2x}$

At $(1, 0)$, $y' = \frac{1}{2}$.

**46.** $h(x) = \ln(2x^2 + 1)$

$h'(x) = \dfrac{1}{2x^2 + 1}(4x) = \dfrac{4x}{2x^2 + 1}$

**48.** $y = x \ln x$

$\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + \ln x = 1 + \ln x$

**50.** $y = \ln\sqrt{x^2 - 4} = \dfrac{1}{2} \ln(x^2 - 4)$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{2x}{x^2 - 4}\right) = \dfrac{x}{x^2 - 4}$

**52.** $f(x) = \ln\left(\dfrac{2x}{x + 3}\right) = \ln 2x - \ln(x + 3)$

$f'(x) = \dfrac{1}{x} - \dfrac{1}{x + 3} = \dfrac{3}{x(x + 3)}$

**54.** $h(t) = \dfrac{\ln t}{t}$

$h'(t) = \dfrac{t(1/t) - \ln t}{t^2} = \dfrac{1 - \ln t}{t^2}$

**56.** $y = \ln(\ln x)$

$\dfrac{dy}{dx} = \dfrac{1/x}{\ln x} = \dfrac{1}{x \ln x}$

**58.** $y = \ln \sqrt[3]{\dfrac{x - 1}{x + 1}} = \dfrac{1}{3}[\ln(x - 1) - \ln(x + 1)]$

$y' = \dfrac{1}{3}\left[\dfrac{1}{x - 1} - \dfrac{1}{x + 1}\right] = \dfrac{1}{3}\,\dfrac{2}{x^2 - 1} = \dfrac{2}{3(x^2 - 1)}$

**60.** $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

$f'(x) = \dfrac{1}{x + \sqrt{4 + x^2}}\left(1 + \dfrac{x}{\sqrt{4 + x^2}}\right)$

$$= \frac{1}{\sqrt{4 + x^2}}$$

**62.** $y = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2+\sqrt{x^2+4}}{x}\right) = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(2+\sqrt{x^2+4}\right) + \dfrac{1}{4}\ln x$

$$\frac{dy}{dx} = \frac{-2x^2\left(x/\sqrt{x^2+4}\right) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4}\left(\frac{1}{2+\sqrt{x^2+4}}\right)\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$

Note that:

$$\frac{1}{2+\sqrt{x^2+4}} = \frac{1}{2+\sqrt{x^2+4}} \cdot \frac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \frac{2-\sqrt{x^2+4}}{-x^2}$$

$$\text{Hence, } \frac{dy}{dx} = \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4}\frac{\left(2-\sqrt{x^2+4}\right)}{-x^2}\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$

$$= \frac{-1 + (1/2)\left(2-\sqrt{x^2+4}\right)}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x}$$

$$= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3}$$

**64.** $y = \ln|\csc x|$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

**66.** $y = \ln|\sec x + \tan x|$

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

**68.** $y = \ln\sqrt{1+\sin^2 x} = \dfrac{1}{2}\ln(1+\sin^2 x)$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)\frac{2\sin x\cos x}{1+\sin^2 x} = \frac{\sin x \cos x}{1+\sin^2 x}$$

**70.** $g(x) = \displaystyle\int_1^{\ln x} (t^2+3)\,dt$

$$g'(x) = \left[(\ln x)^2 + 3\right]\frac{d}{dx}(\ln x) = \frac{(\ln x)^2+3}{x}$$

(Second Fundamental Theorem of Calculus)

**72.** (a) $y = 4 - x^2 - \ln\left(\dfrac{1}{2}x + 1\right), \quad (0, 4)$

$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x+1}\left(\frac{1}{2}\right)$$

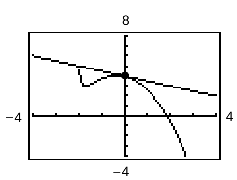
$$= -2x - \frac{1}{x+2}$$

When $x = 0$, $\dfrac{dy}{dx} = -\dfrac{1}{2}$.

Tangent line: $y - 4 = -\dfrac{1}{2}(x-0)$

$$y = -\frac{1}{2}x + 4$$

(b)




**74.** $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y+5xy}{x}\right)$$

**76.**
$$y = x(\ln x) - 4x$$
$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$
$$(x + y) - xy' = x + x\ln x - 4x - x(-3 + \ln x) = 0$$

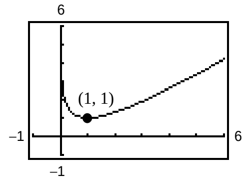
**78.** $y = x - \ln x$

Domain: $x > 0$

$y' = 1 - \dfrac{1}{x} = 0$ when $x = 1$.

$y'' = \dfrac{1}{x^2} > 0$

Relative minimum: $(1, 1)$




**80.** $y = \dfrac{\ln x}{x}$

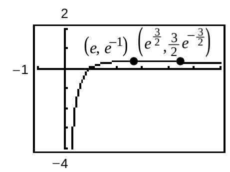
Domain: $x > 0$

$y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

$y'' = \dfrac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \dfrac{2(\ln x) - 3}{x^3} = 0$ when $x = e^{3/2}$.

Relative maximum: $(e, e^{-1})$

Point of inflection: $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right)$




**82.** $y = x^2 \ln \dfrac{x}{4}$. Domain $x > 0$

$y' = x^2\left(\dfrac{1}{x}\right) + 2x\ln\dfrac{x}{4} = x\left(1 + 2\ln\dfrac{x}{4}\right) = 0$ when

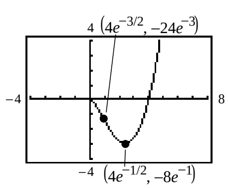
$-1 = 2\ln\dfrac{x}{4} \Rightarrow \ln\dfrac{x}{4} = -\dfrac{1}{2} \Rightarrow x = 4e^{-1/2}$

$y'' = 1 + 2\ln\dfrac{x}{4} + 2x\left(\dfrac{1}{x}\right) = 3 + 2\ln\dfrac{x}{4}$

$y'' = 0$ when $x = 4e^{-3/2}$

Relative minimum: $(4e^{-1/2}, -8e^{-1})$

Point of inflection: $(4e^{-3/2}, -24e^{-3})$




**84.** $f(x) = x\ln x, \quad f(1) = 0$

$f'(x) = 1 + \ln x, \quad f(1) = 1$

$f''(x) = \dfrac{1}{x}, \quad f''(1) = 1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \dfrac{1}{2}f''(1)(x - 1)^2$

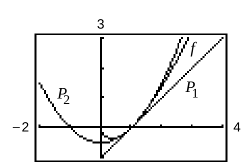
$\quad = (x - 1) + \dfrac{1}{2}(x - 1)^2, \quad P_2(1) = 0$

$P_1{}'(x) = 1, \quad P_1{}'(1) = 1$

$P_2{}'(x) = 1 + (x - 1) = x, \quad P_2{}'(1) = 1$

$P_2''(x) = x, \quad P_2''(1) = 1$

The values of $f$, $P_1$, $P_2$, and their first derivatives agree at $x = 1$. The values of the second derivatives of $f$ and $P_2$ agree at $x = 1$.



**86.** Find $x$ such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

| $n$ | 1 | 2 | 3 |
|---|---|---|---|
| $x_n$ | 2 | 2.2046 | 2.2079 |
| $f(x_n)$ | $-0.3069$ | $-0.0049$ | 0.0000 |

Approximate root: $x = 2.208$

**88.** $y = \sqrt{(x-1)(x-2)(x-3)}$

$$\ln y = \frac{1}{2}[\ln(x-1) + \ln(x-2) + \ln(x-3)]$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right]$$

$$= \frac{1}{2}\left[\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\right]$$

$$\frac{dy}{dx} = \frac{3x^2 - 12x + 11}{2y}$$

$$= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}}$$

**90.** $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

$$\ln y = \frac{1}{2}[\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1}\right]$$

$$\frac{dy}{dx} = \sqrt{\frac{x^2 - 1}{x^2 + 1}}\left[\frac{2x}{x^4 - 1}\right]$$

$$= \frac{(x^2 - 1)^{1/2}2x}{(x^2 + 1)^{1/2}(x^2 - 1)(x^2 + 1)}$$

$$= \frac{2x}{(x^2 + 1)^{3/2}(x^2 - 1)^{1/2}}$$

**92.** $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$

$$\ln y = \ln(x+1) + \ln(x+2) - \ln(x-1) - \ln(x-2)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x-1} - \frac{1}{x-2}$$

$$\frac{dy}{dx} = y\left[\frac{-2}{x^2 - 1} + \frac{-4}{x^2 - 4}\right] = y\left[\frac{-6x^2 + 12}{(x^2 - 1)(x^2 - 4)}\right]$$

$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{-6(x^2 - 2)}{(x+1)(x-1)(x+2)(x-2)}$$

$$= -\frac{6(x^2 - 2)}{(x-1)^2(x-2)^2}$$

**94.** The base of the natural logarithmic function is $e$.

**96.** $g(x) = \ln f(x)$, $f(x) > 0$

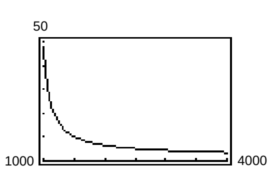
$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of $g$ is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of $f$ is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2 + 1)$ is not concave up.

**98.** $t = \dfrac{5.315}{-6.7968 + \ln x}$, $1000 < x$

(a)




(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280{,}178.40$$

(c) $t(1068.45) \approx 30$ years

$$T = (1068.45)(30)(12) = \$384{,}642.00$$

(d) $\dfrac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2}\left(\dfrac{1}{x}\right)$

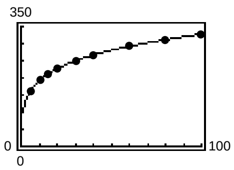
$$= -\frac{5.315}{x(-6.7968 + \ln x)^2}$$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

(e) There are two obvious benefits to paying a higher monthly payment:

1. The term is lower
2. The total amount paid is lower.

**100.** (a)

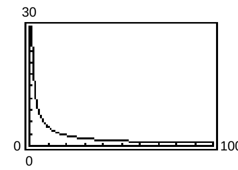



(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75 \text{ deg/lb/in}^2$

$T'(70) \approx 0.97 \text{ deg/lb/in}^2$

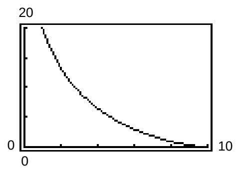
(c)




$\lim\limits_{p\to\infty} T'(p) = 0$

**102.** $y = 10\ln\left(\dfrac{10+\sqrt{100-x^2}}{x}\right) - \sqrt{100-x^2} = 10\left[\ln\left(10+\sqrt{100-x^2}\right) - \ln x\right] - \sqrt{100-x^2}$

(a)




(c) $\lim\limits_{x\to 10^-} \dfrac{dy}{dx} = 0$

(b) $$\frac{dy}{dx} = 10\left[\frac{-x}{\sqrt{100-x^2}\left(10+\sqrt{100-x^2}\right)} - \frac{1}{x}\right] + \frac{x}{\sqrt{100-x^2}}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{-10}{10+\sqrt{100-x^2}}\right] - \frac{10}{x} + \frac{x}{\sqrt{100-x^2}}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{-10}{10+\sqrt{100-x^2}} + 1\right] - \frac{10}{x}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{\sqrt{100-x^2}}{10+\sqrt{100-x^2}}\right] - \frac{10}{x}$$

$$= \frac{x}{10+\sqrt{100-x^2}} - \frac{10}{x}$$

$$= \frac{x\left(10-\sqrt{100-x^2}\right)}{x^2} - \frac{10}{x} = -\frac{\sqrt{100-x^2}}{x}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

**104.** $y = \ln x$

$y' = \dfrac{1}{x} > 0$ for $x > 0$.

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

**106.** False

$\pi$ is a constant.

$\dfrac{d}{dx}[\ln \pi] = 0$

## Section 5.2 The Natural Logarithmic Function: Integration

**2.** $\displaystyle\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

**4.** $u = x - 5$, $du = dx$

$$\int \frac{1}{x-5}\,dx = \ln|x-5| + C$$

**6.** $$\int \frac{1}{3x+2}\,dx = \frac{1}{3}\int \frac{1}{3x+2}(3)\,dx$$

$$= \frac{1}{3}\ln|3x+2| + C$$

**8.** $u = 3 - x^3$, $du = -3x^2\,dx$

$$\int \frac{x^2}{3-x^3}\,dx = -\frac{1}{3}\int \frac{1}{3-x^3}(-3x^2)\,dx$$

$$= -\frac{1}{3}\ln|3-x^3| + C$$

**10.** $u = 9 - x^2,\ du = -2x\,dx$

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int (9-x^2)^{-1/2}(-2x)\,dx = -\sqrt{9-x^2} + C$$

**12.** $$\int \frac{x(x+2)}{x^3+3x^2-4}\,dx = \frac{1}{3}\int \frac{3x^2+6x}{x^3+3x^2-4}\,dx \quad (u = x^3+3x^2-4)$$

$$= \frac{1}{3}\ln|x^3+3x^2-4| + C$$

**14.** $$\int \frac{2x^2+7x-3}{x-2}\,dx = \int\left(2x+11+\frac{19}{x-2}\right)dx$$

$$= x^2 + 11x + 19\ln|x-2| + C$$

**16.** $$\int \frac{x^3-6x-20}{x+5}\,dx = \int\left(x^2-5x+19-\frac{115}{x+5}\right)dx$$

$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115\ln|x+5| + C$$

**18.** $$\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx = \int\left(-3+x+\frac{x}{x^2+3}\right)dx$$

$$= -3x + \frac{x^2}{2} + \frac{1}{2}\ln(x^2+3) + C$$

**20.** $$\int \frac{1}{x\ln(x^3)}\,dx = \frac{1}{3}\int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{3}\ln|\ln|x|| + C$$

**22.** $u = 1 + x^{1/3},\ du = \dfrac{1}{3x^{2/3}}\,dx$

$$\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx = 3\int \frac{1}{1+x^{1/3}}\left(\frac{1}{3x^{2/3}}\right)dx$$

$$= 3\ln|1+x^{1/3}| + C$$

**24.** $$\int \frac{x(x-2)}{(x-1)^3}\,dx = \int \frac{x^2-2x+1-1}{(x-1)^3}\,dx$$

$$= \int \frac{(x-1)^2}{(x-1)^3}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$$

**26.** $u = 1 + \sqrt{3x},\ du = \dfrac{3}{2\sqrt{3x}}dx \implies dx = \dfrac{2}{3}(u-1)\,du$

$$\int \frac{1}{1+\sqrt{3x}}\,dx = \int \frac{1}{u}\,\frac{2}{3}(u-1)\,du$$

$$= \frac{2}{3}\int\left(1-\frac{1}{u}\right)du$$

$$= \frac{2}{3}[u - \ln|u|] + C$$

$$= \frac{2}{3}\left[1+\sqrt{3x} - \ln\left(1+\sqrt{3x}\right)\right] + C$$

$$= \frac{2}{3}\sqrt{3x} - \frac{2}{3}\ln\left(1+\sqrt{3x}\right) + C_1$$

**28.** $u = x^{1/3} - 1, du = \dfrac{1}{3x^{2/3}}\,dx \implies dx = 3(u+1)^2\,du$

$$\begin{aligned}\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx &= \int \frac{u+1}{u}3(u+1)^2\,du\\ &= 3\int \frac{u+1}{u}(u^2+2u+1)\,du\\ &= 3\int\left(u^2+3u+3+\frac{1}{u}\right)du\\ &= 3\left[\frac{u^3}{3}+\frac{3u^2}{2}+3u+\ln|u|\right]+C\\ &= 3\left[\frac{(x^{1/3}-1)^3}{3}+\frac{3(x^{1/3}-1)^2}{2}+3(x^{1/3}-1)+\ln|x^{1/3}-1|\right]+C\\ &= 3\ln|x^{1/3}-1|+\frac{3x^{2/3}}{2}+3x^{1/3}+x+C_1\end{aligned}$$

**30.** $$\begin{aligned}\int \tan 5\theta\,d\theta &= \frac{1}{5}\int\frac{5\sin 5\theta}{\cos 5\theta}d\theta\\ &= -\frac{1}{5}\ln|\cos 5\theta| + C\end{aligned}$$

**32.** $$\begin{aligned}\int \sec\frac{x}{2}\,dx &= 2\int \sec\frac{x}{2}\left(\frac{1}{2}\right)dx\\ &= 2\ln\left|\sec\frac{x}{2}+\tan\frac{x}{2}\right|+C\end{aligned}$$

**34.** $u = \cot t, du = -\csc^2 t\,dt$

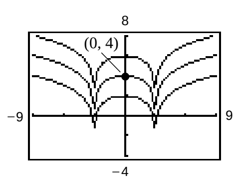
$$\int\frac{\csc^2 t}{\cot t}dt = -\ln|\cot t| + C$$

**36.** $$\begin{aligned}\int(\sec t+\tan t)\,dt &= \ln|\sec t+\tan t| - \ln|\cos t| + C\\ &= \ln\left|\frac{\sec t+\tan t}{\cos t}\right| + C\\ &= \ln|\sec t(\sec t+\tan t)| + C\end{aligned}$$

**38.** $$\begin{aligned}y &= \int\frac{2x}{x^2-9}\,dx\\ &= \ln|x^2-9| + C\end{aligned}$$

$(0, 4)$: $4 = \ln|0-9| + C \implies C = 4 - \ln 9$

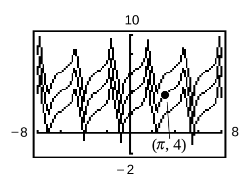
$y = \ln|x^2-9| + 4 - \ln 9$




**40.** $$\begin{aligned}r &= \int\frac{\sec^2 t}{\tan t+1}\,dt\\ &= \ln|\tan t+1| + C\end{aligned}$$

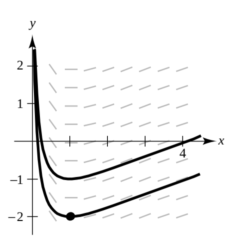
$(\pi, 4)$: $4 = \ln|0+1| + C \implies C = 4$

$r = \ln|\tan t+1| + 4$




**42.** $\dfrac{dy}{dx} = \dfrac{\ln x}{x}$, $(1, -2)$

(a)




(b) $y = \displaystyle\int\frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$

$y(1) = -2 \implies -2 = \dfrac{(\ln 1)^2}{2} + C \implies C = -2$

Hence, $y = \dfrac{(\ln x)^2}{2} - 2.$

**44.** $\displaystyle\int_{-1}^{1} \frac{1}{x+2}\,dx = \Big[\ln|x+2|\Big]_{-1}^{1}$

$= \ln 3 - \ln 1 = \ln 3$

**46.** $u = \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_{e}^{e^2} \frac{1}{x\ln x}\,dx = \int_{e}^{e^2}\left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x||\Big]_{e}^{e^2} = \ln 2$

**48.** $\displaystyle\int_{0}^{1} \frac{x-1}{x+1}\,dx = \int_{0}^{1} 1\,dx + \int_{0}^{1}\frac{-2}{x+1}\,dx$

$= \Big[x - 2\ln|x+1|\Big]_{0}^{1} = 1 - 2\ln 2$

**50.** $\displaystyle\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta = \int_{0.1}^{0.2}(\csc^2 2\theta - 2\csc 2\theta\cot 2\theta + \cot^2 2\theta)\,d\theta$

$\displaystyle= \int_{0.1}^{0.2}(2\csc^2 2\theta - 2\csc 2\theta\cot 2\theta - 1)\,d\theta$

$= \Big[-\cot 2\theta + \csc 2\theta - \theta\Big]_{0.1}^{0.2} \approx 0.0024$

**52.** $\ln|\sin x| + C = \ln\left|\dfrac{1}{\csc x}\right| + C = -\ln|\csc x| + C$

**54.** $-\ln|\csc x + \cot x| + C = -\ln\left|\dfrac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C = -\ln\left|\dfrac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C$

$= -\ln\left|\dfrac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C$

**56.** $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx = -(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - 4\ln(1+\sqrt{x}) + C_1$

$= 4\sqrt{x} - x - 4\ln(1+\sqrt{x}) + C$ where $C = C_1 + 5$.

**58.** $\displaystyle\int \frac{\tan^2 2x}{\sec 2x}\,dx = \frac{1}{2}[\ln|\sec 2x + \tan 2x| - \sin 2x] + C$

**60.** $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \Big[\ln|\sec x + \tan x| - 2\sin x\Big]_{-\pi/4}^{\pi/4}$

$= \ln\left(\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2} \approx -1.066$

**Note: In Exercises 62 and 64, you can use the Second Fundamental Theorem of Calculus or integrate the function.**

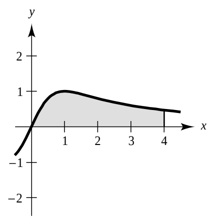
**62.** $\displaystyle F(x) = \int_{0}^{x} \tan t\,dt$

$F'(x) = \tan x$

**64.** $\displaystyle F(x) = \int_{1}^{x^2} \frac{1}{t}\,dt$

$F'(x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

**66.**



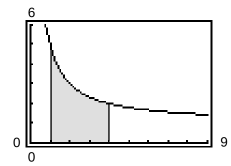

$A \approx 3$
Matches (a)

**68.** $A = \int_1^4 \frac{x+4}{x}\,dx = \int_1^4 \left(1 + \frac{4}{x}\right)dx$

$$= \Big[x + 4\ln x\Big]_1^4$$

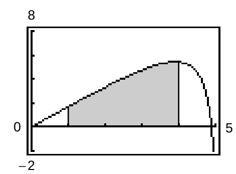
$$= 4 + 4\ln 4 - 1$$

$$= 3 + 4\ln 4 \approx 8.5452$$




**70.** $\int_1^4 (2x - \tan(0.3x))\,dx = \left[x^2 + \frac{10}{3}\ln|\cos(0.3x)|\right]_1^4$

$$= \left[16 + \frac{10}{3}\ln\cos(1.2)\right] - \left[1 + \frac{10}{3}\ln\cos(0.3)\right] \approx 11.7686$$




**72.** Substitution: $(u = x^2 + 4)$ and Power Rule

**74.** Substitution: $(u = \tan x)$ and Log Rule

**76.** Answers will vary.

**78.** Average value $= \frac{1}{4-2}\int_2^4 \frac{4(x+1)}{x^2}\,dx$

$$= 2\int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= 2\left[\ln x - \frac{1}{x}\right]_2^4$$

$$= 2\left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right]$$

$$= 2\left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863$$

**80.** Average value $= \frac{1}{2-0}\int_0^2 \sec\frac{\pi x}{6}\,dx$

$$= \left[\frac{1}{2}\left(\frac{6}{\pi}\right)\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$$

$$= \frac{3}{\pi}\left[\ln\left(2+\sqrt{3}\right) - \ln(1+0)\right]$$

$$= \frac{3}{\pi}\ln\left(2+\sqrt{3}\right)$$

**82.** $t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T-100}\,dT$

$$= \frac{10}{\ln 2}\Big[\ln(T-100)\Big]_{250}^{300} = \frac{10}{\ln 2}[\ln 200 - \ln 150] = \frac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504 \text{ units of time}$$

**84.** $\frac{dS}{dt} = \frac{k}{t}$

$S(t) = \int \frac{k}{t}\,dt = k\ln|t| + C = k\ln t + C$ since $t > 1$.

$S(2) = k\ln 2 + C = 200$

$S(4) = k\ln 4 + C = 300$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

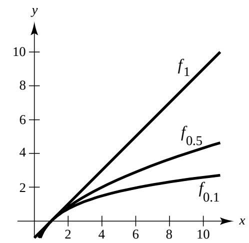
$$S(t) = \frac{100\ln t}{\ln 2} + 100 = 100\left[\frac{\ln t}{\ln 2} + 1\right].$$

**86.** $k = 1$: $f_1(x) = x - 1$

$k = 0.5$: $f_{0.5}(x) = \dfrac{\sqrt{x} - 1}{0.5} = 2\left(\sqrt{x} - 1\right)$

$k = 0.1$: $f_{0.1}(x) = \dfrac{\sqrt[10]{x} - 1}{0.1} = 10\left(\sqrt[10]{x} - 1\right)$

$\lim\limits_{k \to 0^+} f_k(x) = \ln x$




**88.** False

$\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$

**90.** False; the integrand has a nonremovable discontinuity at $x = 0$.

## Section 5.3 Inverse Functions

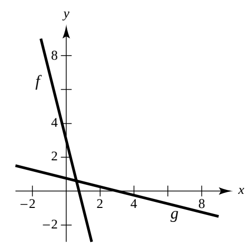
**2.** (a) $f(x) = 3 - 4x$

$g(x) = \dfrac{3 - x}{4}$

$f(g(x)) = f\left(\dfrac{3 - x}{4}\right) = 3 - 4\left(\dfrac{3 - x}{4}\right) = x$

$g(f(x)) = g(3 - 4x) = \dfrac{3 - (3 - 4x)}{4} = x$

(b)




**4.** (a) $f(x) = 1 - x^3$

$g(x) = \sqrt[3]{1 - x}$

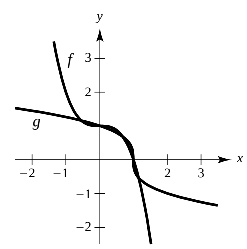
$f(g(x)) = f\left(\sqrt[3]{1 - x}\right) = 1 - \left(\sqrt[3]{1 - x}\right)^3$

$= 1 - (1 - x) = x$

$g(f(x)) = g(1 - x^3)$

$= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x$

(b)




**6.** (a) $f(x) = 16 - x^2,\ x \ge 0$

$g(x) = \sqrt{16 - x}$

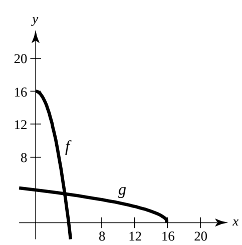
$f(g(x)) = f\left(\sqrt{16 - x}\right) = 16 - \left(\sqrt{16 - x}\right)^2$

$= 16 - (16 - x) = x$

$g(f(x)) = g(16 - x^2) = \sqrt{16 - (16 - x^2)}$

$= \sqrt{x^2} = x$

(b)



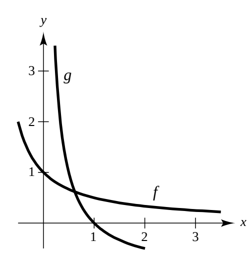

**8.** (a) $f(x) = \dfrac{1}{1 + x},\ x \ge 0$

$g(x) = \dfrac{1 - x}{x},\ 0 < x \le 1$

$f(g(x)) = f\left(\dfrac{1 - x}{x}\right) = \dfrac{1}{1 + \dfrac{1 - x}{x}} = \dfrac{1}{\dfrac{1}{x}} = x$

$g(f(x)) = g\left(\dfrac{1}{1 + x}\right) = \dfrac{1 - \dfrac{1}{1 + x}}{\dfrac{1}{1 + x}} = \dfrac{x}{1 + x} \cdot \dfrac{1 + x}{1} = x$

(b)

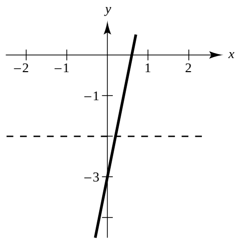


**10.** Matches (b)

**12.** Matches (d)

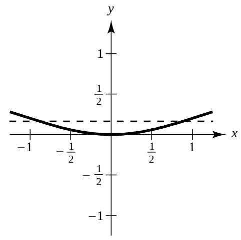
**14.** $f(x) = 5x - 3$

One-to-one; has an inverse



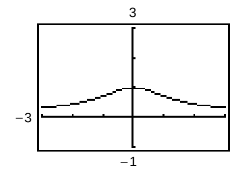

**16.** $F(x) = \dfrac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



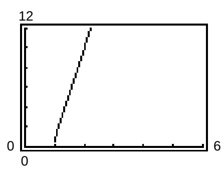

**18.** $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



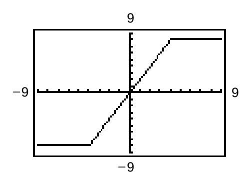

**20.** $f(x) = 5x\sqrt{x - 1}$

One-to-one; has an inverse




**22.** $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse




**24.** $f(x) = \cos\dfrac{3x}{2}$

$f'(x) = -\dfrac{3}{2}\sin\dfrac{3x}{2} = 0$ when $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$

$f$ is not strictly monotonic on $(-\infty, \infty)$. Therefore, $f$ does not have an inverse.

**26.** $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0$ for all $x$.

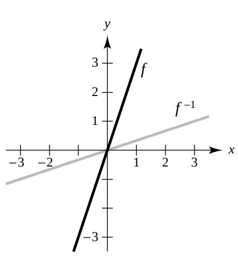
$f$ is increasing on $(-\infty, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**28.** $f(x) = \ln(x - 3),\ x > 3$

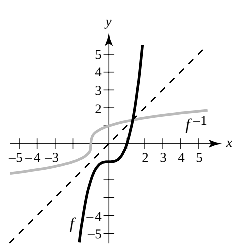
$f'(x) = \dfrac{1}{x - 3} > 0$ for $x > 3$.

$f$ is increasing on $(3, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

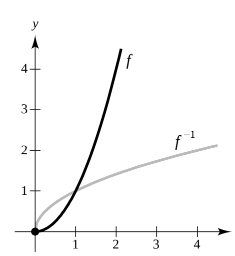
**30.**
$$f(x) = 3x = y$$
$$x = \frac{y}{3}$$
$$y = \frac{x}{3}$$
$$f^{-1}(x) = \frac{x}{3}$$




**32.**
$$f(x) = x^3 - 1 = y$$
$$x = \sqrt[3]{y + 1}$$
$$y = \sqrt[3]{x + 1}$$
$$f^{-1}(x) = \sqrt[3]{x + 1}$$




**34.**
$$f(x) = x^2 = y,\ 0 \leq x$$
$$x = \sqrt{y}$$
$$y = \sqrt{x}$$
$$f^{-1}(x) = \sqrt{x}$$

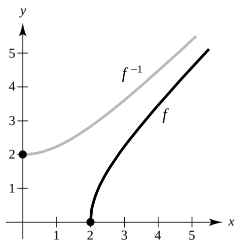


**36.** $f(x) = \sqrt{x^2 - 4} = y,\ x \geq 2$

$x = \sqrt{y^2 + 4}$

$y = \sqrt{x^2 + 4}$

$f^{-1}(x) = \sqrt{x^2 + 4},\ x \geq 0$

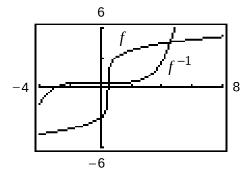



**38.** $f(x) = 3\sqrt[5]{2x - 1} = y$

$x = \dfrac{y^5 + 243}{486}$

$y = \dfrac{x^5 + 243}{486}$

$f^{-1}(x) = \dfrac{x^5 + 243}{486}$



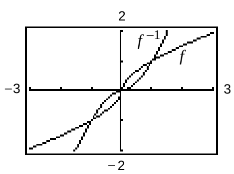

The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

**40.** $f(x) = x^{3/5} = y$

$x = y^{5/3}$

$y = x^{5/3}$

$f^{-1}(x) = x^{5/3}$



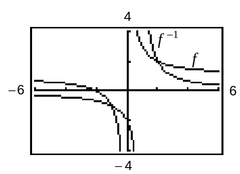

The graphs of f and $f^{-1}$ are reflections of each other across the line $y = x$.

**42.** $f(x) = \dfrac{x + 2}{x} = y$

$x = \dfrac{2}{y - 1}$

$y = \dfrac{2}{x - 1}$

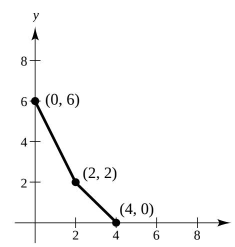
$f^{-1}(x) = \dfrac{2}{x - 1}$




The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

**44.**

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $f^{-1}(x)$ | 6 | 2 | 0 |




**46.** $f(x) = k(2 - x - x^3)$ is one-to-one for all $k \neq 0$. Since $f^{-1}(3) = -2$, $f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \implies k = \frac{1}{4}$.

**48.** $f(x) = |x + 2|$ on $[-2, \infty)$

$f'(x) = \dfrac{|x + 2|}{x + 2}(1) = 1 > 0$ on $(-2, \infty)$

$f$ is increasing on $[-2, \infty)$. Therefore, $f$ is strictly monotonic and has an inverse.

**50.** $f(x) = \cot x$ on $(0, \pi)$

$f'(x) = -\csc^2 x < 0$ on $(0, \pi)$

$f$ is decreasing on $(0, \pi)$. Therefore, $f$ is strictly monotonic and has an inverse.

**52.** $f(x) = \sec x$ on $\left[0, \dfrac{\pi}{2}\right)$

$f'(x) = \sec x \tan x > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

$f$ is increasing on $[0, \pi/2)$. Therefore, $f$ is strictly monotonic and has an inverse.

**54.** $f(x) = 2 - \dfrac{3}{x^2} = y$ on $(0, 10)$

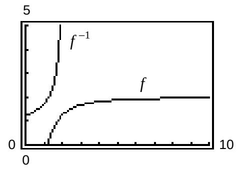
$2x^2 - 3 = x^2 y$

$x^2(2 - y) = 3$

$x = \pm\sqrt{\dfrac{3}{2 - y}}$

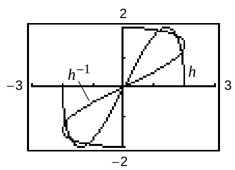
$y = \pm\sqrt{\dfrac{3}{2 - x}}$

$f^{-1}(x) = \sqrt{\dfrac{3}{2 - x}},\ x < 2$



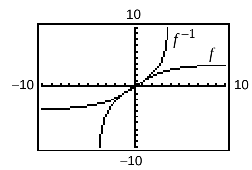

The graphs of $f$ and $f^{-1}$ are reflections of each other across the line $y = x$.

**56.** (a), (b)




(c) $h$ is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

**58.** (a), (b)




(c) Yes, $f$ is one-to-one and has an inverse. The inverse relation is an inverse function.

**60.** $f(x) = -3$

Not one-to-one; does not have an inverse

**62.** $f(x) = ax + b$

$f$ is one-to-one; has an inverse

$ax + b = y$

$x = \dfrac{y - b}{a}$

$y = \dfrac{x - b}{a}$

$f^{-1}(x) = \dfrac{x - b}{a},\ a \neq 0$

**64.** $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$16 - x^4 = y$

$16 - y = x^4$

$\sqrt[4]{16 - y} = x$

$\sqrt[4]{16 - x} = y$

$f^{-1}(x) = \sqrt[4]{16 - x},\ x \leq 16$

**66.** $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$x - 3 = y$

$x = y + 3$

$y = x + 3$

$f^{-1}(x) = x + 3,\ x \geq 0$

**68.** No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

**70.** Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius $r$ corresponding to the area $A$.

**72.** $f(x) = \dfrac{1}{27}(x^5 + 2x^3);\ f(-3) = \dfrac{1}{27}(-243 - 54) = -11 = a.$

$f'(x) = \dfrac{1}{27}(5x^4 + 6x^2)$

$(f^{-1})'(-11) = \dfrac{1}{f'(f^{-1}(-11))} = \dfrac{1}{f'(-3)} = \dfrac{27}{5(-3)^4 + 6(-3)^2} = \dfrac{1}{17}$

**74.** $f(x) = \cos 2x, f(0) = 1 = a$

$f'(x) = -2 \sin 2x$

$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{-2 \sin 0} = \dfrac{1}{0}$ which is undefined.

**76.** $f(x) = \sqrt{x-4}, f(8) = 2 = a$

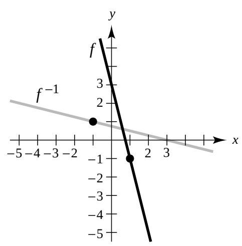
$f'(x) = \dfrac{1}{2\sqrt{x-4}}$

$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(8)} = \dfrac{1}{1/\left(2\sqrt{8-4}\right)} = \dfrac{1}{1/4} = 4$

**78.** (a) Domain $f$ = Domain $f^{-1} = (-\infty, \infty)$

(b) Range $f$ = Range $f^{-1} = (-\infty, \infty)$

(c)




(d) $f(x) = 3 - 4x, \ (1, -1)$

$f'(x) = -4$

$f'(1) = -4$

$f^{-1}(x) = \dfrac{3-x}{4}, \ (-1, 1)$

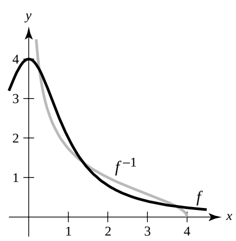
$(f^{-1})'(x) = -\dfrac{1}{4}$

$(f^{-1})'(-1) = -\dfrac{1}{4}$

**80.** (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)




(d) $f(x) = \dfrac{4}{1+x^2}$

$f'(x) = \dfrac{-8x}{(x^2+1)^2}, \ f'(1) = -2$

$f^{-1}(x) = \sqrt{\dfrac{4-x}{x}}$

$(f^{-1})'(x) = \dfrac{-2}{x^2\sqrt{\dfrac{4-x}{x}}}, \ (f^{-1})'(2) = -\dfrac{1}{2}$

**82.** $x = 2 \ln(y^2 - 3)$

$1 = 2\dfrac{1}{y^2-3} 2y \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{y^2-3}{4y}$. At $(0, 4)$, $\dfrac{dy}{dx} = \dfrac{16-3}{16} = \dfrac{13}{16}$.

**In Exercises 84 and 86, use the following.**

$\boldsymbol{f(x) = \frac{1}{8}x - 3}$ **and** $\boldsymbol{g(x) = x^3}$

$\boldsymbol{f^{-1}(x) = 8(x+3)}$ **and** $\boldsymbol{g^{-1}(x) = \sqrt[3]{x}}$

**84.** $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3) = g^{-1}(0) = 0$

**86.** $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}\left(\sqrt[3]{-4}\right)$

$= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$

**In Exercises 88 and 90, use the following.**

$$\boldsymbol{f(x) = x + 4 \text{ and } g(x) = 2x - 5}$$

$$\boldsymbol{f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x+5}{2}}$$

**88.** $$\begin{aligned}(f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x))\\ &= f^{-1}\left(\frac{x+5}{2}\right)\\ &= \frac{x+5}{2} - 4\\ &= \frac{x-3}{2}\end{aligned}$$

**90.** $$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(x+4)\\ &= 2(x+4) - 5\\ &= 2x + 3\end{aligned}$$

Hence, $(g \circ f)^{-1}(x) = \dfrac{x-3}{2}$

(Note: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$)

**92.** The graphs of $f$ and $f^{-1}$ are mirror images with respect to the line $y = x$.

**94.** Theorem 5.9: Let $f$ be differentiable on an interval $I$. If $f$ has an inverse $g$, then $g$ is differentiable at any $x$ for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))},\ f'(g(x)) \neq 0$$

**96.** $f$ is not one-to-one because different $x$-values yield the same $y$-value.

Example: $f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$

Not continuous at $\pm 2$.

**98.** If $f$ has an inverse, then $f$ and $f^{-1}$ are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

**100.** If $f$ has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \implies x_1 = x_2$. Therefore, $f$ is one-to-one. If $f(x)$ is one-to-one, then for every value $b$ in the range, there corresponds exactly one value $a$ in the domain. Define $g(x)$ such that the domain of $g$ equals the range of $f$ and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

**102.** True; if $f$ has a $y$-intercept.

**104.** False

Let $f(x) = x$ or $g(x) = 1/x$.

**106.** From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$\begin{aligned}g''(x) &= \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}\\ &= -\frac{f''(g(x)) \cdot [1/(f'(g(x)))]}{[f'(g(x))]^2}\\ &= -\frac{f''(g(x))}{[f'(g(x))]^3}\end{aligned}$$

If $f$ is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that $g$ is increasing and concave up.

# Section 5.4 Exponential Functions: Differentiation and Integration

**2.** $e^{-2} = 0.1353\ldots$

$\ln 0.1353\ldots = -2$

**4.** $\ln 0.5 = -0.6931\ldots$

$e^{-0.6931\ldots} = \frac{1}{2}$

**6.** $e^{\ln 2x} = 12$

$2x = 12$

$x = 6$

**8.** $4e^x = 83$

$e^x = \dfrac{83}{4}$

$x = \ln\left(\dfrac{83}{4}\right) \approx 3.033$

**10.** $-6 + 3e^x = 8$

$3e^x = 14$

$e^x = \dfrac{14}{3}$

$x = \ln\left(\dfrac{14}{3}\right) \approx 1.540$

**12.** $200e^{-4x} = 15$

$e^{-4x} = \dfrac{15}{200} = \dfrac{3}{40}$

$-4x = \ln\left(\dfrac{3}{40}\right)$

$x = \dfrac{1}{4}\ln\left(\dfrac{40}{3}\right) \approx 0.648$

**14.** $\ln x^2 = 10$

$x^2 = e^{10}$

$x = \pm e^5 \approx \pm 148.4132$

**16.** $\ln 4x = 1$

$4x = e' = e$

$x = \dfrac{e}{4} \approx 0.680$

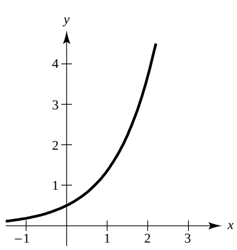
**18.** $\ln(x-2)^2 = 12$

$(x-2)^2 = e^{12}$

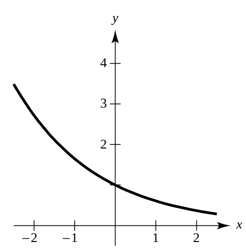
$x - 2 = e^6$

$x = 2 + e^6 \approx 405.429$

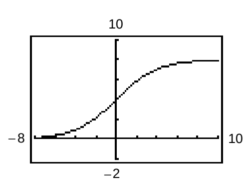
**20.** $y = \frac{1}{2}e^x$




**22.** $y = e^{-x/2}$

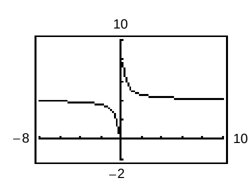



**24.** (a)




Horizontal asymptotes: $y = 0$ and $y = 8$

(b)




Horizontal asymptote: $y = 4$

**26.** $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$

Reflection in the $y$-axis

Matches (d)

**28.** $y = \dfrac{C}{1 + e^{-ax}}$

$$\lim_{x \to \infty} \frac{C}{1 + e^{-ax}} = C$$

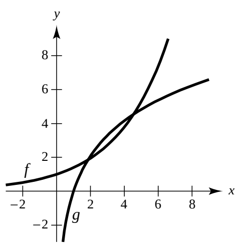
$$\lim_{x \to -\infty} \frac{C}{1 + e^{-ax}} = 0$$

Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

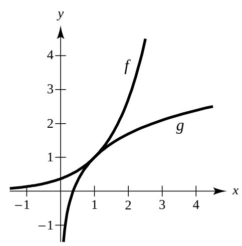
**30.** $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$




**32.** $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$




**34.** In the same way,

$$\lim_{x \to \infty} \left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

**36.** $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040} = 2.71\overline{825396}$

$$e \approx 2.718281828$$

$$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$$

**38.** (a) $y = e^{2x}$

$y' = 2e^{2x}$

At $(0, 1)$, $y' = 2$.

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

At $(0, 1)$, $y' = -2$.

**40.** $f(x) = e^{1-x}$

$f'(x) = -e^{1-x}$

**42.** $y = e^{-x^2}$

$\dfrac{dy}{dx} = -2xe^{-x^2}$

**44.** $y = x^2e^{-x}$

$$\frac{dy}{dx} = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2 - x)$$

**46.** $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \dfrac{6}{t^3e^{3/t^2}}$

**48.** $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$

$= \ln(1 + e^x) - \ln(1 - e^x)$

$$\frac{dy}{dx} = \frac{e^x}{1 + e^x} + \frac{e^x}{1 - e^x} = \frac{2e^x}{1 - e^{2x}}$$

**50.** $y = \ln\left(\dfrac{e^x + e^{-x}}{2}\right)$

$= \ln(e^x + e^{-x}) - \ln 2$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

**52.** $y = \dfrac{e^x - e^{-x}}{2}$

$\dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{2}$

**54.** $y = xe^x - e^x = e^x(x - 1)$

$\dfrac{dy}{dx} = e^x + e^x(x - 1) = xe^x$

**56.** $f(x) = e^3 \ln x$

$$f'(x) = \frac{e^3}{x}$$

**58.** $y = \ln e^x = x$

$$\frac{dy}{dx} = 1$$

**60.**
$$e^{xy} + x^2 - y^2 = 10$$
$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$
$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

**62.** $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$
$$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x$$
$$= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x$$

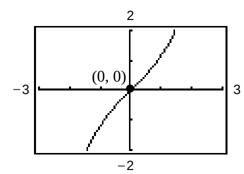
**64.**
$$y = e^x(3\cos 2x - 4\sin 2x)$$
$$y' = e^x(-6\sin 2x - 8\cos 2x) + e^x(3\cos 2x - 4\sin 2x)$$
$$= e^x(-10\sin 2x - 5\cos 2x) = -5e^x(2\sin 2x + \cos 2x)$$
$$y'' = -5e^x(4\cos 2x - 2\sin 2x) - 5e^x(2\sin 2x + \cos 2x) = -5e^x(5\cos 2x) = -25e^x\cos 2x$$
$$y'' - 2y' = -25e^x\cos 2x - 2(-5e^x)(2\sin 2x + \cos 2x) = -5e^x(3\cos 2x - 4\sin 2x) = -5y$$

Therefore, $y'' - 2y' = -5y \Longrightarrow y'' - 2y' + 5y = 0.$

**66.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2} > 0$$
$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$

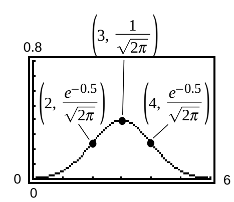



**68.** $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

$$g'(x) = \frac{-1}{\sqrt{2\pi}}(x - 3)e^{-(x-3)^2/2}$$
$$g''(x) = \frac{1}{\sqrt{2\pi}}(x - 2)(x - 4)e^{-(x-3)^2/2}$$

Relative maximum: $\left(3, \dfrac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$

Points of inflection: $\left(2, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$



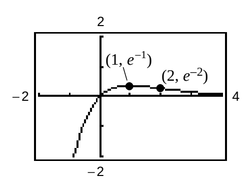

**70.** $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) = 0$ when $x = 1$.

$f''(x) = -e^{-x} + (-e^{-x})(1 - x) = e^{-x}(x - 2) = 0$ when $x = 2$.

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



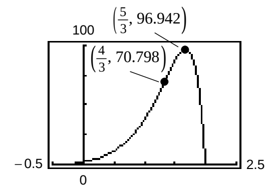
**72.** $f(x) = -2 + e^{3x}(4 - 2x)$

$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0$ when $x = \frac{5}{3}$.

$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0$ when $x = \frac{4}{3}$.

Relative maximum: $\left(\frac{5}{3}, 96.942\right)$

Point of inflection: $\left(\frac{4}{3}, 70.798\right)$




**74.** (a)

$$f(c) = f(c + x)$$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

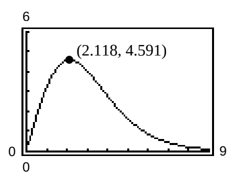
$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

(b) $A(x) = xf(c) = x\left[10\left(\dfrac{x}{e^x - 1}\right)e^{-(x/(e^x-1))}\right]$

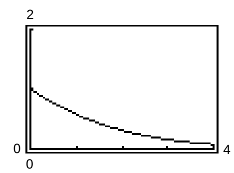
$$= \frac{10x^2}{e^x - 1}e^{x/(1-e^x)}$$

(c) $A(x) = \dfrac{10x^2}{e^x - 1}e^{x/(1-e^x)}$




The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(d) $c = \dfrac{x}{e^x - 1}$




$\lim_{x \to 0^+} c = 1$

$\lim_{x \to \infty} c = 0$

**76.** Let $(x_0, y_0)$ be the desired point on $y = e^{-x}$.

$$y = e^{-x}$$

$$y' = -e^{-x} \quad \text{(Slope of tangent line)}$$

$$-\frac{1}{y'} = e^x \quad \text{(Slope of normal line)}$$

$$y - e^{-x_0} = e^{x_0}(x - x_0)$$

We want $(0, 0)$ to satisfy the equation:

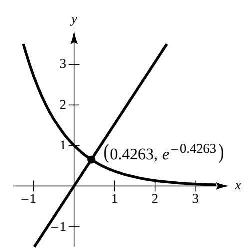
$$-e^{-x_0} = -x_0e^{x_0}$$

$$1 = x_0e^{2x_0}$$

$$x_0e^{2x_0} - 1 = 0$$

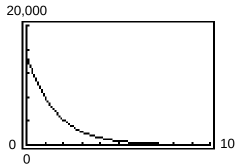
Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$(0.4263, e^{-0.4263})$



**78.** $V = 15{,}000e^{-0.6286t},\ 0 \le t \le 10$

(a)

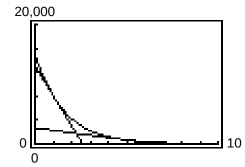



(b) $\dfrac{dV}{dt} = -9429e^{-0.6286t}$

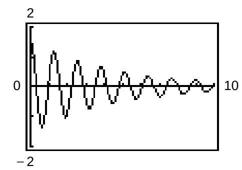
When $t = 1$, $\dfrac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\dfrac{dV}{dt} \approx -406.89$.

(c)

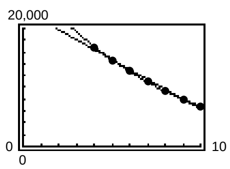



**80.** $1.56e^{-0.22t}\cos 4.9t \le 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \ge 7.79$ seconds.




**82.** (a) $V_1 = -1686.79t + 23{,}181.79$

$V_2 = 109.52t^2 - 3220.12t + 28{,}110.36$




(b) The slope represents the rate of decrease in value of the car.

(c) $V_3 = 31{,}450.77(0.8592)^t = 31{,}450.77e^{-0.1518t}$

(d) Horizontal asymptote: $\lim\limits_{t\to\infty} V_3(t) = 0$

As $t \to \infty$, the value of the car approaches 0.

(e) $\dfrac{dV_3}{dt} = -4774.2e^{-0.1518t}$

For $t = 5$, $\dfrac{dV_3}{dt} \approx -2235$ dollars/year.

For $t = 9$, $\dfrac{dV_3}{dt} \approx -1218$ dollars/year.

**84.** $f(x) = e^{-x^2/2},\ f(0) = 1$

$f'(x) = -xe^{-x^2/2},\ f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1),\ f''(0) = -1$

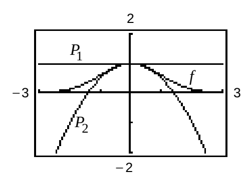
$P_1(x) = 1 + 0(x - 0) = 1,\ P_1(0) = 1$

$P_1'(x) = 0,\ P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \dfrac{1}{2}(x - 0)^2 = 1 - \dfrac{x^2}{2},\ P_2(0) = 1$

$P_2'(x) = -x,\ P_2'(0) = 0$

$P_2''(x) = -1,\ P_2''(0) = -1$




The values of $f$, $P_1$, $P_2$ and their first derivatives agree at $x = 0$. The values of the second derivatives of $f$ and $P_2$ agree at $x = 0$.

**86.** $n^{\text{th}}$ term is $x^n/n!$ in polynomial:

$$y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Conjecture: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

**88.** Let $u = -x^4$, $du = -4x^3\,dx$.

$$\int e^{-x^4}(-4x^3)\,dx = e^{-x^4} + C$$

**90.** $\displaystyle\int_3^4 e^{3-x}\,dx = \Big[-e^{3-x}\Big]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$

**92.** $\displaystyle\int x^2 e^{x^3/2}\,dx = \frac{2}{3}\int e^{x^3/2}\left(\frac{3x^2}{2}\right)dx = \frac{2}{3}e^{x^3/2} + C$

**94.** $\displaystyle\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}\int e^{1/x^2}\left(\frac{-2}{x^3}\right)dx = -\frac{1}{2}e^{1/x^2} + C$

**96.** Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\ln(1 + e^{2x}) + C$$

**98.** Let $u = \dfrac{-x^2}{2}$, $du = -x\,dx$.

$$\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx = -\int_0^{\sqrt{2}} e^{-x^2/2}(-x)\,dx = \Big[-e^{-x^2/2}\Big]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$$

**100.** Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln(e^x + e^{-x}) + C$$

**102.** Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})\,dx$.

$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx = 2\int (e^x + e^{-x})^{-2}(e^x - e^{-x})\,dx$$
$$= \frac{-2}{e^x + e^{-x}} + C$$

**104.** $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx = \int (e^x + 2 + e^{-x})\,dx$

$$= e^x + 2x - e^{-x} + C$$

**106.** $\displaystyle\int e^{\sec 2x}\sec 2x\tan 2x\,dx = \frac{1}{2}e^{\sec 2x} + C$

$(u = \sec 2x,\ du = 2\sec 2x\tan 2x)$

**108.** $\displaystyle\int \ln(e^{2x-1})\,dx = \int (2x - 1)\,dx$

$$= x^2 - x + C$$

**110.** $\displaystyle y = \int (e^x - e^{-x})^2\,dx$

$$= \int (e^{2x} - 2 + e^{-2x})\,dx$$
$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$$

**112.** $\displaystyle f'(x) = \int (\sin x + e^{2x})\,dx = -\cos x + \frac{1}{2}e^{2x} + C_1$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \implies C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1$$

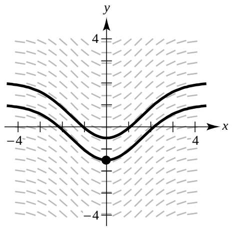
$$f(x) = \int \left(-\cos x + \frac{1}{2}e^{2x} + 1\right)dx$$

$$= -\sin x + \frac{1}{4}e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \implies C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4}e^{2x}$$

**114.** (a)




(b) $\dfrac{dy}{dx} = xe^{-0.2x^2}, \quad \left(0, -\dfrac{3}{2}\right)$

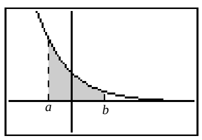
$$y = \int xe^{-0.2x^2}\,dx = \frac{1}{-0.4}\int e^{-0.2x^2}(-0.4x)\,dx$$

$$= -\frac{1}{0.4}e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

$$\left(0, -\frac{3}{2}\right):\ -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \implies C = 1$$

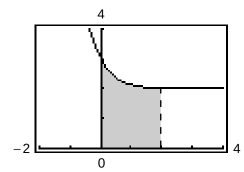
$$y = -2.5e^{-0.2x^2} + 1$$

**116.** $\displaystyle\int_a^b e^{-x}\,dx = \Big[-e^{-x}\Big]_a^b = e^{-a} - e^{-b}$




**118.** $\displaystyle\int_0^2 (e^{-2x} + 2)\,dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$




**120.** (a) $\displaystyle\int_0^4 \sqrt{x}\,e^x\,dx,\ n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing Utility: 92.7437

(b) $\displaystyle\int_0^2 2xe^{-x}\,dx,\ n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing Utility: 1.18799

**122.** $\displaystyle\int_0^x 0.3^{-0.3t}\,dt = \frac{1}{2}$

$$\Big[-e^{-0.3t}\Big]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln\frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

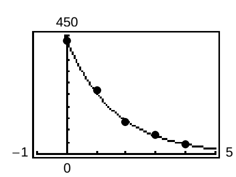
**124.**

| | | | | | |
|---|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 |
| $R$ | 425 | 240 | 118 | 71 | 36 |
| $\ln R$ | 6.052 | 5.481 | 4.771 | 4.263 | 3.584 |

(a) $\ln R = -0.6155t + 6.0609$

$$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$$

(b)




(c) $\displaystyle\int_0^4 R(t)\,dt = \int_0^4 428.78e^{-0.6155t}\,dt \approx 637.2 \text{ liters}$

**126.** The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

**128.** (a) Log Rule: $(u = e^x + 1)$

(b) Substitution: $(u = x^2)$

**130.** $\ln \dfrac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$\ln e^{a-b} = a - b$

Therefore, $\ln \dfrac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\dfrac{e^a}{e^b} = e^{a-b}$.

## Section 5.5 Bases Other than $e$ and Applications

**2.** $y = \left(\dfrac{1}{2}\right)^{t/8}$

At $t_0 = 16$, $y = \left(\dfrac{1}{2}\right)^{16/8} = \dfrac{1}{4}$

**4.** $y = \left(\dfrac{1}{2}\right)^{t/5}$

At $t_0 = 2$, $y = \left(\dfrac{1}{2}\right)^{2/5} \approx 0.7579$

**6.** $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

**8.** $\log_a \dfrac{1}{a} = \log_a 1 - \log_a a = -1$

**10.** (a) $27^{2/3} = 9$

$\log_{27} 9 = \dfrac{2}{3}$

(b) $16^{3/4} = 8$

$\log_{16} 8 = \dfrac{3}{4}$

**12.** (a) $\log_3 \frac{1}{9} = -2$

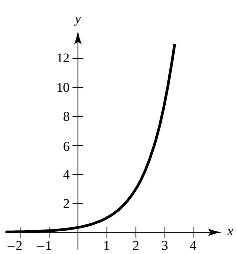
$3^{-2} = \frac{1}{9}$

(b) $49^{1/2} = 7$

$\log_{49} 7 = \frac{1}{2}$

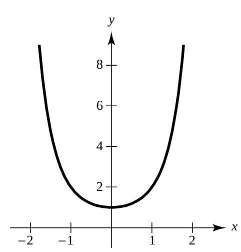
**14.** $y = 3^{x-1}$

| $x$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $y$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 |



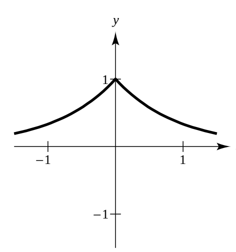
**16.** $y = 2^{x^2}$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 16 | 2 | 1 | 2 | 16 |



**18.** $y = 3^{-|x|}$

| $x$ | 0 | $\pm 1$ | $\pm 2$ |
|---|---|---|---|
| $y$ | 1 | $\frac{1}{3}$ | $\frac{1}{9}$ |



**20.** (a) $\log_3 \dfrac{1}{81} = x$

$3^x = \dfrac{1}{81}$

$x = -4$

(b) $\log_6 36 = x$

$6^x = 36$

$x = 2$

**22.** (a) $\log_b 27 = 3$

$b^3 = 27$

$b = 3$

(b) $\log_b 125 = 3$

$b^3 = 125$

$b = 5$

**24.** (a) $\log_3 x + \log_3(x-2) = 1$

$$\log_3[x(x-2)] = 1$$
$$x(x-2) = 3^1$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x+3) - \log_{10} x = 1$

$$\log_{10} \frac{x+3}{x} = 1$$
$$\frac{x+3}{x} = 10^1$$
$$x + 3 = 10x$$
$$3 = 9x$$
$$x = \frac{1}{3}$$

**26.** $5^{6x} = 8320$

$$6x \ln 5 = \ln 8320$$
$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

**28.** $3(5^{x-1}) = 86$

$$5^{x-1} = \frac{86}{3}$$
$$(x-1)\ln 5 = \ln\left(\frac{86}{3}\right)$$
$$x - 1 = \frac{\ln\left(\frac{86}{3}\right)}{\ln 5}$$
$$x = 1 + \frac{\ln\left(\frac{86}{3}\right)}{\ln 5} \approx 3.085$$

**30.** $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$
$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

**32.** $\log_{10}(t-3) = 2.6$

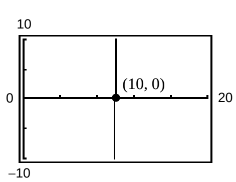
$$t - 3 = 10^{2.6}$$
$$t = 3 + 10^{2.6} \approx 401.107$$

**34.** $\log_5 \sqrt{x-4} = 3.2$

$$\sqrt{x-4} = 5^{3.2}$$
$$x - 4 = (5^{3.2})^2 = 5^{6.4}$$
$$x = 4 + 5^{6.4} \approx 29{,}748.593$$

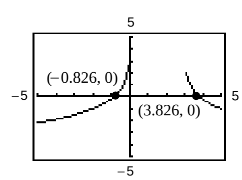
**36.** $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$




**38.** $g(x) = 1 - 2\log_{10}[x(x-3)]$

Zeros: $x \approx -0.826, 3.826$

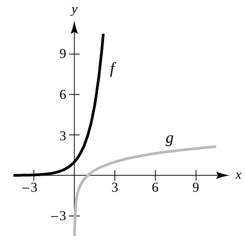


**40.** $f(x) = 3^x$

$g(x) = \log_3 x$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{9}$ | $\frac{1}{3}$ | $1$ | $3$ | $9$ |

| $x$ | $\frac{1}{9}$ | $\frac{1}{3}$ | $1$ | $3$ | $9$ |
|---|---|---|---|---|---|
| $g(x)$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |




**42.** $g(x) = 2^{-x}$

$g'(x) = -(\ln 2)\, 2^{-x}$

**44.** $y = x(6^{-2x})$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}) + 6^{-2x}$$
$$= 6^{-2x}[-2x(\ln 6) + 1]$$
$$= 6^{-2x}(1 - 2x \ln 6)$$

**46.** $f(t) = \dfrac{3^{2t}}{t}$

$$f'(t) = \frac{t(2\ln 3)\, 3^{2t} - 3^{2t}}{t^2}$$
$$= \frac{3^{2t}\,(2t \ln 3 - 1)}{t^2}$$

**48.** $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$g'(\alpha) = 5^{-\alpha/2}\, 2 \cos 2\alpha - \frac{1}{2}(\ln 5)\, 5^{-\alpha/2} \sin 2\alpha$

**50.** $y = \log_{10}(2x) = \log_{10} 2 + \log_{10} x$

$$\frac{dy}{dx} = 0 + \frac{1}{x \ln 10} = \frac{1}{x \ln 10}$$

**52.** $h(x) = \log_3 \dfrac{x\sqrt{x-1}}{2}$

$$= \log_3 x + \frac{1}{2}\log_3(x-1) - \log_3 2$$
$$h'(x) = \frac{1}{x\ln 3} + \frac{1}{2} \cdot \frac{1}{(x-1)\ln 3} - 0$$
$$= \frac{1}{\ln 3}\left[\frac{1}{x} + \frac{1}{2(x-1)}\right]$$
$$= \frac{1}{\ln 3}\left[\frac{3x-2}{2x(x-1)}\right]$$

**54.** $y = \log_{10} \dfrac{x^2-1}{x}$

$$= \log_{10}(x^2-1) - \log_{10} x$$
$$\frac{dy}{dx} = \frac{2x}{(x^2-1)\ln 10} - \frac{1}{x \ln 10}$$
$$= \frac{1}{\ln 10}\left[\frac{2x}{x^2-1} - \frac{1}{x}\right]$$
$$= \frac{1}{\ln 10}\left[\frac{x^2+1}{x(x^2-1)}\right]$$

**56.** $f(t) = t^{3/2}\log_2\sqrt{t+1} = t^{3/2}\,\dfrac{1}{2}\,\dfrac{\ln(t+1)}{\ln 2}$

$$f'(t) = \frac{1}{2\ln 2}\left[t^{3/2}\frac{1}{t+1} + \frac{3}{2}t^{1/2}\ln(t+1)\right]$$

**58.** $y = x^{x-1}$

$$\ln y = (x-1)(\ln x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x-1)\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = y\left[\frac{x-1}{x} + \ln x\right]$$
$$= x^{x-2}\,(x - 1 + x \ln x)$$

**60.** $y = (1+x)^{1/x}$

$$\ln y = \frac{1}{x}\ln(1+x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{1}{1+x}\right) + \ln(1+x)\left(-\frac{1}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1}{x+1} - \frac{\ln(x+1)}{x}\right]$$
$$= \frac{(1+x)^{1/x}}{x}\left[\frac{1}{x+1} - \frac{\ln(x+1)}{x}\right]$$

**62.** $\displaystyle\int 5^{-x}\,dx = \frac{-5^{-x}}{\ln 5} + C$

**64.** $\displaystyle\int_{-2}^{0}(3^3 - 5^2)\,dx = \int_{-2}^{0}(27-25)\,dx$

$$= \int_{-2}^{0} 2\,dx$$
$$= \Big[2x\Big]_{-2}^{0} = 4$$

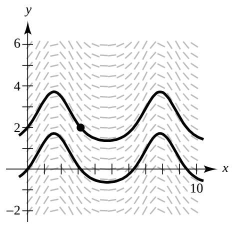
**66.** $\int (3 - x)\, 7^{(3-x)^2}\, dx = -\frac{1}{2}\int -2(3 - x)\, 7^{(3-x)^2}\, dx$

$= -\frac{1}{2 \ln 7}\left[7^{(3-x)^2}\right] + C$

**68.** $\int 2^{\sin x} \cos x\, dx,\; u = \sin x,\; du = \cos x\, dx$

$\frac{1}{\ln 2}\, 2^{\sin x} + C$

**70.** (a)




(b) $\frac{dy}{dx} = e^{\sin x} \cos x \qquad (\pi, 2)$

$y = \int e^{\sin x} \cos x\, dx = e^{\sin x} + C$

$(\pi, 2)$: $2 = e^{\sin \pi} + C = 1 + C \implies C = 1$

$y = e^{\sin x} + 1$

**72.** $\log_b x = \frac{\ln x}{\ln b} = \frac{\log_{10} x}{\log_{10} b}$

**74.** $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$10^y = x$

$f^{-1}(x) = 10^x$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10$

$= \log_{10}(10x)$

$x$ must have been increased by a factor of 10.

(f) $\log_{10}\left(\frac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$= 3n - n = 2n$

Thus, $x_1/x_2 = 10^{2n} = 100^n$.

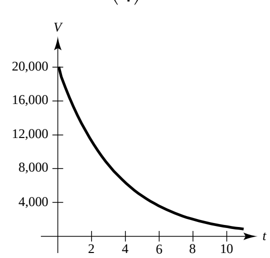
**76.** $f(x) = a^x$

(a) $f(u + v) = a^{u+v} = a^u a^v = f(u) f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

**78.** $V(t) = 20{,}000\left(\frac{3}{4}\right)^t$

(a)



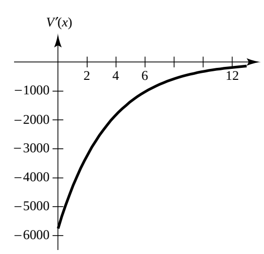

$V(2) = 20{,}000\left(\frac{3}{4}\right)^2 = \$11{,}250$

(b) $\frac{dV}{dt} = 20{,}000\left(\ln \frac{3}{4}\right)\left(\frac{3}{4}\right)^t$

When $t = 1$: $\frac{dV}{dt} \approx -4315.23$

When $t = 4$: $\frac{dV}{dt} \approx -1820.49$

(c)




Horizontal asymptote: $v' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

**80.** $P = \$2500,\ r = 6\% = 0.06,\ t = 20$

$$A = 2500\left(1 + \frac{0.06}{n}\right)^{20n}$$

$$A = 2500e^{(0.06)(20)} = 8300.29$$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | 8017.84 | 8155.09 | 8226.66 | 8275.51 | 8299.47 | 8300.29 |

**82.** $P = \$5000, r = 7\% = 0.07, t = 25$

$$A = 5000\left(1 + \frac{0.07}{n}\right)^{25n}$$

$$A = 5000e^{0.07(25)}$$

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|---|---|
| $A$ | 27,137.16 | 27,924.63 | 28,340.78 | 28,627.09 | 28,768.19 | 28,773.01 |

**84.** $100{,}000 = Pe^{0.06t} \implies P = 100{,}000e^{-0.06t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | 94,176.45 | 54,881.16 | 30,119.42 | 16,529.89 | 9071.80 | 4978.71 |

**86.** $100{,}000 = P\left(1 + \frac{0.07}{365}\right)^{365t} \implies P = 100{,}000\left(1 + \frac{0.07}{365}\right)^{-365t}$

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | 93,240.01 | 49,661.86 | 24,663.01 | 12,248.11 | 6082.64 | 3020.75 |

**88.** Let $P = \$100,\ 0 \le t \le 20.$

(a) $A = 100e^{0.03t}$

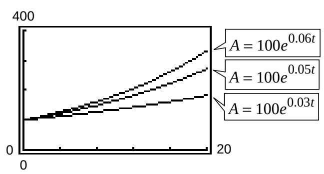
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



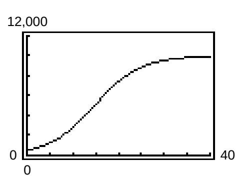

**90.** (a) $\displaystyle\lim_{n\to\infty} \frac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b) $$P' = \frac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})} = \frac{0.215e^{-0.25n}}{(1 + e^{-0.25n})}$$

$P'(3) \approx 0.069$

$P'(10) \approx 0.016$

**92.** (a)




(b) Limiting size: 10,000 fish

(c) $$p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$$

$$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{t/5})^2}\left(\frac{19}{5}\right)(10{,}000)$$

$$= \frac{38{,}000e^{-t/5}}{(1 + 19e^{-t/5})^2}$$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

(d) $$p''(t) = -\frac{38{,}000}{5}(e^{-t/5})\left[\frac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3}\right] = 0$$

$$19e^{-t/5} = 1$$

$$\frac{t}{5} = \ln 19$$

$$t = 5 \ln 19 \approx 14.72$$

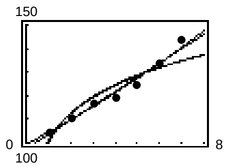
**94.** (a) $y_1 = 6.0536x + 97.5571$

$y_2 = 100.0751 + 17.8148 \ln x$

$y_3 = 99.4557(1.0506)^x$

$y_4 = 101.2875x^{0.1471}$

(b)




$y_3$ seems best.

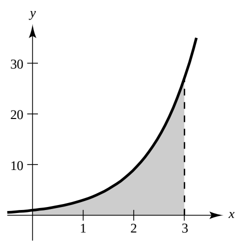
(c) The slope of 6.0536 is the annual rate of change in the amount given to philanthropy.

(d) For 1996, $x = 6$ and $y_1' = 6.0536$, $y_2' \approx 2.9691$,

$y_3' \approx 6.6015$, $y_4' \approx 3.2321$.

$y_3$ is increasing at the greatest rate in 1996.

**96.** $A = \displaystyle\int_0^3 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_0^3 = \frac{26}{\ln 3} \approx 23.666$




**98.**

| $x$ | 1 | $10^{-1}$ | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ |
|---|---|---|---|---|---|
| $(1 + x)^{1/x}$ | 2 | 2.594 | 2.705 | 2.718 | 2.718 |

**100.**

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 600 | 630 | 661.50 | 694.58 | 729.30 |

$y = C(k^t)$

When $t = 0$, $y = 600 \implies C = 600$.

$y = 600(k^t)$

$\dfrac{630}{600} = 1.05,\ \dfrac{661.50}{630} = 1.05,\ \dfrac{694.58}{661.50} \approx 1.05,\ \dfrac{729.30}{694.58} \approx 1.05$

Let $k = 1.05$.

$y = 600(1.05)^t$

**102.** True.

$$\begin{aligned} f(e^{n+1}) - f(e^n) &= \ln e^{n+1} - \ln e^n \\ &= n + 1 - n \\ &= 1 \end{aligned}$$

**104.** True.

$$\begin{aligned} \frac{d^n y}{dx^n} &= Ce^x \\ &= y \text{ for } n = 1, 2, 3, \ldots \end{aligned}$$

**106.** True.

$f(x) = g(x)e^x = 0 \implies$

$g(x) = 0$ since $e^x > 0$ for all $x$.

**108.** $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \sin x\left(\frac{1}{x}\right) + \cos x \cdot \ln x$$

$$y' = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right]$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$y' = \left(\frac{\pi}{2}\right)^{\sin(\pi/2)}\left[\frac{\sin(\pi/2)}{\pi/2} + \cos\left(\frac{\pi}{2}\right)\ln\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{2}{\pi} + 0\right] = 1$$

Tangent line: $y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$$y = x$$

## Section 5.6 Differential Equations: Growth and Decay

**2.** $\frac{dy}{dx} = 4 - x$

$$y = \int (4 - x)\,dx = 4x - \frac{x^2}{2} + C$$

**4.** $\frac{dy}{dx} = 4 - y$

$$\frac{dy}{4 - y} = dx$$

$$\int \frac{-1}{4 - y}\,dy = \int -dx$$

$$\ln|4 - y|\,dy = -x + C_1$$

$$4 - y = e^{-x + C_1} = Ce^{-x}$$

$$y = 4 - Ce^{-x}$$

**6.** $y' = \frac{\sqrt{x}}{3y}$

$$3yy' = \sqrt{x}$$

$$\int 3yy'\,dx = \int \sqrt{x}\,dx$$

$$\frac{3y^2}{2} = \frac{2}{3}x^{3/2} + C_1$$

$$9y^2 - 4x^{3/2} = C$$

**8.** $y' = x(1 + y)$

$$\frac{y'}{1 + y} = x$$

$$\int \frac{y'}{1 + y}\,dx = \int x\,dx$$

$$\int \frac{dy}{1 + y} = \int x\,dx$$

$$\ln(1 + y) = \frac{x^2}{2} + C_1$$

$$1 + y = e^{(x^2/2) + C_1}$$

$$y = e^{C_1}e^{x^2/2} - 1$$

$$= Ce^{x^2/2} - 1$$

**10.** $xy + y' = 100x$

$$y' = 100x + xy = x(100 - y)$$

$$\frac{y'}{100 - y} = x$$

$$\int \frac{y'}{100 - y}\,dx = \int x\,dx$$

$$\int \frac{1}{100 - y}\,dy = \int x\,dx$$

$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$

$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$

$$100 - y = e^{-(x^2/2) - C_1}$$

$$-y = e^{-C_1}e^{-x^2/2} - 100$$

$$y = 100 - Ce^{-x^2/2}$$

**12.** $$\frac{dP}{dt} = k(10 - t)$$

$$\int \frac{dP}{dt}\,dt = \int k(10 - t)\,dt$$

$$\int dP = -\frac{k}{2}(10 - t)^2 + C$$

$$P = -\frac{k}{2}(10 - t)^2 + C$$

**14.** $$\frac{dy}{dx} = kx(L - y)$$

$$\frac{1}{L - y}\frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y}\frac{dy}{dx}\,dx = \int kx\,dx$$

$$\int \frac{1}{L - y}\,dy = \frac{kx^2}{2} + C_1$$

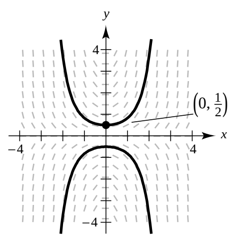
$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1}e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

**16.** (a)




(b) $$\frac{dy}{dx} = xy, \quad \left(0, \frac{1}{2}\right)$$

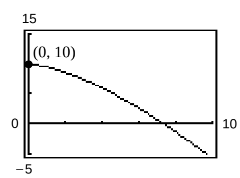
$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right): \ \frac{1}{2} = C_1 e^0 \implies C_1 = \frac{1}{2} \implies y = \frac{1}{2}e^{x^2/2}$$

**18.** $$\frac{dy}{dt} = -\frac{3}{4}\sqrt{t}, \quad (0, 10)$$



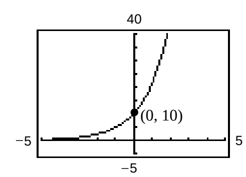

$$\int dy = \int -\frac{3}{4}\sqrt{t}\,dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \implies C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$

**20.** $$\frac{dy}{dt} = \frac{3}{4}y, \quad (0, 10)$$




$$\int \frac{dy}{y} = \int \frac{3}{4}\,dt$$

$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \implies C = 10$$

$$y = 10e^{3t/4}$$

**22.** $$\frac{dN}{dt} = kN$$

$N = Ce^{kt}$ (Theorem 5.16)

$(0, 250)$: $C = 250$

$(1, 400)$: $400 = 250e^k \implies k = \ln\frac{400}{250} = \ln\frac{8}{5}$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}$$

**24.** $$\frac{dP}{dt} = kP$$

$P = Ce^{kt}$ (Theorem 5.16)

$(0, 5000)$: $C = 5000$

$(1, 4750)$: $4750 = 5000e^k \implies k = \ln\left(\frac{19}{20}\right)$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905$$

**26.** $y = Ce^{kt},\ (0, 4),\ \left(5, \frac{1}{2}\right)$

$$C = 4$$
$$y = 4e^{kt}$$
$$\frac{1}{2} = 4e^{5k}$$
$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$
$$y = 4e^{-0.4159t}$$

**28.** $y = Ce^{kt},\ \left(3, \frac{1}{2}\right),\ (4, 5)$

$$\frac{1}{2} = Ce^{3k}$$
$$5 = Ce^{4k}$$
$$2Ce^{3k} = \frac{1}{5}Ce^{4k}$$
$$10e^{3k} = e^{4k}$$
$$10 = e^{k}$$
$$k = \ln 10 \approx 2.3026$$
$$y = Ce^{2.3026t}$$
$$5 = Ce^{2.3026(4)}$$
$$C \approx 0.0005$$
$$y = 0.0005e^{2.3026t}$$

**30.** $y' = \dfrac{dy}{dt} = ky$

**32.** $\dfrac{dy}{dx} = \dfrac{1}{2}x^2y$

$\dfrac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

**34.** Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $1.5 = Ce^{[\ln(1/2)/1620](1000)} \implies C \approx 2.30$ which implies that the initial quantity is 2.30 grams. When $t = 10{,}000$, we have $y = 2.30e^{[\ln(1/2)/1620](10{,}000)} \approx 0.03$ gram.

**36.** Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $2.0 = Ce^{[\ln(1/2)/5730](10{,}000)} \implies C \approx 6.70$ which implies that the initial quantity is 6.70 grams. When $t = 1000$, we have $y = 6.70e^{[\ln(1/2)/5730](1000)} \approx 5.94$ grams.

**38.** Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $3.2 = Ce^{[\ln(1/2)/5730]1000} \implies C \approx 3.61$.

Initial quantity: 3.61 grams.

When $t = 10{,}000$, we have $y \approx 1.08$ grams.

**40.** Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $0.4 = Ce^{[\ln(1/2)/24{,}360](10{,}000)} \implies C \approx 0.53$ which implies that the initial quantity is 0.53 gram. When $t = 1000$, we have $y = 0.53e^{[\ln(1/2)/24{,}360](1000)} \approx 0.52$ gram.

**42.** Since $\dfrac{dy}{dx} = ky,\ y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{5730k}$$
$$k = -\frac{\ln 2}{5730}$$
$$0.15y_0 = y_0e^{(-\ln 2/5730)t}$$
$$\ln 0.15 = -\frac{(\ln 2)t}{5730}$$
$$t = -\frac{5730 \ln 0.15}{\ln 2} \approx 15{,}682.8 \text{ years.}$$

**44.** Since $A = 20{,}000e^{0.055t}$, the time to double is given by $40{,}000 = 20{,}000e^{0.055t}$ and we have

$$2 = e^{0.055t}$$
$$\ln 2 = 0.055t$$
$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years:
$A = 20{,}000e^{(0.055)(10)} \approx \$34{,}665.06$

**46.** Since $A = 10{,}000e^{rt}$ and $A = 20{,}000$ when $t = 5$, we have the following.

$$20{,}000 = 10{,}000e^{5r}$$
$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

Amount after 10 years: $A = 10{,}000e^{[(\ln 2)/5](10)} = \$40{,}000$

**48.** Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$
$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$
$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

**50.** $500{,}000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 500{,}000(1.005)^{-480} \approx \$45{,}631.04$$

**52.** $500{,}000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 500{,}000\left(1 + \frac{0.09}{12}\right)^{-300}$$
$$\approx \$53{,}143.92$$

**54.** (a) $2000 = 1000(1 + 0.6)^t$

$$2 = 1.06^t$$
$$\ln 2 = t \ln 1.06$$
$$t = \frac{\ln 2}{\ln 1.06} \approx 11.90 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.06}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.06}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.06}{12}\right)$$
$$t = \frac{1}{12}\,\frac{\ln 2}{\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.06}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.06}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.06}{365}\right)$$
$$t = \frac{1}{365}\,\frac{\ln 2}{\ln\left(1 + \frac{0.06}{365}\right)} \approx 11.55 \text{ years}$$

(d) $2000 = 1000e^{0.06t}$

$$2 = e^{0.06t}$$
$$\ln 2 = 0.06t$$
$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years}$$

**56.** (a) $2000 = 1000(1 + 0.055)^t$

$$2 = 1.055^t$$
$$\ln 2 = t \ln 1.055$$
$$t = \frac{\ln 2}{\ln 1.055} \approx 12.95 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.055}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.055}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.055}{12}\right)$$
$$t = \frac{1}{12}\,\frac{\ln 2}{\ln\left(1 + \frac{0.055}{12}\right)} \approx 12.63 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.055}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.055}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.055}{365}\right)$$
$$t = \frac{1}{365}\,\frac{\ln 2}{\ln\left(1 + \frac{0.055}{365}\right)} \approx 12.60 \text{ years}$$

(d) $2000 = 1000e^{0.055t}$

$$2 = e^{0.055t}$$
$$\ln 2 = 0.055t$$
$$t = \frac{\ln 2}{0.055} \approx 12.60 \text{ years}$$

**58.** $P = Ce^{kt} = Ce^{0.031t}$

$P(-1) = 11.6 = Ce^{0.031(-1)} \implies C = 11.9652$

$P = 11.9652e^{0.031t}$

$P(10) \approx 16.31$ or 16,310,000 people in 2010

**60.** $P = Ce^{kt} = Ce^{-0.004t}$

$P(-1) = 3.6 = Ce^{-0.004(-1)} \implies C = 3.5856$

$P = 3.5856e^{-0.004t}$

$P(10) \approx 3.45$ or 3,450,000 people in 2010

**62.** (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$

$$1.2455^t = \frac{400}{100.1596} = 3.9936$$

$$t \ln 1.2455 = \ln 3.9936$$

$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.}$$

**64.** $y = Ce^{kt}$, $(0, 742{,}000)$, $(2, 632{,}000)$

$$C = 742{,}000$$

$$632{,}000 = 742{,}000e^{2k}$$

$$k = \frac{\ln(632/742)}{2} \approx -0.0802$$

$$y \approx 742{,}000e^{-0.0802t}$$

When $t = 4$, $y \approx \$538{,}372$.

**66.** (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

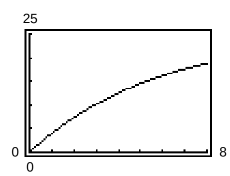
**68.** $S = 25(1 - e^{kt})$

(a) $4 = 25(1 - e^{k(1)}) \implies 1 - e^k = \dfrac{4}{25} \implies e^k = \dfrac{21}{25} \implies k = \ln\left(\dfrac{21}{25}\right) \approx -0.1744$

(b) 25,000 units $\left(\lim_{t\to\infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

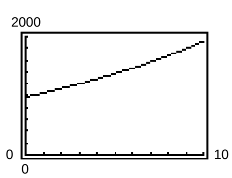
(d)




**70.** (a) $R = 979.3993(1.0694)^t = 979.3993e^{0.0671t}$

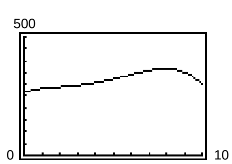
$I = -0.1385t^4 + 2.1770t^3 - 9.9755t^2 + 23.8513t + 266.4923$

(b)

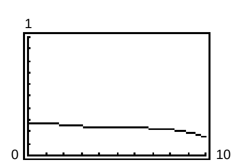



Rate of growth $= R'(t) = 65.7e^{0.0671t}$

(c)




(d) $P(t) = \dfrac{I}{R}$



**72.** $93 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \implies I = 10^{-6.7}$

$80 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \implies I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

**74.** Since $\dfrac{dy}{dt} = k(y - 80)$

$$\int \frac{1}{y - 80}\, dy = \int k\, dt$$

$\ln(y - 80) = kt + C.$

When $t = 0$, $y = 1500$. Thus, $C = \ln 1420$.

When $t = 1$, $y = 1120$. Thus,

$k(1) + \ln 1420 = \ln(1120 - 80)$

$$k = \ln 1040 - \ln 1420 = \ln \frac{104}{142}.$$

Thus, $y = 1420e^{[\ln(104/142)]t} + 80$.

When $t = 5$, $y \approx 379.2°$.

**76.** True

**78.** True

## Section 5.7 Differential Equations: Separation of Variables

**2.** Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy} = \frac{-2xy}{2y^2 - (x^2 + y^2)} = \frac{-2xy}{y^2 - x^2} = \frac{2xy}{x^2 - y^2}$$

**4.** Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

Check:

$$y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$$

$$y'' = 2\, C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$\begin{aligned} y'' + 2y' + 2y &= 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x + \\ &\quad 2(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x) + 2(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x) \\ &= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0 \end{aligned}$$

**6.** $y = \frac{2}{3}(e^{-2x} + e^x)$

$y' = \frac{2}{3}(-2e^{-2x} + e^x)$

$y'' = \frac{2}{3}(4e^{-2x} + e^x)$

Substituting, $y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x) = 2e^x$.

**In Exercises 8–12, the differential equation is $y^{(4)} - 16y = 0$.**

**8.** $y = 3\cos 2x$

$y^{(4)} = 48\cos 2x$

$y^{(4)} - 16y = 48\cos 2x - 48\cos 2x = 0,$ Yes.

**10.** $y = 5\ln x$

$y^{(4)} = -\dfrac{30}{x^4}$

$y^{(4)} - 16y = -\dfrac{30}{x^4} - 80\ln x \neq 0,$ No.

**12.** $y = 3e^{2x} - 4\sin 2x$

$y^{(4)} = 48e^{2x} - 64\sin 2x$

$y^{(4)} - 16y = (24e^{2x} - 64\sin 2x) - 16(3e^{2x} - 4\sin 2x) = 0,$ Yes

**In 14–18, the differential equation is $xy' - 2y = x^3e^x$.**

**14.** $y = x^2e^x,\ y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x,$ Yes.

**16.** $y = \sin x, y' = \cos x$

$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x,$ No.

**18.** $y = x^2e^x - 5x^2, y' = x^2e^x + 2xe^x - 10x$

$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x,$ Yes.

**20.** $y = A\sin\omega t$

$\dfrac{d^2y}{dt^2} = -A\omega^2\sin\omega t$

Since $(d^2y/dt^2) + 16y = 0$, we have

$-A\omega^2\sin\omega t + 16A\sin\omega t = 0.$

Thus, $\omega^2 = 16$ and $\omega = \pm 4$.

**22.** $2x^2 - y^2 = C$ passes through $(3, 4)$

$2(9) - 16 = C \implies C = 2$

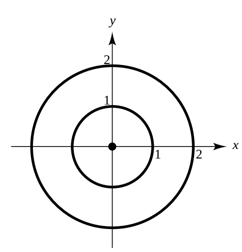
Particular solution: $2x^2 - y^2 = 2$

**24.** Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point

$C = 1, C = 4$, Circles




**26.** Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$6x + 4yy' = 0$

$2(3x + 2yy') = 0$

$3x + 2yy' = 0$

Initial condition:
$y(1) = 3$: $3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

**28.** Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$$y' = C_2\left(\frac{1}{x}\right), y'' = -C_2\left(\frac{1}{x^2}\right)$$

$$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$$

Initial conditions: $y(2) = 0,\ y'(2) = \frac{1}{2}$

$$0 = C_1 + C_2 \ln 2$$

$$y' = \frac{C_2}{x}$$

$$\frac{1}{2} = \frac{C_2}{2} \implies C_2 = 1,\ C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$

**30.** Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2 x)$

$$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2 x) + C_2 e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x\right)$$

$$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2 x\right)$$

$$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2 x\right) - 12(e^{2x/3})\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x\right) + 4(e^{2x/3})(C_1 + C_2 x) = 0$$

Initial conditions: $y(0) = 4,\ y(3) = 0$

$$0 = e^2(C_1 + 3C_2)$$

$$4 = (1)(C_1 + 0) \implies C_1 = 4$$

$$0 = e^2(4 + 3C_2) \implies C_2 = -\frac{4}{3}$$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

**32.** $\dfrac{dy}{dx} = x^3 - 4x$

$$y = \int (x^3 - 4x)\,dx = \frac{x^4}{4} - 2x^2 + C$$

**34.** $\dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x}$

$$y = \int \frac{e^x}{1 + e^x}\,dx = \ln(1 + e^x) + C$$

**36.** $\dfrac{dy}{dx} = x \cos x^2$

$$y = \int x \cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C$$

$(u = x^2,\ du = 2x\,dx)$

**38.** $\dfrac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

**40.** $\dfrac{dy}{dx} = x\sqrt{5 - x}$. Let $u = \sqrt{5 - x}$, $u^2 = 5 - x$, $dx = -2u\,du$

$$\begin{aligned} y = \int x\sqrt{5 - x}\,dx &= \int (5 - u^2)u(-2u)\,du \\ &= \int (-10u^2 + 2u^4)\,du \\ &= \frac{-10u^3}{3} + \frac{2u^5}{5} + C \\ &= -\frac{10}{3}(5 - x)^{3/2} + \frac{2}{5}(5 - x)^{5/2} + C \end{aligned}$$

**42.** $\dfrac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2}\,dx = 5(-2)\int e^{-x/2}\left(-\frac{1}{2}\right)dx$$
$$= -10e^{-x/2} + C$$

**44.** $\dfrac{dy}{dx} = \dfrac{x^2+2}{3y^2}$

$$\int 3y^2\,dy = \int (x^2+2)dx$$
$$y^3 = \frac{x^3}{3} + 2x + C$$

**46.** $\dfrac{dr}{ds} = 0.05s$

$$\int dr = \int 0.05s\,ds$$
$$r = 0.025s^2 + C$$

**48.** $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + \ln C = \ln Cx$$
$$y = Cx$$

**50.** $y\dfrac{dy}{dx} = 6\cos \pi x$

$$\int y\,dy = \int 6\cos \pi x\,dx$$
$$\frac{y^2}{2} = \frac{6}{\pi}\sin \pi x + C_1$$
$$y^2 = \frac{12}{\pi}\sin \pi x + C$$

**52.** $\sqrt{x^2-9}\dfrac{dy}{dx} = 5x$

$$\int dy = \int \frac{5x}{\sqrt{x^2-9}}\,dx$$
$$y = 5(x^2-9)^{1/2} + C$$

**54.** $4y\dfrac{dy}{dx} = 3e^x$

$$\int 4y\,dy = \int 3e^x\,dx$$
$$2y^2 = 3e^x + C$$

**56.** $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}\,dy = -\int x^{1/2}\,dx$$
$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$
$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4,$
$(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

**58.** $2xy' - \ln x^2 = 0$

$$2x\frac{dy}{dx} = 2\ln x$$
$$\int dy = \int \frac{\ln x}{x}\,dx$$
$$y = \frac{(\ln x)^2}{2} + C$$

$y(1) = 2$: $2 = C$

$$y = \frac{1}{2}(\ln x)^2 + 2$$

**60.** $y\sqrt{1-x^2}\dfrac{dy}{dx} = x\sqrt{1-y^2}$

$$\int (1-y^2)^{-1/2}y\,dy = \int (1-x^2)^{-1/2}x\,dx$$
$$-(1-y^2)^{1/2} = -(1-x^2)^{1/2} + C$$

$y(0) = 1$: $0 = -1 + C \implies C = 1$

$$\sqrt{1-y^2} = \sqrt{1-x^2} - 1$$

**62.** $\dfrac{dr}{ds} = e^{r-2s}$

$$\int e^{-r}dr = \int e^{-2s}\,ds$$
$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

$r(0) = 0$: $-1 = -\dfrac{1}{2} + C \implies C = -\dfrac{1}{2}$

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$
$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$
$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1+e^{-2s}}{2}\right)$$
$$r = \ln\left(\frac{2}{1+e^{-2s}}\right)$$

**64.** $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T-70} = -k\int dt$$
$$\ln(T - 70) = -kt + C_1$$
$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140$;
$140 - 70 = 70 = Ce^0 = C$

Particular solution: $T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

**66.** $\dfrac{dy}{dx} = \dfrac{2y}{3x}$

$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$
$$\ln y^3 = \ln x^2 + \ln C$$
$$y^3 = Cx^2$$

Initial condition: $y(8) = 2$, $2^3 = C(8^2)$, $C = \dfrac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \dfrac{1}{2}x^{2/3}$

**68.** $m = \dfrac{dy}{dx} = \dfrac{y-0}{x-0} = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$
$$y = Cx$$

**70.** $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$

Not homogeneous

**72.** $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}}$$
$$= \frac{t^2xy}{t\sqrt{x^2+y^2}} = t\frac{xy}{\sqrt{x^2+y^2}}$$

Homogeneous of degree 1

**74.** $f(x, y) = \tan(x + y)$

$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$

Not homogeneous

**76.** $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

**78.** $y' = \dfrac{(x^3 + y^3)}{xy^2}$

$$xy^2\,dy = (x^3 + y^3)\,dx$$
$$y = vx, \quad dy = x\,dv + v\,dx$$
$$x(vx)^2(x\,dv + v\,dx) = (x^3 + (vx)^3)\,dx$$
$$x^4v^2\,dv + x^3v^3\,dx = x^3\,dx + v^3x^3\,dx$$
$$xv^2\,dv = dx$$
$$\int v^2\,dv = \int \frac{1}{x}\,dx$$
$$\frac{v^3}{3} = \ln|x| + C$$
$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$
$$y^3 = 3x^3\ln|x| + Cx^3$$

**80.** $y' = \dfrac{x^2 + y^2}{2xy}$, $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2v}$$
$$2v\,dx + 2x\,dv = \frac{1 + v^2}{v}\,dx$$
$$\int \frac{2v}{v^2 - 1}\,dv = -\int \frac{dx}{x}$$
$$\ln(v^2 - 1) = -\ln x + \ln C = \ln\frac{C}{x}$$
$$v^2 - 1 = \frac{C}{x}$$
$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$
$$y^2 - x^2 = Cx$$

**82.** $y' = \dfrac{2x + 3y}{x}, y = vx$

$$v + x\frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$

$$x\frac{dv}{dx} = 2 + 2v \implies \int \frac{dv}{1 + v} = 2\int \frac{dx}{x}$$

$$\ln|1 + v| = \ln x^2 + \ln C = \ln x^2C$$

$$1 + v = x^2C$$

$$1 + \frac{y}{x} = x^2C$$

$$\frac{y}{x} = x^2C - 1$$

$$y = Cx^3 - x$$

**84.** $-y^2\,dx + x(x + y)\,dy = 0, y = vx$

$$-x^2v^2\,dx + (x^2 + x^2v)(v\,dx + x\,dv) = 0$$

$$\int \frac{1 + v}{v}\,dv = -\int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

Initial condition: $y(1) = 1, 1 = Ce^{-1} \implies C = e$

Particular solution: $y = e^{1 - y/x}$

**86.** $(2x^2 + y^2)\,dx + xy\,dy = 0$

Let $y = vx, dy = x\,dv + v\,dx$.

$$(2x^2 + v^2x^2)\,dx + x(vx)(x\,dv + v\,dx) = 0$$

$$(2x^2 + 2x^2v^2)\,dx + x^3v\,dv = 0$$

$$(2 + 2v^2)\,dx = -xv\,dv$$

$$\frac{-2}{x}\,dx = \frac{v}{1 + v^2}\,dv$$

$$-2\ln x = \frac{1}{2}\ln(1 + v^2) + C_1$$

$$\ln x^{-2} = \ln(1 + v^2)^{1/2} + \ln C$$

$$x^{-2} = C(1 + v^2)^{1/2}$$

$$\frac{1}{x^2} = C\left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x}(x^2 + y^2)^{1/2}$$

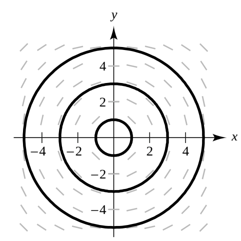
$$\frac{1}{x} = C(x^2 + y^2)^{1/2}$$

$y(1) = 0$: $1 = C(1 + 0) \implies C = 1$

$$\frac{1}{x} = \sqrt{x^2 + y^2}$$

$$1 = x\sqrt{x^2 + y^2}$$

**88.** $\dfrac{dy}{dx} = -\dfrac{x}{y}$




$$y\,dy = -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

**90.** $\dfrac{dy}{dx} = 0.25x(4 - y)$

$$\frac{dy}{4 - y} = 0.25x\,dx$$

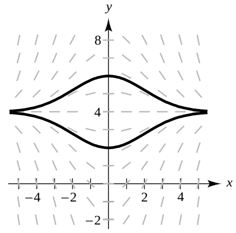
$$\int \frac{dy}{y - 4} = \int -0.25x\,dx$$

$$= \; = -\frac{1}{4}\int x\,dx$$

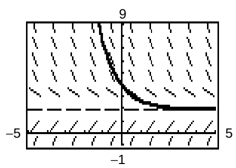
$$\ln|y - 4| = -\frac{1}{8}x^2 + C_1$$

$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$

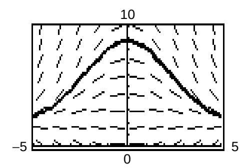
$$y = 4 + Ce^{-(1/8)x^2}$$




**92.** $\dfrac{dy}{dx} = 2 - y,\ y(0) = 4$




**94.** $\dfrac{dy}{dx} = 0.2x(2 - y),\ y(0) = 9$




**96.** $\dfrac{dy}{dt} = ky,\ y = Ce^{kt}$

Initial conditions: $y(0) = 20,\ y(1) = 16$

$$20 = Ce^0 = C$$

$$16 = 20e^k$$

$$k = \ln\frac{4}{5}$$

Particular solution: $y = 20e^{t\ln(4/5)}$

When 75% has been changed:

$$5 = 20e^{t\ln(4/5)}$$

$$\frac{1}{4} = e^{t\ln(4/5)}$$

$$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2 \text{ hr}$$

**98.** $\dfrac{dy}{dx} = k(x - 4)$

The direction field satisfies $(dy/dx) = 0$ along $x = 4$: Matches (b).

**100.** $\dfrac{dy}{dx} = ky^2$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as $y$ increases. Matches (d).

**102.** From Exercise 101,

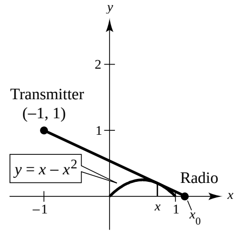
$$w = 1200 - Ce^{-kt},\ k = 1$$

$$w = 1200 - Ce^{-t}$$

$$w(0) = w_0 = 1200 - C \implies C = 1200 - w_0$$

$$w = 1200 - (1200 - w_0)e^{-t}$$

**104.** Let the radio receiver be located at $(x_0, 0)$. The tangent line to $y = x - x^2$ joins $(-1, 1)$ and $(x_0, 0)$.




(a) If $(x, y)$ is the point of tangency on the $y = x - x^2$, then

$$1 - 2x = \frac{y-1}{x+1} = \frac{x - x^2 - 1}{x+1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4+8}}{2}\right) = -1 + \sqrt{3}$$

$$y = x - x^2 = 3\sqrt{3} - 5$$

Then $\dfrac{1-0}{-1-x_0} = \dfrac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \dfrac{6 - 3\sqrt{3}}{-\sqrt{3}}$

$$\sqrt{3} = (1 + x_0)\left(6 - 3\sqrt{3}\right)$$

$$= 6 - 3\sqrt{3} + x_0\left(6 - 3\sqrt{3}\right)$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155$$

(b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y-h}{x+1} = \frac{x - x^2 - h}{x+1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{\left(-2 \pm \sqrt{4 + 4(h+1)}\right)}{2}$$

$$= -1 + \sqrt{2+h}$$

$$y = x - x^2$$

$$= 3\sqrt{2+h} - h - 4$$

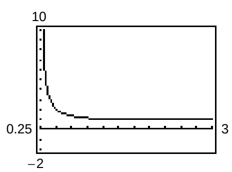
Then, $\dfrac{h-0}{-1-x_0} = \dfrac{h - \left(3\sqrt{2+h} - h - 4\right)}{-1 - \left(-1 + \sqrt{2+h}\right)}$

$$= \frac{2h + 4 - 3\sqrt{2+h}}{-\sqrt{2+h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2+h}}{2h + 4 - 3\sqrt{2+h}}$$

$$x_0 = \frac{h\sqrt{2+h}}{2h + 4 - 3\sqrt{2+h}} - 1$$

(c)




There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

**106.** Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

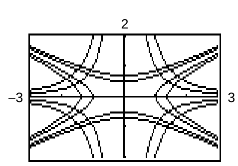
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \dfrac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$

$$\ln y = -2\ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$




**108.** Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

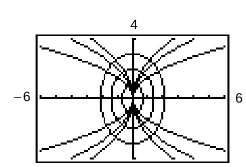
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\dfrac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



**110.** Given family (exponential functions): $y = Ce^x$

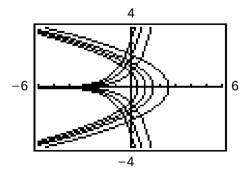
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\dfrac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$




**112.** The number of initial conditions matches the number of constants in the general solution.

**114.** Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

**116.** True

$$\frac{dy}{dx} = (x-2)(y+1)$$

**118.** True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C-y} \qquad \frac{dy}{dx} = \frac{K-x}{y}$$

$$\frac{x}{C-y}\cdot\frac{K-x}{y} = \frac{Kx-x^2}{Cy-y^2} = \frac{2Kx-2x^2}{2Cy-2y^2} = \frac{x^2+y^2-2x^2}{x^2+y^2-2y^2} = \frac{y^2-x^2}{x^2-y^2} = -1$$

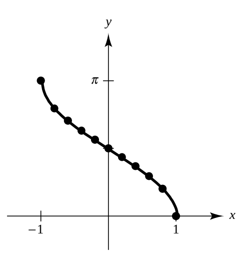
# Section 5.8 Inverse Trigonometric Functions: Differentiation

**2.** $y = \arccos x$

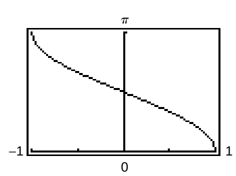
(a)

| $x$ | $-1$ | $-0.8$ | $-0.6$ | $-0.4$ | $-0.2$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 3.142 | 2.499 | 2.214 | 1.982 | 1.772 | 1.571 | 1.369 | 1.159 | 0.927 | 0.634 | 0 |

(b)




(c)




(d) Intercepts: $\left(0, \dfrac{\pi}{2}\right)$ and $(1, 0)$

No symmetry

**4.** $\left(\_\_, \dfrac{\pi}{4}\right) = \left(1, \dfrac{\pi}{4}\right)$

$\left(\_\_, -\dfrac{\pi}{6}\right) = \left(-\dfrac{\sqrt{3}}{3}, -\dfrac{\pi}{6}\right)$

$\left(-\sqrt{3}, \_\_\right) = \left(-\sqrt{3}, -\dfrac{\pi}{3}\right)$

**6.** $\arcsin 0 = 0$

**8.** $\arccos 0 = \dfrac{\pi}{2}$

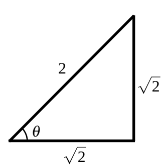
**10.** $\text{arc cot}\left(-\sqrt{3}\right) = \dfrac{5\pi}{6}$

**12.** $\arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$

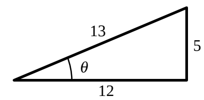
**14.** $\arcsin(-0.39) \approx -0.40$

**16.** $\arctan(-3) \approx -1.25$

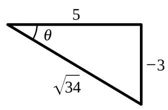
**18.** (a) $\tan\left(\arccos\dfrac{\sqrt{2}}{2}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1$



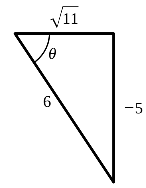

(b) $\cos\left(\arcsin\dfrac{5}{13}\right) = \dfrac{12}{13}$




**20.** (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right] = \dfrac{\sqrt{34}}{5}$



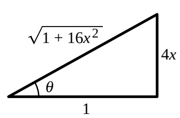

(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right] = -\dfrac{5\sqrt{11}}{11}$




**22.** $y = \sec(\text{arc tan } 4x)$

$\theta = \arctan 4x$

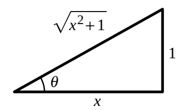
$y = \sec\theta = \sqrt{1 + 16x^2}$




**24.** $y = \cos(\text{arccot } x)$

$\theta = \text{arccot } x$

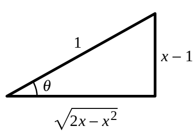
$y = \cos\theta = \dfrac{x}{\sqrt{x^2 + 1}}$




**26.** $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

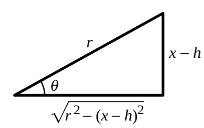
$y = \sec\theta = \dfrac{1}{\sqrt{2x - x^2}}$



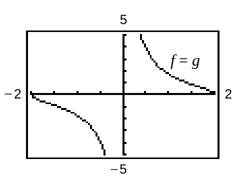

**28.** $y = \cos\left(\arcsin\dfrac{x - h}{r}\right)$

$\theta = \arcsin\dfrac{x - h}{r}$

$y = \cos\theta = \dfrac{\sqrt{r^2 - (x - h)^2}}{r}$




**30.**

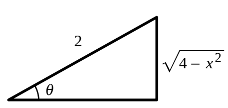



Asymptote: $x = 0$

$\arccos\dfrac{x}{2} = \theta$

$\cos\theta = \dfrac{x}{2}$

$\tan\theta = \dfrac{\sqrt{4 - x^2}}{x}$



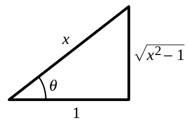

**32.** $\arctan(2x - 5) = -1$

$2x - 5 = \tan(-1)$

$x = \dfrac{1}{2}(\tan(-1) + 5) \approx 1.721$

**34.** $\arccos x = \operatorname{arcsec} x$

$$x = \cos(\operatorname{arcsec} x)$$
$$x = \frac{1}{x}$$
$$x^2 = 1$$
$$x = \pm 1$$




**36.** (a) $\arcsin(-x) = -\arcsin x,\ |x| \le 1.$

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \implies x = -\sin y \implies x = \sin(-y).$$

Thus, $-y = \arcsin x \implies y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x,\ |x| \le 1.$

Let $y = \arccos(-x)$. Then,

$$-x = \cos y \implies x = -\cos y \implies x = \cos(\pi - y).$$

Thus, $\pi - y = \arccos x \implies y = \pi - \arccos x$. Therefore, $\arccos(-x) = \pi - \arccos x$.

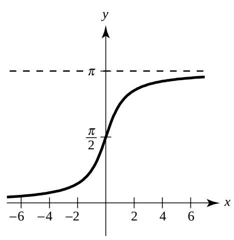
**38.** $f(x) = \arctan x + \dfrac{\pi}{2}$

$$x = \tan\left(y - \frac{\pi}{2}\right)$$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.

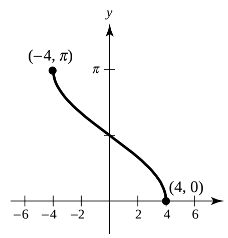



**40.** $f(x) = \arccos\left(\dfrac{x}{4}\right)$

$$\frac{x}{4} = \cos y$$
$$x = 4\cos y$$

Domain: $[-4, 4]$

Range: $[0, \pi]$




**42.** $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1 - t^4}}$$

**44.** $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x|\sqrt{4x^2 - 1}} = \frac{1}{|x|\sqrt{4x^2 - 1}}$$

**46.** $f(x) = \arctan \sqrt{x}$

$$f'(x) = \left(\frac{1}{1 + x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1 + x)}$$

**48.** $h(x) = x^2 \arctan x$

$$h'(x) = 2x \arctan x + \frac{x^2}{1 + x^2}$$

**50.** $f(x) = \arcsin x + \arccos x = \dfrac{\pi}{2}$

$$f'(x) = 0$$

**52.** $y = \ln(t^2 + 4) - \dfrac{1}{2} \arctan \dfrac{t}{2}$

$$y' = \frac{2t}{t^2 + 4} - \frac{1}{2} \cdot \frac{1}{1 + \left(\frac{t}{2}\right)^2}\left(\frac{1}{2}\right)$$
$$= \frac{2t}{t^2 + 4} - \frac{1}{t^2 + 4} = \frac{2t - 1}{t^2 + 4}$$

**54.** $y = \dfrac{1}{2}\left[x\sqrt{4 - x^2} + 4 \arcsin\left(\dfrac{x}{2}\right)\right]$

$$y' = \frac{1}{2}\left[x\frac{1}{2}(4 - x^2)^{-1/2}(-2x) + \sqrt{4 - x^2} + 2\frac{1}{\sqrt{1 - (x/2)^2}}\right]$$
$$= \frac{1}{2}\left[\frac{-x^2}{\sqrt{4 - x^2}} + \sqrt{4 - x^2} + \frac{4}{\sqrt{4 - x^2}}\right]$$
$$= \sqrt{4 - x^2}$$

**56.** $y = x \arctan 2x - \dfrac{1}{4} \ln(1 + 4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1 + 4x^2} + \arctan(2x) - \frac{1}{4}\left(\frac{8x}{1 + 4x^2}\right) = \arctan(2x)$$

**58.** $y = 25 \arcsin \dfrac{x}{5} - x\sqrt{25 - x^2}$

$$\begin{aligned} y' &= 5\frac{1}{\sqrt{1 - (x/2)^2}} - \sqrt{25 - x^2} - x\frac{1}{2}(25 - x^2)^{-1/2}(-2x) \\ &= \frac{25}{\sqrt{25 - x^2}} - \frac{(25 - x^2)}{\sqrt{25 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}} \\ &= \frac{2x^2}{\sqrt{25 - x^2}} \end{aligned}$$

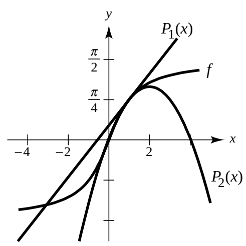
**60.** $y = \arctan \dfrac{x}{2} - \dfrac{1}{2(x^2 + 4)}$

$$\begin{aligned} y' &= \frac{1}{2}\frac{1}{1 + (x/2)^2} + \frac{1}{2}(x^2 + 4)^{-2}(2x) \\ &= \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2} \\ &= \frac{2x^2 + 8 + x}{(x^2 + 4)^2} \end{aligned}$$

**62.** $f(x) = \arctan x,\ a = 1$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$




$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$

**64.** $f(x) = \arcsin x - 2x$

$f'(x) = \dfrac{1}{\sqrt{1 - x^2}} - 2 = 0$ when $\sqrt{1 - x^2} = \dfrac{1}{2}$ or

$$x = \pm\frac{\sqrt{3}}{2}.$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

Relative minimum: $\left(\dfrac{\sqrt{3}}{2}, -0.68\right)$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

Relative maximum: $\left(-\dfrac{\sqrt{3}}{2}, 0.68\right)$

**66.** $f(x) = \arcsin x - 2 \arctan x$

$$\begin{aligned} f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} &= 0 \\ 1 + x^2 &= 2\sqrt{1 - x^2} \\ 1 + 2x^2 + x^4 &= 4(1 - x^2) \\ x^4 + 6x^2 - 3 &= 0 \\ x &= \pm 0.681 \end{aligned}$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

**68.** $\arctan 0 = 0$. $\pi$ is not in the range of $y = \arctan x$.

**70.** The derivatives are algebraic. See Theorem 5.18.

**72.** (a) $\cot \theta = \dfrac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\dfrac{d\theta}{dt} = \dfrac{-3}{x^2 + 9}\dfrac{dx}{dt}$

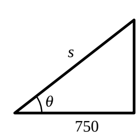
If $x = 10$, $\dfrac{d\theta}{dt} \approx 11.001$ rad/hr.

If $x = 3$, $\dfrac{d\theta}{dt} \approx 66.667$ rad/hr.

A lower altitude results in a greater rate of change of $\theta$.

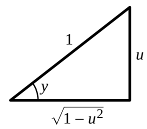
**74.** $\cos \theta = \dfrac{750}{s}$

$$\theta = \arccos\left(\frac{750}{s}\right)$$



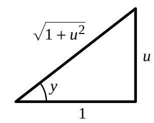

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{d\theta}{ds}\cdot\frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}}\left(\frac{-750}{s^2}\right)\frac{ds}{dt} \\ &= \frac{750}{s\sqrt{s^2 - 750^2}}\frac{ds}{dt} \end{aligned}$$

**76.** (a) Let $y = \arcsin u$. Then



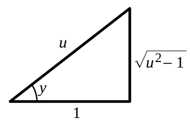

$$\sin y = u$$
$$\cos y \cdot y' = u'$$
$$\frac{dy}{dx} = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}.$$

(b) Let $y = \arctan u$. Then




$$\tan y = u$$
$$\sec^2 y\,\frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}.$$

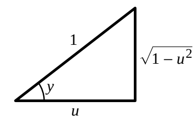
(c) Let $y = \operatorname{arcsec} u$. Then




$$\sec y = u$$
$$\sec y \tan y\,\frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}.$$

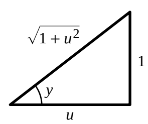
**Note:** The absolute value sign in the formula for the derivative of arcsec $u$ is necessary because the inverse secant function has a positive slope at every value in its domain.

(d) Let $y = \arccos u$. Then



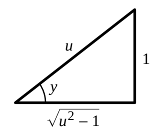

$$\cos y = u$$
$$-\sin y\,\frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}.$$

(e) Let $y = \operatorname{arccot} u$. Then




$$\cot y = u$$
$$-\csc^2 y\,\frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}.$$

(f) Let $y = \operatorname{arccsc} u$. Then




$$\csc y = u$$
$$-\csc y \cot y\,\frac{dy}{dx} = u'$$
$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}$$

**Note:** The absolute value sign in the formula for the derivative of arccsc $u$ is necessary because the inverse cosecant function has a negative slope at every value in its domain.

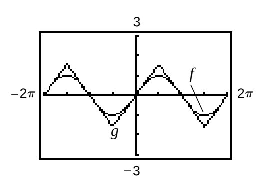
**78.** $f(x) = \sin x$

$g(x) = \arcsin(\sin x)$

(a) The range of $y = \arcsin x$ is $-\pi/2 \le y \le \pi/2$.

(b) Maximum: $\pi/2$

Minimum: $-\pi/2$




**80.** False

The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

**82.** False

$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$

## Section 5.9 Inverse Trigonometric Functions: Integration

**2.** $\displaystyle\int \frac{3}{\sqrt{1-4x^2}}\,dx = \frac{3}{2}\int \frac{2}{\sqrt{1-4x^2}}\,dx = \frac{3}{2}\arcsin(2x) + C$

**4.** $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$

**6.** $\displaystyle\int \frac{4}{1+9x^2}\,dx = \frac{4}{3}\int \frac{3}{1+9x^2}\,dx = \frac{4}{3}\arctan(3x) + C$

**8.** $\displaystyle\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx = \left[\frac{1}{3}\arctan\frac{x}{3}\right]_{\sqrt{3}}^3 = \frac{\pi}{36}$

**10.** $\displaystyle\int \frac{1}{4+(x-1)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$

**12.** $\displaystyle\int \frac{x^4-1}{x^2+1}\,dx = \int (x^2-1)\,dx = \frac{1}{3}x^3 - x + C$

**14.** Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{t^4 + 16}\,dt = \frac{1}{2}\int \frac{1}{(4)^2 + (t^2)^2}(2t)\,dt = \frac{1}{8}\arctan\frac{t^2}{4} + C$$

**16.** Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{1}{x\sqrt{x^4 - 4}}\,dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2 - 2^2}}(2x)\,dx$$

$$= \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$$

**18.** Let $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1 - x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1 - x^2}}\,dx$$

$$= \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$$

**20.** Let $u = 1 + x^2$, $du = 2x\,dx$.

$$\int_{-\sqrt{3}}^0 \frac{x}{1 + x^2}\,dx = \frac{1}{2}\int_{-\sqrt{3}}^0 \frac{1}{1 + x^2}(2x)\,dx$$

$$= \left[\frac{1}{2}\ln(1 + x^2)\right]_{-\sqrt{3}}^0 = -\ln 2$$

**22.** $$\int_1^2 \frac{1}{3 + (x - 2)^2}\,dx = \int_1^2 \frac{1}{\left(\sqrt{3}\right)^2 + (x - 2)^2}\,dx = \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x - 2}{\sqrt{3}}\right)\right]_1^2 = \frac{\sqrt{3}\pi}{18}$$

**24.** $$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \arctan(\sin x)\Big]_0^{\pi/2} = \frac{\pi}{4}$$

**26.** $\displaystyle\int \frac{3}{2\sqrt{x}(1 + x)}\,dx$. $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$, $dx = 2u\,du$

$$\frac{3}{2}\int \frac{2u\,du}{u(1 + u^2)} = 3\int \frac{du}{1 + u^2} = 3\arctan u + C$$

$$= 3\arctan\sqrt{x} + C$$

**28.** $$\int \frac{4x + 3}{\sqrt{1 - x^2}}\,dx = (-2)\int \frac{-2x}{\sqrt{1 - x^2}}\,dx + 3\int \frac{1}{\sqrt{1 - x^2}}\,dx = -4\sqrt{1 - x^2} + 3\arcsin x + C$$

**30.** $$\int \frac{x - 2}{(x + 1)^2 + 4}\,dx = \frac{1}{2}\int \frac{2x + 2}{(x + 1)^2 + 4}\,dx - \int \frac{3}{(x + 1)^2 + 4}\,dx$$

$$= \frac{1}{2}\ln(x^2 + 2x + 5) - \frac{3}{2}\arctan\left(\frac{x + 1}{2}\right) + C$$

**32.** $$\int_{-2}^2 \frac{dx}{x^2 + 4x + 13} = \int_{-2}^2 \frac{dx}{(x + 2)^2 + 9} = \left[\frac{1}{3}\arctan\left(\frac{x + 2}{3}\right)\right]_{-2}^2 = \frac{1}{3}\arctan\left(\frac{4}{3}\right)$$

**34.** $$\int \frac{2x - 5}{x^2 + 2x + 2}\,dx = \int \frac{2x + 2}{x^2 + 2x + 2}\,dx - 7\int \frac{1}{1 + (x + 1)^2}\,dx = \ln\left|x^2 + 2x + 2\right| - 7\arctan(x + 1) + C$$

**36.** $$\int \frac{2}{\sqrt{-x^2 + 4x}}\,dx = \int \frac{2}{\sqrt{4 - (x^2 - 4x + 4)}}\,dx$$

$$= \int \frac{2}{\sqrt{4 - (x - 2)^2}}\,dx$$

$$= 2\arcsin\left(\frac{x - 2}{2}\right) + C$$

**38.** Let $u = x^2 - 2x$, $du = (2x - 2)\,dx$.

$$\int \frac{x - 1}{\sqrt{x^2 - 2x}}\,dx = \frac{1}{2}\int (x^2 - 2x)^{-1/2}(2x - 2)\,dx$$

$$= \sqrt{x^2 - 2x} + C$$

**40.** $$\int \frac{1}{(x - 1)\sqrt{x^2 - 2x}}\,dx = \int \frac{1}{(x - 1)\sqrt{(x - 1)^2 - 1}}\,dx = \operatorname{arcsec}\left|x - 1\right| + C$$

**42.** Let $u = x^2 - 4$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}}\,dx = \frac{1}{2}\arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

**44.** Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u\,du = dx$

$$\int \frac{\sqrt{x - 2}}{x + 1}\,dx = \int \frac{2u^2}{u^2 + 3}\,du = \int \frac{2u^2 + 6 - 6}{u^2 + 3}\,du = 2\int du - 6\int \frac{1}{u^2 + 3}\,du$$

$$= 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3}\arctan\sqrt{\frac{x - 2}{3}} + C$$

**46.** The term is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$: $x^2 + 3x = x^2 + 3x + \frac{9}{4} - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$

**48.** (a) $\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$, $u = x^2$

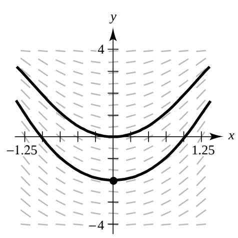
(c) $\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C$, $u = \frac{1}{x}$

**50.** (a) $\int \frac{1}{1 + x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int \frac{x}{1 + x^4}\,dx = \frac{1}{2}\int \frac{2x}{1 + (x^2)^2}\,dx = \frac{1}{2}\arctan(x^2) + C$, $u = x^2$

(c) $\int \frac{x^3}{1 + x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1 + x^4}\,dx = \frac{1}{4}\ln(1 + x^4) + C$, $u = 1 + x^4$

**52.** (a)




(b) $\frac{dy}{dx} = x\sqrt{16 - y^2}$, $(0, -2)$

$$\frac{dy}{\sqrt{16 - y^2}} = x\,dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} + C$$

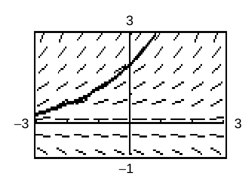
$$(0, -2): \arcsin\left(-\frac{2}{4}\right) = C \implies C = -\frac{\pi}{6}$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} - \frac{\pi}{6}$$

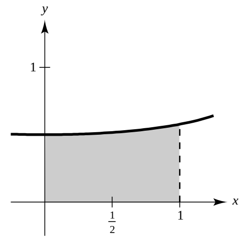
$$\frac{y}{4} = \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

$$y = 4\sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

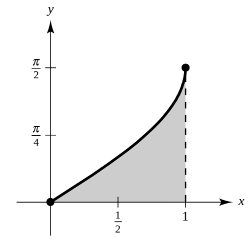
**54.** $\frac{dy}{dx} = \frac{2y}{\sqrt{16 - x^2}}$, $y(0) = 2$



**56.** $A = \int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$




**58.** $\int_0^1 \arcsin x\,dx \approx 0.571$




**60.** $F(x) = \frac{1}{2}\int_x^{x+2} \frac{2}{t^2 + 1}\,dt$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x + 2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

(b) $F(x) = \left[\arctan t\right]_x^{x+2} = \arctan(x + 2) - \arctan x$

$$F'(x) = \frac{1}{1 + (x + 2)^2} - \frac{1}{1 + x^2} = \frac{(1 + x^2) - (x^2 + 4x + 5)}{(x^2 + 1)(x^2 + 4x + 5)} = \frac{-4(x + 1)}{(x^2 + 1)(x^2 + 4x + 5)} = 0 \text{ when } x = -1.$$

**62.** $\int \frac{1}{\sqrt{6x - x^2}}\,dx$

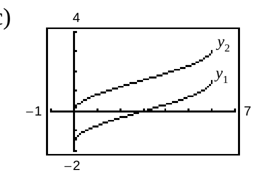
(a) $6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x - 3)^2$

$$\int \frac{1}{\sqrt{6x - x^2}}\,dx = \int \frac{dx}{\sqrt{9 - (x - 3)^2}} = \arcsin\left(\frac{x - 3}{3}\right) + C$$

(b) $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$$\int \frac{1}{\sqrt{6u^2 - u^4}}(2u\,du) = \int \frac{2}{\sqrt{6 - u^2}}\,du = 2\arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

(c)




The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

**64.** Let $f(x) = \arctan x - \frac{x}{1 + x^2}$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1 - x^2}{(1 + x^2)^2} = \frac{2x^2}{(1 + x^2)} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and $f$ is increasing for $x > 0$, $\arctan x - \frac{x}{1 + x^2} > 0$ for $x > 0$. Thus,

$$\arctan x > \frac{x}{1 + x^2}.$$

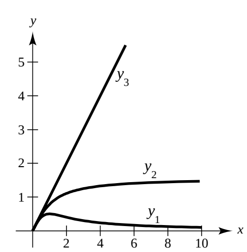
Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and $g$ is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$.

Therefore,

$$\frac{x}{1 + x^2} < \arctan x < x.$$



## Section 5.10 Hyperbolic Functions

**2.** (a) $\cosh(0) = \dfrac{e^0 + e^0}{2} = 1$

(b) $\operatorname{sech}(1) = \dfrac{2}{e + e^{-1}} \approx 0.648$

**4.** (a) $\sinh^{-1}(0) = 0$

(b) $\tanh^{-1}(0) = 0$

**6.** (a) $\operatorname{csch}^{-1}(2) = \ln\left(\dfrac{1 + \sqrt{5}}{2}\right) \approx 0.481$

(b) $\coth^{-1}(3) = \dfrac{1}{2}\ln\left(\dfrac{4}{2}\right) \approx 0.347$

**8.** $\dfrac{1 + \cosh 2x}{2} = \dfrac{1 + (e^{2x} + e^{-2x})/2}{2} = \dfrac{e^{2x} + 2 + e^{-2x}}{4} = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$

**10.** $2 \sinh x \cosh x = 2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

**12.**
$$\begin{aligned}
2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right) &= 2\left[\frac{e^{(x+y)/2} + e^{-(x+y)/2}}{2}\right]\left[\frac{e^{(x-y)/2} + e^{-(x-y)/2}}{2}\right] \\
&= 2\left[\frac{e^x + e^y + e^{-y} + e^{-x}}{4}\right] = \frac{e^x + e^{-x}}{2} + \frac{e^y + e^{-y}}{2} \\
&= \cosh x + \cosh y
\end{aligned}$$

**14.** $\tanh x = \dfrac{1}{2}$

$\left(\dfrac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \implies \operatorname{sech}^2 x = \dfrac{3}{4} \implies \operatorname{sech} x = \dfrac{\sqrt{3}}{2}$

$\cosh x = \dfrac{1}{\sqrt{3}/2} = \dfrac{2\sqrt{3}}{3}$

$\coth x = \dfrac{1}{1/2} = 2$

$\sinh x = \tanh x \cosh x = \left(\dfrac{1}{2}\right)\left(\dfrac{2\sqrt{3}}{3}\right) = \dfrac{\sqrt{3}}{3}$

$\operatorname{csch} x = \dfrac{1}{\sqrt{3}/3} = \sqrt{3}$

Putting these in order:

$\sinh x = \dfrac{\sqrt{3}}{3}$ $\quad \operatorname{csch} x = \sqrt{3}$

$\cosh x = \dfrac{2\sqrt{3}}{3}$ $\quad \operatorname{sech} x = \dfrac{\sqrt{3}}{2}$

$\tanh x = \dfrac{1}{2}$ $\quad \coth x = 2$

**16.** $y = \coth(3x)$

$y' = -3 \operatorname{csch}^2(3x)$

**18.** $g(x) = \ln(\cosh x)$

$g'(x) = \dfrac{1}{\cosh x}(\sinh x) = \tanh x$

**20.** $y = x \cosh x - \sinh x$

$y' = x \sinh x + \cosh x - \cosh x = x \sinh x$

**22.** $h(t) = t - \coth t$

$h'(t) = 1 + \operatorname{csch}^2 t = \coth^2 t$

**24.** $g(x) = \operatorname{sech}^2 3x$

$g'(x) = -2 \operatorname{sech}(3x) \operatorname{sech}(3x) \tanh(3x)(3)$

$= -6 \operatorname{sech}^2 3x \tanh 3x$

**26.** $f(x) = e^{\sinh x}$

$f'(x) = (\cosh x)(e^{\sinh x})$

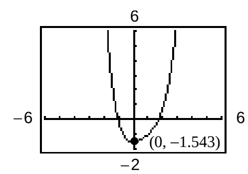
**28.** $y = \operatorname{sech}(x + 1)$

$y' = -\operatorname{sech}(x + 1) \tanh(x + 1)$

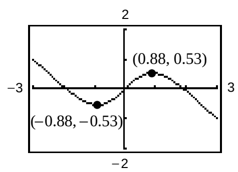
**30.** $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

$f'(x) = x \cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x \cosh(x - 1)$

$f'(x) = 0$ for $x = 0$. By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.




**32.** $h(x) = 2 \tanh x - x$




Relative maximum: $(0.88, 0.53)$

Relative minimum: $(-0.88, -0.53)$

**34.** $y = a \cosh x$

$y' = a \sinh x$

$y'' = a \cosh x$

Therefore, $y'' - y = 0$.

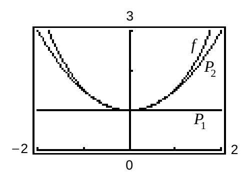
**36.** $f(x) = \cosh x$ $\qquad f(1) = \cosh(0) \approx 1$

$f'(x) = \sinh x$ $\qquad f'(1) = \sinh(0) \approx 0$

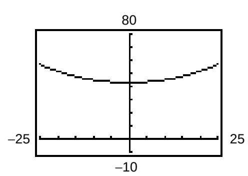
$f''(x) = \cosh x$ $\qquad f''(1) = \cosh(0) \approx 1$

$P_1(x) = f(0) + f'(0)(x - 0) = 1$

$P_2(x) = 1 + \frac{1}{2}x^2$




**38.** (a) $y = 18 + 25 \cosh \dfrac{x}{25}, -25 \le x \le 25$




(b) At $x = \pm 25$, $y = 18 + 25 \cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh \dfrac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$

**40.** Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cosh \sqrt{x}}{\sqrt{x}}\,dx = 2\int \cosh \sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) dx = 2 \sinh \sqrt{x} + C$$

**42.** Let $u = \cosh x$, $du = \sinh x\,dx$.

$$\int \frac{\sinh}{1 + \sinh^2 x}\,dx = \int \frac{\sinh x}{\cosh^2 x}\,dx = \frac{-1}{\cosh x} + C$$

$$= -\operatorname{sech} x + C$$

**44.** Let $u = 2x - 1$, $du = 2\,dx$.

$$\int \operatorname{sech}^2(2x - 1)\,dx = \frac{1}{2}\int \operatorname{sech}^2(2x - 1)(2)\,dx$$

$$= \frac{1}{2} \tanh(2x - 1) + C$$

**46.** Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x \tanh x\,dx$.

$$\int \operatorname{sech}^3 x \tanh x\,dx = -\int \operatorname{sech}^2 x(-\operatorname{sech} x \tanh x)\,dx$$

$$= -\frac{1}{3} \operatorname{sech}^3 x + C$$

**48.** $\displaystyle\int \cosh^2 x\,dx = \int \frac{1+\cosh 2x}{2}\,dx$

$\displaystyle = \frac{1}{2}\left[x + \frac{\sinh 2x}{2}\right] + C$

$\displaystyle = \frac{1}{2}x + \frac{1}{4}\sinh 2x + C$

**50.** $\displaystyle\int_0^4 \frac{1}{\sqrt{25-x^2}}\,dx = \left[\arcsin\frac{x}{5}\right]_0^4 = \arcsin\frac{4}{5}$

**52.** $\displaystyle\int \frac{2}{x\sqrt{1+4x^2}}\,dx = 2\int \frac{1}{(2x)\sqrt{1+(2x)^2}}(2)\,dx = -2\ln\left(\frac{1+\sqrt{1+4x^2}}{|2x|}\right) + C$

**54.** Let $u = \sinh x$, $du = \cosh x\,dx$.

$\displaystyle\int \frac{\cosh x}{\sqrt{9-\sinh^2 x}}\,dx = \arcsin\left(\frac{\sinh x}{3}\right) + C$

$\displaystyle = \arcsin\left(\frac{e^x - e^{-x}}{6}\right) + C$

**56.** $y = \tanh^{-1}\left(\dfrac{x}{2}\right)$

$\displaystyle y' = \frac{1}{1-(x/2)^2}\left(\frac{1}{2}\right) = \frac{2}{4-x^2}$

**58.** $y = \operatorname{sech}^{-1}(\cos 2x)$, $0 < x < \dfrac{\pi}{4}$

$\displaystyle y' = \frac{-1}{\cos 2x\sqrt{1-\cos^2 2x}}(-2\sin 2x) = \frac{2\sin 2x}{\cos 2x|\sin 2x|} = \frac{2}{\cos 2x} = 2\sec 2x,$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

**60.** $y = (\operatorname{csch}^{-1} x)^2$

$\displaystyle y' = 2\operatorname{csch}^{-1} x\left(\frac{-1}{|x|\sqrt{1+x^2}}\right) = \frac{-2\operatorname{csch}^{-1} x}{|x|\sqrt{1+x^2}}$

**62.** $\displaystyle y = x\tanh^{-1} x + \ln\sqrt{1-x^2} = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2)$

$\displaystyle y' = x\left(\frac{1}{1-x^2}\right) + \tanh^{-1} x + \frac{-x}{1-x^2} = \tanh^{-1} x$

**64.** See page 401, Theorem 5.22.

**66.** Equation of tangent line through $P = (x_0, y_0)$:

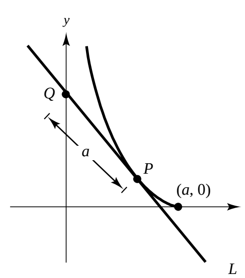
$$y - a\operatorname{sech}^{-1}\frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2 - x_0^2}}{x_0}(x - x_0)$$

When $x = 0$,

$$y = a\operatorname{sech}^{-1}\frac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a\operatorname{sech}^{-1}\frac{x_0}{a}.$$

Hence, $Q$ is the point $[0, a\operatorname{sech}^{-1}(x_0/a)]$.

Distance from $P$ to $Q$: $d = \sqrt{x_0^2 + \left(-\sqrt{a^2 - x_0^2}\right)^2} = a$




**68.** $\displaystyle\int \frac{x}{9-x^4}\,dx = -\frac{1}{2}\int \frac{-2x}{9-(x^2)^2}\,dx = -\frac{1}{2}\left(\frac{1}{6}\right)\ln\left|\frac{3-x^2}{3+x^2}\right| + C$

$\displaystyle = -\frac{1}{12}\ln\left|\frac{3-x^2}{3+x^2}\right| + C$

**70.** Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x}\,dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{1+(x^{3/2})^2}}\left(\frac{3}{2}\sqrt{x}\right)dx = \frac{2}{3}\sinh^{-1}(x^{3/2}) + C = \frac{2}{3}\ln\left(x^{3/2} + \sqrt{1+x^3}\right) + C$$

**72.** $$\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}}$$
$$= -\frac{1}{2}\ln\left(\frac{2+\sqrt{(x+2)^2+4}}{|x+2|}\right) + C$$

**74.** $$\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}}\,dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{(x+1)\sqrt{(x+1)^2+\left(\sqrt{3}\right)^2}}\,dx = -\frac{1}{\sqrt{6}}\ln\left(\frac{\sqrt{3}+\sqrt{(x+1)^2+3}}{x+1}\right) + C$$

**76.** Let $u = 2(x-1)$, $du = 2\,dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}}\,dx = \int \frac{2}{2(x-1)\sqrt{\left(\sqrt{3}\right)^2 - [2(x-1)]^2}}\,dx = -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$$

**78.** $$y = \int \frac{1-2x}{4x-x^2}\,dx = \int \frac{4-2x}{4x-x^2}\,dx + 3\int \frac{1}{(x-2)^2-4}\,dx$$
$$= \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C = \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{x-4}{x}\right| + C$$

**80.** $$A = \int_0^2 \tanh 2x\,dx$$
$$= \int_0^2 \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\,dx$$
$$= \frac{1}{2}\int_0^2 \frac{1}{e^{2x}+e^{-2x}}(2)(e^{2x}-e^{-2x})\,dx$$
$$= \left[\frac{1}{2}\ln(e^{2x}+e^{-2x})\right]_0^2$$
$$= \frac{1}{2}\ln(e^4+e^{-4}) - \frac{1}{2}\ln 2$$
$$= \ln\sqrt{\frac{e^4+e^{-4}}{2}} \approx 1.654$$

**82.** $$A = \int_3^5 \frac{6}{\sqrt{x^2-4}}\,dx$$
$$= \left[6\ln\left(x+\sqrt{x^2-4}\right)\right]_3^5$$
$$= 6\ln\left(5+\sqrt{21}\right) - 6\ln\left(3+\sqrt{5}\right)$$
$$= 6\ln\left(\frac{5+\sqrt{21}}{3+\sqrt{5}}\right) \approx 3.626$$

**84.** (a) $v(t) = -32t$

(b) $s(t) = \int v(t)\,dt = \int (-32t)\,dt = -16t^2 + C$

$s(0) = -16(0)^2 + C = 400 \Rightarrow C = 400$

$s(t) = -16t^2 + 400$

—CONTINUED—

**84. —CONTINUED—**

(c) $$\frac{dv}{dt} = -32 + kv^2$$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = -\int dt$$

Let $u = \sqrt{k}\,v$, then $du = \sqrt{k}\,dv$.

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -t + C$$

Since $v(0) = 0$, $C = 0$.

$$\ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -2\sqrt{32k}\,t$$

$$\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v} = e^{-2\sqrt{32k}\,t}$$

$$\sqrt{32} + \sqrt{k}\,v = e^{-2\sqrt{32k}\,t}\left(\sqrt{32} - \sqrt{k}\,v\right)$$

$$v\left(\sqrt{k} + \sqrt{k}\,e^{-2\sqrt{32k}\,t}\right) = \sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)$$

$$v = \frac{\sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)}{\sqrt{k}\left(e^{-2\sqrt{32k}\,t} + 1\right)} \cdot \frac{e^{\sqrt{32k}\,t}}{e^{\sqrt{32k}\,t}}$$

$$= \frac{\sqrt{32}}{\sqrt{k}}\left[\frac{-\left(e^{\sqrt{32k}\,t} - e^{-\sqrt{32k}\,t}\right)}{e^{\sqrt{32k}\,t} + e^{-\sqrt{32k}\,t}}\right]$$

$$= -\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)$$

(d) $$\lim_{t\to\infty}\left[-\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)\right] = -\frac{\sqrt{32}}{\sqrt{k}}.$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct)\,dt = (1/c)\ln\cosh(ct)$ (which can be verified by differentiation), then

$$s(t) = \int -\frac{\sqrt{32}}{\sqrt{k}}\tanh\left(\sqrt{32k}\,t\right)dt$$

$$= -\frac{\sqrt{32}}{\sqrt{k}}\frac{1}{\sqrt{32k}}\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C$$

$$= -\frac{1}{k}\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C.$$

When $t = 0$,

$$s(0) = C$$

$$= 400 \implies 400 - (1/k)\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right].$$

When $k = 0.01$,

$$s_2(t) = 400 - 100\ln\left(\cosh\sqrt{0.32}\,t\right)$$

$$s_1(t) = -16t^2 + 400.$$

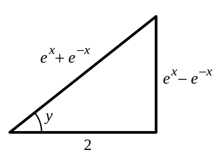
$s_1(t) = 0$ when $t = 5$ seconds.

$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

**86.** Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$




$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$

Thus, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

**88.**
$$y = \operatorname{sech}^{-1} x$$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)} = \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

**90.**
$$y = \sinh^{-1} x$$

$$\sinh y = x$$

$$(\cosh y)y' = 1$$

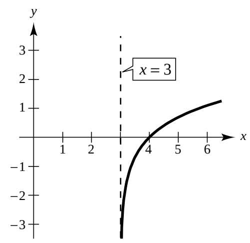
$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

# Review Exercises for Chapter 5

**2.** $f(x) = \ln(x - 3)$

Horizontal shift 3 units to the right

Vertical asymptote: $x = 3$




**4.** $\ln[(x^2 + 1)(x - 1)] = \ln(x^2 + 1) + \ln(x - 1)$

**6.** $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5 = 3\ln x - 6\ln(x^2 + 1) + \ln 5^2 = \ln x^3 - \ln(x^2 + 1)^6 + \ln 25 = \ln\left[\dfrac{25x^3}{(x^2 + 1)^6}\right]$

**8.** 
$$\begin{aligned} \ln x + \ln(x - 3) &= 0 \\ \ln x(x - 3) &= 0 \\ x(x - 3) &= e^0 \\ x^2 - 3x - 1 &= 0 \\ x &= \frac{3 \pm \sqrt{13}}{2} \\ x &= \frac{3 + \sqrt{13}}{2} \text{ only since } \frac{3 - \sqrt{13}}{2} < 0. \end{aligned}$$

**10.** $h(x) = \ln \dfrac{x(x - 1)}{x - 2} = \ln x + \ln(x - 1) - \ln(x - 2)$

$$h'(x) = \frac{1}{x} + \frac{1}{x - 1} - \frac{1}{x - 2} = \frac{x^2 - 4x + 2}{x^3 - 3x^2 + 2x}$$

**12.** $f(x) = \ln[x(x^2 - 2)^{2/3}] = \ln x + \dfrac{2}{3}\ln(x^2 - 2)$

$$f'(x) = \frac{1}{x} + \frac{2}{3}\left(\frac{2x}{x^2 - 2}\right) = \frac{7x^2 - 6}{3x^3 - 6x}$$

**14.** $y = \dfrac{1}{b^2}[a + bx - a\ln(a + bx)]$

$$\frac{dy}{dx} = \frac{1}{b^2}\left(b - \frac{ab}{a + bx}\right) = \frac{x}{a + bx}$$

**16.** 
$$\begin{aligned} y &= -\frac{1}{ax} + \frac{b}{a^2}\ln\frac{a + bx}{x} \\ &= -\frac{1}{ax} + \frac{b}{a^2}[\ln(a + bx) - \ln x] \\ \frac{dy}{dx} &= -\frac{1}{a}\left(-\frac{1}{x^2}\right) + \frac{b}{a^2}\left[\frac{b}{a + bx} - \frac{1}{x}\right] \\ &= \frac{1}{ax^2} + \frac{b}{a^2}\left[\frac{-a}{x(a + bx)}\right] = \frac{1}{ax^2} - \frac{b}{ax(a + bx)} \\ &= \frac{(a + bx) - bx}{ax^2(a + bx)} = \frac{1}{x^2(a + bx)} \end{aligned}$$

**18.** $u = x^2 - 1,\ du = 2x\,dx$

$$\int \frac{x}{x^2 - 1}\,dx = \frac{1}{2}\int \frac{2x}{x^2 - 1}\,dx = \frac{1}{2}\ln|x^2 - 1| + C$$

**20.** $u = \ln x,\ du = \dfrac{1}{x}\,dx$

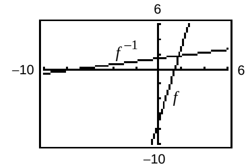
$$\int \frac{\ln\sqrt{x}}{x}\,dx = \frac{1}{2}\int (\ln x)\left(\frac{1}{x}\right)dx = \frac{1}{4}(\ln x)^2 + C$$

**22.** $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_1^e (\ln x)^1\left(\frac{1}{x}\right)dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}$

**24.** 
$$\begin{aligned} \int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx &= \left[\ln\left|\cos\left(\frac{\pi}{4} - x\right)\right|\right]_0^{\pi/4} \\ &= 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\ln 2 \end{aligned}$$

**26.** (a)
$$f(x) = 5x - 7$$
$$y = 5x - 7$$
$$\frac{y+7}{5} = x$$
$$\frac{x+7}{5} = y$$
$$f^{-1}(x) = \frac{x+7}{5}$$

(b)

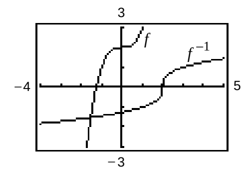



(c) $f^{-1}(f(x)) = f^{-1}(5x - 7) = \dfrac{(5x-7)+7}{5} = x$

$f(f^{-1}(x)) = f\left(\dfrac{x+7}{5}\right) = 5\left(\dfrac{x+7}{5}\right) - 7 = x$

**28.** (a)
$$f(x) = x^3 + 2$$
$$y = x^3 + 2$$
$$\sqrt[3]{y-2} = x$$
$$\sqrt[3]{x-2} = y$$
$$f^{-1}(x) = \sqrt[3]{x-2}$$

(b)

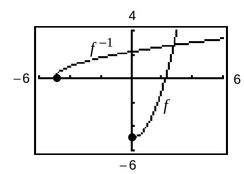



(c) $f^{-1}(f(x)) = f^{-1}(x^3 + 2) = \sqrt[3]{(x^3+2) - 2} = x$

$f(f^{-1}(x)) = f\left(\sqrt[3]{x-2}\right) = \left(\sqrt[3]{x-2}\right)^3 + 2 = x$

**30.** (a)
$$f(x) = x^2 - 5, x \geq 0$$
$$y = x^2 - 5$$
$$\sqrt{y+5} = x$$
$$\sqrt{x+5} = y$$
$$f^{-1}(x) = \sqrt{x+5}$$

(b)




(c) $f^{-1}(f(x)) = f^{-1}(x^2 - 5) = \sqrt{(x^2-5)+5} = x$ for $x \geq 0$.

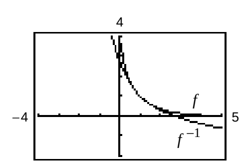
$f(f^{-1}(x)) = f\left(\sqrt{x+5}\right) = \left(\sqrt{x+5}\right)^2 - 5 = x$

**32.**
$$f(x) = x\sqrt{x-3}$$
$$f(4) = 4$$
$$f'(x) = \sqrt{x-3} + \frac{1}{2}x(x-3)^{-1/2}$$
$$f'(4) = 1 + 2 = 3$$
$$(f^{-1})'(4) = \frac{1}{f'(4)} = \frac{1}{3}$$

**34.**
$$f(x) = \ln x$$
$$f^{-1}(x) = e^x$$
$$(f^{-1})'(x) = e^x$$
$$(f^{-1})'(0) = e^0 = 1$$

**36.** (a)
$$f(x) = e^{1-x}$$
$$y = e^{1-x}$$
$$\ln y = 1 - x$$
$$x = 1 - \ln y$$
$$y = 1 - \ln x$$
$$f^{-1}(x) = 1 - \ln x$$

(b)

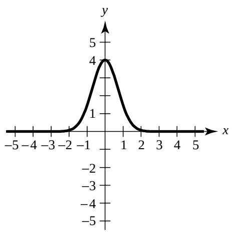



(c) $f^{-1}(f(x)) = f^{-1}\left(e^{1-x}\right) = 1 - \ln(e^{1-x})$

$= 1 - (1 - x) = x$

$f(f^{-1}(x)) = f(1 - \ln x) = e^{1-(1-\ln x)} = e^{\ln x} = x$

**38.** $y = 4e^{-x^2}$




**40.** $g(x) = \ln\left(\dfrac{e^x}{1 + e^x}\right)$

$$= \ln e^x - \ln(1 + e^x) = x - \ln(1 + e^x)$$

$$g'(x) = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x}$$

**42.** $h(z) = e^{-z^2/2}$

$$h'(z) = -ze^{-z^2/2}$$

**44.** $y = 3e^{-3/t}$

$$y' = 3e^{-3/t}(3t^{-2}) = \frac{9e^{-3/t}}{t^2}$$

**46.** $f(\theta) = \dfrac{1}{2}e^{\sin 2\theta}$

$$f'(\theta) = \cos 2\theta\, e^{\sin 2\theta}$$

**48.** $\cos x^2 = xe^y$

$$-2x \sin x^2 = xe^y \frac{dy}{dx} + e^y$$

$$\frac{dy}{dx} = -\frac{2x \sin x^2 + e^y}{xe^y}$$

**50.** Let $u = \dfrac{1}{x}$, $du = \dfrac{-1}{x^2}\,dx$.

$$\int \frac{e^{1/x}}{x^2}\,dx = -\int e^{1/x}\left(-\frac{1}{x^2}\right) dx = -e^{1/x} + C$$

**52.** Let $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - e^{-2x})\,dx$.

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}\,dx$$

$$= \frac{1}{2}\ln(e^{2x} + e^{-2x}) + C$$

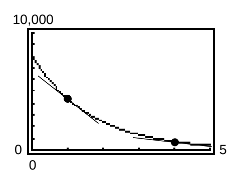
**54.** Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$\int x^2 e^{x^3+1}\,dx = \frac{1}{3}\int e^{x^3+1}(3x^2)\,dx = \frac{1}{3}e^{x^3+1} + C$$

**56.** $\displaystyle\int \frac{e^{2x}}{e^{2x} + 1}\,dx = \frac{1}{2}\int \frac{1}{e^{2x} + 1}2e^{2x}\,dx$

$$= \frac{1}{2}\ln(e^{2x} + 1) + C$$

**58.** (a), (c)




(b) $V = 8000e^{-0.6t},\ 0 \le t \le 5$

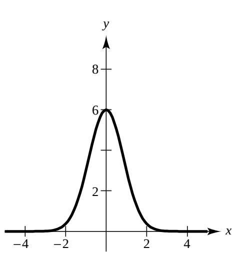
$V'(t) = -4800e^{-0.6t}$

$V'(1) = -2634.3$ dollars/year

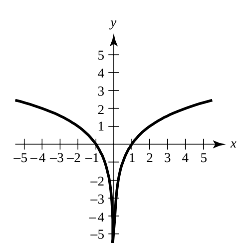
$V'(4) = -435.4$ dollars/year

**60.** Area $= \displaystyle\int_0^2 2e^{-x}\,dx = \Big[-2e^{-x}\Big]_0^2 = -2e^{-2} + 2 = 2 - \frac{2}{e^2} \approx 1.729$

**62.** $g(x) = 6(2^{-x^2})$




**64.** $y = \log_4 x^2$



**66.** $f(x) = 4^x e^x$

$f'(x) = 4^x e^x + (\ln 4)4^x e^x = 4^x e^x(1 + \ln 4)$

**68.** $y = x(4^{-x})$

$y' = 4^{-x} - x \cdot 4^{-x} \ln 4$

**70.** $h(x) = \log_5 \dfrac{x}{x-1} = \log_5 x - \log_5(x-1)$

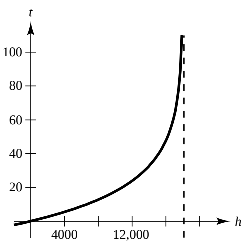
$$h'(x) = \frac{1}{\ln 5}\left[\frac{1}{x} - \frac{1}{x-1}\right] = \frac{1}{\ln 5}\left[\frac{-1}{x(x-1)}\right]$$

**72.** $\displaystyle\int \frac{2^{-1/t}}{t^2}\,dt = \frac{1}{\ln 2}2^{-1/t} + C$

**74.** $t = 50 \log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

(a) Domain: $0 \le h < 18{,}000$

(b)




Vertical asymptote: $h = 18{,}000$

(c) $$t = 50 \log_{10}\left(\frac{18{,}000}{18{,}000 - h}\right)$$

$$10^{t/50} = \frac{18{,}000}{18{,}000 - h}$$

$$18{,}000 - h = 18{,}000(10^{-t/50})$$

$$h = 18{,}000(1 - 10^{-t/50})$$

As $h \to 18{,}000$, $t \to \infty$.

(d) $t = 50 \log_{10} 18{,}000 - 50 \log_{10}(18{,}000 - h)$

$$\frac{dt}{dh} = \frac{50}{(\ln 10)(18{,}000 - h)}$$

$$\frac{d^2t}{dh^2} = \frac{50}{(\ln 10)(18{,}000 - h)^2}$$

No critical numbers

As $t$ increases, the rate of change of the altitude is increasing.

**76.** $2P = Pe^{10r}$

$2 = e^{10r}$

$\ln 2 = 10r$

$r = \dfrac{\ln 2}{10} \approx 6.93\%$

**78.** $y = 5\left(\dfrac{1}{2}\right)^{t/1620}$

$y(600) = 5\left(\dfrac{1}{2}\right)^{600/1620} \approx 3.868$ grams

**80.** (a) $\dfrac{dy}{ds} = -0.012y,\ s > 50$

$$\frac{-1}{0.012}\int \frac{dy}{y} = \int ds$$

$$\frac{-1}{0.012}\ln y = s + C_1$$

$$y = Ce^{-0.012s}$$

When $s = 50$, $y = 28 = Ce^{-0.012(50)} \implies C = 28e^{0.6}$

$y = 28e^{0.6 - 0.012s},\ s > 50$

(b)

| Speed(s) | 50 | 55 | 60 | 65 | 70 |
|---|---|---|---|---|---|
| Miles per Gallon ($y$) | 28 | 26.4 | 24.8 | 23.4 | 22.0 |

**82.** $\dfrac{dy}{dx} = \dfrac{e^{-2x}}{1 + e^{-2x}}$

$$\int dy = \int \frac{e^{-2x}}{1 + e^{-2x}}\,dx = -\frac{1}{2}\int \frac{-2e^{-2x}}{1 + e^{-2x}}\,dx$$

$$y = -\frac{1}{2}\ln(1 + e^{-2x}) + C$$

**84.** $y' - e^y \sin x = 0$

$$\frac{dy}{dx} = e^y \sin x$$

$$\int e^{-y}\,dy = \int \sin x\,dx$$

$$-e^{-y} = -\cos x + C_1$$

$$e^y = \frac{1}{\cos x + C} \qquad (C = -C_1)$$

$$y = \ln\left|\frac{1}{\cos x + C}\right| = -\ln|\cos x + C|$$

**86.** $\dfrac{dy}{dx} = \dfrac{3(x+y)}{x}$ (homogeneous differential equation)

$$3(x+y)\,dx - x\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$3(x+vx)\,dx - x(x\,dv + v\,dx) = 0$$

$$(3x + 2vx)\,dx - x^2\,dv = 0$$

$$(3+2v)\,dx = x\,dv$$

$$\int \frac{1}{x}\,dx = \int \frac{1}{3+2v}\,dv$$

$$\ln|x| = \frac{1}{2}\ln|3+2v| + C_1 = \ln(3+2v)^{1/2} + \ln C_2$$

$$x = C_2(3+2v)^{1/2}$$

$$x^2 = C(3+2v) = C\left(3 + 2\left(\frac{y}{x}\right)\right)$$

$$x^3 = C(3x+2y) = 3Cx + 2Cy$$

$$y = \frac{x^3 - 3Cx}{2C}$$

**88.** $\dfrac{dv}{dt} = kv - 9.8$

(a) $$\int \frac{dv}{kv - 9.8} = \int dt$$

$$\frac{1}{k}\ln|kv - 9.8| = t + C_1$$

$$\ln|kv - 9.8| = kt + C_2$$

$$kv - 9.8 = e^{kt + C_2} = C_3e^{kt}$$

$$v = \frac{1}{k}\left[9.8 + C_3e^{kt}\right]$$

At $t = 0$, $\quad v_0 = \dfrac{1}{k}(9.8 + C_3) \implies C_3 = kv_0 - 9.8$

$$v = \frac{1}{k}[9.8 + (kv_0 - 9.8)e^{kt}]$$

Note that $k < 0$ since the object is moving downward.

(b) $\displaystyle\lim_{t\to\infty} v(t) = \frac{9.8}{k}$

(c) $$s(t) = \int \frac{1}{k}[9.8 + (kv_0 - 9.8)e^{kt}]\,dt$$

$$= \frac{1}{k}\left[9.8t + \frac{1}{k}(kv_0 - 9.8)e^{kt}\right] + C$$

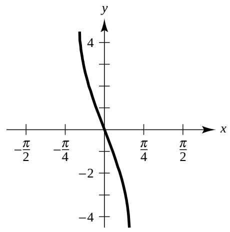
$$= \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + C$$

$$s(0) = \frac{1}{k^2}(kv_0 - 9.8) + C \implies C = s_0 - \frac{1}{k^2}(kv_0 - 9.8)$$

$$s(t) = \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + s_0 - \frac{1}{k^2}(kv_0 - 9.8)$$

$$= \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)(e^{kt} - 1) + s_0$$

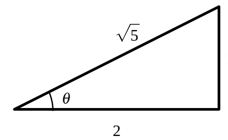
**90.** $h(x) = -3 \arcsin(2x)$




**92.** (a) Let $\theta = \operatorname{arccot} 2$

$\cot \theta = 2$

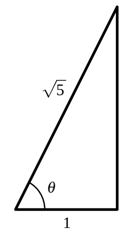
$\tan(\operatorname{arccot} 2) = \tan \theta = \dfrac{1}{2}.$




(b) Let $\theta = \operatorname{arcsec} \sqrt{5}$

$\sec \theta = \sqrt{5}$

$\cos\left(\operatorname{arcsec} \sqrt{5}\right) = \cos \theta = \dfrac{1}{\sqrt{5}}.$




**94.** $y = \arctan(x^2 - 1)$

$$y' = \frac{2x}{1 + (x^2 - 1)^2} = \frac{2x}{x^4 - 2x^2 + 2}$$

**96.** $y = \dfrac{1}{2} \arctan e^{2x}$

$$y' = \frac{1}{2}\left(\frac{1}{1 + e^{4x}}\right)(2e^{2x}) = \frac{e^{2x}}{1 + e^{4x}}$$

**98.** $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \dfrac{x}{2},\ 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2 - 4}} - \frac{1}{\left(|x|/2\right)\sqrt{(x/2)^2 - 1}} = \frac{x}{\sqrt{x^2 - 4}} - \frac{4}{|x|\sqrt{x^2 - 4}} = \frac{x^2 - 4}{|x|\sqrt{x^2 - 4}} = \frac{\sqrt{x^2 - 4}}{x}$$

**100.** Let $u = 5x$, $du = 5\, dx$.

$$\int \frac{1}{3 + 25x^2}\, dx = \frac{1}{5}\int \frac{1}{\left(\sqrt{3}\right)^2 + (5x)^2}(5)\, dx = \frac{1}{5\sqrt{3}} \arctan \frac{5x}{\sqrt{3}} + C$$

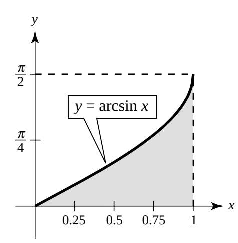
**102.** $\displaystyle\int \frac{1}{16 + x^2}\, dx = \frac{1}{4} \arctan \frac{x}{4} + C$

**104.** $\displaystyle\int \frac{4 - x}{\sqrt{4 - x^2}}\, dx = 4\int \frac{1}{\sqrt{4 - x^2}}\, dx + \frac{1}{2}\int (4 - x^2)^{-1/2}(-2x)\, dx = 4 \arcsin \frac{x}{2} + \sqrt{4 - x^2} + C$

**106.** Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1 - x^2}}\, dx$.

$$\int \frac{\arcsin x}{\sqrt{1 - x^2}}\, dx = \frac{1}{2}(\arcsin x)^2 + C$$

**108.**




Since the area of region $A$ is $1\left(\displaystyle\int_0^{\pi/2} \sin y\, dy\right)$,

the shaded area is $\displaystyle\int_0^1 \arcsin x\, dx = \frac{\pi}{2} - 1 \approx 0.571.$

**110.** $y = x \tanh^{-1} 2x$

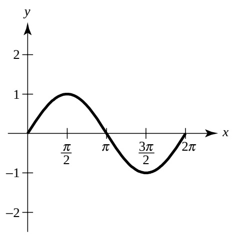
$$y' = x\left(\frac{2}{1 - 4x^2}\right) + \tanh^{-1} 2x = \frac{2x}{1 - 4x^2} + \tanh^{-1} 2x$$

**112.** Let $u = x^3$, $du = 3x^2\, dx$.

$$\int x^2(\operatorname{sech} x^3)^2\, dx = \frac{1}{3}\int (\operatorname{sech} x^3)^2(3x^2)\, dx = \frac{1}{3} \tanh x^3 + C$$

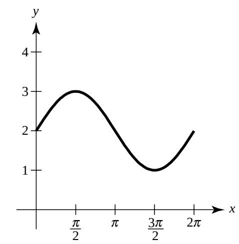
# Problem Solving for Chapter 5

**2.** (a)



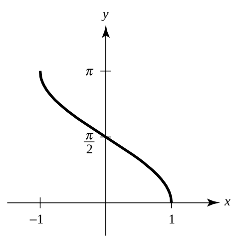

$$\int_0^{\pi} \sin x\, dx = -\int_{\pi}^{2\pi} \sin x\, dx \implies \int_0^{2\pi} \sin x\, dx = 0$$

(b)



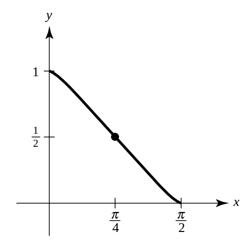

$$\int_0^{2\pi} (\sin x + 2)\, dx = 2(2\pi) = 4\pi$$

(c)




$$\int_{-1}^{1} \arccos x\, dx = 2\left(\frac{\pi}{2}\right) = \pi$$

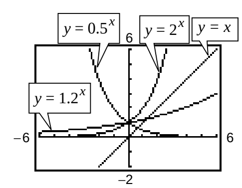
(d)




$y = \dfrac{1}{1 + (\tan x)^{\sqrt{2}}}$ is symmetric with respect to the point $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$. $\displaystyle\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}}\, dx = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$

**4.** $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$.

$y = 2^x$ does not intersect $y = x$.




Suppose $y = x$ is tangent to $y = a^x$ at $(x, y)$.

$a^x = x \implies a = x^{1/x}$.

$y' = a^x \ln a = 1 \implies x \ln x^{1/x} = 1 \implies \ln x = 1 \implies x = e, a = e^{1/e}$

For $0 < a \le e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.

**6.** (a) $y = f(x) = \arcsin x$

$\sin y = x$

$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = -\cos y\Big]_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

(b)
$$\int_{1/2}^{\sqrt{2}/2} \arcsin x\, dx = \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B$$

$$= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3} - \sqrt{2}}{2} - \frac{\pi}{12}$$

$$= \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2} - \sqrt{3}}{2} \approx 0.1346$$

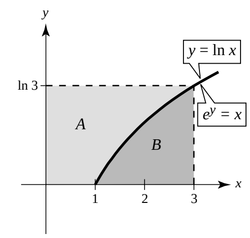
**—CONTINUED—**

**6. —CONTINUED—**

(c) Area $A = \int_0^{\ln 3} e^y\, dy$

$$= e^y\Big]_0^{\ln 3} = 3 - 1 = 2$$

Area $B = \int_1^3 \ln x\, dx = 3(\ln 3) - A = 3 \ln 3 - 2 = \ln 27 - 2 \approx 1.2958$




(d) $\tan y = x$

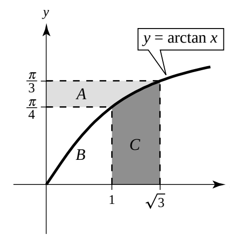
Area $A = \int_{\pi/4}^{\pi/3} \tan y\, dy$

$$= -\ln|\cos y|\Big]_{\pi/4}^{\pi/3}$$

$$= -\ln\frac{1}{2} + \ln\frac{\sqrt{2}}{2} = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

Area $C = \int_1^{\sqrt{3}} \arctan x\, dx = \left(\frac{\pi}{3}\right)\left(\sqrt{3}\right) - \frac{1}{2}\ln 2 - \left(\frac{\pi}{4}\right)(1)$

$$= \frac{\pi}{12}\left(4\sqrt{3} - 3\right) - \frac{1}{2}\ln 2 \approx 0.6818$$




**8.**
$$y = e^x$$
$$y' = e^x$$
$$y - b = e^a(x - a)$$
$$y = e^a x - ae^a + b \quad \text{Tangent line}$$

If $y = 0$,

$$e^a x = ae^a - b$$
$$bx = ab - b \quad (b = e^a)$$
$$x = a - 1$$
$$c = a - 1$$

Thus, $a - c = a - (a - 1) = 1$.

**10.** Let $u = \tan x$, $du = \sec^2 x\, dx$

$$\text{Area} = \int_0^{\pi/4} \frac{1}{\sin^2 x + 4\cos^2 x}\, dx = \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4}\, dx$$

$$= \int_0^1 \frac{du}{u^2 + 4}$$

$$= \left[\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right]_0^1$$

$$= \frac{1}{2}\arctan\left(\frac{1}{2}\right)$$

**12.** (a) $\frac{dy}{dt} = y(1 - y)$, $y(0) = \frac{1}{4}$

$$\int\left(\frac{1}{y} + \frac{1}{1 - y}\right) dy = \int dt$$

$$\ln|y| - \ln|1 - y| = t + C$$

$$\ln\left|\frac{y}{1 - y}\right| = t + C$$

$$\frac{y}{1 - y} = e^{t + C} = C_1 e^t$$

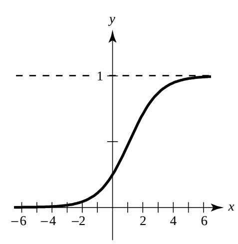
$$y = C_1 e^t - yC_1 e^t$$

$$y = \frac{C_1 e^t}{1 + C_1 e^t} = \frac{1}{1 + C_2 e^{-t}}$$

$$y(0) = \frac{1}{4} = \frac{1}{1 + C_2} \implies C_2 = 3$$

Hence, $y = \dfrac{1}{1 + 3e^{-t}}$.

(b)




$$\frac{dy}{dt} = y(1 - y) = y - y^2$$

$$\frac{d^2y}{dt^2} = y'' = y' - 2yy' \implies y'' = 0 \text{ for } y = \frac{1}{2}$$

$$\frac{d^2y}{dt^2} > 0 \text{ if } 0 < y < \frac{1}{2} \text{ and } \frac{d^2y}{dt^2} < 0 \text{ if } \frac{1}{2} < y < 1.$$

Thus, the rate of growth is maximum at $y = \frac{1}{2}$, the point of inflection.

**—CONTINUED—**

**12. —CONTINUED—**

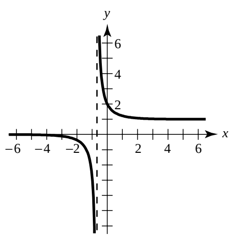
(c) $y' = y(1 - y), y(0) - 2$

As before, $y = \dfrac{1}{1 + C_2 e^{-t}}$

$y(0) = 2 = \dfrac{1}{1 + C_2} \implies C_2 = -\dfrac{1}{2}$

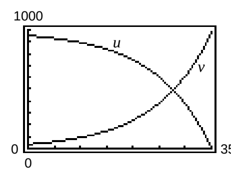
Thus, $y = \dfrac{1}{1 - \frac{1}{2}e^{-t}} = \dfrac{2}{2 - e^{-t}}$.

The graph is different:




**14.** (a) $u = 985.93 - \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$

$v = \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$




(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \implies \frac{du}{dt} = -\frac{dv}{dt}$$

$u'(15) = -v'(15) = -14.06.$

(d) $t = 12.7$ years

Again, the larger part goes for interest.

# PART II

## CHAPTER 6
## Applications of Integration

# C H A P T E R 6
# Applications of Integration

## Section 6.1 Area of a Region Between Two Curves

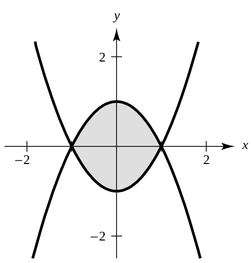
Solutions to Even-Numbered Exercises

**2.** $A = \int_{-2}^{2} [(2x + 5) - (x^2 + 2x + 1)]\, dx = \int_{-2}^{2} (-x^2 + 4)\, dx$

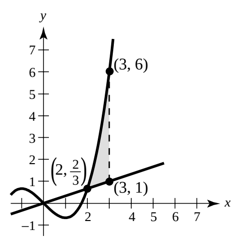
**4.** $A = \int_{0}^{1} (x^2 - x^3)\, dx$

**6.** $A = 2\int_{0}^{1} [(x - 1)^3 - (x - 1)]\, dx$

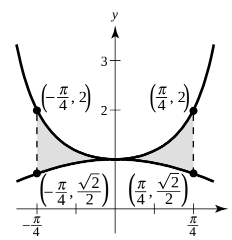
**8.** $\int_{-1}^{1} [(1 - x^2) - (x^2 - 1)]\, dx$




**10.** $\int_{2}^{3} \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$




**12.** $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\, dx$

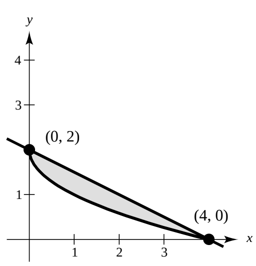



**14.** $f(x) = 2 - \frac{1}{2}x$

$g(x) = 2 - \sqrt{x}$

$A \approx 1$

Matches (a)

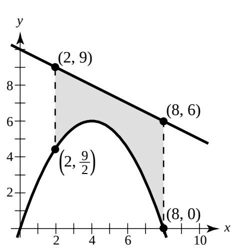



**16.** $A = \int_{2}^{8} \left[\left(10 - \frac{1}{2}x\right) - \left(-\frac{3}{8}x(x - 8)\right)\right] dx$

$= \int_{2}^{8} \left(\frac{3}{8}x^2 - \frac{7}{2}x + 10\right) dx$

$= \left[\frac{x^3}{8} - \frac{7x^2}{4} + 10x\right]_2^8$

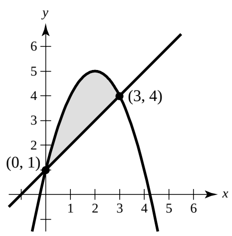
$= (64 - 112 + 80) - (1 - 7 + 20) = 18$



**18.** The points of intersection are given by

$$-x^2 + 4x + 1 = x + 1$$
$$-x^2 + 3x = 0$$
$$x^2 = 3x \text{ when } x = 0, 3$$

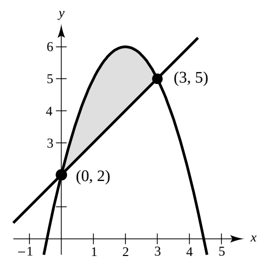
$$A = \int_0^3 [(-x^2 + 4x + 1) - (x + 1)]\, dx$$
$$= \int_0^3 (-x^2 + 3x)\, dx$$
$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3$$
$$= -9 + \frac{27}{2} = \frac{9}{2}$$



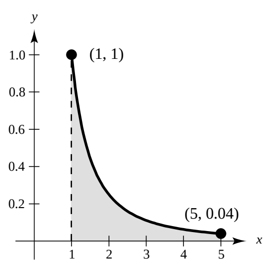

**20.** The points of intersection are given by:

$$-x^2 + 4x + 2 = x + 2$$
$$x(3 - x) = 0 \quad \text{when} \quad x = 0, 3$$

$$A = \int_0^3 [f(x) - g(x)]\, dx$$
$$= \int_0^3 [(-x^2 + 4x + 2) - (x + 2)]\, dx$$
$$= \int_0^3 (-x^2 + 3x)\, dx = \left[\frac{-x^3}{3} + \frac{3}{2}x^2\right]_0^3 = \frac{9}{2}$$



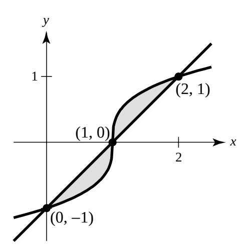

**22.** $A = \int_1^5 \left(\frac{1}{x^2} - 0\right) dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$




**24.** The points of intersection are given by

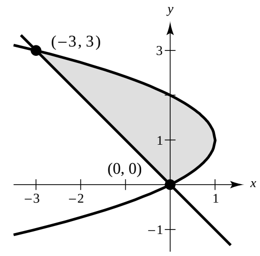
$$\sqrt[3]{x - 1} = x - 1$$
$$x - 1 = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$$
$$x^3 - 3x^2 + 2x = 0$$
$$x(x^2 - 3x + 2) = 0$$
$$x(x - 2)(x - 1) = 0 \implies x = 0, 1, 2$$

$$A = 2\int_0^1 \left[(x - 1) - \sqrt[3]{x - 1}\right] dx$$
$$= 2\left[\frac{x^2}{2} - x - \frac{3}{4}(x - 1)^{4/3}\right]_0^1$$
$$= 2\left[\left(\frac{1}{2} - 1 - 0\right) - \left(-\frac{3}{4}\right)\right] = \frac{1}{2}$$

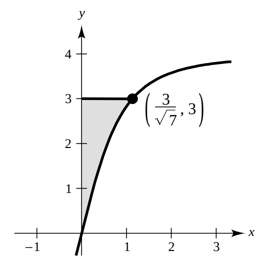


**26.** The points of intersection are given by:

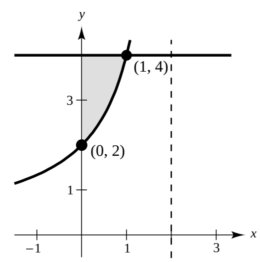
$$2y - y^2 = -y$$
$$y(y-3) = 0 \quad \text{when} \quad y = 0, 3$$
$$A = \int_0^3 [f(y) - g(y)]\, dy$$
$$= \int_0^3 [(2y - y^2) - (-y)]\, dy$$
$$= \int_0^3 (3y - y^2)\, dy = \left[\frac{3}{2}y^2 - \frac{1}{3}y^3\right]_0^3 = \frac{9}{2}$$




**28.** $$A = \int_0^3 [f(y) - g(y)]\, dy$$
$$= \int_0^3 \left[\frac{y}{\sqrt{16 - y^2}} - 0\right] dy$$
$$= -\frac{1}{2}\int_0^3 (16 - y^2)^{-1/2}(-2y)\, dy$$
$$= \left[-\sqrt{16 - y^2}\right]_0^3 = 4 - \sqrt{7} \approx 1.354$$



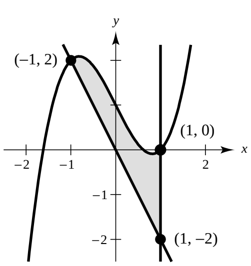

**30.** $$A = \int_0^1 \left(4 - \frac{4}{2 - x}\right) dx$$
$$= \left[4x + 4 \ln|2 - x|\right]_0^1$$
$$= 4 - 4 \ln 2$$
$$\approx 1.227$$




**32.** The point of intersection is given by:

$$x^3 - 2x + 1 = -2x$$
$$x^3 + 1 = 0 \quad \text{when} \quad x = -1$$
$$A = \int_{-1}^1 [f(x) - g(x)]\, dx$$
$$= \int_{-1}^1 [(x^3 - 2x + 1) - (-2x)]\, dx$$
$$= \int_{-1}^1 (x^3 + 1)\, dx = \left[\frac{x^4}{4} + x\right]_{-1}^1 = 2$$

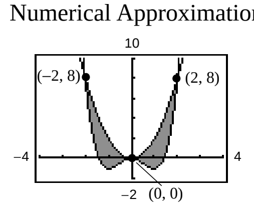
Numerical Approximation: 2.0




**34.** The points of intersection are given by:

$$x^4 - 2x^2 = 2x^2$$
$$x^2(x^2 - 4) = 0 \quad \text{when} \quad x = 0, \pm 2$$
$$A = 2\int_0^2 [2x^2 - (x^4 - 2x^2)]\, dx$$
$$= 2\int_0^2 (4x^2 - x^4)\, dx$$
$$= 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{128}{15}$$

Numerical Approximation: 8.533



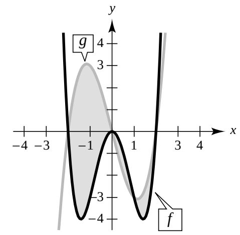
**36.** $f(x) = x^4 - 4x^2, \quad g(x) = x^3 - 4x$

The points of intersection are given by:

$$x^4 - 4x^2 = x^3 - 4x$$

$$x^4 - x^3 - 4x^2 + 4x = 0$$

$$x(x-1)(x+2)(x-2) = 0 \quad \text{when} \quad x = -2, 0, 1, 2$$




$$A = \int_{-2}^{0} [(x^3 - 4x) - (x^4 - 4x^2)]dx + \int_{0}^{1} [(x^4 - 4x^2) - (x^3 - 4x)]dx + \int_{1}^{2} [(x^3 - 4x) - (x^4 - 4x^2)]dx$$

$$= \frac{248}{30} + \frac{37}{60} + \frac{53}{60} = \frac{293}{30}$$

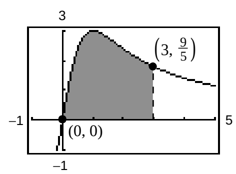
Numerical Approximation: $8.267 + 0.617 + 0.883 \approx 9.767$

**38.** $A = \int_0^3 \left[\frac{6x}{x^2+1} - 0\right] dx$

$$= \left[3\ln(x^2+1)\right]_0^3$$

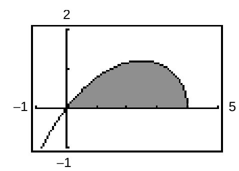
$$= 3 \ln 10$$

$$\approx 6.908$$




Numerical Approximation: 6.908

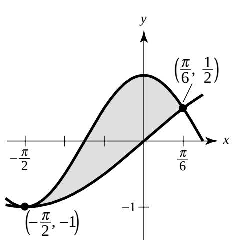
**40.** $A = \int_0^4 x\sqrt{\frac{4-x}{4+x}}\, dx \approx 3.434$



**42.** $A = \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x)\, dx$

$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\pi/2}^{\pi/6}$$

$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - (0) = \frac{3\sqrt{3}}{4} \approx 1.299$$

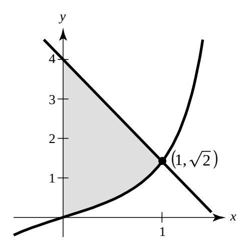



**44.** $A = \int_0^1 \left[\left(\sqrt{2} - 4\right)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right] dx$

$$= \left[\frac{\sqrt{2}-4}{2}x^2 + 4x - \frac{4}{\pi}\sec\frac{\pi x}{4}\right]_0^1$$

$$= \left(\frac{\sqrt{2}-4}{2} + 4 - \frac{4}{\pi}\sqrt{2}\right) - \left(-\frac{4}{\pi}\right)$$

$$= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi}\left(1 - \sqrt{2}\right) \approx 2.1797$$



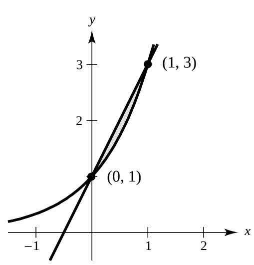

**46.** From the graph we see that $f$ and $g$ intersect twice at $x = 0$ and $x = 1$.

$$A = \int_0^1 [g(x) - f(x)]\, dx$$

$$= \int_0^1 [(2x+1) - 3^x]\, dx$$

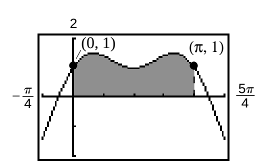
$$= \left[x^2 + x - \frac{1}{\ln 3}(3^x)\right]_0^1$$

$$= 2\left(1 - \frac{1}{\ln 3}\right) \approx 0.180$$



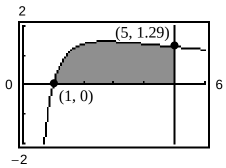

**48.** $A = \int_0^\pi [(2\sin x + \cos 2x) - 0]\, dx$

$$= \left[-2\cos x + \frac{1}{2}\sin 2x\right]_0^\pi = 4$$



**50.** $A = \int_1^5 \left[\frac{4 \ln x}{x} - 0\right] dx$

$= \left[2(\ln x)^2\right]_1^5 = 2(\ln 5)^2 \approx 5.181$

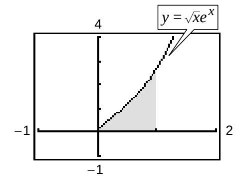



**52.** (a) $y = \sqrt{x}\, e^x,\ y = 0,\ x = 0,\ x = 1$

(b) $A = \int_0^1 \sqrt{x}\, e^x\, dx.$

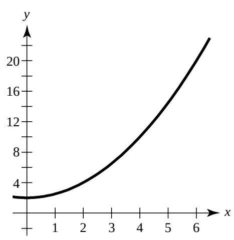
No, it cannot be evaluated by hand.

(c) 1.2556

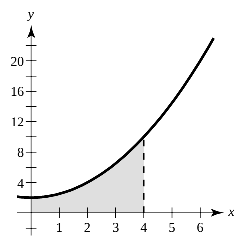



**54.** $F(x) = \int_0^x \left(\frac{1}{2}t^2 + 2\right) dt = \left[\frac{1}{6}t^3 + 2t\right]_0^x = \frac{x^3}{6} + 2x$

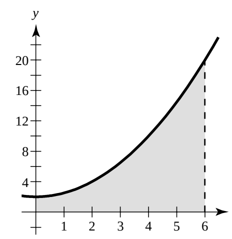
(a) $F(0) = 0$



(b) $F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$

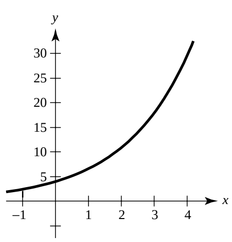


(c) $F(6) = 36 + 12 = 48$

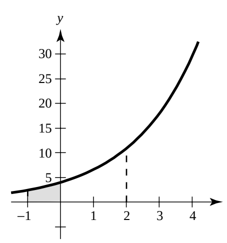


**56.** $F(y) = \int_{-1}^{y} 4e^{x/2}\, dx = \left[8e^{x/2}\right]_{-1}^{y} = 8e^{y/2} - 8e^{-1/2}$

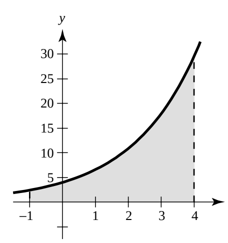
(a) $F(-1) = 0$



(b) $F(0) = 8 - 8e^{-1/2} \approx 3.1478$



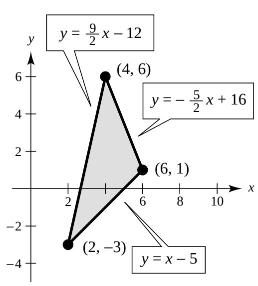
(c) $F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$



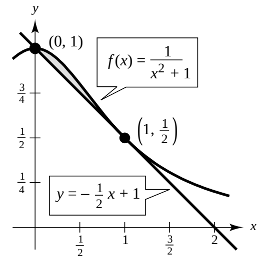
**58.** $A = \int_2^4 \left[\left(\frac{9}{2}x - 12\right) - (x - 5)\right] dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16\right) - (x - 5)\right] dx$

$= \int_2^4 \left(\frac{7}{2}x - 7\right) dx + \int_4^6 \left(-\frac{7}{2}x + 21\right) dx$

$= \left[\frac{7}{4}x^2 - 7x\right]_2^4 + \left[-\frac{7}{4}x^2 + 21x\right]_4^6 = 7 + 7 = 14$



**60.** $f(x) = \dfrac{1}{x^2 + 1}$




$f'(x) = -\dfrac{2x}{(x^2 + 1)^2}$

At $\left(1, \dfrac{1}{2}\right)$, $f'(1) = -\dfrac{1}{2}$.

Tangent line:

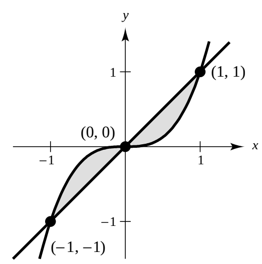
$$y - \frac{1}{2} = -\frac{1}{2}(x - 1) \text{ or } y = -\frac{1}{2}x + 1$$

The tangent line intersects $f(x) = \dfrac{1}{x^2 + 1}$ at $x = 0$.

$$A = \int_0^1 \left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1\right)\right] dx = \left[\arctan x + \frac{x^2}{4} - x\right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$$

**62.** Answers will vary. See page 417.

**64.** $x^3 \geq x$ on $[-1, 0]$




$x^3 \leq x$ on $[0, 1]$

Both functions symmetric to origin

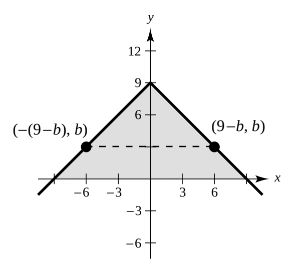
$$\int_{-1}^0 (x^3 - x)\, dx = -\int_0^1 (x^3 - x)\, dx.$$

Thus, $\displaystyle\int_{-1}^1 (x^3 - x)\, dx = 0$.

$$A = 2\int_0^1 (x - x^3)\, dx = 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

**66.** Proposal 2 is better, since the cummulative deficit (the area under the curve) is less.

**68.** $A = 2\displaystyle\int_0^9 (9 - x)\, dx = 2\left[9x - \frac{x^2}{2}\right]_0^9 = 81$




$$2\int_0^{9-b} [(9 - x) - b]\, dx = \frac{81}{2}$$

$$2\int_0^{9-b} [(9 - b) - x]\, dx = \frac{81}{2}$$

$$2\left[(9 - b)x - \frac{x^2}{2}\right]_0^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

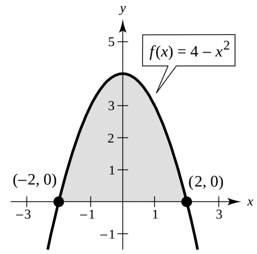
$$9 - b = \frac{9}{\sqrt{2}}$$

$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$

**70.** $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (4 - x_i^2)\,\Delta x$

where $x_i = -2 + \dfrac{4i}{n}$ and $\Delta x = \dfrac{4}{n}$ is the same as

$$\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$




**72.** $$\int_0^5 [(7.21 + 0.26t + 0.02t^2) - (7.21 + 0.1t + 0.01t^2)]\,dt = \int_0^5 (0.01t^2 + 0.16t)\,dt$$
$$= \left[\frac{0.01t^3}{3} + \frac{0.16t^2}{2}\right]_0^5$$
$$= \frac{29}{12} \text{ billion} \approx \$2.417 \text{ billion}$$

**74.** 5% : $P_1 = 893{,}000\, e^{(0.05)t}$

$3\frac{1}{2}$%: $P_2 = 893{,}000\, e^{(0.035)t}$

Difference in profits over 5 years:

$$\int_0^5 [893{,}000e^{0.05t} - 893{,}000e^{0.035t}]\,dt = 893{,}000\left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035}\right]_0^5$$
$$\approx 893{,}000[(25.6805 - 34.0356) - (20 - 28.5714)]$$
$$\approx 893{,}000(0.2163) \approx \$193{,}156$$

**Note:** Using a graphing utility you obtain $193,183.

**76.** The curves intersect at the point where the slope of $y_2$ equals that of $y_1$, 1.

$$y_2 = 0.08x^2 + k \implies y'_2 = 0.16x = 1 \implies x = \frac{1}{.16} = 6.25$$

(a) The value of $k$ is given by
$$y_1 = y_2$$
$$6.25 = (0.08)(6.25)^2 + k$$
$$k = 3.125.$$

(b) $$\text{Area} = 2\int_0^{6.25} (y_2 - y_1)\,dx$$
$$= 2\int_0^{6.25} (0.08x^2 + 3.125 - x)\,dx$$
$$= 2\left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2}\right]_0^{6.25}$$
$$= 2(6.510417) \approx 13.02083$$

**78.** (a) $A \approx 6.031 - 2\left[\pi\left(\dfrac{1}{16}\right)^2\right] - 2\left[\pi\left(\dfrac{1}{8}\right)^2\right] \approx 5.908$

(b) $V = 2A \approx 2(5.908) \approx 11.816 \text{ m}^3$

(c) $5000V \approx 5000(11.816) = 59{,}082$ pounds

**80.** True

## Section 6.2 Volume: The Disk Method

**2.** $V = \pi \int_0^2 (4 - x^2)^2\,dx = \pi \int_0^2 (x^4 - 8x^2 + 16)\,dx = \pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_0^2 = \frac{256\pi}{15}$

**4.** $V = \pi\int_0^3 \left(\sqrt{9 - x^2}\right)^2 dx = \pi\int_0^3 (9 - x^2)\,dx$

$= \pi\left[9x - \frac{x^3}{3}\right]_0^3 = 18\pi$

**6.** $2 = 4 - \frac{x^2}{4}$

$8 = 16 - x^2$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$V = \pi\int_{-2\sqrt{2}}^{2\sqrt{2}} \left[\left(4 - \frac{x^2}{4}\right)^2 - (2)^2\right] dx$

$= 2\pi\int_0^{2\sqrt{2}} \left[\frac{x^4}{16} - 2x^2 + 12\right] dx$

$= 2\pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x\right]_0^{2\sqrt{2}}$

$= 2\pi\left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2}\right]$

$= \frac{448\sqrt{2}}{15}\pi \approx 132.69$

**8.** $y = \sqrt{16 - x^2} \implies x = \sqrt{16 - y^2}$

$V = \pi\int_0^4 \left(\sqrt{16 - y^2}\right)^2 dy = \pi\int_0^4 (16 - y^2)\,dy$

$= \pi\left[16y - \frac{y^3}{3}\right]_0^4 = \frac{128\pi}{3}$

**10.** $V = \pi\int_1^4 (-y^2 + 4y)^2\,dy = \pi\int_1^4 (y^4 - 8y^3 + 16y^2)\,dy$

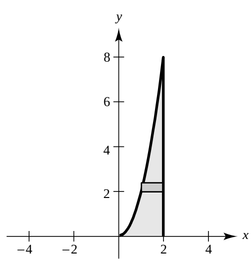
$= \pi\left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3}\right]_1^4$

$= \frac{459\pi}{15} = \frac{153\pi}{5}$

**12.** $y = 2x^2,\ y = 0,\ x = 2$

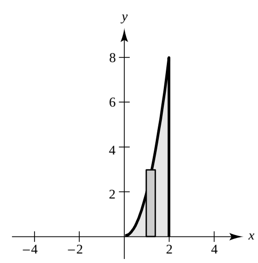
(a) $R(y) = 2,\ r(y) = \sqrt{y/2}$

$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right) dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$




(b) $R(x) = 2x^2,\ r(x) = 0$

$V = \pi\int_0^2 4x^4\,dx = \pi\left[\frac{4x^5}{5}\right]_0^2 = \frac{128\pi}{5}$

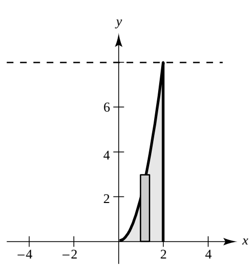



(c) $R(x) = 8,\ r(x) = 8 - 2x^2$

$V = \pi\int_0^2 [64 - (64 - 32x^2 + 4x^4)]\,dx$

$= \pi\int_0^2 (32x^2 - 4x^4)\,dx = 4\pi\int_0^2 (8x^2 - x^4)\,dx$

$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \frac{896\pi}{15}$

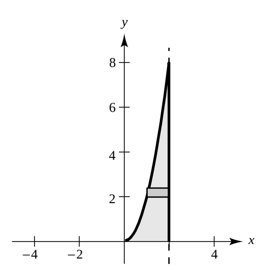



(d) $R(y) = 2 - \sqrt{y/2},\ r(y) = 0$

$V = \pi\int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$

$= \pi\int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$

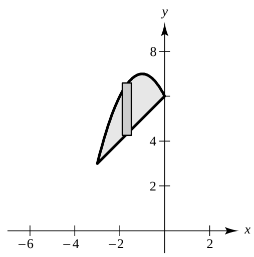
$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8 = \frac{16\pi}{3}$



**14.** $y = 6 - 2x - x^2,\ y = x + 6$ intersect at $(-3, 3)$ and $(0, 6)$.

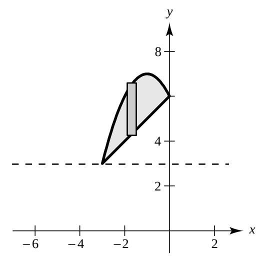
(a) $R(x) = 6 - 2x - x^2,\ r(x) = x + 6$

$$V = \pi\int_{-3}^{0}\left[(6 - 2x - x^2)^2 - (x + 6)^2\right] dx$$
$$= \pi\int_{-3}^{0}(x^4 + 4x^3 - 9x^2 - 36x)\, dx$$
$$= \pi\left[\frac{1}{5}x^5 + x^4 - 3x^3 - 18x^2\right]_{-3}^{0} = \frac{243\pi}{5}$$



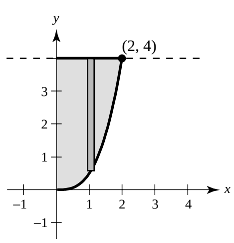

(b) $R(x) = (6 - 2x - x^2) - 3,\ r(x) = (x + 6) - 3$

$$V = \pi\int_{-3}^{0}\left[(3 - 2x - x^2)^2 - (x + 3)^2\right] dx$$
$$= \pi\int_{-3}^{0}(x^4 + 4x^3 - 3x^2 - 18x)\, dx$$
$$= \pi\left[\frac{1}{5}x^5 + x^4 - x^3 - 9x^2\right]_{-3}^{0} = \frac{108\pi}{5}$$



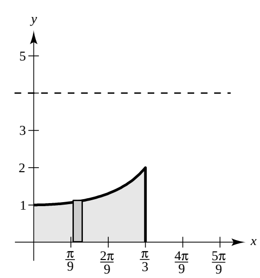

**16.** $R(x) = 4 - \dfrac{x^3}{2}, r(x) = 0$

$$V = \pi\int_{0}^{2}\left(4 - \frac{x^3}{2}\right)^2 dx$$
$$= \pi\int_{0}^{2}\left[16 - 4x^3 + \frac{x^6}{4}\right] dx$$
$$= \pi\left[16x - x^4 + \frac{x^7}{28}\right]_{0}^{2}$$
$$= \pi\left[32 - 16 + \frac{128}{28}\right] = \frac{144}{7}\pi$$



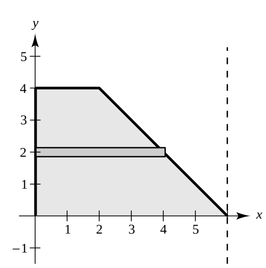

**18.** $R(x) = 4,\ r(x) = 4 - \sec x$

$$V = \pi\int_{0}^{\pi/3}\left[(4)^2 - (4 - \sec x)^2\right] dx$$
$$= \pi\int_{0}^{\pi/3}(8\sec x - \sec^2 x)\, dx$$
$$= \pi\Big[8\ln\left|\sec x + \tan x\right| - \tan x\Big]_{0}^{\pi/3}$$
$$= \pi\left[\left(8\ln\left|2 + \sqrt{3}\right| - \sqrt{3}\right) - \left(8\ln\left|1 + 0\right| - 0\right)\right]$$
$$= \pi\left[8\ln\left(2 + \sqrt{3}\right) - \sqrt{3}\right] \approx 27.66$$



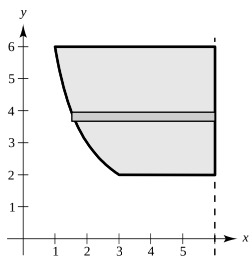

**20.** $R(y) = 6,\ r(y) = 6 - (6 - y) = y$

$$V = \pi\int_{0}^{4}\left[(6)^2 - (y)^2\right] dy$$
$$= \pi\left[36y - \frac{y^3}{3}\right]_{0}^{4} = \frac{368\pi}{3}$$



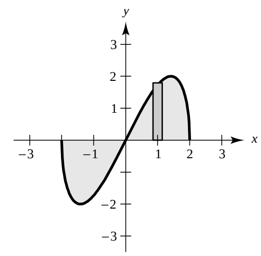
**22.** $R(y) = 6 - \dfrac{6}{y},\ r(y) = 0$

$$V = \pi\int_2^6 \left(6 - \frac{6}{y}\right)^2 dy$$
$$= 36\pi\int_2^6 \left(1 - \frac{2}{y} + \frac{1}{y^2}\right) dy$$
$$= 36\pi\left[y - 2\ln|y| - \frac{1}{y}\right]_2^6$$
$$= 36\pi\left[\left(\frac{35}{6} - 2\ln 6\right) - \left(\frac{3}{2} - 2\ln 2\right)\right]$$
$$= 36\pi\left(\frac{13}{3} + 2\ln\frac{1}{3}\right) = 12\pi(13 - 6\ln 3) \approx 241.59$$



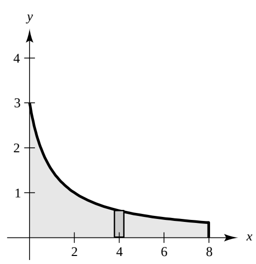
**24.** $R(x) = x\sqrt{4 - x^2},\quad r(x) = 0$

$$V = 2\pi\int_0^2 \left[x\sqrt{4 - x^2}\right]^2 dx$$
$$= 2\pi\int_0^2 (4x^2 - x^4)\, dx$$
$$= 2\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$
$$= \frac{128\pi}{15}$$



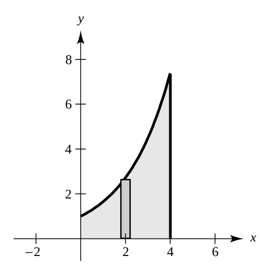
**26.** $R(x) = \dfrac{3}{x + 1},\ r(x) = 0$

$$V = \pi\int_0^8 \left(\frac{3}{x + 1}\right)^2 dx$$
$$= 9\pi\int_0^8 (x + 1)^{-2}\, dx$$
$$= 9\pi\left[-\frac{1}{x + 1}\right]_0^8 = 8\pi$$



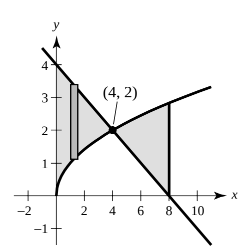
**28.** $R(x) = e^{x/2},\ r(x) = 0$

$$V = \pi\int_0^4 (e^{x/2})^2\, dx$$
$$= \pi\int_0^4 e^x\, dx$$
$$= \left[\pi e^x\right]_0^4$$
$$= \pi(e^4 - 1) \approx 168.38$$



**30.** $V = \pi\displaystyle\int_0^4 \left[\left(4 - \frac{1}{2}x\right)^2 - \left(\sqrt{x}\right)^2\right] dx + \pi\int_4^8 \left[\left(\sqrt{x}\right)^2 - \left(4 - \frac{1}{2}x\right)^2\right] dx$

$$= \pi\int_0^4 \left(\frac{x^2}{4} - 5x + 16\right) dx + \pi\int_4^8 \left(-\frac{x^2}{4} + 5x - 16\right) dx$$
$$= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x\right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x\right]_4^8$$
$$= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi$$



**32.** $y = 9 - x^2,\ y = 0,\ x = 2,\ x = 3$

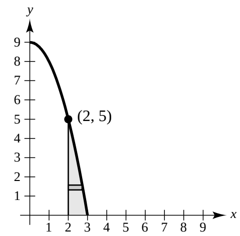
$x = \sqrt{9 - y}$

$$V = \pi\int_0^5 \left[\left(\sqrt{9-y}\right)^2 - 2^2\right] dy$$

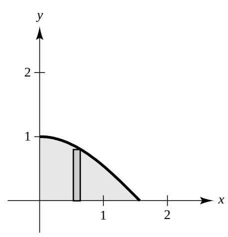
$$= \pi\int_0^5 (5 - y)\, dy$$

$$= \pi\left[5y - \frac{y^2}{2}\right]_0^5$$

$$= \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2}$$




**34.** $V = \pi\displaystyle\int_0^{\pi/2} [\cos x]^2\, dx \approx 2.4674$

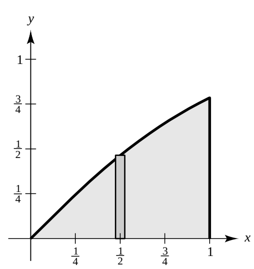


**36.** $V = \pi\displaystyle\int_1^3 [\ln x]^2\, dx \approx 3.2332$

**38.** $V = \pi\displaystyle\int_0^5 [2\arctan(0.2x)]^2\, dx \approx 15.4115$

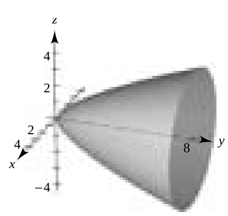
**40.** $A \approx \frac{3}{4}$

Matches (b)

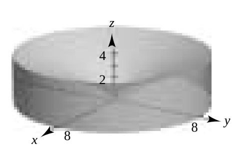


**42.** $V = \displaystyle\int_a^b A(x)\, dx$ or $V = \displaystyle\int_c^d A(y)\, dy$

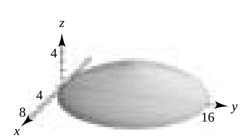
**44.** (a)



(b)



(c)



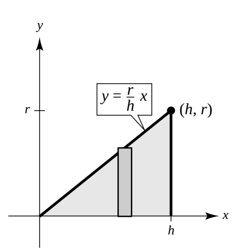
$a < c < b.$

**46.** $R(x) = \dfrac{r}{h}x,\ r(x) = 0$

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\, dx$$

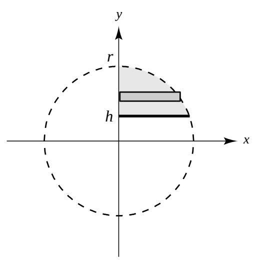
$$= \left[\frac{r^2\pi}{3h^2}x^3\right]_0^h$$

$$= \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$



**48.** $x = \sqrt{r^2 - y^2},\ R(y) = \sqrt{r^2 - y^2},\ r(y) = 0$

$$V = \pi\int_h^r \left(\sqrt{r^2 - y^2}\right)^2 dy$$
$$= \pi\int_h^r (r^2 - y^2)\,dy$$
$$= \pi\left[r^2y - \frac{y^3}{3}\right]_h^r$$
$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2h - \frac{h^3}{3}\right)\right]$$
$$= \pi\left(\frac{2r^3}{3} - r^2h + \frac{h^3}{3}\right)$$
$$= \frac{\pi}{3}(2r^3 - 3r^2h + h^3)$$




**50.** (a) $V = \pi\int_0^4 \left(\sqrt{x}\right)^2 dx = \pi\int_0^4 x\,dx = \left[\frac{\pi x^2}{2}\right]_0^4 = 8\pi$

Let $0 < c < 4$ and set

$$\pi\int_0^c x\,dx = \left[\frac{\pi x^2}{2}\right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$
$$c^2 = 8$$
$$c = \sqrt{8} = 2\sqrt{2}$$

Thus, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

(b) Set $\pi\int_0^c x\,dx = \frac{8\pi}{3}$ (one third of the volume). Then

$$\frac{\pi c^2}{2} = \frac{8\pi}{3},\ c^2 = \frac{16}{3},\ c = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}.$$

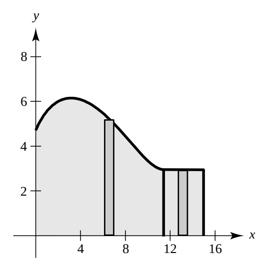
To find the other value, set $\pi\int_0^d x\,dx = \frac{16\pi}{3}$ (two thirds of the volume). Then

$$\frac{\pi d^2}{2} = \frac{16\pi}{3},\ d^2 = \frac{32}{3},\ d = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}.$$

The $x$-values that divide the solid into three parts of equal volume are $x = \left(4\sqrt{3}\right)/3$ and $x = \left(4\sqrt{6}\right)/3$.

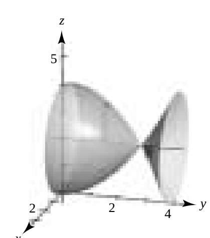
**52.** $y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5 \\ 2.95, & 11.5 < x \le 15 \end{cases}$

$$V = \pi\int_0^{11.5} \left(\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}\right)^2 dx + \pi\int_{11.5}^{15} 2.95^2\,dx$$
$$= \pi\left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x\right]_0^{11.5} + \pi\left[2.95^2x\right]_{11.5}^{15}$$
$$\approx 1031.9016 \text{ cubic centimeters}$$




**54.** (a) First find where $y = b$ intersects the parabola:

$$b = 4 - \frac{x^2}{4}$$
$$x^2 = 16 - 4b = 4(4 - b)$$
$$x = 2\sqrt{4 - b}$$
$$V = \int_0^{2\sqrt{4-b}} \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx + \int_{2\sqrt{4-b}}^4 \pi\left[b - 4 + \frac{x^2}{4}\right]^2 dx$$
$$= \int_0^4 \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx$$
$$= \pi\int_0^4 \left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16\right] dx$$
$$= \pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x\right]_0^4$$
$$= \pi\left[\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64\right] = \pi\left[4b^2 - \frac{64}{3}b + \frac{512}{15}\right]$$

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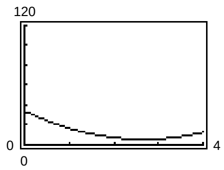
**54. —CONTINUED—**

(b) graph of $V(b) = \pi\left[4b^2 - \dfrac{64}{3}b + \dfrac{512}{15}\right]$




Minimum Volume is 17.87 for $b = 2.67$

(c) $V'(b) = \pi\left[8b - \dfrac{64}{3}\right] = 0 \Rightarrow b = \dfrac{64/3}{8} = \dfrac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \dfrac{8}{3}$ is a relative minimum.

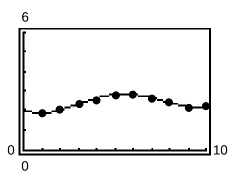
**56.** (a) $V = \displaystyle\int_0^{10} \pi[f(x)]^2\,dx$

Simpson's Rule: $b - a = 10 - 0 = 10, \quad n = 10$

$$V \approx \frac{\pi}{3}[(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2]$$

$$\approx \frac{\pi}{3}[178.405] \approx 186.83 \text{ cm}^3$$

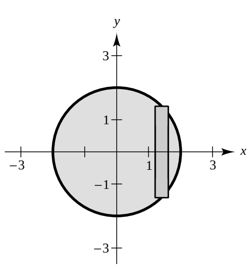
(b) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$




(c) $V \approx \displaystyle\int_0^{10} \pi f(x)^2\,dx \approx 186.35 \text{ cm}^3$

**58.** $V = \dfrac{1}{2}(10)(2)(3) = 30 \text{ m}^3$

**60.**

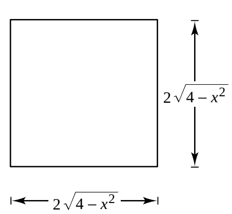



Base of Cross Section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = \left(2\sqrt{4 - x^2}\right)^2$

$$V = \int_{-2}^{2} 4(4 - x^2)\,dx$$

$$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{128}{3}$$

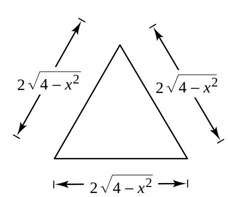



(b) $A(x) = \dfrac{1}{2}bh = \dfrac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{3}\sqrt{4 - x^2}\right)$

$$= \sqrt{3}\,(4 - x^2)$$

$$V = \sqrt{3}\int_{-2}^{2} (4 - x^2)\,dx$$

$$= \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32\sqrt{3}}{3}$$

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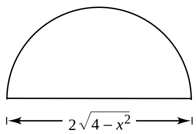
**60. —CONTINUED—**

(c) $A(x) = \frac{1}{2}\pi r^2$

$$= \frac{\pi}{2}\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}(4 - x^2)$$

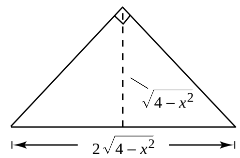
$$V = \frac{\pi}{2}\int_{-2}^{2}(4 - x^2)\,dx = \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{16\pi}{3}$$




(d) $A(x) = \frac{1}{2}bh$

$$= \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{4 - x^2}\right) = 4 - x^2$$

$$V = \int_{-2}^{2}(4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$




**62.** The cross sections are squares. By symmetry, we can set up an integral for an eighth of the volume and multiply by 8.

$$A(y) = b^2 = \left(\sqrt{r^2 - y^2}\right)^2$$

$$V = 8\int_0^r (r^2 - y^2)\,dy$$

$$= 8\left[r^2 y - \frac{1}{3}y^3\right]_0^r$$

$$= \frac{16}{3}r^3$$

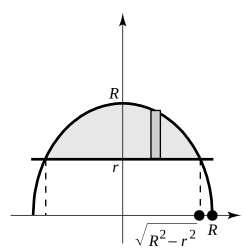

**64.** $V = \pi\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right]dx$

$$= 2\pi\int_0^{\sqrt{R^2-r^2}}(R^2 - r^2 - x^2)\,dx$$

$$= 2\pi\left[(R^2 - r^2)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2-r^2}}$$

$$= 2\pi\left[(R^2 - r^2)^{3/2} - \frac{(R^2 - r^2)^{3/2}}{3}\right]$$

$$= \frac{4}{3}\pi(R^2 - r^2)^{3/2}$$



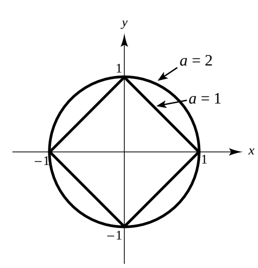

**66.** (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.

(b) $|y| = (1 - |x|^a)^{1/a}$

$$A = 2\int_{-1}^{1}(1 - |x|^a)^{1/a}\,dx = 4\int_0^1 (1 - x^a)^{1/a}\,dx$$

To approximate the volume of the solid, form n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.



## Section 6.3 Volume: The Shell Method

**2.** $p(x) = x$

$h(x) = 1 - x$

$$V = 2\pi \int_0^1 x(1 - x)\,dx$$

$$= 2\pi \int_0^1 (x - x^2)\,dx = 2\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$$

**4.** $p(x) = x$

$h(x) = 8 - (x^2 + 4) = 4 - x^2$

$$V = 2\pi \int_0^2 x(4 - x^2)\,dx$$

$$= 2\pi \int_0^2 (4x - x^3)\,dx$$

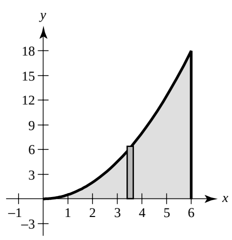
$$= 2\pi\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8\pi$$

**6.** $p(x) = x$

$h(x) = \frac{1}{2}x^2$

$$V = 2\pi \int_0^6 \frac{1}{2}x^3\,dx$$

$$= \left[\pi\frac{x^4}{4}\right]_0^6 = 324\pi$$

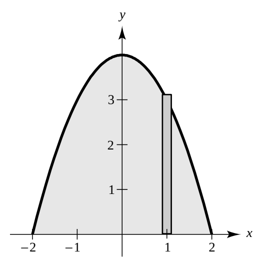



**8.** $p(x) = x$

$h(x) = 4 - x^2$

$$V = 2\pi \int_0^2 (4x - x^3)\,dx$$

$$= 2\pi\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 8\pi$$



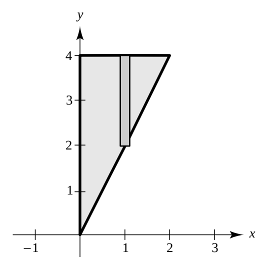

**10.** $p(x) = x$

$h(x) = 4 - 2x$

$$V = 2\pi \int_0^2 x(4 - 2x)\,dx$$

$$= 2\pi \int_0^2 (4x - 2x^2)\,dx$$

$$= 2\pi\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{16\pi}{3}$$

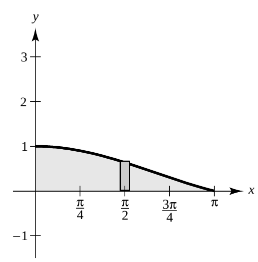



**12.** $p(x) = x$

$h(x) = \frac{\sin x}{x}$

$$V = 2\pi \int_0^\pi x\left[\frac{\sin x}{x}\right]dx$$

$$= 2\pi \int_0^\pi \sin x\,dx = \left[-2\pi\cos x\right]_0^\pi = 4\pi$$




**14.** $p(y) = -y \quad (p(y) \geq 0 \text{ on } [-2, 0])$

$h(y) = 4 - (2 - y) = 2 + y$

$$V = 2\pi \int_{-2}^0 (-y)(2 + y)\,dy$$

$$= 2\pi \int_{-2}^0 (-2y - y^2)\,dy$$

$$= 2\pi\left[-y^2 - \frac{y^3}{3}\right]_{-2}^0 = \frac{8\pi}{3}$$

**16.** $p(y) = y$

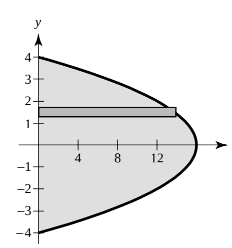
$h(y) = 16 - y^2$

$$V = 2\pi \int_0^4 y(16 - y^2)\,dy$$

$$= 2\pi \int_0^4 (16y - y^3)\,dy$$

$$= 2\pi\left[8y^2 - \frac{y^4}{4}\right]_0^4$$

$$= 2\pi[128 - 64] = 128\pi$$



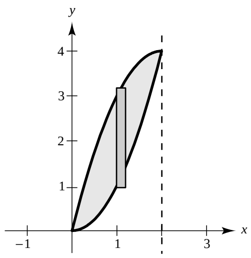
**18.** $p(x) = 2 - x$

$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 (2 - x)(4x - 2x^2)\, dx$$

$$= 2\pi \int_0^2 (8x - 8x^2 + 2x^3)\, dx$$

$$= 2\pi \left[ 4x^2 - \frac{8}{3}x^3 + \frac{1}{2}x^4 \right]_0^2 = \frac{16\pi}{3}$$



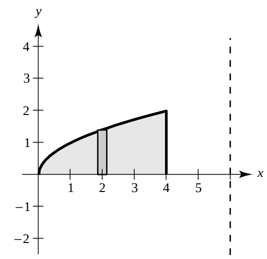
**20.** $p(x) = 6 - x$

$h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 (6 - x)\sqrt{x}\, dx$$

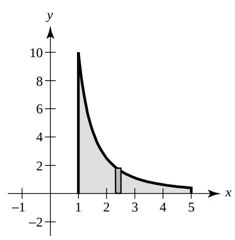
$$= 2\pi \int_0^4 (6x^{1/2} - x^{3/2})\, dx$$

$$= 2\pi \left[ 4x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^4 = \frac{192\pi}{5}$$



**22.** (a) **Disk**

$$R(x) = \frac{10}{x^2},\ r(x) = 0$$



$$V = \pi \int_1^5 \left( \frac{10}{x^2} \right)^2 dx$$

$$= 100\pi \int_1^5 x^{-4}\, dx$$

$$= 100\pi \left[ \frac{x^{-3}}{-3} \right]_1^5$$

$$= -\frac{100\pi}{3} \left[ \frac{1}{125} - 1 \right] = \frac{496}{15}\pi$$

(b) **Shell**

$$R(x) = x,\ r(x) = 0$$

$$V = 2\pi \int_1^5 x \left( \frac{10}{x^2} \right) dx$$

$$= 20\pi \int_1^5 \frac{1}{x}\, dx$$

$$= 20\pi \left[ \ln|x| \right]_1^5 = 20\pi \ln 5$$

(c) **Disk**

$$R(x) = 10,\ r(x) = 10 - \frac{10}{x^2}$$

$$V = \pi \int_1^5 \left[ 10^2 - \left( 10 - \frac{10}{x^2} \right)^2 \right] dx$$

$$= \pi \left[ \frac{100}{3x^3} - \frac{200}{x} \right]_1^5 = \frac{1904}{15}\pi$$

**24.** (a) **Disk**

$$R(x) = (a^{2/3} - x^{2/3})^{3/2}$$

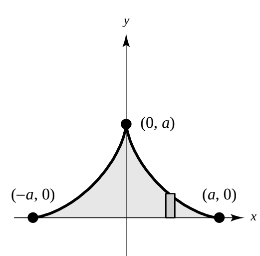
$$r(x) = 0$$

$$V = \pi \int_{-a}^{a} (a^{2/3} - x^{2/3})^3\, dx$$

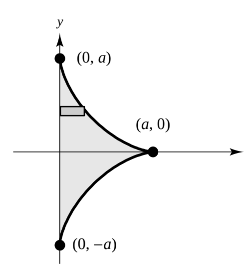
$$= 2\pi \int_0^a (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2)\, dx$$

$$= 2\pi \left[ a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3 \right]_0^a$$

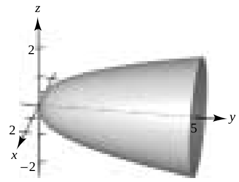
$$= 2\pi \left( a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3 \right) = \frac{32\pi a^3}{105}$$



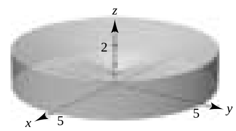

(b) Same as part a by symmetry



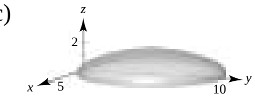
**26.** (a)



(b)



(c)



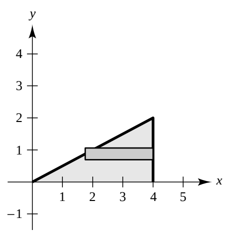
$a < c < b$

**28.** $2\pi \int_0^4 x\left(\frac{x}{2}\right) dx$

represents the volume of the solid generated by revolving the region bounded by $y = x/2$, $y = 0$, and $x = 4$ about the $y$-axis by using the Shell Method.

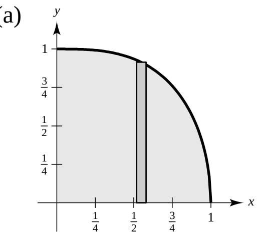
$$\pi \int_0^2 \left[16 - (2y)^2\right] dy = \pi \int_0^2 \left[(4)^2 - (2y)^2\right] dy$$

represents this same volume by using the Disk Method.



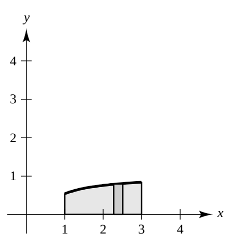
Disk Method

**30.** (a)



(b) $V = 2\pi \int_0^1 x\sqrt{1 - x^3}\, dx \approx 2.3222$

**32.** (a)

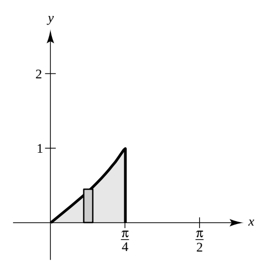


(b) $V = 2\pi \int_1^3 \frac{2x}{1 + e^{1/x}}\, dx \approx 19.0162$

**34.** $y = \tan x$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$

Volume $\approx 1$

Matches (e)



**36.** Total volume of the hemisphere is $\frac{1}{2}\left(\frac{4}{3}\right)\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method, $p(x) = x$, $h(x) = \sqrt{9 - x^2}$. Find $x_0$ such that

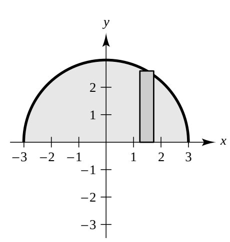
$$6\pi = 2\pi \int_0^{x_0} x\sqrt{9 - x^2}\, dx$$

$$6 = -\int_0^{x_0} (9 - x^2)^{1/2}(-2x)\, dx$$

$$= \left[-\frac{2}{3}(9 - x^2)^{3/2}\right]_0^{x_0} = 18 - \frac{2}{3}(9 - x_0^2)^{3/2}$$

$$(9 - x_0^2)^{3/2} = 18$$

$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460.$$



Diameter: $2\sqrt{9 - 18^{2/3}} \approx 2.920$

**38.** $V = 4\pi \int_{-r}^{r} (R - x)\sqrt{r^2 - x^2}\, dx$

$= 4\pi R \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx - 4\pi \int_{-r}^{r} x\sqrt{r^2 - x^2}\, dx$

$= 4\pi R\left(\dfrac{\pi r^2}{2}\right) + \left[2\pi\left(\dfrac{2}{3}\right)(r^2 - x^2)^{3/2}\right]_{-r}^{r}$

$= 2\pi^2 r^2 R$

**40.** (a) Area region $= \int_0^b [ab^n - ax^n]\, dx$

$= \left[ab^n x - a\dfrac{x^{n+1}}{n+1}\right]_0^b$

$= ab^{n+1} - a\dfrac{b^{n+1}}{n+1}$

$= ab^{n+1}\left(1 - \dfrac{1}{n+1}\right) = ab^{n+1}\left(\dfrac{n}{n+1}\right)$

$R_1(n) = \dfrac{ab^{n+1}\left(\dfrac{n}{n+1}\right)}{(ab^n)b} = \dfrac{n}{n+1}$

(b) $\lim\limits_{n\to\infty} R_1(n) = \lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$

$\lim\limits_{n\to\infty} (ab^n)b = \infty$

(c) **Disk Method:**

$V = 2\pi \int_0^b x(ab^n - ax^n)\, dx$

$= 2\pi a \int_0^b (xb^n - x^{n+1})\, dx$

$= 2\pi a\left[\dfrac{b^n}{2}x^2 - \dfrac{x^{n+2}}{n+2}\right]_0^b$

$= 2\pi a\left[\dfrac{b^{n+2}}{2} - \dfrac{b^{n+2}}{n+2}\right] = \pi ab^{n+2}\left(\dfrac{n}{n+2}\right)$

$R_2(n) = \dfrac{\pi ab^{n+2}\left(\dfrac{n}{n+2}\right)}{(\pi b^2)(ab^n)} = \left(\dfrac{n}{n+2}\right)$

(d) $\lim\limits_{n\to\infty} R_2(n) = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+2}\right) = 1$

$\lim\limits_{n\to\infty} (\pi b^2)(ab^n) = \infty$

(e) As $n \to \infty$, the graph approaches the line $x = 1$.

**42.** (a) $V = 2\pi \int_0^4 x f(x)\, dx$

$= \dfrac{2\pi(40)}{3(4)}[0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0]$

$= \dfrac{20\pi}{3}[5800] \approx 121{,}475$ cubic feet

(b) Top line: $y - 50 = \dfrac{40 - 50}{20 - 0}(x - 0) = -\dfrac{1}{2}x \implies y = -\dfrac{1}{2}x + 50$

Bottom line: $y - 40 = \dfrac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \implies y = -2x + 80$

$V = 2\pi \int_0^{20} x\left(-\dfrac{1}{2}x + 50\right) dx + 2\pi \int_{20}^{40} x(-2x + 80)\, dx$

$= 2\pi \int_0^{20} \left(-\dfrac{1}{2}x^2 + 50x\right) dx + 2\pi \int_{20}^{40} (-2x^2 + 80x)\, dx$

$= 2\pi\left[-\dfrac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi\left[-\dfrac{2x^3}{3} + 40x^2\right]_{20}^{40}$

$= 2\pi\left[\dfrac{26{,}000}{3}\right] + 2\pi\left[\dfrac{32{,}000}{3}\right]$

$\approx 121{,}475$ cubic feet

(Note that Simpson's Rule is exact for this problem.)

## Section 6.4 Arc Length and Surfaces of Revolution

**2.** (1, 2), (7, 10)

(a) $d = \sqrt{(7-1)^2 + (10-2)^2} = 10$

(b) $y = \dfrac{4}{3}x + \dfrac{2}{3}$

$$y' = \frac{4}{3}$$

$$s = \int_1^7 \sqrt{1 + \left(\frac{4}{3}\right)^2}\,dx = \left[\frac{5}{3}x\right]_1^7 = 10$$

**4.** $y = 2x^{3/2} + 3$

$y' = 3x^{1/2}$, $[0, 9]$

$$s = \int_0^9 \sqrt{1 + 9x}\,dx = \left[\frac{2}{27}(1+9x)^{3/2}\right]_0^9 = \frac{2}{27}(82^{3/2} - 1) \approx 54.929$$

**6.** $y = \dfrac{3}{2}x^{2/3} + 4$

$y' = x^{-1/3}$, $[1, 27]$

$$s = \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\,dx = \int_1^{27} \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx = \frac{3}{2}\int_1^{27} \sqrt{x^{2/3}+1}\left(\frac{2}{3x^{1/3}}\right)dx = \left[\frac{3}{2}\cdot\frac{2}{3}(x^{2/3}+1)^{3/2}\right]_1^{27} = 10^{3/2} - 2^{3/2} \approx 28.794$$

**8.**
$$y = \frac{x^5}{10} + \frac{1}{6x^3}$$

$$y' = \frac{1}{2}x^4 - \frac{1}{2x^4}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2, [1, 2]$$

$$s = \int_a^b \sqrt{1 + (y')^2}\,dx = \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2}\,dx = \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)dx = \left[\frac{1}{10}x^5 - \frac{1}{6x^3}\right]_1^2 = \frac{779}{240} \approx 3.246$$

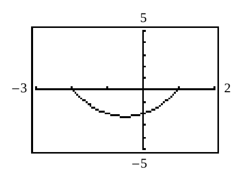
**10.**
$$y = \frac{1}{2}(e^x + e^{-x})$$

$$y' = \frac{1}{2}(e^x - e^{-x}), [0, 2]$$

$$1 + (y')^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2, [0, 2]$$

$$s = \int_0^2 \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2}\,dx = \frac{1}{2}\int_0^2 (e^x + e^{-x})\,dx = \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$$

**12.** (a) $y = x^2 + x - 2,\ -2 \le x \le 1$



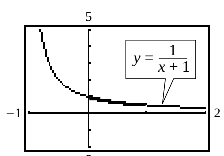

(b) $y' = 2x + 1$

$$1 + (y')^2 = 1 + 4x^2 + 4x + 1$$

$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2}\,dx$$

(c) $L \approx 5.653$

**14.** (a) $y = \dfrac{1}{1+x},\ 0 \le x \le 1$



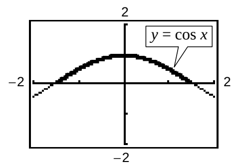

(b) $y' = -\dfrac{1}{(1+x)^2}$

$$1 + (y')^2 = 1 + \frac{1}{(1+x)^4}$$

$$L = \int_0^1 \sqrt{1 + \frac{1}{(1+x)^4}}\,dx$$

(c) $L \approx 1.132$

**16.** (a) $y = \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



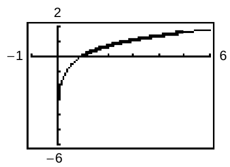

(b) $y' = -\sin x$

$1 + (y')^2 = 1 + \sin^2 x$

$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

(c) 3.820

**18.** (a) $y = \ln x, 1 \le x \le 5$




(b) $y' = \frac{1}{x}$

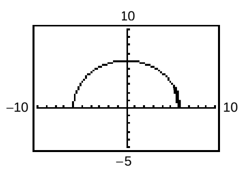
$1 + (y')^2 = 1 + \frac{1}{x^2}$

$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 4.367$

**20.** (a) $x = \sqrt{36 - y^2}, 0 \le y \le 3$

$y = \sqrt{36 - x^2}, 3\sqrt{3} \le x \le 6$




(b) $\frac{dx}{dy} = \frac{1}{2}(36 - y^2)^{-1/2}(-2y)$

$$= \frac{-y}{\sqrt{36 - y^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}}\, dy$$

$$= \int_0^3 \frac{6}{\sqrt{36 - y^2}}\, dy$$

(c) $L \approx 3.142$ ($\pi$!)

Alternatively, you can convert to a function of $x$.

$$y = \sqrt{36 - x^2}$$

$$y' = \frac{dy}{dx} = -\frac{x}{\sqrt{36 - x^2}}$$

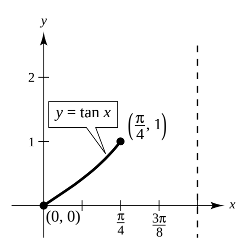
$$L = \int_{3\sqrt{3}}^6 \sqrt{1 + \frac{x^2}{36 - x^2}}\, dx = \int_{3\sqrt{3}}^6 \frac{6}{\sqrt{36 - x^2}}\, dx$$

Although this integral is undefined at $x = 0$, a graphing utility still gives $L \approx 3.142$.

**22.** $\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\, dx$

$s \approx 1$

Matches (e)




**24.** $f(x) = (x^2 - 4)^2, [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (144 - 16)^2} \approx 128.062$

(b) $d = \sqrt{(1 - 0)^2 + (9 - 16)^2} + \sqrt{(2 - 1)^2 + (0 - 9)^2} + \sqrt{(3 - 2)^2 + (25 - 0)^2} + \sqrt{(4 - 3)^2 + (144 - 25)^2}$

$\approx 160.151$

(c) $s = \int_0^4 \sqrt{1 + [4x(x^2 - 4)]^2}\, dx \approx 159.087$

(d) 160.287

**26.** Let $y = \ln x$, $1 \le x \le e$, $y' = \dfrac{1}{x}$ and $L_1 = \displaystyle\int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx$.

Equivalently, $x = e^y$, $0 \le y \le 1$, $\dfrac{dx}{dy} = e^y$, and $L_2 = \displaystyle\int_0^1 \sqrt{1 + e^{2y}}\, dy = \int_0^1 \sqrt{1 + e^{2x}}\, dx$.

Numerically, both integrals yield $L = 2.0035$

**28.**
$$y = 31 - 10(e^{x/20} + e^{-x/20})$$
$$y' = -\frac{1}{2}(e^{x/20} - e^{-x/20})$$
$$1 + (y')^2 = 1 + \frac{1}{4}(e^{x/10} - 2 + e^{-x/10}) = \left[\frac{1}{2}(e^{x/20} + e^{-x/20})\right]^2$$
$$s = \int_{-20}^{20} \sqrt{\left[\frac{1}{2}(e^{x/20} + e^{-x/20})\right]^2}\, dx$$
$$= \frac{1}{2}\int_{-20}^{20} (e^{x/20} + e^{-x/20})\, dx = \Big[10(e^{x/20} - e^{-x/20})\Big]_{-20}^{20} = 20\left(e - \frac{1}{e}\right) \approx 47 \text{ ft}$$

Thus, there are $100(47) = 4700$ square feet of roofing on the barn.

**30.** $y = 693.8597 - 68.7672 \cosh 0.0100333x$

$y' = -0.6899619478 \sinh 0.0100333x$

$$s = \int_{-299.2239}^{299.2239} \sqrt{1 + (-0.6899619478 \sinh 0.0100333x)^2}\, dx \approx 1480$$

(Use Simpson's Rule with $n = 100$ or a graphing utility.)

**32.**
$$y = \sqrt{25 - x^2}$$
$$y' = \frac{-x}{\sqrt{25 - x^2}}$$
$$1 + (y')^2 = \frac{25}{25 - x^2}$$
$$s = \int_{-3}^{4} \sqrt{\frac{25}{25 - x^2}}\, dx$$
$$= \int_{-3}^{4} \frac{5}{\sqrt{25 - x^2}}\, dx$$
$$= \left[5 \arcsin \frac{x}{5}\right]_{-3}^{4}$$
$$= 5\left[\arcsin \frac{4}{5} - \arcsin\left(-\frac{3}{5}\right)\right] \approx 7.8540$$

$\dfrac{1}{4}[2\pi(5)] \approx 7.8540 = s$

**34.** $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}, [4, 9]$$
$$S = 2\pi\int_4^9 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx$$
$$= 4\pi\int_4^9 \sqrt{x + 1}\, dx$$
$$= \frac{8}{3}\pi(x + 1)^{3/2}\Big]_4^9$$
$$= \frac{8\pi}{3}(10^{3/2} - 5^{3/2}) \approx 171.258$$

**36.**
$$y = \frac{x}{2}$$
$$y' = \frac{1}{2}$$
$$1 + (y')^2 = \frac{5}{4}, [0, 6]$$
$$S = 2\pi \int_0^6 \frac{x}{2}\sqrt{\frac{5}{4}}\,dx$$
$$= \left[\frac{2\pi\sqrt{5}}{8}x^2\right]_0^6 = 9\sqrt{5}\,\pi$$

**38.** $y = 9 - x^2, [0, 3]$
$$y' = -2x$$
$$S = 2\pi\int_0^3 x\sqrt{1 + 4x^2}\,dx$$
$$= \frac{\pi}{4}\int_0^3 (1 + 4x^2)^{1/2}(8x)\,dx$$
$$= \left[\frac{\pi}{6}(1 + 4x^2)^{3/2}\right]_0^3$$
$$= \frac{\pi}{6}(37^{3/2} - 1) \approx 117.319$$

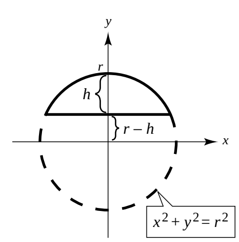
**40.**
$$y = \ln x$$
$$y' = \frac{1}{x}$$
$$1 + (y')^2 = \frac{x^2 + 1}{x^2}, [1, e]$$
$$S = 2\pi\int_1^e x\sqrt{\frac{x^2 + 1}{x^2}}\,dx$$
$$= 2\pi\int_1^e \sqrt{x^2 + 1}\,dx \approx 22.943$$

**42.** The precalculus formula is the distance formula between two points. The representative element is
$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta xi.$$

**44.** The surface of revolution given by $f_1$ will be larger. $r(x)$ is larger for $f_1$.

**46.** $y = \sqrt{r^2 - x^2}$
$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$
$$1 + (y')^2 = \frac{r^2}{r^2 - x^2}$$
$$S = 2\pi\int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\,dx$$
$$= 2\pi\int_{-r}^{r} r\,dx = \Big[2\pi r x\Big]_{-r}^{r} = 4\pi r^2$$

**48.** From Exercise 47 we have:
$$S = 2\pi\int_0^a \frac{rx}{\sqrt{r^2 - x^2}}\,dx$$
$$= -r\pi\int_0^a \frac{-2x\,dx}{\sqrt{r^2 - x^2}}$$
$$= \left[-2r\pi\sqrt{r^2 - x^2}\right]_0^a$$
$$= 2r^2\pi - 2r\pi\sqrt{r^2 - a^2}$$
$$= 2r\pi\left(r - \sqrt{r^2 - a^2}\right)$$
$$= 2\pi r h \text{ (where } h \text{ is the height of the zone)}$$



**50.** (a) We approximate the volume by summing 6 disks of thickness 3 and circumference $C_i$ equal to the average of the given circumferences:

$$V \approx \sum_{i=1}^{6} \pi r_i^2(3) = \sum_{i=1}^{6} \pi\left(\frac{C_i}{2\pi}\right)^2(3) = \frac{3}{4\pi}\sum_{i=1}^{6} C_i^2$$

$$= \frac{3}{4\pi}\left[\left(\frac{50+65.5}{2}\right)^2 + \left(\frac{65.5+70}{2}\right)^2 + \left(\frac{70+66}{2}\right)^2 + \left(\frac{66+58}{2}\right)^2 + \left(\frac{58+51}{2}\right)^2 + \left(\frac{51+48}{2}\right)^2\right]$$

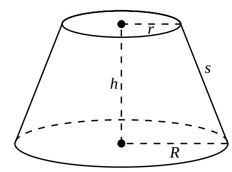
$$= \frac{3}{4\pi}[57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2]$$

$$= \frac{3}{4\pi}[21813.625] = 5207.62 \text{ cubic inches}$$

(b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum,

$$S_1 \approx \pi\left[3^2 + \left(\frac{65.5-50}{2\pi}\right)^2\right]^{1/2}\left[\frac{50}{2\pi} + \frac{65.5}{2\pi}\right]$$

$$= \left(\frac{50+65.5}{2}\right)\left[9 + \left(\frac{65.5-50}{2\pi}\right)^2\right]^{1/2}.$$




Adding the six frustums together,

$$S \approx \left(\frac{50+65.5}{2}\right)\left[9 + \left(\frac{15.5}{2\pi}\right)^2\right]^{1/2} + \left(\frac{65.5+70}{2}\right)\left[9 + \left(\frac{4.5}{2\pi}\right)^2\right]^{1/2} +$$

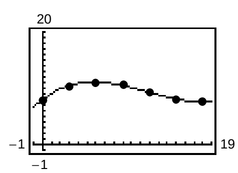
$$\left(\frac{70+66}{2}\right)\left[9 + \left(\frac{4}{2\pi}\right)^2\right]^{1/2} + \left(\frac{66+58}{2}\right)\left[9 + \left(\frac{8}{2\pi}\right)^2\right]^{1/2} +$$

$$\left(\frac{58+51}{2}\right)\left[9 + \left(\frac{7}{2\pi}\right)^2\right]^{1/2} + \left(\frac{51+48}{2}\right)\left[9 + \left(\frac{3}{2\pi}\right)^2\right]^{1/2}$$

$$\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37$$

$$= 1168.64$$

(c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$




(d) $V = \displaystyle\int_0^{18} \pi r^2\,dy \approx 5275.9$ cubic inches

$$S = \int_0^{18} 2\pi r(y)\sqrt{1 + r'(y)^2}\,dy$$

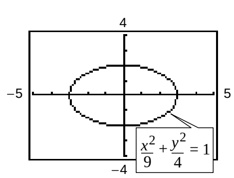
$$\approx 1179.5 \text{ square inches}$$

**52.** Individual project, see Exercise 50, 51.

**54.** (a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \dfrac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \dfrac{x^2}{9}}$




(b) $y = 2\sqrt{1 - \dfrac{x^2}{9}}, 0 \le x \le 3$

$$y' = 2\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{9}\right)^{-1/2}\left(\frac{-2x}{9}\right)$$

$$= \frac{-2x}{9\sqrt{1 - \dfrac{x^2}{9}}} = \frac{-2x}{3\sqrt{9 - x^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\,dx$$

(c) You cannot evaluate this definite integral, since the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

**56.** Essay

## Section 6.5 Work

**2.** $W = Fd = (2800)(4) = 11{,}200 \text{ ft} \cdot \text{lb}$

**4.** $W = Fd = [9(2000)]\left[\frac{1}{2}(5280)\right] = 47{,}520{,}000 \text{ ft} \cdot \text{lb}$

**6.** $W = \int_a^b F(x)\,dx$ is the work done by a force $F$ moving an object along a straight line from $x = a$ to $x = b$.

**8.** (a) $W = \int_0^9 6\,dx = 54 \text{ ft} \cdot \text{lbs}$

(b) $W = \int_0^7 20\,dx + \int_7^9 (-10x + 90)\,dx = 140 + 20$

$= 160 \text{ ft} \cdot \text{lbs}$

(c) $W = \int_0^9 \frac{1}{27}x^2\,dx = \frac{x^3}{81}\Big]_0^9 = 9 \text{ ft} \cdot \text{lbs}$

(d) $W = \int_0^9 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^9 = \frac{2}{3}(27) = 18 \text{ ft} \cdot \text{lbs}$

**10.** $W = \int_0^{10} \frac{5}{4}x\,dx = \left[\frac{5}{8}x^2\right]_6^{10}$

$= 40 \text{ in} \cdot \text{lb} \approx 3.33 \text{ ft} \cdot \text{lb}$

**12.** $F(x) = kx$

$800 = k(70) \implies k = \frac{80}{7}$

$W = \int_0^{70} F(x)\,dx = \int_0^{70} \frac{80}{7}x\,dx = \frac{40x^2}{7}\Big]_0^{70}$

$= 28000 \text{ n} \cdot \text{cm} = 280 \text{ Nm}$

**14.** $F(x) = kx$

$15 = k(1) = k$

$W = 2\int_0^4 15x\,dx = \left[15x^2\right]_0^4$

$= 240 \text{ ft} \cdot \text{lb}$

**16.** $W = 7.5 = \int_0^{1/6} kx\,dx = \frac{kx^2}{2}\Big]_0^{1/6} = \frac{k}{72} \implies k = 540$

$W = \int_{1/6}^{5/24} 540x\,dx = 270x^2\Big]_{1/6}^{5/24} = 4.21875 \text{ ft} \cdot \text{lbs}$

**18.** $W = \int_{4000}^{h} \frac{80{,}000{,}000}{x^2}\,dx = \left[-\frac{80{,}000{,}000}{x}\right]_{4000}^{h}$

$= \frac{-80{,}000{,}000}{h} + 20{,}000$

$\lim_{h\to\infty} W = 20{,}000 \text{ mi/ton} \approx 2.1 \times 10^{11} \text{ ft} \cdot \text{lb}$

**20.** Weight on surface of moon: $\frac{1}{6}(12) = 2$ tons

Weight varies inversely as the square of distance from the center of the moon. Therefore,

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

$$W = \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2}\,dx = \left[\frac{-2.42 \times 10^6}{x}\right]_{1100}^{1150} = 2.42 \times 10^6\left(\frac{1}{1100} - \frac{1}{1150}\right)$$

$$\approx 95.652 \text{ mi} \cdot \text{ton} \approx 1.01 \times 10^9 \text{ ft} \cdot \text{lb}$$

**22.** The bottom half had to be pumped a greater distance then the top half.

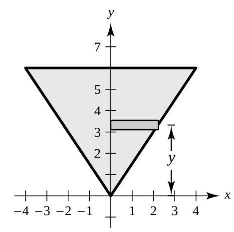
**24.** Volume of disk: $4\pi \, \Delta y$

Weight of disk: $9800(4\pi)\, \Delta y$

Distance the disk of water is moved: $y$

$$W = \int_{10}^{12} y(9800)(4\pi)\, dy = 39{,}200\pi \left[\frac{y^2}{2}\right]_{10}^{12}$$
$$= 39{,}200\pi(22)$$
$$= 862{,}400\pi \text{ newton–meters}$$

**26.** Volume of disk: $\pi\left(\frac{2}{3}y\right)^2 \Delta y$



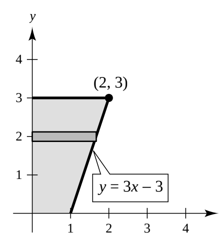

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance: $y$

(a) $W = \frac{4}{9}(62.4)\pi \int_0^2 y^3\, dy = \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb}$

(b) $W = \frac{4}{9}(62.4)\pi \int_4^6 y^3\, dy = \left[\frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right)\right]_4^6 \approx 7210.7\pi \text{ ft} \cdot \text{lb}$

**28.** Volume of each layer: $\frac{y+3}{3}(3)\, \Delta y = (y+3)\, \Delta y$

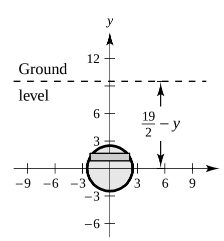



Weight of each layer: $55.6(y+3)\, \Delta y$

Distance: $6 - y$

$$W = \int_0^3 55.6(6-y)(y+3)\, dy = 55.6\int_0^3 (18 + 3y - y^2)\, dy$$
$$= 55.6\left[18y + \frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$$
$$= 3252.6 \text{ ft} \cdot \text{lb}$$

**30.** Volume of layer: $V = 12(2)\sqrt{(25/4) - y^2}\, \Delta y$




Weight of layer: $W = 42(24)\sqrt{(25/4) - y^2}\, \Delta y$

Distance: $\frac{19}{2} - y$

$$W = \int_{-2.5}^{2.5} 42(24)\sqrt{\frac{25}{4} - y^2}\left(\frac{19}{2} - y\right) dy$$
$$= 1008\left[\frac{19}{2}\int_{-2.5}^{2.5}\sqrt{\frac{25}{4} - y^2}\, dy + \int_{-2.5}^{2.5}\sqrt{\frac{25}{4} - y^2}(-y)\, dy\right]$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{5}{2}$. Thus, the work is

$$W = 1008\left(\frac{19}{2}\right)\pi\left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right) = 29{,}925\pi \text{ ft} \cdot \text{lb} \approx 94{,}012.16 \text{ ft} \cdot \text{lb}.$$

**32.** The lower 10 feet of chain are raised 5 feet with a constant force.

$$W_1 = 3(10)5 = 150 \text{ ft} \cdot \text{lb}$$

The top 5 feet will be raised with variable force.

Weight of section: $3\,\Delta y$

Distance: $5 - y$

$$W_2 = 3\int_0^5 (5 - y)\,dy = \left[-\frac{3}{2}(5 - y)^2\right]_0^5 = \frac{75}{2} \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 150 + \frac{75}{2} = \frac{375}{2} \text{ ft} \cdot \text{lb}$$

**34.** The work required to lift the chain is 337.5 ft · lb (from Exercise 31). The work required to lift the 500-pound load is $W = (500)(15) = 7500$. The work required to lift the chain with a 100-pound load attached is

$$W = 337.5 + 7500 = 7837.5 \text{ ft} \cdot \text{lbs}$$

**36.** $W = 3\int_0^6 (12 - 2y)\,dy = \left[-\frac{3}{4}(12 - 2y)^2\right]_0^6 = \frac{3}{4}(12)^2 = 108 \text{ ft} \cdot \text{lb}$

**38.** Work to pull up the ball: $W_1 = 500(40) = 20{,}000 \text{ ft} \cdot \text{lb}$

Work to pull up the cable: force is variable

Weight per section: $1\,\Delta y$

Distance: $40 - x$

$$W_2 = \int_0^{40} (40 - x)\,dx = \left[-\frac{1}{2}(40 - x)^2\right]_0^{40}$$

$$= 800 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 20{,}000 + 800 = 20{,}800 \text{ ft} \cdot \text{lb}$$

**40.** $p = \dfrac{k}{V}$

$$2500 = \frac{k}{1} \implies k = 2500$$

$$W = \int_1^3 \frac{2500}{V}\,dV = \Big[2500 \ln V\Big]_1^3$$

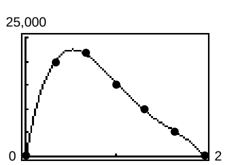
$$= 2500 \ln 3 \approx 2746.53 \text{ ft} \cdot \text{lb}$$

**42.** (a) $W = FD = (8000\pi)(2) = 16{,}000\pi \text{ ft} \cdot \text{lbs}$

(b) $W \approx \dfrac{2 - 0}{3(6)}[0 + 4(20{,}000) + 2(22{,}000) + 4(15{,}000) + 2(10{,}000) + 4(5000) + 0]$

$\approx 24{,}88.889 \text{ ft} \cdot \text{lb}$

(c) $F(x) = -16{,}261.36x^4 + 85{,}295.45x^3 - 157{,}738.64x^2 + 104{,}386.36x - 32.4675$




(d) $F(x) = 0$ when $x \approx 0.524$ feet. $F(x)$ is a maximum when $x \approx 0.524$ feet.

(e) $W = \displaystyle\int_0^2 F(x)\,dx \approx 25{,}180.5 \text{ ft} \cdot \text{lbs}$

**44.** $W = \displaystyle\int_0^4 \left(\frac{e^{x^2} - 1}{100}\right)dx \approx 11{,}494 \text{ ft} \cdot \text{lb}$

**46.** $W = \displaystyle\int_0^2 1000 \sinh x\,dx \approx 2762.2 \text{ ft} \cdot \text{lb}$

## Section 6.6 Moments, Centers of Mass, and Centroids

**2.** $\bar{x} = \dfrac{7(-3) + 4(-2) + 3(5) + 8(6)}{7 + 4 + 3 + 8} = \dfrac{17}{11}$

**4.** $\bar{x} = \dfrac{12(-6) + 1(-4) + 6(-2) + 3(0) + 11(8)}{12 + 1 + 6 + 3 + 11} = 0$

**6.** The center of mass is translated $k$ units as well.

**8.** $200x = 550(5 - x)$ (Person on left)

$200x = 2750 - 550x$

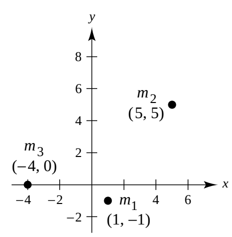
$750x = 2750$

$x = 3\frac{2}{3}$ feet

**10.** $\bar{x} = \dfrac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0$

$\bar{y} = \dfrac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0$

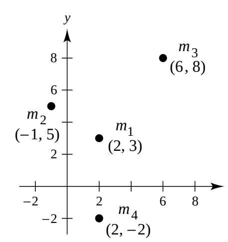
$(\bar{x}, \bar{y}) = (0, 0)$




**12.** $\bar{x} = \dfrac{12(2) + 6(-1) + \frac{15}{2}(6) + 15(2)}{12 + 6 + \frac{15}{2} + 15} = \dfrac{93}{40.5} = \dfrac{62}{27}$

$\bar{y} = \dfrac{12(3) + 6(5) + \frac{15}{2}(8) + 15(-2)}{12 + 6 + \frac{15}{2} + 15} = \dfrac{96}{40.5} = \dfrac{64}{27}$

$(\bar{x}, \bar{y}) = \left(\dfrac{62}{27}, \dfrac{64}{27}\right)$




**14.** $m = \rho\displaystyle\int_0^2 \frac{1}{2}x^2\,dx = \left[\rho\frac{x^3}{6}\right]_0^2 = \frac{4}{3}\rho$

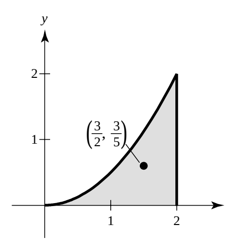
$M_x = \rho\displaystyle\int_0^2 \frac{1}{2}\left(\frac{1}{2}x^2\right)\left(\frac{1}{2}x^2\right)dx = \frac{\rho}{8}\int_0^2 x^4\,dx = \left[\frac{\rho}{40}x^5\right]_0^2 = \frac{32}{40}\rho = \frac{4}{5}\rho$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{\frac{4}{5}\rho}{\frac{4}{3}\rho} = \dfrac{3}{5}$

$M_y = \rho\displaystyle\int_0^2 x\left(\frac{1}{2}x^2\right)dx = \frac{1}{2}\rho\int_0^2 x^3\,dx = \left[\frac{\rho}{8}x^4\right]_0^2 = 2\rho$

$\bar{x} = \dfrac{M_y}{m} = \dfrac{2\rho}{\frac{4}{3}\rho} = \dfrac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\dfrac{3}{2}, \dfrac{3}{5}\right)$



**16.** $m = \rho\int_0^1 \left(\sqrt{x} - x\right) dx = \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = \frac{\rho}{6}$

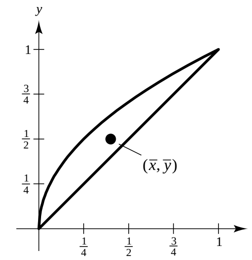
$M_x = \rho\int_0^1 \frac{\left(\sqrt{x} + x\right)}{2}\left(\sqrt{x} - x\right) dx = \frac{\rho}{2}\int_0^1 (x - x^2)\, dx = \frac{\rho}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{\rho}{12}$

$\bar{y} = \frac{M_x}{m} = \frac{\rho}{12}\left(\frac{6}{\rho}\right) = \frac{1}{2}$

$M_y = \rho\int_0^1 x\left(\sqrt{x} - x\right) dx = \rho\int_0^1 (x^{3/2} - x^2)\, dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{x^3}{3}\right]_0^1 = \frac{\rho}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{\rho}{15}\left(\frac{6}{\rho}\right) = \frac{2}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{2}{5}, \frac{1}{2}\right)$




**18.** $m = \rho\int_0^9 \left[\left(\sqrt{x} + 1\right) - \left(\frac{1}{3}x + 1\right)\right] dx = \rho\int_0^9 \left(\sqrt{x} - \frac{1}{3}x\right) dx$

$= \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{6}\right]_0^9 = \rho\left(18 - \frac{27}{2}\right) = \frac{9}{2}\rho$

$M_x = \rho\int_0^9 \frac{\sqrt{x} + 1 + \frac{1}{3}x + 1}{2}\left(\sqrt{x} + 1 - \frac{1}{3}x - 1\right) dx = \frac{\rho}{2}\int_0^9 \left(\sqrt{x} + \frac{1}{3}x + 2\right)\left(\sqrt{x} - \frac{1}{3}x\right) dx$

$= \frac{\rho}{2}\int_0^9 \left(x - \frac{1}{3}x^{3/2} + \frac{1}{3}x^{3/2} - \frac{1}{9}x^2 + 2\sqrt{x} - \frac{2}{3}x\right) dx = \frac{\rho}{2}\int_0^9 \left(\frac{1}{3}x - \frac{1}{9}x^2 + 2\sqrt{x}\right) dx$

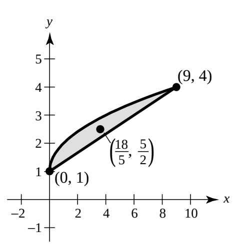
$= \frac{\rho}{2}\left[\frac{x^2}{6} - \frac{x^3}{27} + \frac{4}{3}x^{3/2}\right]_0^9 = \frac{\rho}{2}\left[\frac{27}{2} - 27 + 36\right] = \frac{45}{3}\rho$

$M_y = \rho\int_0^9 x\left[\sqrt{x} + 1 - \frac{1}{3}x - 1\right] dx = \rho\int_0^9 \left(x^{3/2} - \frac{1}{3}x^2\right) dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{1}{9}x^3\right]_0^9$

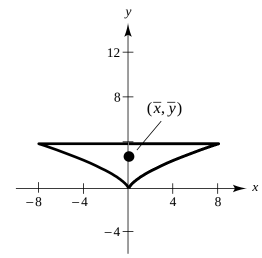
$= \rho\left[\frac{486}{5} - 81\right] = \frac{81}{5}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{\frac{81}{5}\rho}{\frac{9}{2}\rho} = \frac{18}{5}; \bar{y} = \frac{M_x}{m} = \frac{\frac{45}{4}\rho}{\frac{9}{2}\rho} = \frac{5}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{18}{5}, \frac{5}{2}\right)$



**20.** $m = 2\rho\int_0^8 (4 - x^{2/3})\,dx = 2\rho\left[4x - \frac{3}{5}x^{5/3}\right]_0^8 = \frac{128\rho}{5}$



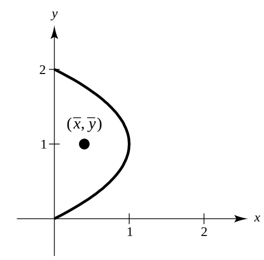

By symmetry, $M_y$ and $\bar{x} = 0$.

$$M_x = 2\rho\int_0^8 \left(\frac{4 + x^{2/3}}{2}\right)(4 - x^{2/3})\,dx = \rho\left[16x - \frac{3}{7}x^{7/3}\right]_0^8 = \frac{512\rho}{7}$$

$$\bar{y} = \frac{512\rho}{7}\left(\frac{5}{128\rho}\right) = \frac{20}{7}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{20}{7}\right)$$

**22.** $m = \rho\int_0^2 (2y - y^2)\,dy = \rho\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{4\rho}{3}$




$$M_y = \rho\int_0^2 \left(\frac{2y - y^2}{2}\right)(2y - y^2)\,dy = \frac{\rho}{2}\left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right]_0^2 = \frac{8\rho}{15}$$

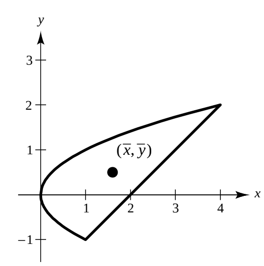
$$\bar{x} = \frac{M_y}{m} = \frac{8\rho}{15}\left(\frac{3}{4\rho}\right) = \frac{2}{5}$$

$$M_x = \rho\int_0^2 y(2y - y^2)\,dy = \rho\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = \frac{4\rho}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4\rho}{3}\left(\frac{3}{4\rho}\right) = 1$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{5}, 1\right)$$

**24.** $m = \rho\int_{-1}^2 [(y + 2) - y^2]\,dy = \rho\left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^2 = \frac{9\rho}{2}$




$$M_y = \rho\int_{-1}^2 \frac{[(y + 2) + y^2]}{2}[(y + 2) - y^2]\,dy$$

$$= \frac{\rho}{2}\int_{-1}^2 [(y + 2)^2 - y^4]\,dy = \frac{\rho}{2}\left[\frac{(y + 2)^3}{3} - \frac{y^5}{5}\right]_{-1}^2 = \frac{36\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{8}{5}$$

$$M_x = \rho\int_{-1}^2 y[(y + 2) - y^2]\,dy$$

$$= \rho\int_{-1}^2 (2y + y^2 - y^3)\,dy = \rho\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_{-1}^2 = \frac{9\rho}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{1}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$$

**26.** $A = \int_1^4 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^4 = \ln 4$

$$M_x = \frac{1}{2}\int_1^4 \frac{1}{x^2}\,dx = \left[\frac{1}{2}\left(-\frac{1}{x}\right)\right]_1^4 = \left(-\frac{1}{8} + \frac{1}{2}\right) = \frac{3}{8}$$

$$M_y = \int_1^4 x\left(\frac{1}{x}\right)dx = \left[x\right]_1^4 = 3$$

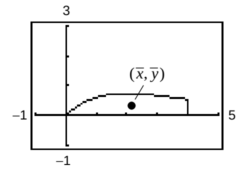
**28.** $A = \int_{-2}^{2} -(x^2 - 4)\,dx = 2\int_{0}^{2} (4 - x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$

$$M_x = \frac{1}{2}\int_{-2}^{2} (x^2 - 4)(4 - x^2)\,dx = -\frac{1}{2}\int_{-2}^{2} (x^4 - 8x^2 + 16)\,dx$$

$$= -\frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-2}^{2} = -\left[\frac{32}{5} - \frac{64}{3} + 32\right] = -\frac{256}{15}$$

$M_y = 0$ by symmetry.

**30.** $m = \rho\int_0^4 xe^{-x/2}\,dx \approx 2.3760\rho$




$$M_x = \rho\int_0^4 \left(\frac{xe^{-x/2}}{2}\right)(xe^{-x/2})\,dx = \frac{\rho}{2}\int_0^4 x^2e^{-x}\,dx \approx 0.7619\rho$$

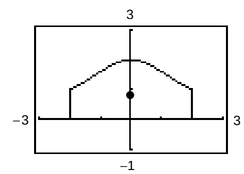
$$M_y = \rho\int_0^4 x^2e^{-x/2}\,dx \approx 5.1732\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 2.2$$

$$\bar{y} = \frac{M_x}{m} \approx 0.3$$

Therefore, the centroid is $(2.2, 0.3)$.

**32.** $m = \rho\int_{-2}^{2} \frac{8}{x^2 + 4}\,dx \approx 6.2832\rho$

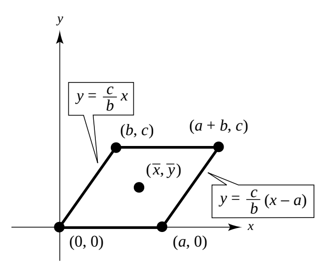



$$M_x = \rho\int_{-2}^{2} \frac{1}{2}\left(\frac{8}{x^2 + 4}\right)\left(\frac{8}{x^2 + 4}\right)dx = 32\rho\int_{-2}^{2} \frac{1}{(x^2 + 4)^2}\,dx \approx 5.14149\rho$$

$$\bar{y} = \frac{M_x}{m} \approx 0.8$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 0.8)$.

**34.** $A = bh = ac$




$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \frac{1}{ac}\frac{1}{2}\int_0^c \left[\left(\frac{b}{c}y + a\right)^2 - \left(\frac{b}{c}y\right)^2\right]dy$$

$$= \frac{1}{2ac}\int_0^c \left(\frac{2ab}{c}y + a^2\right)dy$$

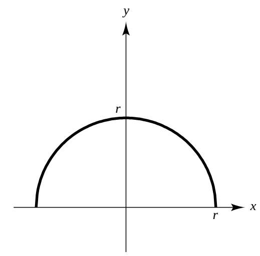
$$= \frac{1}{2ac}\left[\frac{ab}{c}y^2 + a^2y\right]_0^c = \frac{1}{2ac}[abc + a^2c] = \frac{1}{2}(b + a)$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b}{c}y + a\right) - \left(\frac{b}{c}y\right)\right]dy = \left[\frac{1}{c}\frac{y^2}{2}\right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b + a}{2}, \frac{c}{2}\right)$$

This is the point of intersection of the diagonals.

**36.** $\bar{x} = 0$ by symmetry




$$A = \frac{1}{2}\pi r^2$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

$$\bar{y} = \frac{2}{\pi r^2}\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{1}{\pi r^2}\left[\frac{4r^3}{3}\right] = \frac{4r}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$$

**38.** $$A = \int_0^1 [1 - (2x - x^2)]\, dx = \frac{1}{3}$$

$$\frac{1}{A} = 3$$

$$\bar{x} = 3\int_0^1 x[1 - (2x - x^2)]\, dx = 3\int_0^1 [x - 2x^2 + x^3]\, dx = 3\left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\bar{y} = 3\int_0^1 \frac{[1 + (2x - x^2)]}{2}[1 - (2x - x^2)]\, dx = \frac{3}{2}\int_0^1 [1 - (2x - x^2)^2]\, dx$$

$$= \frac{3}{2}\int_0^1 [1 - 4x^2 + 4x^3 - x^4]\, dx = \frac{3}{2}\left[x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5}\right]_0^1 = \frac{7}{10}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{4}, \frac{7}{10}\right)$$

**40.** (a) $M_y = 0$ by symmetry

$$M_y = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} x(b - x^{2n})\, dx = 0$$

because $bx - x^{2n+1}$ is an odd function.

(b) $\bar{y} > \dfrac{b}{2}$ because there is more area above $y = \dfrac{b}{2}$ than below.

(c) $$M_x = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{(b + x^{2n})(b - x^{2n})}{2}\, dx = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} \frac{1}{2}(b^2 - x^{4n})\, dx$$

$$= \frac{1}{2}\left(b^2 x - \frac{x^{4n+1}}{4n+1}\right)\Bigg]_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}}$$

$$= b^2 b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n+1} = \frac{4n}{4n+1}b^{(4n+1)/2n}$$

$$A = \int_{-\sqrt[2n]{b}}^{\sqrt[2n]{b}} (b - x^{2n})\, dx = 2\left[bx - \frac{x^{2n+1}}{2n+1}\right]_0^{\sqrt[2n]{b}}$$

$$= 2\left[b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n+1}\right] = \frac{4n}{2n+1}b^{(2n+1)/2n}$$

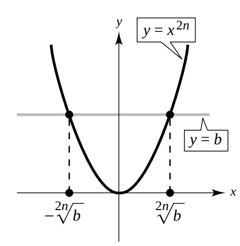
$$\bar{y} = \frac{M_x}{A} = \frac{4n\, b^{(4n+1)/2n}/(4n+1)}{4n\, b^{(24n+1)/2n}/(2n+1)} = \frac{2n+1}{4n+1}b$$

(d)

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\bar{y}$ | $\frac{3}{5}b$ | $\frac{5}{9}b$ | $\frac{7}{13}b$ | $\frac{9}{17}b$ |

(e) $$\lim_{n\to\infty} \bar{y} = \lim_{n\to\infty} \frac{2n+1}{4n+1}b = \frac{1}{2}b$$

(f) As $n \to \infty$, the figure gets narrower.



**42.** Let $f(x)$ be the top curve, given by $l + d$. The bottom curve is $d(x)$.

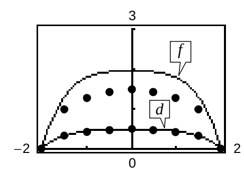
| $x$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $f$ | 2.0 | 1.93 | 1.73 | 1.32 | 0 |
| $d$ | 0.50 | 0.48 | 0.43 | 0.33 | 0 |

(a) $$\text{Area} = 2\int_0^2 [f(x) - d(x)]\,dx$$
$$\approx 2\frac{2}{3(4)}[1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0]$$
$$= \frac{1}{3}[13.86] = 4.62$$
$$M_x = \int_{-2}^{2} \frac{f(x) + d(x)}{2}(f(x) - d(x))\,dx$$
$$= \int_0^2 [f(x)^2 - d(x)^2]\,dx$$
$$= \frac{2}{3(4)}[3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0]$$
$$= \frac{1}{6}[29.878] = 4.9797$$
$$\bar{y} = \frac{M_x}{A} = \frac{4.9797}{4.62} = 1.078$$
$$(\bar{x}, \bar{y}) = (0, 1.078)$$

(b) $f(x) = -0.1061x^4 - 0.06126x^2 + 1.9527$

$d(x) = -0.02648x^4 - 0.01497x^2 + .4862$

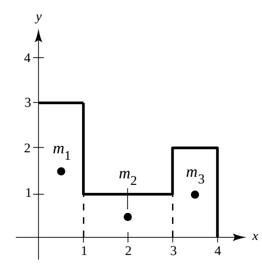
(c) $$\bar{y} = \frac{M_x}{A} \approx \frac{4.9133}{4.59998} = 1.068$$
$$(\bar{x}, \bar{y}) = (0, 1.068)$$




**44.** Centroids of the given regions: $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{7}{2}, 1\right)$

Area: $A = 3 + 2 + 2 = 7$

$$\bar{x} = \frac{3(1/2) + 2(2) + 2(7/2)}{7} = \frac{25/2}{7} = \frac{25}{14}$$
$$\bar{y} = \frac{3(3/2) + 2(1/2) + 2(1)}{7} = \frac{15/2}{7} = \frac{15}{14}$$
$$(\bar{x}, \bar{y}) = \left(\frac{25}{14}, \frac{15}{14}\right)$$

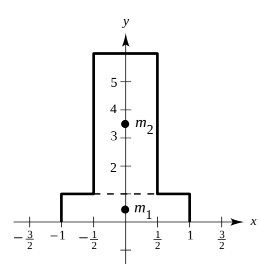



**46.** $m_1 = \frac{7}{8}(2) = \frac{7}{4}, P_1 = \left(0, \frac{7}{16}\right)$

$m_2 = \frac{7}{8}\left(6 - \frac{7}{8}\right) = \frac{287}{64}, P_2 = \left(0, \frac{55}{16}\right)$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16{,}569}{6384} = \frac{5523}{2128}$$
$$(\bar{x}, \bar{y}) = \left(0, \frac{5523}{2128}\right) \approx (0, 2.595)$$



**48.** Centroids of the given regions: $(3, 0)$ and $(1, 0)$

Mass: $8 + \pi$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8 + 3\pi}{8 + \pi}, 0\right) \approx (1.56, 0)$$

**50.** $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

**52.** $A = \int_2^6 2\sqrt{x - 2}\, dx = \frac{4}{3}(x - 2)^{3/2}\Big]_2^6 = \frac{32}{3}$

$M_y = \int_2^6 (x)2\sqrt{x - 2}\, dx = 2\int_2^6 x\sqrt{x - 2}\, dx$

Let $u = x - 2$, $x = u + 2$, $du = dx$:

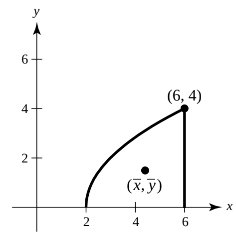
$$M_y = 2\int_0^4 (u + 2)\sqrt{u}\, du = 2\int_0^4 (u^{3/2} + 2u^{1/2})\, du = 2\left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^4$$

$$= 2\left[\frac{64}{5} + \frac{32}{3}\right] = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

$$V = 2\pi rA = 2\pi\left(\frac{22}{5}\right)\left(\frac{32}{3}\right) = \frac{1408\pi}{15} \approx 294.89$$



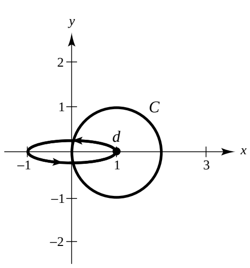

**54.** A planar lamina is a thin flat plate of constant density. The center of mass $(\bar{x}, \bar{y})$ is the balancing point on the lamina.

**56.** Let $R$ be a region in a plane and let $L$ be a line such that $L$ does not intersect the interior of $R$. If $r$ is the distance between the centroid of $R$ and $L$, then the volume $V$ of the solid of revolution formed by revolving $R$ about $L$ is

$$V = 2\pi rA$$

where $A$ is the area of $R$.

**58.** The centroid of the circle is $(1, 0)$. The distance traveled by the centroid is $2\pi$. The arc length of the circle is also $2\pi$. Therefore, $S = (2\pi)(2\pi) = 4\pi^2$.



# Section 6.7 Fluid Pressure and Fluid Force

**2.** $F = PA = [62.4(5)](16) = 4992$ lb

**4.** $F = 62.4(h + 4)(48) - (62.4)(h)(48)$

$= 62.4(4)(48) = 11{,}980.8$ lb

**6.** $h(y) = 3 - y$

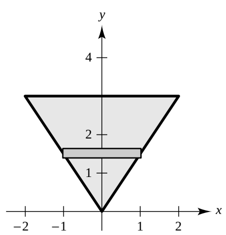
$L(y) = \frac{4}{3}y$

$$F = 62.4 \int_0^3 (3 - y)\left(\frac{4}{3}y\right) dy$$

$$= \frac{4}{3}(62.4) \int_0^3 (3y - y^2)\, dy$$

$$= \frac{4}{3}(62.4)\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = 374.4 \text{ lb}$$

Force is one-third that of Exercise 5.

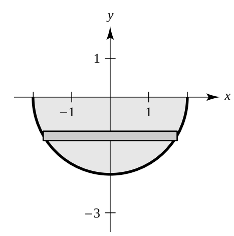



**8.** $h(y) = -y$

$L(y) = 2\sqrt{4 - y^2}$

$$F = 62.4 \int_{-2}^0 (-y)(2)\sqrt{4 - y^2}\, dy$$

$$= \left[62.4\left(\frac{2}{3}\right)(4 - y^2)^{3/2}\right]_{-2}^0 = 332.8 \text{ lb}$$



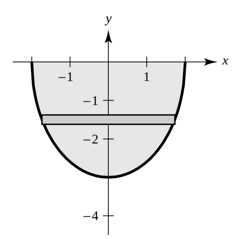

**10.** $h(y) = -y$

$L(y) = \frac{4}{3}\sqrt{9 - y^2}$

$$F = 62.4 \int_{-3}^0 (-y)\frac{4}{3}\sqrt{9 - y^2}\, dy$$

$$= 62.4\left(\frac{2}{3}\right) \int_{-3}^0 (9 - y^2)^{1/2}(-2y)\, dy$$

$$= \left[62.4\left(\frac{4}{9}\right)(9 - y^2)^{3/2}\right]_{-3}^0 = 748.8 \text{ lb}$$




**12.** $h(y) = \left(1 + 3\sqrt{2}\right) - y$

$L_1(y) = 2y$ (lower part)

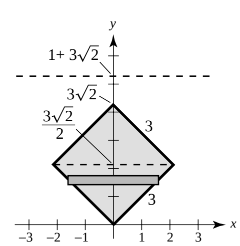
$L_2(y) = 2\left(3\sqrt{2} - y\right)$ (upper part)

$$F = 2(9800)\left[\int_0^{3\sqrt{2}/2} \left(1 + 3\sqrt{2} - y\right)y\, dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}} \left(1 + 3\sqrt{2} - y\right)\left(3\sqrt{2} - y\right) dy\right]$$

$$= 19{,}600\left[\left[\frac{y^2}{2} - 3\sqrt{2}y - \frac{y^3}{3}\right]_0^{3\sqrt{2}/2} + \left[3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2} + 1}{2}y\right]_{3\sqrt{2}/2}^{3\sqrt{2}}\right]$$

$$= 19{,}600\left[\frac{9\left(2\sqrt{2} + 1\right)}{4} + \frac{9\left(\sqrt{2} + 1\right)}{4}\right]$$

$$= 44{,}100\left(3\sqrt{2} + 2\right) \text{ Newtons}$$

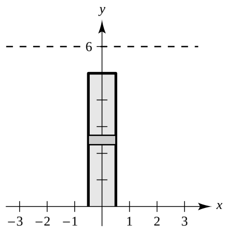


**14.** $h(y) = 6 - y$

$L(y) = 1$

$$F = 9800 \int_0^5 1(6 - y)\, dy$$

$$= 9800\left[6y - \frac{y^2}{2}\right]_0^5 = 171{,}500 \text{ Newtons}$$



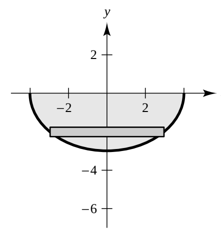

**16.** $h(y) = -y$

$$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$$

$$F = 140.7 \int_{-3}^0 (-y)(2)\left(\frac{4}{3}\sqrt{9 - y^2}\right) dy$$

$$= \frac{(140.7)(4)}{3} \int_{-3}^0 \sqrt{9 - y^2}\,(-2y)\, dy$$

$$= \left[\frac{(140.7)(4)}{3}\left(\frac{2}{3}\right)(9 - y^2)^{3/2}\right]_{-3}^0 = 3376.8 \text{ lb}$$




**18.** $h(y) = -y$

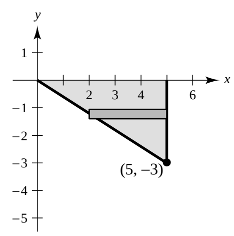
$$L(y) = 5 + \frac{5}{3}y$$

$$F = 140.7 \int_{-3}^0 (-y)\left(5 + \frac{5}{3}y\right) dy$$

$$= 140.7\int_{-3}^0 \left(-5y - \frac{5}{3}y^2\right) dy$$

$$= 140.7\left[-\frac{5}{2}y^2 - \frac{5}{9}y^3\right]_{-3}^0$$

$$= 140.7\left[\frac{45}{2} - 15\right] = 1055.25 \text{ lb}$$




**20.** $h(y) = \frac{3}{2} - y$

$$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$$

$$F = 42 \int_{-3/2}^{3/2} \left(\frac{3}{2} - y\right)\sqrt{9 - 4y^2}\, dy = 63 \int_{-3/2}^{3/2} \sqrt{9 - 4y^2}\, dy + \frac{21}{4} \int_{-3/2}^{3/2} \sqrt{9 - 4y^2}\,(-8y)\, dy$$

The second integral is zero since it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius $\frac{3}{2}$.

$$\left(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2}\right)$$

Thus, the force is $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32$ lb.

**22.** (a) $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi$ lbs

(b) $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi$ lbs

**24.** (a) $F = wkhb = (62.4)\left(\frac{11}{2}\right)(3)(5) = 5148$ lbs

(b) $F = wkhb = (62.4)\left(\frac{17}{5}\right)(5)(10) = 10{,}608$ lbs

**26.** From Exercise 21:

$$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98 \text{ lb}$$

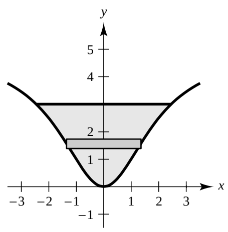
**28.** $h(y) = 3 - y$

Solving $y = 5x^2/(x^2 + 4)$ for $x$, you obtain
$x = \sqrt{4y/(5 - y)}$.

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

$$F = 62.4(2)\int_0^3 (3 - y)\sqrt{\frac{4y}{5 - y}}\,dy$$

$$= 2(124.8)\int_0^3 (3 - y)\sqrt{\frac{y}{5 - y}}\,dy \approx 546.265 \text{ lb}$$

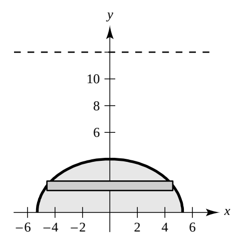



**30.** $h(y) = 12 - y$

$$L(y) = 2\frac{\sqrt{7(16 - y^2)}}{2} = \sqrt{7(16 - y^2)}$$

$$F = 62.4\int_0^4 (12 - y)\sqrt{7(16 - y^2)}\,dy$$

$$= 62.4\sqrt{7}\int_0^4 (12 - y)\sqrt{16 - y^2}\,dy \approx 21373.7 \text{ lb}$$



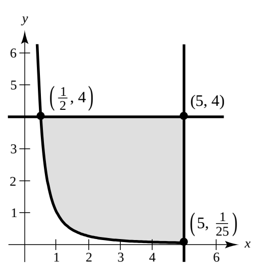

**32.** Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

**34.** The left window experiences the greater fluid force because its centroid is lower.

## Review Exercises for Chapter 6

**2.** $A = \displaystyle\int_{1/2}^5 \left(4 - \frac{1}{x^2}\right) dx$

$$= \left[4x + \frac{1}{x}\right]_{1/2}^5 = \frac{81}{5}$$

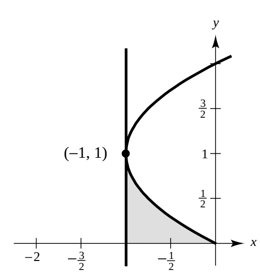



**4.** $A = \displaystyle\int_0^1 [(y^2 - 2y) - (-1)]\,dy$

$$= \int_0^1 (y^2 - 2y + 1)\,dy$$

$$= \int_0^1 (y - 1)^2\,dy$$

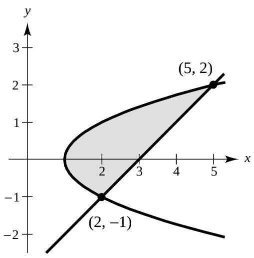
$$= \left[\frac{(y - 1)^3}{3}\right]_0^1 = \frac{1}{3}$$



**6.** $A = \int_{-1}^{2} [(y + 3) - (y^2 + 1)]\, dy$

$= \int_{-1}^{2} (2 + y - y^2)\, dy$

$= \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-1}^{2} = \frac{9}{2}$

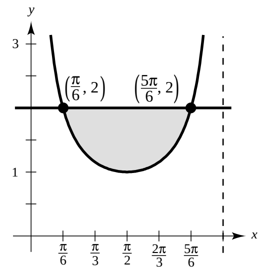



**8.** $A = 2\int_{\pi/6}^{\pi/2} (2 - \csc x)\, dx$

$= 2\Big[2x - \ln|\csc x - \cot x|\Big]_{\pi/6}^{\pi/2}$

$= 2\left([\pi - 0] - \left[\frac{\pi}{3} - \ln\left(2 - \sqrt{3}\right)\right]\right)$

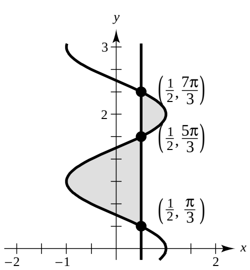
$= 2\left[\frac{2\pi}{3} + \ln\left(2 - \sqrt{3}\right)\right] \approx 1.555$




**10.** $A = \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y\right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2}\right) dy$

$= \left[\frac{y}{2} - \sin y\right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2}\right]_{5\pi/3}^{7\pi/3}$

$= \frac{\pi}{3} + 2\sqrt{3}$



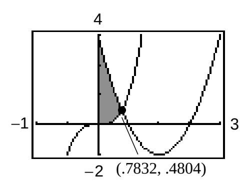

**12.** Point of intersection is given by:

$x^3 - x^2 + 4x - 3 = 0 \implies x \approx 0.783.$

$A \approx \int_{0}^{0.783} (3 - 4x + x^2 - x^3)\, dx$

$= \left[3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{0}^{0.783}$

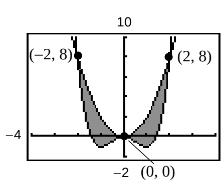
$\approx 1.189$




**14.** $A = 2\int_{0}^{2} [2x^2 - (x^4 - 2x^2)]\, dx$

$= 2\int_{0}^{2} (4x^2 - x^4)\, dx$

$= 2\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{0}^{2} = \frac{128}{15} \approx 8.5333$



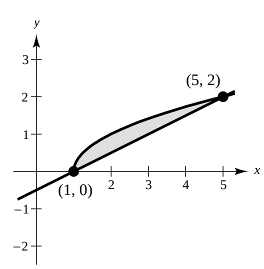

**16.** $y = \sqrt{x - 1} \implies x = y^2 + 1$

$y = \frac{x - 1}{2} \implies x = 2y + 1$

$A = \int_{0}^{2} [(2y + 1) - (y^2 + 1)]\, dy$

$= \int_{1}^{5} \left[\sqrt{x - 1} - \frac{x - 1}{2}\right] dx$

$= \left[\frac{2}{3}(x - 1)^{3/2} - \frac{1}{4}(x - 1)^2\right]_{1}^{5} = \frac{4}{3}$

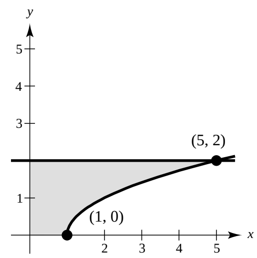


**18.** $A = \int_0^1 2\,dx + \int_1^5 \left[2 - \sqrt{x-1}\right] dx$

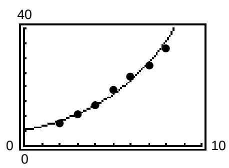
$x = y^2 + 1$

$$A = \int_0^2 (y^2 + 1)\,dy$$

$$= \left[\frac{1}{3}y^3 + y\right]_0^2 = \frac{14}{3}$$




**20.** (a) $R_1(t) = 5.2834(1.2701)^t = 5.2834\,e^{0.2391t}$

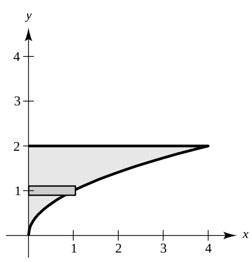



(b) $R_2(t) = 10 + 5.28\,e^{0.2t}$

$$\text{Difference} = \int_{10}^{15} \left[\mathrm{R}_1(t) - \mathrm{R}_2(t)\right] dt \approx 171.25 \text{ billion dollars}$$

**22.** (a) **Shell**

$$V = 2\pi \int_0^2 y^3\,dy = \left[\frac{\pi}{2}y^4\right]_0^2 = 8\pi$$

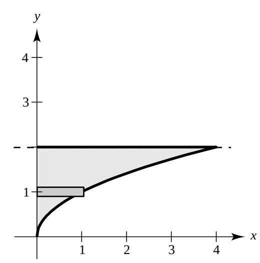



(b) **Shell**

$$V = 2\pi \int_0^2 (2 - y)y^2\,dy$$

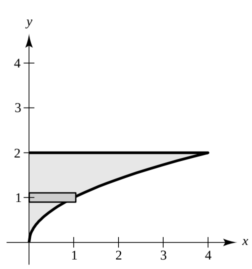
$$= 2\pi \int_0^2 (2y^2 - y^3)\,dy$$

$$= 2\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 = \frac{8\pi}{3}$$




(c) **Disk**

$$V = \pi \int_0^2 y^4\,dy = \left[\frac{\pi}{5}y^5\right]_0^2 = \frac{32\pi}{5}$$

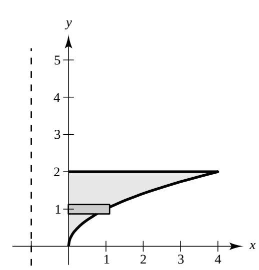



(d) **Disk**

$$V = \pi \int_0^2 \left[(y^2 + 1)^2 - 1^2\right] dy$$

$$= \pi \int_0^2 (y^4 + 2y^2)\,dy$$

$$= \pi\left[\frac{1}{5}y^5 + \frac{2}{3}y^3\right]_0^2 = \frac{176\pi}{15}$$

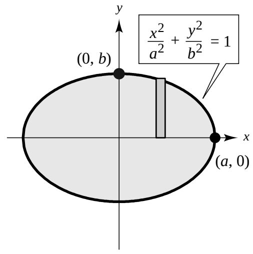


**24.** (a) **Shell**

$$V = 4\pi \int_0^a (x)\frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{-2\pi b}{a}\int_0^a (a^2 - x^2)^{1/2}(-2x)\,dx$$

$$= \left[\frac{-4\pi b}{3a}(a^2 - x^2)^{3/2}\right]_0^a = \frac{4}{3}\pi a^2 b$$

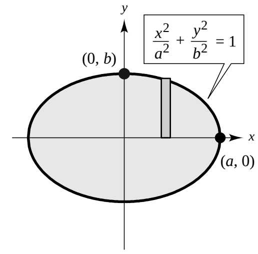



(b) **Disk**

$$V = 2\pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a$$

$$= \frac{4}{3}\pi a b^2$$



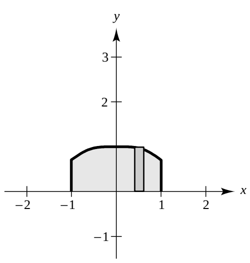

**26.** **Disk**

$$V = 2\pi \int_0^1 \left[\frac{1}{\sqrt{1 + x^2}}\right]^2 dx$$

$$= \left[2\pi \arctan x\right]_0^1$$

$$= 2\pi\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi^2}{2}$$

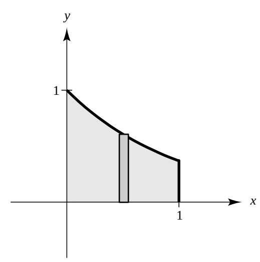



**28.** **Disk**

$$V = \pi \int_0^1 (e^{-x})^2\,dx$$

$$= \pi \int_0^1 e^{-2x}\,dx = \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$

$$= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2}\right) = \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$$

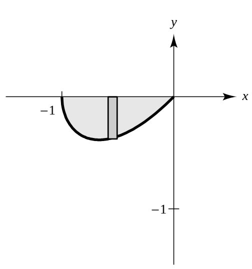



**30.** (a) **Disk**

$$V = \pi \int_{-1}^0 x^2(x + 1)\,dx$$

$$= \pi \int_{-1}^0 (x^3 + x^2)\,dx$$

$$= \pi\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^0 = \frac{\pi}{12}$$




(b) **Shell**

$$u = \sqrt{x + 1}$$

$$x = u^2 - 1$$

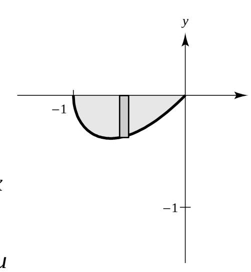
$$dx = 2u\,du$$

$$V = 2\pi \int_{-1}^0 x^2\sqrt{x + 1}\,dx$$

$$= 4\pi \int_0^1 (u^2 - 1)^2 u^2 du$$

$$= 4\pi \int_0^1 (u^6 - 2u^4 + u^2)\,du$$

$$= 4\pi\left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_0^1 = \frac{32\pi}{105}$$



**32.** $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{a^2 - x^2}\right)\left(\sqrt{3}\sqrt{a^2 - x^2}\right)$

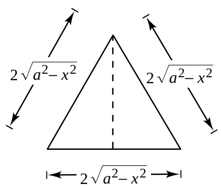
$= \sqrt{3}(a^2 - x^2)$

$$V = \sqrt{3}\int_{-a}^{a} (a^2 - x^2)\,dx = \sqrt{3}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \sqrt{3}\left(\frac{4a^3}{3}\right)$$

Since $\left(4\sqrt{3}\,a^3\right)/3 = 10$, we have $a^3 = \left(5\sqrt{3}\right)/2$. Thus,

$$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$




**34.** $y = \frac{x^3}{6} + \frac{1}{2x}$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2$$

$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)dx = \left[\frac{1}{6}x^3 - \frac{1}{2x}\right]_1^3 = \frac{14}{3}$$

**36.** Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This length is a little over 1 unit. Answers (b).

**38.** $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi\int_0^3 2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx = 4\pi\int_0^3 \sqrt{x+1}\,dx$$

$$= 4\pi\left[\left(\frac{2}{3}\right)(x+1)^{3/2}\right]_0^3 = \frac{56\pi}{3}$$

**40.** $F = kx$

$$50 = k(9) \implies k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$W = \int_0^9 \frac{50}{9}x\,dx = \left[\frac{25}{9}x^2\right]_0^9$$

$$= 225 \text{ in} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}$$

**42.** We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3\text{/min}$$

$$V = \pi r^2 h = \pi\left(\frac{1}{9}\right)h$$

$$\frac{dV}{dt} = \frac{\pi}{9}\left(\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} = \frac{9}{\pi}\left(\frac{dV}{dt}\right) = \frac{9}{\pi}\left(-\frac{8}{7.481}\right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49$ minutes

$(49)(12) = 588$ gallons pumped

Volume of water pumped in Exercise 41: 391.7 gallons

$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi$ ft $\cdot$ ton

**44.** (a) Weight of section of cable: $4\,\Delta x$

Distance: $200 - x$

$$W = 4\int_0^{200}(200 - x)\,dx = \Big[-2(200 - x)^2\Big]_0^{200} = 80{,}000 \text{ ft} \cdot \text{lb} = 40 \text{ ft} \cdot \text{ton}$$

(b) Work to move 300 pounds 200 feet vertically: $200(300) = 60{,}000 \text{ ft} \cdot \text{lb} = 30 \text{ ft} \cdot \text{ton}$

Total work $=$ work for drawing up the cable $+$ work of lifting the load

$= 40 \text{ ft} \cdot \text{ton} + 30 \text{ ft} \cdot \text{ton} = 70 \text{ ft} \cdot \text{ton}$

**46.** $W = \displaystyle\int_a^b F(x)\,dx$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \le x \le 9 \\ -(4/3)x + 16, & 9 \le x \le 12 \end{cases}$$

$$W = \int_0^9 \left(-\frac{2}{9}x + 6\right)dx + \int_9^{12}\left(-\frac{4}{3}x + 16\right)dx$$

$$= \left[-\frac{1}{9}x^2 + 6x\right]_0^9 + \left[-\frac{2}{3}x^2 + 16x\right]_9^{12}$$

$$= (-9 + 54) + (-96 + 192 + 54 - 144) = 51 \text{ ft} \cdot \text{lbs}$$

**48.** $A = \displaystyle\int_{-1}^{3}\left[(2x + 3) - x^2\right]dx = \left[x^2 + 3x - \frac{1}{3}x^3\right]_{-1}^{3} = \frac{32}{3}$

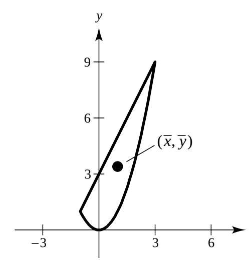
$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32}\int_{-1}^{3} x(2x + 3 - x^2)\,dx = \frac{3}{32}\int_{-1}^{3}(3x + 2x^2 - x^3)\,dx = \frac{3}{32}\left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{-1}^{3} = 1$$

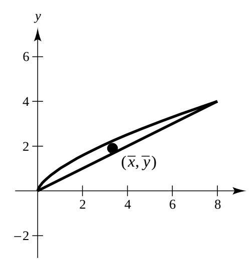
$$\bar{y} = \left(\frac{3}{32}\right)\frac{1}{2}\int_{-1}^{3}\left[(2x + 3)^2 - x^4\right]dx = \frac{3}{64}\int_{-1}^{3}(9 + 12x + 4x^2 - x^4)\,dx$$

$$= \frac{3}{64}\left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{3} = \frac{17}{5}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5}\right)$$




**50.** $A = \displaystyle\int_0^8\left(x^{2/3} - \frac{1}{2}x\right)dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2\right]_0^8 = \frac{16}{5}$




$$\frac{1}{A} = \frac{5}{16}$$

$$\bar{x} = \frac{5}{16}\int_0^8 x\left(x^{2/3} - \frac{1}{2}x\right)dx$$

$$= \;= \frac{5}{16}\left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3\right]_0^8 = \frac{10}{3}$$

$$\bar{y} = \left(\frac{5}{16}\right)\frac{1}{2}\int_0^8\left(x^{4/3} - \frac{1}{4}x^2\right)dx$$

$$= \frac{1}{2}\left(\frac{5}{16}\right)\left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3\right]_0^8 = \frac{40}{21}$$

$$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21}\right)$$

**52.** Wall at shallow end:

$$F = 62.4\int_0^5 y(20)\,dy = \left[(1248)\frac{y^2}{2}\right]_0^5 = 15{,}600 \text{ lb}$$

Wall at deep end:

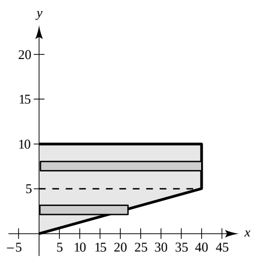
$$F = 62.4\int_0^{10} y(20)\,dy = \Big[(624)y^2\Big]_0^{10} = 62{,}400 \text{ lb}$$

Side wall:

$$F_1 = 62.4\int_0^5 y(40)\,dy = \Big[(1248)y^2\Big]_0^5 = 31{,}200 \text{ lb}$$

$$F_2 = 62.4\int_0^5 (10 - y)8y\,dy = 62.4\int_0^5 (80y - 8y^2)\,dy$$

$$F = F_1 + F_2 = 72{,}800 \text{ lb}$$




**54.** $F = 62.4(16\pi)5 = 4992\pi$ lb

## Problem Solving for Chapter 6

**2.** $R = \displaystyle\int_0^1 x(1 - x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Let $(c, mc)$ be the intersection of the line and the parabola.

Then, $mc = c(1 - c) \implies m = 1 - c$ or $c = 1 - m$.

$$\frac{1}{2}\left(\frac{1}{6}\right) = \int_0^{1-m} (x - x^2 - mx)\,dx$$

$$\frac{1}{12} = \left[\frac{x^2}{2} - \frac{x^3}{3} - m\frac{x^2}{2}\right]_0^{1-m}$$

$$= \frac{(1 - m)^2}{2} - \frac{(1 - m)^3}{3} - m\frac{(1 - m)^2}{2}$$

$$1 = 6(1 - m)^2 - 4(1 - m)^3 - 6m(1 - m)^2$$

$$= (1 - m)^2(6 - 4(1 - m) - 6m)$$

$$= (1 - m)^2(2 - 2m)$$

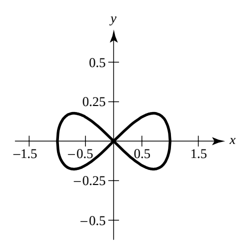
$$\frac{1}{2} = (1 - m)^3$$

$$\left(\frac{1}{2}\right)^{1/3} = 1 - m$$

$$m = 1 - \left(\frac{1}{2}\right)^{1/3} \approx 0.2063$$

**4.** $8y^2 = x^2(1 - x^2)$

$$y = \pm\frac{|x|\sqrt{1 - x^2}}{2\sqrt{2}}$$




For $x > 0$, $y' = \dfrac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}$

$$S = 2(2\pi)\int_0^1 x\sqrt{1 + \left(\frac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\right)^2}\,dx$$

$$= \frac{5\sqrt{2}\pi}{3}$$

**6.** By the Theorem of Pappus,

$$V = 2\pi r A$$
$$= 2\pi\left[d + \frac{1}{2}\sqrt{w^2 + l^2}\right] lw$$

**8.** $f'(x)^2 = e^x$

$$f'(x) = e^{x/2}$$
$$f(x) = 2e^{x/2} + C$$
$$f(0) = 0 \implies C = -2$$
$$f(x) = 2e^{x/2} - 2$$

**10.** Let $\rho_f$ be the density of the fluid and $\rho_0$ the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^{0} A(y)\,dy$$

where $A(y)$ is a typical cross section and $g$ is the acceleration due to gravity. The weight of the object is

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\,dy.$$

$$F = W$$

$$\rho_f g \int_{-h}^{0} A(y)\,dy = \rho_0 g \int_{-h}^{L-h} A(y)\,dy$$

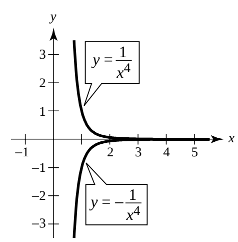
$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}} = \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

**12.** (a) $\bar{y} = 0$ by symmetry

$$M_y = 2\int_1^6 x\frac{1}{x^4}\,dx = 2\int_1^6 \frac{1}{x^3}\,dx = \frac{35}{36}$$

$$m = 2\int_1^6 \frac{1}{x^4}\,dx = \frac{215}{324}$$

$$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \qquad (\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0\right)$$




(b) $M_y = 2\int_1^b \frac{1}{x^3}\,dx = \frac{b^2 - 1}{b^2}$

$$m = 2\int_1^b \frac{1}{x^4}\,dx = \frac{2(b^3 - 1)}{3b^3}$$

$$\bar{x} = \frac{(b^2 - 1)/b^2}{2(b^3 - 1)/3b^3} = \frac{3b(b + 1)}{2(b^2 + b + 1)} \qquad (\bar{x}, \bar{y}) = \left(\frac{3b(b + 1)}{2(b^2 + b + 1)}, 0\right)$$

$$\lim_{b\to\infty} \bar{x} = \frac{3}{2} \qquad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, 0\right)$$

**14.** (a) Trapezoidal: Area $\approx \frac{160}{2(8)}[0 + 2(50) + 2(54) + 2(82) + 2(82) + 2(73) + 2(75) + 2(80) + 0] = 9920$ sq ft

(b) Simpson's: Area $\approx \frac{160}{3(8)}[0 + 4(50) + 2(54) + 4(82) + 2(82) + 4(73) + 2(75) + 4(80) + 0] = 10{,}413\frac{1}{3}$ sq ft

**16.** Point of equilibrium: $1000 - 0.4x^2 = 42x$

$$x = 20,\ p = 840$$

$(P_0, x_0) = (840, 20)$

Consumer surplus $= \int_0^{20} [(1000 - 0.4x^2) - 840]\,dx = 2133.33$

Producer surplus $= \int_0^{20} [840 - 42x]\,dx = 8400$

# CHAPTER 7
# Integration Techniques, L'Hôpital's Rule, and Improper Integrals

# CHAPTER 7
# Integration Techniques, L'Hôpital's Rule, and Improper Integrals

## Section 7.1 Basic Integration Rules

Solutions to Even-Numbered Exercises

**2.** (a) $\frac{d}{dx}\left[\ln\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$

(b) $\frac{d}{dx}\left[\frac{2x}{(x^2+1)^2}+C\right] = \frac{(x^2+1)^2(2)-(2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$

(c) $\frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}[\ln(x^2+1)+C] = \frac{2x}{x^2+1}$

$\int \frac{x}{x^2+1}\,dx$ matches (a).

**4.** (a) $\frac{d}{dx}[2x\sin(x^2+1)+C) = 2x[\cos(x^2+1)(2x)] + 2\sin(x^2+1) = 2[2x^2\cos(x^2+1)+\sin(x^2+1)]$

(b) $\frac{d}{dx}\left[-\frac{1}{2}\sin(x^2+1)+C\right] = -\frac{1}{2}\cos(x^2+1)(2x) = -x\cos(x^2+1)$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sin(x^2+1)+C\right] = \frac{1}{2}\cos(x^2+1)(2x) = x\cos(x^2+1)$

(d) $\frac{d}{dx}[-2x\sin(x^2+1)+C] = -2x[\cos(x^2+1)(2x)] - 2\sin(x^2+1) = -2[2x^2\cos(x^2+1)+\sin(x^2+1)]$

$\int x\cos(x^2+1)\,dx$ matches (c).

**6.** $\int \frac{2t-1}{t^2-t+2}\,dt$

$u = t^2 - t + 2,\ du = (2t-1)\,dt$

Use $\int \frac{du}{u}$.

**8.** $\int \frac{2}{(2t-1)^2+4}\,dt$

$u = 2t-1,\ du = 2dt,\ a = 2$

Use $\int \frac{du}{u^2+a^2}$.

**10.** $\int \frac{-2x}{\sqrt{x^2-4}}\,dx$

$u = x^2-4,\ du = 2x\,dx,\ n = -\frac{1}{2}$

Use $\int u^n\,du$.

**12.** $\int \sec 3x \tan 3x\,dx$

$u = 3x,\ du = 3\,dx$

Use $\int \sec u \tan u\,du$.

**14.** $\int \frac{1}{x\sqrt{x^2-4}}\,dx$

$u = x,\ du = dx,\ a = 2$

Use $\int \frac{du}{u\sqrt{u^2-a^2}}$.

**16.** Let $u = x - 4$, $du = dx$.

$$\int 6(x-4)^5\,dx = 6\int (x-4)^5\,dx = 6\frac{(x-4)^6}{6} + C$$
$$= (x-4)^6 + C$$

**18.** Let $u = t - 9$, $du = dt$.

$$\int \frac{2}{(t-9)^2}\,dt = 2\int (t-9)^{-2}\,dt = \frac{-2}{t-9} + C$$

**20.** Let $u = 4 - 2x^2$, $du = -4x\,dx$.

$$\int x\sqrt{4-2x^2}\,dx = -\frac{1}{4}\int (4-2x^2)^{1/2}(-4x)dx$$
$$= -\frac{1}{6}(4-2x^2)^{3/2} + C$$

**22.** $$\int\left[x - \frac{3}{(2x+3)^2}\right]dx = \int x\,dx - \frac{3}{2}\int (2x+3)^{-2}(2)\,dx$$
$$= \frac{x^2}{2} - \frac{3}{2}\frac{(2x+3)^{-1}}{-1} + C$$
$$= \frac{x^2}{2} + \frac{3}{2(2x+3)} + C$$

**24.** Let $u = x^2 + 2x - 4$, $du = 2(x+1)\,dx$.

$$\int \frac{x+1}{\sqrt{x^2+2x-4}}\,dx = \frac{1}{2}\int (x^2+2x-4)^{-1/2}(2)(x+1)\,dx$$
$$= \sqrt{x^2+2x-4} + C$$

**26.** $$\int \frac{2x}{x-4}\,dx = \int 2\,dx + \int \frac{8}{x-4}\,dx = 2x + 8\ln|x-4| + C$$

**28.** $$\int\left(\frac{1}{3x-1} - \frac{1}{3x+1}\right)dx = \frac{1}{3}\int \frac{1}{3x-1}(3)\,dx - \frac{1}{3}\int \frac{1}{3x+1}(3)\,dx$$
$$= \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C = \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

**30.** $$\int x\left(1+\frac{1}{x}\right)^3 = \int x\left(1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3}\right)dx = \int\left(x + 3 + \frac{3}{x} + \frac{1}{x^2}\right)dx = \frac{1}{2}x^2 + 3x + 3\ln|x| - \frac{1}{x} + C$$

**32.** $$\int \sec 4x\,dx = \frac{1}{4}\int \sec(4x)(4)\,dx$$
$$= \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$$

**34.** Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \frac{\sin x}{\sqrt{\cos x}}\,dx = -\int (\cos x)^{-1/2}(-\sin x)\,dx$$
$$= -2\sqrt{\cos x} + C$$

**36.** Let $u = \cot x$, $du = -\csc^2 x\,dx$.

$$\int \csc^2 x e^{\cot x}\,dx = -\int e^{\cot x}(-\csc^2 x)\,dx = -e^{\cot x} + C$$

**38.** $$\int \frac{5}{3e^x - 2}\,dx = 5\int\left(\frac{1}{3e^x-2}\right)\left(\frac{e^{-x}}{e^{-x}}\right)dx$$
$$= 5\int \frac{e^{-x}}{3-2e^{-x}}\,dx$$
$$= \frac{5}{2}\int \frac{1}{3-2e^{-x}}(2e^{-x})\,dx$$
$$= \frac{5}{2}\ln|3-2e^{-x}| + C$$

**40.** Let $u = \ln(\cos x)$, $du = \dfrac{-\sin x}{\cos x}\,dx$

$$= -\tan x\,dx$$

$$\int (\tan x)(\ln \cos x)\,dx = -\int (\ln \cos x)(-\tan x)\,dx$$

$$= \frac{-[\ln(\cos x)]^2}{2} + C$$

**42.** $$\int \frac{1 + \cos \alpha}{\sin \alpha}\,d\alpha = \int \csc \alpha\,d\alpha + \int \cot \alpha\,d\alpha$$

$$= -\ln\lvert\csc \alpha + \cot \alpha\rvert + \ln\lvert\sin \alpha\rvert + C$$

**44.** $$\int \frac{2}{3(\sec x - 1)}\,dx = \frac{2}{3}\int \frac{1}{\sec x - 1}\cdot\left(\frac{\sec x + 1}{\sec x + 1}\right)dx$$

$$= \frac{2}{3}\int \frac{\sec x + 1}{\tan^2 x}\,dx$$

$$= \frac{2}{3}\int \frac{\sec x}{\tan^2 x}\,dx + \frac{2}{3}\int \cot^2 x\,dx$$

$$= \frac{2}{3}\int \frac{\cos x}{\sin^2 x}\,dx + \frac{2}{3}\int (\csc^2 x - 1)\,dx$$

$$= \frac{2}{3}\left(-\frac{1}{\sin x}\right) - \frac{2}{3}\cot x - \frac{2}{3}x + C$$

$$= -\frac{2}{3}[\csc x + \cot x + x] + C$$

**46.** $$\int \frac{3}{t^2 + 1}\,dt = 3\arctan t + C$$

**48.** Let $u = \sqrt{3}x$, $du = \sqrt{3}\,dx$.

$$\int \frac{1}{4 + 3x^2}\,dx = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{4 + (\sqrt{3}x)^2}\,dx$$

$$= \frac{1}{2\sqrt{3}}\arctan\left(\frac{\sqrt{3}x}{2}\right) + C$$

**50.** Let $u = \dfrac{1}{t}$, $du = \dfrac{-1}{t^2}\,dt$.

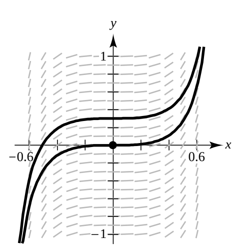
$$\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$$

**52.** $$\int \frac{1}{(x - 1)\sqrt{4x^2 - 8x + 3}}\,dx = \int \frac{2}{[2(x - 1)]\sqrt{[2(x - 1)]^2 - 1}}\,dx = \operatorname{arcsec}\lvert 2(x - 1)\rvert + C$$

**54.** $$\int \frac{1}{\sqrt{1 - 4x - x^2}}\,dx = \int \frac{1}{\sqrt{5 - (x^2 + 4x + 4)}}\,dx = \int \frac{1}{\sqrt{5 - (x + 2)^2}}\,dx = \arcsin\left(\frac{x + 2}{\sqrt{5}}\right) + C \quad (a = \sqrt{5})$$

**56.** $\dfrac{dy}{dx} = \tan^2(2x)$, $(0, 0)$

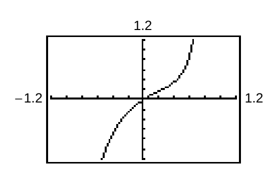
(a)



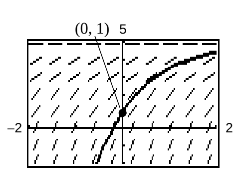

(b) $$\int \tan^2(2x)dx = \int (\sec^2(2x) - 1)\,dx = \frac{1}{2}\tan(2x) - x + C$$

$(0, 0)$: $0 = C$

$$y = \frac{1}{2}\tan(2x) - x$$




**58.**




**60.** $$r = \int \frac{(1 + e^t)^2}{e^t}\,dt = \int \frac{1 + 2e^t + e^{2t}}{e^t}\,dt$$

$$= \int (e^{-t} + 2 + e^t)\,dt = -e^{-t} + 2t + e^t + C$$

**62.** Let $u = 2x$, $du = 2\,dx$.

$$y = \int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx$$
$$= \operatorname{arcsec}|2x| + C$$

**64.** Let $u = \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi} \sin^2 t \cos t\,dt = \left[\frac{1}{3}\sin^3 t\right]_0^{\pi} = 0$$

**66.** Let $u = 1 - \ln x$, $du = \dfrac{-1}{x}\,dx$.

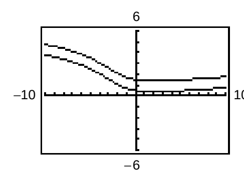
$$\int_1^e \frac{1 - \ln x}{x}\,dx = -\int_1^e (1 - \ln x)\left(\frac{-1}{x}\right)dx$$
$$= \left[-\frac{1}{2}(1 - \ln x)^2\right]_1^e = \frac{1}{2}$$

**68.** $$\int_1^2 \frac{x - 2}{x}\,dx = \int_1^2 \left(1 - \frac{2}{x}\right)dx$$
$$= \Big[x - 2\ln x\Big]_1^2 = 1 - \ln 4 \approx -0.386$$

**70.** $$\int_0^4 \frac{1}{\sqrt{25 - x^2}}\,dx = \left[\arcsin \frac{x}{5}\right]_0^4 = \arcsin \frac{4}{5} \approx 0.927$$

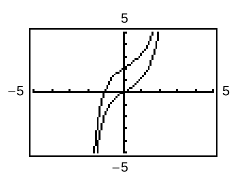
**72.** $$\int \frac{x - 2}{x^2 + 4x + 13}\,dx = \frac{1}{2}\ln(x^2 + 4x + 13) - \frac{4}{3}\arctan\left(\frac{x + 2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.




**74.** $$\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx = \frac{1}{24}\left[e^{3x} + 9e^x - 9e^{-x} - e^{-3x}\right] + C$$

The antiderivatives are vertical translations of each other.




**76.** $$\int \sec u \tan u\,du = \sec u + C$$

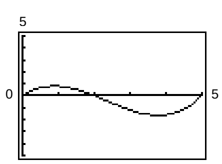
**78.** Arctan Rule: $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{4}{a}\right) + C$

**80.** They differ by a constant:

$$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C.$$

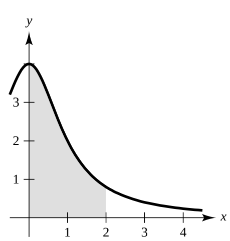
**82.** $f(x) = \dfrac{1}{5}(x^3 - 7x^2 + 10x)$

$\displaystyle\int_0^5 f(x)\,dx < 0$ because more area is below the $x$-axis than above.



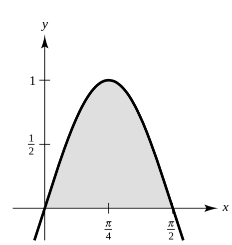

**84.** $\displaystyle\int_0^2 \frac{4}{x^2 + 1}\,dx \approx 4$

Matches (d).




**86.** $A = \displaystyle\int_0^{\pi/2} \sin 2x\,dx$

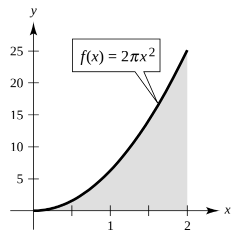
$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = 1$$



**88.** $\int_0^2 2\pi x^2\,dx$

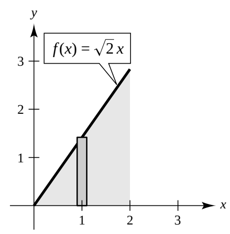
(a) Let $f(x) = 2\pi x^2$ over the interval $[0, 2]$.

$$A = \int_0^2 (2\pi x^2)\,dx$$



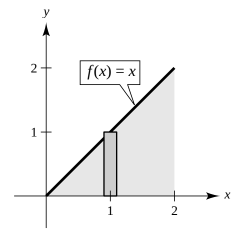

(b) Let $f(x) = \sqrt{2}\,x$ over the interval $[0, 2]$. Revolve this region about the $x$-axis.

$$V = \pi\int_0^2 \left(\sqrt{2}\,x\right)^2 dx$$
$$= \int_0^2 2\pi x^2\,dx$$




(c) Let $f(x) = x$ over the interval $[0, 2]$. Revolve this region about the $y$-axis.

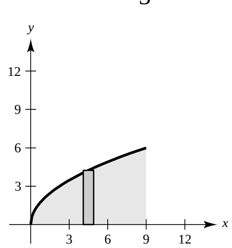
$$V = 2\pi\int_0^2 x(x)\,dx$$
$$= \int_0^2 2\pi\,x^2\,dx$$




**90.** (a) $\dfrac{1}{(\pi/n) - 0}\displaystyle\int_0^{\pi/n} \sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi/n} \sin(nx)(n)\,dx = \left[-\frac{1}{\pi}\cos(nx)\right]_0^{\pi/n} = \frac{2}{\pi}$

(b) $\dfrac{1}{3 - (-3)}\displaystyle\int_{-3}^{3} \frac{1}{1 + x^2}\,dx = \left[\frac{1}{6}\arctan x\right]_{-3}^{3} = \frac{1}{3}\arctan 3$

**92.**
$$y = 2\sqrt{x}$$
$$y' = \frac{1}{\sqrt{x}}$$
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x + 1}{x}$$
$$S = 2\pi\int_0^9 2\sqrt{x}\sqrt{\frac{x + 1}{x}}\,dx$$
$$= 2\pi\int_0^9 2\sqrt{x + 1}\,dx$$
$$= \left[4\pi\left(\frac{2}{3}\right)(x + 1)^{3/2}\right]_0^9$$
$$= \frac{8\pi}{3}\left(10\sqrt{10} - 1\right) \approx 256.545$$




**94.**
$$y = x^{2/3}$$
$$y' = \frac{2}{3x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$
$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dx \approx 7.6337$$

## Section 7.2 Integration by Parts

**2.** $\dfrac{d}{dx}\left[x^2\sin x + 2x\cos x - 2\sin x\right] = x^2\cos x + 2x\sin x - 2x\sin x + 2\cos x - 2\cos x = x^2\cos x$. Matches (d)

**4.** $\dfrac{d}{dx}[-x + x\ln x] = -1 + x\left(\dfrac{1}{x}\right) + \ln x = \ln x$. Matches (a)

**6.** $\displaystyle\int x^2 e^{2x}\,dx$

$u = x^2,\ dv = e^{2x}\,dx$

**8.** $\displaystyle\int \ln 3x\,dx$

$u = \ln 3x,\ dv = dx$

**10.** $\displaystyle\int x^2 \cos x\,dx$

$u = x^2,\ dv = \cos x\,dx$

**12.** $dv = e^{-x}\,dx \implies v = \displaystyle\int e^{-x}\,dx = -e^{-x}$

$u = x \implies du = dx$

$$2\int \frac{x}{e^x}\,dx = 2\int xe^{-x}\,dx$$
$$= 2\left[-xe^{-x} - \int -e^{-x}\,dx\right] = 2[-xe^{-x} - e^{-x}] + C$$
$$= -2xe^{-x} - 2e^{-x} + C$$

**14.** $\displaystyle\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$

**16.** $dv = x^4\,dx \implies v = \dfrac{x^5}{5}$

$u = \ln x \implies du = \dfrac{1}{x}\,dx$

$$\int x^4 \ln x\,dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\left(\frac{1}{x}\right)dx = \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4\,dx$$
$$= \frac{x^5}{5}\ln x - \frac{1}{25}x^5 + C \quad = \frac{x^5}{25}(5\ln x - 1) + C$$

**18.** Let $u = \ln x,\ du = \dfrac{1}{x}\,dx$.

$$\int \frac{1}{x(\ln x)^3}\,dx = \int (\ln x)^{-3}\left(\frac{1}{x}\right)dx = \frac{-1}{2(\ln x)^2} + C$$

**20.** $dv = \dfrac{1}{x^2}\,dx \implies v = \displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln x \implies du = \dfrac{1}{x}\,dx$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

**22.** $dv = \dfrac{x}{(x^2+1)^2}\,dx \implies v = \displaystyle\int (x^2+1)^{-2}\,x\,dx = -\frac{1}{2(x^2+1)}$

$u = x^2e^{x^2} \implies du = (2x^3e^{x^2} + 2xe^{x^2})\,dx = 2xe^{x^2}(x^2+1)\,dx$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2+1)} + C$$

**24.** $dv = \dfrac{1}{x^2}\,dx \implies v = \displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln 2x \implies du = \dfrac{1}{x}\,dx$

$$\int \frac{\ln(2x)}{x^2}\,dx = -\frac{\ln(2x)}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln(2x)}{x} - \frac{1}{x} + C = -\frac{\ln(2x) + 1}{x} + C$$

**26.** $dv = \dfrac{1}{\sqrt{2+3x}}\,dx \implies v = \displaystyle\int (2+3x)^{-1/2}\,dx = \frac{2}{3}\sqrt{2+3x}$

$u = x \implies du = dx$

$$\int \frac{x}{\sqrt{2+3x}}\,dx = \frac{2x\sqrt{2+3x}}{3} - \frac{2}{3}\int \sqrt{2+3x}\,dx$$

$$= \frac{2x\sqrt{2+3x}}{3} - \frac{4}{27}(2+3x)^{3/2} + C = \frac{2\sqrt{2+3x}}{27}[9x - 2(2+3x)] + C = \frac{2\sqrt{2+3x}}{27}(3x-4) + C$$

**28.** $dv = \sin x\,dx \implies v = -\cos x$

$u = x \implies du = dx$

$$\int x \sin dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

**30.** Use integration by parts twice.

(1) $u = x^2, du = 2x\,dx, dv = \cos x\,dx, v = \sin x$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\int x \sin x\,dx$$

(2) $u = x, du = dx, dv = \sin x\,dx, v = -\cos x$

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\left[-x\cos x + \int \cos x\,dx\right]$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

**32.** $dv = \sec\theta\tan\theta\,d\theta \implies v = \displaystyle\int \sec\theta\tan\theta\,d\theta = \sec\theta$

$u = \theta \implies du = d\theta$

$$\int \theta\sec\theta\tan\theta\,d\theta = \theta\sec\theta - \int \sec\theta\,d\theta$$

$$= \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$$

**34.** $dv = dx \implies v = \displaystyle\int dx = x$

$u = \arccos x \implies du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$

$$4\int \arccos x\,dx = 4\left[x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx\right]$$

$$= 4\left[x\arccos x - \sqrt{1-x^2}\right] + C$$

**36.** Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \displaystyle\int e^x\,dx = e^x$

$u = \cos 2x \implies du = -2\sin 2x\,dx$

(2) $dv = e^x\,dx \implies v = \displaystyle\int e^x\,dx = e^x$

$u = \sin 2x \implies du = 2\cos 2x\,dx$

$$\int e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx = e^x\cos 2x + 2\left(e^x\sin 2x - 2\int e^x\cos 2x\,dx\right)$$

$$5\int e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x$$

$$\int e^x\cos 2x\,dx = \frac{e^x}{5}(\cos 2x + 2\sin 2x) + C$$

**38.** $dv = dx \implies v = x$

$u = \ln x \implies du = \dfrac{1}{x}\,dx$

$y' = \ln x$

$$y = \int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - x + C = x(-1 + \ln x) + C$$

**40.** Use integration by parts twice.

(1) $dv = \sqrt{x-1}\,dx \implies v = \int (x-1)^{1/2}\,dx = \frac{2}{3}(x-1)^{3/2}$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = (x-1)^{3/2}dx \implies v = \int (x-1)^{3/2}\,dx = \frac{2}{5}(x-1)^{5/2}$

$u = x \implies du = dx$

$$y = \int x^2\sqrt{x-1}\,dx$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\int x(x-1)^{3/2}\,dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\left[\frac{2}{5}x(x-1)^{5/2} - \frac{2}{5}\int (x-1)^{5/2}\,dx\right]$$

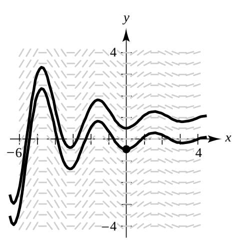
$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C = \frac{2(x-1)^{3/2}}{105}(15x^2 + 12x + 8) + C$$

**42.** $dv = dx \implies v = \int dx = x$

$u = \arctan\frac{x}{2} \implies du = \frac{1}{1+(x/2)^2}\left(\frac{1}{2}\right)dx = \frac{2}{4+x^2}\,dx$

$$y = \int \arctan\frac{x}{2}\,dx = x\arctan\frac{x}{2} - \int \frac{2x}{4+x^2}\,dx = x\arctan\frac{x}{2} - \ln(4+x^2) + C$$

**44.** (a)




(b) $\frac{dy}{dx} = e^{-x/3}\sin 2x, \left(0, -\frac{18}{37}\right)$

$y = \int e^{-x/3}\sin 2x\,dx$

Use integration by parts twice.

(1) $u = \sin 2x,\ du = 2\cos 2x$

$dv = e^{-x/3}\,dx, v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + \int 6e^{-x/3}\cos 2x\,dx$$

(2) $u = \cos 2x,\ du = -2\sin 2x$

$dv = e^{-x/3}\,dx, v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + 6\left[-3e^{-x/3}\cos 2x - \int 6e^{-x/3}\sin 2x\,dx\right] + C$$

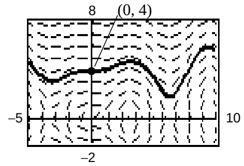
$$37\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x + C$$

$$y = \int e^{-x/3}\sin 2x\,dx = \frac{1}{37}\left[-3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x\right] + C$$

$$\left(0, \frac{-18}{37}\right): \frac{-18}{37} = \frac{1}{37}[0 - 18] + C \implies C = 0$$

$$y = \frac{-1}{37}[3e^{-x/3}\sin 2x + 18e^{-x/3}\cos 2x]$$

**46.** $\dfrac{dy}{dx} = \dfrac{x}{y}\sin x,\ y(0) = 4$




**48.** See Exercise 3.

$$\int_0^1 x^2e^x\,dx = \Big[x^2e^x - 2xe^x + 2e^x\Big]_0^1 = e - 2 \approx 0.718$$

**50.** $dv = \sin 2x\,dx \implies v = \displaystyle\int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

$u = x \implies du = dx$

$$\int x\sin 2x\,dx = \frac{-1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{-1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$
$$= \frac{1}{4}(\sin 2x - 2x\cos 2x) + C$$

Thus, $\displaystyle\int_0^{\pi} x\sin 2x\,dx = \left[\frac{1}{4}(\sin 2x - 2x\cos 2x)\right]_0^{\pi} = -\frac{\pi}{2}.$

**52.** $dv = x\,dx \implies v = \displaystyle\int x\,dx = \frac{x^2}{2}$

$u = \arcsin x^2 \implies du = \dfrac{2x}{\sqrt{1-x^4}}\,dx$

$$\int x\arcsin x^2\,dx = \frac{x^2}{2}\arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}}\,dx$$
$$= \frac{x^2}{2}\arcsin x^2 + \frac{1}{4}(2)(1-x^4)^{1/2} + C$$
$$= \frac{1}{2}\Big[x^2\arcsin x^2 + \sqrt{1-x^4}\Big] + C$$

Thus, $\displaystyle\int_0^1 x\arcsin x^2 dx = \frac{1}{2}\Big[x^2\arcsin x^2 + \sqrt{1-x^4}\Big]_0^1$

$$= \frac{1}{4}(\pi - 2).$$

**54.** Use integration by parts twice.

(1) $dv = e^{-x},\ v = -e^{-x},\ u = \cos x,\ du = -\sin x\,dx$

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int e^{-x}\sin x\,dx$$

(2) $dv = e^{-x}\,dx,\ v = -e^{-x},\ u = \sin x,\ du = \cos x\,dx$

$$\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \left[-e^{-x}\sin x + \int e^{-x}\cos x\,dx\right] \implies 2\int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x$$

Thus,

$$\int_0^2 e^{-x}\cos x\,dx = \left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_0^2$$
$$= \frac{-e^{-2}}{2}[\sin 2 - \cos 2] + \frac{1}{2}$$

**56.** $dv = dx \implies v = \displaystyle\int dx = x$

$u = \ln(1+x^2) \implies du = \dfrac{2x}{1+x^2}\,dx$

$$\int \ln(1+x^2)dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$
$$= x\ln(1+x^2) - 2\int\left[1 - \frac{1}{1+x^2}\right]dx = x\ln(1+x^2) - 2x + 2\arctan x + C$$

Thus, $\displaystyle\int_0^1 \ln(1+x^2)\,dx = \Big[x\ln(1+x^2) - 2x + 2\arctan x\Big]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$

**58.** $u = x,\ du = dx,\ dv = \sec^2 x\,dx,\ v = \tan x$

$$\int x \sec^2 x\,dx = x \tan x - \int \tan x\,dx$$

Hence,

$$\int_0^{\pi/4} x \sec^2 x\,dx = \Big[x \tan x + \ln|\cos x|\Big]_0^{\pi/4}$$
$$= \left(\frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}\right) - 0$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

**60.** $$\int x^3 e^{-2x}\,dx = x^3\left(-\frac{1}{2}e^{-2x}\right) - 3x^2\left(\frac{1}{4}e^{-2x}\right) + 6x\left(-\frac{1}{8}e^{-2x}\right) - 6\left(\frac{1}{16}e^{-2x}\right) + C$$
$$= -\frac{1}{8}e^{-2x}(4x^3 + 6x^2 + 6x + 3) + C$$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^3$ | $e^{-2x}$ |
| − | $3x^2$ | $-\frac{1}{2}e^{-2x}$ |
| + | $6x$ | $\frac{1}{4}e^{-2x}$ |
| − | $6$ | $-\frac{1}{8}e^{-2x}$ |
| + | $0$ | $\frac{1}{16}e^{-2x}$ |

**62.** $$\int x^3 \cos 2x\,dx = x^3\left(\frac{1}{2}\sin 2x\right) - 3x^2\left(-\frac{1}{4}\cos 2x\right) + 6x\left(-\frac{1}{8}\sin 2x\right) - 6\left(\frac{1}{16}\cos 2x\right) + C$$
$$= \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x - \frac{3}{4}x \sin 2x - \frac{3}{8}\cos 2x + C$$
$$= \frac{1}{8}[4x^3 \sin 2x + 6x^2 \cos 2x - 6x \sin 2x - 3\cos 2x] + C$$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^3$ | $\cos 2x$ |
| − | $3x^2$ | $\frac{1}{2}\sin 2x$ |
| + | $6x$ | $-\frac{1}{4}\cos 2x$ |
| − | $6$ | $-\frac{1}{8}\sin 2x$ |
| + | $0$ | $\frac{1}{16}\cos 2x$ |

**64.** $$\int x^2(x-2)^{3/2}dx = \frac{2}{5}x^2(x-2)^{5/2} - \frac{8}{35}x(x-2)^{7/2} + \frac{16}{315}(x-2)^{9/2} + C$$
$$= \frac{2}{315}(x-2)^{5/2}(35x^2 + 40x + 32) + C$$

| Alternate signs | $u$ and its derivatives | $v'$ and its antiderivatives |
|---|---|---|
| + | $x^2$ | $(x-2)^{3/2}$ |
| − | $2x$ | $\frac{2}{5}(x-2)^{5/2}$ |
| + | $2$ | $\frac{4}{35}(x-2)^{7/2}$ |
| − | $0$ | $\frac{8}{315}(x-2)^{9/2}$ |

**66.** Answers will vary. See pages 488, 493.

**68.** Yes. $u = \ln x,\ dv = x\,dx$

**70.** No. Substitution.

**72.** No. Substitution.

**74.** $$\int \alpha^4 \sin \pi\alpha\,d\alpha = \frac{1}{\pi^5}[-(\alpha\pi)^4 \cos \pi\alpha + 4(\alpha\pi)^3 \sin \pi\alpha + 12(\alpha\pi)^2 \cos \pi\alpha - 24(\alpha\pi)\sin \pi\alpha - 24\cos \pi\alpha] + C$$

**76.** $$\int_0^5 x^4(25 - x^2)^{3/2}\,dx = \left[\frac{1{,}171{,}875 \arcsin(x/5)}{128} - \frac{x(2x^2+25)(25-x^2)^{5/2}}{16} + \frac{625x(25-x^2)^{3/2}}{64} + \frac{46{,}875x\sqrt{25-x^2}}{128}\right]_0^5$$
$$\approx 14{,}381.0699$$

**78.** (a) $dv = \sqrt{4+x}\,dx \implies v = \int (4+x)^{1/2}\,dx = \frac{2}{3}(4+x)^{3/2}$

$u = x \implies du = dx$

$$\int x\sqrt{4+x}\,dx = \frac{2}{3}x(4+x)^{3/2} - \frac{2}{3}\int (4+x)^{3/2}\,dx$$

$$= \frac{2}{3}x(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C$$

(b) $u = 4 + x \implies x = u - 4$ and $dx = du$

$$\int x\sqrt{4+x}\,dx = \int (u-4)u^{1/2}du = \int (u^{3/2} - 4u^{1/2})\,du$$

$$= \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{15}u^{3/2}(3u-20) + C$$

$$= \frac{2}{15}(4+x)^{3/2}[3(4+x) - 20] + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C$$

**80.** (a) $dv = \sqrt{4-x}\,dx \implies v = \int (4-x)^{1/2}\,dx$

$$= -\frac{2}{3}(4-x)^{3/2}$$

$u = x \implies du = dx$

$$\int x\sqrt{4-x}\,dx = -\frac{2}{3}x(4-x)^{3/2} + \frac{2}{3}\int (4-x)^{3/2}\,dx$$

$$= -\frac{2}{3}x(4-x)^{3/2} - \frac{4}{15}(4-x)^{5/2} + C$$

$$= -\frac{2}{15}(4-x)^{3/2}[5x + 2(4-x)] + C$$

$$= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C$$

(b) $u = 4 - x \implies x = 4 - u$ and $dx = -du$

$$\int x\sqrt{4-x}\,dx = -\int (4-u)\sqrt{u}\,du$$

$$= -\int (4u^{1/2} - u^{3/2})\,du$$

$$= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$$

$$= -\frac{2}{15}u^{3/2}(20 - 3u) + C$$

$$= -\frac{2}{15}(4-x)^{3/2}[20 - 3(4-x)] + C$$

$$= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C$$

**82.** $n = 0$: $\int e^x\,dx = e^x + C$

$n = 1$: $\int xe^x\,dx = xe^x - e^x + C = xe^x - \int e^x\,dx$

$n = 2$: $\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$

$$= x^2e^x - 2\int xe^x\,dx$$

$n = 3$: $\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

$$= x^3e^x - 3\int x^2e^x\,dx$$

$n = 4$: $\int x^4e^x\,dx$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$$

$$= x^4e^x - 4\int x^3e^x\,dx$$

In general, $\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$.

(See Exercise 86)

**84.** $dv = \cos x\,dx \implies v = \sin x$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$$

**86.** $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

**88.** Use integration by parts twice.

(1) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = \cos bx \implies du = -b\sin bx$

(2) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = \sin bx \implies du = b\cos bx$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left[\frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx\right]$$

$$= \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2}$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C.$$

**90.** $n = 2$ (Use formula in Exercise 84.)

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx \text{ (Use formula in Exercise 83.) } (n = 1)$$

$$= x^2\sin x - 2\left[-x\cos x + \int\cos x\,dx\right] = x^2\sin x + 2x\cos x - 2\sin x + C$$

**92.** $n = 3, a = 2$ (Use formula in Exercise 86 three times.)

$$\int x^3e^{2x}\,dx = \frac{x^3e^{2x}}{2} - \frac{3}{2}\int x^2e^{2x}\,dx \quad (n = 3, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2e^{2x}}{2} - \int xe^{2x}\,dx\right] \quad (n = 2, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3}{2}\left[\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx\right] = \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3e^{2x}}{8} + C \quad (n = 1, a = 2)$$

$$= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3) + C$$

**94.** $dv = e^{-x/3}\,dx \implies v = -3e^{-x/3}$

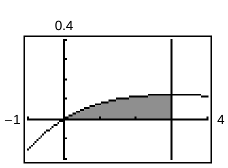
$u = x \implies du = dx$

$$A = \frac{1}{9}\int_0^3 xe^{-x/3}\,dx$$

$$= \frac{1}{9}\left(\left[-3xe^{-x/3}\right]_0^3 + 3\int_0^3 e^{-x/3}\,dx\right)$$

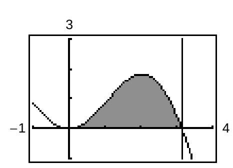
$$= \frac{1}{9}\left(\frac{-9}{e} - \left[9e^{-x/3}\right]_0^3\right)$$

$$= -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e} \approx 0.264$$

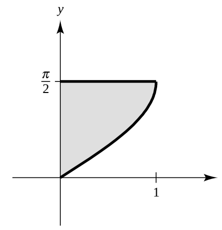



**96.** $A = \int_0^{\pi} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{\pi}$

$= \pi$ (See Exercise 83.)



**98.** In Example 6, we showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$.




You can also solve this problem directly.

$$A = \int_0^1 \left(\frac{\pi}{2} - \arcsin x\right) dx = \left[\frac{\pi}{2}x - x\arcsin x - \sqrt{1 - x^2}\right]_0^1 \text{ (Example 3)}$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{2} - 0\right) - (-1) = 1$$

$$\bar{x} = \frac{M_y}{A} = \int_0^1 x\left[\frac{\pi}{2} - \arcsin x\right] dx = \frac{\pi}{8}$$

$$\bar{y} = \frac{M_x}{A} = \int_0^1 \frac{(\pi/2) + \arcsin x}{2}\left[\frac{\pi}{2} - \arcsin x\right] dx = 1$$

**100.** (a) Average $= \displaystyle\int_1^2 (1.6t \ln t + 1)\,dt = \left[0.8t^2 \ln t - 0.4t^2 + t\right]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$

(b) Average $= \displaystyle\int_3^4 (1.6t \ln t + 1)\,dt = \left[0.8t^2 \ln t - 0.4t^2 + t\right]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$

**102.** $c(t) = 30{,}000 + 500t,\ r = 7\%,\ t_1 = 5$

$$P\int_0^5 (30{,}000 + 500t)e^{-0.07t}\,dt = 500\int_0^5 (60 + t)e^{-0.07t}\,dt$$

Let $u = 60 + t,\ dv = e^{-0.07t}\,dt,\ du = dt,\ v = -\dfrac{100}{7}e^{-0.07t}$.

$$P = 500\left\{\left[(60 + t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 + \frac{100}{7}\int_0^5 e^{-0.07t}\,dt\right\}$$

$$= 500\left\{\left[(60 + t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 - \left[\frac{10{,}000}{49}e^{-0.07t}\right]_0^5\right\} \approx \$131{,}528.68$$

**104.** $$\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{2\pi}{n^2}\cos n\pi + \frac{2\pi}{n^2}\cos(-n\pi)$$

$$= \frac{4\pi}{n^2}\cos n\pi$$

$$= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases}$$

$$= \frac{(-1)^n 4\pi}{n^2}$$

**106.** For any integrable function, $\int f(x)\,dx = C + \int f(x)\,dx$, but this cannot be used to imply that $C = 0$.

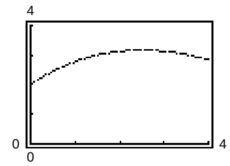
**108.** On $\left[0, \dfrac{\pi}{2}\right]$, $\sin x \le 1 \implies x\sin x \le x \implies \displaystyle\int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\,dx$.

**110.** $f'(x) = \cos\sqrt{x}, f(0) = 2$

(a) It cannot be solved by integration.

(b) You obtain the points

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0 | 2 |
| 1 | 0.05 | 2.05 |
| 2 | 0.10 | 2.098755 |
| 3 | 0.15 | 2.146276 |
| ⋮ | ⋮ | ⋮ |
| 80 | 4.0 | 2.8403565 |




## Section 7.3 Trigonometric Integrals

**2.** (a) $y = \sec x \implies y' = \sec x \tan x = \sin x \sec^2 x.$

Matches (iii)

(b) $y = \cos x + \sec x \implies y' = -\sin x + \sec x \tan x$

$$= -\sin x + \sec^2 x \sin x$$
$$= \sin x(-1 + \sec^2 x)$$
$$= \sin x \tan^2 x \quad \text{Matches (i)}$$

(c) $y = x - \tan x + \frac{1}{3}\tan^3 x \implies y' = 1 - \sec^2 x + \tan^2 x \sec^2 x$

$$= -\tan^2 x + \tan^2 x(1 + \tan^2 x)$$
$$= \tan^4 x \quad \text{Matches (iv)}$$

(d) $y = 3x + 2\sin x\cos^3 x + 3\sin x\cos x \implies$

$$y' = 3 + 2\cos x(\cos^3 x) + 6\sin x\cos^2 x(-\sin x) + 3\cos^2 x - 3\sin^2 x$$
$$= 3 + 2\cos^4 x - 6\cos^2 x(1 - \cos^2 x) + 3\cos^2 x - 3(1 - \cos^2 x)$$
$$= 8\cos^4 x \quad \text{Matches (ii)}$$

**4.** $$\int \cos^3 x \sin^4 x\,dx = \int \cos x(1 - \sin^2 x)\sin^4 x\,dx$$
$$= \int (\sin^4 x - \sin^6 x)\cos x\,dx$$
$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

**6.** Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \sin^3 x\,dx = \int \sin x(1 - \cos^2 x)\,dx$$
$$= \int \cos^2 x(-\sin x)\,dx + \int \sin x\,dx$$
$$= \frac{1}{3}\cos^3 x - \cos x + C$$

**8.** Let $u = \sin\frac{x}{3}$, $du = \frac{1}{3}\cos\frac{x}{3}\,dx$.

$$\int \cos^3 \frac{x}{3}\,dx = \int \left(\cos\frac{x}{3}\right)\left(1 - \sin^2\frac{x}{3}\right)dx$$
$$= 3\int \left(1 - \sin^2\frac{x}{3}\right)\left(\frac{1}{3}\cos\frac{x}{3}\right)dx$$
$$= 3\left(\sin\frac{x}{3} - \frac{1}{3}\sin^3\frac{x}{3}\right) + C$$
$$= 3\sin\frac{x}{3} - \sin^3\frac{x}{3} + C$$

**10.** $\displaystyle\int \frac{\sin^5 t}{\sqrt{\cos t}}\,dt = \int \sin t(1 - \cos^2 t)^2(\cos t)^{-1/2}\,dt$

$\displaystyle = \int \sin t(1 - 2\cos^2 t + \cos^4 t)(\cos t)^{-1/2}\,dt$

$\displaystyle = \int [(\cos t)^{-1/2} - 2(\cos t)^{3/2} + (\cos t)^{7/2}]\sin t\,dt = -2(\cos t)^{1/2} + \frac{4}{5}(\cos t)^{5/2} - \frac{2}{9}(\cos t)^{9/2} + C$

**12.** $\displaystyle\int \sin^2 2x\,dx = \int \frac{1 - \cos 4x}{2}\,dx = \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + C$

$\displaystyle = \frac{1}{8}(4x - \sin 4x) + C$

**14.** $\displaystyle\int \sin^4 2\theta\,d\theta = \int \frac{1 - \cos 4\theta}{2} \cdot \frac{1 - \cos 4\theta}{2}\,d\theta$

$\displaystyle = \frac{1}{4}\int (1 - 2\cos 4\theta + \cos^2 4\theta)\,d\theta$

$\displaystyle = \frac{1}{4}\int \left(1 - 2\cos 4\theta + \frac{1 + \cos 8\theta}{2}\right)d\theta$

$\displaystyle = \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 4\theta + \frac{1}{2}\cos 8\theta\right)d\theta$

$\displaystyle = \frac{1}{4}\left[\frac{3}{2}\theta - \frac{1}{2}\sin 4\theta + \frac{1}{16}\sin 8\theta\right] + C$

$\displaystyle = \frac{3}{8}\theta - \frac{1}{8}\sin 4\theta + \frac{1}{64}\sin 8\theta + C$

**16.** Use integration by parts twice.

$\displaystyle dv = \sin^2 x\,dx = \frac{1 - \cos 2x}{2} \implies v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$

$u = x^2 \implies du = 2x\,dx$

$\displaystyle dv = \sin 2x\,dx \implies v = -\frac{1}{2}\cos 2x$

$u = x \implies du = dx$

$\displaystyle\int x^2\sin^2 x\,dx = \frac{1}{4}x^2(2x - \sin 2x) - \frac{1}{2}\int (2x^2 - x\sin 2x)\,dx$

$\displaystyle = \frac{1}{2}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{3}x^3 + \frac{1}{2}\int x\sin 2x\,dx$

$\displaystyle = \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{2}\left[-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx\right]$

$\displaystyle = \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$

$\displaystyle = \frac{1}{24}(4x^3 - 6x^2\sin 2x - 6x\cos 2x + 3\sin 2x) + C$

**18.** Let $u = \sin x$, $du = \cos x\,dx$.

$\displaystyle\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (1 - \sin^2 x)^2\cos x\,dx$

$\displaystyle = \int_0^{\pi/2} (1 - 2\sin^2 x + \sin^4 x)\cos x\,dx$

$\displaystyle = \left[\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right]_0^{\pi/2}$

$\displaystyle = \frac{8}{15}$

**20.** $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2x)\,dx$

$\displaystyle = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{4}$

**22.** $\int \sec^2(2x - 1)\,dx = \frac{1}{2}\tan(2x - 1) + C$

**24.** $\int \sec^6 3x\,dx = \int (1 + \tan^2 3x)^2 \sec^2 3x\,dx$

$= \int (1 + 2\tan^2 3x + \tan^4 3x)\sec^2 3x\,dx$

$= \frac{1}{3}\tan 3x + \frac{2}{9}\tan^3 3x + \frac{1}{15}\tan^5 3x + C$

**26.** $\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \tan x - x + C$

**28.** $\int \tan^3 \frac{\pi x}{2}\sec^2 \frac{\pi x}{2}\,dx = \frac{1}{2\pi}\tan^4 \frac{\pi x}{2} + C$

**30.** Let $u = \sec 2t$, $du = 2\sec 2t\tan 2t$

$\int \tan^3 2t \cdot \sec^3 2t\,dt = \int (\sec^2 2t - 1)\sec^3 2t \cdot \tan 2t\,dt$

$= \int (\sec^4 2t - \sec^2 2t)(\sec 2t\tan 2t)\,dt$

$= \frac{\sec^5 2t}{10} - \frac{\sec^3 2t}{6} + C$

**32.** $\int \tan^5 2x\sec^2 2x\,dx = \frac{1}{12}\tan^6 2x + C$

**34.** $\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\sec \frac{x}{2}\tan \frac{x}{2}\right)dx$

$= \sec^2 \frac{x}{2} + C$

or $\int \sec^2 \frac{x}{2}\tan \frac{x}{2}\,dx = 2\int \tan \frac{x}{2}\left(\frac{1}{2}\sec^2 \frac{x}{2}\right)dx$

$= \tan^2 \frac{x}{2} + C$

**36.** $\int \tan^3 3x\,dx = \int (\sec^2 3x - 1)\tan 3x\,dx$

$= \frac{1}{3}\int \tan 3x(3\sec^2 3x)\,dx + \frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$

$= \frac{1}{6}\tan^2 3x + \frac{1}{3}\ln\left|\cos 3x\right| + C$

**38.** $\int \frac{\tan^2 x}{\sec^5 x} = \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x\,dx$

$= \int \sin^2 x \cdot \cos^3 x\,dx$

$= \int \sin^2 x(1 - \sin^2 x)\cos x\,dx$

$= \int (\sin^2 x - \sin^4 x)\cos x\,dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

**40.** $s = \int \sin^2 \frac{\alpha}{2}\cos^2 \frac{\alpha}{2}\,d\alpha$

$= \int \left(\frac{1 - \cos\alpha}{2}\right)\left(\frac{1 + \cos\alpha}{2}\right)d\alpha = \int \frac{1 - \cos^2\alpha}{4}\,d\alpha$

$= \frac{1}{4}\int \sin^2\alpha\,d\alpha = \frac{1}{8}\int (1 - \cos 2\alpha)\,d\alpha$

$= \frac{1}{8}\left[\theta - \frac{\sin 2\alpha}{2}\right] + C$

$= \frac{1}{16}(2\alpha - \sin 2\alpha) + C$

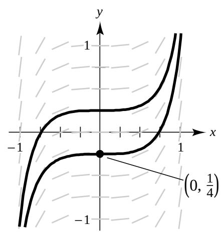
**42.** $y = \int \sqrt{\tan x}\sec^4 x\,dx$

$= \int \tan^{1/2} x(\tan^2 x + 1)\sec^2 x\,dx$

$= \int (\tan^{5/2} x + \tan^{1/2} x)\sec^2 x\,dx$

$= \frac{2}{7}\tan^{7/2} x + \frac{2}{3}\tan^{3/2} x + C$

**44.** (a)



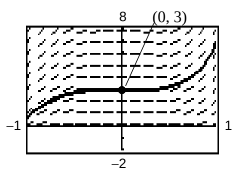

(b) $\dfrac{dy}{dx} = \sec^2 x \tan^2 x, \left(0, -\dfrac{1}{4}\right)$

$$y = \int \sec^2 x \tan^2 x\, dx \quad u = \tan x, du = \sec^2 x\, dx$$

$$y = \frac{\tan^3 x}{3} + C$$

$$\left(0, -\frac{1}{4}\right): \; -\frac{1}{4} = C \implies y = \frac{1}{3}\tan^3 x - \frac{1}{4}$$

**46.** $\dfrac{dy}{dx} = 3\sqrt{y}\tan^2 x, y(0) = 3$




**48.** $\displaystyle\int \cos 4\theta \cos(-3\theta)\, d\theta = \int \cos 4\theta \cos 3\theta\, d\theta$

$$= \frac{1}{2}\int(\cos 7\theta + \cos \theta)\, d\theta$$

$$= \frac{\sin 7\theta}{14} + \frac{\sin \theta}{2} + C$$

**50.** $\displaystyle\int \sin(-4x)\cos 3x\, dx = -\int \sin 4x \cos 3x\, dx$

$$= -\frac{1}{2}\int(\sin x + \sin 7x)dx$$

$$= -\frac{1}{2}\left[-\cos x - \frac{1}{7}\cos 7x\right] + C$$

$$= \frac{1}{14}[7\cos x + \cos 7x] + C$$

**52.** Let $u = \tan\dfrac{x}{2}, du = \dfrac{1}{2}\sec^2\dfrac{x}{2}\, dx.$

$$\int \tan^4\frac{x}{2}\sec^4\frac{x}{2}\, dx = \int \tan^4\frac{x}{2}\left(\tan^2\frac{x}{2} + 1\right)\sec^2\frac{x}{2}\, dx$$

$$= 2\int\left(\tan^6\frac{x}{2} + \tan^4\frac{x}{2}\right)\left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx$$

$$= \frac{2}{7}\tan^7\frac{x}{2} + \frac{2}{5}\tan^5\frac{x}{2} + C$$

**54.** $u = \cot 3x, du = -3\csc^2 3x\, dx$

$$\int \csc^2 3x \cot 3x\, dx = -\frac{1}{3}\int \cot 3x(-3\csc^2 3x)\, dx$$

$$= -\frac{1}{6}\cot^2 3x + C$$

**56.** $\displaystyle\int \frac{\cot^3 t}{\csc t}\, dt = \int \frac{\cos^3 t}{\sin^2 t}\, dt = \int \frac{(1 - \sin^2 t)\cos t}{\sin^2 t}\, dt$

$$= \int \frac{\cos t}{\sin^2 t}\, dt - \int \cos t\, dt$$

$$= \frac{-1}{\sin t} - \sin t + C$$

$$= -\csc t - \sin t + C$$

**58.** $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\cos x}\, dx = \int \frac{1 - 2\cos^2 x}{\cos x}\, dx = \int(\sec x - 2\cos x)\, dx = \ln|\sec x + \tan x| - 2\sin x + C$

**60.** $\displaystyle\int \frac{1 - \sec t}{\cos t - 1}\, dt = \int \frac{\cos t - 1}{(\cos t - 1)\cos t}\, dt$

$$= \int \sec t\, dt = \ln|\sec t + \tan t| + C$$

**62.** $\displaystyle\int_0^{\pi/3} \tan^2 x\, dx = \int_0^{\pi/3}(\sec^2 x - 1)\, dx$

$$= \Big[\tan x - x\Big]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}$$

**64.** Let $u = \tan t, du = \sec^2 t\, dt.$

$$\int_0^{\pi/4}\sec^2 t\sqrt{\tan t}\, dt = \left[\frac{2}{3}\tan^{3/2} t\right]_0^{\pi/4} = \frac{2}{3}$$

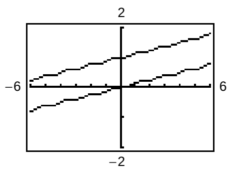
**66.** $\displaystyle\int_{-\pi}^{\pi}\sin 3\theta \cos\theta\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi}(\sin 4\theta + \sin 2\theta)\, d\theta$

$$= -\frac{1}{2}\left[\frac{1}{4}\cos 4\theta + \frac{1}{2}\cos 2\theta\right]_{-\pi}^{\pi} = 0$$

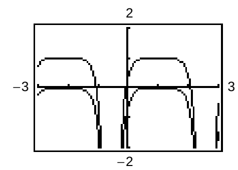
**68.** $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1)\,dx = \int_{-\pi/2}^{\pi/2} \left(\frac{1 - \cos 2x}{2} + 1\right) dx$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{1}{2}\cos 2x\right) dx = \left[\frac{3}{2}x - \frac{1}{4}\sin 2x\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

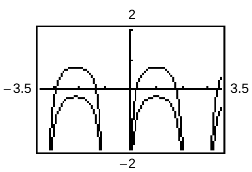
**70.** $\displaystyle\int \sin^2 x \cos^2 x\,dx = \frac{1}{32}[4x - \sin 4x] + C$




**72.** $\displaystyle\int \tan^3(1 - x)\,dx = -\frac{\tan^2(1 - x)}{2} - \ln|\cos(1 - x)| + C$




**74.** $\displaystyle\int \sec^4(1 - x)\tan(1 - x)\,dx = -\frac{\sec^4(1 - x)}{4} + C$




**76.** $\displaystyle\int_0^{\pi/2} (1 - \cos\theta)^2\,d\theta = \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$$= \frac{3\pi}{4} - 2$$

**78.** $\displaystyle\int_0^{\pi/2} \sin^6 x\,dx = \frac{1}{8}\left[\frac{5x}{2} - 2\sin 2x + \frac{3}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right]_0^{\pi/2} = \frac{5\pi}{32}$

**80.** See guidelines on page 500.

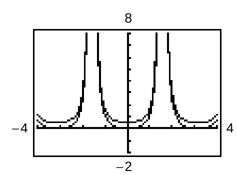
**82.** (a) Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$\int \sec^2 x \tan x\,dx = \frac{1}{2}\tan^2 x + C_1$$

Or let $u = \sec x$, $du = \sec x \tan x\,dx$.

$$\int \sec x(\sec x \tan x)\,dx = \frac{1}{2}\sec^2 x + C$$

(b)




(c) $\displaystyle\frac{1}{2}\sec^2 x + C = \frac{1}{2}(\tan^2 x + 1) + C = \frac{1}{2}\tan^2 x + \left(\frac{1}{2} + C\right) = \frac{1}{2}\tan^2 x + C_2$

**84. Disks**

$R(x) = \tan x$

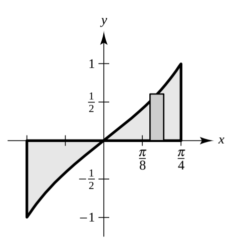
$r(x) = 0$

$$V = 2\pi\int_0^{\pi/4} \tan^2 x\,dx$$

$$= 2\pi\int_0^{\pi/4} (\sec^2 x - 1)\,dx$$

$$= 2\pi\Big[\tan x - x\Big]_0^{\pi/4}$$

$$= 2\pi\left(1 - \frac{\pi}{4}\right) \approx 1.348$$



**86.** (a) $V = \pi\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi/2}(1 + \cos 2x)\,dx = \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi^2}{4}$

(b) $A = \int_0^{\pi/2} \cos x\,dx = \left[\sin x\right]_0^{\pi/2} = 1$

Let $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$.

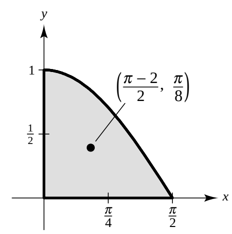
$$\bar{x} = \int_0^{\pi/2} x\cos x\,dx = \left[x\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx = \left[x\sin x + \cos x\right]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$\bar{y} = \frac{1}{2}\int_0^{\pi/2}\cos^2 x\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2}(1 + \cos 2x)\,dx$$

$$= \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8}\right)$$




**88.** $dv = \cos x\,dx \implies v = \sin x$

$u = \cos^{n-1} x \implies du = -(n-1)\cos^{n-2} x\sin x\,dx$

$$\int\cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int\cos^{n-2} x\sin^2 x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int\cos^{n-2} x(1 - \cos^2 x)\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int\cos^{n-2} x\,dx - (n-1)\int\cos^n x\,dx$$

Therefore, $n\int\cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int\cos^{n-2} x\,dx$

$$\int\cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int\cos^{n-2} x\,dx.$$

**90.** Let $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x\tan x\,dx$, $dv = \sec^2 x\,dx$, $v = \tan x$.

$$\int\sec^n x\,dx = \sec^{n-2} x\tan x - \int(n-2)\sec^{n-2} x\tan^2 x\,dx$$

$$= \sec^{n-2} x\tan x - (n-2)\int\sec^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x\tan x - (n-2)\left[\int\sec^n x\,dx - \int\sec^{n-2} x\,dx\right]$$

$$(n-1)\int\sec^n x\,dx = \sec^{n-2} x\tan x + (n-2)\int\sec^{n-2} x\,dx$$

$$\int\sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int\sec^{n-2} x\,dx$$

**92.** $\int\cos^4 x\,dx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\int\cos^2 x\,dx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\left[\frac{\cos x\sin x}{2} + \frac{1}{2}\int dx\right]$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C = \frac{1}{8}\left[2\cos^3 x\sin x + 3\cos x\sin x + 3x\right] + C$$

**94.** $$\int \sin^4 x \cos^2 x\,dx = -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\int \cos^2 x \sin^2 x\,dx$$
$$= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\left[-\frac{\cos^3 x \sin x}{4} + \frac{1}{4}\int \cos^2 x\,dx\right]$$
$$= -\frac{1}{6}\cos^3 x \sin^3 x - \frac{1}{8}\cos^3 x \sin x + \frac{1}{8}\left[\frac{\cos x \sin x}{2} + \frac{x}{2}\right] + C$$
$$= -\frac{1}{48}[8\cos^3 x \sin^3 x + 6\cos^3 x \sin x - 3\cos x \sin x - 3x] + C$$

**96.** (a) $n$ is odd and $n \geq 3$.

$$\int_0^{\pi/2} \cos^n x\,dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$
$$= \frac{n-1}{n}\left[\left[\frac{\cos^{n-3} x \sin x}{n-2}\right]_0^{\pi/2} + \frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\,dx\right]$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\left[\left[\frac{\cos^{n-5} x \sin x}{n-4}\right]_0^{\pi/2} + \frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx\right]$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos x\,dx$$
$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots(\sin x)\right]_0^{\pi/2}$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots 1 \qquad \text{(Reverse the order)}$$
$$= (1)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$
$$= \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

(b) $n$ is even and $n \geq 2$.

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos^2 x\,dx \qquad \text{(From part (a).)}$$
$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\left(\frac{x}{2} + \frac{1}{4}\sin 2x\right)\right]_0^{\pi/2}$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{\pi}{4} \qquad \text{(Reverse the order)}$$
$$= \left(\frac{\pi}{2}\cdot\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)$$
$$= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

## Section 7.4 Trigonometric Substitution

**2.** $$\frac{d}{dx}\left[8\ln\left|\sqrt{x^2-16}+x\right|+\frac{1}{2}x\sqrt{x^2-16}+C\right]=8\left[\frac{\left(x/\sqrt{x^2-16}\right)+1}{\sqrt{x^2-16}+x}\right]+\frac{1}{2}x\left(\frac{x}{\sqrt{x^2-16}}\right)+\frac{1}{2}\sqrt{x^2-16}$$
$$=\frac{8\left(x+\sqrt{x^2-16}\right)}{\sqrt{x^2-16}\left(\sqrt{x^2-16}+x\right)}+\frac{x^2}{2\sqrt{x^2+16}}+\frac{\sqrt{x^2-16}}{2}$$
$$=\frac{16+x^2+x^2-16}{2\sqrt{x^2-16}}$$
$$=\frac{x^2}{\sqrt{x^2-16}}$$

Indefinite integral: $\displaystyle\int\frac{x^2}{\sqrt{x^2-16}}$ Matches (d)

**4.** $$\frac{d}{dx}\left[8\arcsin\frac{x-3}{4}+\frac{(x-3)\sqrt{7+6x-x^2}}{2}+C\right]=8\left[\frac{1}{\sqrt{1-[(x-3)/4]^2}}\cdot\frac{1}{4}\right]+\frac{1}{2}(x-3)\frac{3-x}{\sqrt{7+6x-x^2}}+\frac{1}{2}\sqrt{7+6x-x^2}$$
$$=\frac{8}{\sqrt{16-(x-3)^2}}-\frac{(x-3)^2}{2\sqrt{16-(x-3)^2}}+\frac{\sqrt{16-(x-3)^2}}{2}$$
$$=\frac{16-(x^2-6x+9)+16-(x^2-6x+9)}{2\sqrt{16-(x-3)^2}}$$
$$=\frac{2[16-(x-3)^2]}{2\sqrt{16-(x-3)^2}}$$
$$=\sqrt{16-(x-3)^2}$$
$$=\sqrt{7+6x-x^2}$$

Indefinite integral: $\displaystyle\int\sqrt{7+6x-x^2}\,dx$ Matches (c)

**6.** Same substitution as in Exercise 5.

$$\int\frac{10}{x^2\sqrt{25-x^2}}\,dx=10\int\frac{5\cos\theta\,d\theta}{(25\sin^2\theta)(5\cos\theta)}=\frac{2}{5}\int\csc^2\theta\,d\theta=-\frac{2}{5}\cot\theta+C=\frac{-2\sqrt{25-x^2}}{5x}+C$$

**8.** Same substitution as in Exercise 5

$$\int\frac{x^2}{\sqrt{25-x^2}}\,dx=\int\frac{25\sin^2\theta}{5\cos\theta}(5\cos\theta)\,d\theta=\frac{25}{2}\int(1-\cos 2\theta)\,d\theta$$
$$=\frac{25}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)+C=\frac{25}{2}(\theta-\sin\theta\cos\theta)+C$$
$$=\frac{25}{2}\left[\arcsin\left(\frac{x}{5}\right)-\left(\frac{x}{5}\right)\left(\frac{\sqrt{25-x^2}}{5}\right)\right]+C=\frac{1}{2}\left[25\arcsin\left(\frac{x}{5}\right)-x\sqrt{25-x^2}\right]+C$$

**10.** Same substitution as in Exercise 9

$$\int\frac{\sqrt{x^2-4}}{x}\,dx=\int\frac{2\tan\theta}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta=2\int\tan^2\theta\,d\theta=2\int(\sec^2\theta-1)\,d\theta$$
$$=2(\tan\theta-\theta)+C=2\left[\frac{\sqrt{x^2-4}}{2}-\operatorname{arcsec}\left(\frac{x}{2}\right)\right]+C=\sqrt{x^2-4}-2\operatorname{arcsec}\left(\frac{x}{2}\right)+C$$

**12.** Same substitution as in Exercise 9

$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{8\sec^3\theta}{2\tan\theta}(2\sec\theta\tan\theta)\,d\theta = 8\int \sec^4\theta\,d\theta$$

$$= 8\int (1+\tan^2\theta)\sec^2\theta\,d\theta = 8\left(\tan\theta + \frac{\tan^3\theta}{3}\right) + C = \frac{8}{3}\tan\theta(3+\tan^2\theta) + C$$

$$= \frac{8}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)\left(3+\frac{x^2-4}{4}\right) + C = \frac{1}{3}\sqrt{x^2-4}\,(12+x^2-4) + C = \frac{1}{3}\sqrt{x^2-4}\,(x^2+8) + C$$

**14.** Same substitution as in Exercise 13.

$$\int \frac{9x^3}{\sqrt{1+x^2}}\,dx = 9\int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta = 9\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta = 9\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 3\sec\theta(\sec^2\theta - 3) + C = 3\sqrt{1+x^2}[(1+x^2)-3] + C = 3\sqrt{1+x^2}(x^2-2) + C$$

**16.** Same substitution as in Exercise 13

$$\int \frac{x^2}{(1+x^2)^2}dx = \int \frac{x^2}{\left(\sqrt{1+x^2}\right)^4}\,dx = \int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec^4\theta} = \int \sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int (1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{1}{2}[\theta - \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left[\arctan x - \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C = \frac{1}{2}\left[\arctan x - \frac{x}{1+x^2}\right] + C$$

**18.** Let $u = x$, $a = 1$, and $du = dx$.

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left|x + \sqrt{1+x^2}\right|\right) + C$$

**20.** $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int (9-x^2)^{-1/2}(-2x)\,dx$

$\displaystyle= -(9-x^2)^{1/2} + C$ (Power Rule)

**22.** $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\frac{x}{5} + C$

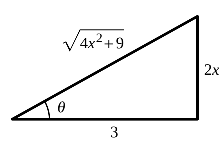
**24.** Let $u = 16 - 4x^2$, $du = -8x\,dx$.

$$\int x\sqrt{16-4x^2}\,dx = -\frac{1}{8}\int (16-4x^2)^{1/2}(-8x)\,dx = \left[-\frac{1}{12}(16-4x^2)^{3/2}\right] + C = -\frac{2}{3}(4-x^2)^{3/2} + C$$

**26.** Let $u = 1 - t^2$, $du = -2t\,dt$.

$$\int \frac{t}{(1-t^2)^{3/2}}\,dt = -\frac{1}{2}\int (1-t^2)^{-3/2}(-2t)\,dt = \frac{1}{\sqrt{1-t^2}} + C$$

**28.** Let $2x = 3\tan\theta$, $dx = \dfrac{3}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+9} = 3\sec\theta$.




$$\int \frac{\sqrt{4x^2+9}}{x^4}\,dx = \int \frac{3\sec\theta[(3/2)\sec^2\theta\,d\theta]}{(3/2)^4\tan^4\theta}$$

$$= \frac{8}{9}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

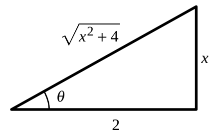
$$= \frac{-8}{27\sin^3\theta} + C$$

$$= -\frac{8}{27}\csc^3\theta + C$$

$$= \frac{-(4x^2+9)^{3/2}}{27x^3} + C$$

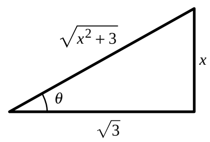
**30.** Let $2x = 4\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, $\sqrt{4x^2+16} = 4\sec\theta$.

$$\int \frac{1}{x\sqrt{4x^2+16}}\,dx = \int \frac{2\sec^2\theta\,d\theta}{2\tan\theta(4\sec\theta)}$$
$$= \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{4}\int \csc\theta\,d\theta$$
$$= -\frac{1}{4}\ln\left|\csc\theta + \cot\theta\right| + C = -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+4}+2}{x}\right| + C$$




**32.** Let $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$, $x^2+3 = 3\sec^2\theta$.

$$\int \frac{1}{(x^2+3)^{3/2}}\,dx = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{3\sqrt{3}\sec^3\theta}$$
$$= \frac{1}{3}\int \cos\theta\,d\theta$$
$$= \frac{1}{3}\sin\theta + C$$
$$= \frac{x}{3\sqrt{x^2+3}} + C$$

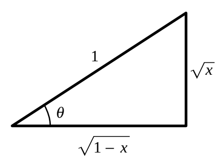



**34.** Let $u = x^2 + 2x + 2$, $du = (2x+2)\,dx$.

$$\int (x+1)\sqrt{x^2+2x+2}\,dx = \frac{1}{2}\int (x^2+2x+2)^{1/2}(2x+2)\,dx = \frac{1}{3}(x^2+2x+2)^{3/2} + C$$

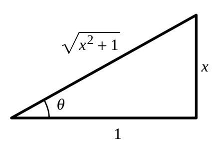
**36.** Let $\sqrt{x} = \sin\theta$, $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$, $\sqrt{1-x} = \cos\theta$.

$$\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx = \int \frac{\cos\theta(2\sin\theta\cos\theta\,d\theta)}{\sin\theta}$$
$$= 2\int \cos^2\theta\,d\theta$$
$$= \int (1+\cos 2\theta)\,d\theta$$
$$= (\theta + \sin\theta\cos\theta) + C$$
$$= \arcsin\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$




**38.** Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $x^2+1 = \sec^2\theta$.

$$\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx = \frac{1}{4}\int \frac{4x^3+4x}{x^4+2x^2+1}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$
$$= \frac{1}{4}\ln(x^4+2x^2+1) + \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{2}\left[\ln(x^2+1) + \arctan x + \frac{x}{x^2+1}\right] + C$$



**40.** $u = \arcsin x, \implies du = \dfrac{1}{\sqrt{1-x^2}}\,dx, dv = x\,dx \implies v = \dfrac{x^2}{2}$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta, dx = \cos\theta\,d\theta, \sqrt{1-x^2} = \cos\theta$

$$\int x \arcsin x\,dx = \frac{x^2}{2}\arcsin x = \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int(1-\cos 2\theta)\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}[\theta - \sin\theta\cos\theta] + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\arcsin x - x\sqrt{1-x^2}\right] + C = \frac{1}{4}\left[(2x^2-1)\arcsin x + x\sqrt{1-x^2}\right] + C$$

**42.** Let $x - 1 = \sin\theta, dx = \cos\theta\,d\theta, \sqrt{1-(x-1)^2} = \sqrt{2x-x^2} = \cos\theta.$

$$\int \frac{x^2}{\sqrt{2x-x^2}}\,dx = \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx$$

$$= \int \frac{(1+\sin\theta)^2(\cos\theta\,d\theta)}{\cos\theta}$$

$$= \int (1 + 2\sin\theta + \sin^2\theta)\,d\theta$$

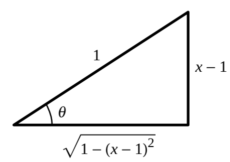
$$= \int \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{3}{2}\arcsin(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + C$$

$$= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}\sqrt{2x-x^2}(x+3) + C$$




**44.** Let $x - 3 = 2\sec\theta, dx = 2\sec\theta\tan\theta\,d\theta, \sqrt{(x-3)^2-4} = 2\tan\theta.$

$$\int \frac{x}{\sqrt{x^2-6x+5}}\,dx = \int \frac{x}{\sqrt{(x-3)^2-4}}\,dx = \int \frac{(2\sec\theta+3)}{2\tan\theta}(2\sec\theta\tan\theta)\,d\theta$$

$$= \int (2\sec^2\theta + 3\sec\theta)\,d\theta$$

$$= 2\tan\theta + 3\ln|\sec\theta + \tan\theta| + C_1$$

$$= 2\left(\frac{\sqrt{(x-3)^3-4}}{2}\right) + 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2-4}}{2}\right| + C_1$$

$$= \sqrt{x^2-6x+5} + 3\ln\left|(x-3) + \sqrt{x^2-6x+5}\right| + C$$

**46.** Same substitution as in Exercise 45

(a) $$\int \frac{1}{(1-t^2)^{5/2}}\,dt = \int \frac{\cos\theta\,d\theta}{\cos^5\theta} = \int \sec^4\theta\,d\theta = \int (\tan^2\theta + 1)\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\tan^3\theta + \tan\theta + C = \frac{1}{3}\left(\frac{t}{\sqrt{1-t^2}}\right)^3 + \frac{t}{\sqrt{1-t^2}} + C$$

Thus, $$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{t^3}{3(1-t^2)^{3/2}} + \frac{t}{\sqrt{1-t^2}}\right]_0^{\sqrt{3}/2}$$

$$= \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/3} = \frac{1}{3}\left(\sqrt{3}\right)^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

**48.** (a) Let $5x = 3\sin\theta$, $dx = \frac{3}{5}\cos\theta\,d\theta$, $\sqrt{9-25x^2} = 3\cos\theta$.

$$\int \sqrt{9-25x^2}\,dx = \int (3\cos\theta)\frac{3}{5}\cos\theta\,d\theta$$

$$= \frac{9}{5}\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{10}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{9}{10}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right] + C$$

Thus, $$\int_0^{3/5} \sqrt{9-25x^2}\,dx = \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9}\right]_0^{3/5} = \frac{9}{10}\left[\frac{\pi}{2}\right] = \frac{9\pi}{20}.$$

(b) When $x = 0$, $\theta = 0$. When $x = \frac{3}{5}$, $\theta = \frac{\pi}{2}$.

Thus, $$\int_0^{3/5} \sqrt{9-25x^2}\,dx = \left[\frac{9}{10}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2} = \frac{9}{10}\left(\frac{\pi}{2}\right) = \frac{9\pi}{20}.$$

**50.** (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$\int \frac{\sqrt{x^2-9}}{x^2}\,dx = \int \frac{3\tan\theta}{9\sec^2\theta}3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \int (\sec\theta - \cos\theta)\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x} + C$$

**—CONTINUED—**

**50. —CONTINUED—**

Hence, $\displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \left[\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x}\right]_3^6 = \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2}.$

(b) When $x = 3$, $\theta = 0$; when $x = 6$, $\theta = \dfrac{\pi}{3}$.

Hence, $\displaystyle\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \Big[\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\Big]_0^{\pi/3} = \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2}.$

**52.** $\displaystyle\int (x^2+2x+11)^{3/2}\,dx = \frac{1}{4}(x+1)(x^2+2x+26)\sqrt{x^2+2x+11} + \frac{75}{2}\ln\left|\sqrt{x^2+2x+11} + (x+1)\right| + C$

**54.** $\displaystyle\int x^2\sqrt{x^2-4}\,dx = \frac{1}{4}x^3\sqrt{x^2-4} - \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left|x + \sqrt{x^2-4}\right| + C$

**56.** (a) Substitution: $u = x^2+1$, $du = 2x\,dx$

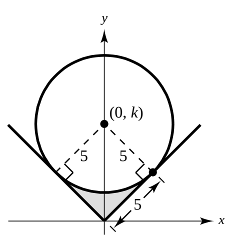
(b) Trigonometric substitution: $x = \sec\theta$

**58.** (a) $x^2 + (y-k)^2 = 25$

Radius of circle $= 5$

$k^2 = 5^2 + 5^2 = 50$

$k = 5\sqrt{2}$




(b) Area $=$ square $- \dfrac{1}{4}$(circle)

$= 25 - \dfrac{1}{4}\pi(5)^2 = 25\left(1 - \dfrac{\pi}{4}\right)$

(c) Area $= r^2 - \dfrac{1}{4}\pi r^2 = r^2\left(1 - \dfrac{\pi}{4}\right)$

**60.** (a) Place the center of the circle at $(0, 1)$; $x^2 + (y-1)^2 = 1$. The depth $d$ satisfies $0 \le d \le 2$. The volume is

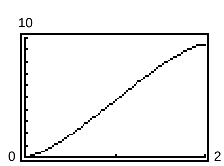
$$V = 3\cdot 2\int_0^d \sqrt{1-(y-1)^2}\,dy$$

$$= 6\cdot\frac{1}{2}\left[\arcsin(y-1) + (y-1)\sqrt{1-(y-1)^2}\right]_0^d \text{ (Theorem 7.2 (1))}$$

$$= 3\left[\arcsin(d-1) + (d-1)\sqrt{1-(d-1)^2} - \arcsin(-1)\right]$$

$$= \frac{3\pi}{2} + 3\arcsin(d-1) + 3(d-1)\sqrt{2d-d^2}.$$

(b)




(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

$$y = \frac{3\pi}{4},\ y = \frac{3\pi}{2},\ y = \frac{9\pi}{4}$$

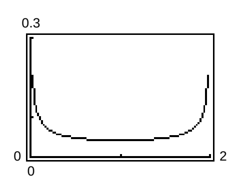
intersect the curve at $d = 0.596, 1.0, 1.404$. The dipstick would have these markings on it.

(d) $\displaystyle V = 6\int_0^d \sqrt{1-(y-1)^2}\,dy$

$$\frac{dV}{dt} = \frac{dV}{dd}\cdot\frac{dd}{dt} = 6\sqrt{1-(d-1)^2}\cdot d'(t) = \frac{1}{4}$$

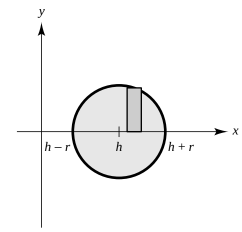
$$\Longrightarrow d'(t) = \frac{1}{24\sqrt{1-(d-1)^2}}$$

(e)




The minimum occurs at $d = 1$, which is the widest part of the tank.

**62.** Let $x - h = r \sin \theta$, $dx = r \cos \theta \, d\theta$, $\sqrt{r^2 - (x-h)^2} = r \cos \theta$.



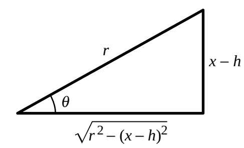

**Shell Method:**

$$V = 4\pi \int_{h-r}^{h+r} x\sqrt{r^2 - (x-h)^2}\,dx$$

$$= 4\pi \int_{-\pi/2}^{\pi/2} (h + r \sin \theta) r \cos \theta (r \cos \theta)\,d\theta = 4\pi r^2 \int_{-\pi/2}^{\pi/2} (h + r\sin\theta)\cos^2\theta\,d\theta$$

$$= 4\pi r^2 \left[\frac{h}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\,d\theta + r\int_{-\pi/2}^{\pi/2} \sin\theta\cos^2\theta\,d\theta\right]$$

$$= 2\pi r^2 h\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} - \left[4\pi r^3\left(\frac{\cos^3\theta}{3}\right)\right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h$$



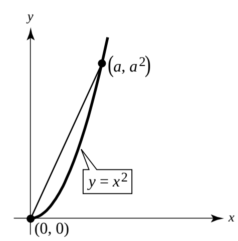

**64.** $y = \frac{1}{2}x^2$, $y' = x$, $1 + (y')^2 = 1 + x^2$

$$s = \int_0^4 \sqrt{1 + x^2}\,dx = \left[\frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right)\right]_0^4 \quad \text{(Theorem 7.2)}$$

$$= \frac{1}{2}\left[4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right] \approx 9.2936$$

**66.** (a) Along line: $d_1 = \sqrt{a^2 + a^4} = a\sqrt{1 + a^2}$

Along parabola: $y = x^2$, $y' = 2x$




$$d_2 = \int_0^a \sqrt{1 + 4x^2}\,dx$$

$$= \frac{1}{4}\left[2x\sqrt{4x^2+1} + \ln\left|2x + \sqrt{4x^2+1}\right|\right]_0^a \quad \text{(Theorem 7.2)}$$

$$= \frac{1}{4}\left[2a\sqrt{4a^2+1} + \ln\left(2a + \sqrt{4a^2+1}\right)\right]$$

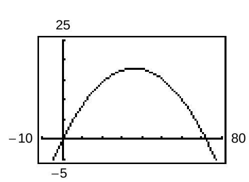
(b) For $a = 1$, $d_1 = \sqrt{2}$ and $d_2 = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(2 + \sqrt{5}\right) \approx 1.4789$.

For $a = 10$, $d_1 = 10\sqrt{101} \approx 100.4988$

$d_2 \approx 101.0473.$

(c) As $a$ increases, $d_2 - d_1 \to 0$.

**68.** (a)




(b) $y = 0$ for $x = 72$

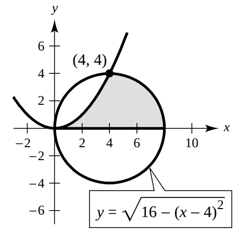
(c) $y = x - \frac{x^2}{72}$, $y' = 1 - \frac{x}{36}$, $1 + (y')^2 = 1 + \left(1 - \frac{x}{36}\right)^2$

$$s = \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2}\,dx = -36\int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2}\left(-\frac{1}{36}\right)dx$$

$$= -\frac{36}{2}\left[\left(1 - \frac{x}{36}\right)\sqrt{1 + \left(1 - \frac{x}{36}\right)^2} + \ln\left|\left(1 - \frac{x}{36}\right) + \sqrt{1 + \left(1 - \frac{x}{36}\right)^2}\right|\right]_0^{72}$$

$$= -18\left[\left(-\sqrt{2} + \ln\left|-1 + \sqrt{2}\right|\right) - \left(\sqrt{2} + \ln\left|1 + \sqrt{2}\right|\right)\right] = 36\sqrt{2} + 18\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 82.641$$

**70.** First find where the curves intersect.




$$y^2 = 16 - (x - 4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x - 4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x - 4)(x^2 + 4x + 32) \implies x = 0, 4$$

$$A = \int_0^4 \frac{1}{4}x^2 dx + \frac{1}{4}\pi(4)^2 = \frac{1}{12}x^3\Big]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

$$M_y = \int_0^4 x\left[\frac{1}{4}x^2\right]dx + \int_4^8 x\sqrt{16 - (x - 4)^2}\,dx$$

$$= \frac{x^4}{16}\Big]_0^4 + \int_4^8 (x - 4)\sqrt{16 - (x - 4)^2}\,dx + \int_4^8 4\sqrt{16 - (x - 4)^2}\,dx$$

$$= 16 + \left[\frac{-1}{3}(16 - (x - 4)^2)^{3/2}\right]_4^8 + 2\left[16 \arcsin\frac{x - 4}{4} + (x - 4)\sqrt{16 - (x - 4)^2}\right]_4^8$$

$$= 16 + \frac{1}{3}16^{3/2} + 2\left[16\left(\frac{\pi}{2}\right)\right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi$$

$$M_x = \int_0^4 \frac{1}{2}\left(\frac{1}{4}x^2\right)^2 dx + \int_4^8 \frac{1}{2}(16 - (x - 4)^2)\,dx$$

$$= \left[\frac{1}{32}\cdot\frac{x^5}{5}\right]_0^4 + \left[8x - \frac{(x - 4)^3}{6}\right]_4^8$$

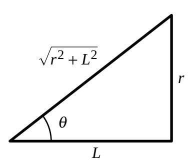
$$= \frac{32}{5} + \left(64 - \frac{64}{6}\right) - 32 = \frac{416}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$$

**72.** Let $r = L\tan\theta$, $dr = L\sec^2\theta\,d\theta$, $r^2 + L^2 = L^2\sec^2\theta$.




$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}}\,dr = \frac{2mL}{R}\int_a^b \frac{L\sec^2\theta\,d\theta}{L^3\sec^3\theta}$$

$$= \frac{2m}{RL}\int_a^b \cos\theta\,d\theta$$

$$= \left[\frac{2m}{RL}\sin\theta\right]_a^b$$

$$= \left[\frac{2m}{RL}\frac{r}{\sqrt{r^2 + L^2}}\right]_0^R$$

$$= \frac{2m}{L\sqrt{R^2 + L^2}}$$

**74.** (a) $F_{\text{inside}} = 48\int_{-1}^{0.8}(0.8 - y)(2)\sqrt{1 - y^2}\,dy$

$$= 96\left[0.8\int_{-1}^{0.8}\sqrt{1 - y^2}\,dy - \int_{-1}^{0.8} y\sqrt{1 - y^2}\,dy\right]$$

$$= 96\left[\frac{0.8}{2}\left(\arcsin y + y\sqrt{1 - y^2}\right) + \frac{1}{3}(1 - y^2)^{3/2}\right]_{-1}^{0.8} \approx 96(1.263) \approx 121.3 \text{ lbs}$$

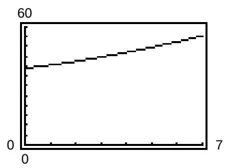
(b) $F_{\text{outside}} = 64\int_{-1}^{0.4}(0.4 - y)(2)\sqrt{1 - y^2}\,dy$

$$= 128\left[0.4\int_{-1}^{0.4}\sqrt{1 - y^2}\,dy - \int_{-1}^{0.4} y\sqrt{1 - y^2}\,dy\right]$$

$$= 128\left[\frac{0.4}{2}\left(\arcsin y + y\sqrt{1 - y^2}\right) + \frac{1}{3}(1 - y^2)^{3/2}\right]_{-1}^{0.4} \approx 92.98$$

**76.** $S = \sqrt{1520.4 + 111.2t + 15.8t^2}$

(a)




(b) $S'(t) = \frac{1}{2}(1520.4 + 111.2t + 15.8t^2)^{-1/2}(111.2 + 31.6t)$

$S'(5) \approx 2.71$

(c) Average value $= \frac{1}{2}\int_{10}^{12} S(t)\,dt \approx 68.24$

**78.** False

$$\int \frac{\sqrt{x^2 - 1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}(\sec\theta\tan\theta\,d\theta)$$

$$= \int \tan^2\theta\,d\theta$$

**80.** True

$$\int_{-1}^{1} x^2\sqrt{1 - x^2}\,dx = 2\int_0^1 x^2\sqrt{1 - x^2}\,dx = 2\int_0^{\pi/2}(\sin^2\theta)(\cos\theta)(\cos\theta\,d\theta) = 2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

## Section 7.5 Partial Fractions

**2.** $\dfrac{4x^2 + 3}{(x - 5)^3} = \dfrac{A}{x - 5} + \dfrac{B}{(x - 5)^2} + \dfrac{C}{(x - 5)^3}$

**4.** $\dfrac{x - 2}{x^2 + 4x + 3} = \dfrac{x - 2}{(x + 1)(x + 3)} = \dfrac{A}{x + 1} + \dfrac{B}{x + 3}$

**6.** $\dfrac{2x - 1}{x(x^2 + 1)^2} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1} + \dfrac{Dx + E}{(x^2 + 1)^2}$

**8.** $\dfrac{1}{4x^2 - 9} = \dfrac{1}{(2x - 3)(2x + 3)} = \dfrac{A}{2x - 3} + \dfrac{B}{2x + 3}$

$$1 = A(2x + 3) + B(2x - 3)$$

When $x = \frac{3}{2}$, $1 = 6A$, $A = \frac{1}{6}$.
When $x = -\frac{3}{2}$, $1 = -6B$, $B = -\frac{1}{6}$.

$$\int \frac{1}{4x^2 - 9}\,dx = \frac{1}{6}\left[\int \frac{1}{2x - 3}\,dx - \int \frac{1}{2x + 3}\,dx\right]$$

$$= \frac{1}{12}\left[\ln|2x - 3| - \ln|2x + 3|\right] + C$$

$$= \frac{1}{12}\ln\left|\frac{2x - 3}{2x + 3}\right| + C$$

**10.** $\displaystyle\int \frac{x + 1}{x^2 + 4x + 3}\,dx = \int \frac{(x + 1)}{(x + 1)(x + 3)}\,dx$

$$= \int \frac{1}{x + 3}\,dx = \ln|x + 3| + C$$

**12.** $\dfrac{5x^2 - 12x - 12}{x(x-2)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$

$5x^2 - 12x - 12 = A(x^2 - 4) + Bx(x+2) + Cx(x-2)$

When $x = 0$, $-12 = -4A \implies A = 3$. When $x = 2$, $-16 = 8B \implies B = -2$. When $x = -2$, $32 = 8C \implies C = 4$.

$$\int \frac{5x^2 - 12x - 12}{x^3 - 4x}\,dx = \int \frac{3}{x}\,dx + \int \frac{-2}{x-2}\,dx + \int \frac{4}{x+2}\,dx = 3\ln|x| - 2\ln|x-2| + 4\ln|x+2| + C$$

**14.** $\dfrac{x^3 - x + 3}{x^2 + x - 2} = x - 1 + \dfrac{2x+1}{(x+2)(x-1)} = x - 1 + \dfrac{A}{x+2} + \dfrac{B}{x-1}$

$2x + 1 = A(x-1) + B(x+2)$

When $x = -2$, $-3 = -3A$, $A = 1$. When $x = 1$, $3 = 3B$, $B = 1$.

$$\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx = \int \left[x - 1 + \frac{1}{x+2} + \frac{1}{x-1}\right]dx$$

$$= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2 + x - 2| + C$$

**16.** $\dfrac{x+2}{x(x-4)} = \dfrac{A}{x-4} + \dfrac{B}{x}$

$x + 2 = Ax + B(x-4)$

When $x = 4$, $6 = 4A$, $A = \frac{3}{2}$.
When $x = 0$, $2 = -4B$, $B = -\frac{1}{2}$.

$$\int \frac{x+2}{x^2 - 4x}\,dx = \int \left[\frac{3/2}{x-4} - \frac{1/2}{x}\right]dx$$

$$= \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x| + C$$

**18.** $\dfrac{2x-3}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$2x - 3 = A(x-1) + B$

When $x = 1$, $B = -1$. When $x = 0$, $A = 2$.

$$\int \frac{2x-3}{(x-1)^2}\,dx = \int \left[\frac{2}{x-1} - \frac{1}{(x-1)^2}\right]dx$$

$$= 2\ln|x-1| + \frac{1}{x-1} + C$$

**20.** $\dfrac{4x^2}{x^3 + x^2 - x - 1} = \dfrac{4x^2}{x^2(x+1) - (x+1)} = \dfrac{4x^2}{(x^2-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

When $x = -1$, $4 = -2C \implies C = -2$. When $x = 1$, $4 = 4A \implies A = 1$. When $x = 0$, $0 = 1 - B + 2 \implies B = 3$.

$$\int \frac{4x^2}{x^3 + x^2 - x - 1}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{3}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx$$

$$= \ln|x-1| + 3\ln|x+1| + \frac{2}{(x+1)} + C$$

**22.** $\dfrac{6x}{x^3 - 8} = \dfrac{6x}{(x-2)(x^2+2x+4)} = \dfrac{A}{x-2} + \dfrac{Bx + C}{x^2 + 2x + 4}$

$6x = A(x^2 + 2x + 4) + (Bx + C)(x - 2)$

When $x = 2$, $12 = 12A \implies A = 1$. When $x = 0$, $0 = 4 - 2C \implies C = 2$. When $x = 1$, $6 = 7 + (B+2)(-1) \implies B = -1$.

$$\int \frac{6x}{x^3 - 8}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{-x+2}{x^2+2x+4}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{-x-1}{x^2+2x+4}\,dx + \int \frac{3}{(x^2+2x+1)+3}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+2x+4| + \frac{3}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+2x+4| + \sqrt{3}\arctan\left(\frac{\sqrt{3}(x+1)}{3}\right) + C$$

**24.** $\dfrac{x^2 - x + 9}{(x^2 + 9)^2} = \dfrac{Ax + B}{x^2 + 9} + \dfrac{Cx + D}{(x^2 + 9)^2}$

$$x^2 - x + 9 = (Ax + B)(x^2 + 9) + Cx + D$$
$$= Ax^3 + Bx^2 + (9A + C)x + (9B + D)$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $D = 0$, and $C = -1$.

$$\int \frac{x^2 - x - 9}{(x^2 + 9)^2} = \int \frac{1}{x^2 + 9}\,dx - \int \frac{x}{(x^2 + 9)^2}\,dx$$
$$= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + \frac{1}{2(x^2 + 9)} + C$$

**26.** $\dfrac{x^2 - 4x + 7}{(x + 1)(x^2 - 2x + 3)} = \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 - 2x + 3}$

$$x^2 - 4x + 7 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

When $x = -1$, $12 = 6A$. When $x = 0$, $7 = 3A + C$. When $x = 1$, $4 = 2A + 2B + 2C$. Solving these equations we have $A = 2$, $B = -1$, $C = 1$.

$$\int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3}\,dx = 2\int \frac{1}{x + 1}\,dx + \int \frac{-x + 1}{x^2 - 2x + 3}\,dx$$
$$= 2\ln|x + 1| - \frac{1}{2}\ln|x^2 - 2x + 3| + C$$

**28.** $\dfrac{x^2 + x + 3}{(x^2 + 3)^2} = \dfrac{Ax + B}{x^2 + 3} + \dfrac{Cx + D}{(x^2 + 3)^2}$

$$x^2 + x + 3 = (Ax + B)(x^2 + 3) + Cx + D$$
$$= Ax^3 + Bx^2 + (3A + C)x + (3B + D)$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $3A + C = 1$, $3B + D = 3$. Solving these equations we have $A = 0$, $B = 1$, $C = 1$, $D = 0$.

$$\int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9}\,dx = \int \left[\frac{1}{x^2 + 3} + \frac{x}{(x^2 + 3)^2}\right] dx$$
$$= \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} - \frac{1}{2(x^2 + 3)} + C$$

**30.** $\dfrac{x - 1}{x^2(x + 1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x + 1}$

$$x - 1 = Ax(x + 1) + B(x + 1) + Cx^2$$

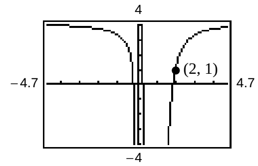
When $x = 0$, $B = -1$. When $x = -1$, $C = -2$. When $x = 1$, $0 = 2A + 2B + C$. Solving these equations we have $A = 2$, $B = -1$, $C = -2$.

$$\int_1^5 \frac{x - 1}{x^2(x + 1)}\,dx = 2\int_1^5 \frac{1}{x}\,dx - \int_1^5 \frac{1}{x^2}\,dx - 2\int_1^5 \frac{1}{x + 1}\,dx$$
$$= \left[2\ln|x| + \frac{1}{x} - 2\ln|x + 1|\right]_1^5$$
$$= \left[2\ln\left|\frac{x}{x + 1}\right| + \frac{1}{x}\right]_1^5$$
$$= 2\ln\frac{5}{3} - \frac{4}{5}$$

**32.** $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx = \int_0^1 dx - \int_0^1 \frac{2x+1}{x^2+x+1}\,dx = \Big[x - \ln|x^2 + x + 1|\Big]_0^1 = 1 - \ln 3$

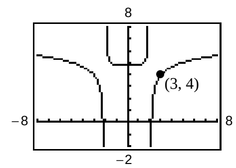
**34.** $\displaystyle\int \frac{6x^2+1}{x^2(x-1)^3}\,dx = 3\ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + \frac{2}{x-1} - \frac{7}{2(x-1)^2} + C$

$(2, 1)$: $\displaystyle 3\ln\left|\frac{1}{2}\right| + \frac{1}{2} + \frac{2}{1} - \frac{7}{2} + C = 1 \Rightarrow C = 2 - 3\ln\frac{1}{2}$



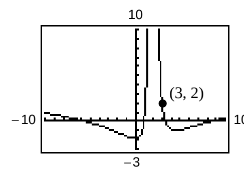

**36.** $\displaystyle\int \frac{x^3}{(x^2-4)^2}\,dx = \frac{1}{2}\ln|x^2-4| - \frac{2}{x^2-4} + C$

$(3, 4)$: $\displaystyle \frac{1}{2}\ln 5 - \frac{2}{5} + C = 4 \Rightarrow C = \frac{22}{5} - \frac{1}{2}\ln 5$



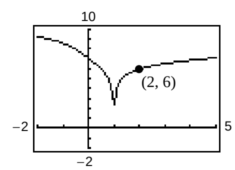

**38.** $\displaystyle\int \frac{x(2x-9)}{x^3 - 6x^2 + 12x - 8}\,dx = 2\ln|x-2| + \frac{1}{x-2} + \frac{5}{(x-2)^2} + C$

$(3, 2)$: $0 + 1 + 5 + C = 2 \Rightarrow C = -4$




**40.** $\displaystyle\int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1}\,dx = -\arctan x + \ln|x-1| + C$

$(2, 6)$: $-\arctan 2 + 0 + C = 6 \Rightarrow C = 6 + \arctan 2$




**42.** Let $u = \cos x$, $du = \sin x\,dx$.

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$. When $u = -1$, $B = -1$, $u = \cos x$. $du = -\sin dx$.

$$\begin{aligned}\int \frac{\sin x}{\cos x + \cos^2 x}\,dx &= -\int \frac{1}{u(u+1)}\,du\\ &= \int \frac{1}{u+1}\,du - \int \frac{1}{u}\,du\\ &= \ln|u+1| - \ln|u| + C\\ &= \ln\left|\frac{u+1}{u}\right| + C\\ &= \ln\left|\frac{\cos x + 1}{\cos x}\right| + C\\ &= \ln|1 + \sec x| + C\end{aligned}$$

**44.** $\displaystyle\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$, $u = \tan x$, $du = \sec^2 x\,dx$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.
When $u = -1$, $1 = -B \Rightarrow B = -1$.

$$\begin{aligned}\int \frac{\sec^2 x\,dx}{\tan x(\tan x + 1)} &= \int \frac{1}{u(u+1)}\,du\\ &= \int\left(\frac{1}{u} - \frac{1}{u+1}\right)du\\ &= \ln|u| - \ln|u+1| + C\\ &= \ln\left|\frac{u}{u+1}\right| + C\\ &= \ln\left|\frac{\tan x}{\tan x + 1}\right| + C\end{aligned}$$

**46.** Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u^2+1)(u-1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+1}$$

$$1 = A(u^2+1) + (Bu+C)(u-1)$$

When $u = 1$, $A = \frac{1}{2}$.

When $u = 0$, $1 = A - C$.

When $u = -1$, $1 = 2A + 2B - 2C$. Solving these equations we have $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$, $u = e^x$, $du = e^x\,dx$.

$$\begin{aligned}\int \frac{e^x}{(e^{2x}+1)(e^x-1)}\,dx &= \int \frac{1}{(u^2+1)(u-1)}\,du \\ &= \frac{1}{2}\left(\int \frac{1}{u-1}\,du - \int \frac{u+1}{u^2+1}\,du\right) \\ &= \frac{1}{2}\left(\ln|u-1| - \frac{1}{2}\ln|u^2+1| - \arctan u\right) + C \\ &= \frac{1}{4}\left(2\ln|e^x-1| - \ln|e^{2x}+1| - 2\arctan e^x\right) + C\end{aligned}$$

**48.** $\dfrac{1}{a^2-x^2} = \dfrac{A}{a-x} + \dfrac{B}{a+x}$

$$1 = A(a+x) + B(a-x)$$

When $x = a$, $A = 1/2a$.
When $x = -a$, $B = 1/2a$.

$$\begin{aligned}\int \frac{1}{a^2-x^2}\,dx &= \frac{1}{2a}\int\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx \\ &= \frac{1}{2a}\left(-\ln|a-x| + \ln|a+x|\right) + C \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C\end{aligned}$$

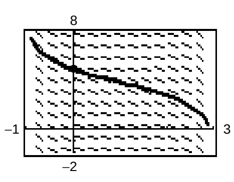
**50.** $\dfrac{1}{x^2(a+bx)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{a+bx}$

$$1 = Ax(a+bx) + B(a+bx) + Cx^2$$

When $x = 0$, $1 = Ba \implies B = 1/a$.
When $x = -a/b$, $1 = C(a^2/b^2) \implies C = b^2/a^2$.
When $x = 1$, $1 = (a+b)A + (a+b)B + C \implies$
$A = -b/a^2$.

$$\begin{aligned}\int \frac{1}{x^2(a+bx)}\,dx &= \int\left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a+bx}\right)dx \\ &= -\frac{b}{a^2}\ln|x| - \frac{1}{ax} + \frac{b}{a^2}\ln|a+bx| + C \\ &= -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C \\ &= -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C\end{aligned}$$

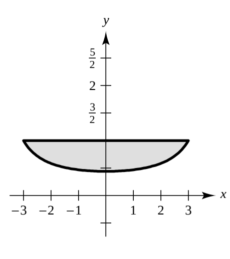
**52.** $\dfrac{dy}{dx} = \dfrac{4}{(x^2-2x-3)}$, $y(0) = 5$



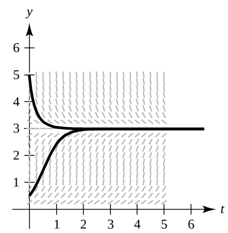

**54.** (a) $\dfrac{N(x)}{D(x)} = \dfrac{A_1}{px+q} + \dfrac{A_2}{(px+q)^2} + \cdots + \dfrac{A_m}{(px+q)^m}$

(b) $\dfrac{N(x)}{D(x)} = \dfrac{A_1+B_1x}{(ax^2+bx+c)} + \cdots + \dfrac{A_n+B_nx}{(ax^2+bx+c)^n}$

**56.**
$$\begin{aligned}A &= 2\int_0^3\left(1 - \frac{7}{16-x^2}\right)dx \\ &= 2\int_0^3 dx - 14\int_0^3 \frac{1}{16-x^2}\,dx \\ &= \left[2x - \frac{14}{8}\ln\left|\frac{4+x}{4-x}\right|\right]_0^3 \qquad \text{(From Exercise 46)} \\ &= 6 - \frac{7}{4}\ln 7 \approx 2.595\end{aligned}$$



**58.** (a)




(b) The slope is negative because the function is decreasing.

(c) For $y > 0$, $\lim_{t\to\infty} y(t) = 3$.

(d) $$\frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$1 = A(L-y) + By \implies A = \frac{1}{L}, B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k\,dt$$

$$\frac{1}{L}\left[\int \frac{1}{y}dy + \int \frac{1}{L-y}dy\right] = \int k\,dt$$

$$\frac{1}{L}\left[\ln|y| - \ln|L-y|\right] = kt + C_1$$

$$\ln\left|\frac{y}{L-y}\right| = kLt + LC_1$$

$$C_2 e^{kLt} = \frac{y}{L-y}$$

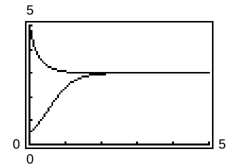
When $t = 0$, $\dfrac{y_0}{L-y_0} = C_2 \implies \dfrac{y}{L-y} = \dfrac{y_0}{L-y_0}e^{kLt}$.

Solving for $y$, you obtain $y = \dfrac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}$.

(e) $k = 1, L = 3$

(*i*) $y(0) = 5$: $\quad y = \dfrac{15}{5 - 2e^{-3t}}$

(*ii*) $y(0) = \dfrac{1}{2}$: $\quad y = \dfrac{3/2}{(1/2) + (5/2)e^{-3t}} = \dfrac{3}{1 + 5e^{-3t}}$




(f) $$\frac{dy}{dt} = ky(L-y)$$

$$\frac{d^2y}{dt^2} = k\left[y\left(\frac{-dy}{dt}\right) + (L-y)\frac{dy}{dt}\right] = 0$$

$$\implies y\frac{dy}{dt} = (L-y)\frac{dy}{dt}$$

$$\implies \quad y = \frac{L}{2}$$

From the first derivative test, this is a maximum.

**60.** (a) $$V = \pi\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx = 4\pi\int_0^3 \frac{x^2}{(x^2+1)^2}\,dx$$

$$= 4\pi\int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right) dx \qquad \text{(partial fractions)}$$

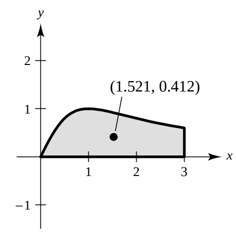
$$= 4\pi\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3 \qquad \text{(trigonometric substitution)}$$

$$= 2\pi\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = 2\pi\left[\arctan 3 - \frac{3}{10}\right] \approx 5.963$$

**—CONTINUED—**

**60. —CONTINUED—**

(b) $A = \int_0^3 \frac{2x}{x^2+1}\,dx = \Big[\ln(x^2+1)\Big]_0^3 = \ln 10$




$$\bar{x} = \frac{1}{A}\int_0^3 \frac{2x^2}{x^2+1}\,dx = \frac{1}{\ln 10}\int_0^3\left(2 - \frac{2}{x^2+1}\right)dx$$

$$= \frac{1}{\ln 10}\Big[2x - 2\arctan x\Big]_0^3 = \frac{2}{\ln 10}[3 - \arctan 3] \approx 1.521$$

$$\bar{y} = \frac{1}{A}\left(\frac{1}{2}\right)\int_0^3\left(\frac{2x}{x^2+1}\right)^2 dx = \frac{2}{\ln 10}\int_0^3 \frac{x^2}{(x^2+1)^2}\,dx$$

$$= \frac{2}{\ln 10}\int_0^3\left(\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right)dx \qquad \text{(partial fractions)}$$

$$= \frac{2}{\ln 10}\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^3 \qquad \text{(trigonometric substitution)}$$

$$= \frac{2}{\ln 10}\left[\frac{1}{2}\arctan x - \frac{x}{2(x^2+1)}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan x - \frac{x}{x^2+1}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan 3 - \frac{3}{10}\right] \approx 0.412$$

$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$

**62.** (a) $\frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x}, A = \frac{1}{z_0 - y_0}, B = -\frac{1}{z_0 - y_0}$ (Assume $y_0 \neq z_0$)

$$\frac{1}{z_0 - y_0}\int\left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x}\right)dx = kt + C$$

$$\frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0}\ln\frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0}\left[\ln\left|\frac{z_0 - x}{y_0 - x}\right| - \ln\left(\frac{z_0}{y_0}\right)\right] = kt$$

$$\ln\left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)}\right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0\left[e^{(z_0 - y_0)kt} - 1\right]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

(b) (1) If $y_0 < z_0$, $\lim_{t\to\infty} x = y_0$.

(2) If $y_0 > z_0$, $\lim_{t\to\infty} x = z_0$.

(c) If $y_0 = z_0$, then the original equation is

$$\int \frac{1}{(y_0 - x)^2}\,dx = \int k\,dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \implies \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kt\,y_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kt\,y_0 + 1}$$

$$x = y_0 - \frac{y_0}{kt\,y_0 + 1}$$

As $t \to \infty$, $x \to y_0 = x_0$.

## Section 7.6 Integration by Tables and Other Integration Techniques

**2.** By Formula 13: $(b = 2, a = -5)$

$$\frac{2}{3}\int \frac{1}{x^2(2x-5)^2}\,dx = \frac{2}{3}\left(\frac{-1}{25}\right)\left[\frac{-5+4x}{x(-5+2x)} + \frac{4}{-5}\ln\left|\frac{x}{2x-5}\right|\right] + C$$

$$= \frac{8}{375}\ln\left|\frac{x}{2x-5}\right| - \frac{2}{75}\,\frac{(4x-5)}{x(2x-5)} + C$$

**4.** By Formula 29: $(a = 3)$

$$\frac{1}{3}\int \frac{\sqrt{x^2-9}}{x}\,dx = \frac{1}{3}\sqrt{x^2-9} - \operatorname{arcsec}\frac{|x|}{3} + C$$

**6.** By Formula 41: $\displaystyle\int \frac{x}{\sqrt{9-x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{3^2-(x^2)^2}}\,dx$

$$= \frac{1}{2}\arcsin\frac{x^2}{3} + C$$

**8.** By Formulas 51 and 47: $\displaystyle\int \frac{\cos^3\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos^3\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx$

$$= 2\left[\frac{\cos^2\sqrt{x}\,\sin\sqrt{x}}{3} + \frac{2}{3}\int \cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx\right] = \frac{2}{3}\sin\sqrt{x}\left(\cos^2\sqrt{x} + 2\right) + C$$

$$u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx$$

**10.** By Formula 71:

$$\int \frac{1}{1-\tan 5x}\,dx = \frac{1}{5}\int \frac{1}{1-\tan 5x}(5)\,dx$$

$$= \frac{1}{5}\left(\frac{1}{2}\right)\left(u - \ln|\cos u - \sin u|\right) + C$$

$$= \frac{1}{10}\left(5x - \ln|\cos 5x - \sin 5x|\right) + C$$

$$u = 5x,\ du = 5\,dx$$

**12.** By Formula 85: $\left(a = -\frac{1}{2}, b = 2\right)$

$$\int e^{-x/2}\sin 2x\,dx = \frac{e^{-x/2}}{(1/4)+4}\left(-\frac{1}{2}\sin 2x - 2\cos 2x\right) + C$$

$$= \frac{4}{17}e^{-x/2}\left(-\frac{1}{2}\sin 2x - 2\cos 2x\right) + C$$

**14.** By Formulas 90 and 91: $\displaystyle\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx$

$$= x(\ln x)^3 - 3x[2 - 2\ln x + (\ln x)^2] + C$$

$$= x[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6] + C$$

**16.** (a) By Formula 89: $\displaystyle\int x^4\ln x\,dx = \frac{x^5}{5^2}[-1 + (4+1)\ln x] + C = \frac{-x^5}{25} + \frac{1}{5}x^5\ln x + C$

(b) Integration by parts: $u = \ln x,\ du = \dfrac{1}{x}\,dx,\ dv = x^4\,dx,\ v = \dfrac{x^5}{5}$

$$\int x^4\ln x\,dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\,\frac{1}{x}\,dx = \frac{x^5}{5}\ln x - \frac{x^5}{25} + C$$

**18.** (a) By Formula 24: $a = \sqrt{75}$, $x = u$, and

$$\int \frac{1}{x^2 - 75}\,dx = \frac{1}{2\sqrt{75}} \ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$
$$= \frac{\sqrt{3}}{30} \ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2 - 75} = \frac{A}{x - \sqrt{75}} + \frac{B}{x + \sqrt{75}}$$
$$1 = A\left(x + \sqrt{75}\right) + B\left(x - \sqrt{75}\right)$$

$$x = \sqrt{75}:\ 1 = 2A\sqrt{75} \Rightarrow A = \frac{1}{2\sqrt{75}} = \frac{1}{10\sqrt{3}} = \frac{\sqrt{3}}{30}$$

$$x = -\sqrt{75}:\ 1 = -2B\sqrt{75} \Rightarrow B = -\frac{\sqrt{3}}{30}$$

$$\int \frac{1}{x^2 - 75}\,dx = \int \left[\frac{\sqrt{3}/30}{x - \sqrt{75}} - \frac{\sqrt{3}/30}{x + \sqrt{75}}\right] dx$$
$$= \frac{\sqrt{3}}{30} \ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

**20.** By Formula 21: $\displaystyle\int \frac{x}{\sqrt{1 + x}}\,dx = -\frac{2}{3}(2 - x)\sqrt{1 + x} + C$

**22.** By Formula 79: $\displaystyle\int \operatorname{arcsec} 2x\,dx = \frac{1}{2}\left[2x \operatorname{arcsec} 2x - \ln\left|2x + \sqrt{4x^2 - 1}\right|\right] + C$

$u = 2x$, $du = 2\,dx$

**24.** By Formula 52: $\displaystyle\int x \sin x\,dx = \sin x - x \cos x + C$

**26.** By Formula 7: $\displaystyle\int \frac{x^2}{(3x - 5)^2}\,dx = \frac{1}{27}\left(3x - \frac{25}{3x - 5} + 10 \ln|3x - 5|\right) + C$

**28.** By Formula 14: $\displaystyle\int \frac{1}{x^2 + 2x + 2}\,dx = \frac{2}{\sqrt{4}} \arctan\left(\frac{2x + 2}{2}\right) + C = \arctan(x + 1) + C$

**30.** By Formula 56:

$$\int \frac{\theta^2}{1 - \sin \theta^3}\,d\theta = \frac{1}{3}\int \frac{1}{1 - \sin \theta^3} 3\theta^2\,d\theta$$
$$= \frac{1}{3}(\tan \theta^3 + \sec \theta^3) + C$$

**32.** By Formula 71:

$$\int \frac{e^x}{1 - \tan e^x}\,dx = \frac{1}{2}\left(e^x - \ln\left|\cos e^x - \sin e^x\right|\right) + C$$

$u = e^x$, $du = e^x\,dx$

**34.** By Formula 23: $\displaystyle\int \frac{1}{t[1 + (\ln t)^2]}\,dt = \int \frac{1}{1 + (\ln t)^2}\left(\frac{1}{t}\right) dt = \arctan(\ln t) + C$

$u = \ln t$, $du = \dfrac{1}{t}\,dt$

**36.** By Formula 26: $\displaystyle\int \sqrt{3 + x^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2 + 3} + 3 \ln\left|x + \sqrt{x^2 + 3}\right|\right) + C$

**38.** By Formula 27: $\displaystyle\int x^2\sqrt{2 + (3x)^2}\,dx = \frac{1}{27}\int (3x)^2\sqrt{\left(\sqrt{2}\right)^2 + (3x)^2}\,3\,dx$

$$= \frac{1}{8(27)}\left[3x(18x^2 + 2)\sqrt{2 + 9x^2} - 4 \ln\left|3x + \sqrt{2 + 9x^2}\right|\right] + C$$

**40.** By Formula 77: $\displaystyle\int \sqrt{x}\arctan(x^{3/2})\,dx = \frac{2}{3}\int \arctan(x^{3/2})\left(\frac{3}{2}\sqrt{x}\right)dx$

$$= \frac{2}{3}\left[x^{3/2}\arctan(x^{3/2}) - \ln\sqrt{1 + x^3}\right] + C$$

**42.** By Formula 45: $\displaystyle\int \frac{e^x}{(1 - e^{2x})^{3/2}}\,dx = \frac{e^x}{\sqrt{1 - e^{2x}}} + C$

$u = e^x,\ du = e^x\,dx$

**44.** By Formula 27:

$$\int (2x - 3)^2\sqrt{(2x - 3)^2 + 4}\,dx = \frac{1}{2}\int (2x - 3)^2\sqrt{(2x - 3)^2 + 4}(2)\,dx$$

$$= \frac{1}{8}(2x - 3)[(2x - 3)^2 + 2]\sqrt{(2x - 3)^2 + 4} - \ln\left|2x - 3 + \sqrt{(2x - 3)^2 + 4}\right| + C$$

$u = 2x - 3,\ du = 2\,dx$

**46.** By Formula 31: $\displaystyle\int \frac{\cos x}{\sqrt{\sin^2 x + 1}}\,dx = \ln\left|\sin x + \sqrt{\sin^2 x + 1}\right| + C$

$u = \sin x,\ du = \cos x\,dx$

**48.** $\displaystyle\int \sqrt{\frac{3 - x}{3 + x}}\,dx = \int \frac{3 - x}{\sqrt{9 - x^2}}\,dx$

$$= 3\int \frac{1}{\sqrt{9 - x^2}}\,dx + \int \frac{-x}{\sqrt{9 - x^2}}\,dx$$

$$= 3\arcsin\frac{x}{3} + \sqrt{9 - x^2} + C$$

**50.** By Formula 67:

$$\int \tan^3\theta\,d\theta = \frac{\tan^2\theta}{2} - \int \tan\theta\,d\theta$$

$$= \frac{\tan^2\theta}{2} + \ln|\cos x| + C$$

**52.** Integration by parts: $w = u^n,\ dw = nu^{n-1}\,du,\ dv = \dfrac{du}{\sqrt{a + bu}},\ v = \dfrac{2}{b}\sqrt{a + bu}$

$$\int \frac{u^n}{\sqrt{a + bu}}\,du = \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a + bu}\,du$$

$$= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a + bu}\cdot\frac{\sqrt{a + bu}}{\sqrt{a + bu}}\,du$$

$$= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2n}{b}\int \frac{au^{n-1} + bu^n}{\sqrt{a + bu}}\,du$$

$$= \frac{2u^n}{b}\sqrt{a + bu} - \frac{2na}{b}\int \frac{u^{n-1}}{\sqrt{a + bu}}\,du - 2n\int \frac{u^n}{\sqrt{a + bu}}\,du$$

Therefore, $\displaystyle(2n + 1)\int \frac{u^n}{\sqrt{a + bu}}\,du = \frac{2}{b}\left[u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\,du\right]$ and

$$\int \frac{u^n}{\sqrt{a + bu}} = \frac{2}{(2n + 1)b}\left[u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\,du\right].$$

**54.** $\displaystyle\int u^n(\cos u)\,du = u^n\sin u - n\int u^{n-1}(\sin u)\,du$

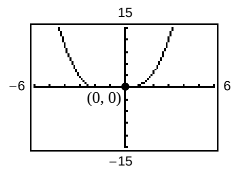
$w = u^n,\ dv = \cos u\,du,\ dw = nu^{n-1}\,du,\ v = \sin u$

**56.** $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - \int n(\ln u)^{n-1}\left(\frac{1}{u}\right)u\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

$w = (\ln u)^n,\ dv = du,\ dw = n(\ln u)^{n-1}\left(\dfrac{1}{u}\right)du,\ v = u$

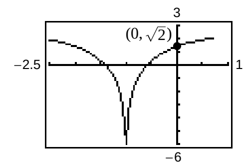
**58.** $\displaystyle\int x\sqrt{x^2+2x}\,dx = \frac{1}{6}\left[2(x^2+2x)^{3/2} - 3(x+1)\sqrt{x^2+2x} + 3\ln\left|x+1+\sqrt{x^2+2x}\right|\right] + C$

$(0, 0)$: $\dfrac{1}{6}[3\ln|1|] + C = 0 \Rightarrow C = 0$



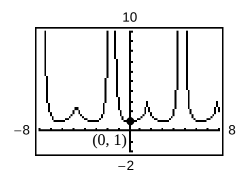

**60.** $\displaystyle\int \frac{\sqrt{2-2x-x^2}}{x+1}\,dx = \sqrt{2-2x-x^2} - \sqrt{3}\ln\left|\frac{\sqrt{3}+\sqrt{2-2x-x^2}}{x+1}\right| + C$

$(0, \sqrt{2})$: $\sqrt{2} - \sqrt{3}\ln\left(\sqrt{3}+\sqrt{2}\right) + C = \sqrt{2} \Rightarrow C = \sqrt{3}\ln\left(\sqrt{3}+\sqrt{2}\right)$




**62.** $\displaystyle\int \frac{\sin\theta}{(\cos\theta)(1+\sin\theta)}\,d\theta = \frac{1}{2}\left[\frac{-\sin\theta}{1+\sin\theta} + \ln\left|\frac{1+\sin\theta}{\cos\theta}\right|\right] + C$

$(0, 1)$: $C = 1 \Rightarrow y = \dfrac{1}{2}\left[\dfrac{-\sin\theta}{1+\sin\theta} + \ln\left|\dfrac{1+\sin\theta}{\cos\theta}\right|\right] + 1$




**64.** $\displaystyle\int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta = -\int \frac{-\sin\theta}{1+(\cos\theta)^2}\,d\theta$

$\quad = -\arctan(\cos\theta) + C$

**66.** $\displaystyle\int_0^{\pi/2} \frac{1}{3-2\cos\theta}\,d\theta = \int_0^1 \left[\frac{\dfrac{2u}{1+u^2}}{3-\dfrac{2(1-u^2)}{1+u^2}}\right]$

$\displaystyle\quad = 2\int_0^1 \frac{1}{5u^2+1}\,du$

$\displaystyle\quad = \left[\frac{2}{\sqrt{5}}\arctan\left(\sqrt{5}\,u\right)\right]_0^1$

$\displaystyle\quad = \frac{2}{\sqrt{5}}\arctan\sqrt{5}$

$u = \tan\dfrac{\theta}{2}$

**68.** $\displaystyle\int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\,d\theta$

$\displaystyle\quad = \int \frac{\cos\theta-\cos^2\theta}{\sin^2\theta}\,d\theta$

$\displaystyle\quad = \int (\csc\theta\cot\theta - \cot^2\theta)\,d\theta$

$\displaystyle\quad = \int (\csc\theta\cot\theta - (\csc^2\theta - 1))\,d\theta$

$\quad = -\csc\theta + \cot\theta + \theta + C$

**70.** $\displaystyle\int \frac{1}{\sec\theta-\tan\theta}\,d\theta = \int \frac{1}{(1/\cos\theta)-(\sin\theta/\cos\theta)}\,d\theta$

$\displaystyle\quad = -\int \frac{-\cos\theta}{1-\sin\theta}\,d\theta$

$\quad = -\ln|1-\sin\theta| + C$

$u = 1 - \sin\theta,\ du = -\cos\theta\,d\theta$

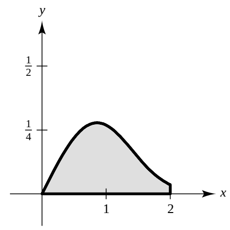
**72.** $A = \int_0^2 \frac{x}{1+e^{x^2}}\,dx$

$= \frac{1}{2}\int_0^2 \frac{2x\,dx}{1+e^{x^2}}$

$= \frac{1}{2}\left[x^2 - \ln(1+e^{x^2})\right]_0^2$

$= \frac{1}{2}\left[4 - \ln(1+e^4)\right] + \frac{1}{2}\ln 2$

$\approx 0.337$ square units




**74.** Log Rule: $\int \frac{1}{u}\,du,\ u = e^x + 1$

**76.** Integration by parts

**78.** Formula 16 with $u = e^{2x}$

**80.** A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formula 50, 54.

**82.** $W = \int_0^5 \frac{500x}{\sqrt{26-x^2}}\,dx$

$= -250\int_0^5 (26-x^2)^{-1/2}(-2x)\,dx$

$= \left[-500\sqrt{26-x^2}\right]_0^5$

$= 500\left(\sqrt{26}-1\right)$

$\approx 2049.51$ ft · lbs

**84.** $\frac{1}{2-0}\int_0^2 \frac{5000}{1+e^{4.8-1.9t}}\,dt = \frac{2500}{-1.9}\int_0^2 \frac{-1.9\,dt}{1+e^{4.8-1.9t}}$

$= -\frac{2500}{1.9}\left[(4.8-1.9t) - \ln(1+e^{4.8-1.9t})\right]_0^2$

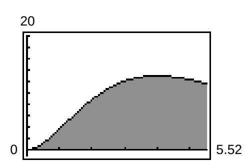
$= -\frac{2500}{1.9}[(1-\ln(1+e)) - (4.8 - \ln(1+e^{4.8}))]$

$= \frac{2500}{1.9}\left[3.8 + \ln\left(\frac{1+e}{1+e^{4.8}}\right)\right] \approx 401.4$

**86.** (a) $\int_0^k 6x^2e^{-x/2}\,dx = 50$

By trial and error, $k = 5.51897$.

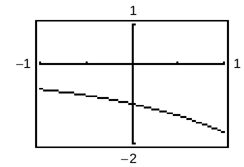
(b) $\int_0^{5.51897} 6x^2e^{-x/2}\,dx$




**88.** True

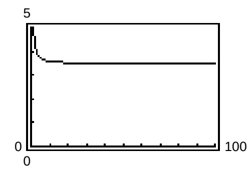
## Section 7.7 Indeterminate Forms and L'Hôpital's Rule

**2.** $\lim_{x\to 0} \frac{1-e^x}{x} \approx -1$




| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.9516$ | $-0.9950$ | $-0.9995$ | $-1.00005$ | $-1.005$ | $-1.0517$ |

**4.** $\lim_{x\to\infty} \frac{6x}{\sqrt{3x^2-2x}} \approx 3.4641 \left(\text{exact: } \frac{6}{\sqrt{3}}\right)$




| $x$ | 1 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 6 | 3.5857 | 3.4757 | 3.4653 | 3.4642 | 3.4641 |

**6.** (a) $\lim_{x\to -1} \frac{2x^2-x-3}{x+1} = \lim_{x\to -1} \frac{(2x-3)(x+1)}{x+1} = \lim_{x\to -1}(2x-3) = -5$

(b) $\lim_{x\to -1} \frac{2x^2-x-3}{x+1} = \lim_{x\to -1} \frac{(d/dx)[2x^2-x-3]}{(d/dx)[x+1]} = \lim_{x\to -1} \frac{4x-1}{1} = -5$

**8.** (a) $\lim_{x\to 0} \frac{\sin 4x}{2x} = \lim_{x\to 0} 2\left(\frac{\sin 4x}{4x}\right) = 2(1) = 2$

(b) $\lim_{x\to 0} \frac{\sin 4x}{2x} = \lim_{x\to 0} \frac{(d/dx)[\sin 4x]}{(d/dx)[2x]} = \lim_{x\to 0} \frac{4\cos 4x}{2} = 2$

**10.** (a) $\lim_{x\to\infty} \frac{2x+1}{4x^2+x} = \lim_{x\to\infty} \frac{(2/x)+(1/x^2)}{4+(1/x)} = \frac{0}{4} = 0$

(b) $\lim_{x\to\infty} \frac{2x+1}{4x^2+x} = \lim_{x\to\infty} \frac{(d/dx)[2x+1]}{(d/dx)[4x^2+x]} = \lim_{x\to\infty} \frac{2}{8x+1} = 0$

**12.** $\lim_{x\to -1} \frac{x^2-x-2}{x+1} = \lim_{x\to -1} \frac{2x-1}{1} = -3$

**14.** $\lim_{x\to 2^-} \frac{\sqrt{4-x^2}}{x-2} = \lim_{x\to 2^-} \frac{-x/\sqrt{4-x^2}}{1}$

$= \lim_{x\to 2^-} \frac{-x}{\sqrt{4-x^2}} = -\infty$

**16.** $\lim_{x\to 0^+} \frac{e^x-(1+x)}{x^3} = \lim_{x\to 0^+} \frac{e^x-1}{3x^2}$

$= \lim_{x\to 0^+} \frac{e^x}{6x} = \infty$

**18.** $\lim_{x\to 1} \frac{\ln x^2}{x^2-1} = \lim_{x\to 1} \frac{2\ln x}{x^2-1}$

$= \lim_{x\to 1} \frac{2/x}{2x}$

$= \lim_{x\to 1} \frac{1}{x^2} = 1$

**20.** $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a}{b}$

**22.** $\lim_{x\to 1} \frac{\arctan x - (\pi/4)}{x-1} = \lim_{x\to 1} \frac{1/(1+x^2)}{1} = \frac{1}{2}$

**24.** $\lim_{x\to\infty} \frac{x-1}{x^2+2x+3} = \lim_{x\to\infty} \frac{1}{2x+2} = 0$

**26.** $\lim_{x\to\infty} \frac{x^3}{x+1} = \lim_{x\to\infty} \frac{3x^2}{1} = \infty$

**28.** $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

**30.** $\displaystyle\lim_{x\to\infty}\frac{x^2}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{x}{\sqrt{1+(1/x)^2}} = \infty$

**32.** $\displaystyle\lim_{x\to\infty}\frac{\sin x}{x-\pi} = 0$

**Note:** Use the Squeeze Theorem for $x > \pi$.

$$-\frac{1}{x-\pi} \le \frac{\sin x}{x-\pi} \le \frac{1}{x-\pi}$$

**34.** $\displaystyle\lim_{x\to\infty}\frac{\ln x^4}{x^3} = \lim_{x\to\infty}\frac{4\ln x}{x^3} = \lim_{x\to\infty}\frac{4/x}{3x^2}$

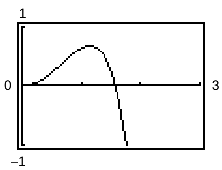
$$= \lim_{x\to\infty}\frac{4}{3x^3} = 0$$

**36.** $\displaystyle\lim_{x\to\infty}\frac{e^{x/2}}{x} = \lim_{x\to\infty}\frac{(1/2)e^{x/2}}{1} = \infty$

**38.** (a) $\displaystyle\lim_{x\to0^+} x^3\cot x = (0)(\infty)$

(b) $\displaystyle\lim_{x\to0^+} x^3\cot x = \lim_{x\to0^+}\frac{x^3}{\tan x} = \lim_{x\to0^+}\frac{3x^2}{\sec^2 x} = 0$

(c)



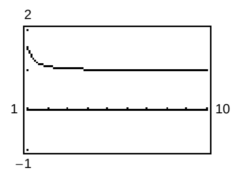

**40.** (a) $\displaystyle\lim_{x\to\infty}\left(x\tan\frac{1}{x}\right) = (\infty)(0)$

(b) $\displaystyle\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x}$

$$= \lim_{x\to\infty}\frac{-(1/x^2)\sec^2(1/x)}{-(1/x^2)}$$

$$= \lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right) = 1$$

(c)




**42.** (a) $\displaystyle\lim_{x\to0^+}(e^x+x)^{2/x} = 1^\infty$

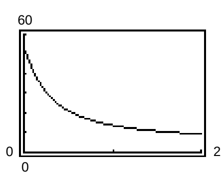
(b) Let $y = \displaystyle\lim_{x\to0^+}(e^x+x)^{2/x}$.

$$\ln y = \lim_{x\to0^+}\frac{2\ln(e^x+x)}{x}$$

$$= \lim_{x\to0^+}\frac{2(e^x+1)/(e^x+x)}{1} = 4$$

Thus, $\ln y = 4 \implies y = e^4 \approx 54.598$.

(c)




**44.** (a) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 1^\infty$

(b) Let $y = \displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

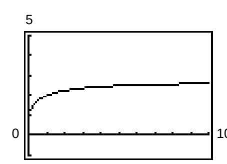
$$\ln y = \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln[1+(1/x)]}{1/x}$$

$$= \lim_{x\to\infty}\frac{\left[\dfrac{(-1/x^2)}{1+(1/x)}\right]}{(-1/x^2)} = \lim_{x\to\infty}\frac{1}{1+(1/x)} = 1$$

Thus, $\ln y = 1 \implies y = e^1 = e$. Therefore,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

(c)



**46.** (a) $\lim_{x \to \infty} (1 + x)^{1/x} = \infty^0$

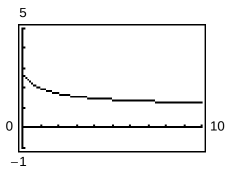
(b) Let $y = \lim_{x \to \infty} (1 + x)^{1/x}$.

$$\ln y = \lim_{x \to \infty} \frac{\ln(1 + x)}{x}$$

$$= \lim_{x \to \infty} \left(\frac{1/(1 + x)}{1}\right) = 0$$

Thus, $\ln y = 0 \implies y = e^0 = 1$.
Therefore, $\lim_{x \to \infty} (1 + x)^{1/x} = 1$.

(c)




**48.** (a) $\lim_{x \to 4^+} [3(x - 4)]^{x-4} = 0^0$

(b) Let $y = \lim_{x \to 4^+} [3(x - 4)]^{x-4}$.

$$\ln y = \lim_{x \to 4^+} (x - 4)\ln[3(x - 4)]$$

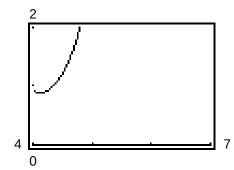
$$= \lim_{x \to 4^+} \frac{\ln[3(x - 4)]}{1/(x - 4)}$$

$$= \lim_{x \to 4^+} \frac{1/(x - 4)}{-1/(x - 4)^2}$$

$$= \lim_{x \to 4^+} [-(x - 4)] = 0$$

Hence, $\lim_{x \to 4^+} [3(x - 4)]^{x-4} = 1$.

(c)




**50.** (a) $\lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 0^0$

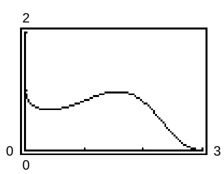
(b) Let $y = \lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x$.

$$\ln y = \lim_{x \to 0^+} x \ln\left[\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= 0 \cdot 0 = 0$$

Hence, $\lim_{x \to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 1$.

(c)



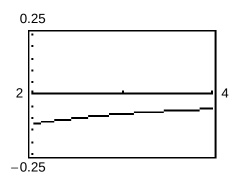

**52.** (a) $\lim_{x \to 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right) = \infty - \infty$

(b) $\lim_{x \to 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right) = \lim_{x \to 2^+} \frac{1 - \sqrt{x - 1}}{x^2 - 4}$

$$= \lim_{x \to 2^+} \frac{-1/\left(2\sqrt{x - 1}\right)}{2x}$$

$$= \lim_{x \to 2^+} \frac{-1}{4x\sqrt{x - 1}} = \frac{-1}{8}$$

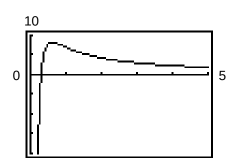
(c)



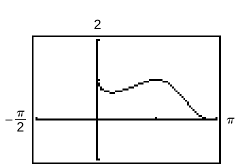

**54.** (a) $\lim_{x \to 0^+} \left(\frac{10}{x} - \frac{3}{x^2}\right) = \infty - \infty$

(b) $\lim_{x \to 0^+} \left(\frac{10}{x} - \frac{3}{x^2}\right) = \lim_{x \to 0^+} \left(\frac{10x - 3}{x^2}\right) = -\infty$

(c)




**56.** (a)

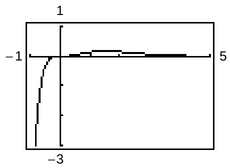



(b) Let $y = (\sin x)^x$, then $\ln y = x \ln(\sin x)$.

$$\lim_{x \to 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x \to 0^+} \frac{\cos x/\sin x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{\tan x} = \lim_{x \to 0^+} \frac{-2x}{\sec^2 x} = 0$$

Therefore, since $\ln y = 0$, $y = 1$ and $\lim_{x \to 0^+} (\sin x)^x = 1$.

**58.** (a)




(b) $\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$

**60.** See Theorem 7.4.

**62.** Let $f(x) = x + 25$ and $g(x) = x$.

**64.** $\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$

**66.**
$$\lim_{x\to\infty} \frac{(\ln x)^2}{x^3} = \lim_{x\to\infty} \frac{(2\ln x)/x}{3x^2}$$
$$= \lim_{x\to\infty} \frac{2\ln x}{3x^3}$$
$$= \lim_{x\to\infty} \frac{2/x}{9x^2} = \lim_{x\to\infty} \frac{2}{9x^3} = 0$$

**68.**
$$\lim_{x\to\infty} \frac{x^m}{e^{nx}} = \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}$$
$$= \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}}$$
$$= \cdots = \lim_{x\to\infty} \frac{m!}{n^m e^{nx}} = 0$$

**70.**

| $x$ | 1 | 5 | 10 | 20 | 30 | 40 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|
| $\frac{e^x}{x^5}$ | 2.718 | 0.047 | 0.220 | 151.614 | $4.40 \times 10^5$ | $2.30 \times 10^9$ | $1.66 \times 10^{13}$ | $2.69 \times 10^{33}$ |

**72.** $y = x^x, x > 0$

$\lim_{x\to\infty} x^x = \infty$ and $\lim_{x\to 0^+} x^x = 1$

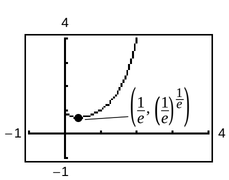
No horizontal asymptotes

$$\ln y = x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = x^x(1 + \ln x) = 0$$

Critical number: $x = e^{-1}$

| | | |
|---|---|---|
| Intervals: | $(0, e^{-1})$ | $(e^{-1}, 0)$ |
| Sign of $dy/dx$: | $-$ | $+$ |
| $y = f(x)$: | Decreasing | Increasing |

Relative minimum: $(e^{-1}, (e^{-1})^{e^{-1}}) = \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$




**74.** $y = \frac{\ln x}{x}$

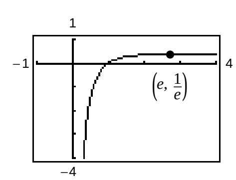
Horizontal asymptote: $y = 0$ (See Exercise 29)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

Critical number: $x = e$

| | | |
|---|---|---|
| Intervals: | $(0, e)$ | $(e, \infty)$ |
| Sign of $dy/dx$: | $+$ | $-$ |
| $y = f(x)$: | Increasing | Decreasing |

Relative maximum: $\left(e, \frac{1}{e}\right)$




**76.** $\lim_{x\to\infty} \frac{\sin \pi x - 1}{x} = 0$ (Numerator is bounded)

Limit is not of the form $0/0$ or $\infty/\infty$.
L'Hôpital's Rule does not apply.

**78.** $\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

Limit is not of the form $0/0$ or $\infty/\infty$.
L'Hôpital's Rule does not apply.

**80.** $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$\ln A = \ln P + nt \ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}\right] = \lim_{n\to\infty}\left[\frac{-\frac{r}{n^2}\left(\frac{1}{1 + (r/n)}\right)}{-\left(\frac{1}{n^2 t}\right)}\right] = \lim_{n\to\infty}\left[rt\left(\frac{1}{1 + \frac{r}{n}}\right)\right] = rt$$

Since $\lim_{n\to\infty} \ln A = \ln P + rt$, we have $\lim_{n\to\infty} A = e^{(\ln P + rt)} = e^{\ln P}e^{rt} = Pe^{rt}$. Alternatively,

$$\lim_{n\to\infty} A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = Pe^{rt}.$$

**82.** Let $N$ be a fixed value for $n$. Then

$$\lim_{x\to\infty} \frac{x^{N-1}}{e^x} = \lim_{x\to\infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x\to\infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \ldots = \lim_{x\to\infty}\left[\frac{(N-1)!}{e^x}\right] = 0. \qquad \text{(See Exercise 68)}$$

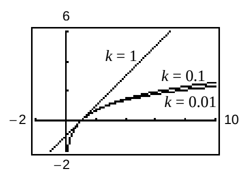
**84.** $f(x) = \dfrac{x^k - 1}{k}$

$k = 1, \quad f(x) = x - 1$

$k = 0.1, \quad f(x) = \dfrac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$

$k = 0.01, \quad f(x) = \dfrac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$

$$\lim_{k\to 0^+} \frac{x^k - 1}{k} = \lim_{k\to 0^+} \frac{x^k(\ln x)}{1} = \ln x$$




**86.** $f(x) = \dfrac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{-1/2}{3} = \frac{-1/c^2}{2c}$$

$$-\frac{1}{6} = -\frac{1}{2c^3}$$

$$2c^3 = 6$$

$$c = \sqrt[3]{3}$$

**88.** $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\frac{f(4) - f(1)}{g(4) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\ln 4}{63} = \frac{1/c}{3c^2} = \frac{1}{3c^3}$$

$$3c^3 \ln 4 = 63$$

$$c^3 = \frac{21}{\ln 4}$$

$$c = \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474$$

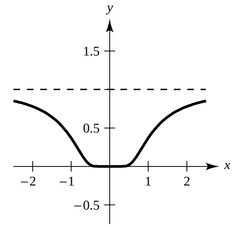
**90.** False. If $y = e^x/x^2$, then

$$y' = \frac{x^2e^x - 2xe^x}{x^4} = \frac{xe^x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}.$$

**92.** False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$$\lim_{x\to\infty} \frac{x}{x+1} = 1, \text{ but } \lim_{x\to\infty} [x - (x+1)] = -1.$$

**94.** $g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$




$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x}$$

Let $y = \dfrac{e^{-1/x^2}}{x}$, then $\ln y = \ln\left(\dfrac{e^{-1/x^2}}{x}\right) = -\dfrac{1}{x^2} - \ln x = \dfrac{-1 - x^2 \ln x}{x^2}$. Since

$$\lim_{x \to 0} x^2 \ln x = \lim_{x \to 0} \frac{\ln x}{1/x^2} = \lim_{x \to 0} \frac{1/x}{-2/x^3} = \lim_{x \to 0} \left(-\frac{x^2}{2}\right) = 0$$

we have $\displaystyle\lim_{x \to 0} \left(\frac{-1 - x^2 \ln x}{x^2}\right) = -\infty$. Thus, $\displaystyle\lim_{x \to 0} y = e^{-\infty} = 0 \implies g'(0) = 0$.

**Note:** The graph appears to support this conclusion—the tangent line is horizontal at $(0, 0)$.

**96.** $\displaystyle\lim_{x \to a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \to a} g(x) \ln f(x) = (-\infty)(-\infty) = \infty$$

As $x \to a$, $\ln y \implies \infty$, and hence $y = \infty$. Thus,

$$\lim_{x \to a} f(x)^{g(x)} = \infty.$$

**98.** $\displaystyle\lim_{x \to 0^+} x^{\ln 2/(1 + \ln x)}$

Let $y = x^{\ln 2/(1 + \ln x)}$, then:

$$\ln y = \frac{\ln 2}{1 + \ln x} \cdot \ln x = \frac{(\ln 2)(\ln x)}{1 + \ln x}$$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{(\ln 2)(\ln x)}{1 + \ln x} = \lim_{x \to 0^+} \frac{(\ln 2)/x}{1/x}$$

$$= \lim_{x \to 0^+} (\ln 2) = \ln 2$$

Thus, $\displaystyle\lim_{x \to \infty} y = e^{\ln 2} = 2$.

## Section 7.8 Improper Integrals

**2.** Infinite discontinuity at $x = 3$.

$$\begin{aligned} \int_3^4 \frac{1}{(x - 3)^{3/2}}\,dx &= \lim_{b \to 3^+} \int_b^4 (x - 3)^{-3/2}\,dx \\ &= \lim_{b \to 3^+} \Big[-2(x - 3)^{-1/2}\Big]_b^4 \\ &= \lim_{b \to 3^+} \left[-2 + \frac{2}{\sqrt{b - 3}}\right] = \infty \end{aligned}$$

Diverges

**4.** Infinite discontinuity at $x = 1$.

$$\begin{aligned} \int_0^2 \frac{1}{(x - 1)^{2/3}}\,dx &= \int_0^1 \frac{1}{(x - 1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x - 1)^{2/3}}\,dx \\ &= \lim_{b \to 1^-} \int_0^b \frac{1}{(x - 1)^{2/3}}\,dx + \lim_{c \to 1^+} \int_c^2 \frac{1}{(x - 1)^{2/3}}\,dx \\ &= \lim_{b \to 1^-} \Big[3\sqrt[3]{x - 1}\Big]_0^b + \lim_{c \to 1^+} \Big[3\sqrt[3]{x - 1}\Big]_c^2 = (0 + 3) + (3 - 0) = 6 \end{aligned}$$

Converges

**6.** Infinite limit of integration.

$$\int_{-\infty}^{0} e^{2x}\,dx = \lim_{b\to-\infty}\int_{b}^{0} e^{2x}\,dx = \lim_{b\to-\infty}\left[\frac{1}{2}e^{2x}\right]_{b}^{0} = \frac{1}{2} - 0 = \frac{1}{2}$$

Converges

**8.** $\int_{0}^{\infty} e^{-x}\,dx \neq 0$. You need to evaluate the limit.

$$\lim_{b\to\infty}\int_{0}^{b} e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_{0}^{b} = \lim_{b\to\infty}\left[-e^{-b} + 1\right] = 1$$

**10.** $\displaystyle\int_{1}^{\infty}\frac{5}{x^3}\,dx = \lim_{b\to\infty}\int_{1}^{b}\frac{5}{x^3}\,dx = \lim_{b\to\infty}\left[-\frac{5}{2}x^{-2}\right]_{1}^{b} = \frac{5}{2}$

**12.** $\displaystyle\int_{1}^{\infty}\frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\int_{1}^{b}4x^{-1/4}\,dx = \lim_{b\to\infty}\left[\frac{16}{3}x^{3/4}\right]_{1}^{b} = \infty$ Diverges

**14.** $\displaystyle\int_{0}^{\infty} xe^{-x/2}\,dx = \lim_{b\to\infty}\int_{0}^{b} xe^{-x/2}\,dx = \lim_{b\to\infty}\left[e^{-x/2}(-2x - 4)\right]_{0}^{b} = \lim_{b\to\infty} e^{-b/2}(-2b - 4) + 4 = 4$

**16.** $\displaystyle\int_{0}^{\infty}(x-1)e^{-x}\,dx = \lim_{b\to\infty}\int_{0}^{b}(x-1)e^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x}\right]_{0}^{b} = \lim_{b\to\infty}\left(\frac{-b}{e^b} + 0\right) = 0$ by L'Hôpital's Rule.

**18.** $$\int_{0}^{\infty} e^{-ax}\sin bx\,dx = \lim_{c\to\infty}\left[\frac{e^{-ax}(-a\sin bx - b\cos bx)}{a^2+b^2}\right]_{0}^{c} = 0 - \frac{-b}{a^2+b^2} = \frac{b}{a^2+b^2}$$

**20.** $$\int_{1}^{\infty}\frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_{1}^{b}\frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left[\frac{(\ln x)^2}{2}\right]_{1}^{b} = \infty \quad \text{Diverges}$$

**22.** $$\int_{0}^{\infty}\frac{x^3}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_{0}^{b}\frac{x}{x^2+1}\,dx - \lim_{b\to\infty}\int_{0}^{b}\frac{x}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)}\right]_{0}^{b} = \infty - \frac{1}{2}$$

Diverges

**24.** $\displaystyle\int_{0}^{\infty}\frac{e^x}{1+e^x}\,dx = \lim_{b\to\infty}\left[\ln(1+e^x)\right]_{0}^{b} = \infty - \ln 2$

Diverges

**26.** $\displaystyle\int_{0}^{\infty}\sin\frac{x}{2}\,dx = \lim_{b\to\infty}\left[-2\cos\frac{x}{2}\right]_{0}^{b}$

Diverges since $\cos\frac{x}{2}$ does not approach a limit as $x\to\infty$.

**28.** $\displaystyle\int_{0}^{4}\frac{8}{x}\,dx = \lim_{b\to0^+}\int_{b}^{4}\frac{8}{x}\,dx = \lim_{b\to0^+}\left[8\ln x\right]_{b}^{4} = \infty$

Diverges

**30.** $$\int_{0}^{6}\frac{4}{\sqrt{6-x}}\,dx = \lim_{b\to6^-}\int_{0}^{b}4(6-x)^{-1/2}\,dx = \lim_{b\to6^-}\left[-8(6-x)^{1/2}\right]_{0}^{b} = -8(0) + 8\sqrt{6} = 8\sqrt{6}$$

**32.** $$\int_{0}^{e}\ln x^2\,dx = \lim_{b\to0^+}\int_{0}^{e}2\ln x\,dx = \lim_{b\to0^+}\left[2x\ln x - 2x\right]_{b}^{e} = \lim_{b\to0^+}[(2e - 2e) - (2b\ln b - 2b)] = 0$$

**34.** $\displaystyle\int_{0}^{\pi/2}\sec\theta\,d\theta = \lim_{b\to(\pi/2)}\left[\ln|\sec\theta + \tan\theta|\right]_{0}^{b} = \infty,$

Diverges

**36.** $\displaystyle\int_{0}^{2}\frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to2^-}\left[\arcsin\left(\frac{x}{2}\right)\right]_{0}^{b} = \frac{\pi}{2}$

**38.** $\displaystyle\int_0^2 \frac{1}{4-x^2}\,dx = \lim_{b\to 2^-}\int_0^b \frac{1}{4}\left(\frac{1}{2+x}+\frac{1}{2-x}\right)dx = \lim_{b\to 2^-}\left[\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|\right]_0^b = \infty - 0$

Diverges

**40.** $$\int_1^3 \frac{2}{(x-2)^{8/3}}\,dx = \int_1^2 2(x-2)^{-8/3}\,dx + \int_2^3 2(x-2)^{-8/3}\,dx$$
$$= \lim_{b\to 2^-}\int_1^b 2(x-2)^{-8/3}\,dx + \lim_{c\to 2^+}\int_c^3 2(x-2)^{-8/3}\,dx$$
$$= \lim_{b\to 2^-}\left[-\frac{6}{5}(x-2)^{-5/3}\right]_1^b + \lim_{c\to 2^+}\left[-\frac{6}{5}(x-2)^{-5/3}\right]_c^3 = \infty$$

Diverges

**42.** $\displaystyle\int \frac{1}{x\ln x}\,dx = \ln|\ln|x|| + C$

Thus,

$$\int_1^\infty \frac{1}{x\ln x}\,dx = \int_1^e \frac{1}{x\ln x}\,dx + \int_e^\infty \frac{1}{x\ln x}\,dx$$
$$= \lim_{b\to 1^+}\Big[\ln(\ln x)\Big]_1^e + \lim_{c\to\infty}\Big[\ln(\ln x)\Big]_e^\infty.$$

Diverges

**44.** If $p = 1$, $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\Big[\ln x\Big]_a^1 = \lim_{a\to 0^+} -\ln a = \infty.$

Diverges. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_a^1 = \lim_{a\to 0^+}\left[\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right].$$

This converges to $\dfrac{1}{1-p}$ if $1-p > 0$ or $p < 1$.

**46.** (a) Assume $\displaystyle\int_a^\infty g(x)\,dx = L$ (converges).

Since $0 \le f(x) \le g(x)$ on $[a, \infty)$, $\displaystyle 0 \le \int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx = L$ and $\displaystyle\int_a^\infty f(x)\,dx$ converges.

(b) $\displaystyle\int_a^\infty g(x)\,dx$ diverges, because otherwise, by part (a), if $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.

**48.** $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \frac{1}{1-(1/3)} = \frac{3}{2}$ converges.

(See Exercise 44, $p = \frac{1}{3}$.)

**50.** $\displaystyle\int_0^\infty x^4 e^{-x}\,dx$ converges.

(See Exercise 45.)

**52.** Since $\dfrac{1}{\sqrt{x-1}} \ge \dfrac{1}{x}$ on $[2, \infty)$ and $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges by Exercise 43, $\displaystyle\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$ diverges.

**54.** Since $\dfrac{1}{\sqrt{x}(1+x)} \le \dfrac{1}{x^{3/2}}$ on $[1, \infty)$ and $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges by Exercise 43, $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$ converges.

**56.** $\dfrac{1}{\sqrt{x}\ln x} \ge \dfrac{1}{x}$ since $\sqrt{x}\ln x < x$ on $[2, \infty)$. Since $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges by Exercise 43, $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}\ln x}\,dx$ diverges.

**58.** See the definitions, pages 540, 543.

**60.** Answers will vary.

(a) $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

Converges (Example 4)

(b) $\displaystyle\int_{-\infty}^\infty x\,dx$

Diverges

**62.** $f(t) = t$

$$F(s) = \int_0^\infty te^{-st}\,dt = \lim_{b\to\infty}\left[\frac{1}{s^2}(-st-1)e^{-st}\right]_0^b$$
$$= \frac{1}{s^2},\ s > 0$$

**64.** $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{t(a-s)}\,dt$$
$$= \lim_{b\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^b$$
$$= 0 - \frac{1}{a-s} = \frac{1}{s-a},\ s > a$$

**66.** $f(t) = \sin at$

$$F(s) = \int_0^\infty e^{-st}\sin at\,dt$$
$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^b$$
$$= 0 + \frac{a}{s^2+a^2} = \frac{a}{s^2+a^2},\ s > 0$$

**68.** $f(t) = \sinh at$

$$F(s) = \int_0^\infty e^{-st}\sinh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt = \frac{1}{2}\int_0^\infty \left[e^{t(-s+a)} - e^{t(-s-a)}\right]dt$$
$$= \lim_{b\to\infty}\frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} - \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right]$$
$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right] = \frac{a}{s^2-a^2},\ s > |a|$$

**70.** (a) $A = \displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = 1$

(b) **Disk:**

$$V = \pi\int_1^\infty \frac{1}{x^4}\,dx = \lim_{b\to\infty}\left[-\frac{\pi}{3x^3}\right]_1^b = \frac{\pi}{3}$$

(c) **Shell:**

$$V = 2\pi\int_1^\infty x\left(\frac{1}{x^2}\right)dx = \lim_{b\to\infty}\left[2\pi(\ln x)\right]_1^b = \infty$$

Diverges

**72.** $(x-2)^2 + y^2 = 1$

$2(x-2) + 2yy' = 0$

$$y' = \frac{-(x-2)}{y}$$

$$\sqrt{1+(y')^2} = \sqrt{1+[(x-2)^2/y^2]} = \frac{1}{y}\ \text{(Assume } y > 0.)$$

$$S = 4\pi\int_1^3 \frac{x}{y}\,dx = 4\pi\int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\,dx = 4\pi\int_1^3\left[\frac{x-2}{\sqrt{1-(x-2)^2}} + \frac{2}{\sqrt{1-(x-2)^2}}\right]dx$$
$$= \lim_{\substack{a\to1^+\\ b\to3^-}}\left\{4\pi\left[-\sqrt{1-(x-2)^2} + 2\arcsin(x-2)\right]_a^b\right\} = 4\pi[0 + 2\arcsin(1) - 2\arcsin(-1)] = 8\pi^2$$

**74.** (a) $F(x) = \dfrac{K}{x^2}$, $5 = \dfrac{K}{(4000)^2}$, $K = 80{,}000{,}000$

$$W = \int_{4000}^\infty \frac{80{,}000{,}000}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-80{,}000{,}000}{x}\right]_{4000}^b = 20{,}000 \text{ mi-ton}$$

(b) $$\frac{W}{2} = 10{,}000 = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^b = \frac{-80{,}000{,}000}{b} + 20{,}000$$

$$\frac{80{,}000{,}000}{b} = 10{,}000$$

$$b = 8000$$

Therefore, 4000 miles *above* the earth's surface.

**76.** (a) $\displaystyle\int_{-\infty}^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \int_{0}^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \lim_{b\to\infty}\left[-e^{-2t5}\right]_0^b = 1$

(b) $\displaystyle\int_0^4 \frac{2}{5}e^{-2t/5}\,dt = \left[-e^{-2t/5}\right]_0^4 = -e^{-8/5} + 1$

$\approx 0.7981 = 79.81\%$

(c) $\displaystyle\int_0^{\infty} t\left[\frac{2}{5}e^{-2t/5}\right]dt = \lim_{b\to\infty}\left[-te^{2t/5} - \frac{5}{2}e^{-2t/5}\right]_0^b = \frac{5}{2}$

**78.** (a) $\displaystyle C = 650{,}000 + \int_0^5 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$\displaystyle = 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^5 \approx \$778{,}512.58$

(b) $\displaystyle C = 650{,}000 + \int_0^{10} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$\displaystyle = 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^{10} \approx \$905{,}718.14$

(c) $\displaystyle C = 650{,}000 + \int_0^{\infty} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

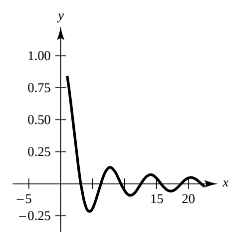
$\displaystyle = 650{,}000 + 25{,}000\lim_{b\to\infty}\left[-\frac{t}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^b \approx \$1{,}622{,}222.22$

**80.** (a) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \infty$

$\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1$

$\displaystyle\int_1^{\infty} \frac{1}{x^n}\,dx$ will converge if $n > 1$ and will diverge if $n \le 1$.

(b) It would appear to converge.




(c) Let $dv = \sin x\,dx \implies v = -\cos x$

$\displaystyle u = \frac{1}{x} \implies du = -\frac{1}{x^2}\,dx$

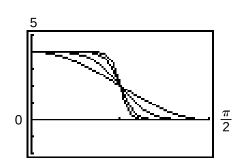
$\displaystyle\int_1^{\infty} \frac{\sin x}{x}\,dx = \lim_{b\to 0}\left[-\frac{\cos x}{x}\right]_1^b - \int_1^{\infty}\frac{\cos x}{x^2}\,dx$

$\displaystyle = \cos 1 - \int_1^{\infty}\frac{\cos x}{x^2}\,dx$

Converges

**82.** (a) Yes, the integral is not defined at $x = \pi/2$.

(b)




(c) As $n \to \infty$, the integral approaches $4(\pi/4) = \pi$.

(d) $\displaystyle I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n}\,dx$

$I_2 \approx 3.14159$

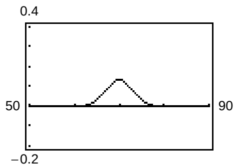
$I_4 \approx 3.14159$

$I_8 \approx 3.14159$

$I_{12} \approx 3.14159$

**84.** (a) $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$\int_{50}^{90} f(x)\,dx \approx 1.0$




(b) $P(72 \le x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \le x \le 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$P(70 \le x < \infty) = 0.5$

and

$0.5 = P(70 \le x < \infty)$

$= P(70 \le x \le 72) + P(72 \le x < \infty).$

**86.** False. This is equivalent to Exercise 85.

**88.** True

## Review Exercises for Chapter 7

**2.** $\int xe^{x^2-1}\,dx = \frac{1}{2}\int e^{x^2-1}(2x)\,dx$

$= \frac{1}{2}e^{x^2-1} + C$

**4.** $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$

$= -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C$

$= -\sqrt{1-x^2} + C$

**6.** $\int 2x\sqrt{2x-3}\,dx = \int (u^4 + 3u^2)\,du = \frac{u^5}{5} + u^3 + C$

$= \frac{2(2x-3)^{3/2}}{5}(x+1) + C$

$u = \sqrt{2x-3},\ x = \frac{u^2+3}{2},\ dx = u\,du$

**8.** $\frac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} = 1 + \frac{x}{(x^2+1)^2}$

$\int \frac{x^4 + 2x^2 + x + 1}{(x^2+1)^2}\,dx = \int dx + \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx$

$= x - \frac{1}{2(x^2+1)} + C$

**10.** $\int (x^2-1)e^x\,dx = (x^2-1)e^x - 2\int xe^x\,dx = (x^2-1)e^x - 2xe^x + 2\int e^x\,dx = e^x(x^2-2x+1) + 1$

(1) $dv = e^x\,dx \implies v = e^x$

$u = x^2 - 1 \implies du = 2x\,dx$

(2) $dv = e^x\,dx \implies v = e^x$

$u = x \implies du = dx$

**12.** $u = \arctan 2x,\ du = \frac{2}{1+4x^2}\,dx,\ dv = dx, v = x$

$\int \arctan 2x\,dx = x\arctan 2x - \int \frac{2x}{1+4x^2}\,dx$

$= x\arctan 2x - \frac{1}{4}\ln(1+4x^2) + C$

**14.** $\int \ln\sqrt{x^2-1}\,dx = \frac{1}{2}\int \ln(x^2-1)\,dx$

$= \frac{1}{2}x\ln|x^2-1| - \int \frac{x^2}{x^2-1}\,dx$

$= \frac{1}{2}x\ln|x^2-1| - \int dx - \int \frac{1}{x^2-1}\,dx$

$= \frac{1}{2}x\ln|x^2-1| - x - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

$dv = dx \implies v = x$

$u = \ln(x^2-1) \implies du = \frac{2x}{x^2-1}\,dx$

**16.** $\displaystyle\int e^x \arctan(e^x)\,dx = e^x \arctan(e^x) - \int \frac{e^{2x}}{1+e^{2x}}\,dx$

$\displaystyle = e^x \arctan(e^x) - \frac{1}{2}\ln(1+e^{2x}) + C$

$dv = e^x\,dx \implies v = e^x$

$\displaystyle u = \arctan e^x \implies du = \frac{e^x}{1+e^{2x}}\,dx$

**18.** $\displaystyle\int \sin^2 \frac{\pi x}{2}\,dx = \int \frac{1}{2}(1-\cos \pi x)\,dx = \frac{1}{2}\left[x - \frac{1}{\pi}\sin \pi x\right] + C = \frac{1}{2\pi}[\pi x - \sin \pi x] + C$

**20.** $\displaystyle\int \tan\theta \sec^4\theta\,d\theta = \int (\tan^3\theta + \tan\theta)\sec^2\theta\,d\theta = \frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta + C_1$

or

$\displaystyle\int \tan\theta \sec^4\theta\,d\theta = \int \sec^3\theta(\sec\theta\tan\theta)\,d\theta = \frac{1}{4}\sec^4\theta + C_2$

**22.** $\displaystyle\int \cos 2\theta(\sin\theta + \cos\theta)^2\,d\theta = \int (\cos^2\theta - \sin^2\theta)(\sin\theta+\cos\theta)^2\,d\theta$

$\displaystyle = \int (\sin\theta+\cos\theta)^3(\cos\theta - \sin\theta)\,d\theta = \frac{1}{4}(\sin\theta + \cos\theta)^4 + C$

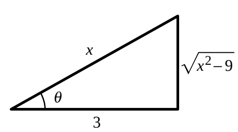
**24.** $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta)$

$\displaystyle = 3\int \tan^2\theta\,d\theta$

$\displaystyle = 3\int (\sec^2\theta - 1)\,d\theta$

$= 3(\tan\theta - \theta) + C$

$\displaystyle = \sqrt{x^2-9} - 3\,\mathrm{arcsec}\left(\frac{x}{3}\right) + C$




$x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-9} = 3\tan\theta$

**26.** $\displaystyle\int \sqrt{9-4x^2}\,dx = \frac{1}{2}\int \sqrt{9-(2x)^2}\,(2)\,dx$

$\displaystyle = \frac{1}{2}\cdot\frac{1}{2}\left[9\arcsin\frac{2x}{3} + 2x\sqrt{9-4x^2}\right] + C$

$\displaystyle = \frac{9}{4}\arcsin\frac{2x}{3} + \frac{x}{2}\sqrt{9-4x^2} + C$

**28.** $\displaystyle\int \frac{\sin\theta}{1+2\cos^2\theta}\,d\theta = \frac{-1}{\sqrt{2}}\int \frac{1}{1+2\cos^2\theta}\left(-\sqrt{2}\sin\theta\right)d\theta$

$\displaystyle = \frac{-1}{\sqrt{2}}\arctan\left(\sqrt{2}\cos\theta\right) + C$

$u = \sqrt{2}\cos\theta,\ du = -\sqrt{2}\sin\theta\,d\theta$

**30.** (a) $$\int x\sqrt{4+x}\,dx = 64\int \tan^3\theta\sec^3\theta\,d\theta$$
$$= 64\int(\sec^4\theta - \sec^2\theta)\sec\theta\tan\theta\,d\theta$$
$$= \frac{64\sec^3\theta}{15}(3\sec^3\theta - 5) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$x = 4\tan^2\theta$, $dx = 8\tan\theta\sec^2\theta\,d\theta$,

$\sqrt{4+x} = 2\sec\theta$

(b) $$\int x\sqrt{4+x}\,dx = 2\int(u^4 - 4u^2)\,du$$
$$= \frac{2u^3}{15}(3u^2-20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u^2 = 4+x$, $dx = 2u\,du$

(c) $$\int x\sqrt{4+x}\,dx = \int(u^{3/2} - 4u^{1/2})\,du$$
$$= \frac{2u^{3/2}}{15}(3u-20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u = 4+x$, $du = dx$

(d) $$\int x\sqrt{4+x}\,dx = \frac{2x}{3}(4+x)^{3/2} - \frac{2}{3}\int(4+x)^{3/2}\,dx$$
$$= \frac{2x}{3}(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$dv = \sqrt{4+x}\,dx \implies v = \frac{2}{3}(4+x)^{3/2}$

$u = x \implies du = dx$

**32.** $$\frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} = 2x - 3 + \frac{4}{x} - \frac{3}{x-1}$$

$$\int\frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x}\,dx = \int\left(2x - 3 + \frac{4}{x} - \frac{3}{x-1}\right)dx = x^2 - 3x + 4\ln|x| - 3\ln|x-1| + C$$

**34.** $$\frac{4x-2}{3(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$4x - 2 = 3A(x-1) + 3B$$

Let $x = 1$: $2 = 3B \implies B = \frac{2}{3}$

Let $x = 2$: $6 = 3A + 3B \implies A = \frac{4}{3}$

$$\int\frac{4x-2}{3(x-1)^2}\,dx = \frac{4}{3}\int\frac{1}{x-1}\,dx + \frac{2}{3}\int\frac{1}{(x-1)^2}\,dx = \frac{4}{3}\ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3}\left(2\ln|x-1| - \frac{1}{x-1}\right) + C$$

**36.** $$\int\frac{\sec^2\theta}{\tan\theta(\tan\theta - 1)}\,d\theta = \int\frac{1}{u(u-1)}\,du = \int\frac{1}{u-1}\,du - \int\frac{1}{u}\,du$$
$$= \ln|u-1| - \ln|u| + C = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C = \ln|1 - \cot\theta| + C$$

$u = \tan\theta$, $du = \sec^2\theta\,d\theta$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

Let $u = 0$: $1 = -A \implies A = -1$

Let $u = 1$: $1 = B$

**38.** $\displaystyle\int \frac{x}{\sqrt{2+3x}}\,dx = \frac{-2(4-3x)}{27}\sqrt{2+3x} + C$ (Formula 21)

$$= \frac{6x-8}{27}\sqrt{2+3x} + C$$

**40.** $\displaystyle\int \frac{x}{1+e^{x^2}}\,dx = \frac{1}{2}\int \frac{1}{1+e^u}\,du$ $\quad (u = x^2)$

$$= \frac{1}{2}\left[u - \ln(1+e^u)\right] + C \quad \text{(Formula 84)}$$

$$= \frac{1}{2}\left[x^2 - \ln(1+e^{x^2})\right] + C$$

**42.** $\displaystyle\int \frac{3}{2x\sqrt{9x^2-1}}\,dx = \frac{3}{2}\int \frac{1}{3x\sqrt{(3x)^2-1}}\,3\,dx$ $\quad (u = 3x)$

$$= \frac{3}{2}\operatorname{arcsec}|3x| + C \quad \text{(Formula 33)}$$

**44.** $\displaystyle\int \frac{1}{1+\tan \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{1+\tan \pi x}(\pi)dx$ $\quad (u = \pi x)$

$$= \frac{1}{\pi}\frac{1}{2}\left[\pi x + \ln|\cos \pi x + \sin \pi x|\right] + C \quad \text{(Formula 71)}$$

**46.** $\displaystyle\int \tan^n x\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

**48.** $\displaystyle\int \frac{\csc\sqrt{2x}}{\sqrt{x}}\,dx = \sqrt{2}\int \csc\sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)dx$

$$= -\sqrt{2}\ln\left|\csc\sqrt{2x} + \cot\sqrt{2x}\right| + C$$

$u = \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx$

**50.** $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx = \int u(4u^3 - 4u)\,du = \int (4u^4 - 4u^2)\,du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15}\left(1+\sqrt{x}\right)^{3/2}\left(3\sqrt{x} - 2\right) + C$

$u = \sqrt{1+\sqrt{x}},\ x = u^4 - 2u^2 + 1,\ dx = (4u^3 - 4u)\,du$

**52.** $\dfrac{3x^3+4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$3x^3 + 4x = (Ax+B)(x^2+1) + Cx + D$

$\qquad = Ax^3 + Bx^2 + (A+C)x + (B+D)$

$A = 3, B = 0, A + C = 4 \implies C = 1,$

$B + D = 0 \implies D = 0$

$$\int \frac{3x^3+4x}{(x^2+1)^2}\,dx = 3\int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \frac{3}{2}\ln(x^2+1) - \frac{1}{2(x^2+1)} + C$$

**54.** $\displaystyle\int (\sin\theta + \cos\theta)^2\,d\theta = \int (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\,d\theta$

$$= \int (1 + \sin 2\theta)\,d\theta = \theta - \frac{1}{2}\cos 2\theta + C = \frac{1}{2}(2\theta - \cos 2\theta) + C$$

**56.** $y = \displaystyle\int \frac{\sqrt{4 - x^2}}{2x}\,dx = \int \frac{2\cos\theta(2\cos\theta)\,d\theta}{4\sin\theta}$

$$= \int (\csc\theta - \sin\theta)\,d\theta$$

$$= \left[-\ln\left|\csc\theta + \cos\theta\right| + \cos\theta\right] + C$$

$$= -\ln\left|\frac{2 + \sqrt{4 - x^2}}{x}\right| + \frac{\sqrt{4 - x^2}}{2} + C$$

$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4 - x^2} = 2\cos\theta$

**58.** $y = \displaystyle\int \sqrt{1 - \cos\theta}\,d\theta = \int \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\,d\theta = -\int (1 + \cos\theta)^{-1/2}(-\sin\theta)d\theta = -2\sqrt{1 + \cos\theta} + C$

$u = 1 + \cos\theta,\ du = -\sin\theta\,d\theta$

**60.** $\displaystyle\int_0^1 \frac{x}{(x - 2)(x - 4)}\,dx = \Big[2\ln|x - 4| - \ln|x - 2|\Big]_0^1$

$$= 2\ln 3 - 2\ln 4 + \ln 2$$

$$= \ln\frac{9}{8} \approx 0.118$$

**62.** $\displaystyle\int_0^2 xe^{3x}\,dx = \left[\frac{e^{3x}}{9}(3x - 1)\right]_0^2 = \frac{1}{9}(5e^6 + 1) \approx 224.238$

**64.** $\displaystyle\int_0^3 \frac{x}{\sqrt{1 + x}}\,dx = \left[\frac{-2(2 - x)}{3}\sqrt{1 + x}\right]_0^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

**66.** $A = \displaystyle\int_0^4 \frac{1}{25 - x^2}\,dx$

$$= \left[-\frac{1}{10}\ln\left|\frac{x - 5}{x + 5}\right|\right]_0^4 = -\frac{1}{10}\ln\frac{1}{9} = \frac{1}{10}\ln 9 \approx 0.220$$

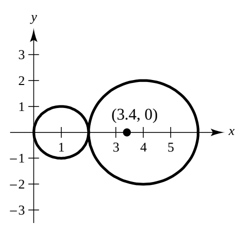
**68.** By symmetry, $\bar{y} = 0$.

$A = \pi + 4\pi = 5\pi$

$$\bar{x} = \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi}$$

$$= \frac{17\pi}{5\pi} = 3.4$$

$(\bar{x}, \bar{y}) = (3.4, 0)$




**70.** $s = \displaystyle\int_0^{\pi} \sqrt{1 + \sin^2 2x}\,dx \approx 3.82$

**72.** $\displaystyle\lim_{x \to 0} \frac{\sin \pi x}{\sin 2\pi x} = \lim_{x \to 0} \frac{\pi\cos\pi x}{2\pi\cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$

**74.** $\displaystyle\lim_{x \to \infty} xe^{-x^2} = \lim_{x \to \infty} \frac{x}{e^{x^2}} = \lim_{x \to \infty} \frac{1}{2xe^{x^2}} = 0$

**76.** $y = \displaystyle\lim_{x \to 1^+} (x - 1)^{\ln x}$

$\ln y = \displaystyle\lim_{x \to 1^+} \left[(\ln x)\ln(x - 1)\right]$

$$= \lim_{x \to 1^+}\left[\frac{\ln(x - 1)}{\frac{1}{\ln x}}\right] = \lim_{x \to 1^+}\left[\frac{\frac{1}{x - 1}}{\left(\frac{1}{x}\right)\frac{-1}{\ln^2 x}}\right] = \lim_{x \to 1^+}\left[\frac{\frac{-\ln^2 x}{x - 1}}{x}\right] = \lim_{x \to 1^+}\left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}}\right]$$

$$= \lim_{x \to 1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.

**78.** $$\lim_{x\to 1^+}\left(\frac{2}{\ln x}-\frac{2}{x-1}\right)=\lim_{x\to 1^+}\left[\frac{2x-2-2\ln x}{(\ln x)(x-1)}\right]$$
$$=\lim_{x\to 1^+}\left[\frac{2-(2/x)}{(x-1)(1/x)+\ln x}\right]$$
$$=\lim_{x\to 1^+}\frac{2x-2}{(x-1)+x\ln x}=\lim_{x\to 1^+}\frac{2}{1+1+\ln x}=1$$

**80.** $$\int_0^1 \frac{6}{x-1}\,dx=\lim_{b\to 1^-}\Big[6\ln|x-1|\Big]_0^b=-\infty$$

Diverges

**82.** $$\int_0^\infty \frac{e^{-1/x}}{x^2}\,dx=\lim_{\substack{a\to 0^+\\ b\to\infty}}\Big[e^{-1/x}\Big]_a^b=1-0=1$$

**84.** $$V=\pi\int_0^\infty (xe^{-x})^2dx$$
$$=\pi\int_0^\infty x^2e^{-2x}\,dx$$
$$=\lim_{b\to\infty}\left[-\frac{\pi e^{-2x}}{4}(2x^2+2x+1)\right]_0^b=\frac{\pi}{4}$$

**86.** $$\int_2^\infty\left[\frac{1}{x^5}+\frac{1}{x^{10}}+\frac{1}{x^{15}}\right]dx<\int_2^\infty\frac{1}{x^5-1}\,dx<\int_2^\infty\left[\frac{1}{x^5}+\frac{1}{x^{10}}+\frac{2}{x^{15}}\right]dx$$
$$\lim_{b\to\infty}\left[-\frac{1}{4x^4}-\frac{1}{9x^9}-\frac{1}{14x^{14}}\right]_2^b<\int_2^\infty\frac{1}{x^5-1}\,dx<\lim_{b\to\infty}\left[-\frac{1}{4x^4}-\frac{1}{9x^9}-\frac{1}{7x^{14}}\right]_2^b$$
$$0.015846<\int_2^\infty\frac{1}{x^5-1}\,dx<0.015851$$

# Problem Solving for Chapter 7

**2.** (a) $$\int_0^1 \ln x\,dx=\lim_{b\to 0^+}\Big[x\ln-x\Big]_b^1$$
$$=(-1)-\lim_{b\to 0^+}(b\ln b-b)=-1$$

**Note:** $\lim_{b\to 0^+} b\ln b=\lim_{b\to 0^+}\frac{\ln b}{b^{-1}}=\lim_{b\to 0^+}\frac{1/b}{-1/b^2}=0$

$$\int_0^1(\ln x)^2\,dx=\lim_{b\to 0^+}\Big[x(\ln x)^2-2x\ln x+2x\Big]_b^1$$
$$=2-\lim_{b\to 0^+}(b(\ln b)^2-2b\ln b+2b)=2$$

(b) Note first that $\lim_{b\to 0^+} b(\ln b)^n=0$ (Mathematical induction).

Also, $\int(\ln x)^{n+1}\,dx=x(\ln x)^{n+1}-(n+1)\int(\ln x)^n\,dx.$

Assume $\int_0^1(\ln x)^n\,dx=(-1)^n n!.$

Then, $$\int_0^1(\ln x)^{n+1}\,dx=\lim_{b\to 0^+}\Big[x(\ln x)^{n+1}\Big]_b^1-(n+1)\int_0^1(\ln x)^n\,dx$$
$$=0-(n+1)(-1)^n n!=(-1)^{n+1}(n+1)!.$$

**4.**
$$\lim_{x\to\infty}\left(\frac{x-c}{x+c}\right)^x = \frac{1}{4}$$
$$\lim_{x\to\infty} x\ln\left(\frac{x-c}{x+c}\right) = \ln\frac{1}{4}$$
$$\lim_{x\to\infty}\frac{\ln(x-c)-\ln(x+c)}{1/x} = -\ln 4$$
$$\lim_{x\to\infty}\frac{\dfrac{1}{x-c}-\dfrac{1}{x+c}}{-\dfrac{1}{x^2}} = -\ln 4$$
$$\lim_{x\to\infty}\frac{2c}{(x-c)(x+c)}(-x^2) = -\ln 4$$
$$\lim_{x\to\infty}\frac{2cx^2}{x^2-c^2} = \ln 4$$
$$2c = \ln 4$$
$$2x = 2\ln 2$$
$$c = \ln 2$$

**6.** $\sin\theta = BD$, $\cos\theta = OD$

$$\text{Area } \triangle DAB = \frac{1}{2}(DA)(BD) = \frac{1}{2}(1-\cos\theta)\sin\theta$$

$$\text{Shaded area} = \frac{\theta}{2} - \frac{1}{2}(1)(BD) = \frac{\theta}{2} - \frac{1}{2}\sin\theta$$

$$R = \frac{\triangle DAB}{\text{Shaded area}} = \frac{1/2(1-\cos\theta)\sin\theta}{1/2(\theta-\sin\theta)}$$

$$\lim_{\theta\to 0^+} R = \lim_{\theta\to 0^+}\frac{(1-\cos\theta)\sin\theta}{\theta-\sin\theta} = \lim_{\theta\to 0^+}\frac{(1-\cos\theta)\cos\theta+\sin^2\theta}{1-\cos\theta}$$
$$= \lim_{\theta\to 0^+}\frac{(1-\cos\theta)(-\sin\theta)+\cos\theta\sin\theta+2\sin\theta\cos\theta}{\sin\theta}$$
$$= \lim_{\theta\to 0^+}\frac{-\sin\theta-4\cos\theta\sin\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{4\cos\theta-1}{1} = 3$$

**8.** $u = \tan\dfrac{x}{2}$, $\cos x = \dfrac{1-u^2}{1+u^2}$, $2+\cos x = 2+\dfrac{1-u^2}{1+u^2} = \dfrac{3+u^2}{1+u^2}$

$$dx = \frac{2\,du}{1+u^2}$$

$$\int_0^{\pi/2}\frac{1}{2+\cos x}\,dx = \int_0^1\left(\frac{1+u^2}{3+u^2}\right)\left(\frac{2}{1+u^2}\right)du$$
$$= \int_0^1\frac{2}{3+u^2}\,du$$
$$= \left[2\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_0^1$$
$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{2}{\sqrt{3}}\frac{\pi}{6} = \frac{\pi\sqrt{3}}{9} \approx 0.6046$$

**10.** Let $u = cx$, $du = c\,dx$.

$$\int_0^b e^{-c^2x^2}\,dx = \int_0^{cb} e^{-u^2}\,\frac{du}{c} = \frac{1}{c}\int_0^{cb} e^{-u^2}\,du$$

As $b \to \infty$, $cb \to \infty$. Hence, $\displaystyle\int_0^\infty e^{-c^2x^2}\,dx = \frac{1}{c}\int_0^\infty e^{-x^2}\,dx$.

$\bar{x} = 0$ by symmetry.

$$\bar{y} = \frac{M_x}{m} = \frac{2\displaystyle\int_0^\infty \frac{(e^{-c^2x^2})}{2}\,dx}{2\displaystyle\int_0^\infty e^{-c^2x^2}\,dx}$$

$$= \frac{1}{2}\,\frac{\displaystyle\int_0^\infty e^{-2c^2x^2}\,dx}{\displaystyle\int_0^\infty e^{-c^2x^2}\,dx}$$

$$= \frac{1}{2}\,\frac{\dfrac{1}{\sqrt{2}c}\displaystyle\int_0^\infty e^{-x^2}\,dx}{\dfrac{1}{c}\displaystyle\int_0^\infty e^{-x^2}\,dx}$$

$$= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

Thus, $(\bar{x}, \bar{y}) = \left(0, \dfrac{\sqrt{2}}{4}\right)$.

**12.** (a) Let $y = f^{-1}(x)$, $f(y) = x$, $dx = f'(y)\,dy$.

$$\int f^{-1}(x)\,dx = \int y f'(y)\,dy$$

$$= y f(y) - \int f(y)\,dy \qquad \left[\begin{array}{l} u = y,\ du = dy \\ dv = f'(y)\,dy,\ v = f(y) \end{array}\right]$$

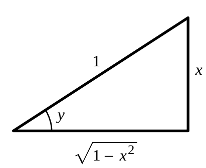
$$= x f^{-1}(x) - \int f(y)\,dy$$

(b) $f^{-1}(x) = \arcsin x = y$, $f(x) = \sin x$

$$\int \arcsin x\,dx = x \arcsin x - \int \sin y\,dy$$

$$= x \arcsin x + \cos y + C$$

$$= x \arcsin x + \sqrt{1 - x^2} + C$$




(c) $f(x) = e^x$, $f^{-1}(x) = \ln x = y \qquad x = 1 \Leftrightarrow y = 0;\ x = e \Leftrightarrow y = 1$

$$\int_1^e \ln x\,dx = \Big[x \ln x\Big]_1^e - \int_0^1 e^y\,dy$$

$$= e - \Big[e^y\Big]_0^1$$

$$= e - (e - 1) = 1$$

**14.** (a) Let $x = \frac{\pi}{2} - u$, $dx = du$.

$$I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx = \int_{\pi/2}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)}(-du)$$

$$= \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u}\,du$$

Hence,

$$2I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\pi/2} 1\,dx = \frac{\pi}{2} \implies I = \frac{\pi}{4}.$$

(b) $$I = \int_{\pi/2}^0 \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\cos^n\left(\frac{\pi}{2} - u\right) + \sin^n\left(\frac{\pi}{2} - u\right)}(-du)$$

$$= \int_0^{\pi/2} \frac{\cos^n u}{\sin^n u + \cos^n u}\,du$$

Thus, $2I = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2} \implies I = \frac{\pi}{4}$.

**16.** $$\frac{N(x)}{D(x)} = \frac{P_1}{x - c_1} + \frac{P_2}{x - c_2} + \cdots + \frac{P_n}{x - c_n}$$

$$N(x) = P_1(x - c_2)(x - c_3)\ldots(x - c_n) + P_2(x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + P_n(x - c_1)(x - c_2)\ldots(x - c_{n-1})$$

Let $x = c_1$: $N(c_1) = P_1(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$

$$P_1 = \frac{N(c_1)}{(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)}$$

Let $x = c_2$: $N(c_2) = P_2(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$

$$P_2 = \frac{N(c_2)}{(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)}$$

$\vdots \qquad\qquad \vdots$

Let $x = c_n$: $N(c_n) = P_n(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})$

$$P_n = \frac{N(c_n)}{(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})}$$

If $D(x) = (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_n)$, then by the Product Rule

$$D'(x) = (x - c_2)(x - c_3)\ldots(x - c_n) + (x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_{n-1})$$

and

$$D'(c_1) = (c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$$

$$D'(c_2) = (c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$$

$$\vdots$$

$$D'(c_n) = (c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1}).$$

Thus, $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \ldots, n$.

**18.** 
$$\begin{aligned}
s(t) &= \int\left[-32t + 12{,}000 \ln \frac{50{,}000}{50{,}000 - 400t}\right] dt \\
&= -16t^2 + 12{,}000\int[\ln 50{,}000 - \ln(50{,}000 - 400t)]\, dt \\
&= 16t^2 + 12{,}000t \ln 50{,}000 - 12{,}000\left[t \ln(50{,}000 - 400t) - \int \frac{-400t}{50{,}000 - 400t}\, dt\right] \\
&= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t\int\left[1 - \frac{50{,}000}{50{,}000 - 400t}\right] dt \\
&= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t + 1{,}500{,}000 \ln(50{,}000 - 400t) + C
\end{aligned}$$

$$s(0) = 1{,}500{,}000 \ln 50{,}000 + C = 0$$

$$C = -1{,}500{,}000 \ln 50{,}000$$

$$s(t) = -16t^2 + 12{,}000t\left[1 + \ln \frac{50{,}000}{50{,}000 - 400t}\right] + 1{,}500{,}000 \ln \frac{50{,}000 - 400t}{50{,}000}$$

When $t = 100$, $s(100) \approx 557{,}168.626$ feet

**20.** Let $u = (x - a)(x - b)$, $du = [(x - a) + (x - b)]\, dx$, $dv = f''(x)\, dx$, $v = f'(x)$.

$$\begin{aligned}
\int_a^b (x - a)(x - b)\, dx &= \Big[(x - a)(x - b) f'(x)\Big]_a^b - \int_a^b [(x - a) + (x - b)] f'(x)\, dx \\
&= -\int_a^b (2x - a - b) f'(x)\, dx \qquad \begin{pmatrix} u = 2x - a - b \\ dv = f'(x)\, dx \end{pmatrix} \\
&= \Big[-(2x - a - b) f(x)\Big]_a^b + \int_a^b 2f(x)\, dx \\
&= 2\int_a^b f(x)\, dx
\end{aligned}$$

# CHAPTER 8
# Infinite Series

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## Section 8.1 Sequences

Solutions to Even-Numbered Exercises

**2.** $a_n = \dfrac{2n}{n+3}$

$a_1 = \dfrac{2}{4} = \dfrac{1}{2}$

$a_2 = \dfrac{4}{5}$

$a_3 = \dfrac{6}{6} = 1$

$a_4 = \dfrac{8}{7}$

$a_5 = \dfrac{10}{8} = \dfrac{5}{4}$

**4.** $a_n = \left(-\dfrac{2}{3}\right)^n$

$a_1 = -\dfrac{2}{3}$

$a_2 = \dfrac{4}{9}$

$a_3 = -\dfrac{8}{27}$

$a_4 = \dfrac{16}{81}$

$a_5 = -\dfrac{32}{243}$

**6.** $a_n = \cos\dfrac{n\pi}{2}$

$a_1 = \cos\dfrac{\pi}{2} = 0$

$a_2 = \cos\pi = -1$

$a_3 = \cos\dfrac{3\pi}{2} = 0$

$a_4 = \cos 2\pi = 1$

$a_5 = \cos\dfrac{5\pi}{2} = 0$

**8.** $a_n = (-1)^{n+1}\left(\dfrac{2}{n}\right)$

$a_1 = \dfrac{2}{1} = 2$

$a_2 = -\dfrac{2}{2} = -1$

$a_3 = \dfrac{2}{3}$

$a_4 = -\dfrac{2}{4} = -\dfrac{1}{2}$

$a_5 = \dfrac{2}{5}$

**10.** $a_n = 10 + \dfrac{2}{n} + \dfrac{6}{n^2}$

$a_1 = 10 + 2 + 6 = 18$

$a_2 = 10 + 1 + \dfrac{3}{2} = \dfrac{25}{2}$

$a_3 = 10 + \dfrac{2}{3} + \dfrac{2}{3} = \dfrac{34}{3}$

$a_4 = 10 + \dfrac{1}{2} + \dfrac{3}{8} = \dfrac{87}{8}$

$a_5 = 10 + \dfrac{2}{5} + \dfrac{6}{25} = \dfrac{266}{25}$

**12.** $a_n = \dfrac{3n!}{(n-1)!} = 3n$

$a_1 = 3(1) = 3$

$a_2 = 3(2) = 6$

$a_3 = 3(3) = 9$

$a_4 = 3(4) = 12$

$a_5 = 3(5) = 15$

**14.** $a_1 = 4,\ a_{k+1} = \left(\dfrac{k+1}{2}\right)a_k$

$a_2 = \left(\dfrac{1+1}{2}\right)a_1 = 4$

$a_3 = \left(\dfrac{2+1}{2}\right)a_2 = 6$

$a_4 = \left(\dfrac{3+1}{2}\right)a_3 = 12$

$a_5 = \left(\dfrac{4+1}{2}\right)a_4 = 30$

**16.** $a_1 = 6,\ a_{k+1} = \dfrac{1}{3}a_k^{\,2}$

$a_2 = \dfrac{1}{3}a_1^{\,2} = \dfrac{1}{3}(6^2) = 12$

$a_3 = \dfrac{1}{3}a_2^{\,2} = \dfrac{1}{3}(12^2) = 48$

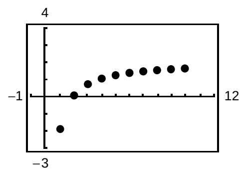
$a_4 = \dfrac{1}{3}a_3^{\,2} = \dfrac{1}{3}(48^2) = 768$

$a_5 = \dfrac{1}{3}a_4^{\,2} = \dfrac{1}{3}(768)^2 = 196{,}608$

**18.** Because the sequence tends to 8 as $n$ tends to infinity, it matches (a).

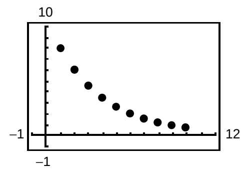
**20.** This sequence increases for a few terms, then decreases $a_2 = \frac{16}{2} = 8$. Matches (b).

**22.**



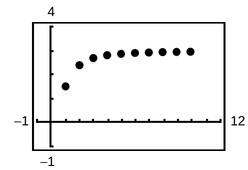
$a_n = 2 - \dfrac{4}{n}, n = 1, . . . , 10$

**24.**



$a_n = 8(0.75)^{n-1}, n = 1, 2, . . . , 10$

**26.**



$a_n = \dfrac{3n^2}{n^2 + 1}, n = 1, . . . , 10$

**28.** $a_n = \dfrac{n + 6}{2}$

$$a_5 = \frac{5 + 6}{2} = \frac{11}{2}$$

$$a_6 = \frac{6 + 6}{2} = 6$$

**30.** $a_{n+1} = 2a_n,\ a_1 = 5$

$$a_5 = 2(40) = 80$$

$$a_6 = 2(80) = 160$$

**32.** $\dfrac{25!}{23!} = \dfrac{23!(24)(25)}{23!}$

$$= (24)(25) = 600$$

**34.** $\dfrac{(n + 2)!}{n!} = \dfrac{n!(n + 1)(n + 2)}{n!}$

$$= (n + 1)(n + 2)$$

**36.** $\dfrac{(2n + 2)!}{(2n)!} = \dfrac{(2n)!(2n + 1)(2n + 2)}{(2n)!}$

$$= (2n + 1)(2n + 2)$$

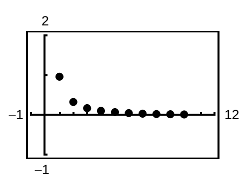
**38.** $\lim\limits_{n\to\infty} \left(5 - \dfrac{1}{n^2}\right) = 5 - 0 = 5$

**40.** $\lim\limits_{n\to\infty} \dfrac{5n}{\sqrt{n^2 + 4}} = \lim\limits_{n\to\infty} \dfrac{5}{\sqrt{1 + (4/n^2)}}$

$$= \frac{5}{1} = 5$$

**42.** $\lim\limits_{n\to\infty} \cos\left(\dfrac{2}{n}\right) = 1$

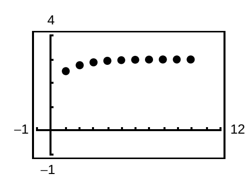
**44.**



The graph seems to indicate that the sequence converges to 0. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{3/2}} = \lim_{x\to\infty} \frac{1}{x^{3/2}} = 0.$$

**46.**



The graph seems to indicate that the sequence converges to 3. Analytically,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(3 - \frac{1}{2^n}\right) = 3 - 0 = 3.$$

**48.** $\lim\limits_{n\to\infty} [1 + (-1)^n]$

does not exist, (alternates between 0 and 2), diverges.

**50.** $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$, converges

**52.** $\lim\limits_{n\to\infty} \dfrac{1 + (-1)^n}{n^2} = 0$, converges

**54.** $\lim\limits_{n\to\infty} \dfrac{\ln\sqrt{n}}{n} = \lim\limits_{n\to\infty} \dfrac{1/2 \ln n}{n}$

$= \lim\limits_{n\to\infty} \dfrac{1}{2n} = 0$, converges

(L'Hôpital's Rule)

**56.** $\lim\limits_{n\to\infty} (0.5)^n = 0$, converges

**58.** $\lim\limits_{n\to\infty} \dfrac{(n - 2)!}{n!} = \lim\limits_{n\to\infty} \dfrac{1}{n(n - 1)} = 0$, converges

**60.** $\lim_{n\to\infty}\left(\frac{n^2}{2n+1}-\frac{n^2}{2n-1}\right)=\lim_{n\to\infty}\frac{-2n^2}{4n^2-1}=-\frac{1}{2}$, converges

**62.** $a_n = n\sin\frac{1}{n}$

Let $f(x) = x\sin\frac{1}{x}$.

$$\lim_{x\to\infty} x\sin\frac{1}{x}=\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{x\to\infty}\frac{(-1/x^2)\cos(1/x)}{-1/x^2}=\lim_{x\to\infty}\cos\frac{1}{x}=\cos 0=1 \text{ (L'Hôpital's Rule)}$$

or,

$$\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{y\to 0^+}\frac{\sin(y)}{y}=1. \text{ Therefore } \lim_{n\to\infty} n\sin\frac{1}{n}=1.$$

**64.** $\lim_{n\to\infty}2^{1/n}=2^0=1$, converges

**66.** $\lim_{n\to\infty}\frac{\cos\pi n}{n^2}=0$, converges

**68.** $a_n = 4n-1$

**70.** $a_n=\frac{(-1)^{n-1}}{n^2}$

**72.** $a_n=\frac{n+2}{3n-1}$

**74.** $a_n=(-1)^n\frac{3^{n-2}}{2^{n-1}}$

**76.** $a_n = 1+\frac{2^n-1}{2^n}$

$=\frac{2^{n+1}-1}{2^n}$

**78.** $a_n=\frac{1}{n!}$

**80.** $a_n=\frac{x^{n-1}}{(n-1)!}$

**82.** Let $f(x)=\frac{3x}{x+2}$. Then $f'(x)=\frac{6}{(x+2)^2}$.

Thus, $f$ is increasing which implies $\{a_n\}$ is increasing.

$|a_n| < 3$, bounded

**84.** $a_n = ne^{-n/2}$

$a_1 = 0.6065$

$a_2 = 0.7358$

$a_3 = 0.6694$

Not monotonic; $|a_n| \le 0.7358$, bounded

**86.** $a_n=\left(-\frac{2}{3}\right)^n$

$a_1=-\frac{2}{3}$

$a_2=\frac{4}{9}$

$a_3=-\frac{8}{27}$

Not monotonic; $|a_n|\le\frac{2}{3}$, bounded

**88.** $a_n=\left(\frac{3}{2}\right)^n<\left(\frac{3}{2}\right)^{n+1}=a_{n+1}$

Monotonic; $\lim_{n\to\infty}a_n=\infty$, not bounded

**90.** $a_n=\frac{\cos n}{n}$

$a_1 = 0.5403$

$a_2 = -0.2081$

$a_3 = -0.3230$

$a_4 = -0.1634$

Not monotonic; $|a_n|\le 1$, bounded

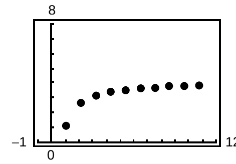
**92.** (a) $a_n = 4 - \dfrac{3}{n}$

$\left|4 - \dfrac{3}{n}\right| < 4 \implies$ bounded

$a_n = 4 - \dfrac{3}{n} < 3 - \dfrac{4}{n+1} = a_{n+1} \implies$ monotonic

Therefore, $\{a_n\}$ converges.

(b)




$$\lim_{n\to\infty}\left(4 - \frac{3}{n}\right) = 4$$

**94.** (a) $a_n = 4 + \dfrac{1}{2^n}$

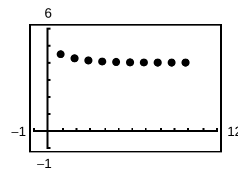
$\left|4 + \dfrac{1}{2^n}\right| \le 4.5 \implies \{a_n\}$ bounded

$a_n = 4 + \dfrac{1}{2^n} > 4 + \dfrac{1}{2^{n+1}}$

$= a_{n+1} \implies \{a_n\}$ monotonic

Therefore, $\{a_n\}$ converges.

(b)




$$\lim_{n\to\infty}\left(4 + \frac{1}{2^n}\right) = 4$$

**96.** $A_n = 100(101)[(1.01)^n - 1]$

(a) $A_1 = \$101.00$

$A_2 = \$203.01$

$A_3 = \$306.04$

$A_4 = \$410.10$

$A_5 = \$515.20$

$A_6 = \$621.35$

(b) $A_{60} = \$8248.64$

(c) $A_{240} = \$99{,}914.79$

**98.** The first sequence because every other point is below the $x$-axis.

**100.** Impossible. The sequence converges by Theorem 8.5.

**102.** Impossible. An unbounded sequence diverges.

**104.** $P_n = 16{,}000(1.045)^n$

$P_1 = \$16{,}720.00$

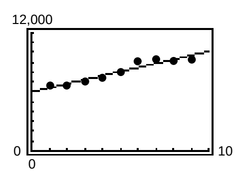
$P_2 = \$17{,}472.40$

$P_3 \approx \$18{,}258.66$

$P_4 \approx \$19{,}080.30$

$P_5 \approx \$19{,}938.91$

**106.** (a) $a_n = 410.9212n + 6003.8545$




(b) For 2004, $n = 14$ and $a_n = 11{,}757$, or $11,757,000,000.

**108.** $a_n = \left(1 + \dfrac{1}{n}\right)^n$

$a_1 = 2.0000$

$a_2 = 2.2500$

$a_3 \approx 2.3704$

$a_4 \approx 2.4414$

$a_5 \approx 2.4883$

$a_6 \approx 2.5216$

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$$

**110.** Since

$$\lim_{n\to\infty} s_n = L > 0,$$

there exists for each $\epsilon > 0$, an integer $N$ such that $|s_n - L| < \epsilon$ for every $n > N$. Let $\epsilon = L > 0$ and we have,

$$|s_n - L| < L,\ -L < s_n - L < L, \text{ or } 0 < s_n < 2L$$

for each $n > N$.

**112.** If $\{a_n\}$ is bounded, monotonic and nonincreasing, then $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. Then

$$-a_1 \leq -a_2 \leq -a_3 \leq \cdots \leq -a_n \leq \cdots$$

is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Since $\{-a_n\}$ converges, then so does $\{a_n\}$.

**114.** True

**116.** True

**118.** $x_0 = 1, x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, n = 1, 2, \ldots$

$x_1 = 1.5$ $\quad x_6 = 1.414214$

$x_2 = 1.41667$ $\quad x_7 = 1.414214$

$x_3 = 1.414216$ $\quad x_8 = 1.414114$

$x_4 = 1.414214$ $\quad x_9 = 1.414214$

$x_5 = 1.414214$ $\quad x_{10} = 1.414214$

The limit of the sequence appears to be $\sqrt{2}$. In fact, this sequence is Newton's Method applied to $f(x) = x^2 - 2$.

## Section 8.2 Series and Convergence

**2.** $S_1 = \frac{1}{6} \approx 0.1667$

$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$

$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$

$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$

$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$

**4.** $S_1 = 1$

$S_2 = 1 + \frac{1}{3} \approx 1.3333$

$S_3 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.5333$

$S_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} \approx 1.6444$

$S_5 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} \approx 1.7354$

**6.** $S_1 = 1$

$S_2 = 1 - \frac{1}{2} = 0.5$

$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$

$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$

$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$

**8.** $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$ Geometric series

$r = \frac{4}{3} > 1$

Diverges by Theorem 8.6

**10.** $\sum_{n=0}^{\infty} 2(-1.03)^n$ Geometric series

$|r| = 1.03 > 1$

Diverges by Theorem 8.6

**12.** $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

Diverges by Theorem 8.9

**14.** $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$

Diverges by Theorem 8.9

**16.** $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$\lim_{n\to\infty} \frac{n!}{2^n} = \infty$

Diverges by Theorem 8.9

**18.** $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$

$S_0 = 1, S_1 = \frac{5}{3}, S_2 \approx 2.11, \ldots$

Matches graph (b).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1 - 2/3} = \frac{1}{1/3} = 3$$

**20.** $\sum_{n=0}^{\infty}\frac{17}{3}\left(-\frac{8}{9}\right)^n = \frac{17}{3}\left[1 - \frac{8}{9} + \frac{64}{81} - \cdots\right]$

$S_0 = \frac{17}{3}, S_1 \approx 0.63, S_3 \approx 5.1, \ldots$

Matches (d).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty}\frac{17}{3}\left(-\frac{8}{9}\right)^n = \frac{17/3}{1-(-8/9)} = \frac{17/3}{17/9} = 3$$

**22.** $\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) = \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

**24.** $\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$

Geometric series with $|r| = \left|-\frac{1}{2}\right| < 1$.

Converges by Theorem 8.6

**26.** $\sum_{n=0}^{\infty}(-0.6)^n$

Geometric series with $|r| = |-0.6| < 1$.

Converges by Theorem 8.6

**28.** (a) $\sum_{n=1}^{\infty}\frac{4}{n(n+4)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+4}\right)$

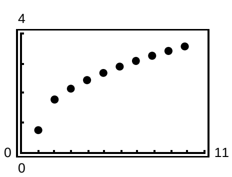
$$= \left(1 - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{7}}\right) + \left(\frac{1}{4} - \cancel{\frac{1}{8}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833$$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 1.5377 | 1.7607 | 1.9051 | 2.0071 | 2.0443 |

(c)



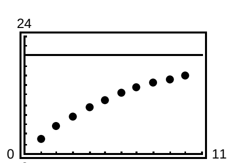

(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

**30.** (a) $\sum_{n=1}^{\infty} 3(0.85)^{n-1} = \frac{3}{1 - 0.85} = 20$ (Geometric series)

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 11.1259 | 16.0625 | 19.2248 | 19.9941 | 19.999998 |

(c)

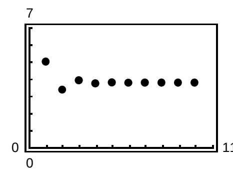


**32.** (a) $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1} = \frac{5}{1-(-1/3)} = \frac{15}{4} = 3.75$

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 3.7654 | 3.7499 | 3.7500 | 3.7500 | 3.7500 |

(c)




(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

**34.** $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = 2\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots\right] = 2\left(1+\frac{1}{2}\right) = 3$

**36.** $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) = \frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{9}\right) + \ldots\right] = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$

**38.** $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{6}{1-(4/5)} = 30$ (Geometric)

**40.** $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{2}{1-(-2/3)} = \frac{6}{5}$

**42.** $\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1-(3/4)} = 32$

**44.** $\sum_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \frac{4}{1-(-1/2)} = \frac{8}{3}$

**46.** $\sum_{n=1}^{\infty} \left[(0.7)^n + (0.9)^n\right] = \sum_{n=0}^{\infty}\left(\frac{7}{10}\right)^n + \sum_{n=0}^{\infty}\left(\frac{9}{10}\right)^n - 2 = \frac{1}{1-(7/10)} + \frac{1}{1-(9/10)} - 2 = \frac{10}{3} + 10 - 2 = \frac{34}{3}$

**48.** $0.81\overline{81} = \sum_{n=0}^{\infty} \frac{81}{100}\left(\frac{1}{100}\right)^n$

Geometric series with $a = \frac{81}{100}$ and $r = \frac{1}{100}$

$S = \frac{a}{1-r} = \frac{81/100}{1-(1/100)} = \frac{81}{99} = \frac{9}{11}$

**50.** $0.215\overline{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200}\left(\frac{1}{100}\right)^n$

Geometric series with $a = \frac{3}{200}$ and $r = \frac{1}{100}$

$S = \frac{1}{5} + \frac{a}{1-r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$

**52.** $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

$\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$

Diverges by Theorem 8.9

**54.** $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

$$= \frac{1}{3}\left[\left(1-\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{3}-\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{6}-\frac{1}{9}\right) + \cdots\right]$$

$$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}, \text{ converges}$$

**56.** $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{(\ln 2)3^n}{3n^2} = \lim_{n\to\infty} \frac{(\ln 2)^2 3^n}{6\mathrm{n}} = \lim_{n\to\infty} \frac{(\ln n)^3 3^n}{6} = \infty$

(by L'Hôpital's Rule) Diverges by Theorem 8.9

**58.** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$

Geometric series with $r = \frac{1}{4}$

Converges by Theorem 8.6

**60.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{100}$

Geometric series with $r = 2$

Diverges by Theorem 8.6

**62.** $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$

$$\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k \neq 0$$

Diverges by Theorem 8.9

**64.** $\lim_{n\to\infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.

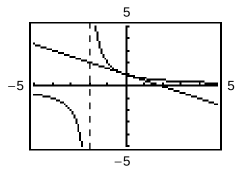
$\displaystyle\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

**66.** If $\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

**68.** (a) $(-x/2)$ is the common ratio.

(c) $y_1 = \dfrac{2}{2 + x}$

$y_2 = 1 - \dfrac{x}{2}$




(b) $$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} = \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{1}{1 - (-x/2)}$$

$$= \frac{2}{2 + x},\ |x| < 2$$

Geometric series:

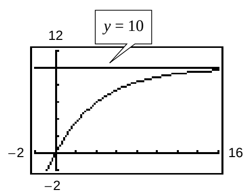
$$a = 1,\ r = -\frac{x}{2},\ \left|-\frac{x}{2}\right| < 1 \implies |x| < 2$$

**70.** $f(x) = 2\left[\dfrac{1 - 0.8^x}{1 - 0.8}\right]$

Horizontal asymptote: $y = 10$

$\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

$S = \dfrac{2}{1 - (4/5)} = 10$




The horizontal asymptote is the sum of the series. $f(n)$ is the $n^{\text{th}}$ partial sum.

**72.** $\dfrac{1}{2^n} < 0.0001$

$10{,}000 < 2^n$

This inequality is true when $n = 14$.

$(0.01)^n < 0.0001$

$10{,}000 < 10^n$

This inequality is true when $n = 5$. This series converges at a faster rate.

**74.** $V(t) = 225{,}000(1 - 0.3)^n = (0.7)^n(225{,}000)$

$V(5) = (0.7)^5(225{,}000) = \$37{,}815.75$

**76.** $$\sum_{i=0}^{n-1} 100(0.60)^i = \frac{100[1 - 0.6^n]}{1 - 0.6}$$

$$= 250(1 - 0.6^n) \quad \text{million dollars}$$

Sum $= 250$ million dollars

**78.** The ball in Exercise 77 takes the following times for each fall.

$s_1 = -16t^2 + 16$ $\qquad$ $s_1 = 0$ if $t = 1$

$s_2 = -16t^2 + 16(0.81)$ $\qquad$ $s_2 = 0$ if $t = 0.9$

$s_3 = -16t^2 + 16(0.81)^2$ $\qquad$ $s_3 = 0$ if $t = (0.9)^2$

$\vdots$ $\qquad$ $\vdots$

$s_n = -16t^2 + 16(0.81)^{n-1}$ $\qquad$ $s_n = 0$ if $t = (0.9)^{n-1}$

Beginning with $s_2$, the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n = -1 + 2\sum_{n=0}^{\infty}(0.9)^n$$

$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$

**80.** $P(n) = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$

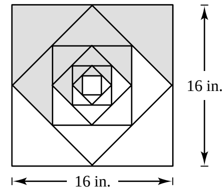
$P(2) = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^2 = \dfrac{4}{27}$

$$\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$$

**82.** (a) $64 + 32 + 16 + 8 + 4 + 2 = 126$ in.$^2$

(b) $\displaystyle\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1 - (1/2)} = 128$ in.$^2$

**Note**: This is one-half of the area of the original square!




**84.** Surface area $= 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\frac{1}{9}\right)^2 + \cdots = 4[\pi + \pi + \cdots] = \infty$

**86.** $$\sum_{n=0}^{12t-1} P\left(1 + \frac{r}{12}\right)^n = \frac{P\left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]}{1 - \left(1 + \frac{r}{12}\right)}$$
$$= P\left(-\frac{12}{r}\right)\left[\left(1 - \left(1 + \frac{r}{12}\right)^{12t}\right]\right.$$
$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$
$$\sum_{n=0}^{12t-1} P(e^{r/12})^n = \frac{P(1 - (e^{r/12})^{12t})}{1 - e^{r/12}} = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

**88.** $P = 75, r = 0.05, t = 25$

(a) $A = 75\left(\frac{12}{0.05}\right)\left[\left(1 + \frac{0.05}{12}\right)^{12(25)} - 1\right] \approx \$44{,}663.23$

(b) $A = \dfrac{75(e^{0.05(25)} - 1)}{e^{0.05/12} - 1} \approx \$44{,}732.85$

**90.** $P = 20, r = 0.06, t = 50$

(a) $A = 20\left(\frac{12}{0.06}\right)\left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right] \approx \$75{,}743.82$

(b) $A = \dfrac{20(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$76{,}151.45$

**92.** $$T = 40{,}000 + 40{,}000(1.04) + \cdots + 40{,}000(1.04)^{39}$$
$$= \sum_{n=0}^{39} 40{,}000(1.04)^n$$
$$= 40{,}000\left(\frac{1 - 1.04^{40}}{1 - 1.04}\right)$$
$$\approx \$3{,}801{,}020$$

**94.** $$x = 0.a_1a_2a_3 \ldots a_k\overline{a_1a_2a_3 \ldots a_k}$$
$$= 0.a_1a_2a_3 \ldots a_k\left[1 + \frac{1}{10^k} + \left(\frac{1}{10^k}\right)^2 + \left(\frac{1}{10^k}\right)^3 + \cdots\right]$$
$$= 0.a_1a_2a_3 \ldots a_k \sum_{n=0}^{\infty}\left(\frac{1}{10^k}\right)^n$$
$$= 0.a_1a_2a_3 \ldots a_k\left[\frac{1}{1 - (1/10^k)}\right] = \text{a rational number}$$

**96.** Let $\{S_n\}$ be the sequence of partial sums for the convergent series $\displaystyle\sum_{n=1}^{\infty} a_n = L$. Then

$\displaystyle\lim_{n\to\infty} S_n = L$ and since $R_n = \displaystyle\sum_{k=n+1}^{\infty} a_k = L - S_n$,

we have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(L - S_n) = \lim_{n\to\infty} L - \lim_{n\to\infty} S_n = L - L = 0.$$

**98.** If $\Sigma(a_n + b_n)$ converged, then $\Sigma(a_n + b_n) - \Sigma a_n = \Sigma b_n$ would converge, which is a contradiction.

Thus, $\Sigma(a_n + b_n)$ diverges.

**100.** True

**102.** True; $\lim_{n\to\infty} \dfrac{n}{1000(n+1)} = \dfrac{1}{1000} \neq 0$

**104.** $\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \dfrac{1}{r}\left(\dfrac{1}{r}\right)^n = \dfrac{1/r}{1-(1/r)} = \dfrac{1}{r-1} \quad \left(\text{since } \left|\dfrac{1}{r}\right| < 1\right)$

This is a geometric series which converges if $\left|\dfrac{1}{r}\right| < 1 \iff |r| > 1$.

## Section 8.3 The Integral Test and *p*-Series

**2.** $\sum_{n=1}^{\infty} \dfrac{2}{3n+5}$

Let $f(x) = \dfrac{2}{3x+5}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{2}{3x+5}\,dx = \left[\frac{2}{3}\ln(3x+5)\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**4.** $\sum_{n=1}^{\infty} ne^{-n/2}$

Let $f(x) = xe^{-x/2}$.

$f$ is positive, continuous, and decreasing for $x \geq 3$.

Since $f'(x) = \dfrac{2-x}{2e^{x/2}} < 0$ for $x \geq 3$.

$$\int_3^{\infty} xe^{-x/2}\,dx = \left[-2(x+2)e^{-x/2}\right]_3^{\infty} = 10e^{-3/2}$$

Converges by Theorem 8.10

**6.** $\sum_{n=1}^{\infty} \dfrac{1}{2n+1}$

Let $f(x) = \dfrac{1}{2x+1}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x+1}\,dx = \left[\ln\sqrt{2x+1}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**8.** $\sum_{n=1}^{\infty} \dfrac{n}{n^2+3}$

Let $f(x) = \dfrac{x}{x^2+3}$.

$f(x)$ is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{3-x^2}{(x^2+3)} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{x}{x^2+3}\,dx = \left[\ln\sqrt{x^2+3}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**10.** $\sum_{n=1}^{\infty} n^k e^{-n}$

Let $f(x) = \dfrac{x^k}{e^x}$.

$f$ is positive, continuous, and decreasing for $x > k$ since

$$f'(x) = \frac{x^{k-1}(k-x)}{e^x} < 0$$

for $x > k$. We use integration by parts.

$$\int_1^{\infty} x^k e^{-x}\,dx = \left[-x^k e^{-x}\right]_1^{\infty} + k\int_1^{\infty} x^{k-1} e^{-x}\,dx$$
$$= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e}$$

Converges by Theorem 8.10

**12.** $\sum_{n=1}^{\infty} \dfrac{1}{n^{1/3}}$

Let $f(x) = \dfrac{1}{x^{1/3}}$.

$f$ is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^{1/3}}\,dx = \left[\frac{3}{2}x^{2/3}\right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

**14.** $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Convergent $p$-series with $p = \frac{5}{3} > 1$

**16.** $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Convergent $p$-series with $p = 2 > 1$

**18.** $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Divergent $p$-series with $p = \frac{2}{3} < 1$

**20.** $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Convergent $p$-series with $p = \pi > 1$

**22.** $\sum_{n=1}^{\infty} \frac{2}{n} = \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \cdots$

$S_1 = 2$

$S_2 = 3$

$S_3 \approx 3.67$

Matches (d)

Diverges—harmonic series

**24.** $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 + \frac{2}{2^2} + \frac{2}{3^2} + \cdots$

$S_1 = 2$

$S_2 = 2.5$

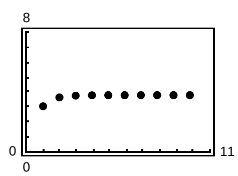
$S_3 \approx 2.722$

Matches (c)

Converges—$p$-series with $p = 2 > 1$.

**26.** (a)

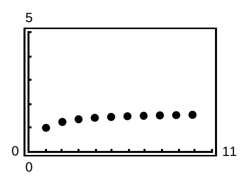
| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 3.7488 | 3.75 | 3.75 | 3.75 | 3.75 |




The partial sums approach the sum 3.75 very rapidly.

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 1.4636 | 1.5498 | 1.5962 | 1.6251 | 1.635 |




The partial sums approach the sum $\frac{\pi^2}{6} \approx 1.6449$ slower than the series in part (a).

**28.** $\xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$

Converges for $x > 1$ by Theorem 8.11.

**30.** $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{\ln x}{x^p}\,dx = \int_2^{\infty} x^{-p} \ln x\,dx = \left[\frac{x^{-p+1}}{(-p+1)^2}\left[-1 + (-p+1)\ln x\right]\right]_2^{\infty}.$$ (Use Integration by Parts.)

Converges for $-p + 1 < 0$ or $p > 1$.

**32.** A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a $p$-series, $p > 0$.

The $p$-series converges if $p > 1$ and diverges if $0 < p \leq 1$.

**34.** The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

**36.** From Exercise 35, we have:

$$0 \le S - S_N \le \int_N^{\infty} f(x)\,dx$$

$$S_N \le S \le S_N + \int_N^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{N} a_n \le S \le \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx$$

**38.** $S_4 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363$

$$R_4 \le \int_4^{\infty} \frac{1}{x^5}\,dx = \left[-\frac{1}{4x^4}\right]_4^{\infty} \approx 0.0010$$

$$1.0363 \le \sum_{n=1}^{\infty} \frac{1}{n^5} \le 1.0363 + 0.0010 = 1.0373$$

**40.** $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$$R_{10} \le \int_{10}^{\infty} \frac{1}{(x+1)[\ln(x+1)]^3}\,dx = \left[-\frac{1}{2[\ln(x+1)]^2}\right]_{10}^{\infty} = \frac{1}{2(\ln 11)^3} \approx 0.0870$$

$$1.9821 \le \sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3} \le 1.9821 + 0.0870 = 2.0691$$

**42.** $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$$R_4 \le \int_4^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_4^{\infty} \approx 0.0183$$

$$0.5713 \le \sum_{n=0}^{\infty} e^{-n} \le 0.5713 + 0.0183 = 0.5896$$

**44.** $0 \le R_N \le \int_N^{\infty} \frac{1}{x^{3/2}}\,dx = \left[-\frac{2}{x^{1/2}}\right]_N^{\infty} = \frac{2}{\sqrt{N}} < 0.001$

$$N^{-1/2} < 0.0005$$

$$\sqrt{N} > 2000$$

$$N \ge 4{,}000{,}000$$

**46.** $R_N \le \int_N^{\infty} e^{-x/2}\,dx = \left[-2e^{-x/2}\right]_N^{\infty} = \frac{2}{e^{N/2}} < 0.001$

$$\frac{2}{e^{N/2}} < 0.001$$

$$e^{N/2} > 2000$$

$$\frac{N}{2} > \ln 2000$$

$$N > 2\ln 2000 \approx 15.2$$

$$N \ge 16$$

**48.** $R_n \le \int_N^{\infty} \frac{2}{x^2+5}\,dx = 2\left[\frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)\right]_N^{\infty}$

$$= \frac{2}{\sqrt{5}}\left(\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right)\right) < 0.001$$

$$\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) < 0.001118$$

$$1.56968 < \arctan\left(\frac{N}{\sqrt{5}}\right)$$

$$\frac{N}{\sqrt{5}} > \tan 1.56968$$

$$N \ge 2004$$

**50.** (a) $\int_{10}^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_{10}^{\infty} = \frac{1}{(p-1)10^{p-1}}, p > 1$

(b) $f(x) = \frac{1}{x^p}$

$$R_{10}(p) = \sum_{n=11}^{\infty} \frac{1}{n^p} \le \text{Area under the graph of } f \text{ over the interval } [10, \infty)$$

(c) The horizontal asymptote is $y = 0$. As $n$ increases, the error decreases.

**52.** $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^{\infty} [\ln(n+1) + \ln(n-1) - 2\ln n]$

$$= \cancel{\ln 3} + \cancel{\ln 1} - 2\ln 2) + (\cancel{\ln 4} + \ln 2 - \cancel{2\ln 3}) + (\cancel{\ln 5} + \cancel{\ln 3} - \cancel{2\ln 4}) + (\cancel{\ln 6} + \cancel{\ln 4} - \cancel{2\ln 5})$$

$$+ (\cancel{\ln 7} + \cancel{\ln 5} - \cancel{2\ln 6}) + (\cancel{\ln 8} + \cancel{\ln 6} - \cancel{2\ln 7}) + (\cancel{\ln 9} + \cancel{\ln 7} - \cancel{2\ln 8}) + \cdots = -\ln 2$$

**54.** $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Let $f(x) = \frac{1}{x\sqrt{x^2-1}}$.

$f$ is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \Big[\operatorname{arcsec} x\Big]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 8.10

**56.** $3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

$p$-series with $p = 0.95$

Diverges by Theorem 8.11

**58.** $\sum_{n=0}^{\infty} (1.075)^n$

Geometric series with $r = 1.075$

Diverges by Theorem 8.6

**60.** $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$

Since these are both convergent $p$-series, the difference is convergent.

**62.** $\sum_{n=2}^{\infty} \ln(n)$

$\lim_{n\to\infty} \ln(n) = \infty$

Diverges by Theorem 8.9

**64.** $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let $f(x) = \frac{\ln x}{x^3}$.

$f$ is positive, continuous, and decreasing for $x \geq 2$ since $f'(x) = \frac{1 - 3\ln x}{x^4} < 0$ for $x \geq 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_2^{\infty} + \frac{1}{2}\int_2^{\infty} \frac{1}{x^3}\,dx = \frac{\ln 2}{8} + \left[-\frac{1}{4x^2}\right]_2^{\infty} = \frac{\ln 2}{8} + \frac{1}{16} \text{ (Use Integration by Parts.)}$$

Converges by Theorem 8.10. See Exercise 14.

## Section 8.4 Comparisons of Series

**2.** (a) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \cdots \quad S_1 = 2$

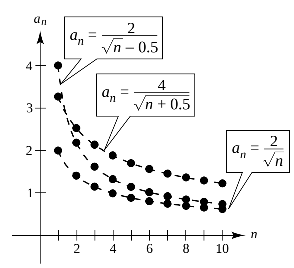
$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \cdots \quad S_1 = 4$

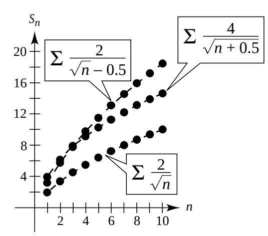
$\sum_{n=1}^{\infty} \frac{4}{\sqrt{n + 0.5}} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \cdots \quad S_1 \approx 3.3$

(b) The first series is a $p$-series. It diverges $\left(p = \frac{1}{2} < 1\right)$.

(c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent $p$-series. Hence, the other two series diverge.

(d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.



**4.** $\dfrac{1}{3n^2 + 2} < \dfrac{1}{3n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$$

converges by comparison with the convergent $p$-series

$$\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

**6.** $\dfrac{1}{\sqrt{n} - 1} > \dfrac{1}{\sqrt{n}}$ for $n \geq 2$.

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$

diverges by comparison with the divergent $p$-series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

**8.** $\dfrac{3^n}{4^n + 5} < \left(\dfrac{3}{4}\right)^n$

Therefore,

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n + 5}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n.$$

**10.** $\dfrac{1}{\sqrt{n^3 + 1}} < \dfrac{1}{n^{3/2}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$$

converges by comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

**12.** $\dfrac{1}{4\sqrt[3]{n} - 1} > \dfrac{1}{4\sqrt[4]{n}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n} - 1}$$

diverges by comparison with the divergent $p$-series

$$\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

**14.** $\dfrac{4^n}{3^n - 1} > \dfrac{4^n}{3^n}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n.$$

**16.** $\displaystyle\lim_{n\to\infty} \frac{2/(3^n - 5)}{1/3^n} = \lim_{n\to\infty} \frac{2 \cdot 3^n}{3^n - 5} = 2$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2}{3^n - 5}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

**18.** $\displaystyle\lim_{n\to\infty} \frac{3/\sqrt{n^2 - 4}}{1/n} = \lim_{n\to\infty} \frac{3n}{\sqrt{n^2 - 4}} = 3$

Therefore,

$$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2 - 4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=3}^{\infty} \frac{1}{n}.$$

**20.** $\displaystyle\lim_{n\to\infty} \frac{\dfrac{5n - 3}{n^2 - 2n + 5}}{1/n} = \lim_{n\to\infty} \frac{5n^2 - 3n}{n^2 - 2n + 5} = 5$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5n - 3}{n^2 - 2n + 5}$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**22.** $\displaystyle\lim_{n\to\infty} \frac{\dfrac{1}{n(n^2 + 1)}}{1/n^3} = \lim_{n\to\infty} \frac{n^3}{n^3 + n} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2 + 1)}$$

converges by a limit comparison with the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

**24.** $\lim_{n\to\infty} \frac{n/[(n+1)2^{n-1}]}{1/(2^{n-1})} = \lim_{n\to\infty} \frac{n}{n+1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

**26.** $\lim_{n\to\infty} \frac{5/\left(n+\sqrt{n^2+4}\right)}{1/n} = \lim_{n\to\infty} \frac{5n}{n+\sqrt{n^2+4}} = \frac{5}{2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{n+\sqrt{n^2+4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**28.** $\lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1/n^2)\sec^2(1/n)}{-1/n^2} = \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right) = 1$

Therefore,

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

**30.** $\sum_{n=0}^{\infty} 5\left(-\frac{1}{5}\right)^n$

Converges

Geometric series with $r = -\frac{1}{5}$

**32.** $\sum_{n=4}^{\infty} \frac{1}{3n^2-2n-15}$

Converges

Limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

**34.** $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots = \frac{1}{2}$

Converges; telescoping series

**36.** $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Converges; telescoping series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

**38.** If $j < k-1$, then $k - j > 1$. The $p$-series with $p = k - j$ converges and since

$$\lim_{n\to\infty} \frac{P(n)/Q(n)}{1/n^{k-j}} = L > 0,$$

the series $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ converges by the limit comparison test. Similarly, if $j \ge k-1$, then $k - j \le 1$ which implies that

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

diverges by the limit comparison test.

**40.** $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$,

which converges since the degree of the numerator is two less than the degree of the denominator.

**42.** $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

diverges since the degree of the numerator is only one less than the degree of the denominator.

**44.** $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty \neq 0$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ diverges.}$$

**46.** See Theorem 8.13, page 585.
One example is

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \text{ diverges because } \lim_{n\to\infty} \frac{1/\sqrt{n-1}}{1/\sqrt{n}} = 1$$

and

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges } (p\text{-series}).$$

**48.** This is not correct. The beginning terms do not affect the convergence or divergence of a series.

In fact,

$$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

and

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } (p\text{-series}).$$

**50.** $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \cdots = \sum_{n=0}^{\infty} \frac{1}{200 + 10n}$,

diverges

**52.** $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200 + n^3}$,

converges

**54.** (a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$

converges since the degree of the numerator is two less than the degree of the denominator. (See Exercise 38.)

(b)

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | 1.1839 | 1.02087 | 1.2212 | 1.2287 | 1.2312 |

(c) $\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$

(d) $\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$

**56.** True

**58.** False. Let $a_n = 1/n$, $b_n = 1/n$, $c_n = 1/n^2$. Then, $a_n \le b_n + c_n$, but $\sum_{n=1}^{\infty} c_n$ converges.

**60.** Since $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$ converges by Exercise 59.

**62.** $\sum \frac{1}{n^2}$ converge, and hence so does $\sum \left(\frac{1}{n^2}\right)^2 = \sum \frac{1}{n^4}$.

**64.** (a) $\sum a_n = \sum \frac{1}{n^3}$ and $\sum b_n = \sum \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^3}{1/n^2} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \sum \frac{1}{n^2}$$

converges, so does $\sum \frac{1}{n^3}$.

(b) $\sum a_n = \sum \frac{1}{\sqrt{n}}$ and $\sum b_n = \sum \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1/n} = \lim_{n\to\infty} \sqrt{n} = \infty \quad \text{and} \quad \sum \frac{1}{n}$$

diverges, so does $\sum \frac{1}{\sqrt{n}}$.

## Section 8.5 Alternating Series

**2.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{n^2} = \frac{6}{1} - \frac{6}{4} + \frac{6}{9} - \cdots$

$S_1 = 6, S_2 = 4.5$

Matches (d)

**4.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10}{n2^n} = \frac{10}{2} - \frac{10}{8} + \cdots$

$S_1 = 5, S_2 = 3.75$

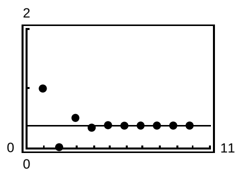
Matches (a)

**6.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0 | 0.5 | 0.3333 | 0.375 | 0.3667 | 0.3681 | 0.3679 | 0.3679 | 0.3679 |

(b)




(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

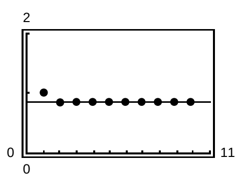
(d) The distance in part (c) is always less than the magnitude of the next series.

**8.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$

(a)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 1 | 0.8333 | 0.8417 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 |

(b)




(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

**10.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$

$\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$

Diverges by the $n$th Term Test.

**12.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$

$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

Converges by Theorem 8.14

**14.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2+1}$

$a_{n+1} = \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} = a_n$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

Converges by Theorem 8.14

**16.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{n^2+5}$

$\lim_{n\to\infty} \frac{n^2}{n^2+5} = 1$

Diverges by $n$th Term Test

**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

$$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1} \text{ for } n \geq 2$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = \lim_{n\to\infty} \frac{1/(n+1)}{1} = 0$$

Converges by Theorem 8.14

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Converges; (see Exercise 9)

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Converges; (see Exercise 9)

**24.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$$

$$\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$$

Converges by Theorem 8.14

**26.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

$$\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n\to\infty} n^{1/6} = \infty$$

Diverges by the $n$th Term Test

**28.** $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n}+1}$

Let $f(x) = \dfrac{2e^x}{e^{2x}+1}$. Then

$$f'(x) = \frac{2e^{2x}(1-e^{2x})}{(e^{2x}+1)^2} < 0 \text{ for } x > 0.$$

Thus, $f(x)$ is decreasing for $x > 0$ which implies $a_{n+1} < a_n$.

$$\lim_{n\to\infty} \frac{2e^n}{e^{2n}+1} = \lim_{n\to\infty} \frac{2e^n}{2e^{2n}} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 8.14.

**30.** $\displaystyle S_6 = \sum_{n=1}^{6} \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$

$$|R_6| = |S - S_6| \leq a_7 = \frac{4}{\ln 8} \approx 1.9236; 0.7831 \leq S \leq 4.6303$$

**32.** $\displaystyle S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}n}{2^n} = 0.1875$

$$|R_6| = |S - S_6| \leq a_7 = \frac{7}{2^7} \approx 0.05469; 0.1328 \leq S \leq 0.2422$$

**34.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$

(a) By Theorem 8.15,

$$|R_n| \leq a_{N+1} = \frac{1}{2^{N+1}(N+1)!} < 0.001.$$

This inequality is valid when $N = 4$.

(b) We may approximate the series by

$$\sum_{n=0}^{4} \frac{(-1)^n}{2^n n!} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{348} \approx 0.607.$$

(5 terms. Note that the sum begins with $n = 0$.)

**36.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(2N+2)!} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=0}^{3} \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.540.$$

(4 terms. Note that the sum begins with $n = 0$.)

**38.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}$

(a) By Theorem 8.15,

$$|R_N| \le a_{N+1} = \frac{1}{4^{N+1}(N+1)} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=1}^{3} \frac{(-1)^{n+1}}{4^n n} = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} \approx 0.224.$$

(3 terms)

**40.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

By Theorem 8.15, $|R_N| \le a_{N+1} = \dfrac{1}{(N+1)^4} < 0.001.$

This inequality is valid when $N = 5$.

**42.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

diverges by the Integral Test. Therefore, the series converge conditionally.

**44.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent $p$-series.

Therefore, the given series converges absolutely.

**46.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

$\displaystyle\lim_{n\to\infty} \frac{2n+3}{n+10} = 2$ Therefore, the series diverges by the $n$th Term Test.

**48.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by a comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

Therefore, the given series converges absolutely.

**50.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent $p$-series.

Therefore, the given series converge absolutely.

**52.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

diverges by a limit comparison to the divergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

**54.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

$\displaystyle\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \ne 0$ Therefore, the series diverges by the $n$th Term Test.

**56.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left|\frac{\sin[(2n-1)\pi/2]}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent $p$-series. Therefore, the series converges conditionally.

**58.** $|S - S_n| = |R_n| \leq a_{n+1}$ (Theorem 8.15)

**60.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ (Alternating Harmonic Series)

**62.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

If $p = 0$, then

$$\lim_{n\to\infty} \frac{1}{n^p} = 1$$

and the series diverges. If $p > 0$, then

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \text{ and } \frac{1}{(n+1)^p} < \frac{1}{n^p}.$$

Therefore, the series converge by the Alternating Series Test.

**64.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

**66.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

converges absolutely (by comparison) for

$$-1 < x < 1,$$

since

$$\left|\frac{x^n}{n}\right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for $-1 < x < 1$.

(b) When $x = -1$, we have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When $x = 1$, we have the divergent harmonic series

$$\frac{1}{n}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges conditionally for $x = -1$.

**68.** True, equivalent to Theorem 8.16

**70.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+5}$

converges by limit comparison to convergent $p$-series

$$\sum \frac{1}{n^2}.$$

**72.** Converges by limit comparison to convergent geometric series $\displaystyle\sum \frac{1}{2^n}$.

**74.** Diverges by $n$th Term Test. $\displaystyle\lim_{n\to\infty} a_n = \frac{3}{2}$

**76.** Converges (conditionally) by Alternating Series Test.

**78.** Diverges by comparison to Divergent Harmonic Series:

$$\frac{\ln n}{n} > \frac{1}{n} \text{ for } n \geq 3.$$

## Section 8.6 The Ratio and Root Tests

**2.** $\dfrac{(2k-2)!}{(2k)!} = \dfrac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \dfrac{1}{(2k)(2k-1)}$

**4.** Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid since $\dfrac{1}{1} = \dfrac{2^3 3!(3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} = \frac{2^n n!(2n-3)(2n-1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}.$$

To do this, note that:

$$\begin{aligned}
\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} \cdot \frac{1}{(2n-3)} \\
&= \frac{2^n n!\cancel{(2n-3)}(2n-1)}{(2n)!} \cdot \frac{1}{\cancel{(2n-3)}} \\
&= \frac{2^n n!(2n-1)}{(2n)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)} \\
&= \frac{2^n (2)(n+1)n!(2n-1)(2n+1)}{(2n)!(2n+1)(2n+2)} \\
&= \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}
\end{aligned}$$

The formula is valid for all $n \geq 3$.

**6.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \cdots$

$S_1 = \dfrac{3}{4}, S_2 \approx 1.03$

Matches (c)

**8.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$

$S_1 = 2$

Matches (b)

**10.** $\displaystyle\sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$

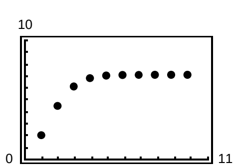
$S_1 = 4$

Matches (e)

**12.** (a) Ratio Test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\dfrac{(n+1)^2+1}{(n+1)!}}{\dfrac{n^2+1}{n!}} = \lim_{n\to\infty} \left(\frac{n^2+2n+2}{n^2+1}\right)\left(\frac{1}{n+1}\right) = 0 < 1$. Converges

(b)

| $n$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 7.0917 | 7.1548 | 7.1548 | 7.1548 | 7.1548 |

(c) 


(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

**14.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right| = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

**16.** $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)3^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n3^n}\right| = \lim_{n\to\infty} \frac{3(n+1)}{2n} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

$$\lim_{n\to\infty} \left|\frac{a_n + 1}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3/2^{n+1}}{n^3/2^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{2n^3}\right| = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \le \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n\to\infty} \frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 8.14, the series converges.

**Note:** The Ratio Test is inconclusive since $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1.$

The series converges conditionally.

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(3/2)^{n+1}}{n^2+2n+1} \cdot \frac{n^2}{(3/2)^n}\right| = \lim_{n\to\infty} \frac{3n^2}{2(n^2+2n+1)} = \frac{3}{2} > 1$$

Therefore, by the Ratio Test, the series diverges.

**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!}\right| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty$$

Therefore, by the Ratio Test, the series diverges.

**26.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right| = \lim_{n\to\infty} \frac{(n+1)(n+1)^n n!}{(n+1)n!n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e > 1$$

Therefore, by the Ratio Test, the series diverges.

**28.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{[(n+1)!]^2}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^2}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0$$

Therefore, by the Ratio Test, the series converges.

**30.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{4n+4}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{4n}}\right| = \lim_{n\to\infty} \frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n}\right| = \lim_{n\to\infty} \frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$

**34.** (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^4} \cdot \frac{n^4}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^4 = 1$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)^p} \cdot \frac{n^p}{1}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$$

**36.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Therefore, by the Root Test, the series diverges.

**38.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|}$$

$$= \lim_{n\to\infty} \left(\frac{3n}{2n+1}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Therefore, by the Root Test, the series diverges.

**40.** $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{e^n}} = \frac{1}{e}$$

Therefore, by the Root Test, the series converges.

**42.** $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n+1}{3^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n+1}}{3}$$

$$\text{Let}\quad y = \lim_{n\to\infty} \sqrt[x]{x+1}$$

$$\ln y = \lim_{n\to\infty} \left(\ln \sqrt[x]{x+1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{x} \ln(x+1)$$

$$= \lim_{n\to\infty} \frac{\ln(x+1)}{x} = \frac{1}{x+1} = 0.$$

Since $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

**44.** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n} = 5\sum_{n=1}^{\infty} \frac{1}{n}$

This is the divergent harmonic series.

**46.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\pi}{4}\right)^n$

Since $\pi/4 < 1$,this is convergent geometric series.

**48.** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2n^2+1}$

$$\lim_{n\to\infty}\frac{n/(2n^2+1)}{1/n} = \lim_{n\to\infty}\frac{n^2}{2n^2+1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n}.$$

**50.** $\displaystyle\sum_{n=1}^{\infty}\frac{10}{3\sqrt{n^3}}$

$$\lim_{n\to\infty}\frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a limit comparison test with the $p$-series

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}.$$

**52.** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{4n^2-1}$

$$\lim_{n\to\infty}\frac{2^n}{4n^2-1} = \lim_{n\to\infty}\frac{(\ln 2)2^n}{8n} = \lim_{n\to\infty}\frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the $n$th Term Test for Divergence.

**54.** $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)}$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \le \frac{1}{n\ln(n)} = a_n$$

$$\lim_{n\to\infty}\frac{1}{n\ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

**56.** $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}$

$$\frac{\ln(n)}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the $p$-series

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}.$$

**58.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{3^n}\right| = \lim_{n\to\infty}\frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

**60.** $\displaystyle\sum_{n=1}^{\infty}\frac{3\cdot5\cdot7\cdots(2n+1)}{18^n(2n-1)n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!}\cdot\frac{18^n(2n-1)n!}{3\cdot5\cdot7\cdots(2n+1)}\right| = \lim_{n\to\infty}\frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{2}{18} = \frac{1}{9}$$

Therefore, by the Ratio Test, the series converge.

**62.** (b) and (c)

$$\sum_{n=0}^{\infty}(n+1)\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty}n\left(\frac{3}{4}\right)^{n-1}$$

$$= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots$$

**64.** (a) and (b) are the same.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)2^{n-1}} = \frac{1}{2} - \frac{1}{2\cdot2^2} + \frac{1}{3\cdot2^3} - \cdots$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n} = \frac{1}{2} - \frac{1}{2\cdot2^2} + \frac{1}{3\cdot2^3} - \cdots$$

**66.** Replace $n$ with $n + 2$.

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!}$$

**68.** $$\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \, . \, . \, . \, (2k+1)} = \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)} = \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \approx 0.40967$$

(See Exercise 3 and use 10 terms, $k = 9$.)

**70.** See Theorem 8.18.

**72.** One example is $\displaystyle\sum_{n=1}^{\infty}\left(-100 + \frac{1}{n}\right)$.

**74.** Assume that

$\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$ or that $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$.

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$$|a_{n+1}| > |a_n|, \quad n > N \Longrightarrow \lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum a_n \text{ diverges}$$

**76.** The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and $f(x)$ is a real function such that $f(1/n) = U_n$ for all positive integers $n$ and $d^2 f/dx^2$ exists at $x = 0$, then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise. Below are some examples.

| Convergent Series | Divergent Series |
|---|---|
| $\sum \frac{1}{n^3}, f(x) = x^3$ | $\sum \frac{1}{n}, f(x) = x$ |
| $\sum\left(1 - \cos\frac{1}{n}\right), f(x) = 1 - \cos x$ | $\sum \sin\frac{1}{n}, f(x) = \sin x$ |

## Section 8.7 Taylor Polynomials and Approximations

**2.** $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

$y$-axis symmetry

Three relative extrema

Matches (c)

**4.** $y = e^{-1/2}\left[\frac{1}{3}(x-1)^3 - (x-1) + 1\right]$

Cubic

Matches (b)

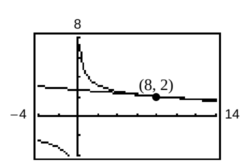
**6.** $f(x) = \dfrac{4}{\sqrt[3]{x}} = 4x^{-1/3}$ $\qquad f(8) = 2$

$f'(x) = -\dfrac{4}{3}x^{-4/3}$ $\qquad f'(8) = -\dfrac{1}{12}$

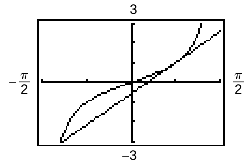
$P_1(x) = f(8) + f'(8)(x - 8)$

$\qquad = 2 + \left(-\dfrac{1}{12}\right)(x - 8)$

$P_1(x) = -\dfrac{1}{12}x + \dfrac{8}{3}$



**8.** $f(x) = \tan x \qquad f\left(\frac{\pi}{4}\right) = 1$



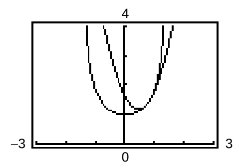

$f'(x) = \sec^2 x \qquad f'\left(\frac{\pi}{4}\right) = 2$

$$P_1 = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = 2x + 1 - \frac{\pi}{2}$$

**10.** $f(x) = \sec x \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$




$f'(x) = \sec x \tan x \qquad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

| $x$ | $-2.15$ | 0.585 | 0.685 | $\pi/4$ | 0.885 | 0.985 | 1.785 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-1.8270$ | 1.1995 | 1.2913 | 1.4142 | 1.5791 | 1.8088 | $-4.7043$ |
| $P_2(x)$ | 15.5414 | 1.2160 | 1.2936 | 1.4142 | 1.5761 | 1.7810 | 4.9475 |

**12.** $f(x) = x^2e^x, f(0) = 0$

(a) $f'(x) = (x^2 + 2x)e^x \qquad f'(0) = 0$

$f''(x) = (x^2 + 4x + 2)e^x \qquad f''(0) = 2$

$f'''(x) = (x^2 + 6x + 6)e^x \qquad f'''(0) = 6$

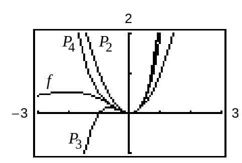
$f^{(4)}(x) = (x^2 + 8x + 12)e^x \qquad f^{(4)}(0) = 12$

$$P_2(x) = \frac{2x^2}{2!} = x^2$$

$$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$$

$$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$$

(b)




(c) $f''(0) = 2 = P_2''(0)$

$f'''(0) = 6 = P_3'''(0)$

$f^{(4)}(0) = 12 = P_4^{(4)}(0)$

(d) $f^{(n)}(0) = P_n^{(n)}(0)$

**14.** $f(x) = e^{-x} \qquad f(0) = 1$

$f'(x) = -e^{-x} \qquad f'(0) = -1$

$f''(x) = e^{-x} \qquad f''(0) = 1$

$f'''(x) = -e^{-x} \qquad f'''(0) = -1$

$f^{(4)}(x) = e^{-x} \qquad f^{(4)}(0) = 1$

$f^{(5)}(x) = -e^{-x} \qquad f^{(5)}(0) = -1$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

**16.** $f(x) = e^{3x}$ $\quad f(0) = 1$

$f'(x) = 3e^{3x}$ $\quad f'(0) = 3$

$f''(x) = 9e^{3x}$ $\quad f''(0) = 9$

$f'''(x) = 27e^{3x}$ $\quad f'''(0) = 27$

$f^{(4)}(x) = 81e^{3x}$ $\quad f^{(4)}(0) = 81$

$$P_4(x) = 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

**18.** $f(x) = \sin \pi x$ $\quad f(0) = 0$

$f'(x) = \pi \cos \pi x$ $\quad f'(0) = \pi$

$f''(x) = -\pi^2 \sin \pi x$ $\quad f''(0) = 0$

$f'''(x) = -\pi^3 \cos \pi x$ $\quad f'''(0) = -\pi^3$

$$P_3(x) = 0 + \pi x + \frac{0}{2!}x^2 + \frac{-\pi^3}{3!}x^3 = \pi x - \frac{\pi^3}{6}x^3$$

**20.** $f(x) = x^2e^{-x}$ $\quad f(0) = 0$

$f'(x) = 2xe^{-x} - x^2e^{-x}$ $\quad f'(0) = 0$

$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$ $\quad f''(0) = 2$

$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x}$ $\quad f'''(0) = -6$

$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x}$ $\quad f^{(4)}(0) = 12$

$$P_4(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4$$
$$= x^2 - x^3 + \frac{1}{2}x^4$$

**22.** $f(x) = \dfrac{x}{x+1} = \dfrac{x+1-1}{x+1} = 1 - (x+1)^{-1}$ $\quad f(0) = 0$

$f'(x) = (x+1)^{-2}$ $\quad f'(0) = 1$

$f''(x) = -2(x+1)^{-3}$ $\quad f''(0) = -2$

$f'''(x) = 6(x+1)^{-4}$ $\quad f'''(0) = 6$

$f^{(4)}(x) = -24(x+1)^{-5}$ $\quad f^{(4)}(0) = -24$

$$P_4(x) = 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 = x - x^2 + x^3 - x^4$$

**24.** $f(x) = \tan x$ $\quad f(0) = 0$

$f'(x) = \sec^2 x$ $\quad f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''(0) = 0$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ $\quad f'''(0) = 2$

$$P_3(x) = 0 + 1(x) + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3$$

**26.** $f(x) = 2x^{-2}$ $\quad f(2) = \dfrac{1}{2}$

$f'(x) = -4x^{-3}$ $\quad f'(2) = -\dfrac{1}{2}$

$f''(x) = 12x^{-4}$ $\quad f''(2) = \dfrac{3}{4}$

$f'''(x) = -48x^{-5}$ $\quad f'''(x) = -\dfrac{3}{2}$

$f^{(4)}(x) = 240x^{-6}$ $\quad f^{(4)}(x) = \dfrac{15}{4}$

$$P_4(x) = \frac{1}{2} - \frac{1}{2}(x-2) + \frac{3}{8}(x-2)^2 - \frac{1}{4}(x-2)^3 + \frac{5}{32}(x-2)^4$$

**28.** $f(x) = x^{1/3}$ $\quad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\quad f''(8) = -\frac{1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3}$ $\quad f'''(8) = \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20{,}736}(x-8)^3$$

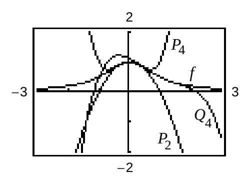
**30.** $f(x) = x^2 \cos x$ $\quad f(\pi) = -\pi^2$

$f'(x) = \cos x - x^2 \sin x$ $\quad f'(\pi) = -2\pi$

$f''(x) = 2\cos x - 4x \sin x - x^2 \cos x$ $\quad f''(\pi) = -2 + \pi^2$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{(\pi^2-2)}{2}(x-\pi)^2$$

**32.** $f(x) = \dfrac{1}{x^2+1}$




$f'(x) = \dfrac{-2x}{(x^2+1)^2}$

$f''(x) = \dfrac{2(3x^2-1)}{(x^2+1)^3}$

$f'''(x) = \dfrac{24x(1-x^2)}{(x^2+1)^4}$

$f^{(4)}(x) = \dfrac{24(5x^4-10x^2+1)}{(x^2+1)^5}$

(a) $n = 2, c = 0$

$$P_2(x) = 1 + 0x + \frac{-2}{2!}x^2 = 1 - x^2$$

(b) $n = 4, c = 0$

$$P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$$

(c) $n = 4, c = 1$

$$Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x-1) + \frac{1/2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4 = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4$$

**34.** $f(x) = \ln x$

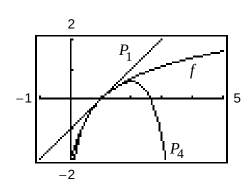
$P_1(x) = x - 1$

$P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

(a)

| $x$ | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
|---|---|---|---|---|---|
| $\ln x$ | 0.0000 | 0.2231 | 0.4055 | 0.5596 | 0.6931 |
| $P_1(x)$ | 0.0000 | 0.2500 | 0.5000 | 0.7500 | 1.0000 |
| $P_4(x)$ | 0.0000 | 0.2230 | 0.4010 | 0.5303 | 0.5833 |

(b)




(c) As the distance increases, the accuracy decreases.

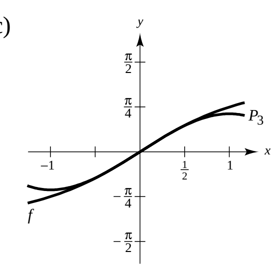
**36.** (a) $f(x) = \arctan x$

$P_3(x) = x - \dfrac{x^3}{3}$

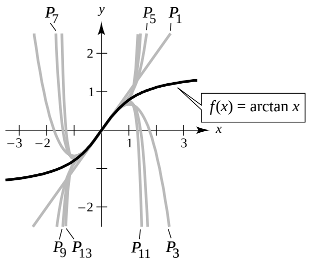
(b)

| $x$ | $-0.75$ | $-0.50$ | $-0.25$ | 0 | 0.25 | 0.50 | 0.75 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-0.6435$ | $-0.4636$ | $-0.2450$ | 0 | 0.2450 | 0.4636 | 0.6435 |
| $P_3(x)$ | $-0.6094$ | $-0.4583$ | $-0.2448$ | 0 | 0.2448 | 0.4583 | 0.6094 |

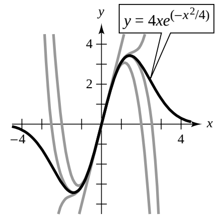
(c)



**38.** $f(x) = \arctan x$




**40.** $f(x) = 4xe^{-x^2/4}$




**42.** $f(x) = x^2e^{-x} \approx x^2 - x^3 + \dfrac{1}{2}x^4$

$f\left(\dfrac{1}{5}\right) \approx 0.0328$

**44.** $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x - \pi) + \left(\dfrac{\pi^2 - 2}{2}\right)(x - \pi)^2$

$f\left(\dfrac{7\pi}{8}\right) \approx -6.7954$

**46.** $f(x) = e^x; f^{(6)}(x) = e^x \implies$ Max on $[0, 1]$ is $e^1$.

$R_5(x) \le \dfrac{e^1}{6!}(1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$

**48.** $f(x) = \arctan x; f^{(4)}(x) = \dfrac{24x(x^2+1)}{(1-x^2)^4}$

$\implies$Max on $[0, 0.4]$ is $f^{(4)}(0.4) \approx 22.3672$.

$R_3(x) \le \dfrac{22.3672}{4!}(0.4)^4 \approx 0.0239$

**50.** $f(x) = e^x$

$f^{(n+1)}(x) = e^x$

Max on $[0, 0.6]$ is $e^{0.6} \approx 1.8221$.

$R_n \le \dfrac{1.8221}{(n+1)!}(0.6)^{n+1} < 0.001$

By trial and error, $n = 5$.

**52.** $f(x) = \cos(\pi x^2)$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$f(x) = g(\pi x^2)$$

$$= 1 - \frac{(\pi x^2)^2}{2!} + \frac{(\pi x^2)^4}{4!} - \frac{(\pi x^2)^6}{6!} + \cdots$$

$$= 1 - \frac{\pi^2 x^4}{2!} + \frac{\pi^4 x^8}{4!} - \frac{\pi^6 x^{12}}{6!} + \cdots$$

$$f(0.6) = 1 - \frac{\pi^2}{2!}(0.6)^4 + \frac{\pi^4}{4!}(0.6)^8 - \frac{\pi^6}{6!}(0.6)^{12} + \cdots$$

Since this is an alternating series,

$$R_n \le a_{n+1} = \frac{\pi^{2n}}{(2n)!}(0.6)^{4n} < 0.0001.$$

By trial and error, $n = 4$. Using 4 terms $f(0.6) \approx 0.4257$.

**54.** $f(x) = \sin x \approx x - \dfrac{x^3}{3!}$

$$|R_3(x)| = \left|\frac{\sin z}{4!}x^4\right| \le \frac{|x^4|}{4!} < 0.001$$

$$x^4 < 0.024$$

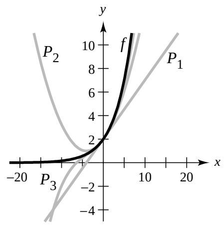
$$|x| < 0.3936$$

$$-0.3936 < x < 0.3936$$

**56.** $f(c) = P_2(c)$, $f'(c) = P_2'(c)$, and $f''(c) = P_2''(c)$

**58.** See Theorem 8.19, page 611.

**60.**




**62.** (a) $P_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ for $f(x) = \sin x$

$$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) $Q_6(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$ for $\cos x$

$$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c) $R(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

**64.** Let $f$ be an odd function and $P_n$ be the $n^{\text{th}}$ Maclaurin polynomial for $f$. Since $f$ is odd, $f'$ is even:

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h\to 0} \frac{f(x+(-h)) - f(x)}{-h} = f'(x).$$

Similarly, $f''$ is odd, $f'''$ is even, etc. Therefore, $f, f'', f^{(4)}$, etc. are all odd functions, which implies that $f(0) = f''(0) = \cdots = 0$. Hence, in the formula

$P_n(x) = f(0) + f'(0)x + \dfrac{f''(0)x^2}{2!} + \cdots$ all the coefficients of the even power of $x$ are zero.

**66.** Let $P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$ where $a_i = \dfrac{f^{(i)}(c)}{i!}$.

$$P_n(c) = a_0 = f(c)$$

For $1 \le k \le n$, $\quad P_n^{(k)}(c) = a_n k! = \left(\dfrac{f^{(k)}(c)}{k!}\right)k! = f^{(k)}(c)$.

## Section 8.8 Power Series

**2.** Centered at 0

**4.** Centered at $\pi$

**6.** $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

$$L = \lim_{n\to\infty} \left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(2x)^{n+1}}{(2x)^n}\right| = 2|x|$$

$$2|x| < 1 \implies R = \frac{1}{2}$$

**8.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

$$L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n x^n}\right|$$

$$= \frac{1}{2}|x|$$

$$\frac{1}{2}|x| < 1 \implies R = 2$$

**10.** $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

$$L = \lim_{n\to\infty}\left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+2}/(n+1)!}{(2n)!x^{2n}/n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x^2}{(n+1)}\right| = \infty$$

The series only converges at $x = 0$. $R = 0$.

**12.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{k}\right)^n$

Since the series is geometric, it converges only if $|x/k| < 1$ or $-k < x < k$.

**14.** $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{u_n + 1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)x}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the series $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)$ diverges.

When $x = -1$, the series $\displaystyle\sum_{n=0}^{\infty}-(n+1)$ diverges.

Therefore, the interval of convergence is $-1 < x < 1$.

**16.** $\displaystyle\sum_{n=0}^{\infty}\frac{(3x)^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+1)!}\cdot\frac{(2n)!}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

**18.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(n+1)(n+2)}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x}{n+3}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)}$ converges.

When $x = -1$, the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}$ converges by limit comparison to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Therefore, the interval of convergence is $-1 \le x \le 1$.

**20.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n!(x-4)^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!(x-4)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(-1)^n n!(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-4)}{3}\right| = \infty$$

$R = 0$

Center: $x = 4$

Therefore, the series converges only for $x = 4$.

**22.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-2)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{4(n+2)}\right| = \frac{1}{4}|x-2|$$

$R = 4$

Center: $x = 2$

Interval: $-4 < x - 2 < 4$ or $-2 < x < 6$

When $x = -2$, the alternating series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$ converges.

When $x = 6$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

Therefore, the interval of convergence is $-2 \le x < 6$.

**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}} \cdot \frac{nc^n}{(-1)^{n+1}(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-c)}{c(n+1)}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the $p$-series $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 2c$, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Therefore, the interval of convergence is $0 < x \le 2c$.

**26.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+3)}x^2\right| = |x^2|$$

$R = 1$

Interval: $-1 < x < 1$

When $x = 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

When $x = -1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

Therefore, the interval of convergence is $-1 \le x \le 1$.

**28.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

**30.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} (x^{2n+1})$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{2 \cdot 4 \cdots (2n)(2n+2)x^{2n+3}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)x^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)x^2}{(2n+3)}\right| = |x^2|$$

$R = 1$

When $x = \pm 1$, the series diverges by comparing it to

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

which diverges. Therefore, the interval of convergence is $-1 < x < 1$.

**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!(x-c)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!(x-c)}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-c)}{2n+1}\right| = \frac{1}{2}|x-c|$$

$R = 2$

Interval: $-2 < x - c < 2$ or $c - 2 < x < c + 2$

The series diverges at the endpoints. Therefore, the interval of convergence is $c - 2 < x < c + 2$.

$$\left[\frac{n!(c+2-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{n!\,2^2}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1\right]$$

**36.** (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$, $0 < x \le 10$

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}$, $0 < x < 10$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}$, $0 < x < 10$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}$, $0 \le x \le 10$

**38.** (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$, $1 < x \le 3$

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}$, $1 < x < 3$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}$, $1 < x < 3$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}$, $1 \le x \le 3$

**40.** $g(2) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$

$S_1 = 1$, $S_2 = 1.67$. Matches (a)

**42.** $g(-2) = \displaystyle\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$ alternating. Matches (d)

**44.** The set of all values of $x$ for which the power series converges is the interval of convergence. If the power series converges for all $x$, then the radius of convergence is $R = \infty$. If the power series converges at only $c$, then $R = 0$. Otherwise, according to Theorem 8.20, there exists a real number $R > 0$ (radius of convergence) such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.

**46.** You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

**48.** (a) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $-\infty < x < \infty$ (See Exercise 11)

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(c) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f(0) = 1$

(d) $f(x) = e^x$

**50.**

$$y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n 4nx^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y'' = \sum_{n=1}^{\infty} \frac{(-1)^n 4n\cancel{(4n-1)}x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots \cancel{(4n-1)}} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)}$$

$$y'' + x^2 y = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0$$

**52.** $J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1}k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$

(a) $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3}(k+1)!(k+2)!} \cdot \frac{2^{2k+1}k!(k+1)!}{(-1)^k x^{2k+1}}\right| = \lim_{k\to\infty} \left|\frac{(-1)x^2}{2^2 (k+2)(k+1)}\right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1!)}$

$$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k}}{2^{2k+1}k!(k+1)!}$$

$$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k-1}}{2^{2k+1}k!(k+1)!}$$

$$x^2 J_1'' + xJ_1' + (x^2-1)J_1 = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

$$+ \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

$$= \left[\sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)(2k)x^{2k+1}}{2^{2k+1}k!(k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k+1}k!(k+1)!}\right.$$

$$\left. - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right] + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}[(2k+1)(2k) + (2k+1) - 1]}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

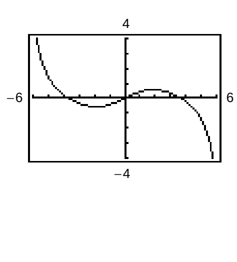
$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1}(k-1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1}k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1}k!(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}[(-1)+1]}{2^{2k+1}k!(k+1)!} = 0$$

(c) $P_7(x) = \frac{x}{2} - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18{,}432}x^7$



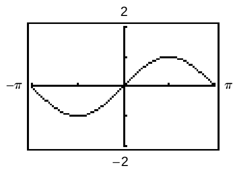

(d) $J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2(k+1)x^{2k+1}}{2^{2k+2}(k+1)!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1}k!(k+1)!}$

$$-J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

**Note:** $J_0'(x) = -J_1(x)$

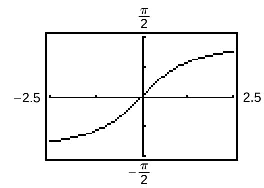
**54.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = \sin x$

(See Exercise 47.)




**56.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = \arctan x, \; -1 \le x \le 1$

(See Exercise 38 in Section 8.7.)




**58.** $\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \dfrac{x^{2n-1}}{(2n-1)!}$

Replace $n$ with $n - 1$.

**60.** True; if

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $x = 2$, then we know that it must converge on $(-2, 2]$.

**62.** True

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}\right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

## Section 8.9 Representation of Functions by Power Series

**2.** (a) $f(x) = \dfrac{4}{5-x} = \dfrac{4/5}{1-x/5} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{4x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

(b)
$$\begin{array}{r l}
 & \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4x^3}{625} + \cdots \\
5 - x \,\big) & 4 \\
 & \underline{4 - \frac{4}{5}x} \\
 & \frac{4}{5}x \\
 & \underline{\frac{4}{5}x - \frac{4}{25}x^2} \\
 & \frac{4}{25}x^2 \\
 & \underline{\frac{4}{25}x^2 - \frac{4x^3}{125}} \\
 & \frac{4x^3}{125} \\
 & \underline{\frac{4x^3}{125} - \frac{4x^4}{625}} \\
 & \vdots
\end{array}$$

**4.** (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

This series converges on $(-1, 1)$.

(b)
$$\begin{array}{r l}
 & 1 - x + x^2 - x^3 + \cdots \\
1 + x \,\big) & 1 \\
 & \underline{1 + x} \\
 & -x \\
 & \underline{-x - x^2} \\
 & x^2 \\
 & \underline{x^2 + x^3} \\
 & -x^3 \\
 & \underline{-x^3 - x^4} \\
 & \vdots
\end{array}$$

**6.** Writing $f(x)$ in the form $\dfrac{a}{1-r}$, we have

$$\frac{4}{5-x} = \frac{4}{7-(x+2)} = \frac{4/7}{1-1/7(x+2)} = \frac{a}{1-r}.$$

Therefore, the power series for $f(x)$ is given by

$$\frac{4}{5-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{4}{7}\left(\frac{1}{7}(x+2)\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{4(x+2)^n}{7^{n+1}}.$$

$|x+2| < 7$ or $-5 < x < 9$

**8.** Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = (-2/3)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{3}{2x-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left[-\frac{2}{3}(x-2)\right]^n,$$
$$= \sum_{n=0}^{\infty} \frac{(-2)^n(x-2)^n}{3^n},$$

$|x-2| < \dfrac{3}{2}$ or $\dfrac{1}{2} < x < \dfrac{7}{2}$.

**10.** Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{1}{-5+2x} = \frac{-1/5}{1-(2/5)x} = \frac{a}{1-r}$$

which implies that $a = -1/5$ and $r = (2/5)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)\left(\frac{2}{5}x\right)^n = -\sum_{n=0}^{\infty}\frac{2^n x^n}{5^{n+1}},$$

$|x| < \dfrac{5}{2}$ or $-\dfrac{5}{2} < x < \dfrac{5}{2}$.

**12.** Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{4}{3x+2} = \frac{4}{8+3(x-2)} = \frac{1/2}{1+(3/8)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1/2$ and $r = (-3/8)(x-2)$. Therefore, the power series for $f(x)$ is given by

$$\frac{4}{3x+2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\frac{1}{2}\left[-\frac{3}{8}(x-2)\right]^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-3)^n(x-2)^n}{8^n},$$

$|x-2| < \dfrac{8}{3}$ or $-\dfrac{2}{3} < x < \dfrac{14}{3}$.

**14.** $$\frac{4x-7}{2x^2+3x-2} = \frac{3}{x+2} - \frac{2}{2x-1} = \frac{3}{2+x} - \frac{2}{-1+2x} = \frac{3/2}{1+(1/2)x} + \frac{2}{1-2x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{4x-7}{2x^2+3x-2} = \sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} 2(2x)^n = \sum_{n=0}^{\infty}\left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1}\right]x^n,\ |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

**16.** First finding the power series for $4/(4+x)$, we have

$$\frac{1}{1+(1/4)x} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n}$$

Now replace $x$ with $x^2$.

$$\frac{4}{4+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n}.$$

The interval of convergence is $|x^2| < 4$ or $-2 < x < 2$ since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{4^{n+1}}\cdot\frac{4^n}{(-1)^n x^{2n}}\right| = \left|-\frac{x^2}{4}\right| = \frac{|x^2|}{4}.$$

**18.** $$h(x) = \frac{x}{x^2-1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{1}{2}\sum_{n=0}^{\infty}x^n$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}[(-1)^n - 1]x^n = \frac{1}{2}[0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots]$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2)x^{2n+1} = -\sum_{n=0}^{\infty}x^{2n+1},\ 1 < x < 1$$

**20.** By taking the second derivative, we have $\dfrac{d^2}{dx^2}\left[\dfrac{1}{x+1}\right] = \dfrac{2}{(x+1)^3}$. Therefore,

$$\begin{aligned}\frac{2}{(x+1)^3} &= \frac{d^2}{dx^2}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] \\ &= \frac{d}{dx}\left[\sum_{n=1}^{\infty}(-1)^n n x^{n-1}\right] = \sum_{n=2}^{\infty}(-1)^n n(n-1)x^{n-2} = \sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n,\ -1 < x < 1.\end{aligned}$$

**22.** By integrating, we have

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C_1 \text{ and } \int \frac{1}{1-x}\,dx = -\ln(1-x) + C_2.$$

$f(x) = \ln(1-x^2) = \ln(1+x) - [-\ln(1-x)]$. Therefore,

$$\begin{aligned}\ln(1-x^2) &= \int \frac{1}{1+x}\,dx - \int \frac{1}{1-x}\,dx \\ &= \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx - \int\left[\sum_{n=0}^{\infty} x^n\right]dx = \left[C_1 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}\right] - \left[C_2 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right] \\ &= C + \sum_{n=0}^{\infty}\frac{[(-1)^n - 1]x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n+1}\end{aligned}$$

To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1-x^2) = -\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1},\ -1 < x < 1$$

**24.** $$\begin{aligned}\frac{2x}{x^2+1} &= 2x\sum_{n=0}^{\infty}(-1)^n x^{2n} \text{ (See Exercise 23.)} \\ &= \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\end{aligned}$$

Since $\dfrac{d}{dx}(\ln(x^2+1)) = \dfrac{2x}{x^2+1}$, we have

$$\ln(x^2+1) = \int\left[\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1},\ -1 \le x \le 1.$$

To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x^2+1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1},\ -1 \le x \le 1.$$

**26.** Since $\displaystyle\int \frac{1}{4x^2+1}\,dx = \frac{1}{2}\arctan(2x)$, we can use the result of Exercise 25 to obtain

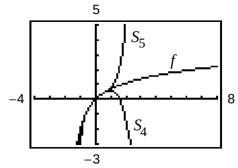
$$\arctan(2x) = 2\int \frac{1}{4x^2+1}\,dx = 2\int\left[\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\right]dx = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1},\ -\frac{1}{2} < x \le \frac{1}{2}.$$

To solve for $C$, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\arctan(2x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1},\ -\frac{1}{2} < x \le \frac{1}{2}.$$

**28.** $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \le \ln(x+1)$

$$\le x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$




| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ | 0.0 | 0.18227 | 0.33493 | 0.45960 | 0.54827 | 0.58333 |
| $\ln(x+1)$ | 0.0 | 0.18232 | 0.33647 | 0.47000 | 0.58779 | 0.69315 |
| $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ | 0.0 | 0.18233 | 0.33698 | 0.47515 | 0.61380 | 0.78333 |

In Exercise 35–38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

**30.** $g(x) = x - \frac{x^3}{3}$, cubic with 3 zeros.

Matches (d)

**32.** $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$,

Matches (b)

**34.** The approximations of degree 3, 7, 11, . . . $(4n-1, n = 1, 2, \ldots)$ have relative extrema.

**In Exercises 36 and 38,** $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

**36.**
$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$
$$\int \arctan x^2\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$
$$\int_0^{3/4} \arctan x^2\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$
$$= \frac{27}{192} - \frac{2187}{344{,}064} + \frac{177{,}147}{230{,}686{,}720}$$

Since $177{,}147/230{,}686{,}720 < 0.001$, we can approximate the series by its first two terms: 0.13427

**38.**
$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$
$$\int x^2 \arctan x\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$
$$\int_0^{1/2} x^2 \arctan x\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \cdots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} x^2 \arctan x\,dx \approx 0.015625$.

**In Exercises 40 and 42,** $\dfrac{\mathbf{1}}{\mathbf{1-x}} = \displaystyle\sum_{n=0}^{\infty} x^n,\ |x|1.$

**40.** Replace $n$ with $n + 1$.

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

**42.** (a) $\displaystyle \frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{2}{9}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1} = \frac{2}{9}\frac{1}{[1-(2/3)]^2} = 2$

(b) $\displaystyle \frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n = \frac{9}{100}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^{n-1}$

$$= \frac{9}{100}\cdot\frac{1}{[1-(9/10)]^2} = 9$$

**44.** Replace $x$ with $x^2$.

**46.** Integrate the series and multiply by $(-1)$.

**48.** (a) From Exercise 47, we have

$$\arctan\frac{120}{119} - \arctan\frac{1}{239} = \arctan\frac{120}{119} + \arctan\left(-\frac{1}{239}\right)$$

$$= \arctan\left[\frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)}\right] = \arctan\left(\frac{28{,}561}{28{,}561}\right) = \arctan 1 = \frac{\pi}{4}$$

(b) $\displaystyle 2\arctan\frac{1}{5} = \arctan\frac{1}{5} + \arctan\frac{1}{5} = \arctan\left[\frac{2(1/5)}{1-(1/5)^2}\right] = \arctan\frac{10}{24} = \arctan\frac{5}{12}$

$$4\arctan\frac{1}{5} = 2\arctan\frac{1}{5} + 2\arctan\frac{1}{5} = \arctan\frac{5}{12} + \arctan\frac{5}{12} = \arctan\left[\frac{2(5/12)}{1-(5/12)^2}\right] = \arctan\frac{120}{119}$$

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

**50.** (a) $\displaystyle \arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left[\frac{(1/2)+(1/3)}{1-(1/2)(1/3)}\right] = \arctan\left(\frac{5/6}{5/6}\right) = \frac{\pi}{4}$

(b) $\displaystyle \pi = 4\left[\arctan\frac{1}{2} + \arctan\frac{1}{3}\right]$

$$= 4\left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7}\right] + 4\left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7}\right]$$

$$\approx 4(0.4635) + 4(0.3217) \approx 3.14$$

**52.** From Exercise 51, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^n n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/3)^n}{n}$$

$$= \ln\left(\frac{1}{3}+1\right) = \ln\frac{4}{3} \approx 0.2877.$$

**54.** From Example 5, we have $\displaystyle \arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}.$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+1}$$

$$= \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

**56.** From Exercise 54, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(1/3)^{2n+1}}{2n+1}$$

$$= \arctan\frac{1}{3} \approx 0.3218.$$

**58.** From Example 5, we have $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n}(2n+1)} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
&= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1} \\
&= \sqrt{3} \arctan \frac{1}{\sqrt{3}} \\
&= \sqrt{3}\left(\frac{\pi}{6}\right) = \frac{\pi}{2\sqrt{3}}
\end{aligned}$$

## Section 8.10 Taylor and Maclaurin Series

**2.** For $c = 0$, we have

$$\begin{aligned}
f(x) &= e^{3x} \\
f^{(n)}(x) &= 3^n e^{3x} \implies f^{(n)}(0) = 3^n \\
e^{3x} &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}
\end{aligned}$$

**4.** For $c = \pi/4$, we have:

$$\begin{aligned}
f(x) &= \sin x & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\
f'(x) &= \cos x & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\
f''(x) &= -\sin x & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\
f'''(x) &= -\cos x & f'''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\
f^{(4)}(x) &= \sin x & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}
\end{aligned}$$

and so on. Therefore we have:

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\
&= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \cdots\right] \\
&= \frac{\sqrt{2}}{2}\left\{\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (\pi/4)]^{n+1}}{(n+1)!} + 1\right\}
\end{aligned}$$

**6.** For $c = 1$, we have:

$$\begin{aligned}
f(x) &= e^x \\
f^{(n)}(x) &= e^x \implies f^{(n)}(1) = e \\
e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}
\end{aligned}$$

**8.** For $c = 0$, we have:

$$f(x) = \ln(x^2 + 1) \qquad f(0) = 0$$

$$f'(x) = \frac{2x}{x^2 + 1} \qquad f'(0) = 0$$

$$f''(x) = \frac{2 - 2x^2}{(x^2 + 1)^2} \qquad f''(0) = 2$$

$$f'''(x) = \frac{4x(x^2 - 3)}{(x^2 + 1)^3} \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \frac{12(-x^4 + 6x^2 - 1)}{(x^2 + 1)^4} \qquad f^{(4)}(0) = -12$$

$$f^{(5)}(x) = \frac{48x(x^4 - 10x^2 + 5)}{(x^2 + 1)^5} \qquad f^{(5)}(0) = 0$$

$$f^{(6)}(x) = \frac{-240(5x^6 - 15x^4 + 15x^2 - 1)}{(x^2 + 1)^6} \qquad f^{(6)}(0) = 240$$

and so on. Therefore, we have:

$$\ln(x^2 + 1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n + 1}$$

**10.** For $c = 0$, we have;

$$f(x) = \tan(x) \qquad f(0) = 0$$

$$f'(x) = \sec^2(x) \qquad f'(0) = 1$$

$$f''(x) = 2\sec^2(x)\tan(x) \qquad f''(0) = 0$$

$$f'''(x) = 2[\sec^4(x) + 2\sec^2(x)\tan^2(x)] \qquad f'''(0) = 2$$

$$f^{(4)}(x) = 8[\sec^4(x)\tan(x) + \sec^2(x)\tan^3(x)] \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 8[2\sec^6(x) + 11\sec^4(x)\tan^2(x) + 2\sec^2(x)\tan^4(x)] \qquad f^{(5)}(0) = 16$$

$$\tan(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

**12.** The Maclaurin Series for $f(x) = e^{-2x}$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$.

$f^{(n+1)}(x) = (-2)^{n+1}e^{-2x}$. Hence, by Taylor's Theorem,

$$0 \le |Rn(x)| = \left|\frac{f^{(n+1)}(z)}{(n + 1)!}x^{n+1}\right| = \left|\frac{(-2)^{n+1}e^{-2z}}{(n + 1)!}x^{n+1}\right|.$$

Since $\displaystyle\lim_{n\to\infty} \left|\frac{(-2)^{n+1}x^{n+1}}{(n + 1)!}\right| = \lim_{n\to\infty} \left|\frac{(2x)^{n+1}}{(n + 1)!}\right| = 0$, it follows that $Rn(x) \to 0$ as $n \to \infty$.

Hence, the Maclaurin Series for $e^{-2x}$ converges to $e^{-2x}$ for all $x$.

**14.** Since $(1+x)^{-k} = 1 - kx + \dfrac{k(k+1)x^2}{2!} - \dfrac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\left[1+(-x)\right]^{-1/2} = 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \cdots$$

$$= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$$

**16.** Since $(1+x)^k = 1 + kx + \dfrac{k(k-1)x^2}{2!} + \dfrac{k(k-1)(k-2)x^3}{3!} + \cdots$, we have

$$(1+x)^{1/3} = 1 + \left(\frac{1}{3}\right)x + \frac{(1/3)(-2/3)x^2}{2!} + \frac{(1/3)(-2/3)(-5/3)x^3}{3!} + \cdots$$

$$= 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots$$

$$= 1 + \frac{x}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2 \cdot 5 \cdot 8 \cdots (3n-4)}{3^n n!}.$$

**18.** Since $(1+x)^{1/2} = 1 + \dfrac{x}{2} + \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n n!}$ (Exercise 14)

we have $(1+x^3)^{1/2} = 1 + \dfrac{x^3}{2} + \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{3n}}{2^n n!}.$

**20.**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots$$

**22.**

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos 4x = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$$

$$= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \cdots$$

**24.**

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$2\sin x^3 = 2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$

$$= 2\left[x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right]$$

$$= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \cdots$$

**26.**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$$

$$2\cos h(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2\frac{x^{2n}}{(2n)!}$$

**28.** The formula for the binomial series gives $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n\, n!}$, which implies that

$$(1+x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n\, n!}$$

$$\begin{aligned}\ln\left(x + \sqrt{x^2+1}\right) &= \int \frac{1}{\sqrt{x^2+1}}\,dx \\ &= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \ldots (2n-1)^{2n+1}}{2^n(2n+1)n!} \\ &= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.\end{aligned}$$

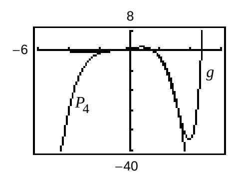
**30.** $$\begin{aligned}x \cos x &= x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\ &= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}\end{aligned}$$

**32.** $$\begin{aligned}\frac{\arcsin x}{x} &= \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} \cdot \frac{1}{x} \\ &= \sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{(2^n n!)^2(2n+1)},\ x \neq 0\end{aligned}$$

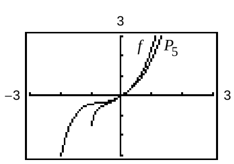
**34.** $e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots$ (See Exercise 33.)

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

**36.** $$\begin{aligned}g(x) &= e^x \cos x \\ &= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \\ &= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{x^3}{2}\right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}\right) + \cdots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots\end{aligned}$$



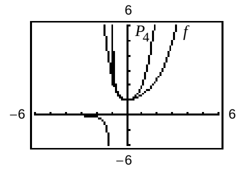

**38.** $$\begin{aligned}f(x) &= e^x \ln(1+x) \\ &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\ &= x + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6}\right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots\end{aligned}$$



**40.** $f(x) = \dfrac{e^x}{1+x}$. Divide the series for $e^x$ by $(1+x)$.

$$
\begin{array}{r l}
 & 1 + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{3x^4}{8} + \cdots \\
1 + x \,\big) & 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120} + \cdots \\
 & \underline{1 + x} \\
 & \quad 0 + \dfrac{x^2}{2} + \dfrac{x^3}{6} \\
 & \quad \underline{\dfrac{x^2}{2} + \dfrac{x^3}{2}} \\
 & \qquad -\dfrac{x^3}{3} + \dfrac{x^4}{24} \\
 & \qquad \underline{-\dfrac{x^3}{3} - \dfrac{x^4}{3}} \\
 & \qquad\quad \dfrac{3x^4}{8} + \dfrac{x^5}{120} \\
 & \qquad\quad \underline{\dfrac{3x^4}{8} + \dfrac{3x^5}{8}} \\
 & \qquad\qquad \vdots
\end{array}
$$

$f(x) = 1 + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \dfrac{3x^4}{8} - \cdots$




**42.** $y = x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} = x\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}\right) \approx x \cos x.$

Matches (b)

**44.** $y = x^2 - x^3 + x^4 = x^2(1 - x + x^2) \approx x^2\left(\dfrac{1}{1+x}\right).$

Matches (d)

**46.**
$$
\begin{aligned}
\int_0^x \sqrt{1+t^3}\,dt &= \int_0^x \left[1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n}}{2^n n!}\right] dt \\
&= \left[t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n+1}}{(3n+1)2^n n!}\right]_0^x \\
&= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) x^{3n+1}}{(3n+1)2^n n!}
\end{aligned}
$$

**48.** Since $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, we have

$\sin(1) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415.$ (4 terms)

**50.** Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$, we have $e^{-1} = 1 - 1 + \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \dfrac{1}{5!} + \cdots$

and $\dfrac{e-1}{e} = 1 - e^{-1} = 1 - \dfrac{1}{2!} + \dfrac{1}{3!} - \dfrac{1}{4!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321.$ (6 terms)

**52.** Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

we have $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$

**54.** $\displaystyle\int_0^{1/2} \frac{\arctan x}{x}\,dx = \int_0^{1/2}\left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right)dx = \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^{1/2}$

Since $1/(9^2 2^9) < 0.0001$, we have

$$\int_0^{1/2} \frac{\arctan x}{x}\,dx \approx \left(\frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9}\right) \approx 0.4872.$$

**Note:** We are using $\displaystyle\lim_{x\to 0^+} \frac{\arctan x}{x} = 1$.

**56.** $\displaystyle\int_{0.5}^{1} \cos\sqrt{x}\,dx = \int_{0.5}^{1}\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right)dx = \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots\right]_{0.5}^{1}$

Since $\frac{1}{210,600}(1 - 0.5^5) < 0.0001$, we have

$$\int_{0.5}^{1} \cos\sqrt{x}\,dx \approx \left[(1 - 0.5) - \frac{1}{4}(1 - 0.5^2) + \frac{1}{72}(1 - 0.5^3) - \frac{1}{2880}(1 - 0.5^4) + \frac{1}{201,600}(1 - 0.5)^5\right] \approx 0.3243.$$

**58.** $\displaystyle\int_0^{1/4} x\ln(x+1)\,dx = \int_0^{1/4}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right)dx$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4\cdot 2} + \frac{x^5}{5\cdot 3} - \frac{x^6}{6\cdot 4} + \cdots\right]_0^{1/4}$$

Since $\dfrac{(1/4)^5}{15} < 0.0001$,

$$\int_0^{1/4} x\ln(x+1)\,dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472.$$

**60.** From Exercise 19, we have

$$\frac{1}{\sqrt{2\pi}}\int_1^2 e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_1^2 \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\right]_1^2$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(2^{n+1}-1)}{2^n n!(2n+1)}$$

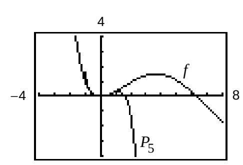
$$\approx \frac{1}{\sqrt{2\pi}}\left[1 - \frac{7}{2\cdot 1\cdot 3} + \frac{31}{2^2\cdot 2!\cdot 5} - \frac{127}{2^3\cdot 3!\cdot 7} + \frac{511}{2^4\cdot 4!\cdot 9} - \frac{2047}{2^5\cdot 5!\cdot 11}\right.$$

$$\left. + \frac{8191}{2^6\cdot 6!\cdot 13} - \frac{32,767}{2^7\cdot 7!\cdot 15} + \frac{131,071}{2^8\cdot 8!\cdot 17} - \frac{524,287}{2^9\cdot 9!\cdot 19}\right] \approx 0.1359.$$

**62.** $f(x) = \sin\dfrac{x}{2}\ln(1+x)$

$P_5(x) = \dfrac{x^2}{2} - \dfrac{x^3}{4} + \dfrac{7x^4}{48} - \dfrac{11x^5}{96}$

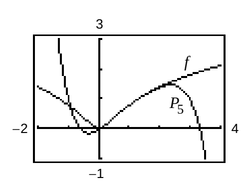
The polynomial is a reasonable approximation on the interval $(-0.60, 0.73)$.




**64.** $f(x) = \sqrt[3]{x}\cdot\arctan x,\ c = 1$

$$P_5(x) \approx 0.7854 + 0.7618(x-1) - 0.3412\left[\frac{(x-1)^2}{2!}\right] - 0.0424\left[\frac{(x-1)^3}{3!}\right]$$

$$+ 1.3025\left[\frac{(x-1)^4}{4!}\right] - 5.5913\left[\frac{(x-1)^5}{5!}\right]$$




The polynomial is a reasonable approximation on the interval $(0.48, 1.75)$.

**66.** $a_{2n+1} = 0$ (odd coefficients are zero)

**68.** Answers will vary.

**70.** $\theta = 60°, v_0 = 64, k = \frac{1}{16}, g = -32$

$$y = \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots$$

$$= \sqrt{3}x - 32\left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^3 16} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots\right]$$

$$= \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{2^n x^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32\sum_{n=2}^{\infty}\frac{x^n}{n(32)^n(16)^{n-2}}$$

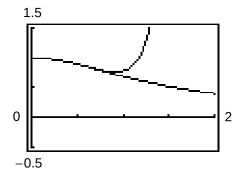
**72.** (a) $f(x) = \dfrac{\ln(x^2+1)}{x^2}$.

From Exercise 8, you obtain

$$P = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$$

(b)




(c) $F(x) = \displaystyle\int_0^x \frac{\ln(t^2+1)}{t^2}\,dt$

$G(x) = \displaystyle\int_0^x P_8(t)\,dt$

| $x$ | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 |
|---|---|---|---|---|---|---|
| $F(x)$ | 0.2475 | 0.4810 | 0.6920 | 0.8776 | 1.1798 | 1.4096 |
| $G(x)$ | 0.2475 | 0.4810 | 0.6920 | 0.8805 | 5.3064 | 652.21 |

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.

**74.** Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N!\left[e - \left(1 + 1 + \ldots + \frac{1}{N!}\right)\right]$, a positive integer. But,

$$a = N!\left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots\right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots$$

$$= \frac{1}{N+1}\left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \ldots\right] = \frac{1}{N+1}\left[\frac{1}{1 - \left(\frac{1}{N+1}\right)}\right] = \frac{1}{N}, \text{ a contradiction.}$$

## Review Exercises for Chapter 8

**2.** $a_n = \dfrac{n}{n^2+1}$

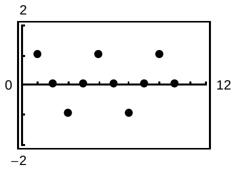
**4.** $a_n = 4 - \dfrac{n}{2}$: 3.5, 3, . . .

Matches (c)

**6.** $a_n = 6\left(-\dfrac{2}{3}\right)^{n-1}$: 6, −4, . . .

Matches (b)

**8.** $a_n = \sin \dfrac{n\pi}{2}$




The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}:\ 1, 0, -1, 0, 1, 0, \ldots$$

**10.** $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Converges

**12.** $\displaystyle\lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{n\to\infty} \frac{1}{1/n} = \infty$

Diverges

**14.** $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{k\to\infty} \left[\left(1 + \frac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$

Converges; $k = 2n$

**16.** Let $\quad y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{b^n + c^n}(b^n \ln b + c^n \ln c)$$

Assume $b \geq c$ and note that the terms

$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as $n \to \infty$. Hence $a_n$ converges.

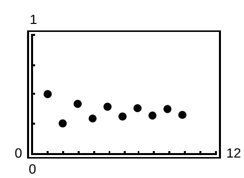
**18.** (a) $V_n = 120{,}000(0.70)^n$, $n = 1, 2, 3, 4, 5$

(b) $V_5 = 120{,}000(0.70)^5 = \$20{,}168.40$

**20.** (a)

| $k$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_k$ | 0.3917 | 0.3228 | 0.3627 | 0.3344 | 0.3564 |

(b)

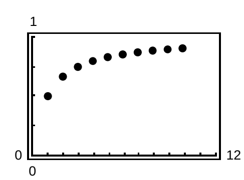



(c) The series converges by the Alternating Series Test.

**22.** (a)

| $k$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_k$ | 0.8333 | 0.9091 | 0.9375 | 0.9524 | 0.9615 |

(b)




(c) The series converges, by the limit comparison test with $\sum \dfrac{1}{n^2}$.

**24.** Diverges. Geometric series, $r = 1.82 > 1$.

**26.** Diverges. $n$th Term Test, $\displaystyle\lim_{n\to\infty} a_n = \frac{2}{3}$.

**28.** $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

**30.** 
$$\sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right] = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$
$$= \frac{1}{1-(2/3)} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots\right] = 3 - 1 = 2$$

**32.** $0.\overline{923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots]$

$$= \sum_{n=0}^{\infty} (0.923076)(0.000001)^n = \frac{0.923076}{1 - 0.000001} = \frac{923{,}076}{999{,}999} = \frac{12(76{,}923)}{13(76{,}923)} = \frac{12}{13}$$

**34.** $S = \sum_{n=0}^{39} 32{,}000(1.055)^n = \dfrac{32{,}000(1 - 1.055^{40})}{1 - 1.055}$

$\approx \$4{,}371{,}379.65$

**36.** See Exercise 86 in Section 8.2.

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 100\left(\frac{12}{0.065}\right)\left[\left(1 + \frac{0.065}{12}\right)^{120} - 1\right]$$

$$\approx \$16{,}840.32$$

**38.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$

Divergent $p$-series, $p = \frac{3}{4} < 1$

**40.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$

The first series is a convergent $p$-series and the second series is a convergent geometric series. Therefore, their difference converges.

**42.** $\displaystyle\sum_{n=1}^{\infty} \frac{n + 1}{n(n + 2)}$

$$\lim_{n\to\infty} \frac{(n + 1)/n(n + 2)}{1/n} = \lim_{n\to\infty} \frac{n + 1}{n + 2} = 1$$

By a limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

**44.** Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

**46.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n + 1}$

$$a_{n+1} = \frac{\sqrt{n + 1}}{n + 2} \le \frac{\sqrt{n}}{n + 1} = a_n$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n + 1} = 0$$

By the Alternating Series Test, the series converges.

**48.** Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3\ln(n + 1)}{n + 1} < \frac{3\ln n}{n} = a_n,\ \lim_{n\to\infty} \frac{3\ln n}{n} = 0$$

**50.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n + 1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right|$$

$$= \lim_{n\to\infty} \frac{n + 1}{e} = \infty$$

By the Ratio Test, the series diverges.

**52.** $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 5 \cdot 8 \cdots (3n - 1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1 \cdot 3 \cdots (2n - 1)(2n + 1)}{2 \cdot 5 \cdots (3n - 1)(3n + 2)} \cdot \frac{2 \cdot 5 \cdots (3n - 1)}{1 \cdot 3 \cdots (2n - 1)}\right| = \lim_{n\to\infty} \frac{2n + 1}{3n + 2} = \frac{2}{3}$$

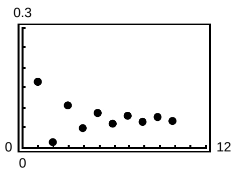
By the Ratio Test, the series converges.

**54.** (a) The series converges by the Alternating Series Test.

(b)

| $x$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_n$ | 0.0871 | 0.0669 | 0.0734 | 0.0702 | 0.0721 |

(c)




(d) The sum is approximately 0.0714.

**56.** No. Let $a_n = \dfrac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{3937.5}{n^2}$ is a convergent $p$-series.

**58.**

$f(x) = \tan x \qquad f\left(-\dfrac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x \qquad f'\left(-\dfrac{\pi}{4}\right) = 2$

$f''(x) = 2\sec^2 x \tan x \qquad f''\left(-\dfrac{\pi}{4}\right) = -4$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \qquad f'''\left(-\dfrac{\pi}{4}\right) = 16$

$$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$$

**60.** $\cos(0.75) \approx 1 - \dfrac{(0.75)^2}{2!} + \dfrac{(0.75)^4}{4!} - \dfrac{(0.75)^6}{6!} \approx 0.7317$

**62.** $e^{-0.25} \approx 1 - 0.25 + \dfrac{(0.25)^2}{2!} - \dfrac{(0.25)^3}{3!} + \dfrac{(0.25)^4}{4!} \approx 0.779$

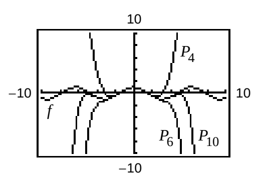
**64.**

$f(x) = \cos x$

$P_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

$P_6(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$

$P_{10}(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \dfrac{x^{10}}{10!}$




**66.** $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

**68.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n(x-2)^n}\right|$$
$$= 3|x-2|$$

$R = \dfrac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \le x < \frac{7}{3}$.

**70.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty}\left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left|\frac{x-2}{2}\right| < 1 \quad \text{or} \quad 0 < x < 4.$$

**72.**

$$y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)x^{2n+1}}{2^{n+1}(n+1)!}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!}$$

$$y'' + 3xy' + 3y = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}(2n+2)x^{2n+2}}{2^{n+1}(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+1}(2n+2)x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}[-(2n+1)+1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}(-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+2}x^{2n+2}}{2^n n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n+1}x^{2n}}{2^n n!}(2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1}(n-1)!} \cdot \frac{2n}{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!}[-2n+2n] = 0$$

**74.** $\dfrac{3}{2+x} = \dfrac{3/2}{1+(x/2)} = \dfrac{3/2}{1-(-x/2)} = \dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

**76.** Integral: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$

**78.** $8 - 2(x-3) + \dfrac{1}{2}(x-3)^2 - \dfrac{1}{8}(x-3)^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} 8\left[\frac{-(x-3)}{4}\right]^n = \frac{8}{1-[-(x-3)/4]}$

$$= \frac{32}{4+(x-3)} = \frac{32}{1+x}, \quad -1 < x < 7$$

**80.**

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x+(\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\cdot 3!}\left(x+\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2\cdot 4!}\left(x+\frac{\pi}{4}\right)^4 + \cdots$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x+\frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2}[x+(\pi/4)]^{n+1}}{(n+1)!}\right]$$

**82.**

$$f(x) = \csc(x)$$

$$f'(x) = -\csc(x)\cot(x)$$

$$f''(x) = \csc^3(x) + \csc(x)\cot^2(x)$$

$$f'''(x) = -5\csc^3(x)\cot(x) - \csc(x)\cot^3(x)$$

$$f^{(4)}(x) = 5\csc^5(x) + 15\csc^3(x)\cot^2(x) + \csc(x)\cot^4(x)$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x-(\pi/2)]^n}{n!} = 1 + \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{5}{4!}\left(x-\frac{\pi}{2}\right)^4 + \cdots$$

**84.** $f(x) = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \cdots$$

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!}$$

$$= 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \cdots$$

$$= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}$$

**86.** $h(x) = (1+x)^{-3}$

$$h'(x) = -3(1+x)^{-4}$$

$$h''(x) = 12(1+x)^{-5}$$

$$h'''(x) = -60(1+x)^{-6}$$

$$h^{(4)}(x) = 360(1+x)^{-7}$$

$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x}{2}$$

**88.** $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$

$$\ln\left(\frac{6}{5}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5) - 1}{n}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823$$

**90.** $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

**92.** $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

**94.** $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \cdots$$

$$\int_0^1 xe^x \, dx = \Big[xe^x - e^x\Big]_0^1 = (e - e) - (0 - 1) = 1$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \, dx = \sum_{n=0}^{\infty} \left[\frac{x^{n+2}}{(n+2)n!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$$

**96.** (a)

$$\begin{aligned}
f(x) &= \sin 2x & f(0) &= 0\\
f'(x) &= 2\cos 2x & f'(0) &= 2\\
f''(x) &= -4\sin 2x & f''(0) &= 0\\
f'''(x) &= -8\cos 2x & f'''(0) &= -8\\
f^{(4)}(x) &= 16\sin 2x & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= 32\cos 2x & f^{(5)}(0) &= 32\\
f^{(6)}(x) &= -64\sin 2x & f^{(6)}(0) &= 0\\
f^{(7)}(x) &= -128\cos 2x & f^{(7)}(0) &= -128
\end{aligned}$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

(b) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
\sin 2x &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\\
&= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

(c) $$\begin{aligned}
\sin 2x &= 2\sin x\cos x\\
&= 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)\\
&= 2\left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120}\right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040}\right) + \cdots\right]\\
&= 2\left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

**98.**

$$\begin{aligned}
\cos t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}\\
\cos\frac{\sqrt{t}}{2} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!}\\
\int_0^x \cos\frac{\sqrt{t}}{2}\,dt &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)}\right]_0^x\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}
\end{aligned}$$

**100.**

$$\begin{aligned}
e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\\
e^t - 1 &= \sum_{n=1}^{\infty} \frac{t^n}{n!}\\
\frac{e^t - 1}{t} &= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}\\
\int_0^x \frac{e^t - 1}{t}\,dt &= \left[\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!}\right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}
\end{aligned}$$

**102.**

$$\arcsin x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2\cdot 3} + \frac{1\cdot 3x^4}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^6}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$$

By L'Hôpital's Rule, $\displaystyle\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{1} = 1.$

# Problem Solving for Chapter 8

**2.** Let $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Then $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

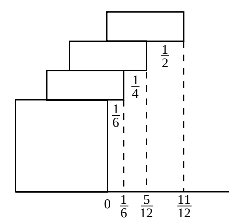
$$= S + \frac{1}{2^2} + \frac{1}{4^2} + \cdots$$

$$= S + \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$= S + \frac{1}{2^2}\left(\frac{\pi^2}{6}\right).$$

Thus, $S = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{6}\left(\frac{3}{4}\right) = \frac{\pi^2}{8}.$

**4.** (a) Position the three blocks as indicated in the figure. The bottom block extends $1/6$ over the edge of the table, the middle block extends $1/4$ over the edge of the bottom block, and the top block extends $1/2$ over the edge of the middle block.




The centers of gravity are located at

bottom block: $\frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$

middle block: $\frac{1}{6} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{12}$

top block: $\frac{1}{6} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{5}{12}.$

The center of gravity of the top 2 blocks is

$$\left(-\frac{1}{12} + \frac{5}{12}\right)/2 = \frac{1}{6},$$

which lies over the bottom block. The center of gravity of the 3 blocks is

$$\left(-\frac{1}{3} - \frac{1}{12} + \frac{5}{12}\right)/3 = 0$$

which lies over the table. Hence, the far edge of the top block lies

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$$

beyond the edge of the table.

(b) Yes. If there are $n$ blocks, then the edge of the top block lies $\sum_{c=1}^{n} \frac{1}{2i}$ from the edge of the table. Using 4 blocks,

$$\sum_{c=1}^{4} \frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$$

which shows that the top block extends beyond the table.

(c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{c=1}^{\infty} \frac{1}{2i} = \frac{1}{2}\sum_{c=1}^{\infty} \frac{1}{i} = \infty.$$

**6.** $a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$

If $a = b$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

If $a \neq b$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n}$ diverges.

No values of $a$ and $b$ give absolute convergence. $a = b$ implies conditional convergence.

**8.**

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{12}}{6!} + \cdots$$

$$\frac{f^{(12)}(0)}{12!} = \frac{1}{6!} \implies f^{(12)}(0) = \frac{12!}{6!} = 665{,}280$$

**10.** (a) If $p = 1$, $\int_2^{\infty} \frac{1}{x \ln x}\,dx = \ln \ln x \Big]_2^{\infty}$ diverges.

If $p > 1$, $\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$ converges.

If $p < 1$, diverges.

(b) $\sum_{n=4}^{\infty} \frac{1}{n \ln(n^2)} = \frac{1}{2}\sum_{n=4}^{\infty} \frac{1}{n \ln n}$ diverges by part (a).

**12.** Let $b_n = a_n r^n$.

$$(bn)^{1/n} = (a_n r^n)^{1/n} = a_n^{1/n} \cdot r \to Lr \text{ as } n \to \infty.$$

$$Lr < \frac{1}{r} r = 1.$$

By the Root Test, $\sum b_n$ converges $\implies$ $\sum a_n r^n$ converges.

**14.** (a) $\frac{1}{0.99} = \frac{1}{1 - 0.01} = \sum_{n=0}^{\infty} (0.01)^n$

$$= 1 + 0.01 + (0.01)^2 + \cdots$$

$$= 1.010101\ldots$$

(b) $\frac{1}{0.98} = \frac{1}{1 - 0.02} = \sum_{n=0}^{\infty} (0.02)^n$

$$= 1 + 0.02 + (0.02)^2 + \cdots$$

$$= 1 + 0.02 + 0.0004 + \cdots$$

$$= 1.0204081632\ldots$$

**16.** (a) Height $= 2\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\right]$

$$= 2\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty \quad \left(p\text{-series}, p = \frac{1}{2} < 1\right)$$

(b) $S = 4\pi\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots\right] 4\pi\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

(c) $W = \frac{4}{3}\pi\left[1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots\right]$

$$= \frac{4}{3}\pi\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}$$

# CHAPTER 9
# Conics, Parametric Equations, and Polar Coordinates

# CHAPTER 9
# Conics, Parametric Equations, and Polar Coordinates

## Section 9.1 Conics and Calculus

Solutions to Even-Numbered Exercises

**2.** $x^2 = 8y$

Vertex: $(0, 0)$

$p = 2 > 0$

Opens upward
Matches graph (a).

**4.** $\dfrac{(x-2)^2}{16} + \dfrac{(y+1)^2}{4} = 1$

Center: $(2, -1)$
Ellipse
Matches (b)

**6.** $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$

Circle radius 3.
Matches (g)

**8.** $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{4} = 1$

Hyperbola

Center: $(-2, 0)$

Horizontal transverse axis.
Matches (d)

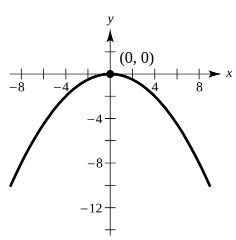
**10.** $x^2 + 8y = 0$

$$x^2 = 4(-2)y$$

Vertex: $(0, 0)$

Focus: $(0, -2)$

Directrix: $y = 2$



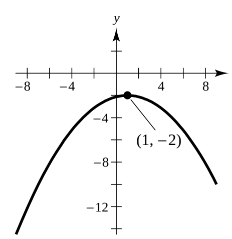

**12.** $(x-1)^2 + 8(y+2) = 0$

$$(x-1)^2 = 4(-2)(y+2)$$

Vertex: $(1, -2)$

Focus: $(1, -4)$

Directrix: $y = 0$




**14.** $y^2 + 6y + 8x + 25 = 0$

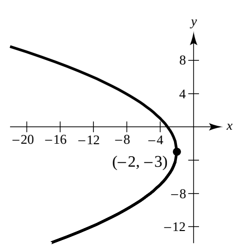
$$y^2 + 6y + 9 = -8x - 25 + 9$$

$$(y+3)^2 = 4(-2)(x+2)$$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$




**16.** $y^2 + 4y + 8x - 12 = 0$

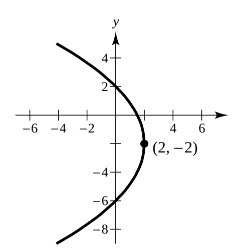
$$y^2 + 4y + 4 = -8x + 12 + 4$$

$$(y+2)^2 = 4(-2)(x-2)$$

Vertex: $(2, -2)$

Focus: $(0, -2)$

Directrix: $x = 4$



**18.** $y = -\frac{1}{6}(x^2 - 8x + 6) = -\frac{1}{6}(x^2 - 8x + 16 - 10)$

$-6y = (x-4)^2 - 10$

$-6y + 10 = (x-4)^2$

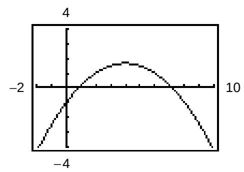
$(x-4)^2 = -6\left(y - \frac{5}{3}\right)$

$(x-4)^2 = 4\left(-\frac{3}{2}\right)\left(y - \frac{5}{3}\right)$

Vertex: $\left(4, \frac{5}{3}\right)$

Focus: $\left(4, \frac{1}{6}\right)$

Directrix: $y = \frac{19}{6}$




**20.** $x^2 - 2x + 8y + 9 = 0$

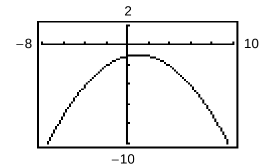
$x^2 - 2x + 1 = -8y - 9 + 1$

$(x-1)^2 = 4(-2)(y+1)$

Vertex: $(1, -1)$

Focus: $(1, -3)$

Directrix: $y = 1$




**22.** $(x+1)^2 = 4(-2)(y-2)$

$x^2 + 2x + 8y - 15 = 0$

**24.** Vertex: $(0, 2)$

$(y-2)^2 = 4(2)(x-0)$

$y^2 - 8x - 4y + 4 = 0$

**26.** $y = 4 - (x-2)^2 = 4x - x^2$

$x^2 - 4x + y = 0$

**28.** From Example 2: $4p = 8$ or $p = 2$

Vertex: $(4, 0)$

$(x-4)^2 = 8(y-0)$

$x^2 - 8x - 8y + 16 = 0$

**30.** $5x^2 + 7y^2 = 70$

$\frac{x^2}{14} + \frac{y^2}{10} = 1$

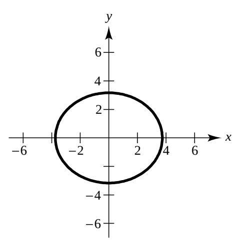
$a^2 = 14, b^2 = 10, c^2 = 4$

Center: $(0, 0)$

Foci: $(\pm 2, 0)$

Vertices: $\left(\pm\sqrt{14}, 0\right)$

$e = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$



**32.** $\frac{(x+2)^2}{1} + \frac{(y+4)^2}{1/4} = 1$

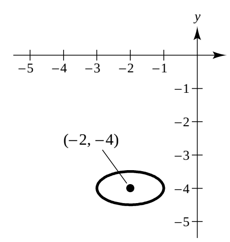
$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$

Center: $(-2, -4)$

Foci: $\left(-2 \pm \frac{\sqrt{3}}{2}, -4\right)$

Vertices: $(-1, -4), (-3, -4)$

$e = \frac{\sqrt{3}}{2}$




**34.** $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

$16(x^2 - 4x + 4) + 25(y^2 + 6y + 0) = -279 + 64 + 225$

$= 10$

$\frac{(x-2)^2}{(5/8)} + \frac{(y+3)^2}{(2/5)} = 1$

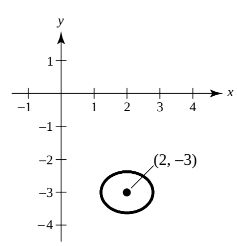
$a^2, \frac{5}{8}, b^2 = \frac{2}{5}, c^2 = a^2 - b^2 = \frac{9}{40}$

Center: $(2, -3)$

Foci: $\left(2 \pm \frac{3\sqrt{10}}{20}, -3\right)$

Vertices: $\left(2 \pm \frac{\sqrt{10}}{4}, -3\right)$

$e = \frac{c}{a} = \frac{3}{5}$



**36.** $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9(y^2 - 4y + 4) = -43 + 16 + 36$$

$$= 9$$

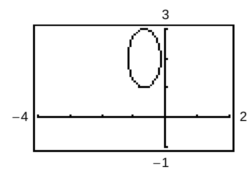
$$\frac{[x + (2/3)]^2}{1/4} + \frac{(y - 2)^2}{1} = 1$$

$a^2 = 1,\ b^2 = \frac{1}{4},\ c^2 = \frac{3}{4}$

Center: $\left(-\frac{2}{3}, 2\right)$

Foci: $\left(-\frac{2}{3}, 2 \pm \frac{\sqrt{3}}{2}\right)$

Vertices: $\left(-\frac{2}{3}, 3\right), \left(-\frac{2}{3}, 1\right)$




Solve for $y$:

$$9(y^2 - 4y + 4) = -36x^2 - 48x - 43 + 36$$

$$(y - 2)^2 = \frac{-(36x^2 + 48x + 7)}{9}$$

$$y = 2 \pm \frac{1}{3}\sqrt{-(36x^2 + 48x + 7)}$$ (Graph each of these separately.)

**38.** $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

$$50x^2 + 25y^2 + 120x - 160y + 78 = 0$$

$$50\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) + 25\left(y^2 - \frac{32}{5}y + \frac{256}{25}\right) = -78 + 72 + 256 = 250$$

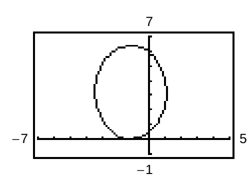
$$\frac{[x + (6/5)]^2}{5} + \frac{[y - (16/5)]^2}{10} = 1$$

$a^2 = 10,\ b^2 = 5,\ c^2 = 5$

Center: $\left(-\frac{6}{5}, \frac{16}{5}\right)$

Foci: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{5}\right)$

Vertices: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{10}\right)$




Solve for $y$: $(y^2 - 6.4y + 10.24) = -2x^2 - 4.8x - 3.12 + 10.24$

$$(y - 3.2)^2 = 7.12 - 4x - 2x^2$$

$$y = 3.2 \pm \sqrt{7.12 - 4x - 2x^2}$$ (Graph each of these separately.)

**40.** Vertices: $(0, 2), (4, 2)$

Eccentricity: $\frac{1}{2}$

Horizontal major axis

Center: $(2, 2)$

$a = 2,\ c = 1 \implies b = \sqrt{3}$

$$\frac{(x - 2)^2}{4} + \frac{(y - 2)^2}{3} = 1$$

**42** Foci: $(0, \pm 5)$

Major axis length: 14

Vertical major axis

Center: $(0, 0)$

$c = 5,\ a = 7 \implies b = \sqrt{24}$

$$\frac{x^2}{24} + \frac{y^2}{49} = 1$$

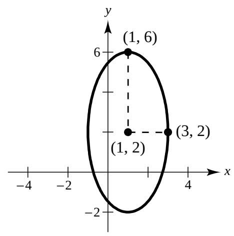
**44.** Center: $(1, 2)$

Vertical major axis

Points on ellipse: $(1, 6), (3, 2)$

From the sketch, we can see that $h = 1, k = 2, a = 4, b = 2$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$




**46.** $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

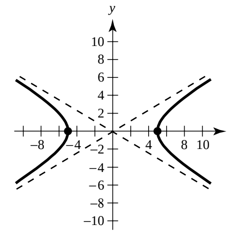
$a = 5, b = 3, c = \sqrt{a^2 + b^2} = \sqrt{34}$

Center: $(0, 0)$

Vertices: $(\pm 5, 0)$

Foci: $(\pm\sqrt{34}, 0)$

Asymptotes: $y = \pm\dfrac{3}{5}x$

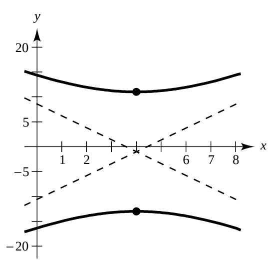



**48.** $\dfrac{(y+1)^2}{12^2} - \dfrac{(x-4)^2}{5^2} = 1$

$a = 12, b = 5, c = \sqrt{a^2 + b^2} = 13$

Center: $(4, -1)$
Vertices: $(4, 11), (4, -13)$
Foci: $(4, -14), (4, 12)$

Asymptotes: $y = -1 \pm \dfrac{12}{5}(x - 4)$




**50.** $y^2 - 9x^2 + 36x - 72 = 0$

$$y^2 - 9(x^2 - 4x + 4) = 72 - 36 = 36$$

$$\frac{y^2}{36} - \frac{(x-2)^2}{4} = 1$$

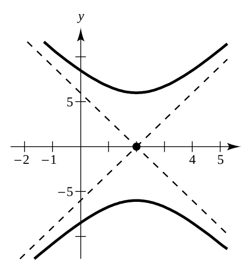
$a = 6, b = 2, c = \sqrt{a^2 + b^2} = 2\sqrt{10}$

Center: $(2, 0)$

Vertices: $(2, 6), (2, -6)$

Foci: $(2, 2\sqrt{10}), (2, -2\sqrt{10})$

Asymptotes: $y = \pm 3(x - 2)$




**52.** $9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$

$$9(x+3)^2 - 4(y-1)^2 = -1$$

$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

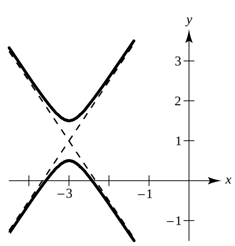
$a = \dfrac{1}{2}, b = \dfrac{1}{3}, c = \dfrac{\sqrt{13}}{6}$

Center: $(-3, 1)$

Vertices: $\left(-3, \dfrac{1}{2}\right), \left(-3, \dfrac{3}{2}\right)$

Foci: $\left(-3, 1 \pm \dfrac{1}{6}\sqrt{13}\right)$

Asymptotes: $y = 1 \pm \dfrac{3}{2}(x + 3)$




**54.**
$$9x^2 - y^2 + 54x + 10y + 55 = 0$$

$$9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25 = 1$$

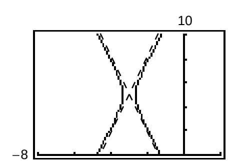
$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

$a = \dfrac{1}{3}, b = 1, c = \dfrac{\sqrt{10}}{3}$

Center: $(-3, 5)$

Vertices: $\left(-3 \pm \dfrac{1}{3}, 5\right)$

Foci: $\left(-3 \pm \dfrac{\sqrt{10}}{3}, 5\right)$




Solve for $y$:

$$y^2 - 10y + 25 = 9x^2 + 54x + 55 + 25$$

$$(y-5)^2 = 9x^2 + 54x + 80$$

$$y = 5 \pm \sqrt{9x^2 + 54x + 80}$$

(Graph each curve separately.)

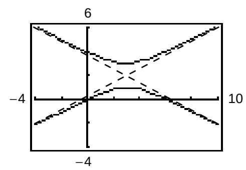
**56.** $$3y^2 - x^2 + 6x - 12y = 0$$
$$3(y^2 - 4y + 4) - (x^2 - 6x + 9) = 0 + 12 - 9 = 3$$
$$\frac{(y-2)^2}{1} - \frac{(x-3)^2}{3} = 1$$

$a = 1, b = \sqrt{3}, c = 2$

Center: $(3, 2)$

Vertices: $(3, 1), (3, 3)$

Foci: $(3, 0), (3, 4)$




Solve for $y$:

$$3(y^2 - 4y + 4) = x^2 - 6x + 12$$
$$(y-2)^2 = \frac{x^2 - 6x + 12}{3}$$
$$y = 2 \pm \sqrt{\frac{x^2 - 6x + 12}{3}}$$

(Graph each curve separately.)

**58.** Vertices: $(0, \pm 3)$

Asymptotes: $y = \pm 3x$

Vertical transverse axis

$a = 3$

Slopes of asymptotes: $\pm\frac{a}{b} = \pm 3$

Thus, $b = 1$. Therefore,

$$\frac{y^2}{9} - \frac{x^2}{1} = 1.$$

**60.** Vertices: $(2, \pm 3)$

Foci: $(2, \pm 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

Therefore, $\frac{y^2}{9} - \frac{(x-2)^2}{16} = 1$.

**62.** Center: $(0, 0)$

Vertex: $(3, 0)$

Focus: $(5, 0)$

Horizontal transverse axis

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

Therefore, $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

**64.** Focus: $(10, 0)$

Asymptotes: $y = \pm\frac{3}{4}x$

Horizontal transverse axis

Center: $(0, 0)$ since asymptotes intersect at the origin.

$c = 10$

Slopes of asymptotes: $\pm\frac{b}{a} = \pm\frac{3}{4}$ and $b = \frac{3}{4}a$

$c^2 = a^2 + b^2 = 100$

Solving these equations, we have $a^2 = 64$ and $b^2 = 36$. Therefore, the equation is

$$\frac{x^2}{64} - \frac{y^2}{36} = 1.$$

**66.** (a) $\frac{y^2}{4} - \frac{x^2}{2} = 1, y^2 - 2x^2 = 4, 2yy' - 4x = 0,$

$$y' = \frac{4x}{2y} = \frac{2x}{y}$$

At $x = 4$: $y = \pm 6, y' = \frac{\pm 2(4)}{6} = \pm\frac{4}{3}$

At $(4, 6)$: $y - 6 = -\frac{4}{3}(x - 4)$ or $4x - 3y + 2 = 0$

At $(4, -6)$: $y + 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y + 2 = 0$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 3/4$.

At $(4, 6)$: $y - 6 = -\frac{3}{4}(x - 4)$ or $3x + 4y - 36 = 0$

At $(4, -6)$: $y + 6 = \frac{3}{4}(x - 4)$ or $3x - 4y - 36 = 0$

**68.** $4x^2 - y^2 - 4x - 3 = 0$

$A = 4, C = -1$

$AC < 0$

Hyperbola

**70.** $25x^2 - 10x - 200y - 119 = 0$

$A = 25, C = 0$

Parabola

**72.** $y^2 - x - 4y - 5 = 0$

$A = 0, C = 1$

Parabola

**74.** $2x^2 - 2xy = 3y - y^2 - 2xy$

$2x^2 + y^2 - 3y = 0$

$A = 2, C = 1, AC > 0$

Ellipse

**76.** $9x^2 + 54x + 81 = 36 - 4(y^2 - 4y + 4)$

$9x^2 + 4y^2 + 54x - 16y + 61 = 0$

$A = 9, C = 4, AC > 0$

Ellipse

**78.** (a) An ellipse is the set of all points $(x, y)$, the sum of whose distance from two distinct fixed points (foci) is constant.

(b) $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ or $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

**80.** $e = \dfrac{c}{a}, c = \sqrt{a^2 - b^2} \quad 0 < e < 1$

For $e \approx 0$, the ellipse is nearly circular.

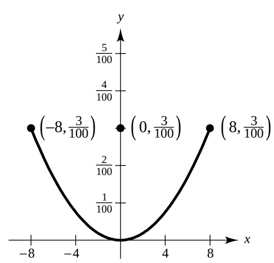
For $e \approx 1$, the ellipse is elongated.

**82.** Assume that the vertex is at the origin.

(a)
$$x^2 = 4py$$
$$8^2 = 4p\left(\frac{3}{100}\right)$$
$$\frac{1600}{3} = p$$
$$x^2 = 4\left(\frac{1600}{3}\right)y = \frac{6400}{3}y$$

(b) The deflection is 1 cm when

$$y = \frac{2}{100} \implies x = \pm\sqrt{\frac{128}{3}} \approx \pm 6.53 \text{ meters.}$$




**84.** (a) Without loss of generality, place the coordinate system so that the equation of the parabola is $x^2 = 4py$ and, hence,

$$y' = \left(\frac{1}{2p}\right)x.$$

Therefore, for distinct tangent lines, the slopes are unequal and the lines intersect.

(b)
$$x^2 - 4x - 4y = 0$$
$$2x - 4 - 4\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(0, 0)$, the slope is $-1$: $y = -x$. At $(6, 3)$, the slope is 2: $y = 2x - 9$. Solving for $x$,

$$-x = 2x - 9$$
$$-3x = -9$$
$$x = 3$$
$$y = -3.$$

Point of intersection: $(3, -3)$

**86.** The focus of $x^2 = 8y = 4(2)y$ is $(0, 2)$. The distance from a point on the parabola, $(x, x^2/8)$, and the focus, $(0, 2)$, is

$$d = \sqrt{(x-0)^2 + \left(\frac{x^2}{8} - 2\right)^2}.$$

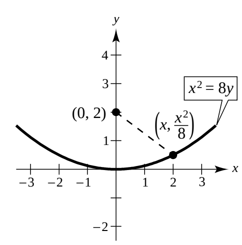
Since $d$ is minimized when $d^2$ is minimized, it is sufficient to minimize the function

$$f(x) = x^2 + \left(\frac{x^2}{8} - 2\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{x^2}{8} - 2\right)\left(\frac{x}{4}\right) = \frac{x^3}{16} + x.$$

$f'(x) = 0$ implies that

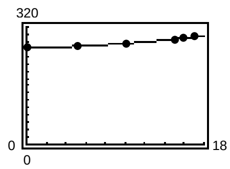
$$\frac{x^3}{16} + x = x\left(\frac{x^2}{16} + 1\right) = 0 \implies x = 0.$$



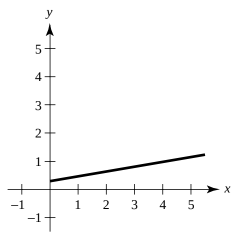

This is a minimum by the First Derivative Test. Hence, the closest point to the focus is the vertex, $(0, 0)$.

**88.** (a) $C = 0.0853t^2 + 0.2917t + 263.3559$

(b)




(c) $\dfrac{dC}{dt} = 0.1706t + 0.2971$




The consumption of fruits is increasing at a rate of 0.1706 pounds/year.

**90.**

$$x = \frac{1}{4}y^2$$

$$x' = \frac{1}{2}y$$

$$1 + (x')^2 = 1 + \frac{y^2}{4}$$

$$s = \int_0^4 \sqrt{1 + \left(\frac{y^2}{4}\right)}\, dy = \frac{1}{2}\int_0^4 \sqrt{4 + y^2}\, dy$$

$$= \frac{1}{4}\left[ y\sqrt{4 + y^2} + 4\ln\left|y + \sqrt{4 + y^2}\right|\right]_0^4$$

$$= \frac{1}{4}\left[4\sqrt{20} + 4\ln\left|4 + \sqrt{20}\right| - 4\ln 2\right]$$

$$= 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

**92.** $x^2 = 20y$

$$y = \frac{x^2}{20}$$

$$y' = \frac{x}{10}$$

$$S = 2\pi\int_0^r x\sqrt{1 + \left(\frac{x}{10}\right)^2}\, dx \quad = 2\pi\int_0^r \frac{x\sqrt{100 + x^2}}{10}\, dx$$

$$= \left[\frac{\pi}{10} \cdot \frac{2}{3}(100 + x^2)^{3/2}\right]_0^r = \frac{\pi}{15}\left[(100 + r^2)^{3/2} - 1000\right]$$

**94.** $A = 2\displaystyle\int_0^h \sqrt{4py}\, dy$

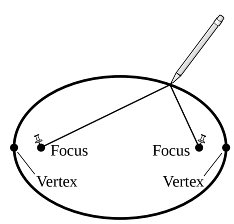
$$= 4\sqrt{p}\int_0^h y^{1/2}\, dy$$

$$= \left[4\sqrt{p}\left(\frac{2}{3}\right)y^{3/2}\right]_0^h$$

$$= \frac{8}{3}\sqrt{p}h^{3/2}$$

**96.** (a) At the vertices we notice that the string is horizontal and has a length of $2a$.

(b) The thumbtacks are located at the foci and the length of string is the constant sum of the distances from the foci.




**98.**

$$e = \frac{c}{a}$$

$$0.0167 = \frac{c}{149{,}570{,}000}$$

$$c \approx 2{,}497{,}819$$

Least distance: $a - c = 147{,}072{,}181$ km

Greatest distance: $a + c = 152{,}067{,}819$ km

**100.** $e = \dfrac{A - P}{A + P}$

$$= \frac{(122{,}000 + 4000) - (119 + 4000)}{(122{,}000 + 4000) + (119 + 4000)}$$

$$= \frac{121{,}881}{130{,}119} \approx 0.9367$$

**102.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(b^2/a^2)} = 1$$

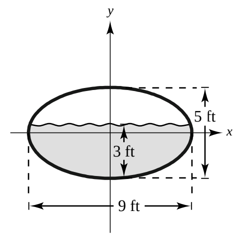
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(a^2 - c^2)/a^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

As $e \to 0$, $1 - e^2 \to 1$ and we have

$\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$ or the circle $x^2 + y^2 = a^2$.

**104.** $\dfrac{x^2}{(4.5)^2} + \dfrac{y^2}{(2.5)^2} = 1$




$$x^2 = (4.5)^2\left[1 - \frac{y^2}{(2.5)^2}\right]$$

$$x = \pm\frac{9}{5}\sqrt{(2.5)^2 - y^2}$$

$V = $ (Area of bottom)(Length) $+$ (Area of top)(Length)

$$V = \left[\frac{\pi(4.5)(2.5)}{2}\right](16) + 16\int_0^{0.5} \frac{9}{5}\sqrt{(2.5)^2 - y^2}\,dy \quad \text{(Recall: Area of ellipse is } \pi ab.\text{)}$$

$$= 90\pi + \frac{144}{5}\cdot\frac{1}{2}\left[y\sqrt{(2.5)^2 - y^2} + (2.5)^2 \arcsin\frac{y}{2.5}\right]_0^{0.5} = 90\pi + \frac{72}{5}\left[0.5\sqrt{6} + (2.5)^2\arcsin\frac{1}{5}\right] \approx 318.5 \text{ ft}^3$$

**106.** $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$18x + 8yy' + 36 - 24y' = 0$$

$$(8y - 24)y' = -(18x + 36)$$

$$y' = \frac{-(18x + 36)}{8y - 24}$$

$y' = 0$ when $x = -2$. $y'$ undefined when $y = 3$.
At $x = -2$, $y = 0$ or $6$.

Endpoints of major axis: $(-2, 0)$, $(-2, 6)$

At $y = 3$, $x = 0$ or $-4$.

Endpoints of minor axis: $(0, 3)$, $(-4, 3)$

**Note:** Equation of ellipse is $\dfrac{(x + 2)^2}{4} + \dfrac{(y - 3)^2}{9} = 1$

**108.** (a) $A = 4\displaystyle\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,dx = \frac{3}{2}\left[x\sqrt{16 - x^2} + 16\arcsin\frac{x}{4}\right]_0^4 = 12\pi$

(b) **Disk:** $V = 2\pi\displaystyle\int_0^4 \frac{9}{16}(16 - x^2)\,dx = \frac{9\pi}{8}\left[\left(16x - \frac{1}{3}x^3\right)\right]_0^4 = 48\pi$

$$y = \frac{3}{4}\sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16(16 - x^2)}}$$

$$S = 2(2\pi)\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\sqrt{\frac{16(16 - x^2) + 9x^2}{16(16 - x^2)}}\,dx = 4\pi\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,\frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}}\,dx = \frac{3\pi}{4}\int_0^4 \sqrt{256 - 7x^2}\,dx$$

$$= \frac{3\pi}{8\sqrt{7}}\left[\sqrt{7}x\sqrt{256 - 7x^2} + 256\arcsin\frac{\sqrt{7}x}{16}\right]_0^4 = \frac{3\pi}{8\sqrt{7}}\left(48\sqrt{7} + 256\arcsin\frac{\sqrt{7}}{4}\right) \approx 138.93$$

**—CONTINUED—**

**108. —CONTINUED—**

(c) **Shell:** $V = 4\pi\int_0^4 x\left[\frac{3}{4}\sqrt{16-x^2}\right]dx = 3\pi\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(16-x^2)^{3/2}\right]_0^4 = 64\pi$

$$x = \frac{4}{3}\sqrt{9-y^2}$$

$$x' = \frac{-4y}{3\sqrt{9-y^2}}$$

$$\sqrt{1+(x')^2} = \sqrt{1+\frac{16y^2}{9(9-y^2)}}$$

$$S = 2(2\pi)\int_0^3 \frac{4}{3}\sqrt{9-y^2}\sqrt{\frac{9(9-y^2)+16y^2}{9(9-y^2)}}\,dy$$

$$= 4\pi\int_0^3 \frac{4}{9}\sqrt{81+7y^2}\,dy$$

$$= \frac{16}{9}\left(\frac{\pi}{2\sqrt{7}}\right)\left[\sqrt{7}y\sqrt{81+7y^2} + 81\ln\left|\sqrt{7}y + \sqrt{81+7y^2}\right|\right]_0^3$$

$$= \frac{8\pi}{9\sqrt{7}}3\sqrt{7}(12) + 81\ln\left(3\sqrt{7}+12\right) - 81\ln 9\Big] \approx 168.53$$

**110.** (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$y' = -\frac{xb^2}{ya^2}$$

At $P$, $y' = -\frac{b^2}{a^2}\cdot\frac{x_0}{y_0} = m$.

(b) Slope of line through $(-c, 0)$ and $(x_0, y_0)$: $m_1 = \frac{y_0}{x_0+c}$

Slope of line through $(c, 0)$ and $(x_0, y_0)$: $m_2 = \frac{y_0}{x_0-c}$

(c) $\tan\alpha = \frac{m_2 - m}{1+m_2m} = \frac{\frac{y_0}{x_0-c} - \left(-\frac{b^2x_0}{a^2y_0}\right)}{1+\left(\frac{y_0}{x_0-c}\right)\left(-\frac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0{}^2 + b^2x_0(x_0-c)}{a^2y_0(x_0-c) - b^2x_0y_0}$

$$= \frac{a^2y_0{}^2 + b^2x_0{}^2 - b^2x_0c}{x_0y_0(a^2-b^2) - a^2y_0c} = \frac{a^2b^2 - b^2x_0c}{x_0y_0c^2 - a^2y_0c} = \frac{b^2(a^2-x_0c)}{y_0c(x_0c-a^2)} = -\frac{b^2}{y_0c}$$

$$\alpha = \arctan\left(-\frac{b^2}{y_0c}\right) = -\arctan\left(\frac{b^2}{y_0c}\right)$$

$$\tan\beta = \frac{m_1 - m}{1+m_1m} = \frac{\frac{y_0}{x_0+c} - \left(-\frac{b^2x_0}{a^2y_0}\right)}{1+\left(\frac{y_0}{x_0+c}\right)\left(-\frac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0{}^2 + b^2x_0(x_0+c)}{a^2y_0(x_0+c) - b^2x_0y_0}$$

$$= \frac{a^2y_0{}^2 + b^2x_0{}^2 + b^2x_0c}{a^2x_0y_0 + a^2cy_0 - b^2x_0y_0} = \frac{a^2b^2 + b^2x_0c}{x_0y_0(a^2-b^2) + a^2cy_0} = \frac{b^2(a^2+x_0c)}{y_0c(x_0c+a^2)} = \frac{b^2}{y_0c}$$

$$\beta = \arctan\left(\frac{b^2}{y_0c}\right)$$

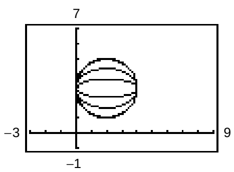
Since $|\alpha| = |\beta|$, the tangent line to an ellipse at a point $P$ makes equal angles with the lines through $P$ and the foci.

**112.** (a) $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$. Hence,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

(b) $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{4(1-e^2)} = 1$




(c) As $e$ approaches 0, the ellipse approaches a circle.

**114.** The transverse axis is vertical since $(-3, 0)$ and $(-3, 3)$ are the foci.

Center: $\left(-3, \dfrac{3}{2}\right)$

$c = \dfrac{3}{2}, 2a = 2, b^2 = c^2 - a^2 = \dfrac{5}{4}$

Therefore, the equation is

$$\frac{[y - (3/2)]^2}{1} - \frac{(x+3)^2}{5/4} = 1.$$

**116.** Center: $(0, 0)$

Horizontal transverse axis

Foci: $(\pm c, 0)$

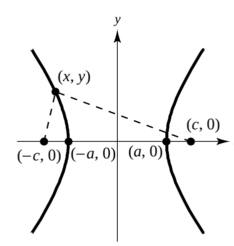
Vertices: $(\pm a, 0)$

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is

$$(a + c) - (c - a) = 2a.$$

Now, for any point $(x, y)$ on the hyperbola, the difference of the distances between $(x, y)$ and the two foci must also be $2a$.

$$\begin{aligned}
\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} &= 2a \\
\sqrt{(x-c)^2 + y^2} &= 2a + \sqrt{(x+c)^2 + y^2} \\
(x-c)^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\
-4xc - 4a^2 &= 4a\sqrt{(x+c)^2 + y^2} \\
-(xc + a^2) &= a\sqrt{(x+c)^2 + y^2} \\
x^2c^2 + 2a^2cx + a^4 &= a^2[x^2 + 2cx + c^2 + y^2] \\
x^2(c^2 - a^2) - a^2y^2 &= a^2(c^2 - a^2) \\
\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} &= 1
\end{aligned}$$




Since $a^2 + b^2 = c^2$, we have $(x^2/a^2) - (y^2/b^2) = 1$.

**118.** $c = 150, 2a = 0.001(186{,}000), a = 93,$
$b = \sqrt{150^2 - 93^2} = \sqrt{13{,}851}$

$$\frac{x^2}{93^2} - \frac{y^2}{13{,}851} = 1$$

When $y = 75$, we have

$$x^2 = 93^2\left(1 + \frac{75^2}{13{,}851}\right)$$

$$x \approx 110.3 \text{ miles.}$$

**120.** $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$

$$\begin{aligned}
y - y_0 &= \frac{b^2x_0}{a^2y_0}(x - x_0) \\
a^2y_0y - a^2y_0{}^2 &= b^2x_0x - b^2x_0{}^2 \\
b^2x_0{}^2 - a^2y_0{}^2 &= b^2x_0x - a^2y_0y \\
a^2b^2 &= b^2x_0x - a^2y_0y
\end{aligned}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

**122.** $Ax^2 + Cy^2 + Dx + Ey + F = 0$ (Assume $A \neq 0$ and $C \neq 0$; see (b) below)

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

(a) If $A = C$, we have

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$

which is the standard equation of a circle.

(c) If $AC > 0$, we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = 1$$

which is the equation of an ellipse.

(b) If $C = 0$, we have

$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$

If $A = 0$, we have

$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$

These are the equations of parabolas.

(d) If $AC < 0$, we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} - \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = \pm 1$$

which is the equation of a hyperbola.

**124.** True

**126.** False. The $y^4$ term should be $y^2$.

**128.** True

## Section 9.2 Plane Curves and Parametric Equations

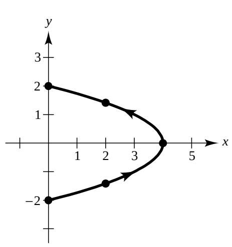
**2.** $x = 4\cos^2\theta$ $\quad y = 2\sin\theta$

$0 \le x \le 4$ $\quad -2 \le y \le 2$

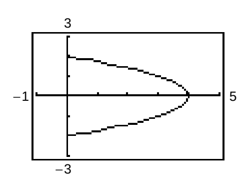
(a)

| $\theta$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $x$ | 0 | 2 | 4 | 2 | 0 |
| $y$ | $-2$ | $-\sqrt{2}$ | 0 | $\sqrt{2}$ | 2 |

(b)




(c)




(d) $$\frac{x}{4} = \cos^2\theta$$

$$\frac{y^2}{4} = \sin^2\theta$$

$$\frac{x}{4} + \frac{y^2}{4} = 1$$

$$x = 4 - y^2, -2 \le y \le 2$$

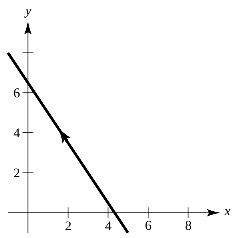
(e) The graph would be oriented in the opposite direction.

**4.** $x = 3 - 2t$

$y = 2 + 3t$

$y = 2 + 3\left(\dfrac{3 - x}{2}\right)$

$2y + 3x - 13 = 0$



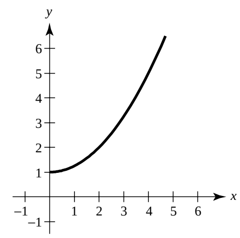

**6.** $x = 2t^2$

$y = t^4 + 1$

$y = \left(\dfrac{x}{2}\right)^2 + 1 = \dfrac{x^2}{4} + 1, x \geq 0$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.



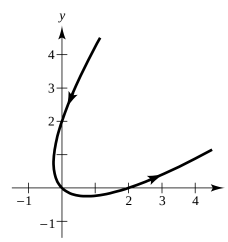

**8.** $x = t^2 + t,\ y = t^2 - t$

Subtracting the second equation from the first, we have

$$x - y = 2t \quad \text{or} \quad t = \frac{x - y}{2}$$

$$y = \frac{(x - y)^2}{4} - \frac{x - y}{2}$$

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $x$ | 2 | 0 | 0 | 2 | 6 |
| $y$ | 6 | 2 | 0 | 0 | 2 |




Since the discriminant is

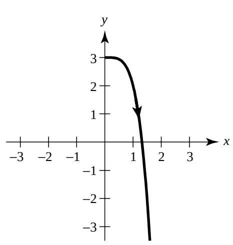
$$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$$

the graph is a rotated parabola.

**10.** $x = \sqrt[4]{t}, t \geq 0$

$y = 3 - t$

$y = 3 - x^4, x \geq 0$

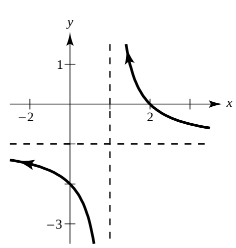



**12.** $x = 1 + \dfrac{1}{t}$

$y = t - 1$

$x = 1 + \dfrac{1}{t}$ implies $t = \dfrac{1}{x - 1}$

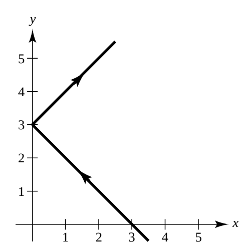
$y = \dfrac{1}{x - 1} - 1$




**14.** $x = |t - 1|$

$y = t + 2$

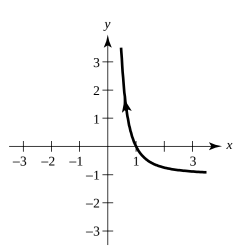
$x = |(y - 2) - 1| = |y - 3|$




**16.** $x = e^{-t}, x > 0$

$y = e^{2t} - 1$

$y = x^{-2} - 1 = \dfrac{1}{x^2} - 1, x > 0$



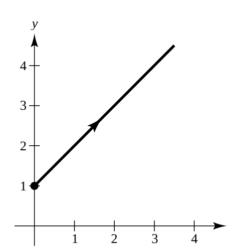

**18.** $x = \tan^2 \theta$

$y = \sec^2 \theta$

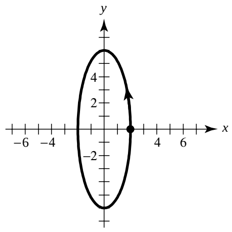
$\sec^2 \theta = \tan^2 \theta + 1$

$y = x + 1$

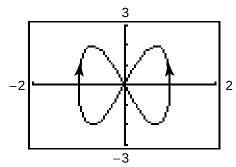
$x \geq 0$



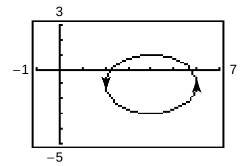
**20.**
$$x = 2\cos\theta$$
$$y = 6\sin\theta$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{6}\right)^2 = \cos^2 + \sin^2\theta = 1$$
$$\frac{x^2}{4} + \frac{y^2}{36} = 1 \text{ ellipse}$$



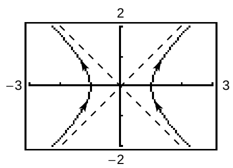

**22.**
$$x = \cos\theta$$
$$y = 2\sin 2\theta$$
$$y = 4\sin\theta\cos\theta$$
$$1 - x^2 = \sin^2\theta$$
$$y = \pm 4x\sqrt{1 - x^2}$$



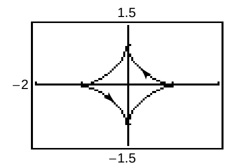

**24.**
$$x = 4 + 2\cos\theta$$
$$y = -1 + 2\sin\theta$$
$$(x - 4)^2 = 4\cos^2\theta$$
$$(y + 1)^2 = 4\sin^2\theta$$
$$(x - 4)^2 + (y + 1)^2 = 4$$



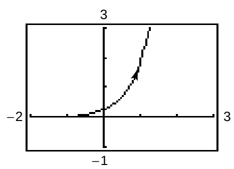

**26.**
$$x = \sec\theta$$
$$y = \tan\theta$$
$$x^2 = \sec^2\theta$$
$$y^2 = \tan^2\theta$$



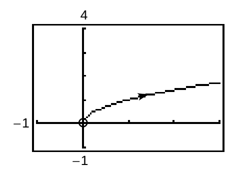

**28.**
$$x = \cos^3\theta$$
$$y = \sin^3\theta$$
$$x^{2/3} = \cos^2\theta$$
$$y^{2/3} = \sin^2\theta$$




**30.**
$$x = \ln 2t$$
$$y = t^2$$
$$t = \frac{e^x}{2}$$
$$y = \frac{e^{2x}}{r} = \frac{1}{4}e^{2x}$$

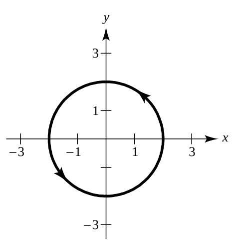



**32.**
$$x = e^{2t}$$
$$y = e^t$$
$$y^2 = x$$
$$y > 0$$
$$y = \sqrt{x}, x > 0$$



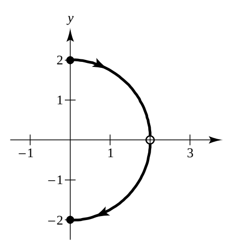

**34.** By eliminating the parameters in (a) – (d), we get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

(a) $x = 2\cos\theta,\ y = 2\sin\theta$



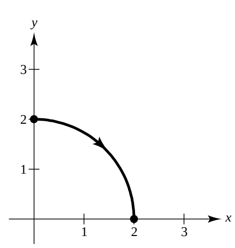

(b) $x = \dfrac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \dfrac{1}{t^2}}$ $\quad y = \dfrac{1}{t}$

$x \geq 0,\ x \neq 2$ $\quad y \neq 0$



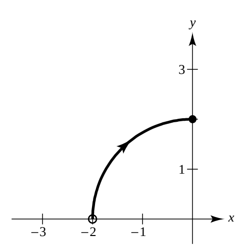

(c) $x = \sqrt{t}$ $\quad y = \sqrt{4 - t}$

$x \geq 0$ $\quad y \geq 0$




(d) $x = -\sqrt{4 - e^{2t}}$ $\quad y = e^t$

$-2 < x \leq 0$ $\quad y > 0$



**36.** The orientations are reversed. The graphs are the same. They are both smooth.

**38.** The set of points $(x, y)$ corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

**40.**
$$x = h + r\cos\theta$$
$$y = k + r\sin\theta$$
$$\cos\theta = \frac{x-h}{r}$$
$$\sin\theta = \frac{y-k}{r}$$
$$\cos^2\theta + \sin^2\theta = \frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$
$$(x-h)^2 + (y-k)^2 = r^2$$

**42.**
$$x = h + a\sec\theta$$
$$y = k + b\tan\theta$$
$$\frac{x-h}{a} = \sec\theta$$
$$\frac{y-k}{b} = \tan\theta$$
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

**44.** From Exercise 39 we have

$x = 1 + 4t$

$y = 4 - 6t.$

Solution not unique

**46.** From Exercise 40 we have

$x = -3 + 3\cos\theta$

$y = 1 + 3\sin\theta.$

Solution not unique

**48.** From Exercise 41 we have

$a = 5, c = 3 \implies b = 4$

$x = 4 + 5\cos$

$y = 2 + 4\sin\theta.$

Center: $(4, 2)$
Solution not unique

**50.** From Exercise 42 we have

$a = 1, c = 2 \implies b = \sqrt{3}$

$x = \sqrt{3}\tan\theta$

$y = \sec\theta.$

Center: $(0, 0)$
Solution not unique
The transverse axis is vertical, therefore, $x$ and $y$ are interchanged.

**52.** $y = \dfrac{2}{x-1}$

Example

$x = t, y = \dfrac{2}{t-1}$

$x = -t, y = \dfrac{2}{-t-1}$

**54.** $y = x^2$

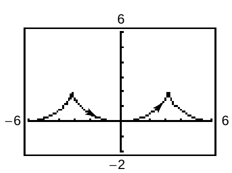
Example

$x = t, \quad y = t^2$

$x = t^3, \quad y = t^6$

**56.** $x = \theta + \sin\theta$

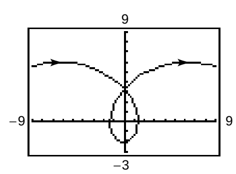
$y = 1 - \cos\theta$




Not smooth at $x = (2n - 1)\pi$

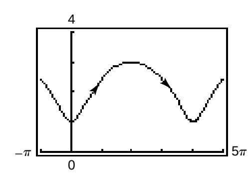
**58.** $x = 2\theta - 4\sin\theta$

$y = 2 - 4\cos\theta$




**60.** $x = 2\theta - \sin\theta$

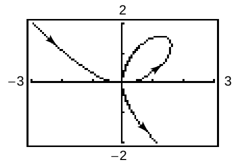
$y = 2 - \cos\theta$




Smooth everywhere

**62.** $x = \dfrac{3t}{1 + t^3}$

$y = \dfrac{3t^2}{1 + t^3}$




Smooth everywhere

**64.** Each point $(x, y)$ in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each $t$, plot $(x, y)$. As $t$ increases, the curve is traced out in a specific direction called the orientation of the curve.

**66.** (a) Matches (ii) because $-1 \le x \le 0$ and $1 \le y \le 2$.

(b) Matches (i) because $x = (y - 2)^2 - 1$ for all $y$.

**68.** $x = \cos^3 \theta$

$y = 2 \sin^2 \theta$

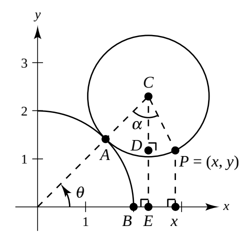
Matches (a)

**70.** $x = \cot \theta$

$y = 4 \sin \theta \cos \theta$

Matches (c)

**72.** Let the circle of radius 1 be centered at $C$. $A$ is the point of tangency on the line OC. $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle $\theta$ changes $\overset{\frown}{AB} = \overset{\frown}{AP}$. $\overset{\frown}{AB} = 2\theta$ and $\overset{\frown}{AP} = \alpha \implies \alpha = 2\theta$. Form the right triangle $\triangle CDP$. The angle $OCE = (\pi/2) - \theta$ and




$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3 \sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3 \cos \theta - \cos 3\theta$$

$$y = EC - CD = 3 \sin \theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3 \sin \theta - \sin 3\theta$$

Hence, $x = 3 \cos \theta - \cos 3\theta$, $y = 3 \sin \theta - \sin 3\theta$.

**74.** False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and $y$ is not a function of $x$.

**76.** (a) $x = (v_0 \cos \theta)t$

$y = h + (v_0 \sin \theta)t - 16t^2$

$$t = \frac{x}{v_0 \cos \theta} \implies y = h + (v_0 \sin \theta)\frac{x}{v_0 \cos \theta} - 16\left(\frac{x}{v_0 \cos \theta}\right)^2$$

$$y = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2}x^2$$

(b) $y = 5 + x - 0.005x^2 = h + (\tan \theta)x - \dfrac{16 \sec^2 \theta}{v_0^2}x^2$

$h = 5$, $\tan \theta = 1 \implies \theta = \dfrac{\pi}{4}$, and

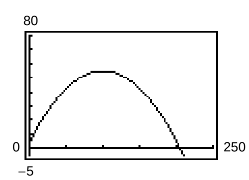
$$0.005 = \frac{16 \sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \implies v_0 = 80.$$

Hence, $x = (80 \cos(45°))t$

$y = 5 + (80 \sin(45°))t - 16t^2$.

(c)




(d) Maximum height: $y = 55$ (at $x = 100$)

Range: 204.88

## Section 9.3 Parametric Equations and Calculus

**2.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$

**4.** $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(-1/2)e^{-\theta/2}}{2e^{\theta}} = -\dfrac{1}{4}e^{-3\theta/2} = \dfrac{-1}{4e^{3\theta/2}}$

**6.** $x = \sqrt{t},\ y = 3t - 1$

$$\frac{dy}{dx} = \frac{3}{1/\left(2\sqrt{t}\right)} = 6\sqrt{t} = 6 \quad \text{when } t = 1.$$

$$\frac{d^2y}{dx^2} = \frac{3/\sqrt{t}}{1/\left(2\sqrt{t}\right)} = 6 \ \text{ concave upwards}$$

**8.** $x = t^2 + 3t + 2, y = 2t$

$$\frac{dy}{dx} = \frac{2}{2t+3} = \frac{2}{3} \text{ when } t = 0.$$

$$\frac{d^2y}{dx^2} = \frac{-2(2)/(2t+3)}{2t+3} = \frac{-4}{(2t+3)^2} = \frac{-4}{9} \text{ when t} = 0.$$

concave downward

**10.** $x = \cos\theta,\ y = 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta \cdot \frac{dy}{dx} \text{ is undefined when } \theta = 0.$$

$$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} = \frac{-3}{\sin^3\theta} \cdot \frac{d^2y}{dx^2} \text{ is undefined when } \theta = 0.$$

**12.** $x = \sqrt{t},\ y = \sqrt{t-1}$

$$\frac{dy}{dx} = \frac{1/\left(2\sqrt{t-1}\right)}{1/\left(2\sqrt{t}\right)}$$

$$= \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\frac{d^2y}{dx^2} = \frac{\left[\sqrt{t-1}/\left(2\sqrt{t}\right) - \sqrt{t}\left(1/2\sqrt{t-1}\right)\right]/(t-1)}{1/\left(2\sqrt{t}\right)}$$

$$= \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2.$$

concave downward

**14.** $x = \theta - \sin\theta,\ y = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{[(1-\cos\theta)\cos\theta - \sin^2\theta]}{(1-\cos\theta)^2}}{(1-\cos\theta)}$$

$$= \frac{-1}{(1-\cos\theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

concave downward

**16.** $x = 2 - 3\cos\theta, y = 3 + 2\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$$

At $(-1, 3),\ \theta = 0,$ and $\dfrac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5),\ \theta = \dfrac{\pi}{2},$ and $\dfrac{dy}{dx} = 0.$

Tangent line: $y = 5$

At $\left(\dfrac{4 + 3\sqrt{3}}{2}, 2\right),\ \theta = \dfrac{7\pi}{6},$ and $\dfrac{dy}{dx} = \dfrac{2\sqrt{3}}{3}.$

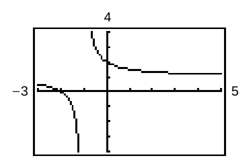
Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3}\left(x - \frac{4 + 3\sqrt{3}}{2}\right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

**18.** $x = t - 1,\ y = \dfrac{1}{t} + 1,\ t = 1$

(a)



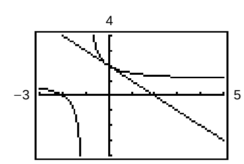

(b) At $t = 1,\ (x, y) = (0, 2),$ and

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = -1,\ \frac{dy}{dx} = -1$$

(c) $\dfrac{dy}{dx} = -1.$ At $(0, 2),\ y - 2 = -1(x - 0)$

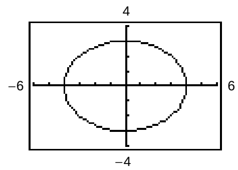
$$y = -x + 2$$

(d)



**20.** $x = 4\cos\theta,\ y = 3\sin\theta,\ \theta = \dfrac{3\pi}{4}$

(a)



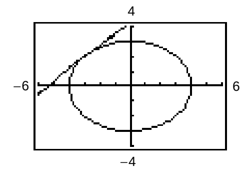

(b) At $\theta = \dfrac{3\pi}{4},\ (x, y) = \left(\dfrac{-4}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$, and

$$\frac{dx}{dt} = -2\sqrt{2},\ \frac{dy}{dt} = -\frac{3\sqrt{2}}{2},\ \frac{dy}{dx} = \frac{3}{4}$$

(c) $\dfrac{dy}{dx} = \dfrac{3}{4}$. At $\left(\dfrac{-4}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$, $\quad y - \dfrac{3}{\sqrt{2}} = \dfrac{3}{4}\left(x + \dfrac{4}{\sqrt{2}}\right)$

$$y = \frac{3}{4}x + 3\sqrt{2}$$

(d)




**22.** $x = t^2 - t, y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1, \dfrac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2, \dfrac{dy}{dt} = \dfrac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$. Tangent Line

**24.** $x = 2\theta,\ y = 2(1 - \cos\theta)$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\sin\theta = 0$ when $\theta = 0,\ \pm\pi,\ \pm 2\pi, \ldots$.

Points: $(4n\pi, 0),\ (2[2n - 1]\pi, 4)$ where $n$ is an integer.

Points shown: $(0, 0),\ (2\pi, 4),\ (4\pi, 0)$

Vertical tangents: $\dfrac{dx}{d\theta} = 2 \neq 0$; none

**26.** $x = t + 1,\ y = t^2 + 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 2t + 3 = 0$ when $t = -\dfrac{3}{2}$.

Point: $\left(-\dfrac{1}{2}, -\dfrac{9}{4}\right)$

Vertical tangents: $\dfrac{dx}{dt} = 1 \neq 0$; none

**28.** $x = t^2 - t + 2,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2), (4, 2)$

Vertical tangents: $\dfrac{dx}{dt} = 2t - 1 = 0$ when $t = \dfrac{1}{2}$.

Point: $\left(\dfrac{7}{4}, -\dfrac{11}{8}\right)$

**30.** $x = \cos\theta, y = 2\sin 2\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 4\cos 2\theta = 0$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

Points: $\left(\dfrac{\sqrt{2}}{2}, 2\right), \left(-\dfrac{\sqrt{2}}{2}, -2\right), \left(-\dfrac{\sqrt{2}}{2}, 2\right), \left(\dfrac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\dfrac{dx}{d\theta} = -\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0), (-1, 0)$

**32.** $x = 4\cos^2\theta,\ y = 2\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Since $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -8\cos\theta\sin\theta = 0$ when

$\theta = 0, \pi$.

Point: $(4, 0)$

**34.** $x = \cos^2\theta,\ y = \cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = -\sin\theta = 0$ when $x = 0, \pi$.

Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -2\cos\theta\sin\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

(Exclude $0, \pi$.)

Point: $(0, 0)$

**36.** $x = t^2 + 1, y = 4t^3 + 3, -1 \le t \le 0$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 12t^2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 144t^4$$

$$s = \int_{-1}^{0} \sqrt{4t^2 + 144t^4}\,dt = \int_{-1}^{0} -2t\sqrt{1 + 36t^2}\,dt$$

$$= \left[\frac{-(1 + 36t^2)^{3/2}}{54}\right]_{-1}^{0} = \frac{-1}{54}(1 - 37^{3/2}) \approx 4.149$$

**38.** $x = \arcsin t, y = \ln\sqrt{1 - t^2}, 0 \le t \le \dfrac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}}, \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1 - t^2}\right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{(1 - t^2)^2}}\,dt = \int_0^{1/2} \frac{1}{1 - t^2}\,dt$$

$$= \left[-\frac{1}{2}\ln\left|\frac{t - 1}{t + 1}\right|\right]_0^{1/2}$$

$$= -\frac{1}{2}\ln\left(\frac{1}{3}\right) = \frac{1}{2}\ln(3) \approx 0.549$$

**40.** $x = t,\ y = \dfrac{t^5}{10} + \dfrac{1}{6t^3},\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = \dfrac{t^4}{2} - \dfrac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\,dt =$$

$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\,dt$$

$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right)dt$$

$$= \left[\frac{t^5}{10} - \frac{1}{6t^3}\right]_1^2 = \frac{779}{240}$$

**42.** $x = a\cos\theta,\ y = a\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = a\cos\theta$

$$S = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta$$

$$= 4a\int_0^{\pi/2} d\theta = \Big[4a\theta\Big]_0^{\pi/2} = 2\pi a$$

**44.** $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta,\ \dfrac{dx}{d\theta} = \theta\cos\theta$

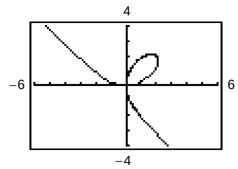
$\dfrac{dy}{d\theta} = \theta\sin\theta$

$$S = \int_0^{2\pi} \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi} \theta\,d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi} = 2\pi^2$$

**46.** $x = \dfrac{4t}{1 + t^3},\ y = \dfrac{4t^2}{1 + t^3}$

(a) $x^3 + y^3 = 4xy$




(b) $\dfrac{dy}{dt} = \dfrac{(1 + t^3)(8t) - 4t^2(3t^2)}{(1 + t^3)^2}$

$= \dfrac{4t(2 - t^3)}{(1 + t^3)^2} = 0$ when $t = 0$ or $t = \sqrt[3]{2}$.

Points: $(0, 0), \left(\dfrac{4\sqrt[3]{2}}{3}, \dfrac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

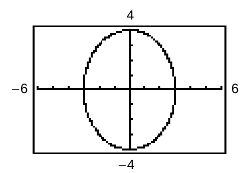
(c) $s = 2\displaystyle\int_0^1 \sqrt{\left[\frac{4(1 - 2t^3)}{(1 + t^3)^2}\right]^2 + \left[\frac{4t(2 - t^3)}{(1 + t^3)^2}\right]^2}\, dt = 2\int_0^1 \sqrt{\frac{16}{(1 + t^3)^4}\left[t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1\right]}\, dt$

$= 8\displaystyle\int_0^1 \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1 + t^3)^2}\, dt \approx 6.557$

**48.** $x = 3\cos\theta,\ y = 4\sin\theta$

$\dfrac{dx}{d\theta} = -3\sin\theta,\ \dfrac{dy}{d\theta} = 4\cos\theta$

$s = \displaystyle\int_0^{2\pi} \sqrt{9\sin^2\theta + 16\cos^2\theta}\, d\theta \approx 22.1$




**50.** $x = t,\ y = 4 - 2t,\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = -2$

(a) $S = 2\pi\displaystyle\int_0^2 (4 - 2t)\sqrt{1 + 4}\, dt$

$= \left[2\sqrt{5}\pi(4t - t^2)\right]_0^2 = 8\pi\sqrt{5}$

(b) $S = 2\pi\displaystyle\int_0^2 t\sqrt{1 + 4}\, dt = \left[\sqrt{5}\pi t^2\right]_0^2 = 4\pi\sqrt{5}$

**52.** $x = \dfrac{1}{3}t^3, y = t + 1, 1 \le t \le 2$, $y$-axis

$\dfrac{dx}{dt} = t^2, \dfrac{dy}{dt} = 1$

$S = 2\pi\displaystyle\int_1^2 \frac{1}{3}t^3\sqrt{t^4 + 1}\, dt = \frac{\pi}{9}\left[(x^4 + 1)^{3/2}\right]_1^2$

$= \dfrac{\pi}{9}(17^{3/2} - 2^{3/2}) \approx 23.48$

**54.** $x = a\cos\theta,\ y = b\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = b\cos\theta$

(a) $S = 4\pi\displaystyle\int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

$= 4\pi\displaystyle\int_0^{\pi/2} ab\sin\theta\sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right)\cos^2\theta}\, d\theta = \frac{-4ab\pi}{e}\int_0^{\pi/2} (-e\sin\theta)\sqrt{1 - e^2\cos^2\theta}\, d\theta$

$= \dfrac{-2ab\pi}{e}\left[e\cos\theta\sqrt{1 - e^2\cos^2\theta} + \arcsin(e\cos\theta)\right]_0^{\pi/2} = \dfrac{-ab\pi}{e}\left[e\sqrt{1 - e^2} + \arcsin(e)\right]$

$= 2\pi b^2 + \left(\dfrac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\right)\arcsin\left(\dfrac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi\left(\dfrac{ab}{e}\right)\arcsin(e)$

$\left(e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{c}{a}\text{: eccentricity}\right)$

**—CONTINUED—**

**54. —CONTINUED—**

(b) $S = 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$

$$= 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta = \frac{4a\pi}{c}\int_0^{\pi/2} c\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta$$

$$= \frac{2a\pi}{c}\left[c\sin\theta\sqrt{b^2 + c^2\sin^2\theta} + b^2\ln\left|c\sin\theta + \sqrt{b^2 + c^2\sin^2\theta}\right|\right]_0^{\pi/2}$$

$$= \frac{2a\pi}{c}\left[c\sqrt{b^2 + c^2} + b^2\ln\left|c + \sqrt{b^2 + c^2}\right| - b^2\ln b\right]$$

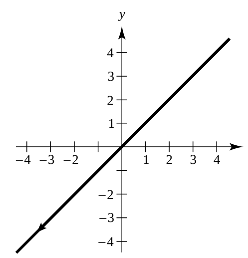
$$= 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}}\ln\left|\frac{a + \sqrt{a^2 - b^2}}{b}\right| = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left|\frac{1 + e}{1 - e}\right|$$

**56.** (a) 0

(b) 4

**58.** One possible answer is the graph given by

$x = -t, y = -t.$



**60.** (a) $S = 2\pi\int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

(b) $S = 2\pi\int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

**62.** Let $y$ be a continuous function of $x$ on $a \le x \le b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t)\,dt$ and

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)f'(t)\,dt.$$

**64.** $x = \sqrt{4 - t},\ y = \sqrt{t},\ \frac{dx}{dt} = -\frac{1}{2\sqrt{4 - t}},\ 0 \le t \le 4$

$$A = \int_4^0 \sqrt{t}\left(-\frac{1}{2\sqrt{4 - t}}\right)dt = \int_0^2 \sqrt{4 - u^2}\,du = \frac{1}{2}\left[u\sqrt{4 - u^2} + 4\arcsin\frac{u}{2}\right]_0^2 = \pi$$

Let $u = \sqrt{4 - t}$, then $du = -1/\left(2\sqrt{4 - t}\right)dt$ and $\sqrt{t} = \sqrt{4 - u^2}$.

$$\bar{x} = \frac{1}{\pi}\int_4^0 \sqrt{4 - t}\sqrt{t}\left(-\frac{1}{2\sqrt{4 - t}}\right)dt = -\frac{1}{2\pi}\int_4^0 \sqrt{t}\,dt = \left[-\frac{1}{2\pi}\frac{2}{3}t^{3/2}\right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi}\int_4^0 \left(\sqrt{t}\right)^2\left(-\frac{1}{2\sqrt{4 - t}}\right)dt = -\frac{1}{4\pi}\int_4^0 \frac{t}{\sqrt{4 - t}}\,dt = -\frac{1}{4\pi}\left[\frac{-2(8 + t)}{3}\sqrt{4 - t}\right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

**66.** $x = \cos\theta,\ y = 3\sin\theta,\ \frac{dx}{d\theta} = -\sin\theta$

$$V = 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-\sin\theta)\,d\theta$$

$$= -18\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta = -18\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 12\pi$$

**68.** $x = 2\cot\theta,\ y = 2\sin^2\theta,\ \dfrac{dx}{d\theta} = -2\csc^2\theta$

$$A = 2\int_{\pi/2}^{0} (2\sin^2\theta)(-2\csc^2\theta)\,d\theta = -8\int_{\pi/2}^{0} d\theta = \Big[-8\theta\Big]_{\pi/2}^{0} = 4\pi$$

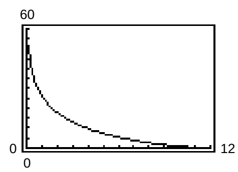
**70.** $\frac{3}{8}\pi a^2$ is area of asteroid (b).

**72.** $2\pi a^2$ is area of deltoid (c).

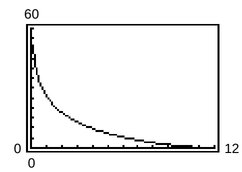
**74.** $2\pi ab$ is area of teardrop (e).

**76.** (a) $y = -12\ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$

$0 < x \leq 12$




(b) $x = 12\operatorname{sech}\dfrac{t}{12},\ y = t - 12\tanh\dfrac{t}{12},\ 0 \leq t$




Same as the graph in (a), but has the advantage of showing the position of the object and any given time $t$.

(c) $\dfrac{dy}{dx} = \dfrac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12)\tanh(t/12)} = -\sinh\dfrac{t}{12}$

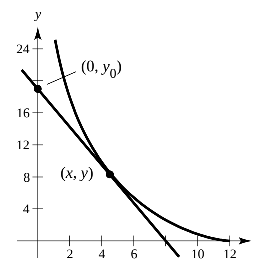
Tangent line: $y - \left(t_0 - 12\tanh\dfrac{t_0}{12}\right) = -\sinh\dfrac{t_0}{12}\left(x - 12\operatorname{sech}\dfrac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh\frac{t_0}{12}\right)x$$

$y$-intercept: $(0, t_0)$

Distance between $(0, t_0)$ and $(x, y)$: $d = \sqrt{\left(12\operatorname{sech}\dfrac{t_0}{12}\right)^2 + \left(-12\tanh\dfrac{t_0}{12}\right)^2} = 12$

$$d = 12 \text{ for any } t \geq 0.$$




**78.** False. Both $dx/dt$ and $dy/dt$ are zero when $t = 0$. By eliminating the parameter, we have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

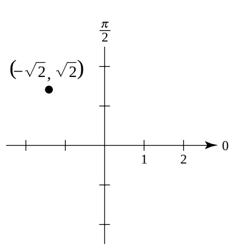
## Section 9.4 Polar Coordinates and Polar Graphs

**2.** $\left(-2, \dfrac{7\pi}{4}\right)$

$x = -2\cos\left(\dfrac{7\pi}{4}\right) = -\sqrt{2}$

$y = -2\sin\left(\dfrac{7\pi}{4}\right) = \sqrt{2}$

$(x, y) = \left(-\sqrt{2}, \sqrt{2}\right)$

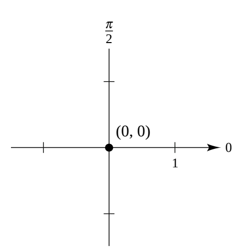



**4.** $\left(0, -\dfrac{7\pi}{6}\right)$

$x = 0\cos\left(-\dfrac{7\pi}{6}\right) = 0$

$y = 0\sin\left(-\dfrac{7\pi}{6}\right) = 0$

$(x, y) = (0, 0)$

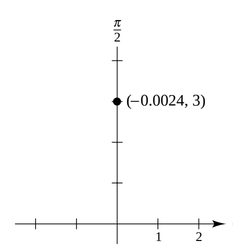



**6.** $(-3, -1.57)$

$x = -3\cos(-1.57) \approx -0.0024$

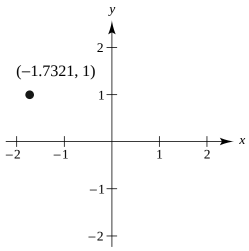
$y = -3\sin(-1.57) \approx 3$

$(x, y) = (-0.0024, 3)$



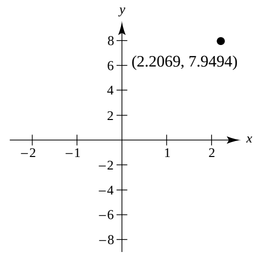
**8.** $(r, \theta) = \left(-2, \dfrac{11\pi}{6}\right)$

$(x, y) = (-1.7321, 1)$




**10.** $(r, \theta) = (8.25, 1.3)$

$(x, y) = (2.2069, 7.9494)$

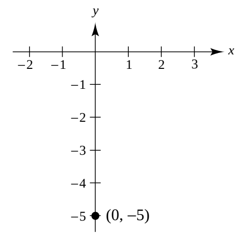



**12.** $(x, y) = (0, -5)$

$r = \pm 5$

$\tan\theta$ undefined

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \left(5, \dfrac{3\pi}{2}\right), \left(-5, \dfrac{\pi}{2}\right)$



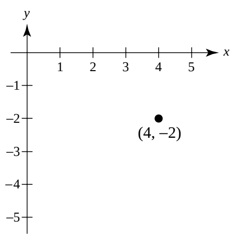

**14.** $(x, y) = (4, -2)$

$r = \pm\sqrt{16 + 4} = \pm 2\sqrt{5}$

$\tan\theta = -\dfrac{2}{4} = -\dfrac{1}{2}$

$\theta \approx -0.464$

$\left(2\sqrt{5}, -0.464\right), \left(-2\sqrt{5}, 2.678\right)$




**16.** $(x, y) = \left(3\sqrt{2}, 3\sqrt{2}\right)$

$(r, \theta) = (6, 0.785)$

**18.** $(x, y) = (0, -5)$

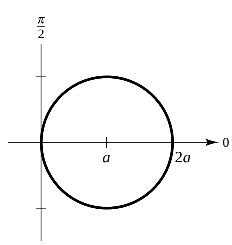
$(r, \theta) = (5, -1.571)$

**20.** (a) Moving horizontally, the $x$-coordinate changes. Moving vertically, the $y$-coordinate changes.

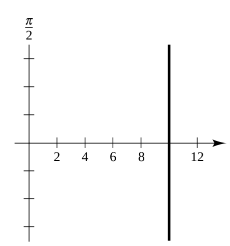
(b) Both $r$ and $\theta$ values change.

(c) In polar mode, horizontal (or vertical) changes result in changes in both $r$ and $\theta$.

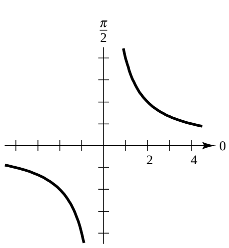
**22.**
$$x^2 + y^2 - 2ax = 0$$
$$r^2 - 2ar\cos\theta = 0$$
$$r(r - 2a\cos\theta) = 0$$
$$r = 2a\cos\theta$$



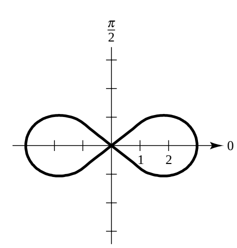
**24.**
$$x = 10$$
$$r\cos\theta = 10$$
$$r = 10\sec\theta$$



**26.**
$$xy = 4$$
$$(r\cos\theta)(r\sin\theta) = 4$$
$$r^2 = 4\sec\theta\csc\theta$$
$$= 8\csc 2\theta$$



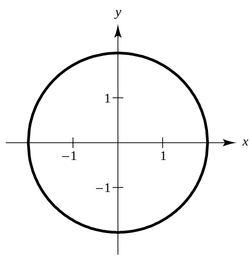
**28.**
$$(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$$
$$(r^2)^2 - 9(r^2\cos^2\theta - r^2\sin^2\theta) = 0$$
$$r^2[r^2 - 9(\cos 2\theta)] = 0$$
$$r^2 = 9\cos 2\theta$$



**30.** $r = -2$

$r^2 = 4$

$x^2 + y^2 = 4$



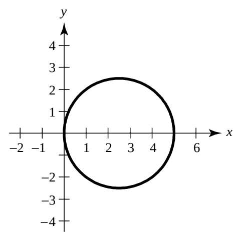

**32.** $r = 5 \cos \theta$

$r^2 = 5r \cos \theta$

$x^2 + y^2 = 5x$

$x^2 - 5x + \dfrac{25}{4} + y^2 = \dfrac{25}{4}$

$\left(x - \dfrac{5}{2}\right)^2 + y^2 = \left(\dfrac{5}{2}\right)^2$

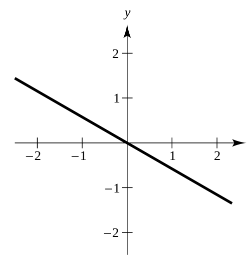



**34.** $\theta = \dfrac{5\pi}{6}$

$\tan \theta = \tan \dfrac{5\pi}{6}$

$\dfrac{y}{x} = -\dfrac{\sqrt{3}}{3}$

$y = -\dfrac{\sqrt{3}}{3}x$

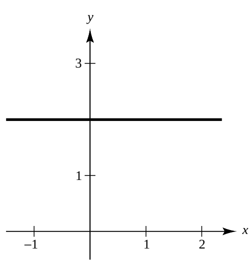



**36.** $r = 2 \csc \theta$

$r \sin \theta = 2$

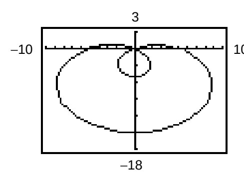
$y = 2$

$y - 2 = 0$



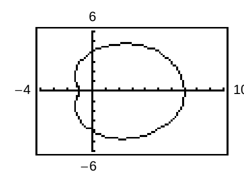

**38.** $r = 5(1 - 2 \sin \theta)$

$0 \le \theta < 2\pi$



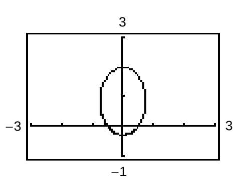

**40.** $r = 4 + 3 \cos \theta$

$0 \le \theta < 2\pi$



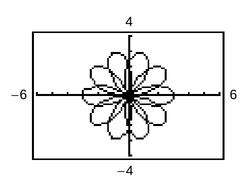

**42.** $r = \dfrac{2}{4 - 3 \sin \theta}$

Traced out once on $0 \le \theta \le 2\pi$




**44.** $r = 3 \sin\left(\dfrac{5\theta}{2}\right)$

$0 \le \theta < 4\pi$

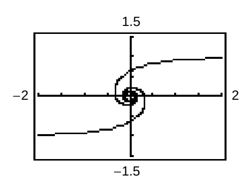



**46.** $r^2 = \dfrac{1}{\theta}$.

Graph as

$r_1 = \dfrac{1}{\sqrt{\theta}}, r_2 = -\dfrac{1}{\sqrt{\theta}}$.

It is traced out once on $[0, \infty)$.



**48.** (a) The rectangular coordinates of $(r_1, \theta_1)$ are $(r_1 \cos \theta_1, r_1 \sin \theta_1)$. The rectangular coordinates of $(r_2, \theta_2)$ are $(r_2 \cos \theta_2, r_2 \sin \theta_2)$.

$$\begin{aligned}
d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\
&= (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2 \\
&= r_2^2 \cos^2 \theta_2 - 2r_1r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1 + r_2^2 \sin^2 \theta_2^2 - 2r_1r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1 \\
&= r_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2) + r_1^2 (\cos^2 \theta_1 + \sin^2 \theta_1) - 2\, r_1r_2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\
&= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) \\
d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}
\end{aligned}$$

(b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned}
d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(0)} \\
&= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|
\end{aligned}$$

(c) If $\theta_1 - \theta_2 = 90°$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

(d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = (2, \pi/2)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2) \cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

Using $(r_1, \theta_1) = (-1, \pi)$ and $(r_2, \theta_2) = [2, (5\pi/2)]$, $d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2) \cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}$.

You always obtain the same distance.

**50.** $\left(10, \frac{7\pi}{6}\right), (3, \pi)$

$$\begin{aligned}
d &= \sqrt{10^2 + 3^2 - 2(10)(3) \cos\left(\frac{7\pi}{6} - \pi\right)} \\
&= \sqrt{109 - 60 \cos \frac{\pi}{6}} = \sqrt{109 - 30\sqrt{3}} \approx 7.6
\end{aligned}$$

**52.** $(4, 2.5),\ (12, 1)$

$$\begin{aligned}
d &= \sqrt{4^2 + 12^2 - 2(4)(12) \cos(2.5 - 1)} \\
&= \sqrt{160 - 96 \cos 1.5} \approx 12.3
\end{aligned}$$

**54.** $r = 2(1 - \sin \theta)$

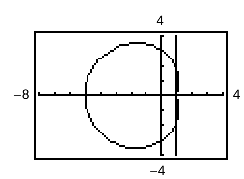
$$\frac{dy}{dx} = \frac{-2 \cos \theta \sin \theta + 2 \cos \theta(1 - \sin \theta)}{-2 \cos \theta \cos \theta - 2 \sin \theta(1 - \sin \theta)}$$

At $(2, 0)$, $\dfrac{dy}{dx} = -1$.

At $\left(3, \dfrac{7\pi}{6}\right)$, $\dfrac{dy}{dx}$ is undefined.

At $\left(4, \dfrac{3\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

**56.** (a), (b) $r = 3 - 2 \cos \theta$

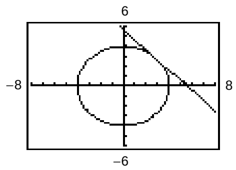



$(r, \theta) = (1, 0) \implies (x, y) = (1, 0)$

Tangent line: $x = 1$

(c) At $\theta = 0$, $\dfrac{dy}{dx}$ does not exist (vertical tangent).

**58.** (a), (b) $r = 4$




at $(r, \theta) = \left(4, \dfrac{\pi}{4}\right) \Rightarrow (x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

Tangent line: $y - 2\sqrt{2} = -1\left(x - 2\sqrt{2}\right)$

$$y = -x + 4\sqrt{2}$$

(c) At $\theta = \dfrac{\pi}{4}, \dfrac{dy}{dx} = -1.$

**60.** $r = a \sin \theta$

$$\frac{dy}{d\theta} = a \sin \theta \cos \theta + a \cos \theta \sin \theta$$

$$= 2a \sin \theta \cos \theta = 0$$

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a \sin^2 \theta + a \cos^2 \theta = a(1 - 2 \sin^2 \theta) = 0$$

$$\sin \theta = \pm\frac{1}{\sqrt{2}}, \ \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Horizontal: $(0, 0), \left(a, \dfrac{\pi}{2}\right)$

Vertical: $\left(\dfrac{a\sqrt{2}}{2}, \dfrac{\pi}{4}\right), \left(\dfrac{a\sqrt{2}}{2}, \dfrac{3\pi}{4}\right)$

**62.** $r = a \sin \theta \cos^2 \theta$

$$\frac{dy}{d\theta} = a \sin \theta \cos^3 \theta + [-2a \sin^2 \theta \cos \theta + a \cos^3 \theta] \sin \theta$$

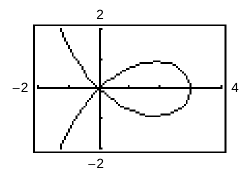
$$= 2a[\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta]$$

$$= 2a \sin \theta \cos \theta(\cos^2 \theta - \sin^2 \theta) = 0$$

$$\theta = 0, \ \tan^2 \theta = 1, \ \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

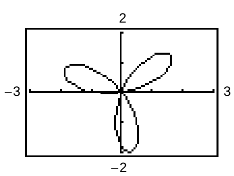
Horizontal: $\left(\dfrac{\sqrt{2}a}{4}, \dfrac{\pi}{4}\right), \left(\dfrac{\sqrt{2}a}{4}, \dfrac{3\pi}{4}\right), (0, 0)$

**64.** $r = 3 \cos 2\theta \sec \theta$




Horizontal tangents: $(2.133, \pm 0.4352)$

**66.** $r = 2 \cos(3\theta - 2)$




Horizontal tangents:

$(1.894, 0.776), (1.755, 2.594), (1.998, -1.442)$

**68.**

$$r = 3 \cos \theta$$

$$r^2 = 3r \cos \theta$$

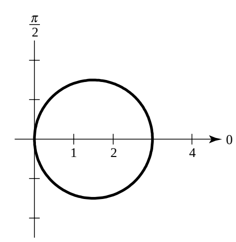
$$x^2 + y^2 = 3x$$

$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$

Circle: $r = \dfrac{3}{2}$

Center: $\left(\dfrac{3}{2}, 0\right)$

Tangent at pole: $\theta = \dfrac{\pi}{2}$

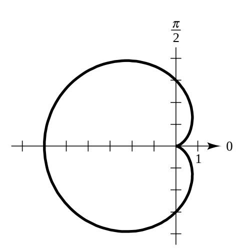



**70.** $r = 3(1 - \cos \theta)$

Cardioid

Symmetric to polar axis since $r$ is a function of $\cos \theta$.

| $\theta$ | $0$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $\frac{3}{2}$ | $3$ | $\frac{9}{2}$ | $6$ |



**72.** $r = -\sin(5\theta)$

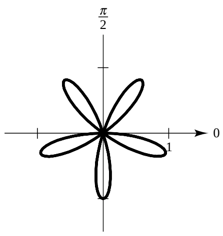
Rose curve with five petals

Symmetric to $\theta = \dfrac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5\cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{5}, \dfrac{2\pi}{5}, \dfrac{3\pi}{5}, \dfrac{4\pi}{5}$




**74.** $r = 3\cos 2\theta$

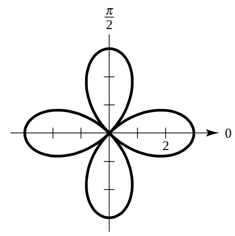
Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(3, 0), \left(-3, \dfrac{\pi}{2}\right), (3, \pi), \left(-3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

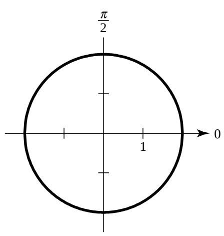
$\theta = \dfrac{5\pi}{4}$ and $\dfrac{7\pi}{4}$ given the same tangents.




**76.** $r = 2$

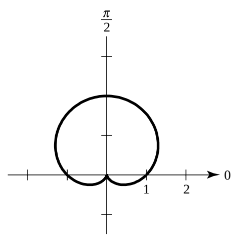
Circle radius: 2

$x^2 + y^2 = 4$




**78.** $r = 1 + \sin\theta$

Cardioid

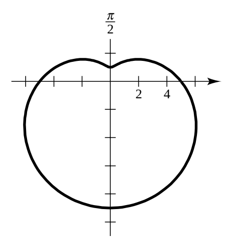



**80.** $r = 5 - 4\sin\theta$

Limaçon

Symmetric to $\theta = \dfrac{\pi}{2}$

| $\theta$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{6}$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $r$ | 9 | 7 | 5 | 3 | 1 |

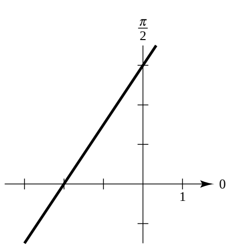



**82.** $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$

$$2r\sin\theta - 3r\cos\theta = 6$$

$$2y - 3x = 6$$

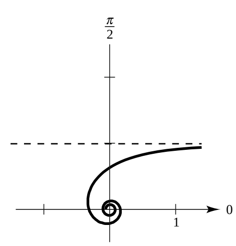
Line




**84.** $r = \dfrac{1}{\theta}$

Hyperbolic spiral

| $\theta$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|---|
| $r$ | $\frac{4}{\pi}$ | $\frac{2}{\pi}$ | $\frac{4}{3\pi}$ | $\frac{1}{\pi}$ | $\frac{4}{5\pi}$ | $\frac{2}{3\pi}$ |



**86.** $r^2 = 4 \sin \theta$

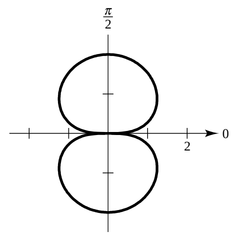
Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \dfrac{\pi}{2}\right)$

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $\pm\sqrt{2}$ | $\pm 2$ | $\pm\sqrt{2}$ | $0$ |

Tangent at the pole: $\theta = 0$




**88.** Since

$$r = 2 + \csc \theta = 2 + \frac{1}{\sin \theta},$$

the graphs has symmetry with respect to $\theta = \pi/2$. Furthermore,

$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow 0^+$$

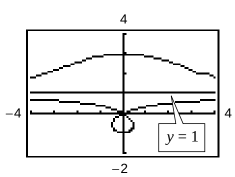
$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow \pi^-.$$

Also, $r = 2 + \dfrac{1}{\sin \theta} = 2 + \dfrac{r}{\sin \theta} = 2 + \dfrac{r}{y}$

$$ry = 2y + r$$

$$r = \frac{2y}{y - 1}.$$

Thus, $r \Longrightarrow \pm\infty$ as $y \Longrightarrow 1$.




**90.** $r = 2 \cos 2\theta \sec \theta$

Strophoid

$$r \Longrightarrow -\infty \text{ as } \theta \Longrightarrow \frac{\pi^-}{2}$$

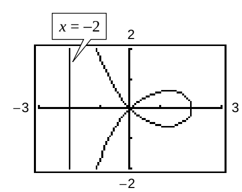
$$r \Longrightarrow \infty \text{ as } \theta \Longrightarrow \frac{-\pi^+}{2}$$

$$r = 2 \cos 2\theta \sec \theta = 2(2 \cos^2 \theta - 1) \sec \theta$$

$$r \cos \theta = 4 \cos^2 \theta - 2$$

$$x = 4 \cos^2 \theta - 2$$

$$\lim_{\theta \to \pm\pi/2} (4 \cos^2 \theta - 2) = -2$$




**92.** $x = r \cos \theta, y = r \sin \theta$

$$x^2 + y^2 = r^2, \tan \theta = \frac{y}{x}$$

**94.** Slope of tangent line to graph of $r = f(\theta)$ at $(r, \theta)$ is

$$\frac{dy}{dx} = \frac{f(\theta)\cos \theta + f'(\theta)\sin \theta}{-f(\theta)\sin \theta + f'(\theta)\cos \theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

**96.** $r = 4 \cos 2\theta$

Rose curve

Matches (b)

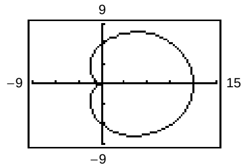
**98.** $r = 2 \sec \theta$

Line

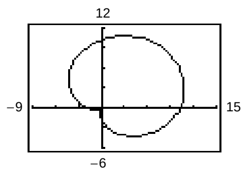
Matches (d)

**100.** $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0, r = 6[1 + \cos\theta]$



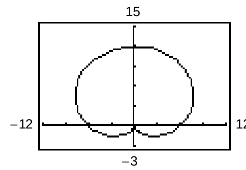

(b) $\theta = \dfrac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \dfrac{\pi}{4}\right)\right]$




The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \dfrac{\pi}{2}$

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right]$$
$$= 6\left[1 + \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}\right]$$
$$= 6[1 + \sin\theta]$$




The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/2$.

**102.** (a) $\sin\left(\theta - \dfrac{\pi}{2}\right) = \sin\theta\cos\left(\dfrac{\pi}{2}\right) - \cos\theta\sin\left(\dfrac{\pi}{2}\right)$

$$= -\cos\theta$$
$$r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$$
$$= f(-\cos\theta)$$

(b) $\sin(\theta - \pi) = \sin\theta\cos\pi - \cos\theta\sin\pi$

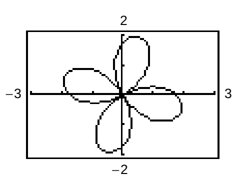
$$= -\sin\theta$$
$$r = f[\sin(\theta - \pi)]$$
$$= f(-\sin\theta)$$

(c) $\sin\left(\theta - \dfrac{3\pi}{2}\right) = \sin\theta\cos\left(\dfrac{3\pi}{2}\right) - \cos\theta\sin\left(\dfrac{3\pi}{2}\right)$

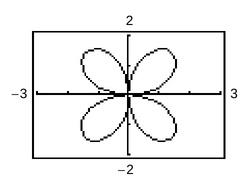
$$= \cos\theta$$
$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos\theta)$$

**104.** $r = 2\sin 2\theta = 4\sin\theta\cos\theta$

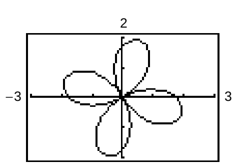
(a) $r = 4\sin\left(\theta - \dfrac{\pi}{6}\right)\cos\left(\theta - \dfrac{\pi}{6}\right)$



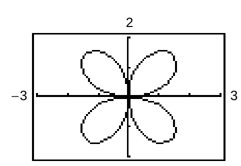

(b) $r = 4\sin\left(\theta - \dfrac{\pi}{2}\right)\cos\left(\theta - \dfrac{\pi}{2}\right) = -4\sin\theta\cos\theta$




(c) $r = 4\sin\left(\theta - \dfrac{2\pi}{3}\right)\cos\left(\theta - \dfrac{2\pi}{3}\right)$




(d) $r = 4\sin(\theta - \pi)\cos(\theta - \pi) = 4\sin\theta\cos\theta$

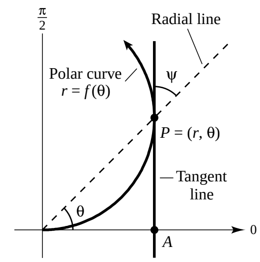


**106.** By Theorem 9.11, the slope of the tangent line through $A$ and $P$ is

$$\frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta}$$

This is equal to

$$\tan(\theta+\psi) = \frac{\tan\theta + \tan\psi}{1-\tan\theta\tan\psi} = \frac{\sin\theta + \cos\theta\tan\psi}{\cos\theta - \sin\theta\tan\psi}.$$




Equating the expressions and cross-multiplying, you obtain

$$(f\cos\theta + f'\sin\theta)(\cos\theta - \sin\theta\tan\psi) = (\sin\theta + \cos\theta\tan\psi)(-f\sin\theta + f'\cos\theta)$$

$$f\cos^2\theta - f\cos\theta\sin\theta\tan\psi + f'\sin\theta\cos\theta - f'\sin^2\theta\tan\psi = -f\sin^2\theta - f\sin\theta\cos\theta\tan\psi + f'\sin\theta\cos\theta + f'\cos^2\theta\tan\psi$$

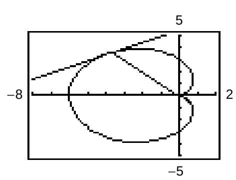
$$f(\cos^2\theta + \sin^2\theta) = f'\tan\psi(\cos^2\theta + \sin^2\theta)$$

$$\tan\psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$

**108.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{3(1-\cos\theta)}{3\sin\theta}$

At $\theta = \dfrac{3\pi}{4}$, $\tan\psi = \dfrac{1 + \left(\sqrt{2}/2\right)}{\sqrt{2}/2} = \dfrac{2+\sqrt{2}}{\sqrt{2}}$.

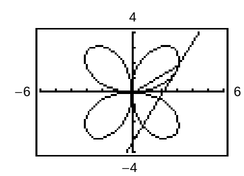
$$\psi = \arctan\left(\frac{2+\sqrt{2}}{\sqrt{2}}\right) \approx 1.041 (\approx 59.64°)$$



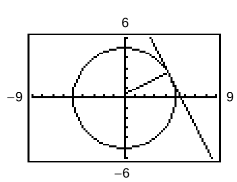

**110.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{4\sin 2\theta}{8\cos 2\theta}$

At $\theta = \dfrac{\pi}{6}$, $\tan\psi = \dfrac{\sin(\pi/3)}{2\cos(\pi/3)} = \dfrac{\sqrt{3}}{2}$.

$$\psi = \arctan\left(\frac{\sqrt{3}}{2}\right) \approx 0.7137 (\approx 40.89°)$$




**112.** $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{5}{0}$ undefined $\implies \psi = \dfrac{\pi}{2}$.

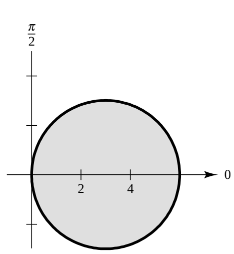



**114.** True

**116.** True

## Section 9.5 Area and Arc Length in Polar Coordinates

**2.** (a) $r = 3\cos\theta$




$$A = \pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$$

(b) $A = 2\left(\dfrac{1}{2}\right)\displaystyle\int_0^{\pi/2} \left[3\cos\theta\right]^2 d\theta$

$$= 9\int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{9}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{9\pi}{4}$$

**4.** $A = 2\left[\frac{1}{2}\int_0^{\pi/4} (6\sin 2\theta)^2\,d\theta\right] = 36\int_0^{\pi/4} \sin^2 2\theta\,d\theta$

$= 36\int_0^{\pi/4} \frac{1-\cos 4\theta}{2}\,d\theta$

$= 18\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$

$= 18\left[\frac{\pi}{4}\right] = \frac{9\pi}{2}$

**6.** $A = 2\left[\frac{1}{2}\int_0^{\pi/10} (\cos 5\theta)^2\,d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{10}\sin(10\,\theta)\right]_0^{\pi/10} = \frac{\pi}{20}$

**8.** $A = 2\left[\frac{1}{2}\int_0^{\pi/2} (1-\sin\theta)^2\,d\theta\right]$

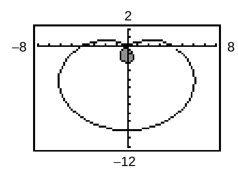
$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi - 8}{4}$

**10.** $A = 2\left[\frac{1}{2}\int_{\arcsin(2/3)}^{\pi/2} (4 - 6\sin\theta)^2\,d\theta\right.$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\sin^2\theta\right] d\theta$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$

$= \left[34\theta + 48\cos\theta - 9\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$



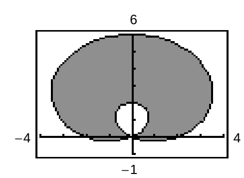

**12.** Four times the area in Exercise 11, $A = 4\left(\pi + 3\sqrt{3}\right)$. More specifically, we see that the area inside the outer loop is

$2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/2} (2(1+2\sin\theta))^2\,d\theta\right] = \int_{-\pi/6}^{\pi/2} (4 + 16\sin\theta + 16\sin^2\theta)\,d\theta = 8\pi + 6\sqrt{3}.$

The area inside the inner loop is

$2\,\frac{1}{2}\left[\int_{7\pi/6}^{3\pi/2} (2(1+2\sin\theta))^2\,d\theta\right] = 4\pi - 6\sqrt{3}.$

Thus, the area between the loops is $\left(8\pi + 6\sqrt{3}\right) - \left(4\pi - 6\sqrt{3}\right) = 4\pi + 12\sqrt{3}.$




**14.** $r = 3(1 + \sin\theta)$

$r = 3(1 - \sin\theta)$

Solving simultaneously,

$3(1 + \sin\theta) = 3(1 - \sin\theta)$

$2\sin\theta = 0$

$\theta = 0, \pi.$

Replacing $r$ by $-r$ and $\theta$ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin\theta) = 3(1 - \sin\theta)$, $\sin\theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0)$, $(3, \pi)$, $(0, 0)$

**16.** $r = 2 - 3\cos\theta$

$r = \cos\theta$

Solving simultaneously,

$2 - 3\cos\theta = \cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$, $(0, 0)$

**18.** $r = 1 + \cos\theta$

$r = 3\cos\theta$

Solving simultaneously,

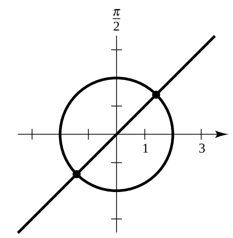
$$1 + \cos\theta = 3\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, 0)$

**20.** $\theta = \dfrac{\pi}{4}$

$r = 2$



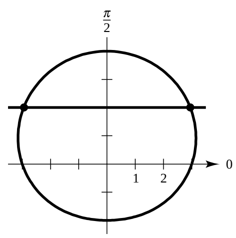

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection:

$$\left(2, \frac{\pi}{4}\right), \left(-2, \frac{\pi}{4}\right)$$

**22.** $r = 3 + \sin\theta$

$r = 2\csc\theta$




The graph of $r = 3 + \sin\theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2\csc\theta$ is the horizontal line $y = 2$. Therefore, there are two points of intersection. Solving simultaneously,

$$3 + \sin\theta = 2\csc\theta$$
$$\sin^2\theta + 3\sin\theta - 2 = 0$$
$$\sin\theta = \frac{-3 \pm \sqrt{17}}{2}$$
$$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596.$$

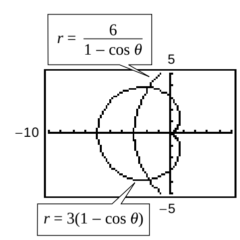
Points of intersection:

$$\left(\frac{\sqrt{17} + 3}{2}, \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$
$$\left(\frac{\sqrt{17} + 3}{2}, \pi - \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$(3.56, 0.596), (3.56, 2.545)$

**24.** $r = 3(1 - \cos\theta)$

$r = \dfrac{6}{1 - \cos\theta}$




The graph of $r = 3(1 - \cos\theta)$ is a cardioid with polar axis symmetry. The graph of

$$r = 6/(1 - \cos\theta)$$

is a parabola with focus at the pole, vertex$(3, \pi)$, and polar axis symmetry. Therefore, there are two points of intersection. Solving simultaneously,

$$3(1 - \cos\theta) = \frac{6}{1 - \cos\theta}$$
$$(1 - \cos\theta)^2 = 2$$
$$\cos\theta = 1 \pm \sqrt{2}$$
$$\theta = \arccos\left(1 - \sqrt{2}\right).$$

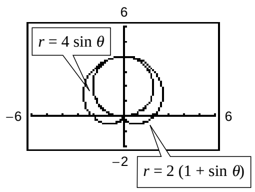
Points of intersection: $\left(3\sqrt{2}, \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 1.998), \left(3\sqrt{2}, 2\pi - \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 4.285)$

**26.** $r = 4 \sin \theta$

$r = 2(1 + \sin \theta)$

Points of intersection: $(0, 0), \left(4, \dfrac{\pi}{2}\right)$

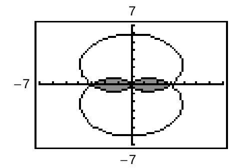
The graphs reach the pole at different times ($\theta$ values).




**28.** $A = 4\left[\dfrac{1}{2}\displaystyle\int_0^{\pi/2} 9(1 - \sin \theta)^2 \, d\theta\right]$

$$= 18 \int_0^{\pi/2} (1 - \sin \theta)^2 \, d\theta = \frac{9}{2}(3\pi - 8)$$

(from Exercise 14)



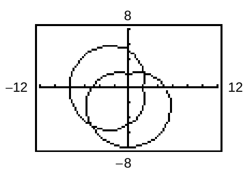

**30.** $r = 5 - 3 \sin \theta$ and $r = 5 - 3 \cos \theta$ intersect at $\theta = \pi/4$ and $\pi = 5\pi/4$.

$$A = 2\left[\frac{1}{2}\int_{\pi/4}^{5\pi/4} (5 - 3 \sin \theta)^2 \, d\theta\right]$$

$$= \left[\frac{59}{2}\theta + 30 \cos \theta - \frac{9}{4} \sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{59}{2}\left(\frac{5\pi}{4}\right) - 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right) - \left(\frac{59}{2}\left(\frac{\pi}{4}\right) + 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right)$$

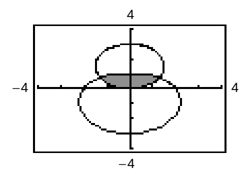
$$= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251$$




**32.** $A = 2\left[\dfrac{1}{2}\displaystyle\int_{\pi/6}^{\pi/2} (3 \sin \theta)^2 \, d\theta - \dfrac{1}{2}\displaystyle\int_{\pi/6}^{\pi/2} (2 - \sin \theta)^2 \, d\theta\right]$

$$= \int_{\pi/6}^{\pi/2} (-4 \cos 2\theta + 4 \sin \theta) \, d\theta$$

$$= \left[-2 \sin(2\theta) - 4 \cos \theta\right]_{\pi/6}^{\pi/2} = 3\sqrt{3}$$



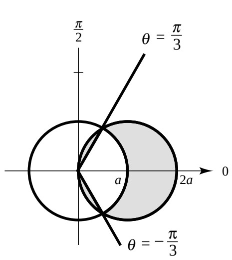

**34.** Area = Area of $r = 2a \cos \theta$ − Area of sector − twice area between $r = 2a \cos \theta$ and the lines $\theta = \dfrac{\pi}{3}, \theta = \dfrac{\pi}{2}$.

$$A = \pi a^2 - \left(\frac{\pi}{3}\right)a^2 - 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2} (2a \cos \theta)^2 \, d\theta\right]$$

$$= \frac{2\pi a^2}{3} - 2a^2 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$= \frac{2\pi a^2}{3} - 2a^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{2\pi a^2}{3} - 2a^2 \left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \frac{2\pi a^2 + 3\sqrt{3}a^2}{6}$$




**36.** $r = a \cos \theta, r = a \sin \theta$

$\tan \theta = 1, \theta = \pi/4$

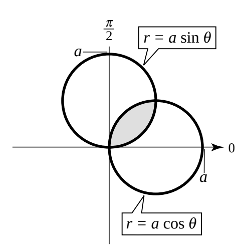
$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} (a \cos \theta)^2 \, d\theta\right]$$

$$= a^2 \int_0^{\pi/4} \frac{1 + \cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}a^2\left[\frac{\pi}{4} + \frac{1}{2}\right]$$

$$= \frac{1}{4}a^2 + \frac{1}{8}a^2\pi$$



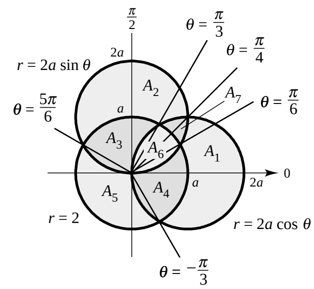
**38.** By symmetry, $A_1 = A_2$ and $A_3 = A_4$.

$$A_1 = A_2 = \frac{1}{2}\int_{-\pi/3}^{\pi/6}\left[(2a\cos\theta)^2 - (a)^2\right]d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4}\left[(2a\cos\theta)^2 - (2a\sin\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/3}^{\pi/6}(4\cos^2\theta - 1)\,d\theta + 2a^2\int_{\pi/6}^{\pi/4}\cos 2\theta\,d\theta$$

$$= \frac{a^2}{2}\Big[\theta + \sin 2\theta\Big]_{-\pi/3}^{\pi/6} + a^2\Big[\sin 2\theta\Big]_{\pi/6}^{\pi/4} = \frac{a^2}{2}\left(\frac{\pi}{2} + \sqrt{3}\right) + a^2\left(1 - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{4} + 1\right)$$

$$A_3 = A_4 = \frac{1}{2}\left(\frac{\pi}{2}\right)a^2 = \frac{\pi a^2}{4}$$




$$A_5 = \frac{1}{2}\left(\frac{5\pi}{6}\right)a^2 - 2\left(\frac{1}{2}\right)\int_{5\pi/6}^{\pi}(2a\sin\theta)^2\,d\theta$$

$$= \frac{5\pi a^2}{12} - 2a^2\int_{5\pi/6}^{\pi}(1 - \cos 2\theta)\,d\theta$$

$$= \frac{5\pi a^2}{12} - a^2\Big[2\theta - \sin 2\theta\Big]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$$

$$A_6 = 2\left(\frac{1}{2}\right)\int_0^{\pi/6}(2a\sin\theta)^2\,d\theta + 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}a^2\,d\theta$$

$$= 2a^2\int_0^{\pi/6}(1 - \cos 2\theta)\,d\theta + \Big[a^2\theta\Big]_{\pi/6}^{\pi/4}$$

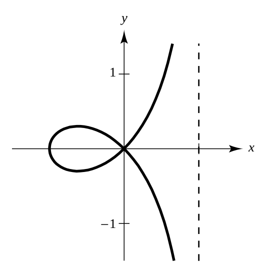
$$= a^2\Big[2\theta - \sin 2\theta\Big]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi a^2}{12} = a^2\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$$

$$A_7 = 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}\left[(2a\sin\theta)^2 - (a)^2\right]d\theta$$

$$= a^2\int_{\pi/6}^{\pi/4}(4\sin^2\theta - 1)\,d\theta = a^2\Big[\theta - \sin 2\theta\Big]_{\pi/6}^{\pi/4} = a^2\left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right)$$

[**Note:** $A_1 + A_6 + A_7 + A_4 = \pi a^2 =$ area of circle of radius $a$]

**40.** $$r = \sec\theta - 2\cos\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$




$$r\cos\theta = 1 - 2\cos^2\theta$$

$$x = 1 - 2\frac{r^2\cos^2\theta}{r^2} = 1 - 2\left(\frac{x^2}{x^2 + y^2}\right)$$

$$(x^2 + y^2)x = x^2 + y^2 - 2x^2$$

$$y^2(x - 1) = -x^2 - x^3$$

$$y^2 = \frac{x^2(1 + x)}{1 - x}$$

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4}(\sec\theta - 2\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/4}(\sec^2\theta - 4 + 4\cos^2\theta)\,d\theta = \int_0^{\pi/4}(\sec^2\theta - 4 + 2(1 + \cos 2\theta))\,d\theta = \Big[\tan\theta - 2\theta + \sin 2\theta\Big]_0^{\pi/4} = 2 - \frac{\pi}{2}$$

**42.** $r = 2a\cos\theta$

$r' = -2a\sin\theta$

$$s = \int_{-\pi/2}^{\pi/2} \sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2}\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2a\, d\theta = \Big[2\theta\Big]_{-\pi/2}^{\pi/2} = 2\pi a$$

**44.** $r = 8(1 + \cos\theta),\ 0 \le \theta \le 2\pi$

$r' = -8\sin\theta$

$$s = 2\int_0^{\pi} \sqrt{[8(1+\cos\theta)]^2 + (-8\sin\theta)^2}\, d\theta$$

$$= 16\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\, d\theta$$

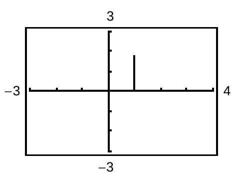
$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta} \cdot \left(\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}\right) d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1-\cos\theta}}\, d\theta$$

$$= \Big[32\sqrt{2}\sqrt{1-\cos\theta}\Big]_0^{\pi}$$

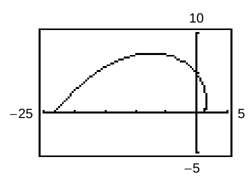
$$= 64$$

**46.** $r = \sec\theta,\ 0 \le \theta \le \dfrac{\pi}{3}$



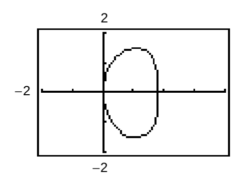

Length $\approx 1.73$ (exact $\sqrt{3}$)

**48.** $r = e^{\theta},\ 0 \le \theta \le \pi$




Length $\approx 31.31$

**50.** $r = 2\sin(2\cos\theta),\ 0 \le \theta \le \pi$




Length $\approx 7.78$

**52.** $r = a\cos\theta$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi/2} a\cos\theta\,(\cos\theta)\sqrt{a^2\cos\theta + a^2\sin^2\theta}\, d\theta$$

$$= 2\pi a^2\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi a^2\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= \left[\pi a^2\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi^2 a^2}{2}$$

**54.** $r = a(1 + \cos\theta)$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta = 2\pi a^2\int_0^{\pi} \sin\theta(1+\cos\theta)\sqrt{2 + 2\cos\theta}\, d\theta$$

$$= -2\sqrt{2}\pi a^2\int_0^{\pi} (1+\cos\theta)^{3/2}(-\sin\theta)\, d\theta = -\frac{4\sqrt{2}\pi a^2}{5}\Big[(1+\cos\theta)^{5/2}\Big]_0^{\pi} = \frac{32\pi a^2}{5}$$

**56.** $r = \theta$

$r' = 1$

$$S = 2\pi\int_0^{\pi} \theta\sin\theta\sqrt{\theta^2 + 1}\ d\theta \approx 42.32$$

**58.** The curves might intersect for different values of $\theta$:

See page 696.

**60.** (a) $S = 2\pi \int_\alpha^\beta f(\theta) \sin\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

(b) $S = 2\pi \int_\alpha^\beta f(\theta) \cos\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

**62.** $r = 8\cos\theta,\ 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^\pi r^2\, d\theta = \frac{1}{2}\int_0^\pi 64\cos^2\theta\, d\theta = 32\int_0^\pi \frac{1 + \cos 2\theta}{2}\, d\theta = 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = 16\pi$

(Area circle $= \pi r^2 = \pi 4^2 = 16\pi$)

(b)

| $\theta$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|---|
| $A$ | 6.32 | 12.14 | 17.06 | 20.80 | 23.27 | 24.60 | 25.08 |

(c), (d) For $\frac{1}{4}$ of area $(4\pi \approx 12.57)$: 0.42
For $\frac{1}{2}$ of area $(8\pi \approx 25.13)$: 1.57 $(\pi/2)$
For $\frac{3}{4}$ of area $(12\pi \approx 37.70)$: 2.73

(e) No, it does not depend on the radius.

**64.** False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

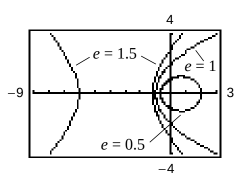
## Section 9.6 Polar Equations of Conics and Kepler's Laws

**2.** $r = \dfrac{2e}{1 - e\cos\theta}$

(a) $e = 1,\ r = \dfrac{2}{1 - \cos\theta}$, parabola

(b) $e = 0.5,\ r = \dfrac{1}{1 - 0.5\cos\theta} = \dfrac{2}{2 - \cos\theta}$, ellipse

(c) $e = 1.5,\ r = \dfrac{3}{1 - 1.5\cos\theta} = \dfrac{6}{2 - 3\cos\theta}$, hyperbola

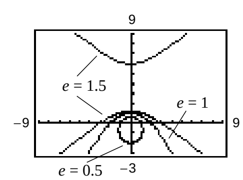



**4.** $r = \dfrac{2e}{1 + e\sin\theta}$

(a) $e = 1,\ r = \dfrac{2}{1 + \sin\theta}$, parabola

(b) $e = 0.5,\ r = \dfrac{1}{1 + 0.5\sin\theta} = \dfrac{2}{2 + \sin\theta}$, ellipse

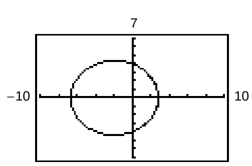
(c) $e = 1.5,\ r = \dfrac{3}{1 + 1.5\sin\theta} = \dfrac{6}{2 + 3\sin\theta}$, hyperbola

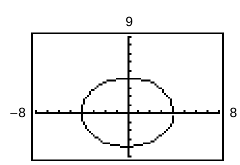



**6.** $r = \dfrac{4}{1 - 0.4\cos\theta}$

(a) Because $e = 0.4 < 1$, the conic is an ellipse with vertical directrix to the left of the pole.

(c)




(b) $r = \dfrac{4}{1 + 0.4\cos\theta}$

The ellipse is shifted to the left. The vertical directrix is to the right of the pole

$$r = \frac{4}{1 - 0.4\sin\theta}.$$

The ellipse has a horizontal directrix below the pole.

**8.** Ellipse; Matches (f)

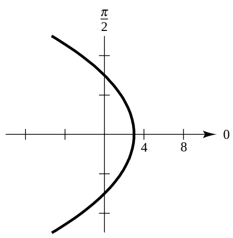
**10.** Parabola; Matches (e)

**12.** Hyperbola; Matches (d)

**14.** $r = \dfrac{6}{1 + \cos\theta}$

Parabola since $e = 1$

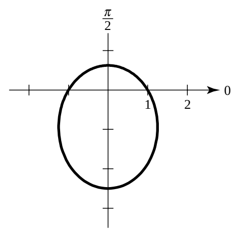
Vertex: $(3, 0)$




**16.** $r = \dfrac{5}{5 + 3\sin\theta} = \dfrac{1}{1 + (3/5)\sin\theta}$

Ellipse since $e = \dfrac{3}{5} < 1$

Vertices: $\left(\dfrac{5}{8}, \dfrac{\pi}{2}\right), \left(\dfrac{5}{2}, \dfrac{3\pi}{2}\right)$



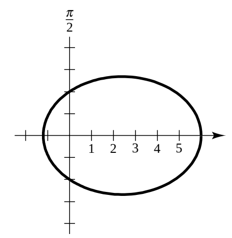

**18.** $r(3 - 2\cos\theta) = 6$

$$r = \frac{6}{3 - 2\cos\theta}$$

$$= \frac{2}{1 - (2/3)\cos\theta}$$

Ellipse since $e = \dfrac{2}{3} < 1$

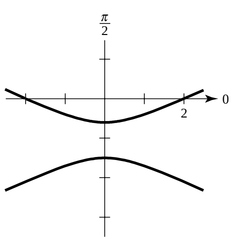
Vertices: $(6, 0), \left(\dfrac{6}{5}, \pi\right)$




**20.** $r = \dfrac{-6}{3 + 7\sin\theta} = \dfrac{-2}{1 + (7/3)\sin\theta}$

Hyperbola since $e = \dfrac{7}{3} > 1$.

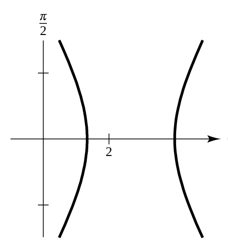
Vertices: $\left(-\dfrac{3}{5}, \dfrac{\pi}{2}\right), \left(\dfrac{3}{2}, \dfrac{3\pi}{2}\right)$

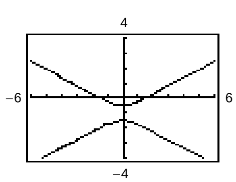



**22.** $r = \dfrac{4}{1 + 2\cos\theta}$

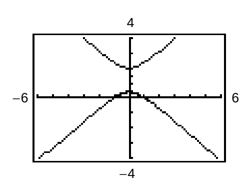
Hyperbola since $e = 2 > 1$

Vertices: $\left(\dfrac{4}{3}, 0\right), (-4, \pi)$




**24.** 


Hyperbola

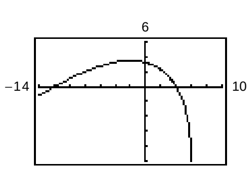
**26.** 


Hyperbola

**28.** $r = \dfrac{6}{1 + \cos\left(\theta - \dfrac{\pi}{3}\right)}$

Rotate the graph of $r = \dfrac{6}{1 + \cos\theta}$

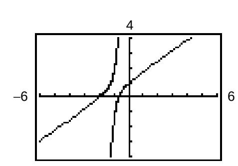
counterclockwise through the angle $\dfrac{\pi}{3}$.




**30.** $r = \dfrac{-6}{3 + 7\sin(\theta + (2\pi/3))}$

Rotate graph of $r = \dfrac{-6}{3 + 7\sin\theta}$.

Clockwise through angle of $2\pi/3$.



**32.** Change $\theta$ to $\theta - \frac{\pi}{6}$: $r = \dfrac{2}{1 + \sin\left(\theta - \frac{\pi}{6}\right)}$

**34.** Parabola

$e = 1, y = 1, d = 1$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{1}{1 + \sin\theta}$$

**36.** Ellipse

$e = \frac{3}{4}, y = -2, d = 2$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{2(3/4)}{1 - (3/4)\sin\theta} = \frac{6}{4 - 3\sin\theta}$$

**38.** Hyperbola

$e = \frac{3}{2}, x = -1, d = 1$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{3/2}{1 - (3/2)\cos\theta} = \frac{3}{2 - 3\cos\theta}$$

**40.** Parabola

Vertex: $(5, \pi)$

$e = 1, d = 10$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{10}{1 - \cos\theta}$$

**42.** Ellipse

Vertices: $\left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$

$e = \frac{1}{3}, d = 8$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{8/3}{1 + (1/3)\sin\theta} = \frac{8}{3 + \sin\theta}$$

**44.** Hyperbola

Vertices: $(2, 0), (10, 0)$

$e = \frac{3}{2}, d = \frac{10}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{5}{1 + (3/2)\cos\theta} = \frac{10}{2 + 3\cos\theta}$$

**46.** $r = \dfrac{4}{1 + \sin\theta}$ is a parabola with horizontal directrix above the pole.

(a) Parabola with vertical directrix to left pole.

(b) Parabola with horizontal directrix below pole.

(c) Parabola with vertical directrix to right of pole.

(d) Parabola (b) rotated counterclockwise $\pi/4$.

**48.** (a)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2b^2 + y^2a^2 = a^2b^2$$

$$b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$$

$$r^2[b^2\cos^2\theta + a^2(1 - \cos^2\theta)] = a^2b^2$$

$$r^2[a^2 + \cos^2\theta(b^2 - a^2)] = a^2b^2$$

$$r^2 = \frac{a^2b^2}{a^2 + (b^2 - a^2)\cos^2\theta} = \frac{a^2b^2}{a^2 - c^2\cos^2\theta}$$

$$= \frac{b^2}{1 - (c/a)^2\cos^2\theta} = \frac{b^2}{1 - e^2\cos^2\theta}$$

(b)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x^2b^2 - y^2a^2 = a^2b^2$$

$$b^2r^2\cos^2\theta - a^2r^2\sin^2\theta = a^2b^2$$

$$r^2[b^2\cos^2\theta - a^2(1 - \cos^2\theta) = a^2b^2$$

$$r^2[-a^2 + \cos^2\theta(a^2 + b^2)] = a^2b^2$$

$$r^2 = \frac{a^2b^2}{-a^2 + c^2\cos^2\theta} = \frac{b^2}{-1 + (c^2/a^2)\cos^2\theta}$$

$$= \frac{-b^2}{1 - e^2\cos^2\theta}$$

**50.** $a = 4, c = 5, b = 3, e = \dfrac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16)\cos^2\theta}$$

**52.** $a = 2, b = 1, c = \sqrt{3}, e = \dfrac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4)\cos^2\theta}$$

**54.** $A = 2\left[\dfrac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/2}\left(\dfrac{2}{3 - 2\sin\theta}\right)^2 d\theta\right] = 4\displaystyle\int_{-\pi/2}^{\pi/2}\dfrac{1}{(3 - 2\sin\theta)^2}\,d\theta \approx 3.37$

**56.** (a) $r = \dfrac{ed}{1 - e\cos\theta}$

When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$
$$a(1 + e)(1 - e) = ed$$
$$a(1 - e^2) = ed.$$

Thus, $r = \dfrac{(1 - e^2)a}{1 - e\cos\theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi,\ r = \dfrac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0,\ r = \dfrac{(1 - e^2)a}{1 - e} = a(1 + e)$.

**58.** $a = 1.427 \times 10^9$ km

$e = 0.0543$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} = \frac{1.422792505 \times 10^9}{1 - 0.0543\cos\theta}$$

Perihelion distance: $a(1 - e) = 1.3495139 \times 10^9$ km

Aphelion distance: $a(1 + e) = 1.5044861 \times 10^9$ km

**60.** $a = 36.0 \times 10^6$ mi, $e = 0.206$

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta} \approx \frac{34.472 \times 10^6}{1 - 0.206\cos\theta}$$

Perihelion distance: $a(1 - e) = 28.582 \times 10^6$ mi

Aphelion distance: $a(1 + e) = 43.416 \times 10^6$ mi

**62.**
$$r = a\sin\theta + b\cos\theta$$
$$r^2 = ar\sin\theta + br\cos\theta$$
$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

## Review Exercises for Chapter 9

**2.** Matches (b) - hyperbola

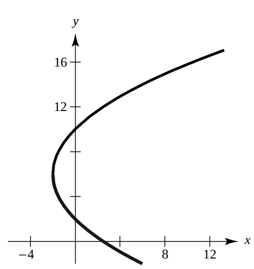
**4.** Matches (c) - hyperbola

**6.** $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$
$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$



**8.** $4x^2 + y^2 - 16x + 15 = 0$

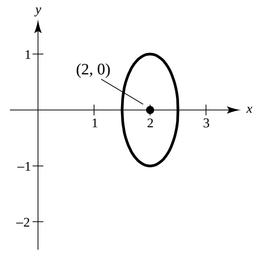
$4(x^2 - 4x + 4) + y^2 = -15 + 16$

$$\frac{(x-2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: $(2, 0)$

Vertices: $(2, \pm 1)$




**10.** $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$$

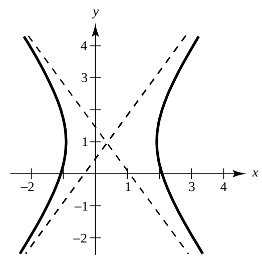
$$\frac{[x - (1/2)]^2}{2} - \frac{(y-1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$




**12.** Vertex: $(4, 2)$

Focus: $(4, 0)$

Parabola opens downward

$p = -2$

$(x-4)^2 = 4(-2)(y-2)$

$x^2 - 8x + 8y = 0$

**14.** Center: $(0, 0)$

Solution points: $(1, 2), (2, 0)$

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, \ 4/b^2 = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

**16.** Foci: $(0, \pm 8)$

Asymptotes: $y = \pm 4x$

Center: $(0, 0)$

Vertical transverse axis

$c = 8$

$y = \dfrac{a}{b}x = 4x$ asymptote $\rightarrow a = 4b$

$b^2 = c^2 - a^2 = 64 - (4b)^2 \implies 17b^2 = 64$

$\implies b^2 = \dfrac{64}{17} \implies a^2 = \dfrac{1024}{17}$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

**18.** $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \dfrac{\sqrt{21}}{5}$

By Example 5 of Section 9.1,

$$C = 20\int_0^{\pi/2} \sqrt{1 - \frac{21}{25}\sin^2\theta}\, d\theta \approx 23.01.$$

**20.** $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: $(0, 50)$

(b)
$$y = \frac{1}{200}x^2$$
$$y' = \frac{1}{100}x$$
$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10{,}000}}$$
$$S = 2\pi\int_0^{100} x\sqrt{1 + \frac{x^2}{10{,}000}}\,dx \approx 38{,}294.49$$

**22.** (a) $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\left(\frac{1}{2}\right)\left[x\sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi\int_0^b \frac{a^2}{b^2}(b^2 - y^2)\,dy = \frac{2\pi a^2}{b^2}\int_0^b (b^2 - y^2)\,dy = \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{1}{3}y^3\right]_0^b = \frac{4}{3}\pi a^2 b$

$$S = 4\pi\int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\left(\frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b\sqrt{b^2 - y^2}}\right)dy$$
$$= \frac{4\pi a}{b^2}\int_0^b \sqrt{b^4 + c^2y^2}\,dy = \frac{2\pi a}{b^2 c}\left[cy\sqrt{b^4 + c^2y^2} + b^4 \ln\left|cy + \sqrt{b^4 + c^2y^2}\right|\right]_0^b$$
$$= \frac{2\pi a}{b^2 c}\left[b^2 c\sqrt{b^2 + c^2} + b^4 \ln\left|cb + b\sqrt{b^2 + c^2}\right| - b^4 \ln(b^2)\right]$$
$$= 2\pi a^2 + \frac{\pi ab^2}{c}\ln\left(\frac{c + a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left(\frac{1 + e}{1 - e}\right)$$

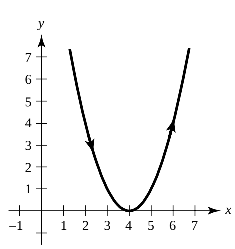
(c) **Disk:** $V = 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx = \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}\pi ab^2$

$$S = 2(2\pi)\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\left(\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}}\right)dx$$
$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - c^2x^2}\,dx = \frac{2\pi b}{a^2 c}\left[cx\sqrt{a^4 - c^2x^2} + a^4 \arcsin\left(\frac{cx}{a^2}\right)\right]_0^a$$
$$= \frac{a\pi b}{a^2 c}\left[a^2 c\sqrt{a^2 - c^2} + a^4 \arcsin\left(\frac{c}{a}\right)\right] = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e)$$

**24.** $x = t + 4, y = t^2$

$t = x - 4 \implies y = (x - 4)^2$

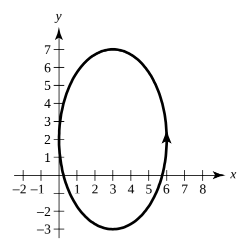
Parabola



**26.** $x = 3 + 3\cos\theta, y = 2 + 5\sin\theta$

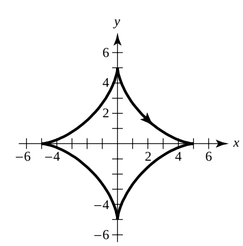
$$\left(\frac{x - 3}{3}\right)^2 + \left(\frac{y - 2}{5}\right)^2 = 1$$
$$\frac{(x - 3)^2}{9} + \frac{(y - 2)^2}{25} = 1$$

Ellipse



**28.** $x = 5\sin^3\theta, y = 5\cos^3\theta$

$$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$$
$$x^{2/3} + y^{2/3} = 5^{2/3}$$



**30.** $(x - h)^2 + (y - k)^2 = r^2$

$(x - 5)^2 + (y - 3)^2 = 2^2 = 4$

**32.** $a = 4,\ c = 5,\ b^2 = c^2 - a^2 = 9,\ \dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

Let $\dfrac{y^2}{16} = \sec^2\theta$ and $\dfrac{x^2}{9} = \tan^2\theta$.

Then $x = 3\tan\theta$ and $y = 4\sec\theta$.

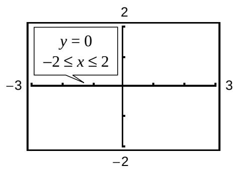
**34.** $x = (a - b)\cos t + b\cos\left(\dfrac{a - b}{b}t\right)$

$y = (a - b)\sin t - b\sin\left(\dfrac{a - b}{b}t\right)$

(a) $a = 2, b = 1$

$x = \cos t + \cos t = 2\cos t$

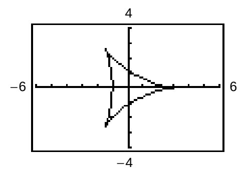
$y = \sin t - \sin t = 0$




(b) $a = 3, b = 1$

$x = 2\cos t + \cos 2t$

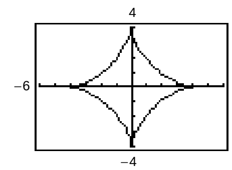
$y = 2\sin t - \sin 2t$




(c) $a = 4, b = 1$

$x = 3\cos t + \cos 3t$

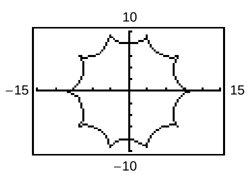
$y = 3\sin t - \sin 3t$




(d) $a = 10, b = 1$

$x = 9\cos t + \cos 9t$

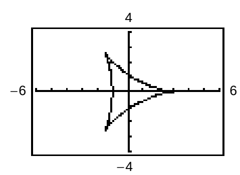
$y = 9\sin t - \sin 9t$




(e) $a = 3, b = 2$

$x = \cos t + 2\cos\dfrac{t}{2}$

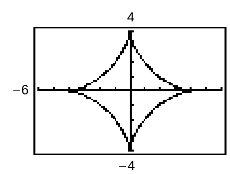
$y = \sin t - 2\sin\dfrac{t}{2}$




(f) $a = 4, b = 3$

$x = \cos t + 3\cos\dfrac{t}{3}$

$y = \sin t - 3\sin\dfrac{t}{3}$

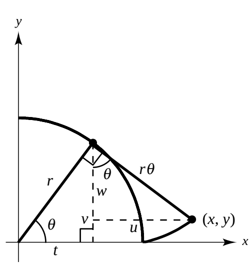



**36.** $x = t + u = r\cos\theta + r\theta\sin\theta$

$= r(\cos\theta + \theta\sin\theta)$

$y = v - w = r\sin\theta - r\theta\cos\theta$

$= r(\sin\theta - \theta\cos\theta)$




**38.** $x = t + 4$

$y = t^2$

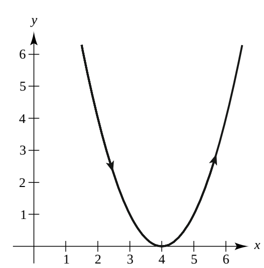
(a) $\dfrac{dy}{dx} = \dfrac{2t}{1} = 2t = 0$ when $t = 0$.

Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$y = (x - 4)^2$

(c)



**40.** $x = \dfrac{1}{t}$

$y = t^2$

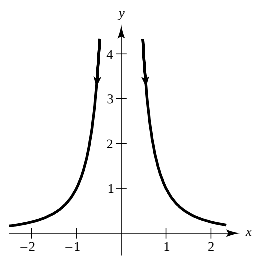
(a) $\dfrac{dy}{dx} = \dfrac{2t}{-1/t^2} = -2t^3$

No horizontal tangents $\quad (t \neq 0)$

(b) $t = \dfrac{1}{x}$

$y = \dfrac{1}{x^2}$

(c)




**42.** $x = 2t - 1$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

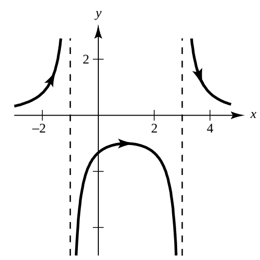
$= \dfrac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \dfrac{x + 1}{2}$

$y = \dfrac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \dfrac{4}{(x - 3)(x + 1)}$

(c)




**44.** $x = 6 \cos \theta$

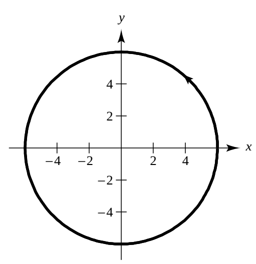
$y = 6 \sin \theta$

(a) $\dfrac{dy}{dx} = \dfrac{6 \cos \theta}{-6 \sin \theta} = -\cot \theta = 0$ when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\dfrac{x}{6}\right)^2 + \left(\dfrac{y}{6}\right)^2 = 1$

(c)




**46.** $x = e^t$

$y = e^{-t}$

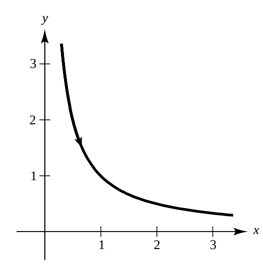
(a) $\dfrac{dy}{dx} = \dfrac{-e^{-t}}{e^t} = -\dfrac{1}{e^{2t}} = -\dfrac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \dfrac{1}{x}, x > 0$

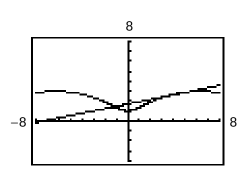
(c)




**48.** $x = 2\theta - \sin \theta$

$y = 2 - \cos \theta$

(a), (c)




(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} \approx 1.134, \left(2 - \dfrac{\sqrt{3}}{2}\right),$

$\dfrac{dy}{dt} = 0.5,$ and $\dfrac{dy}{dx} \approx 0.441$

**50.** $x = 6 \cos \theta$

$y = 6 \sin \theta$

$\dfrac{dx}{d\theta} = -6 \sin \theta$

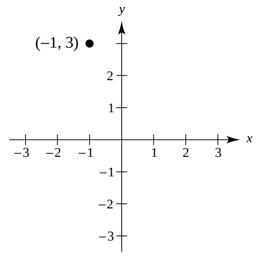
$\dfrac{dy}{d\theta} = 6 \cos \theta$

$$s = \int_0^{\pi} \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta}\, d\theta = \Big[6\theta\Big]_0^{\pi} = 6\pi$$

(one-half circumference of circle)

**52.** $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$
$$\theta = \arctan(-3) \approx 1.89 \; (108.43°)$$
$$(r, \theta) = \left(\sqrt{10}, 1.89\right), \left(-\sqrt{10}, 5.03\right)$$




**54.**
$$r = 10$$
$$r^2 = 100$$
$$x^2 + y^2 = 100$$

**56.**
$$r = \frac{1}{2 - \cos\theta}$$
$$2r - r\cos\theta = 1$$
$$2\left(\pm\sqrt{x^2 + y^2}\right) - x = 1$$
$$4(x^2 + y^2) = (x + 1)^2$$
$$3x^2 + 4y^2 - 2x - 1 = 0$$

**58.**
$$r = 4\sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$
$$= \frac{4}{(1/2)\cos\theta + \left(\sqrt{3}/2\right)\sin\theta}$$
$$r\left(\cos\theta + \sqrt{3}\sin\theta\right) = 8$$
$$x + \sqrt{3}\,y = 8$$

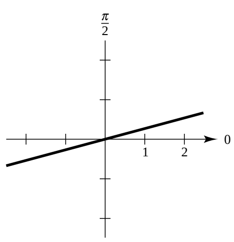
**60.**
$$\theta = \frac{3\pi}{4}$$
$$\tan\theta = -1$$
$$\frac{y}{x} = -1$$
$$y = -x$$

**62.**
$$x^2 + y^2 - 4x = 0$$
$$r^2 - 4r\cos\theta = 0$$
$$r = 4\cos\theta$$

**64.**
$$(x^2 + y^2)\left(\arctan\frac{y}{x}\right)^2 = a^2$$
$$r^2\theta^2 = a^2$$

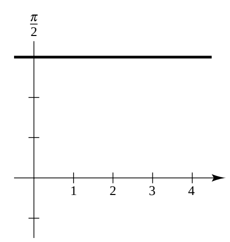
**66.** $\theta = \dfrac{\pi}{12}$

Line



**68.** $r = 3\csc\theta$, $r\sin\theta = 3$, $y = 3$

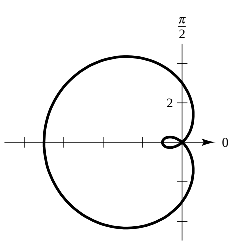
Horizontal line



**70.** $r = 3 - 4\cos\theta$

Limaçon

Symmetric to polar axis

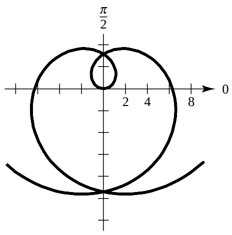


| $\theta$ | $0$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $-1$ | $1$ | $3$ | $5$ | $7$ |

**72.** $r = 2\theta$

Spiral

Symmetric to $\theta = \pi/2$




| $\theta$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $r$ | $0$ | $\frac{\pi}{5}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ | $\frac{5\pi}{2}$ | $3\pi$ |

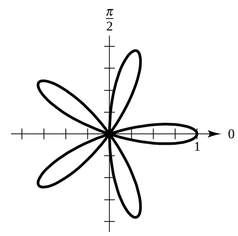
**74.** $r = \cos(5\theta)$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), \left(-1, \frac{\pi}{5}\right), \left(1, \frac{2\pi}{5}\right), \left(-1, \frac{3\pi}{5}\right), \left(1, \frac{4\pi}{5}\right)$

Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$




**76.** $r^2 = \cos(2\theta)$

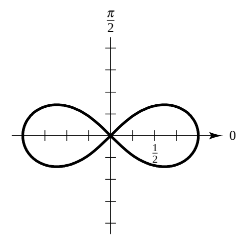
Lemniscate

Symmetric to the polar axis

Relative extrema: $(\pm 1, 0)$

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

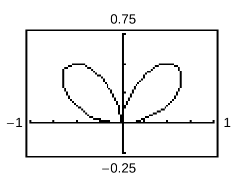
| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ |
|---|---|---|---|
| $r$ | $\pm 1$ | $\pm\frac{\sqrt{2}}{2}$ | $0$ |




**78.** $r = 2 \sin\theta \cos^2\theta$

Bifolium

Symmetric to $\theta = \pi/2$



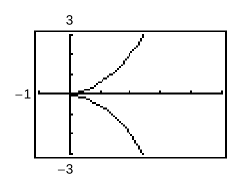

**80.** $r = 4(\sec\theta - \cos\theta)$

Semicubical parabola

Symmetric to the polar axis

$r \implies \infty$ as $\theta \implies \frac{\pi^-}{2}$

$r \implies \infty$ as $\theta \implies \frac{-\pi^+}{2}$



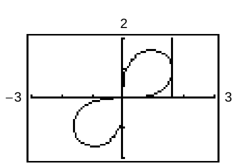

**82.** $r^2 = 4\sin(2\theta)$

(a) $2r\left(\frac{dr}{d\theta}\right) = 8\cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4\cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$

(c)




(b) $$\frac{dy}{dx} = \frac{r\cos\theta + [(4\cos 2\theta \sin\theta)/r]}{-r\sin\theta + [(4\cos 2\theta\cos\theta)/r]}$$

$$= \frac{\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta}$$

Horizontal tangents:

$\frac{dy}{dx} = 0$ when $\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta = 0$,

$\tan\theta = -\tan(2\theta)$, $\theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{3}\right)$

Vertical tangents when $\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$:

$\tan 2\theta \tan\theta = 1$, $\theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{6}\right)$

**84.** False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

**86.** $r = a \sin\theta, r = a\cos\theta$

The points of intersection are $\left(a/\sqrt{2}, \pi/4\right)$ and $(0, 0)$. For $r = a\sin\theta$,

$$m_1 = \frac{dy}{dx} = \frac{a\cos\theta\sin\theta + a\sin\theta\cos\theta}{a\cos^2\theta - a\sin^2\theta} = \frac{2\sin\theta\cos\theta}{\cos 2\theta}.$$

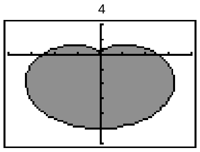
At $\left(a/\sqrt{2}, \pi/4\right)$, $m_1$ is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a\cos\theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a\sin^2\theta + a\cos^2\theta}{-a\sin\theta\cos\theta - a\cos\theta\sin\theta} = \frac{\cos 2\theta}{-2\sin\theta\cos\theta}.$$

At $\left(a/\sqrt{2}, \pi/4\right)$, $m_2 = 0$ and at $(0, 0)$, $m_2$ is undefined. Therefore, the graphs are orthogonal at $\left(a/\sqrt{2}, \pi/4\right)$ and $(0, 0)$.

**88.** $r = 5(1 - \sin\theta)$

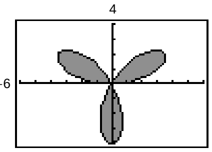
$$A = 2\left[\frac{1}{2}\int_{\pi/2}^{3\pi/2}[5(1 - \sin\theta)]^2\right] d\theta \approx 117.81 \quad \left(75\frac{\pi}{2}\right)$$




**90.** $r = 4\sin 3\theta$

$$A = 3\left[\frac{1}{2}\int_0^{\pi/3}(4\sin 3\theta)^2\, d\theta\right.$$

$$\approx 12.57 \quad (4\pi)$$

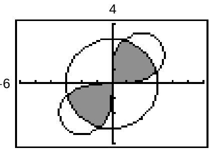



**92.** $r = 3, r^2 = 18\sin 2\theta$

$$9 = r^2 = 18\sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{12}$$

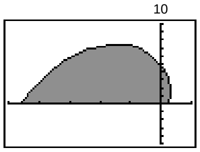



$$A = 2\left[\frac{1}{2}\int_0^{\pi/12} 18\sin 2\theta\, d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} 9\, d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} 18\sin 2\theta\, d\theta\right]$$

$$\approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$

**94.** $r = e^\theta, 0 \le \theta \le \pi$

$$A = \frac{1}{2}\int_0^{\pi}(e^\theta)^2\, d\theta \approx 133.62$$




**96.** $r = a\cos 2\theta, \dfrac{dr}{d\theta} = -2a\sin 2\theta$

$$s = 8\int_0^{\pi/4}\sqrt{a^2\cos^2 2\theta + 4a^2\sin^2 2\theta}\, d\theta$$

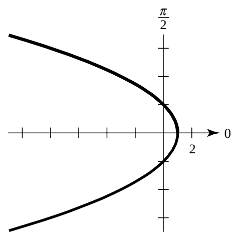
$$= 8a\int_0^{\pi/4}\sqrt{1 + 3\sin^2 2\theta}\, d\theta \quad \text{(Simpson's Rule: } n = 4\text{)}$$

$$\approx \frac{\pi a}{6}[1 + 4(1.1997) + 2(1.5811) + 4(1.8870) + 2]$$

$$\approx 9.69a$$

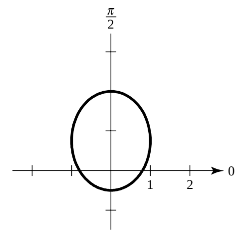
**98.** $r = \dfrac{2}{1 + \cos\theta}, e = 1$

Parabola



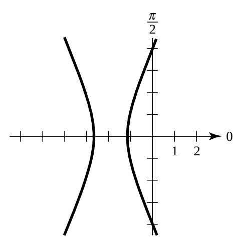

**100.** $r = \dfrac{4}{5 - 3\sin\theta} = \dfrac{4/5}{1 - (3/5)\sin\theta}, e = \dfrac{3}{5}$

Ellipse




**102.** $r = \dfrac{8}{2 - 5\cos\theta} = \dfrac{4}{1 - (5/2)\cos\theta}, e = \dfrac{5}{2}$

Hyperbola




**104.** Line

Slope: $\sqrt{3}$

Solution point: $(0, 0)$

$y = \sqrt{3}\,x, r\sin\theta = \sqrt{3}\,r\cos\theta,$

$\tan\theta = \sqrt{3}, \theta = \dfrac{\pi}{3}$

**106.** Parabola

Vertex: $\left(2, \dfrac{\pi}{2}\right)$

Focus: $(0, 0)$

$e = 1, d = 4$

$r = \dfrac{4}{1 + \sin\theta}$

**108.** Hyperbola

Vertices: $(1, 0), (7, 0)$

Focus: $(0, 0)$

$a = 3, c = 4, e = \dfrac{4}{3}, d = \dfrac{7}{4}$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right)\cos\theta} = \frac{7}{3 + 4\cos\theta}$$

## Problem Solving for Chapter 9

**2.** Assume $p > 0$.

Let $y = mx + p$ be the equation of the focal chord.

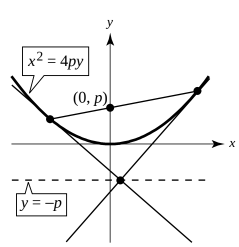
First find $x$-coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py, 2x = 4py' \implies y' = \frac{x}{2p}.$$




(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p}\left[2pm + 2p\sqrt{m^2 + 1}\right]\frac{1}{2p}\left[2pm - 2p\sqrt{m^2 + 1}\right] = \frac{1}{4p^2}[4p^2m^2 - 4p^2(m^2 + 1)] = \frac{1}{4p^2}[-4p^2] = -1$$

**—CONTINUED—**

**2. —CONTINUED—**

(b) Finally, we show that the tangent lines intersect at a point on the directrix $y = -p$.

Let $b = 2pm + 2p\sqrt{m^2+1}$ and $c = 2pm - 2p\sqrt{m^2+1}$.

$$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2+1}$$

$$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2+1}$$

$$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2+1}$$

$$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2+1}$$

Tangent line at $x = b$: $y - \dfrac{b^2}{4p} = \dfrac{b}{2p}(x-b) \implies y = \dfrac{bx}{2p} - \dfrac{b^2}{4p}$

Tangent line at $x = c$: $y - \dfrac{c^2}{4p} = \dfrac{c}{2p}(x-c) \implies y = \dfrac{cx}{2p} - \dfrac{c^2}{4p}$

Intersection of tangent lines:

$$\frac{bx}{2p} - \frac{b^2}{4p} = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$2bx - b^2 = 2cx - c^2$$

$$2x(b-c) = b^2 - c^2$$

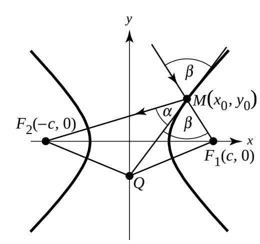
$$2x\left(4p\sqrt{m^2+1}\right) = 16p^2m\sqrt{m^2+1}$$

$$x = 2pm$$

Finally, the corresponding $y$-value is $y - p$, which shows that the intersection point lies on the directrix.

**4.** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,\ a^2 + b^2 = c^2,\ MF_2 - Mf_1 = 2a$

$$y' = \frac{b^2x}{a^2y}$$




Tangent line at $M(x_0, y_0)$:

$$y = y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{x_0x - x_0^2}{a^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

At $x = 0, y = -\dfrac{b^2}{y_0} \implies Q = \left(0, -\dfrac{b^2}{y_0}\right)$.

$$QF_2 = QF_1 = \sqrt{c^2 + \frac{b^4}{y_0^2}} = d$$

$$MQ = \sqrt{x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2} = f$$

By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ)\cos\alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2)\cos\alpha$$

$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta.$$

$$\cos\alpha = \frac{(MF_2)^2f^2 - d^2}{2f(MF_2)},\ \cos\beta = \frac{(MF_1)^2 + f^2 - d^2}{2f(MF_1)}$$

$MF_2 = MF_1 + 2a$. Let $z = MF_1$.

Slopes: $MF_1$: $\dfrac{y_0}{x_0 - c}$; $QF_1$: $\dfrac{-b^2}{y_0c}$; $QF_2$: $\dfrac{b^2}{y_0c}$

**—CONTINUED—**

**4. —CONTINUED—**

To show $\alpha = \beta$, consider

$$
\begin{aligned}
& [(MF_2)^2 + f^2 - d^2][2f(MF_1)] = [(MF_1)^2 + f^2 - d^2][2f(MF_2)] \\
\Leftrightarrow \quad & [(z + 2a)^2 + f^2 - d^2][z] = [z^2 + f^2 - d^2][z + 2a] \\
\Leftrightarrow \quad & z^2 + 2az = f^2 - d^2 \\
\Leftrightarrow \quad & (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right) \\
\Leftrightarrow \quad & az - x_0c + a^2 = 0 \\
\Leftrightarrow \quad & a\sqrt{(x_0 - c)^2 + y_0^2} = x_0c - a^2 \\
\Leftrightarrow \quad & x_0^2b^2 - a^2y_0^2 = a^2b^2 \\
\Leftrightarrow \quad & \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.
\end{aligned}
$$

Thus, $\alpha = \beta$ and the reflective property is verified.

**6.** (a) $y^2 = \dfrac{t^2(1 - t^2)^2}{(1 + t^2)^2}, x^2 = \dfrac{(1 - t^2)^2}{(1 + t^2)^2}$

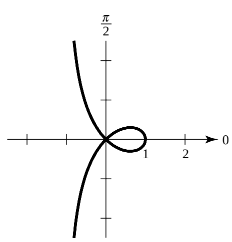
$$\frac{1 - x}{1 + x} = \frac{1 - \left(\dfrac{1 - t^2}{1 + t^2}\right)}{1 + \left(\dfrac{1 - t^2}{1 + t^2}\right)} = \frac{2t^2}{2} = t^2$$

Thus, $y^2 = x^2\left(\dfrac{1 - x}{1 + x}\right)$.

(b)

$$
\begin{aligned}
r^2 \sin^2 \theta &= r^2 \cos^2 \theta\left(\frac{1 - r\cos\theta}{1 + r\cos\theta}\right) \\
\sin^2 \theta(1 + r\cos\theta) &= \cos^2\theta(1 - r\cos\theta) \\
r\cos\theta\sin^2\theta + \sin^2\theta &= \cos^2\theta - r\cos^3\theta \\
r\cos\theta(\sin^2\theta + \cos^2\theta) &= \cos^2\theta - \sin^2\theta \\
r\cos\theta &= \cos 2\theta \\
r &= \cos 2\theta \cdot \sec\theta
\end{aligned}
$$

(c)




(d) $r(\theta) = 0$ for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

Thus, $y = x$ and $y = -x$ are tangent lines to curve at the origin.

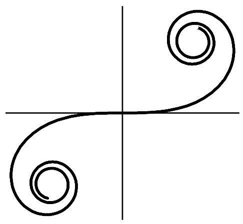
(e) $y'(t) = \dfrac{(1 + t^2)(1 - 3t^2) - (t - t^3)(2t)}{(1 + t^2)^2} = \dfrac{1 - 4t^2 - t^4}{(1 + t^2)^2} = 0$

$$t^4 + 4t^2 - 1 = 0 \implies t^2 = -2 \pm \sqrt{5} \implies x = \frac{1 - \left(-2 \pm \sqrt{5}\right)}{1 + \left(-2 \pm \sqrt{5}\right)} = \frac{3 \mp \sqrt{5}}{-1 \pm \sqrt{5}}$$

$$= \frac{3 - \sqrt{5}}{-1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{2}$$

$$\left(\frac{\sqrt{5} - 1}{2}, \pm\frac{\sqrt{5} - 1}{2}\sqrt{-2 + \sqrt{5}}\right)$$

**8.** (a)



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(b) $(-x, -y) = \left(-\int_0^t \cos \frac{\pi u^2}{2}\, du, -\int_0^t \sin \frac{\pi u^2}{2}\, du\right)$ is on the curve whenever $(x, y)$ is on the curve.

(c) $x'(t) = \cos \frac{\pi t^2}{2}, y'(t) = \sin \frac{\pi t^2}{2}, x'(t)^2 + y'(t)^2 = 1$

Thus, $s = \int_0^a dt = a$.

On $[-\pi, \pi]$, $s = 2\pi$.

**10.** $r = \dfrac{ab}{a \sin \theta + b \cos \theta}, 0 \le \theta \le \dfrac{\pi}{2}$

$$r(a \sin \theta + b \cos \theta) = ab$$

$$ay + bx = ab$$

$$\frac{y}{b} + \frac{x}{a} = 1$$

Line segment

Area $= \dfrac{1}{2}ab$

**12.** Let $(r, \theta)$ be on the graph.

$$\sqrt{r^2 + 1 + 2r \cos \theta}\sqrt{r^2 + 1 - 2r \cos \theta} = 1$$

$$(r^2 + 1)^2 - 4r^2 \cos^2 \theta = 1$$

$$r^4 + 2r^2 + 1 - 4r^2 \cos^2 \theta = 1$$

$$r^2(r^2 - 4 \cos^2 \theta + 2) = 0$$

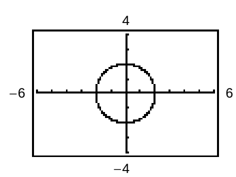
$$r^2 = 4 \cos^2 \theta - 2$$

$$r^2 = 2(2 \cos^2 \theta - 1)$$

$$r^2 = 2 \cos 2\theta$$

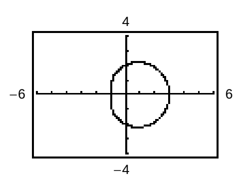
**14.** (a) $r = 2$

Circle radius 2



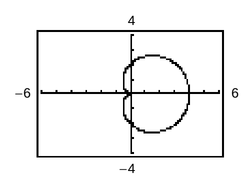

(b) $r = 2 + \cos \theta$

Convex limaçon



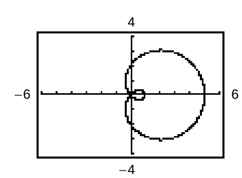

(c) $r = 2 + 2 \cos \theta$

Cardioid




(d) $r = 2 + 3 \cos \theta$

Limaçon with inner loop




**16.** The curve is produced over the interval

$0 \le \theta \le 9\pi$.

# CHAPTER 10
# Vectors and the Geometry of Space

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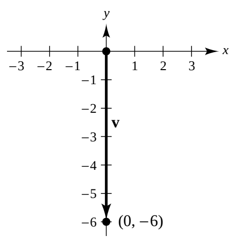
# Vectors and the Geometry of Space

## Section 10.1 Vectors in the Plane

Solutions to Even-Numbered Exercises

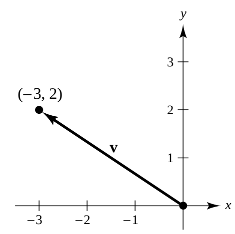
**2.** (a) $\mathbf{v} = \langle 3 - 3, -2 - 4 \rangle = \langle 0, -6 \rangle$

(b)




**4.** (a) $\mathbf{v} = \langle -1 - 2, 3 - 1 \rangle = \langle -3, 2 \rangle$

(b)




**6.** $\mathbf{u} = \langle 1 - (-4), 8 - 0 \rangle = \langle 5, 8 \rangle$

$\mathbf{v} = \langle 7 - 2, 7 - (-1) \rangle = \langle 5, 8 \rangle$

$\mathbf{u} = \mathbf{v}$

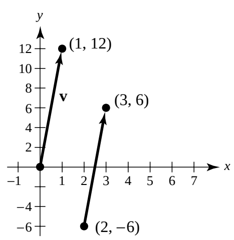
**8.** $\mathbf{u} = \langle 11 - (-4), -4 - (-1) \rangle = \langle 15, -3 \rangle$

$\mathbf{v} = \langle 25 - 0, 10 - 13 \rangle = \langle 15, -3 \rangle$

$\mathbf{u} = \mathbf{v}$

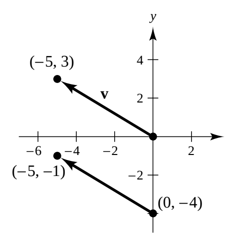
**10.** (b) $\mathbf{v} = \langle 3 - 2, 6 - (-6) \rangle = \langle 1, 12 \rangle$

(a) and (c).



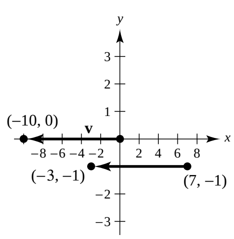

**12.** (b) $\mathbf{v} = \langle -5 - 0, -1 - (-4) \rangle = \langle -5, 3 \rangle$

(a) and (c).



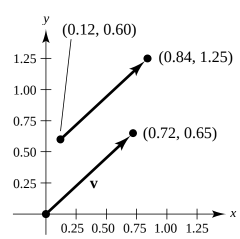

**14.** (b) $\mathbf{v} = \langle -3 - 7, -1 - (-1) \rangle = \langle -10, 0 \rangle$

(a) and (c).

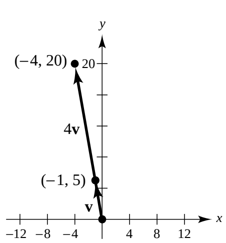



**16.** (b) $\mathbf{v} = \langle 0.84 - 0.12, 1.25 - 0.60 \rangle = \langle 0.72, 0.65 \rangle$

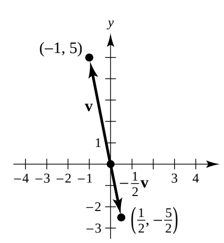
(a) and (c).




**18.** (a) $4\mathbf{v} = \langle -4, 20 \rangle$




(b) $-\frac{1}{2}\mathbf{v} = \left\langle \frac{1}{2}, -\frac{5}{2} \right\rangle$

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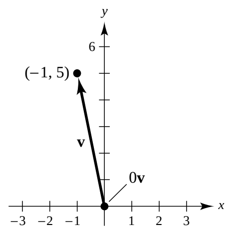
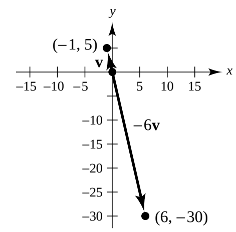
**18. —CONTINUED—**

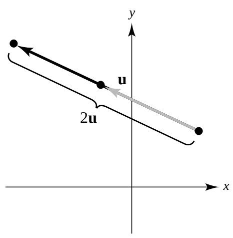
(c) $0\mathbf{v} = \langle 0, 0 \rangle$



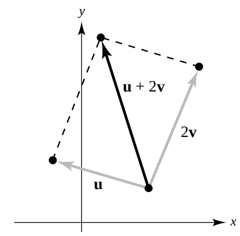

(d) $-6\mathbf{v} = \langle 6, -30 \rangle$




**20.** Twice as long as given vector **u**.




**22.**



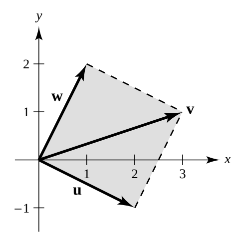

**24.** (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle -3, -8 \rangle = \left\langle -2, -\frac{16}{3} \right\rangle$

(b) $\mathbf{v} - \mathbf{u} = \langle 8, 25 \rangle - \langle -3, -8 \rangle = \langle 11, 33 \rangle$

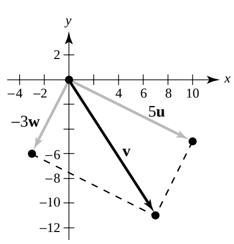
(c) $2\mathbf{u} + 5\mathbf{v} = 2\langle -3, -8 \rangle + 5\langle 8, 25 \rangle = \langle 34, 109 \rangle$

**26.** $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + (\mathbf{i} + 2\mathbf{j})$

$= 3\mathbf{i} + \mathbf{j} = \langle 3, 1 \rangle$




**28.** $\mathbf{v} = 5\mathbf{u} - 3\mathbf{w} = 5\langle 2, -1 \rangle - 3\langle 1, 2 \rangle = \langle 7, -11 \rangle$




**30.** $u_1 - 3 = 4$

$u_2 - 2 = -9$

$u_1 = 7$

$u_2 = -7$

$Q = (7, -7)$

**32.** $\|\mathbf{v}\| = \sqrt{144 + 25} = 13$

**34.** $\|\mathbf{v}\| = \sqrt{100 + 9} = \sqrt{109}$

**36.** $\|\mathbf{v}\| = \sqrt{1 + 1} = \sqrt{2}$

**38.** $\|\mathbf{u}\| = \sqrt{5^2 + 15^2} = \sqrt{250} = 5\sqrt{10}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 5, 15 \rangle}{5\sqrt{10}} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle \text{ unit vector}$$

**40.** $\|\mathbf{u}\| = \sqrt{(-6.2)^2 + (3.4)^2} = \sqrt{50} = 5\sqrt{2}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle -6.2, 3.4 \rangle}{5\sqrt{2}} = \left\langle \frac{-1.24}{\sqrt{2}}, \frac{0.68}{\sqrt{2}} \right\rangle \text{ unit vector}$$

**42.** $\mathbf{u} = \langle 0, 1 \rangle, \mathbf{v} = \langle 3, -3 \rangle$

(a) $\|\mathbf{u}\| = \sqrt{0 + 1} = 1$

(b) $\|\mathbf{v}\| = \sqrt{9 + 9} = 3\sqrt{2}$

(c) $\mathbf{u} + \mathbf{v} = \langle 3, -2 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + 4} = \sqrt{13}$

(d) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 0, 1 \rangle$

$\left\| \dfrac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{3\sqrt{2}} \langle 3, -3 \rangle$

$\left\| \dfrac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f) $\dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \dfrac{1}{\sqrt{13}} \langle 3, -2 \rangle$

$\left\| \dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**44.** $\mathbf{u} = \langle 2, -4 \rangle, \mathbf{v} = \langle 5, 5 \rangle$

(a) $\|\mathbf{u}\| = \sqrt{4 + 16} = 2\sqrt{5}$

(b) $\|\mathbf{v}\| = \sqrt{25 + 25} = 5\sqrt{2}$

(c) $\mathbf{u} + \mathbf{v} = \langle 7, 1 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{49 + 1} = 5\sqrt{2}$

(d) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{2\sqrt{5}} \langle 2, -4 \rangle$

$\left\| \dfrac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e) $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{5\sqrt{2}} \langle 5, 5 \rangle$

$\left\| \dfrac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f) $\dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \dfrac{1}{5\sqrt{2}} \langle 7, 1 \rangle$

$\left\| \dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

**46.**
$$\begin{aligned}
\mathbf{u} &= \langle -3, 2 \rangle \\
\|\mathbf{u}\| &= \sqrt{13} \approx 3.606 \\
\mathbf{v} &= \langle 1, -2 \rangle \\
\|\mathbf{v}\| &= \sqrt{5} \approx 2.236 \\
\mathbf{u} + \mathbf{v} &= \langle -2, 0 \rangle \\
\|\mathbf{u} + \mathbf{v}\| &= 2 \\
\|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\|
\end{aligned}$$

**48.**
$$\begin{aligned}
\frac{\mathbf{u}}{\|\mathbf{u}\|} &= \frac{1}{\sqrt{2}} \langle -1, 1 \rangle \\
4\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) &= 2\sqrt{2} \langle -1, 1 \rangle \\
\mathbf{v} &= \left\langle -2\sqrt{2}, 2\sqrt{2} \right\rangle
\end{aligned}$$

**50.**
$$\begin{aligned}
\frac{\mathbf{u}}{\|\mathbf{u}\|} &= \frac{1}{3} \langle 0, 3 \rangle \\
3\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) &= \langle 0, 3 \rangle \\
\mathbf{v} &= \langle 0, 3 \rangle
\end{aligned}$$

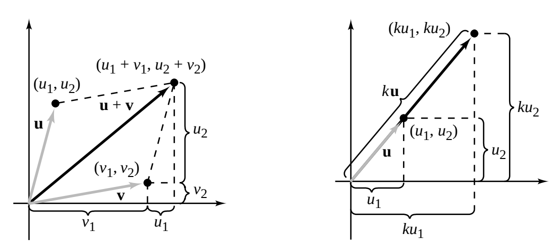
**52.**
$$\begin{aligned}
\text{v} &= 5\left[(\cos 120°)\mathbf{i} + (\sin 120°)\mathbf{j}\right] \\
&= -\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}
\end{aligned}$$

**54.**
$$\begin{aligned}
\mathbf{v} &= (\cos 3.5°)\mathbf{i} + (\sin 3.5°)\mathbf{j} \\
&\approx 0.9981\mathbf{i} + 0.0610\mathbf{j} = \langle 0.9981, 0.0610 \rangle
\end{aligned}$$

**56.**
$$\begin{aligned}
\mathbf{u} &= 4\mathbf{i} \\
\mathbf{v} &= \mathbf{i} + \sqrt{3}\mathbf{j} \\
\mathbf{u} + \mathbf{v} &= 5\mathbf{i} + \sqrt{3}\mathbf{j}
\end{aligned}$$

**58.**
$$\begin{aligned}
\mathbf{u} &= 5[\cos(-0.5)]\mathbf{i} + 5[\sin(-0.5)]\mathbf{j} \\
&= 5[\cos(0.5)]\mathbf{i} - 5[\sin(0.5)]\mathbf{j} \\
\mathbf{v} &= 5[\cos(0.5)]\mathbf{i} + 5[\sin(0.5)]\mathbf{j} \\
\mathbf{u} + \mathbf{v} &= 10[\cos(0.5)]\mathbf{i}
\end{aligned}$$

**60.** See page 718:



**62.** See Theorem 10.1, page 719.

**For Exercises 64–68, $a\mathbf{u} + b\mathbf{w} = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}$.**

**64.** $\mathbf{v} = 3\mathbf{j}$. Therefore, $a + b = 0$, $2a - b = 3$. Solving simultaneously, we have $a = 1$, $b = -1$.

**66.** $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j}$. Therefore, $a + b = 3$, $2a - b = 3$. Solving simultaneously, we have $a = 2$, $b = 1$.

**68.** $\mathbf{v} = -\mathbf{i} + 7\mathbf{j}$. Therefore, $a + b = -1$, $2a - b = 7$. Solving simultaneously, we have $a = 2$, $b = -3$.

**70.** $y = x^3$, $y' = 3x^2 = 12$ at $x = -2$.

(a) $m = 12$. Let $\mathbf{w} = \langle 1, 12\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{\sqrt{145}}\langle 1, 12\rangle.$$

(b) $m = -\frac{1}{12}$. Let $\mathbf{w} = \langle 12, -1\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{\sqrt{145}}\langle 12, -1\rangle.$$

**72.** $f(x) = \tan x$

$f'(x) = \sec^2 x = 2$ at $x = \frac{\pi}{4}$.

(a) $m = 2$. Let $\mathbf{w} = \langle 1, 2\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{\sqrt{5}}\langle 1, 2\rangle.$$

(b) $m = -\frac{1}{2}$. Let $\mathbf{w} = \langle -2, 1\rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm\frac{1}{\sqrt{5}}\langle -2, 1\rangle.$$

**74.**

$$\begin{aligned}\mathbf{u} &= 2\sqrt{3}\mathbf{i} + 2\mathbf{j}\\ \mathbf{u} + \mathbf{v} &= -3\mathbf{i} + 3\sqrt{3}\mathbf{j}\\ \mathbf{v} &= (\mathbf{u} + \mathbf{v}) - \mathbf{u} = \left(-3 - 2\sqrt{3}\right)\mathbf{i} + \left(3\sqrt{3} - 2\right)\mathbf{j}\end{aligned}$$

**76.** magnitude $\approx 63.5$

direction $\approx -8.26°$

**78.**

$$\begin{aligned}\|\mathbf{F}_1\| &= 2,\ \theta_{\mathbf{F}_1} = -10°\\ \|\mathbf{F}_2\| &= 4,\ \theta_{\mathbf{F}_2} = 140°\\ \|\mathbf{F}_3\| &= 3,\ \theta_{\mathbf{F}_3} = 200°\\ \|\mathbf{R}\| &= \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 4.09\\ \theta_{\mathbf{R}} &= \theta_{\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3} \approx 163.0°\end{aligned}$$

**80.**

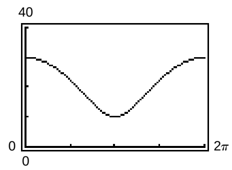
$$\begin{aligned}\mathbf{F}_1 + \mathbf{F}_2 &= (500\cos 30°\mathbf{i} + 500\sin 30°\mathbf{j}) + (200\cos(-45°)\mathbf{i} + 200\sin(-45°)\mathbf{j})\\ &= \left(250\sqrt{3} + 100\sqrt{2}\right)\mathbf{i} + \left(250 - 100\sqrt{2}\right)\mathbf{j}\\ \|\mathbf{F}_1 + \mathbf{F}_2\| &= \sqrt{\left(250\sqrt{3} + 100\sqrt{2}\right)^2 + \left(250 - 100\sqrt{2}\right)^2} \approx 584.6 \text{ lb}\\ \tan\theta &= \frac{250 - 100\sqrt{2}}{250\sqrt{3} + 100\sqrt{2}} \implies \theta \approx 10.7°\end{aligned}$$

**82.**

$$\begin{aligned}\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 &= [400(\cos(-30°)\mathbf{i} + \sin(-30°)\mathbf{j})] + [280(\cos(45°)\mathbf{i} + \sin(45°)\mathbf{j})] + [350(\cos(135°)\mathbf{i} + \sin(135°)\mathbf{j}]\\ &= \left[200\sqrt{3} + 140\sqrt{2} - 175\sqrt{2}\right]\mathbf{i} + \left[-200 + 140\sqrt{2} + 175\sqrt{2}\right]\mathbf{j}\\ \|\mathbf{R}\| &= \sqrt{\left(200\sqrt{3} - 35\sqrt{2}\right)^2 + \left(-200 + 315\sqrt{2}\right)^2} \approx 385.2483 \text{ newtons}\\ \theta_{\mathbf{R}} &= \arctan\left(\frac{-200 + 315\sqrt{2}}{200\sqrt{3} - 35\sqrt{2}}\right) \approx 0.6908 \approx 39.6°\end{aligned}$$

**84.** $\mathbf{F}_1 = \langle 20, 0\rangle, \mathbf{F}_2 = 10\langle \cos\theta, \sin\theta\rangle$

(a) $$\begin{aligned}\|\mathbf{F}_1 + \mathbf{F}_2\| &= \|\langle 20 + 10\cos\theta, 10\sin\theta\rangle\| \\ &= \sqrt{400 + 400\cos\theta + 100\cos^2\theta + 100\sin^2\theta} \\ &= \sqrt{500 + 400\cos\theta}\end{aligned}$$

(b) 


(c) The range is $10 \le \|\mathbf{F}_1 + \mathbf{F}_2\| \le 30$.

The maximum is 30, which occur at $\theta = 0$ and $\theta = 2\pi$.

The minimum is 10 at $\theta = \pi$.

(d) The minimum of the resultant is 10.

**86.** $\mathbf{u} = \langle 7 - 1, 5 - 2\rangle = \langle 6, 3\rangle$

$\frac{1}{3}\mathbf{u} = \langle 2, 1\rangle$

$P_1 = (1, 2) + (2, 1) = (3, 3)$

$P_2 = (1, 2) + 2(2, 1) = (5, 4)$

**88.** $\theta_1 = \arctan\left(\frac{24}{20}\right) \approx 0.8761$ or $50.2°$

$\theta_2 = \arctan\left(\frac{24}{-10}\right) + \pi \approx 1.9656$ or $112.6°$

$\mathbf{u} = \|\mathbf{u}\|(\cos\theta_1\,\mathbf{i} + \sin\theta_1\,\mathbf{j})$

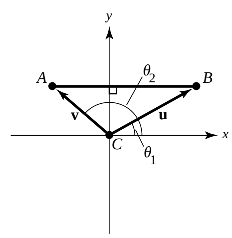
$\mathbf{v} = \|\mathbf{v}\|(\cos\theta_2\,\mathbf{i} + \sin\theta_2\,\mathbf{j})$

Vertical components: $\|\mathbf{u}\|\sin\theta_1 + \|\mathbf{v}\|\sin\theta_2 = 5000$

Horizontal components: $\|\mathbf{u}\|\cos\theta_1 + \|\mathbf{v}\|\cos\theta_2 = 0$

Solving this system, you obtain

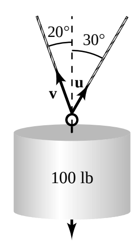
$\|\mathbf{u}\| \approx 2169.4$ and $\|\mathbf{v}\| \approx 3611.2$.




**90.** To lift the weight vertically, the sum of the vertical components of **u** and **v** must be 100 and the sum of the horizontal components must be 0.

$\mathbf{u} = \|\mathbf{u}\|\,(\cos 60°\mathbf{i} + \sin 60°\mathbf{j})$

$\mathbf{v} = \|\mathbf{v}\|\,(\cos 110°\mathbf{i} + \sin 110°\mathbf{j})$




Thus, $\|\mathbf{u}\|\sin 60° + \|\mathbf{v}\|\sin 110° = 100$, or

$$\|\mathbf{u}\|\left(\frac{\sqrt{3}}{2}\right) + \|\mathbf{v}\|\sin 110° = 100.$$

And $\|\mathbf{u}\|\cos 60° + \|\mathbf{v}\|\cos 110° = 0$ or

$$\|\mathbf{u}\|\left(\frac{1}{2}\right) + \|\mathbf{v}\|\cos 110° = 0$$

Multiplying the last equation by $(\sqrt{3})$ and adding to the first equation gives

$$\|\mathbf{u}\|\left(\sin 110° - \sqrt{3}\cos 110°\right) = 100 \implies \|\mathbf{v}\| \approx 65.27 \text{ lb}.$$

Then, $\|\mathbf{u}\|\left(\frac{1}{2}\right) + 65.27\cos 110° = 0$ gives

$\|\mathbf{u}\| \approx 44.65$ lb.

(a) The tension in each rope: $\|\mathbf{u}\| = 44.65$ lb, $\|\mathbf{v}\| = 65.27$ lb.

(b) Vertical components: $\|\mathbf{u}\|\sin 60° \approx 38.67$ lb.

$\|\mathbf{v}\|\sin 110° \approx 61.33$ lb.

**92.** $\mathbf{u} = 400\mathbf{i}$(plane)

$\mathbf{v} = 50(\cos 135°\mathbf{i} + \sin 135°\mathbf{j}) = -25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j}$ (wind)

$\mathbf{u} + \mathbf{v} = \left(400 - 25\sqrt{2}\right)\mathbf{i} + 25\sqrt{2}\mathbf{j} \approx 364.64\mathbf{i} + 35.36\mathbf{j}$

$\tan\theta = \dfrac{35.36}{364.64} \implies \theta \approx 5.54°$

Direction North of East: $\approx$ N 84.46° E

Speed: $\approx$ 336.35 mph

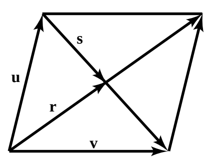
**94.** $\|\mathbf{u}\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1,$

$\|\mathbf{v}\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

**96.** Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors that determine the parallelogram, as indicated in the figure. The two diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. Therefore, $\mathbf{r} = x(\mathbf{u} + \mathbf{v})$, $\mathbf{s} = y(\mathbf{v} - \mathbf{u})$. But,

$$\begin{aligned}\mathbf{u} &= \mathbf{r} - \mathbf{s}\\ &= x(\mathbf{u} + \mathbf{v}) - y(\mathbf{v} - \mathbf{u}) = (x + y)\mathbf{u} + (x - y)\mathbf{v}.\end{aligned}$$

Therefore, $x + y = 1$ and $x - y = 0$. Solving we have $x = y = \frac{1}{2}$.




**98.** The set is a circle of radius 5, centered at the origin.

$\|\mathbf{u}\| = \|\langle x, y\rangle\| = \sqrt{x^2 + y^2} = 5 \implies x^2 + y^2 = 25$

**100.** True

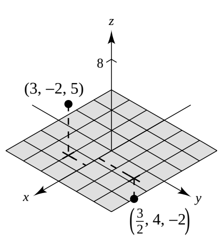
**102.** False

$a = b = 0$

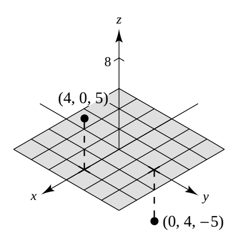
**104.** True

## Section 10.2 Space Coordinates and Vectors in Space

**2.**




**4.**




**6.** $A(2, -3, -1)$

$B(-3, 1, 4)$

**8.** $x = 7, y = -2, z = -1$:

$(7, -2, -1)$

**10.** $x = 0, y = 3, z = 2$: $(0, 3, 2)$

**12.** The $x$-coordinate is 0.

**14.** The point is 2 units in front of the $xz$-plane.

**16.** The point is on the plane $z = -3$.

**18.** The point is behind the $yz$-plane.

**20.** The point is in front of the plane $x = 4$.

**22.** The point $(x, y, z)$ is 4 units above the $xy$-plane, and above either quadrant II or IV.

**24.** The point could be above the $xy$-plane, and thus above quadrants I or III, or below the $xy$-plane, and thus below quadrants II or IV.

**26.** $$\begin{aligned} d &= \sqrt{(2-(-2))^2 + (-5-3)^2 + (-2-2)^2} \\ &= \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6} \end{aligned}$$

**28.** $$\begin{aligned} d &= \sqrt{(4-2)^2 + (-5-2)^2 + (6-3)^2} \\ &= \sqrt{4+49+9} = \sqrt{62} \end{aligned}$$

**30.** $A(5, 3, 4)$, $B(7, 1, 3)$, $C(3, 5, 3)$

$|AB| = \sqrt{4+4+1} = 3$

$|AC| = \sqrt{4+4+1} = 3$

$|BC| = \sqrt{16+16+0} = 4\sqrt{2}$

Since $|AB| = |AC|$, the triangle is isosceles.

**32.** $A(5, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, -3)$

$|AB| = \sqrt{25+4+0} = \sqrt{29}$

$|AC| = \sqrt{25+0+9} = \sqrt{34}$

$|BC| = \sqrt{0+4+9} = \sqrt{13}$

Neither

**34.** The $y$-coordinate is changed by 3 units:

$(5, 6, 4)$, $(7, 4, 3)$, $(3, 8, 3)$

**36.** $$\left(\frac{4+8}{2}, \frac{0+8}{2}, \frac{-6+20}{2}\right) = (6, 4, 7)$$

**38.** Center: $(4, -1, 1)$

Radius: 5

$$(x-4)^2 + (y+1)^2 + (z-1)^2 = 25$$

$$x^2 + y^2 + z^2 - 8x + 2y - 2z - 7 = 0$$

**40.** Center: $(-3, 2, 4)$

$r = 3$

(tangent to $yz$-plane)

$$(x+3)^2 + (y-2)^2 + (z-4)^2 = 9$$

**42.** $$\begin{aligned} x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 &= 0 \\ \left(x^2 + 9x + \frac{81}{4}\right) + (y^2 - 2y + 1) + (z^2 + 10z + 25) &= -19 + \frac{81}{4} + 1 + 25 \\ \left(x + \frac{9}{2}\right)^2 + (y-1)^2 + (z+5)^2 &= \frac{109}{4} \end{aligned}$$

Center: $\left(-\frac{9}{2}, 1, -5\right)$

Radius: $\frac{\sqrt{109}}{2}$

**44.** $$\begin{aligned} 4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 &= 0 \\ x^2 + y^2 + z^2 - x - 8y + 2z + \frac{33}{4} &= 0 \\ \left(x^2 - x + \frac{1}{4}\right) + (y^2 - 8y + 16) + (z^2 + 2z + 1) &= -\frac{33}{4} + \frac{1}{4} + 16 + 1 \\ \left(x - \frac{1}{2}\right)^2 + (y-4)^2 + (z+1)^2 &= 9 \end{aligned}$$

Center: $\left(\frac{1}{2}, 4, -1\right)$

Radius: 3

**46.** $$x^2 + y^2 + z^2 < 4x - 6y + 8z - 13$$

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) + (z^2 - 8z + 16) < 4 + 9 + 16 - 13$$

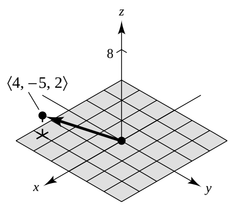
$$(x - 2)^2 + (y + 3)^2 + (z - 4)^2 < 16$$

Interior of sphere of radius 4 centered at $(2, -3, 4)$.

**48.** (a) $\mathbf{v} = (4 - 0)\mathbf{i} + (0 - 5)\mathbf{j} + (3 - 1)\mathbf{k}$

$= 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} = \langle 4, -5, 2\rangle$

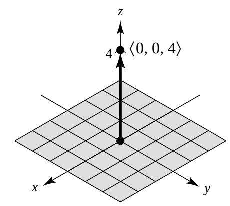
(b)




**50.** (a) $\mathbf{v} = (2 - 2)\mathbf{i} + (3 - 3)\mathbf{j} + (4 - 0)\mathbf{k}$

$= 4\mathbf{k} = \langle 0, 0, 4\rangle$

(b)




**52.** $\langle -1 - 4, 7 - (-5), -3 - 2\rangle = \langle -5, 12, -5\rangle$

$\|\langle -5, 12, -5\rangle\| = \sqrt{25 + 144 + 25} = \sqrt{194}$

Unit vector: $\dfrac{\langle -5, 12, -5\rangle}{\sqrt{194}} = \left\langle \dfrac{-5}{\sqrt{194}}, \dfrac{12}{\sqrt{194}}, \dfrac{-5}{\sqrt{194}}\right\rangle$

**54.** $\langle 2 - 1, 4 - (-2), -2 - 4\rangle = \langle 1, 6, -6\rangle$

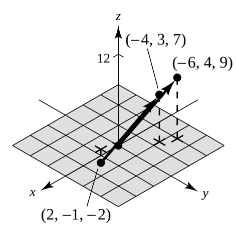
$\|\langle 1, 6, -6\rangle\| = \sqrt{1 + 36 + 36} = \sqrt{73}$

Unit vector: $\left\langle \dfrac{1}{\sqrt{73}}, \dfrac{6}{\sqrt{73}}, \dfrac{-6}{\sqrt{73}}\right\rangle$

**56.** (b) $\mathbf{v} = (-4 - 2)\mathbf{i} + (3 + 1)\mathbf{j} + (7 + 2)\mathbf{k}$

$= -6\mathbf{i} + 4\mathbf{j} + 9\mathbf{k} = \langle -6, 4, 9\rangle$

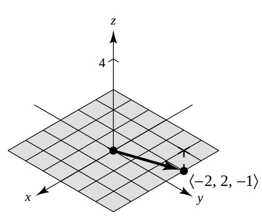
(a) and (c).



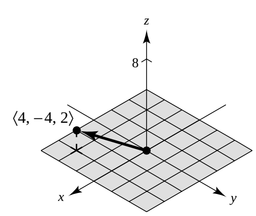

**58.** $(q_1, q_2, q_3) - \left(0, 2, \frac{5}{2}\right) = \left(1, -\frac{2}{3}, \frac{1}{2}\right)$

$Q = \left(1, -\frac{8}{3}, 3\right)$

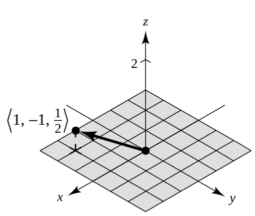
**60.** (a) $-\mathbf{v} = \langle -2, 2, -1\rangle$



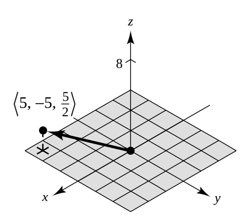

(b) $2\mathbf{v} = \langle 4, -4, 2\rangle$




(c) $\frac{1}{2}\mathbf{v} = \left\langle 1, -1, \frac{1}{2}\right\rangle$




(d) $\frac{5}{2}\mathbf{v} = \left\langle 5, -5, \frac{5}{2}\right\rangle$



**62.** $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w} = \langle 1, 2, 3\rangle - \langle 2, 2, -1\rangle + \langle 8, 0, -8\rangle = \langle 7, 0, -4\rangle$

**64.** $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w} = \langle 5, 10, 15\rangle - \langle 6, 6, -3\rangle - \langle 2, 0, -2\rangle = \langle -3, 4, 20\rangle$

**66.** $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = 2\langle 1, 2, 3\rangle + \langle 2, 2, -1\rangle - \langle 4, 0, -4\rangle + 3\langle z_1, z_2, z_3\rangle = \langle 0, 0, 0\rangle$

$\langle 0, 6, 9\rangle + \langle 3z_1, 3z_2, 3z_3\rangle = \langle 0, 0, 0\rangle$

$0 + 3z_1 = 0 \implies z_1 = 0$

$6 + 3z_2 = 0 \implies z_2 = -2$

$9 + 3z_3 = 0 \implies z_3 = -3$

$\mathbf{z} = \langle 0, -2, -3\rangle$

**68.** (b) and (d) are parallel since $-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k} = -2\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right)$ and $\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k} = \frac{3}{2}\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right)$.

**70.** $\mathbf{z} = \langle -7, -8, 3\rangle$

(b) is parallel since $(-z)\mathbf{z} = \langle 14, 16, -6\rangle$.

**72.** $P(4, -2, 7)$, $Q(-2, 0, 3)$, $R(7, -3, 9)$

$\overrightarrow{PQ} = \langle -6, 2, -4\rangle$

$\overrightarrow{PR} = \langle 3, -1, 2\rangle$

$\langle 3, -1, 2\rangle = -\frac{1}{2}\langle -6, 2, -4\rangle$

Therefore, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.

The points are collinear.

**74.** $P(0, 0, 0)$, $Q(1, 3, -2)$, $R(2, -6, 4)$

$\overrightarrow{PQ} = \langle 1, 3, -2\rangle$

$\overrightarrow{PR} = \langle 2, -6, 4\rangle$

Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not parallel, the points are not collinear.

**76.** $A(1, 1, 3)$, $B(9, -1, -2)$, $C(11, 2, -9)$, $D(3, 4, -4)$

$\overrightarrow{AB} = \langle 8, -2, -5\rangle$

$\overrightarrow{DC} = \langle 8, -2, -5\rangle$

$\overrightarrow{AD} = \langle 2, 3, -7\rangle$

$\overrightarrow{BC} = \langle 2, 3, -7\rangle$

Since $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$, the given points form the vertices of a parallelogram.

**78.** $\|\mathbf{v}\| = \sqrt{1 + 0 + 9} = \sqrt{10}$

**80.** $\mathbf{v} = \langle -4, 3, 7\rangle$

$\|\mathbf{v}\| = \sqrt{16 + 9 + 49} = \sqrt{74}$

**82.** $\mathbf{v} = \langle 1, 3, -2\rangle$

$\|\mathbf{v}\| = \sqrt{1 + 9 + 4} = \sqrt{14}$

**84.** $\mathbf{u} = \langle 6, 0, 8\rangle$

$\|\mathbf{u}\| = \sqrt{36 + 0 + 64} = 10$

(a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \dfrac{1}{10}\langle 6, 0, 8\rangle$

(b) $-\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = -\dfrac{1}{10}\langle 6, 0, 8\rangle$

**86.** $\mathbf{u} = \langle 8, 0, 0\rangle$

$\|\mathbf{u}\| = 8$

(a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 1, 0, 0\rangle$

(b) $-\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \langle -1, 0, 0\rangle$

**88.** (a) $\mathbf{u} + \mathbf{v} = \langle 4, 7.5, -2\rangle$

(b) $\|\mathbf{u} + \mathbf{v}\| \approx 8.732$

(c) $\|\mathbf{u}\| \approx 5.099$

(d) $\|\mathbf{v}\| \approx 9.014$

**90.** $c\mathbf{u} = \langle c, 2c, 3c\rangle$

$\|c\mathbf{u}\| = \sqrt{c^2 + 4c^2 + 9c^2} = 3$

$14c^2 = 9$

$c = \pm\dfrac{3\sqrt{14}}{14}$

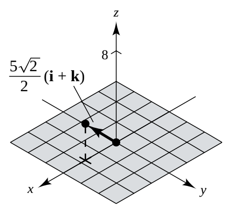
**92.** $\mathbf{v} = 3\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = 3\left\langle \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}} \right\rangle = \left\langle \dfrac{3}{\sqrt{3}}, \dfrac{3}{\sqrt{3}}, \dfrac{3}{\sqrt{3}} \right\rangle$

**94.** $\mathbf{v} = \sqrt{5}\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \sqrt{5}\left\langle \dfrac{-2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}, \dfrac{1}{\sqrt{14}} \right\rangle$

$= \left\langle \dfrac{-\sqrt{70}}{7}, \dfrac{3\sqrt{70}}{14}, \dfrac{\sqrt{70}}{14} \right\rangle$

**96.** $\mathbf{v} = 5(\cos 45°\mathbf{i} + \sin 45°\mathbf{k}) = \dfrac{5\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$ or

$\mathbf{v} = 5(\cos 135°\mathbf{i} + \sin 135°\mathbf{k}) = \dfrac{5\sqrt{2}}{2}(-i + k)$




**98.** $\mathbf{v} = \langle 5, 6, -3 \rangle$

$\frac{2}{3}\mathbf{v} = \left\langle \frac{10}{3}, 4, -2 \right\rangle$

$(1, 2, 5) + \left(\frac{10}{3}, 4, -2\right) = \left(\frac{13}{3}, 6, 3\right)$

**100.** $x_0$ is directed distance to $yz$-plane.

$y_0$ is directed distance to $xz$-plane.

$z_0$ is directed distance to $xy$-plane.

**102.** $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

**104.** A sphere of radius 4 centered at $(x_1, y_1, z_1)$.

$\|\mathbf{v}\| = \|\langle x - x_2, y - y_1, z - z_1 \rangle\|$

$= \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = 4$

$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = 16$ sphere

**106.** As in Exercise 105(c), $x = a$ will be a vertical asymptote. Hence, $\lim\limits_{r_0 \to a^-} T = \infty$.

**108.** $550 = \|c(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k})\|$

$302{,}500 = 18{,}125c^2$

$c^2 = 16.689655$

$c \approx 4.085$

$\mathbf{F} \approx 4.085(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k})$

$\approx 306\mathbf{i} - 204\mathbf{j} - 409\mathbf{k}$

**110.** Let $A$ lie on the $y$-axis and the wall on the $x$-axis. Then $A = (0, 10, 0)$, $B = (8, 0, 6)$, $C = (-10, 0, 6)$ and

$\overrightarrow{AB} = \langle 8, -10, 6 \rangle$, $\overrightarrow{AC} = \langle -10, -10, 6 \rangle$.

$\|\overrightarrow{AB}\| = 10\sqrt{2}$, $\|\overrightarrow{AC}\| = 2\sqrt{59}$

Thus, $\mathbf{F}_1 = 420\dfrac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}$, $\mathbf{F}_2 = 650\dfrac{\overrightarrow{AC}}{\|\overrightarrow{AC}\|}$

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \approx \langle 237.6, -297.0, 178.2 \rangle$

$+ \langle -423.1, -423.1, 253.9 \rangle$

$\approx \langle -185.5, -720.1, 432.1 \rangle$

$\|\mathbf{F}\| \approx 860.0$ lb

## Section 10.3 The Dot Product of Two Vectors

**2.** $\mathbf{u} = \langle 4, 10 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 4(-2) + 10(3) = 22$

(b) $\mathbf{u} \cdot \mathbf{u} = 4(4) + 10(10) = 116$

(c) $\|\mathbf{u}\|^2 = 116$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 22\langle -2, 3 \rangle = \langle -44, 66 \rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(22) = 44$

**4.** $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i}$

(a) $\mathbf{u} \cdot \mathbf{v} = 1$

(b) $\mathbf{u} \cdot \mathbf{u} = 1$

(c) $\|\mathbf{u}\|^2 = 1$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{i}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

**6.** $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + 1(-3) + (-2)(2) = -5$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + 1(1) + (-2)(-2) = 9$

(c) $\|\mathbf{u}\|^2 = 9$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -5(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = -5\mathbf{i} + 15\mathbf{j} - 10\mathbf{k}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-5) = -10$

**8.** $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$

$\mathbf{v} = \langle 2.22, 1.85, 3.25 \rangle$

Increase prices by 4%: $1.04\langle 2.22, 1.85, 3.25 \rangle$.

New total amount: $1.04(\mathbf{u} \cdot \mathbf{v}) = 1.04(17{,}139.05)$

$= \$17{,}824.61$

**10.** $\dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \cos\theta$

$$\mathbf{u} \cdot \mathbf{v} = (40)(25)\cos\frac{5\pi}{6} = -500\sqrt{3}$$

**12.** $\mathbf{u} = \langle 3, 1 \rangle, \mathbf{v} = \langle 2, -1 \rangle$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

**14.**
$$\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

$$= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\theta = \arccos\left[\frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)\right] = 105^\circ$$

**16.** $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{3(2) + 2(-3) + 0}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = 0$$

$$\theta = \frac{\pi}{2}$$

**18.** $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

$$= \frac{9}{\sqrt{14}\sqrt{6}} = \frac{9}{2\sqrt{21}} = \frac{3\sqrt{21}}{14}$$

$$\theta = \arccos\left(\frac{3\sqrt{21}}{14}\right) \approx 10.9^\circ$$

**20.** $\mathbf{u} = \langle 2, 18 \rangle, \mathbf{v} = \left\langle \frac{3}{2}, -\frac{1}{6} \right\rangle$

$\mathbf{u} \neq c\mathbf{v} \implies$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \implies$ orthogonal

**22.** $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j}), \mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

$\mathbf{u} = -\frac{1}{6}\mathbf{v} \implies$ parallel

**24.** $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}, \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \implies$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \implies$ orthogonal

**26.** $\mathbf{u} = \langle \cos\theta, \sin\theta, -1 \rangle$,

$\mathbf{v} = \langle \sin\theta, -\cos\theta, 0 \rangle$

$\mathbf{u} \neq c\mathbf{v} \implies$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \implies$ orthogonal

**28.** $\mathbf{u} = \langle 5, 3, -1 \rangle \quad \|\mathbf{u}\| = \sqrt{35}$

$$\cos\alpha = \frac{5}{\sqrt{35}}$$

$$\cos\beta = \frac{3}{\sqrt{35}}$$

$$\cos\gamma = \frac{-1}{\sqrt{35}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{25}{35} + \frac{9}{35} + \frac{1}{35} = 1$$

**30.** $\mathbf{u} = \langle a, b, c \rangle, \|\mathbf{u}\| = \sqrt{a^2 + b^2 + c^2}$

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1$$

**32.** $\mathbf{u} = \langle -4, 3, 5 \rangle \quad \|\mathbf{u}\| = \sqrt{50} = 5\sqrt{2}$

$$\cos\alpha = \frac{-4}{5\sqrt{2}} \implies \alpha \approx 2.1721 \text{ or } 124.4°$$

$$\cos\beta = \frac{3}{5\sqrt{2}} \implies \beta \approx 1.1326 \text{ or } 64.9°$$

$$\cos\gamma = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \implies \gamma \approx \frac{\pi}{4} \text{ or } 45°$$

**34.** $\mathbf{u} = \langle -2, 6, 1 \rangle \quad \|\mathbf{u}\| = \sqrt{41}$

$$\cos\alpha = \frac{-2}{\sqrt{41}} \implies \alpha \approx 1.8885 \text{ or } 108.2°$$

$$\cos\beta = \frac{6}{\sqrt{41}} \implies \alpha \approx 0.3567 \text{ or } 20.4°$$

$$\cos\gamma = \frac{1}{\sqrt{41}} \implies \alpha \approx 1.4140 \text{ or } 81.0°$$

**36.** $\mathbf{F}_1$: $C_1 = \dfrac{300}{\|\mathbf{F}_1\|} \approx 13.0931$

$\mathbf{F}_2$: $C_2 = \dfrac{100}{\|\mathbf{F}_2\|} \approx 6.3246$

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &\approx 13.0931\langle -20, -10, 5 \rangle + 6.3246\langle 5, 15, 0 \rangle\rangle \\ &= \langle -230.239, -36.062, 65.4655 \rangle \end{aligned}$$

$\|\mathbf{F}\| \approx 242.067$ lb

$$\cos\alpha \approx \frac{-230.239}{\|\mathbf{F}\|} \implies \alpha \approx 162.02°$$

$$\cos\beta \approx \frac{-36.062}{\|\mathbf{F}\|} \implies \beta \approx 98.57°$$

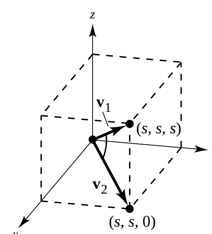
$$\cos\gamma \approx \frac{65.4655}{\|\mathbf{F}\|} \implies \gamma \approx 74.31°$$

**38.**

$$\begin{aligned} \mathbf{v}_1 &= \langle s, s, s \rangle \\ \|\mathbf{v}_1\| &= s\sqrt{3} \\ \mathbf{v}_2 &= \langle s, s, 0 \rangle \\ \|\mathbf{v}_2\| &= s\sqrt{2} \end{aligned}$$

$$\cos\theta = \frac{s\sqrt{2}}{s\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$\theta = \arccos\frac{\sqrt{6}}{3} \approx 35.26°$$




**40.** $\mathbf{F}_1 = C_1(0, 10, 10)$. $\|\mathbf{F}_1\| = 200 = C_1 10\sqrt{2} \implies C_1 = 10\sqrt{2}$

and $\mathbf{F}_1 = \left\langle 0, 100\sqrt{2}, 100\sqrt{2} \right\rangle$

$\mathbf{F}_2 = C_2\langle -4, -6, 10 \rangle$

$\mathbf{F}_2 = C_3\langle 4, -6, 10 \rangle$

$\mathbf{F} = \langle 0, 0, w \rangle$

$\mathbf{F} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$

$-4C_2 + 4C_3 = 0 \implies C_2 = C_3$

$100\sqrt{2} - 6C_2 - 6C_3 = 0 \implies C_2 = C_3 = \dfrac{25\sqrt{2}}{3}N$

**42.** $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 9, 7 \rangle - \langle 3, 9 \rangle = \langle 6, -2 \rangle$

**44.** $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 8, 2, 0 \rangle - \langle 6, 3, -3 \rangle = \langle 2, -1, 3 \rangle$

**46.** $\mathbf{u} = \langle 2, -3\rangle, \mathbf{v} = \langle 3, 2\rangle$

(a) $\mathbf{w}_1 = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = 0\mathbf{v} = \langle 0, 0\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, -3\rangle$

**48.** $\mathbf{u} = \langle 1, 0, 4\rangle, \mathbf{v} = \langle 3, 0, 2\rangle$

(a) $\mathbf{w}_1 = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \dfrac{11}{13}\langle 3, 0, 2\rangle = \left\langle \dfrac{33}{13}, 0, \dfrac{22}{13}\right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 1, 0, 4\rangle - \left\langle \dfrac{33}{13}, 0, \dfrac{22}{13}\right\rangle$

$$= \left\langle -\frac{20}{13}, 0, \frac{30}{13}\right\rangle$$

**50.** The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$.

The angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

**52.** (a) and (b) are defined.

**54.** See figure 10.29, page 739.

**56.** Yes, $\left\|\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right\| = \left\|\dfrac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}\right\|$

$$|\mathbf{u}\cdot\mathbf{v}|\frac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = |\mathbf{v}\cdot\mathbf{u}|\frac{\|\mathbf{u}\|}{\|\mathbf{u}\|^2}$$

$$\frac{1}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{u}\|}$$

$$\|\mathbf{u}\| = \|\mathbf{v}\|$$

**58.** (a) $\|\mathbf{u}\| = 5, \|\mathbf{v}\| \approx 8.602, \theta \approx 91.33°$

(b) $\|\mathbf{u}\| \approx 9.165, \|\mathbf{v}\| \approx 5.745, \theta = 90°$

**60.** (a) $\left\langle \dfrac{64}{17}, \dfrac{16}{17}\right\rangle$

(b) $\left\langle -\dfrac{21}{26}, \dfrac{63}{26}, \dfrac{42}{13}\right\rangle$

**62.** Because $\mathbf{u}$ appears to be a multiple of $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\mathbf{u}$. Analytically,

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\langle -3, -2\rangle\cdot\langle 6, 4\rangle}{\langle 6, 4\rangle\cdot\langle 6, 4\rangle}\langle 6, 4\rangle$$

$$= \frac{-26}{52}\langle 6, 4\rangle = \langle -3, -2\rangle = \mathbf{u}.$$

**64.** $\mathbf{u} = -8\mathbf{i} + 3\mathbf{j}$. Want $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{v} = 3\mathbf{i} + 8\mathbf{j}$ and $-\mathbf{v} = -3\mathbf{i} - 8\mathbf{j}$ are orthogonal to $\mathbf{u}$.

**66.** $\mathbf{u} = \langle 0, -3, 6\rangle$. Want $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{v} = \langle 0, 6, 3\rangle$ and $-\mathbf{v} = \langle 0, -6, -3\rangle$ are orthogonal to $\mathbf{u}$.

**68.** $\overrightarrow{OA} = \langle 10, 5, 20\rangle, \mathbf{v} = \langle 0, 0, 1\rangle$

$$\text{proj}_{\mathbf{v}}\overrightarrow{OA} = \frac{20}{1^2}\langle 0, 0, 1\rangle = \langle 0, 0, 20\rangle$$

$\|\text{proj}_{\mathbf{v}}\overrightarrow{OA}\| = 20$

**70.** $\mathbf{F} = 25(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})$

$\mathbf{v} = 50\mathbf{i}$

$W = \mathbf{F}\cdot\mathbf{v} = 1250\cos 20° \approx 1174.6 \text{ ft}\cdot\text{lb}$

**72.** $\overrightarrow{PQ} = \langle -4, 2, 10\rangle$

$\vec{V} = \langle -2, 3, 6\rangle$

$W = \overrightarrow{PQ}\cdot\vec{V} = 74$

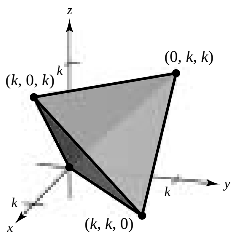
**74.** True

$\mathbf{w}\cdot(\mathbf{u} + \mathbf{v}) = \mathbf{w}\cdot\mathbf{u} + \mathbf{w}\cdot\mathbf{v}$

$= 0 + 0 = 0 \implies \mathbf{w}$

and $\mathbf{u} + \mathbf{v}$ are orthogonal.

**76.** (a)




(b) Length of each edge:

$$\sqrt{k^2 + k^2 + 0^2} = k\sqrt{2}$$

(c) $\cos\theta = \dfrac{k^2}{(k\sqrt{2})(k\sqrt{2})} = \dfrac{1}{2}$

$$\theta = \arccos\left(\frac{1}{2}\right) = 60^\circ$$

(d) $\vec{r}_1 = \langle k, k, 0\rangle - \left\langle \dfrac{k}{2}, \dfrac{k}{2}, \dfrac{k}{2}\right\rangle = \left\langle \dfrac{k}{2}, \dfrac{k}{2}, -\dfrac{k}{2}\right\rangle$

$$\vec{r}_2 = \langle 0, 0, 0\rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2}\right\rangle = \left\langle -\frac{k}{2}, -\frac{k}{2}, -\frac{k}{2}\right\rangle$$

$$\cos\theta = \frac{-\dfrac{k^2}{4}}{\left(\dfrac{k}{2}\right)^2 \cdot 3} = -\frac{1}{3}$$

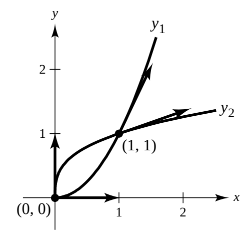
$$\theta = 109.5^\circ$$

**78.** The curves $y_1 = x^2$ and $y_2 = x^{1/3}$ intersect at $(0, 0)$ and at $(1, 1)$.

At $(0, 0)$: $\langle 1, 0\rangle$ is tangent to $y_1$ and $\langle 0, 1\rangle$ is tangent to $y_2$. The angle between these vectors is 90°.

At $(1, 1)$: $(1/\sqrt{5})\langle 1, 2\rangle$ is tangent to $y_1$ and $(3/\sqrt{10})\langle 1, 1/3\rangle = (1/\sqrt{10})\langle 3, 1\rangle$ is tangent to $y_2$. To find the angle between these vectors,

$$\cos\theta = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}}(3 + 2) = \frac{1}{\sqrt{2}} \implies \theta = 45^\circ.$$




**80.**
$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\ |\mathbf{u}\cdot\mathbf{v}| &= \left|\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\right| \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,|\cos\theta| \\ &\le \|\mathbf{u}\|\,\|\mathbf{v}\| \text{ since } |\cos\theta| \le 1. \end{aligned}$$

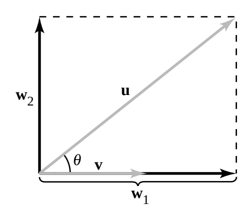
**82.** Let $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$, as indicated in the figure. Because $\mathbf{w}_1$ is a scalar multiple of $\mathbf{v}$, you can write

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{v} + \mathbf{w}_2.$$

Taking the dot product of both sides with $\mathbf{v}$ produces

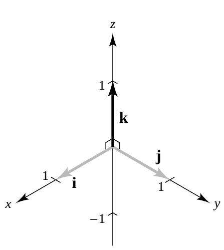
$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= (c\mathbf{v} + \mathbf{w}_2)\cdot\mathbf{v} = c\mathbf{v}\cdot\mathbf{v} + \mathbf{w}_2\cdot\mathbf{v} \\ &= c\|\mathbf{v}\|^2, \text{ since } \mathbf{w}_2 \text{ and } \mathbf{v} \text{ are orthogonol.} \end{aligned}$$

Thus, $\mathbf{u}\cdot\mathbf{v} = c\|\mathbf{v}\|^2 \implies c = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}$ and $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = c\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$.

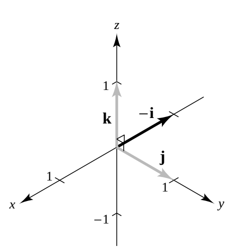



## Section 10.4 The Cross Product of Two Vectors in Space

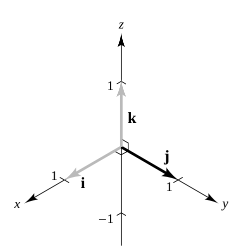
**2.** $\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$




**4.** $\mathbf{k}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$




**6.** $\mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$



**8.** (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 5 \\ 2 & 3 & -2 \end{vmatrix} = \langle -15, 16, 9 \rangle$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle 15, -16, -9 \rangle$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**10.** (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -2 \\ 1 & 5 & 1 \end{vmatrix} = \langle 8, -5, 17 \rangle$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle -8, 5, -17 \rangle$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

**12.** $\mathbf{u} = \langle -1, 1, 2 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - \mathbf{k} = \langle -2, 0, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (-1)(-2) + (1)(0) + (2)(-1) = 0 \implies \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (0)(-2) + (1)(0) + (0)(-1) = 0 \implies \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

**14.** $\mathbf{u} = \langle -10, 0, 6 \rangle$, $\mathbf{v} = \langle 7, 0, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} = 42\mathbf{j} = \langle 0, 42, 0 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (-10)(0) + (0)(42) + 6(0) = 0 \implies \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

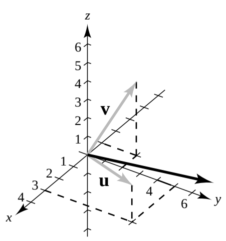
$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 7(0) + (0)(42) + (0)(0) = 0 \implies \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

**16.** $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = 6\mathbf{i} - \mathbf{j} + 13\mathbf{k}$

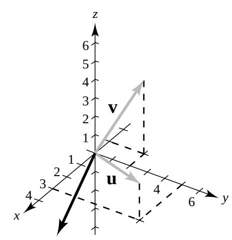
$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(6) + 6(-1) = 0 \implies \mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(6) + 1(-1) + 1(13) = 0 \implies \mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$$

**18.**




**20.**




**22.** $\mathbf{u} = \langle -8, -6, 4 \rangle$

$\mathbf{v} = \langle 10, -12, -2 \rangle$

$\mathbf{u} \times \mathbf{v} = \langle 60, 24, 156 \rangle$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{36\sqrt{22}} \langle 60, 24, 156 \rangle = \left\langle \frac{5}{3\sqrt{22}}, \frac{2}{3\sqrt{22}}, \frac{13}{3\sqrt{22}} \right\rangle$$

**24.** $\mathbf{u} = \dfrac{2}{3}\mathbf{k}$

$\mathbf{v} = \dfrac{1}{2}\mathbf{i} + 6\mathbf{k}$

$\mathbf{u} \times \mathbf{v} = \left\langle 0, \dfrac{1}{3}, 0 \right\rangle$

$\dfrac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \langle 0, 1, 0 \rangle$

**26.** (a) $\mathbf{u} \times \mathbf{v} = \langle -18, -12, 48 \rangle$

$\|\mathbf{u} \times \mathbf{v}\| \approx 52.650$

(b) $\mathbf{u} \times \mathbf{v} = \langle -50, 40, -34 \rangle$

$\|\mathbf{u} \times \mathbf{v}\| \approx 72.498$

**28.** $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{v} = \mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{j} + \mathbf{k}$$

$A = \|\mathbf{u} \times \mathbf{v}\| = \|-\mathbf{j} + \mathbf{k}\| = \sqrt{2}$

**30.** $\mathbf{u} = \langle 2, -1, 0\rangle$

$\mathbf{v} = \langle -1, 2, 0\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = \langle 0, 0, 3\rangle$$

$A = \|\mathbf{u}\times\mathbf{v}\| = \|\langle 0, 0, 3\rangle\| = 3$

**32.** $A(2, -3, 1), B(6, 5, -1), C(3, -6, 4), D(7, 2, 2)$

$\overrightarrow{AB} = \langle 4, 8, -2\rangle, \overrightarrow{AC} = \langle 1, -3, 3\rangle, \overrightarrow{CD} = \langle 4, 8, -2\rangle, \overrightarrow{BD} = \langle 1, -3, 3\rangle$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the figure is a parallelogram.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are adjacent sides and

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & -2 \\ 1 & -3 & 3 \end{vmatrix} = \langle 18, -14, -20\rangle.$$

$\text{Area} = \|\overrightarrow{AB}\times\overrightarrow{AC}\| = \sqrt{920} = 2\sqrt{230}$

**34.** $A(2, -3, 4), B(0, 1, 2), C(-1, 2, 0)$

$\overrightarrow{AB} = \langle -2, 4, -2\rangle, \overrightarrow{AC} = \langle -3, 5, -4\rangle$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = -6\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$A = \frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = \frac{1}{2}\sqrt{44} = \sqrt{11}$

**36.** $A(1, 2, 0), B(-2, 1, 0), C(0, 0, 0)$

$\overrightarrow{AB} = \langle -3, -1, 0\rangle, \overrightarrow{AC} = \langle -1, -2, 0\rangle$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 0 \\ -1 & -2 & 0 \end{vmatrix} = 5\mathbf{k}$$

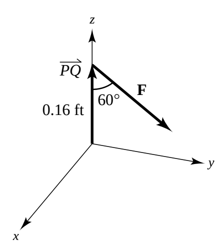
$A = \frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = \frac{5}{2}$

**38.** $\mathbf{F} = -2000(\cos 30°\mathbf{j} + \sin 30°\mathbf{k}) = -1000\sqrt{3}\mathbf{j} - 1000\mathbf{k}$

$\overrightarrow{PQ} = 0.16\mathbf{k}$

$$\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix} = 160\sqrt{3}\mathbf{i}$$

$\|\overrightarrow{PQ}\times\mathbf{F}\| = 160\sqrt{3}$ ft · lb




**40.** (a) $B$ is $-\frac{15}{12} = -\frac{5}{4}$ to the left of $A$, and one foot upwards:

$\overrightarrow{AB} = \frac{-5}{4}\mathbf{j} + \mathbf{k}$

$\mathbf{F} = -200(\cos\theta\mathbf{j} + \sin\theta\mathbf{k})$

(b) $$\overrightarrow{AB}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -5/4 & 1 \\ 0 & -200\cos\theta & -200\sin\theta \end{vmatrix}$$

$$= (250\sin\theta + 200\cos\theta)\mathbf{i}$$

$$\|\overrightarrow{AB}\times\mathbf{F}\| = |250\sin\theta + 200\cos\theta|$$

$$= 25(10\sin\theta + 8\cos\theta)$$

(c) For $\theta = 30°$,

$$\|\overrightarrow{AB}\times\mathbf{F}\| = 25\left(10\left(\frac{1}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= 25\left(5 + 4\sqrt{3}\right) \approx 298.2.$$

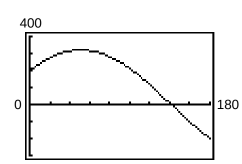
(d) If $T = \|\overrightarrow{AB}\times\mathbf{F}\|$,

$$\frac{dT}{d\theta} = 25(10\cos\theta - 8\sin\theta) = 0 \implies \tan\theta = \frac{5}{4}$$

$$\implies \theta \approx 51.34°.$$

The vectors are orthogonal.

(e) The zero is $\theta \approx 141.34°$, the angle making $\overrightarrow{AB}$ parallel to $\mathbf{F}$.



**42.** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$

**44.** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$

**46.** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix} = -72$

$V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = 72$

**48.** $\mathbf{u} = \langle 1, 1, 0\rangle$

$\mathbf{v} = \langle 1, 0, 2\rangle$

$\mathbf{w} = \langle 0, 1, 1\rangle$

$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -3$

$V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = 3$

**50.** See Theorem 10.8, page 746.

**52.** Form the vectors for two sides of the triangle, and compute their cross product:

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle \times \langle x_3 - x_1, y_3 - y_1, z_3 - z_1\rangle$$

**54.** False, let $\mathbf{u} = \langle 1, 0, 0\rangle$, $\mathbf{v} = \langle 1, 0, 0\rangle$, $\mathbf{w} = \langle -1, 0, 0\rangle$.

Then,

$\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w} = \mathbf{0}$, but $\mathbf{v} \neq \mathbf{w}$.

**56.** $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $\mathbf{w} = \langle w_1, w_2, w_3\rangle$

$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$

$\mathbf{v}\times\mathbf{w} = (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$

$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

**58.** $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $c$ is a scalar.

$$\begin{aligned}(c\mathbf{u})\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (cu_2v_3 - cu_3v_2)\mathbf{i} - (cu_1v_3 - cu_3v_1)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k} \\ &= c[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}] = c(\mathbf{u}\times\mathbf{v})\end{aligned}$$

**60.** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$= w_1(u_2v_3 - v_2u_3) - w_2(u_1v_3 - v_1u_3) + w_3(u_1v_2 - v_1u_2)$

$= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2)$

$= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

**62.** If $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other, $\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

$$\mathbf{u} \times \mathbf{v} = (c\mathbf{v}) \times \mathbf{v} = c(\mathbf{v} \times \mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = 0$. (Assume $\mathbf{u} \neq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$.) Thus, $\sin\theta = 0$, $\theta = 0$, and $\mathbf{u}$ and $\mathbf{v}$ are parallel. Therefore,

$\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

**64.** $\mathbf{u} = \langle a_1, b_1, c_1\rangle$, $\mathbf{v} = \langle a_2, b_2, c_2\rangle$, $\mathbf{w} = \langle a_3, b_3, c_3\rangle$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\mathbf{i} - (a_2c_3 - a_3c_2)\mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{k}$$

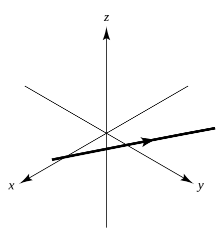
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ (b_2c_3 - b_3c_2) & (a_3c_2 - a_2c_3) & (a_2b_3 - a_3b_2) \end{vmatrix}$$

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= [b_1(a_2b_3 - a_3b_2) - c_1(a_3c_2 - a_2c_3)]\mathbf{i} - [a_1(a_2b_3 - a_3b_2) - c_1(b_2c_3 - b_3c_2)]\mathbf{j} + \\
&\quad [a_1(a_3c_2 - a_2c_3) - b_1(b_2c_3 - b_3c_2)]\mathbf{k} \\
&= [a_2(a_1a_3 + b_1b_3 + c_1c_3) - a_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{i} + \\
&\quad [b_2(a_1a_3 + b_1b_3 + c_1c_3) - b_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{j} + \\
&\quad [c_2(a_1a_3 + b_1b_3 + c_1c_3) - c_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{k} \\
&= (a_1a_3 + b_1b_3 + c_1c_3)\langle a_2, b_2, c_2\rangle - (a_1a_2 + b_1b_2 + c_1c_2)\langle a_3, b_3, c_3\rangle \\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}
\end{aligned}$$

## Section 10.5 Lines and Planes in Space

**2.** $x = 2 - 3t$, $y = 2$, $z = 1 - t$

(a)




(b) When $t = 0$ we have $P = (2, 2, 1)$. When $t = 2$ we have $Q = (-4, 2, -1)$.

$\overrightarrow{PQ} = \langle -6, 0, -2\rangle$

The components of the vector and the coefficients of $t$ are proportional since the line is parallel to $\overrightarrow{PQ}$.

(c) $z = 0$ when $t = 1$. Thus, $x = -1$ and $y = 2$.

Point: $(-1, 2, 0)$

$x = 0$ when $t = \frac{2}{3}$. Point: $\left(0, 2, \frac{1}{3}\right)$

**4.** Point: $(0, 0, 0)$

Direction vector: $\mathbf{v} = \left\langle -2, \frac{5}{2}, 1\right\rangle$

Direction numbers: $-4, 5, 2$

(a) Parametric: $x = -4t$, $y = 5t$, $z = 2t$

(b) Symmetric: $\dfrac{x}{-4} = \dfrac{y}{5} = \dfrac{z}{2}$

**6.** Point: $(-3, 0, 2)$

Direction vector: $\mathbf{v} = \langle 0, 6, 3\rangle$

Direction numbers: $0, 2, 1$

(a) Parametric: $x = -3$, $y = 2t$, $z = 2 + t$

(b) Symmetric: $\dfrac{y}{2} = z - 2$, $x = -3$

**8.** Point: $(-3, 5, 4)$

Directions numbers: $3, -2, 1$

(a) Parametric: $x = -3 + 3t, y = 5 - 2t, z = 4 + t$

(b) Symmetric: $\dfrac{x+3}{3} = \dfrac{y-5}{-2} = z - 4$

**10.** Points: $(2, 0, 2), (1, 4, -3)$

Direction vector: $\langle 1, -4, 5\rangle$

Direction numbers: $1, -4, 5$

(a) Parametric: $x = 2 + t, y = -4t, z = 2 + 5t$

(b) Symmetric: $x - 2 = \dfrac{y}{-4} = \dfrac{z-2}{5}$

**12.** Points: $(0, 0, 25), (10, 10, 0)$

Direction vector: $\langle 10, 10, -25\rangle$

Direction numbers: $2, 2, -5$

(a) Parametric: $x = 2t, y = 2t, z = 25 - 5t$

(b) Symmetric: $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z-25}{-5}$

**14.** Point: $(2, 3, 4)$

Direction vector: $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Direction numbers: $3, 2, -1$

Parametric: $x = 2 + 3t, y = 3 + 2t, z = 4 - t$

**16.** Points: $(2, 0, -3), (4, 2, -2)$

Direction vector: $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Direction numbers: $2, 2, 1$

Parametric: $x = 2 + 2t, y = 2t, z = -3 + t$

Symmetric: $\dfrac{x-2}{2} = \dfrac{y}{2} = \dfrac{z+3}{1}$

(a) Not on line $\left(1 \neq \dfrac{1}{2} \neq 1\right)$

(b) On line

(c) Not on line $\left(\dfrac{-3}{2} = \dfrac{-3}{2} \neq -1\right)$

**18.** $L_1$: $\mathbf{v} = \langle 4, -2, 3\rangle$ $\quad (8, -5, -9)$ on line

$L_2$: $\mathbf{v} = \langle 2, 1, 5\rangle$

$L_3$: $\mathbf{v} = \langle -8, 4, -6\rangle$ $\quad (8, -5, -9)$ on line

$L_4$: $\mathbf{v} = \langle -2, 1, 1.5\rangle$

$L_1$ and $L_2$ are identical.

**20.** By equating like variables, we have

(i) $-3t + 1 = 3s + 1$, (ii) $4t + 1 = 2s + 4$, and (iii) $2t + 4 = -s + 1$.

From (i) we have $s = -t$, and consequently from (ii), $t = \frac{1}{2}$ and from (iii), $t = -3$. The lines do not intersect.

**22.** Writing the equations of the lines in parametric form we have

$x = 2 - 3t \qquad y = 2 + 6t \qquad z = 3 + t$

$x = 3 + 2s \qquad y = -5 + s \qquad z = -2 + 4s.$

By equating like variables, we have $2 - 3t = 3 + 2s,\ 2 + 6t = -5 + s,\ 3 + t = -2 + 4s$. Thus, $t = -1, s = 1$ and the point of intersection is $(5, -4, 2)$.

$\mathbf{u} = \langle -3, 6, 1\rangle$ (First line)

$\mathbf{v} = \langle 2, 1, 4\rangle$ (Second line)

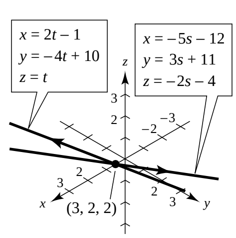
$$\cos\theta = \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{4}{\sqrt{46}\sqrt{21}} = \frac{4}{\sqrt{966}} = \frac{2\sqrt{966}}{483}$$

**24.** $x = 2t - 1 \qquad x = -5s - 12$

$y = -4t + 10 \qquad y = 3s + 11$

$z = t \qquad z = -2s - 4$

Point of intersection: $(3, 2, 2)$



**26.** $2x + 3y + 4z = 4$

$P = (0, 0, 1), Q = (2, 0, 0), R = (3, 2, -2)$

(a) $\overrightarrow{PQ} = \langle 2, 0, -1\rangle, \overrightarrow{PR} = \langle 3, 2, -3\rangle$

(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 3 & 2 & -3 \end{vmatrix} = \langle 2, 3, 4\rangle$

The components of the cross product are proportional (for this choice of $P$, $Q$, and $R$, they are the same) to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

**28.** Point: $(1, 0, -3)$

$\mathbf{n} = \mathbf{k} = \langle 0, 0, 1\rangle$

$0(x - 1) + 0(y - 0) + 1[z - (-3)] = 0$

$z + 3 = 0$

**30.** Point: $(0, 0, 0)$

Normal vector: $\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$

$-3(x - 0) + 0(y - 0) + 2(z - 0) = 0$

$-3x + 2z = 0$

**32.** Point: $(3, 2, 2)$

Normal vector: $\mathbf{v} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$4(x - 3) + (y - 2) - 3(z - 2) = 0$

$4x + y - 3z = 8$

**34.** Let $\mathbf{u}$ be vector from $(2, 3, -2)$ to $(3, 4, 2)$: $\langle 1, 1, 4\rangle$.

Let $\mathbf{v}$ be vector from $(2, 3, -2)$ to $(1, -1, 0)$: $\langle -1, -4, 2\rangle$.

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{vmatrix} = \langle 18, -6, -3\rangle$

$= -3\langle -6, 2, 1\rangle$

$-6(x - 2) + 2(y - 3) + 1(z + 2) = 0$

$-6x + 2y + z = -8$

**36.** $(1, 2, 3)$, Normal vector: $\mathbf{v} = \mathbf{i}$, $1(x - 1) = 0, x = 1$

**38.** The plane passes through the three points $(0, 0, 0)$, $(0, 1, 0)$ $(\sqrt{3}, 0, 1)$.

The vector from $(0, 0, 0)$ to $(0, 1, 0)$: $\mathbf{u} = \mathbf{j}$

The vector from $(0, 0, 0)$ to $(\sqrt{3}, 0, 1)$: $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{k}$

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{vmatrix} = \mathbf{i} - \sqrt{3}\mathbf{k}$

$x - \sqrt{3}z = 0$

**40.** The direction of the line is $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. Choose any point on the line, $[(0, 4, 0)$, for example$]$, and let $\mathbf{v}$ be the vector from $(0, 4, 0)$ to the given point $(2, 2, 1)$:

$\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{k}$

$(x - 2) - 2(z - 1) = 0$

$x - 2z = 0$

**42.** Let $\mathbf{v}$ be the vector from $(3, 2, 1)$ to $(3, 1, -5)$:
$\mathbf{v} = -\mathbf{j} - 6\mathbf{k}$

Let $\mathbf{n}$ be the normal to the given plane: $\mathbf{n} = 6\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Since $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane $P$, the normal vector to $P$ is:

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -6 \\ 6 & 7 & 2 \end{vmatrix} = 40\mathbf{i} - 36\mathbf{j} + 6\mathbf{k}$$

$$= 2(20\mathbf{i} - 18\mathbf{j} + 3\mathbf{k})$$

$20(x - 3) - 18(y - 2) + 3(z - 1) = 0$

$20x - 18y + 3z = 27$

**44.** Let $\mathbf{u} = \mathbf{k}$ and let $\mathbf{v}$ be the vector from $(4, 2, 1)$ to $(-3, 5, 7)$: $\mathbf{v} = -7\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

Since $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane $P$, the normal vector to $P$ is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -7 & 3 & 6 \end{vmatrix} = -3\mathbf{i} - 7\mathbf{j} = -(3\mathbf{i} + 7\mathbf{j})$$

$3(x - 4) + 7(y - 2) = 0$

$3x + 7y = 26$

**46.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 3, 1, -4\rangle$, $\mathbf{n}_2 = \langle -9, -3, 12\rangle$. Since $\mathbf{n}_2 = -3\mathbf{n}_1$, the planes are parallel, but not equal

**48.** The normal vectors to the planes are

$$\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k},$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|3 - 8 - 2|}{\sqrt{14}\sqrt{21}} = \frac{1}{\sqrt{6}}.$$

Therefore, $\theta = \arccos\left(\frac{1}{\sqrt{6}}\right) \approx 65.9°$.

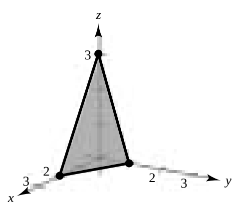
**50.** The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 2, 0, -1\rangle, \mathbf{n}_2 = \langle 4, 1, 8\rangle,$$

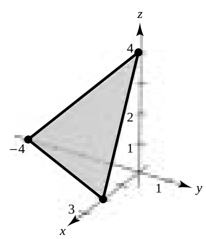
$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = 0.$$

Thus, $\theta = \frac{\pi}{2}$ and the planes are orthogonal.

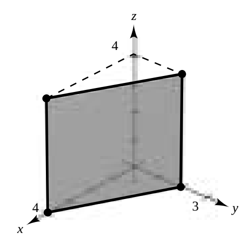
**52.** $3x + 6y + 2z = 6$



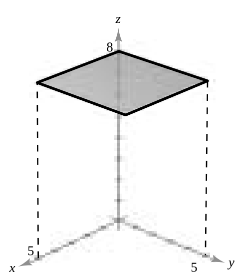
**54.** $2x - y + z = 4$



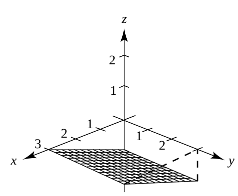
**56.** $x + 2y = 4$



**58.** $z = 8$



**60.** $x - 3z = 3$



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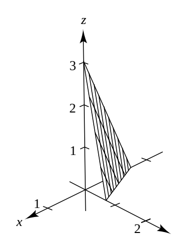
**62.** $2.1x - 4.7y - z + 3 = 0$



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**64.** $P_1$: $\mathbf{n} = \langle -60, 90, 30\rangle$ or $\langle -2, 3, 1\rangle$ $\quad \left(0, 0, \frac{9}{10}\right)$ on plane

$P_2$: $\mathbf{n} = \langle 6, -9, -3\rangle$ or $\langle -2, 3, 1\rangle$ $\quad \left(0, 0, -\frac{2}{3}\right)$ on plane

$P_3$: $\mathbf{n} = \langle -20, 30, 10\rangle$ or $\langle -2, 3, 1\rangle$ $\quad \left(0, 0, \frac{5}{6}\right)$ on plane

$P_4$: $\mathbf{n} = \langle 12, -18, 6\rangle$ or $\langle -2, 3, -1\rangle$

$P_1, P_2$, and $P_3$ are parallel.

**66.** If $c = 0$, $z = 0$ is $xy$-plane.

If $c \neq 0$, $cy + z = 0 \implies y = \frac{-1}{c}z$ is a plane parallel to $x$-axis and passing through the points $(0, 0, 0)$ and $(0, 1, -c)$.

**68.** The normals to the planes are $\mathbf{n}_1 = \langle 6, -3, 1\rangle$. and $\mathbf{n}_2 = \langle -1, 1, 5\rangle$.

The direction vector for the line is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 1 \\ -1 & 1 & 5 \end{vmatrix} = \langle -16, -31, 3\rangle.$$

Now find a point of intersection of the planes.

$$\begin{aligned} 6x - 3y + z &= 5 &\implies& \quad 6x - 3y + z = 5 \\ -x + y + 5z &= 5 &\implies& \quad \underline{-6x + 6y + 30z = 30} \\ &&& \quad 3y + 31z = 35 \end{aligned}$$

Let $y = -9, z = 2 \implies x = -4 \implies (-4, -9, 2)$.

$$x = -4 - 16t, y = -9 - 31t, z = 2 + 3t$$

**70.** Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = 1 + 4t,\ y = 2t,\ z = 3 + 6t$$

$$2(1 + 4t) + 3(2t) = -5,\ t = \frac{-1}{2}$$

Substituting $t = -\frac{1}{2}$ into the parametric equations for the line we have the point of intersection $(-1, -1, 0)$. The line does not lie in the plane.

**72.** Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = 4 + 2t,\ y = -1 - 3t,\ z = -2 + 5t$$

$$5(4 + 2t) + 3(-1 - 3t) = 17,\ t = 0$$

Substituting $t = 0$ into the parametric equations for the line we have the point of intersection $(4, -1, -2)$. The line does not lie in the plane.

**74.** Point: $Q(0, 0, 0)$

Plane: $8x - 4y + z = 8$

Normal to plane: $\mathbf{n} = \langle 8, -4, 1\rangle$

Point in plane: $P\langle 1, 0, 0\rangle$

Vector: $\overrightarrow{PQ} = \langle -1, 0, 0\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-8|}{\sqrt{81}} = \frac{8}{9}$$

**76.** Point: $Q(3, 2, 1)$

Plane: $x - y + 2z = 4$

Normal to plane: $\mathbf{n} = \langle 1, -1, 2\rangle$

Point in plane: $P\langle 4, 0, 0\rangle$

Vector: $\overrightarrow{PQ} = \langle -1, 2, 1\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-1|}{\sqrt{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

**78.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 4, -4, 9\rangle$ and $\mathbf{n}_2 = \langle 4, -4, 9\rangle$. Since $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$P = (-5,0, 3)$ is a point in $4x - 4y + 9z = 7$.

$Q = (0, 0, 2)$ is a point in $4x - 4y + 9z = 18$.

$\overrightarrow{PQ} = \langle 5, 0, -1\rangle$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{11}{\sqrt{113}} = \frac{11\sqrt{113}}{113}$$

**80.** The normal vectors to the planes are $\mathbf{n}_1 = \langle 2, 0, -4\rangle$ and $\mathbf{n}_2 = \langle 2, 0, -4\rangle$. Since $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$P = (2, 0, 0)$ is a point in $2x - 4z = 4$. $Q = (5, 0, 0)$ is a point in $2x - 4z = 10$.

$$\overrightarrow{PQ} = \langle 3, 0, 0\rangle,\ D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{6}{\sqrt{20}} = \frac{3\sqrt{5}}{5}$$

**82.** $\mathbf{u} = \langle 2, 1, 2\rangle$ is the direction vector for the line.

$P = \langle 0, -3, 2\rangle$ is a point on the line (let $t = 0$).

$\overrightarrow{PQ} = \langle 1, 1, 2\rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \langle 0, 2, -1\rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

**84.** The equation of the plane containing $P(x_1, y_1, z_1)$ and having normal vector $\mathbf{n} = \langle a, b, c\rangle$ is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

You need $\mathbf{n}$ and $P$ to find the equation.

**86.** $x = a$: plane parallel to $yz$-plane containing $(a, 0, 0)$

$y = b$: plane parallel to $xz$-plane containing $(0, b, 0)$

$z = c$: plane parallel to $xy$-plane containing $(0, 0, c)$

**88.** (a) $t\mathbf{v}$ represents a line parallel to $\mathbf{v}$.

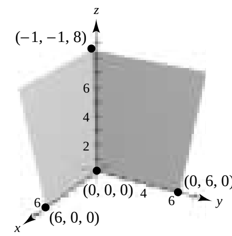
(b) $\mathbf{u} + t\mathbf{v}$ represents a line through the terminal point of $\mathbf{u}$ parallel to $\mathbf{v}$.

(c) $s\mathbf{u} + t\mathbf{v}$ represent the plane containing $\mathbf{u}$ and $\mathbf{v}$.

**90.** On one side we have the points $(0, 0, 0)$, $(6, 0, 0)$, and $(-1, -1, 8)$.

$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 0 \\ -1 & -1 & 8 \end{vmatrix} = -48\mathbf{j} - 6\mathbf{k}$$

On the adjacent side we have the points $(0, 0, 0)$, $(0, 6, 0)$, and $(-1, -1, 8)$.




$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ -1 & -1 & 8 \end{vmatrix} = 48\mathbf{i} + 6\mathbf{k}$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{36}{2340} = \frac{1}{65}$$

$$\theta = \arccos\frac{1}{65} \approx 89.1^\circ$$

**92.** False. They may be skew lines. (See Section Project)

## Section 10.6 Surfaces in Space

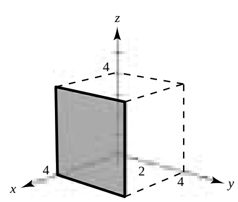
**2.** Hyperboloid of two sheets
Matches graph (e)

**4.** Elliptic cone
Matches graph (b)

**6.** Hyperbolic paraboloid
Matches graph (a)

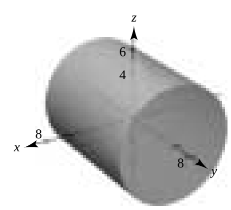
**8.** $x = 4$

Plane parallel to the $yz$-coordinate plane



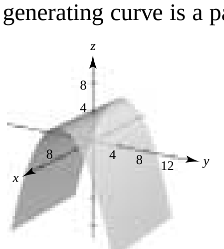
**10.** $x^2 + z^2 = 25$

The $y$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $y$-axis. The generating curve is a circle.



**12.** $z = 4 - y^2$

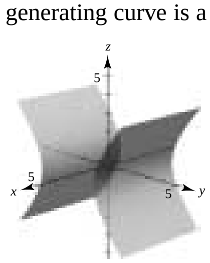
The $x$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is a parabola.



**14.** $y^2 - z^2 = 4$

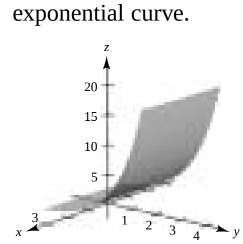
$$\frac{y^2}{4} - \frac{z^2}{4} = 1$$

The $x$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is a hyperbola.



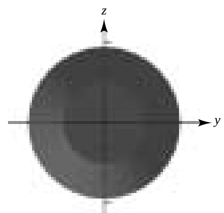
**16.** $z = e^y$

The $x$-coordinate is missing so we have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is the exponential curve.

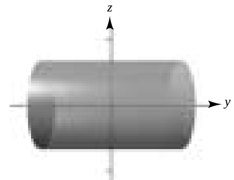


**18.** $y^2 + z^2 = 4$

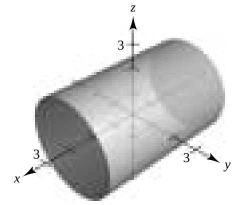
(a) From $(10, 0, 0)$:




(b) From $(0, 10, 0)$:




(c) From $(10, 10, 10)$:



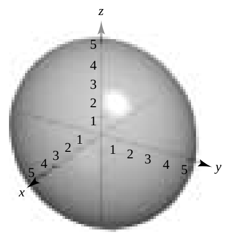

**20.** $\dfrac{x^2}{16} + \dfrac{y^2}{25} + \dfrac{z^2}{25} = 1$

Ellipsoid

$xy$-trace: $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ ellipse

$xz$-trace: $\dfrac{x^2}{16} + \dfrac{z^2}{25} = 1$ ellipse

$yz$-trace: $y^2 + z^2 = 25$ circle




**22.** $z^2 - x^2 - \dfrac{y^2}{4} = 1$

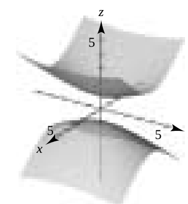
Hyperboloid of two sheets

$xy$-trace: none

$xz$-trace: $z^2 - x^2 = 1$ hyperbola

$yz$-trace: $z^2 - \dfrac{y^2}{4} = 1$ hyperbola

$z = \pm\sqrt{10}$: $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ ellipse



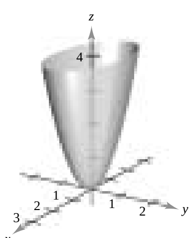

**24.** $z = x^2 + 4y^2$

Elliptic paraboloid

$xy$-trace: point $(0, 0, 0)$

$xz$-trace: $z = x^2$ parabola

$yz$-trace: $z = 4y^2$ parabola



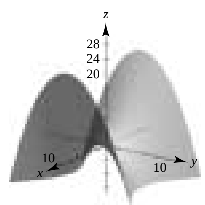

**26.** $3z = -y^2 + x^2$

Hyperbolic paraboloid

$xy$-trace: $y = \pm x$

$xz$-trace: $z = \frac{1}{3}x^2$

$yz$-trace: $z = -\frac{1}{3}y^2$



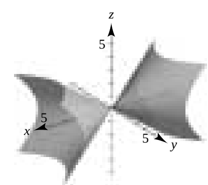

**28.** $x^2 = 2y^2 + 2z^2$

Elliptic Cone

$xy$-trace: $x = \pm\sqrt{2}y$

$xz$-trace: $x = \pm\sqrt{2}z$

$yz$-trace: point: $(0, 0, 0)$




**30.**

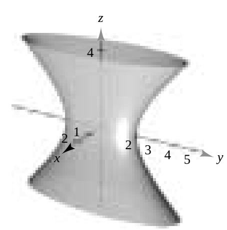
$$9x^2 + y^2 - 9z^2 - 54x - 4y - 54z + 4 = 0$$

$$9(x^2 - 6x + 9) + (y^2 - 4y + 4) - 9(z^2 + 6z + 9) = 81 + 4 - 81$$

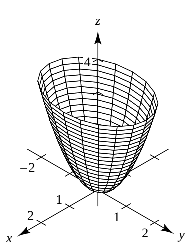
$$9(x - 3)^2 + (y - 2)^2 - 9(z + 3)^2 = 4$$

$$\frac{(x - 3)^2}{4/9} + \frac{(y - 2)^2}{4} - \frac{(z + 3)^2}{4/9} = 1$$

Hyperboloid of one sheet with center $(3, 2, -3)$.

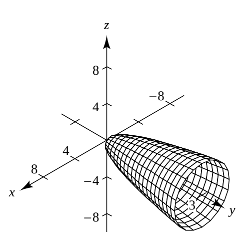



**32.** $z = x^2 + 0.5y^2$



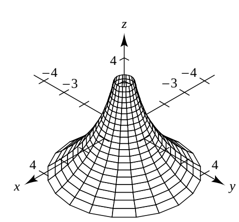

**34.** $z^2 = 4y - x^2$

$z = \pm\sqrt{4y - x^2}$

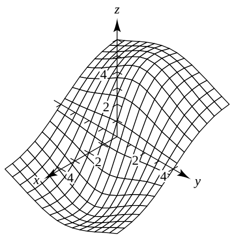



**36.** $x^2 + y^2 = e^{-z}$

$-\ln(x^2 + y^2) = z$

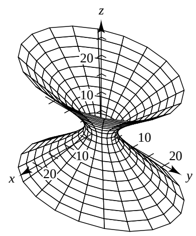


**38.** $z = \dfrac{-x}{8 + x^2 + y^2}$




**40.** $9x^2 + 4y^2 - 8z^2 = 72$

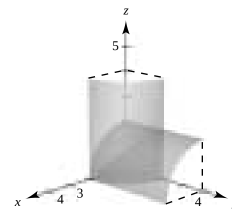
$z = \pm\sqrt{\dfrac{9}{8}x^2 + \dfrac{1}{2}y^2 - 9}$




**42.** $z = \sqrt{4 - x^2}$

$y = \sqrt{4 - x^2}$

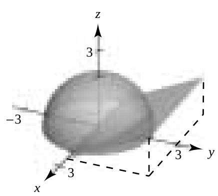
$x = 0, y = 0, z = 0$




**44.** $z = \sqrt{4 - x^2 - y^2}$

$y = 2z$

$z = 0$




**46.** $x^2 + z^2 = [r(y)]^2$ and $z = r(y) = 3y$; therefore,

$x^2 + z^2 = 9y^2$.

**48.** $y^2 + z^2 = [r(x)]^2$ and $z = r(x) = \dfrac{1}{2}\sqrt{4 - x^2}$; therefore,

$y^2 + z^2 = \dfrac{1}{4}(4 - x^2),\ x^2 + 4y^2 + 4z^2 = 4$.

**50.** $x^2 + y^2 = [r(z)]^2$ and $y = r(z) = e^z$; therefore,

$x^2 + y^2 = e^{2z}$.

**52.** $x^2 + z^2 = \cos^2 y$

Equation of generating curve:
$x = \cos y$ or $z = \cos y$

**54.** The trace of a surface is the intersection of the surface with a plane. You find a trace by setting one variable equal to a constant, such as $x = 0$ or $z = 2$.

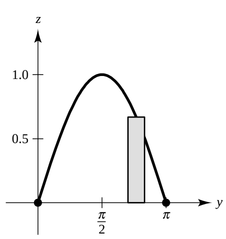
**56.** About $x$-axis: $y^2 + z^2 = [r(x)]^2$

About $y$-axis: $x^2 + z^2 = [r(y)]^2$

About $z$-axis: $x^2 + y^2 = [r(z)]^2$

**58.** $V = 2\pi \displaystyle\int_0^{\pi} y \sin y\, dy$

$= 2\pi\Big[\sin y - y\cos y\Big]_0^{\pi} = 2\pi^2$




**60.** $z = \dfrac{x^2}{2} + \dfrac{y^2}{4}$

(a) When $y = 4$ we have $z = \dfrac{x^2}{2} + 4,\ 4\left(\dfrac{1}{2}\right)(z - 4) = x^2$.

Focus: $\left(0, 4, \dfrac{9}{2}\right)$

(b) When $x = 2$ we have

$z = 2 + \dfrac{y^2}{4},\ 4(z - 2) = y^2$.

Focus: $(2, 0, 3)$

**62.** If $(x, y, z)$ is on the surface, then

$z^2 = x^2 + y^2 + (z - 4)^2$

$z^2 = x^2 + y^2 + z^2 - 8z + 16$

$8z = x^2 + y^2 + 16 \Longrightarrow z = \dfrac{x^2}{8} + \dfrac{y^2}{8} + 2$

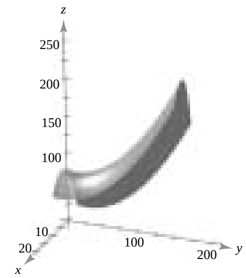
Elliptic paraboloid shifted up 2 units. Traces parallel to $xy$-plane are circles.

**64.** $z = -0.775x^2 + 0.007y^2 + 22.15x - 0.54y - 45.4$

(a)

| Year | 1980 | 1985 | 1990 | 1995 | 1996 | 1997 |
|---|---|---|---|---|---|---|
| $z$ | 37.5 | 72.2 | 111.5 | 185.2 | 200.1 | 214.6 |
| Model | 37.8 | 72.0 | 112.2 | 185.8 | 204.5 | 214.7 |

(b)




(c) For $y$ constant, the traces parallel to the $xz$-plane are concave downward. That is, for fixed $y$ (public assistance), the rate of increase of $z$ (Medicare) is decreasing with respect to $x$ (worker's compensation).

(d) The traces parallel to the $yz$-plane ($x$ constant) are concave upward. That is, for fixed $x$ (worker's compensation), the rate of increase of $z$ (Medicare) is increasing with respect to $y$ (public assistance).

**66.** Equating twice the first equation with the second equation,

$$2x^2 + 6y^2 - 4z^2 + 4y - 8 = 2x^2 + 6y^2 - 4z^2 - 3x - 2$$
$$4y - 8 = -3x - 2$$
$$3x + 4y = 6, \text{ a plane}$$

## Section 10.7 Cylindrical and Spherical Coordinates

**2.** $\left(4, \frac{\pi}{2}, -2\right)$, cylindrical

$x = 4\cos\frac{\pi}{2} = 0$

$y = 4\sin\frac{\pi}{2} = 4$

$z = -2$

$(0, 4, -2)$, rectangular

**4.** $\left(6, -\frac{\pi}{4}, 2\right)$, cylindrical

$x = 6\cos\left(-\frac{\pi}{4}\right) = 3\sqrt{2}$

$y = 6\sin\left(-\frac{\pi}{4}\right) = -3\sqrt{2}$

$z = 2$

$\left(3\sqrt{2}, -3\sqrt{2}, 2\right)$

**6.** $\left(1, \frac{3\pi}{2}, 1\right)$, cylindrical

$x = \cos\frac{3\pi}{2} = 0$

$y = \sin\frac{3\pi}{2} = -1$

$z = 1$

$(0, -1, 1)$, rectangular

**8.** $\left(2\sqrt{2}, -2\sqrt{2}, 4\right)$, rectangular

$r = \sqrt{\left(2\sqrt{2}\right)^2 + \left(-2\sqrt{2}\right)^2} = 4$

$\theta = \arctan(-1) = -\frac{\pi}{4}$

$z = 4$

$\left(4, -\frac{\pi}{4}, 4\right)$, cylindrical

**10.** $\left(2\sqrt{3}, -2, 6\right)$, rectangular

$r = \sqrt{12 + 4} = 4$

$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$

$z = 1$

$\left(4, -\frac{\pi}{6}, 1\right)$, cylindrical

**12.** $(-3, 2, -1)$, rectangular

$r = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

$\theta = \arctan\left(\frac{-2}{3}\right) = -\arctan\frac{2}{3}$

$z = -1$

$\left(\sqrt{13}, -\arctan\frac{2}{3}, -1\right)$, cylindrical

**14.** $z = x^2 + y^2 - 2$ rectangular equation

$z = r^2 - 2$ cylindrical equation

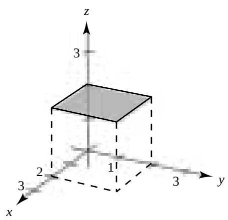
**16.** $x^2 + y^2 = 8x$ rectangular equation

$r^2 = 8r\cos\theta$

$r = 8\cos\theta$ cylindrical equation

**18.** $z = 2$

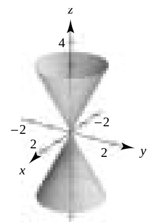
Same




**20.** $r = \dfrac{z}{2}$

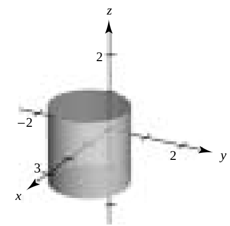
$$\sqrt{x^2 + y^2} = \frac{z}{2}$$

$$x^2 + y^2 - \frac{z^2}{4} = 0$$



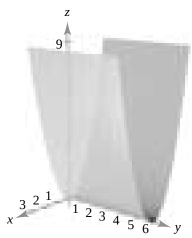

**22.** $r = 2\cos\theta$

$$r^2 = 2r\cos\theta$$
$$x^2 + y^2 = 2x$$
$$x^2 + y^2 - 2x = 0$$
$$(x - 1)^2 + y^2 = 1$$




**24.** $z = r^2\cos^2\theta$

$$z = x^2$$




**26.** $(1, 1, 1)$, rectangular

$$\rho = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\theta = \arctan 1 = \frac{\pi}{4}$$

$$\phi = \arccos\frac{1}{\sqrt{3}}$$

$\left(\sqrt{3}, \dfrac{\pi}{4}, \arccos\dfrac{1}{\sqrt{3}}\right)$, spherical

**28.** $\left(2, 2, 4\sqrt{2}\right)$, rectangular

$$\rho = \sqrt{2^2 + 2^2 + \left(4\sqrt{2}\right)^2} = 2\sqrt{10}$$

$$\theta = \arctan 1 = \frac{\pi}{4}$$

$$\phi = \arccos\frac{2}{\sqrt{5}}$$

$\left(2\sqrt{10}, \dfrac{\pi}{4}, \arccos\dfrac{2}{\sqrt{5}}\right)$, spherical

**30.** $(-4, 0, 0)$, rectangular

$$\rho = \sqrt{(-4)^2 + 0^2 + 0^2} = 4$$

$$\theta = \pi$$

$$\phi = \arccos 0 = \frac{\pi}{2}$$

$\left(4, \pi, \dfrac{\pi}{2}\right)$, spherical

**32.** $\left(12, \dfrac{3\pi}{4}, \dfrac{\pi}{9}\right)$, spherical

$$x = 12\sin\frac{\pi}{9}\cos\frac{3\pi}{4} \approx -2.902$$

$$y = 12\sin\frac{\pi}{9}\sin\frac{3\pi}{4} \approx 2.902$$

$$z = 12\cos\frac{\pi}{9} \approx 11.276$$

$(-2.902, 2.902, 11.276)$, rectangular

**34.** $\left(9, \dfrac{\pi}{4}, \pi\right)$, spherical

$$x = 9\sin\pi\cos\frac{\pi}{4} = 0$$

$$y = 9\sin\pi\sin\frac{\pi}{4} = 0$$

$$z = 9\cos\pi = -9$$

$(0, 0, -9)$, rectangular

**36.** $\left(6, \pi, \dfrac{\pi}{2}\right)$, spherical

$$x = 6\sin\frac{\pi}{2}\cos\pi = -6$$

$$y = 6\sin\frac{\pi}{2}\sin\pi = 0$$

$$z = 6\cos\frac{\pi}{2} = 0$$

$(-6, 0, 0)$, rectangular

**38.** (a) Programs will vary.

(b) $(\rho, \theta, \phi) = (5, 1, 0.5)$

$(x, y, z) = (1.295, 2.017, 4.388)$

**40.** $x^2 + y^2 - 3z^2 = 0$ rectangular equation

$$x^2 + y^2 + z^2 = 4z^2$$

$$\rho^2 = 4\rho^2 \cos^2 \phi$$

$$1 = 4\cos^2 \phi$$

$$\cos \phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{3} \quad \text{(cone) spherical equation}$$

**42.** $x = 10$ rectangular equation

$$\rho \sin \phi \cos \theta = 10$$

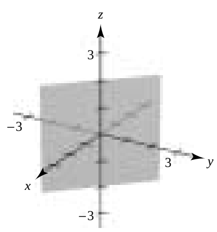
$$\rho = 10 \csc \phi \sec \theta \quad \text{spherical equation}$$

**44.** $\theta = \dfrac{3\pi}{4}$

$$\tan \theta = \frac{y}{x}$$

$$-1 = \frac{y}{x}$$

$$x + y = 0$$



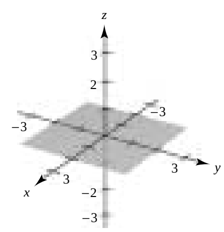

**46.** $\phi = \dfrac{\pi}{2}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$0 = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = 0$$

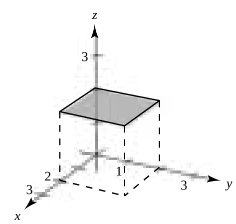
*xy*-plane




**48.** $\rho = 2 \sec \phi$

$$\rho \cos \phi = 2$$

$$z = 2$$

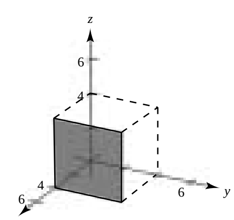



**50.** $\rho = 4 \csc \phi \sec \phi$

$$= \frac{4}{\sin \phi \cos \theta}$$

$$\rho \sin \phi \cos \theta = 4$$

$$x = 4$$




**52.** $\left(3, -\dfrac{\pi}{4}, 0\right)$, cylindrical

$$\rho = \sqrt{3^2 + 0^2} = 3$$

$$\theta = -\frac{\pi}{4}$$

$$\phi = \arccos\left(\frac{0}{9}\right) = \frac{\pi}{2}$$

$\left(3, -\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$, spherical

**54.** $\left(2, \dfrac{2\pi}{3}, -2\right)$, cylindrical

$$\rho = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta = \frac{2\pi}{3}$$

$$\phi = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

$\left(2\sqrt{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}\right)$, spherical

**56.** $\left(-4, \dfrac{\pi}{3}, 4\right)$, cylindrical

$$\rho = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{3}$$

$$\phi = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$\left(4\sqrt{2}, \dfrac{\pi}{3}, \dfrac{\pi}{4}\right)$, spherical

**58.** $\left(4, \dfrac{\pi}{2}, 3\right)$, cylindrical

$$\rho = \sqrt{4^2 + 3^2} = 5$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos \frac{3}{5}$$

$\left(5, \dfrac{\pi}{2}, \arccos \dfrac{3}{5}\right)$, spherical

**60.** $\left(4, \dfrac{\pi}{18}, \dfrac{\pi}{2}\right)$, spherical

$$r = 4 \sin \frac{\pi}{2} = 4$$

$$\theta = \frac{\pi}{18}$$

$$z = 4 \cos \frac{\pi}{2} = 0$$

$\left(4, \dfrac{\pi}{18}, 0\right)$, cylindrical

**62.** $\left(18, \frac{\pi}{3}, \frac{\pi}{3}\right)$, spherical

$r = \rho \sin \phi = 18 \sin \frac{\pi}{3} = 9$

$\theta = \frac{\pi}{3}$

$z = \rho \cos \phi = 18 \cos \frac{\pi}{3} = 9\sqrt{3}$

$\left(9, \frac{\pi}{3}, 9\sqrt{3}\right)$, cylindrical

**64.** $\left(5, -\frac{5\pi}{6}, \pi\right)$, spherical

$r = 5 \sin \pi = 0$

$\theta = -\frac{5\pi}{6}$

$z = 5 \cos \pi = -5$

$\left(0, -\frac{5\pi}{6}, -5\right)$, cylindrical

**66.** $\left(7, \frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$r = 7 \sin \frac{3\pi}{4} = \frac{7\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}$

$z = 7 \cos \frac{3\pi}{4} = -\frac{7\sqrt{2}}{2}$

$\left(\frac{7\sqrt{2}}{2}, \frac{\pi}{4}, -\frac{7\sqrt{2}}{2}\right)$, cylindrical

| | Rectangular | Cylindrical | Spherical |
|---|---|---|---|
| **68.** | $(6, -2, -3)$ | $(6.325, -0.322, -3)$ | $(7.000, -0.322, 2.014)$ |
| **70.** | $(7.317, -6.816, 6)$ | $(10, -0.75, 6)$ | $(11.662, -0.750, 1.030)$ |
| **72.** | $(6.115, 1.561, 4.052)$ | $(6.311, 0.25, 4.052)$ | $(7.5, 0.25, 1)$ |
| **74.** | $(3\sqrt{2}, 3\sqrt{2}, -3)$ | $(6, 0.785, -3)$ | $(6.708, 0.785, 2.034)$ |
| **76.** | $(0, -5, 4)$ | $(5, -1.571, 4)$ | $(6.403, -1.571, 0.896)$ |
| **78.** | $(-1.732, 1, 3)$ | $\left(-2, \frac{11\pi}{6}, 3\right)$ | $(3.606, 2.618, 0.588)$ |
| | $\left[\text{Note: Use the cylindrical coordinate } \left(2, \frac{5\pi}{6}, 3\right)\right]$ | | |
| **80.** | $(2.207, 7.949, -4)$ | $(8.25, 1.3, -4)$ | $(9.169, 1.3, 2.022)$ |

**82.** $\theta = \frac{\pi}{4}$

Plane

Matches graph (e)

**84.** $\phi = \frac{\pi}{4}$

Cone

Matches graph (a)

**86.** $\rho = 4 \sec \phi, z = \rho \cos \phi = 4$

Plane

Matches graph (b)

**88.** $r = a$ Cylinder with $z$-axis symmetry

$\theta = b$ Plane perpendicular to $xy$-plane

$z = c$ Plane parallel to $xy$-plane

**90.** $\rho = a$ Sphere

$\theta = b$ Vertical half-plane

$\phi = c$ Half-cone

**92.** $4(x^2 + y^2) = z^2$

(a) $4r^2 = z^2, 2r = z$

(b) $4(\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta) = \rho^2 \cos^2 \phi,$

$4 \sin^2 \phi = \cos^2 \phi, \tan^2 \phi = \frac{1}{4},$

$\tan \phi = \frac{1}{2}, \phi = \arctan \frac{1}{2}$

**94.** $x^2 + y^2 = z$

(a) $r^2 = z$

(b) $\rho^2 \sin^2 \phi = \rho \cos \phi, \rho \sin^2 \phi = \cos \phi,$

$\rho = \frac{\cos \phi}{\sin^2 \phi}, \rho = \csc \phi \cot \phi$

**96.** $x^2 + y^2 = 16$

(a) $r^2 = 16, r = 4$

(b) $\rho^2 \sin^2 \phi = 16, \rho^2 \sin^2 \phi - 16 = 0,$
$(\rho \sin \phi - 4)(\rho \sin \phi + 4) = 0, \rho = 4 \csc \phi$

**98.** $y = 4$

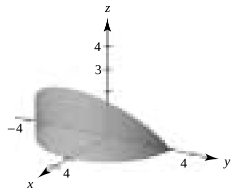
(a) $r \sin \theta = 4, r = 4 \csc \theta$

(b) $\rho \sin \phi \sin \theta = 4, \rho = 4 \csc \phi \csc \theta$

**100.** $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

$0 \le r \le 3$

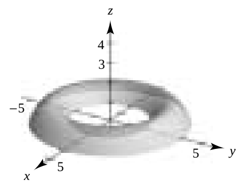
$0 \le z \le r\cos\theta$




**102.** $0 \le \theta \le 2\pi$

$2 \le r \le 4$

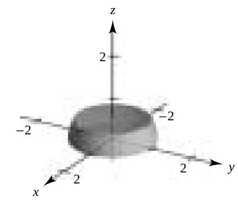
$z^2 \le -r^2 + 6r - 8$




**104.** $0 \le \theta \le 2\pi$

$\dfrac{\pi}{4} \le \phi \le \dfrac{\pi}{2}$

$0 \le \rho \le 1$



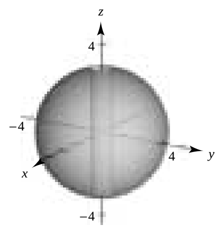

**106.** Cylindrical: $0.75 \le r \le 1.25, z = 8$

**108.** Cylindrical

$\frac{1}{2} \le r \le 3$

$0 \le \theta \le 2\pi$

$-\sqrt{9 - r^2} \le z \le \sqrt{9 - r^2}$




**110.** $\rho = 2\sec\phi \implies \rho\cos\phi = 2 \implies z = 2$ plane

$\rho = 4$ sphere

The intersection of the plane and the sphere is a circle.

# Review Exercises for Chapter 10

**2.** $P = (-2, -1),\ Q = (5, -1)\ R = (2, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 7, 0\rangle = 7\mathbf{i},\ \mathbf{v} = \overrightarrow{PR} = \langle 4, 5\rangle = 4\mathbf{i} + 5\mathbf{j}$

(b) $\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$

(c) $2\mathbf{u} + \mathbf{v} = 14\mathbf{i} + (4\mathbf{i} + 5\mathbf{j}) = 18\mathbf{i} + 5\mathbf{j}$

**4.** $\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j} = \dfrac{1}{2}\cos 225^\circ\,\mathbf{i} + \dfrac{1}{2}\sin 225^\circ\,\mathbf{j}$

$$= -\frac{\sqrt{2}}{4}\mathbf{i} + \frac{\sqrt{2}}{4}\mathbf{j}$$

**6.** (a) The length of cable $POQ$ is $L$.

$$\overrightarrow{OQ} = 9\mathbf{i} - y\mathbf{j}$$

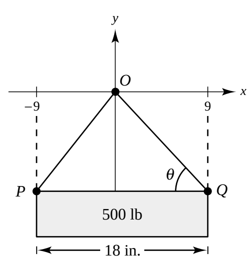
$$L = 2\sqrt{9^2 + y^2} \implies \sqrt{\frac{L^2}{4} - 81} = y$$

Tension: $T = c\|\overrightarrow{OQ}\| = c\sqrt{81 + y^2}$

Also,

$$cy = 250 \implies T = \frac{250}{y}\sqrt{81 + y^2} \implies T = \frac{250}{\sqrt{(L^2/4) - 81}} \cdot \frac{L}{2} = \frac{250L}{\sqrt{L^2 - 324}}$$

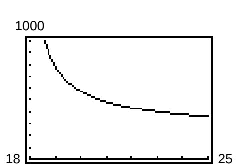
Domain: $L > 18$ inches




(b)

| $L$ | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|
| $T$ | 780.9 | 573.54 | 485.36 | 434.81 | 401.60 | 377.96 | 360.24 |

(c)




(d) The line $T = 400$ intersects the curve at

$L = 23.06$ inches.

(e) $\displaystyle\lim_{L\to\infty} T = 250$

The maximum tension is 250 pounds in each side of the cable since the total weight is 500 pounds.

**8.** $x = z = 0,\ y = -7$: $(0, -7, 0)$

**10.** Looking towards the $xy$-plane from the positive $z$-axis. The point is either in the second quadrant $(x < 0, y > 0)$ or in the fourth quadrant $(x > 0, y < 0)$. The $z$-coordinate can be any number.

**12.** Center: $\left(\dfrac{0+4}{2}, \dfrac{0+6}{2}, \dfrac{4+0}{2}\right) = (2, 3, 2)$

Radius: $\sqrt{(2-0)^2 + (3-0)^2 + (2-4)^2} = \sqrt{4+9+4} = \sqrt{17}$

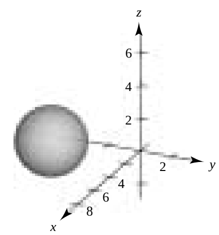
$(x-2)^2 + (y-3)^2 + (z-2)^2 = 17$

**14.** $(x^2 - 10x + 25) + (y^2 + 6y + 9) + (z^2 - 4z + 4) = -34 + 25 + 9 + 4$

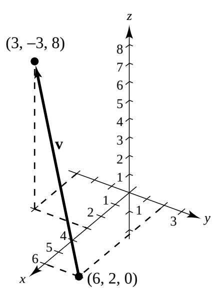
$(x-5)^2 + (y+3)^2 + (z-2)^2 = 4$

Center: $(5, -3, 2)$

Radius: $2$




**16.** $\mathbf{v} = \langle 3-6, -3-2, 8-0\rangle = \langle -3, -5, 8\rangle$




**18.** $\mathbf{v} = \langle 8-5, -5+4, 5-7\rangle = \langle 3, -1, -2\rangle$

$\mathbf{w} = \langle 11-5, 6+4, 3-7\rangle = \langle 6, 10, -4\rangle$

Since $\mathbf{v}$ and $\mathbf{w}$ are not parallel, the points do not lie in a straight line.

**20.** $8\dfrac{\langle 6, -3, 2\rangle}{\sqrt{49}} = \dfrac{8}{7}\langle 6, -3, 2\rangle = \left\langle \dfrac{48}{7}, -\dfrac{24}{7}, \dfrac{16}{7}\right\rangle$

**22.** $P = (2, -1, 3),\ Q = (0, 5, 1),\ R = (5, 5, 0)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -2, 6, -2\rangle = -2\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$,
$\mathbf{v} = \overrightarrow{PR} = \langle 3, 6, -3\rangle = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

(b) $\mathbf{u}\cdot\mathbf{v} = (-2)(3) + (6)(6) + (-2)(-3) = 36$

(c) $\mathbf{v}\cdot\mathbf{v} = 9 + 36 + 9 = 54$

**24.** $\mathbf{u} = \langle -4, 3, -6\rangle,\ \mathbf{v} = \langle 16, -12, 24\rangle$

Since $\mathbf{v} = -4\mathbf{u}$, the vectors are parallel.

**26.** $\mathbf{u} = \langle 4, -1, 5\rangle,\ \mathbf{v} = \langle 3, 2, -2\rangle$

$\mathbf{u}\cdot\mathbf{v} = 0 \implies$ is orthogonal to $\mathbf{v}$.

$\theta = \dfrac{\pi}{2}$

**28.** $\mathbf{u} = \langle 1, 0, -3\rangle$

$\mathbf{v} = \langle 2, -2, 1\rangle$

$\mathbf{u}\cdot\mathbf{v} = -1$

$\|\mathbf{u}\| = \sqrt{10}$

$\|\mathbf{v}\| = 3$

$\cos\theta = \dfrac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{1}{3\sqrt{10}}$

$\theta \approx 83.9^\circ$

**30.** $W = \mathbf{F}\cdot\overrightarrow{PQ} = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\|\cos\theta = (75)(8)\cos 30^\circ$

$= 300\sqrt{3}$ ft $\cdot$ lb

**In Exercises 32–40, $\mathbf{u} = \langle 3, -2, 1\rangle$, $\mathbf{v} = \langle 2, -4, -3\rangle$, $\mathbf{w} = \langle -1, 2, 2\rangle$.**

**32.** $\cos\theta = \dfrac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{11}{\sqrt{14}\sqrt{29}}$

$$\theta = \arccos\left(\frac{11}{\sqrt{14}\sqrt{29}}\right) \approx 56.9^\circ$$

**34.** Work $= |\mathbf{u}\cdot\mathbf{w}| = |-3 - 4 + 2| = 5$

**36.** $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -2 & 1 \end{vmatrix} = -10\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}$

Thus, $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$.

**38.** $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \langle 3, -2, 1\rangle \times \langle 1, -2, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$

$\mathbf{u}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = -6\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$

$(\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w}) = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \mathbf{u}\times(\mathbf{v}+\mathbf{w})$

**40.** Area triangle $= \dfrac{1}{2}\|\mathbf{v}\times\mathbf{w}\| = \dfrac{1}{2}\sqrt{(-2)^2 + (-1)^2} = \dfrac{\sqrt{5}}{2}$ (See Exercise 35)

**42.** $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2(5) = 10$

**44.** Direction numbers: 1, 1, 1

(a) $x = 1 + t,\ y = 2 + t,\ z = 3 + t$

(b) $x - 1 = y - 2 = z - 3$

**46.** $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 1 \\ -3 & 1 & 4 \end{vmatrix} = -21\mathbf{i} - 11\mathbf{j} - 13\mathbf{k}$

Direction numbers: 21, 11, 13

(a) $x = 21t,\ y = 1 + 11t,\ z = 4 + 13t$

(b) $\dfrac{x}{21} = \dfrac{y-1}{11} = \dfrac{z-4}{13}$

**48.** $P = (-3, -4, 2),\ Q = (-3, 4, 1),\ R = (1, 1, -2)$

$\overrightarrow{PQ} = \langle 0, 8, -1\rangle,\ \overrightarrow{PR} = \langle 4, 5, -4\rangle$

$\mathbf{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix} = -27\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$

$-27(x+3) - 4(y+4) - 32(z-2) = 0$

$$27x + 4y + 32z = -33$$

**50.** The normal vectors to the planes are the same,

$$\mathbf{n} = \langle 5, -3, 1 \rangle.$$

Choose a point in the first plane, $P = (0, 0, 2)$. Choose a point in the second plane, $Q = (0, 0, -3)$.

$$\overrightarrow{PQ} = \langle 0, 0, -5 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-5|}{\sqrt{35}} = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7}$$

**52.** $Q(-5, 1, 3)$ point

$\mathbf{u} = \langle 1, -2, -1 \rangle$ direction vector

$P = (1, 3, 5)$ point on line

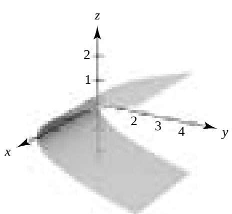
$$\overrightarrow{PQ} = \langle -6, -2, -2 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -2 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \langle -2, -8, 14 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{264}}{\sqrt{6}} = 2\sqrt{11}$$

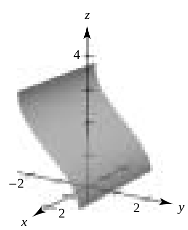
**54.** $y = z^2$

Since the $x$-coordinate is missing, we have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is a parabola in the $yz$-coordinate plane.




**56.** $y = \cos z$

Since the $x$-coordinate is missing, we have a cylindrical surface with rulings parallel to the $x$-axis. The generating curve is $y = \cos z$.




**58.** $16x^2 + 16y^2 - 9z^2 = 0$

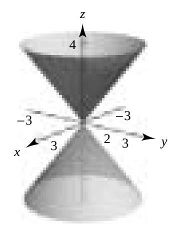
Cone

$xy$-trace: point $(0,0,0)$

$xz$-trace: $z = \pm\dfrac{4x}{3}$

$yz$-trace: $z = \pm\dfrac{4y}{3}$

$z = 4,\ x^2 + y^2 = 9$



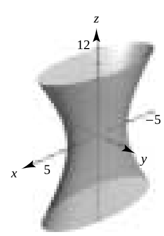

**60.** $\dfrac{x^2}{25} + \dfrac{y^2}{4} - \dfrac{z^2}{100} = 1$

Hyperboloid of one sheet

$xy$-trace: $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$

$xz$-trace: $\dfrac{x^2}{25} - \dfrac{z^2}{100} = 1$

$yz$-trace: $\dfrac{y^2}{4} - \dfrac{z^2}{100} = 1$




**62.** Let $y = r(x) = 2\sqrt{x}$ and revolve the curve about the $x$-axis.

**64.** $\left(\dfrac{\sqrt{3}}{4}, \dfrac{3}{4}, \dfrac{3\sqrt{3}}{2}\right)$, rectangular

(a) $r = \sqrt{\left(\dfrac{\sqrt{3}}{4}\right)^2 + \left(\dfrac{3}{4}\right)^2} = \dfrac{\sqrt{3}}{2},\ \theta = \arctan\sqrt{3} = \dfrac{\pi}{3},\ z = \dfrac{3\sqrt{3}}{2},\ \left(\dfrac{\sqrt{3}}{2}, \dfrac{\pi}{2}, \dfrac{3\sqrt{3}}{2}\right)$, cylindrical

(b) $\rho = \sqrt{\left(\dfrac{\sqrt{3}}{4}\right)^2 + \left(\dfrac{3}{4}\right)^2 + \left(\dfrac{3\sqrt{3}}{2}\right)^2} = \dfrac{\sqrt{30}}{2},\ \theta = \dfrac{\pi}{3},\ \phi = \arccos\dfrac{3}{\sqrt{10}},\ \left(\dfrac{\sqrt{30}}{2}, \dfrac{\pi}{3}, \arccos\dfrac{3}{\sqrt{10}}\right)$, spherical

**66.** $\left(81, -\dfrac{5\pi}{6}, 27\sqrt{3}\right)$, cylindrical

$\rho = \sqrt{6561 + 2187} = 54\sqrt{3}$

$\theta = -\dfrac{5\pi}{6}$

$\phi = \arccos\left(\dfrac{27\sqrt{3}}{54\sqrt{3}}\right) = \arccos\dfrac{1}{2} = \dfrac{\pi}{3}$

$\left(54\sqrt{3}, -\dfrac{5\pi}{6}, \dfrac{\pi}{3}\right)$, spherical

**68.** $\left(12, -\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$, spherical

$r^2 = \left(12 \sin\left(\dfrac{2\pi}{3}\right)\right)^2 \Rightarrow r = 6\sqrt{3}$

$\theta = -\dfrac{\pi}{2}$

$z = \rho\cos\phi = 12\cos\left(\dfrac{2\pi}{3}\right) = -6$

$\left(6\sqrt{3}, -\dfrac{\pi}{2}, -6\right)$, cylindrical

**70.** $x^2 + y^2 + z^2 = 16$

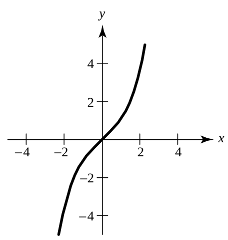
(a) Cylindrical: $r^2 + z^2 = 16$

(b) Spherical: $\rho = 4$

## Problem Solving for Chapter 10

**2.** $f(x) = \displaystyle\int_0^x \sqrt{t^4 + 1}\, dt$

(a)




(b) $f'(x) = \sqrt{x^4 + 1}$

$f'(0) = 1 = \tan\theta$

$\theta = \dfrac{\pi}{4}$

$\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2} \right\rangle$

(c) $\pm\left\langle \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2} \right\rangle$

(d) The line is $y = x$: $x = t, y = t$.

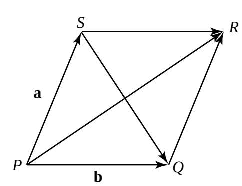
**4.** Label the figure as indicated.

$\overrightarrow{PR} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{SQ} = \mathbf{b} - \mathbf{a}$

$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 = 0$, because

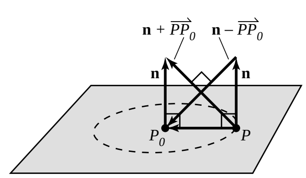
$\|\mathbf{a}\| = \|\mathbf{b}\|$ in a rhombus.




**6.** $(\mathbf{n} + \overrightarrow{PP_0}) \perp (\mathbf{n} - \overrightarrow{PP_0})$

Figure is a square.

Thus, $\|\overrightarrow{PP_0}\| = \|\mathbf{n}\|$ and the points $P$ form a circle of radius $\|\mathbf{n}\|$ in the plane with center at $P$.



**8.** (a) $V = 2\int_0^r \pi(r^2 - x^2)\,dx = 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = \frac{4}{3}\pi r^3$

(b) At height $z = d > 0$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{d^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{d^2}{c^2} = \frac{c^2 - d^2}{c^2}$$

$$\frac{x^2}{\dfrac{a^2(c^2 - d^2)}{c^2}} + \frac{y^2}{\dfrac{b^2(c^2 - d^2)}{c^2}} = 1.$$

$$\text{Area} = \pi\sqrt{\left(\frac{a^2(c^2 - d^2)}{c^2}\right)\left(\frac{b^2(c^2 - d^2)}{c^2}\right)} = \frac{\pi ab}{c^2}(c^2 - d^2)$$

$$V = 2\int_0^c \frac{\pi ab}{c^2}(c^2 - d^2)\,dd$$

$$= \frac{2\pi ab}{c^2}\left[c^2d - \frac{d^3}{3}\right]_0^c$$

$$= \frac{4}{3}\pi abc$$

**10.** (a) $r = 2\cos\theta$

Cylinder

(b) $z = r^2\cos 2\theta$

$z^2 = x^2 - y^2$

Hyperbolic paraboloid

**12.** $x = -t + 3, y = \frac{1}{2}t + 1, z = 2t - 1; Q = (4, 3, s)$

(a) $\mathbf{u} = \langle -2, 1, 4\rangle$ direction vector for line

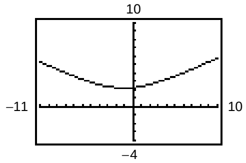
$P = (3, 1, -1)$ point on line

$\overrightarrow{PQ} = \langle 1, 2, s + 1\rangle$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & s+1 \\ -2 & 1 & 4 \end{vmatrix} = (7 - s)\mathbf{i} + (-6 - 2s)\mathbf{j} + 5\mathbf{k}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{(7 - s)^2 + (-6 - 2s)^2 + 25}}{\sqrt{21}}$$

(b)




The minimum is $D \approx 2.2361$ at $s = -1$.

(c) Yes, there are slant asymptotes. Using $s = x$, we have

$$D(s) = \frac{1}{\sqrt{21}}\sqrt{5x^2 + 10x + 110} = \frac{\sqrt{5}}{\sqrt{21}}\sqrt{x^2 + 2x + 22}$$

$$= \frac{\sqrt{5}}{\sqrt{21}}\sqrt{(x + 1)^2 + 21} \rightarrow \pm\sqrt{\frac{5}{21}}(x + 1)$$

$y = \pm\frac{\sqrt{105}}{21}(s + 1)$ slant asymptotes.

**14.** (a) The tension $T$ is the same in each tow line.

$$6000\mathbf{i} = T(\cos 20° + \cos(-20))\mathbf{i} + T(\sin 20° + \sin(-20°))\mathbf{j}$$

$$= 2T\cos 20°\mathbf{i}$$

$$\Longrightarrow T = \frac{6000}{2\cos 20°} \approx 3192.5 \text{ lbs}$$

(b) As in part (a), $6000\mathbf{i} = 2T\cos\theta$

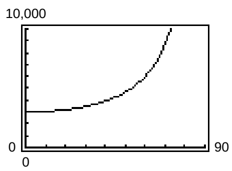
$$\Longrightarrow T = \frac{3000}{\cos\theta}$$

Domain: $0 < \theta < 90°$

(c)

| $\theta$ | 10° | 20° | 30° | 40° | 50° | 60° |
|---|---|---|---|---|---|---|
| $T$ | 3046.3 | 3192.5 | 3464.1 | 3916.2 | 4667.2 | 6000.0 |

(d)




(e) As $\theta$ increases, there is less force applied in the direction of motion.

**16.** (a) Los Angeles: $(4000, -118.24°, 55.95°)$

Rio de Janeiro: $(4000, -43.22°, 112.90°)$

(b) Los Angeles: $x = 4000\sin 55.95°\cos(-118.24°)$

$y = 4000\sin 55.95°\sin(-118.24°)$

$z = 4000\cos 55.95°$

$(-1568.2, -2919.7, 2239.7)$

Rio de Janeiro: $x = (4000\sin 112.90°\cos(-43.22°)$

$y = 4000\sin 112.90°\sin(-43.22°)$

$z = 4000\cos 112.90°$

$(2685.2, -2523.3, -1556.5)$

(c) $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{(-1568.2)(2685.2) + (-2919.7)(-2523.3) + (2239.7)(-1556.5)}{(4000)(4000)}$

$\theta \approx 91.18° \approx 1.59$ radians

(d) $s = 4000(1.59) \approx 6366$ miles

**—CONTINUED—**

**16. —CONTINUED—**

(e) For Boston and Honolulu:

a. Boston: $(4000, -71.06°, 47.64°)$

Honolulu: $(4000, -157.86°, 68.69°)$

b. Boston: $x = 4000 \sin 47.64° \cos(-71.06°)$

$y = 4000 \sin 47.64° \sin(-71.06°)$

$z = 4000 \cos 47.64°$

$(959.4, -2795.7, 2695.1)$

Honolulu: $x = (4000 \sin 68.69° \cos(-157.86°)$

$y = 4000 \sin 68.69° \sin(-157.86°)$

$z = 4000 \cos 68.69°$

$(-3451.7, -1404.4, 1453.7)$

(f) $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{(959.4)(-3451.7) + (-2795.7)(-1404.4) + (2695.1)(1453.7)}{(4000)(4000)}$

$\theta \approx 73.5° \approx 1.28$ radians

(g) $s = 4000(1.28) \approx 5120$ miles

**18.** Assume one of $a$, $b$, $c$, is not zero, say $a$. Choose a point in the first plane such as $(-d_1/a, 0, 0)$. The distance between this point and the second plane is

$$D = \frac{|a(-d_1/a) + b(0) + c(0) + d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

**20.** Essay.

# CHAPTER 11
# Vector-Valued Functions

# CHAPTER 11
# Vector-Valued Functions

## Section 11.1 Vector-Valued Functions

Solutions to Even-Numbered Exercises

**2.** $\mathbf{r}(t) = \sqrt{4 - t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\,\mathbf{k}$

Component functions: $f(t) = \sqrt{4 - t^2}$

$g(t) = t^2$

$h(t) = -6t$

Domain: $[-2, 2]$

**4.** $\mathbf{r}(t) = \sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + t\,\mathbf{k}$

Component functions: $f(t) = \sin t$

$g(t) = 4\cos t$

$h(t) = t$

Domain: $(-\infty, \infty)$

**6.** 
$$\begin{aligned}\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t) &= (\ln t\,\mathbf{i} + 5t\,\mathbf{j} - 3t^2\mathbf{k}) - (\mathbf{i} + 4t\,\mathbf{j} - 3t^2\mathbf{k})\\ &= (\ln t - 1)\mathbf{i} + (5t - 4t)\mathbf{j} + (-3t^2 + 3t^2)\mathbf{k}\\ &= (\ln t - 1)\mathbf{i} + t\,\mathbf{j}\end{aligned}$$

Domain: $(0, \infty)$

**8.** $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t & t \\ \sqrt[3]{t} & \dfrac{1}{t+1} & t+2 \end{vmatrix} = \left(-t(t+2) - \dfrac{t}{t+1}\right)\mathbf{i} - \left(t^3(t+2) - t\sqrt[3]{t}\right)\mathbf{j} + \left(\dfrac{t^3}{t+1} + t\sqrt[3]{t}\right)\mathbf{k}$

Domain: $(-\infty, -1), (-1, \infty)$

**10.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

(a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(c) $\mathbf{r}(\theta - \pi) = \cos(\theta - \pi)\mathbf{i} + 2\sin(\theta - \pi)\mathbf{j} = -\cos\theta\,\mathbf{i} - 2\sin\theta\,\mathbf{j}$

(d) $\mathbf{r}\left(\dfrac{\pi}{6} + \Delta t\right) - \mathbf{r}\left(\dfrac{\pi}{6}\right) = \cos\left(\dfrac{\pi}{6} + \Delta t\right)\mathbf{i} + 2\sin\left(\dfrac{\pi}{6} + \Delta t\right)\mathbf{j} - \left(\cos\left(\dfrac{\pi}{6}\right)\mathbf{i} + 2\sin\dfrac{\pi}{6}\mathbf{j}\right)$

**12.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$

(a) $\mathbf{r}(0) = \mathbf{k}$

(b) $\mathbf{r}(4) = 2\mathbf{i} + 8\mathbf{j} + e^{-1}\mathbf{k}$

(c) $\mathbf{r}(c + 2) = \sqrt{c + 2}\,\mathbf{i} + (c + 2)^{3/2}\mathbf{j} + e^{-[(c+2)/4]}\mathbf{k}$

(d) 
$$\begin{aligned}\mathbf{r}(9 + \Delta t) - \mathbf{r}(9) &= \left(\sqrt{9 + \Delta t}\right)\mathbf{i} + (9 + \Delta t)^{3/2}\mathbf{j} + e^{-[(9+\Delta t)/4]}\mathbf{k} - (3\mathbf{i} + 27\mathbf{j} + e^{-9/4}\mathbf{k})\\ &= \left(\sqrt{9 + \Delta t} - 3\right)\mathbf{i} + ((9 + \Delta t)^{3/2} - 27)\mathbf{j} + (e^{-[(9+\Delta t)/4]} - e^{-9/4})\mathbf{k}\end{aligned}$$

**14.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 3t\,\mathbf{j} - 4t\,\mathbf{k}$

$$\begin{aligned}\|\mathbf{r}(t)\| &= \sqrt{\left(\sqrt{t}\right)^2 + (3t)^2 + (-4t)^2}\\ &= \sqrt{t + 9t^2 + 16t^2} = \sqrt{t(1 + 25t)}\end{aligned}$$

**16.** $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3\cos t)(4\sin t) + (2\sin t)(-6\cos t) + (t-2)(t^2) = t^3 - 2t^2$, a scalar.

The dot product is a scalar-valued function.

**18.** $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \ -1 \le t \le 1$

$x = \cos(\pi t), y = \sin(\pi t), z = t^2$

Thus, $x^2 + y^2 = 1$. Matches (c)

**20.** $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \dfrac{2t}{3}\mathbf{k}, \ 0.1 \le t \le 5$

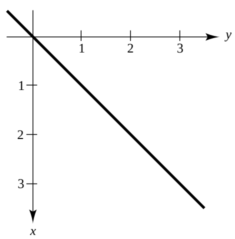
$x = t, \ y = \ln t, \ z = \dfrac{2t}{3}$

Thus, $z = \frac{2}{3}x$ and $y = \ln x$. Matches (a)

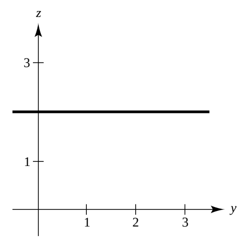
**22.** $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$

$x = t, y = t, z = 2 \implies x = y$

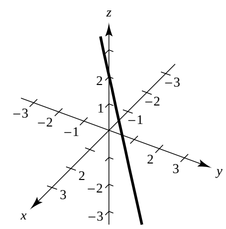
(a) $(0, 0, 20)$



(b) $(10, 0, 0)$



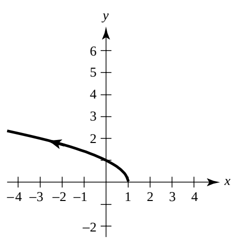
(c) $(5, 5, 5)$



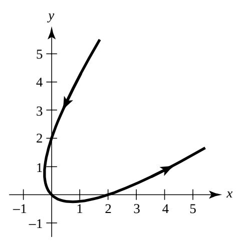
**24.** $x = 1 - t, y = \sqrt{t}$

$y = \sqrt{1 - x}$

Domain: $t \ge 0$



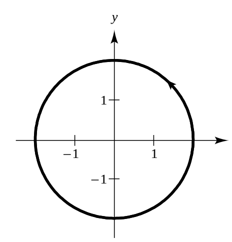
**26.** $x = t^2 + t, y = t^2 - t$



**28.** $x = 2\cos t$

$y = 2\sin t$

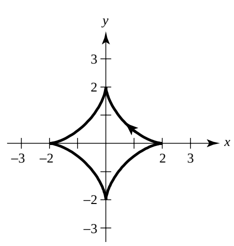
$x^2 + y^2 = 4$



**30.** $x = 2\cos^3 t, y = 2\sin^3 t$

$$\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{2}\right)^{2/3} = \cos^2 t + \sin^2 t = 1$$

$$x^{2/3} + y^{2/3} = 2^{2/3}$$



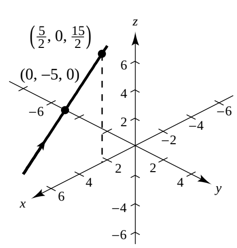
**32.** $x = t$

$y = 2t - 5$

$y = 3t$

Line passing through the points:

$$(0, -5, 0), \left(\frac{5}{2}, 0, \frac{15}{2}\right)$$

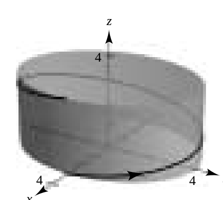



**34.** $x = 3\cos t, \ y = 4\sin t, \ z = \dfrac{t}{2}$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$z = \frac{t}{2}$$

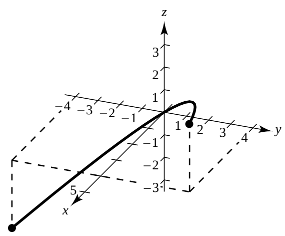
Elliptic helix



**36.** $x = t^2, y = 2t, z = \frac{3}{2}t$

$x = \frac{y^2}{4}, z = \frac{3}{4}y$

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $x$ | $4$ | $1$ | $0$ | $1$ | $4$ |
| $y$ | $-4$ | $-2$ | $0$ | $2$ | $4$ |
| $z$ | $-3$ | $-\frac{3}{2}$ | $0$ | $\frac{3}{2}$ | $3$ |




**38.** $x = \cos t + t \sin t$

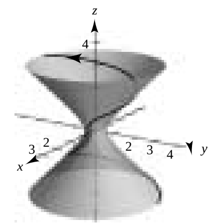
$y = \sin t - t \cos t$

$z = t$

$x^2 + y^2 = 1 + t^2 = 1 + z^2$ or $x^2 + y^2 - z^2 = 1$

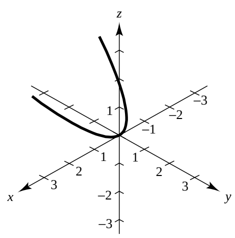
$z = t$

Helix along a hyperboloid of one sheet



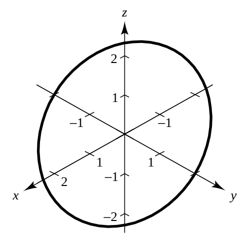

**40.** $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

Parabola

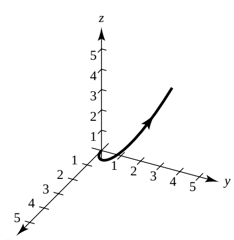



**42.** $\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} + 2\cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}$

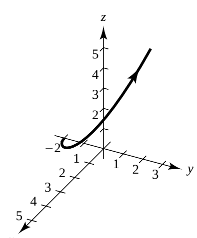
Ellipse



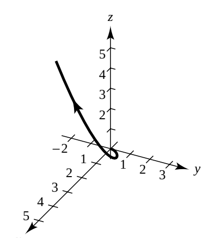

**44.** $r(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{h}$



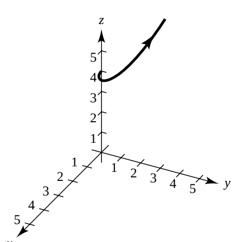
(a) $\mathbf{u}(t) = \mathbf{r}(t) - 2\mathbf{j}$ is a translation 2 units to the left along the $y$-axis.



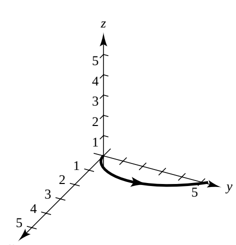
(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$ has the roles of $x$ and $y$ interchanged. The graph is a reflection in the plane $x = y$.



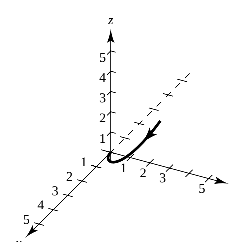
(c) $\mathbf{u}(t) = \mathbf{r}(t) + 4\mathbf{k}$ is an upward shift 4 units.



(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$ shrinks the $z$-value by a factor of 4. The curve rises more slowly.



(e) $\mathbf{u}(t) = \mathbf{r}(-t)$ reverses the orientation.



**46.** $2x - 3y + 5 = 0$

Let $x = t$, then $y = \frac{1}{3}(2t + 5)$.

$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(2\mathbf{t} + 5)\mathbf{j}$$

**48.** $y = 4 - x^2$

Let $x = t$, then $y = 4 - t^2$.

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$$

**50.** $(x - 2)^2 + y^2 = 4$

Let $x - 2 = 2\cos t, y = 2\sin t$.

$$\mathbf{r}(t) = (2 + 2\cos t)\mathbf{i} + 2\sin t\,\mathbf{j}$$

**52.** $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Let $x = 4\cos t, y = 3\sin t$.

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$$

**54.** One possible answer is

$$\mathbf{r}(t) = 1.5\cos t\mathbf{i} + 1.5\sin t\mathbf{j} + \frac{1}{\pi}t\mathbf{k},\ 0 \le t \le 2\pi$$

Note that $\mathbf{r}(2\pi) = 1.5\mathbf{i} + 2\mathbf{k}$.

**56.** $\mathbf{r}_1(t) = t\mathbf{i},\quad 0 \le t \le 10 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(10) = 10\mathbf{i})$

$\mathbf{r}_2(t) = 10(\cos t\mathbf{i} + \sin t\mathbf{j}),\quad 0 \le t \le \frac{\pi}{4} \quad \left(\mathbf{r}_2(0) = 10\mathbf{i}, \mathbf{r}_2\left(\frac{\pi}{4}\right) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}\right)$

$\mathbf{r}_3(t) = 5\sqrt{2}(1 - t)\mathbf{i} + 5\sqrt{2}(1 - t)\mathbf{j},\quad 0 \le t \le 1 \quad (\mathbf{r}_3(0) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}, \mathbf{r}_3(1) = \mathbf{0}$

(Other answers possible)

**58.** $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j},\quad 0 \le t \le 1 \ \left(y = \sqrt{x}\right)$

$\mathbf{r}_2(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j},\quad 0 \le t \le 1 \ (y = x)$

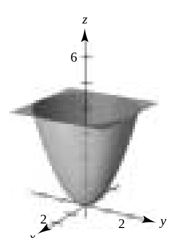
(Other answers possible)

**60.** $z = x^2 + y^2,\ z = 4$

Therefore, $x^2 + y^2 = 4$ or

$$x = 2\cos t, y = 2\sin t, z = 4.$$

$\mathbf{r}(t) = 2\cos t\mathbf{i} + \mathbf{2}\sin t\mathbf{j} + 4\mathbf{k}$

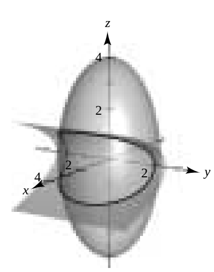



**62.** $4x^2 + 4y^2 + z^2 = 16, x = z^2$

If $z = t$, then $x = t^2$ and $y = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ | $-1.3$ | $-1.2$ | $-1$ | 0 | 1 | 1.2 |
| $x$ | 1.69 | 1.44 | 1 | 0 | 1 | 1.44 |
| $y$ | 0.85 | 1.25 | 1.66 | 2 | 1.66 | 1.25 |
| $z$ | $-1.3$ | $-1.2$ | $-1$ | 0 | 1 | 1.2 |

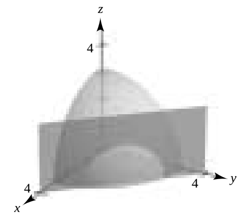
$$\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{2}\sqrt{16 - 4t^4 - t^2}\mathbf{j} + t\mathbf{k}$$



**64.** $x^2 + y^2 + z^2 = 10,\ x + y = 4$

Let $x = 2 + \sin t$, then $y = 2 - \sin t$ and $z = \sqrt{2(1 - \sin^2 t)} = \sqrt{2}\cos t$.

| $t$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{6}$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $x$ | 1 | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | 3 | 2 |
| $y$ | 3 | $\frac{5}{2}$ | 2 | $\frac{3}{2}$ | 1 | 2 |
| $z$ | 0 | $\frac{\sqrt{6}}{2}$ | $\sqrt{2}$ | $\frac{\sqrt{6}}{2}$ | 0 | $-\sqrt{2}$ |




$\mathbf{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2}\cos t\mathbf{k}$

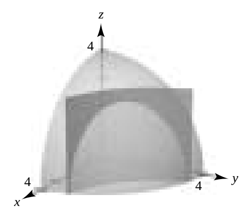
**66.** $x^2 + y^2 + z^2 = 16,\ xy = 4$ (first octant)

Let $x = t$, then

$$y = \frac{4}{t} \quad \text{and} \quad x^2 + y^2 + z^2 = t^2 + \frac{16}{t^2} + z^2 = 16.$$

$$z = \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}$$

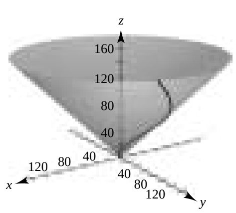
$$\left(\sqrt{8 - 4\sqrt{3}} \le t \le \sqrt{8 + 4\sqrt{3}}\right)$$




| $t$ | $\sqrt{8 + 4\sqrt{3}}$ | 1.5 | 2 | 2.5 | 3.0 | 3.5 | $\sqrt{8 + 4\sqrt{3}}$ |
|---|---|---|---|---|---|---|---|
| $x$ | 1.0 | 1.5 | 2 | 2.5 | 3.0 | 3.5 | 3.9 |
| $y$ | 3.9 | 2.7 | 2 | 1.6 | 1.3 | 1.1 | 1.0 |
| $z$ | 0 | 2.6 | 2.8 | 2.7 | 2.3 | 1.6 | 0 |

$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{t}\mathbf{j} + \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}\mathbf{k}$

**68.** $x^2 + y^2 = (e^{-t}\cos t)^2 + (e^{-t}\sin t)^2 = e^{-2t} = z^2$




**70.** $\lim_{t\to 0}\left[e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right] = \mathbf{i} + \mathbf{j} + \mathbf{k}$

since

$$\lim_{t\to 0}\frac{\sin t}{t} = \lim_{t\to 0}\frac{\cos t}{1} = 1 \quad \text{(L'Hôpital's Rule)}$$

**72.** $\lim_{t\to 1}\left[\sqrt{t}\mathbf{i} + \frac{\ln t}{t^2 - 1}\mathbf{j} + 2t^2\mathbf{k}\right] = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$

since

$$\lim_{t\to 1}\frac{\ln t}{t^2 - 1} = \lim_{t\to 1}\frac{1/t}{2t} = \frac{1}{2}. \quad \text{(L'Hôpital's Rule)}$$

**74.** $\lim_{t\to\infty}\left[e^{-t}\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{t}{t^2 + 1}\mathbf{k}\right] = \mathbf{0}$

since

$$\lim_{t\to\infty} e^{-t} = 0,\ \lim_{t\to\infty}\frac{1}{t} = 0,\ \text{and}\ \lim_{t\to\infty}\frac{t}{t^2 + 1} = 0.$$

**76.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t-1}\,\mathbf{j}$

Continuous on $[1, \infty)$

**78.** $\mathbf{r}(t) = \langle 2e^{-t}, e^{-t}, \ln(t-1)\rangle$

Continuous on $t - 1 > 0$ or $t > 1$: $(1, \infty)$.

**80.** $\mathbf{r}(t) = \left\langle 8, \sqrt{t}, \sqrt[3]{t}\right\rangle$

Continuous on $[0, \infty)$

**82.** No. The graph is the same because $\mathbf{r}(t) = \mathbf{u}(t+2)$. For example, if $\mathbf{r}(0)$ is on the graph of $\mathbf{r}$, then $\mathbf{u}(2)$ is the same point.

**84.** A vector-valued function $\mathbf{r}$ is continuous at $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

The function $\mathrm{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j} & t \ge 0 \\ -\mathbf{i} + \mathbf{j} & t < 0 \end{cases}$ is not continuous at $t = 0$.

**86.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}\lim_{t\to c}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \lim_{t\to c}[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)] \\ &= \lim_{t\to c} x_1(t)\lim_{t\to c} x_2(t) + \lim_{t\to c} y_1(t)\lim_{t\to c} y_2(t) + \lim_{t\to c} z_1(t)\lim_{t\to c} z_2(t) \\ &= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\cdot\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right] \\ &= \lim_{t\to c}\mathbf{r}(t)\cdot\lim_{t\to c}\mathbf{u}(t)\end{aligned}$$

**88.** Let

$$f(t) = \begin{cases} 1, & \text{if } t \ge 0 \\ -1, & \text{if } t < 0 \end{cases}$$

and $\mathbf{r}(t) = f(t)\mathbf{i}$. Then $\mathbf{r}$ is not continuous at $c = 0$, whereas, $\|\mathbf{r}\| = 1$ is continuous for all $t$.

**90.** False. The graph of $x = y = z = t^3$ represents a line.

## Section 11.2 Differentiation and Integration of Vector-Valued Functions

**2.** $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j},\ t_0 = 1$

$x(t) = t,\ y(t) = t^3$

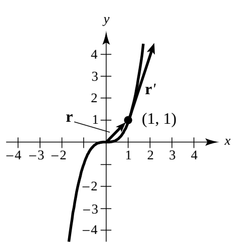
$y = x^3$

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.




**4.** $\mathbf{r}(t) = t^2\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ t_0 = 2$

$x(t) = t^2,\ y(t) = \dfrac{1}{t}$

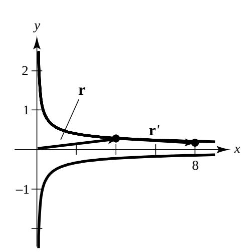
$x = \dfrac{1}{y^2}$

$\mathbf{r}(2) = 4\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

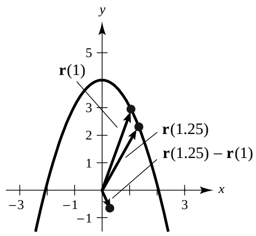
$\mathbf{r}'(2) = 4\mathbf{i} - \dfrac{1}{4}\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



**6.** $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$

(a)




(b) $\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}(1.25) = 1.25\mathbf{i} + 2.4375\mathbf{j}$

$\mathbf{r}(1.25) - \mathbf{r}(1) = 0.25\mathbf{i} - 0.5625\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} - 2\mathbf{j}$

$$\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1} = \frac{0.25\mathbf{i} - 0.5625\mathbf{j}}{0.25} = \mathbf{i} - 2.25\mathbf{j}$$

This vector approximates $\mathbf{r}'(1)$.

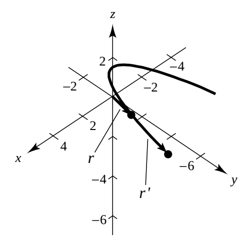
**8.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k},\ t_0 = 2$

$y = x^2,\ z = \frac{3}{2}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$




**10.** $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + 16t\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + 16\mathbf{j} + t\mathbf{k}$

**12.** $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$$\mathbf{r}'(t) = \frac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \frac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \frac{2}{t}\mathbf{k}$$

**14.** $\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t, t^2\rangle$

$\mathbf{r}'(t) = \langle t\sin t, t\cos t, 2t\rangle$

**16.** $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

$$\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1 - t^2}}, -\frac{1}{\sqrt{1 - t^2}}, 0\right\rangle$$

**18.** $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

(a) $\mathbf{r}'(t) = (2t + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t + 1)(2) + (2t - 1)(2) = 8t$

**20.** $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -8\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}''(t) = -8\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-8\sin t)(-8\cos t) + 3\cos t(-3\sin t)$

$= 55\sin t\cos \mathrm{t}$

**22.** $\mathbf{r}(t) = t\mathbf{i} + (2t + 3)\mathbf{j} + (3\mathrm{t} - 5)\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\mathbf{r}''(t) = 0$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$

**24.** $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan(t)\rangle$

(a) $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t\rangle$

$\mathbf{r}''(t) = \langle e^{-t}, 2, 2\sec^2 t\tan t\rangle$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -e^{-2t} + 4t + 2\sec^4 t\tan t$

**26.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k},\ t_0 = \dfrac{1}{4}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 0.75e^{0.75t}\mathbf{k}$

$$\mathbf{r}'\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{2}\mathbf{j} + 0.75e^{0.1875}\mathbf{k} = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{3}{4}e^{3/16}\mathbf{k}$$

$$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}e^{3/16}\right)^2} = \sqrt{\frac{5}{4} + \frac{9}{16}e^{3/8}} = \frac{\sqrt{20 + 9e^{3/8}}}{4}$$

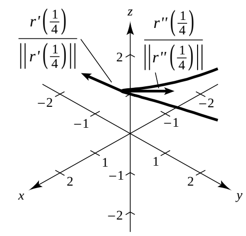
$$\frac{\mathbf{r}'(1/4)}{\|\mathbf{r}'(1/4)\|} = \frac{1}{\sqrt{20 + 9e^{3/8}}}(4\mathbf{i} + 2\mathbf{j} + 3e^{3/16}\mathbf{k})$$

$$\mathbf{r}''(t) = 2\mathbf{i} + \frac{9}{16}e^{0.75t}\mathbf{k}$$

$$\mathbf{r}''\left(\frac{1}{4}\right) = 2\mathbf{i} + \frac{9}{16}e^{3/16\mathbf{k}}$$

$$\left\|\mathbf{r}''\left(\frac{1}{4}\right)\right\| = \sqrt{2^2 + \left(\frac{9}{16}e^{3/16}\right)^2} = \sqrt{4 + \frac{81}{256}e^{3/8}} = \frac{\sqrt{1024 + 81e^{3/8}}}{16}$$

$$\frac{\mathbf{r}''(1/4)}{\|\mathbf{r}'(1/4)\|} = \frac{1}{\sqrt{1024 + 81e^{3/8}}}(32\mathbf{j} + 9e^{3/16}\mathbf{k})$$




**28.** $\mathbf{r}(t) = \dfrac{1}{t - 1}\mathbf{i} + 3t\mathbf{j}$

$\mathbf{r}'(t) = -\dfrac{1}{(t - 1)^2}\mathbf{i} + 3\mathbf{j}$

Not continuous when $t = 1$

Smooth on $(-\infty, 1),\ (1, \infty)$

**30.** $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 + \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$

$\mathbf{r}'((2n - 1)\pi) = \mathbf{0}$, $n$ any integer

Smooth on $((2n - 1)\pi, (2n + 1)\pi)$

**32.** $\mathbf{r}(t) = \dfrac{2t}{8 + t^3}\mathbf{i} + \dfrac{2t^2}{8 + t^3}\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{16 - 4t^3}{(t^3 + 8)^2}\mathbf{i} + \dfrac{32t - 2t^4}{(t^3 + 8)^2}\mathbf{j}$

$\mathbf{r}'(t) \neq \mathbf{0}$ for any value of $t$.

$\mathbf{r}$ is not continuous when $t = -2$.

Smooth on $(-\infty, -2), (-2, \infty)$.

**34.** $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}'(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t$: $(-\infty, \infty)$

**36.** $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \dfrac{1}{4}t\mathbf{k}$

$\mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \dfrac{1}{4}\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t > 0$: $(0, \infty)$

**38.** $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{u}(t) = \dfrac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(b) $\mathbf{r}''(t) = -2\sin t\mathbf{j} - 2\cos t\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 1 + 4\sin^2 t + 4\cos^2 t = 5$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0,\ t \neq 0$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = \left(3t - \dfrac{1}{t}\right)\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k}$

$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = \left(3 - \dfrac{1}{t^2}\right)\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$

**—CONTINUED—**

**38. —CONTINUED—**

(e) $$\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2\sin t & 2\cos t \\ 1/t & 2\sin t & 2\cos t \end{vmatrix}$$

$$= 2\cos t\left(\frac{1}{t} - t\right)\mathbf{j} + 2\sin t\left(t - \frac{1}{t}\right)\mathbf{k}$$

$$D_t[\mathbf{r}(t) - \mathbf{u}(t)] = \left[-2\sin t\left(\frac{1}{t} - t\right) + 2\cos t\left(-\frac{1}{t^2} - 1\right)\right]\mathbf{j}$$

$$+ \left[2\cos t\left(t - \frac{1}{t}\right) + 2\sin t\left(1 + \frac{1}{t^2}\right)\right]\mathbf{k}$$

(f) $\|\mathbf{r}(t)\| = \sqrt{t^2 + 4}$

$$D_t(\|\mathbf{r}(t)\|) = \frac{1}{2}(t^2 + 4)^{-1/2}(2t) = \frac{t}{\sqrt{t^2 + 4}}$$

**40.** $$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$$

$$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 2t^3 + t$$

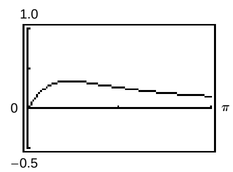
$$\|\mathbf{r}(t)\| = \sqrt{t^4 + t^2}, \ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$

$$\cos\theta = \frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$

$$\theta = \arccos\frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$

$\theta = 0.340$ ($\approx 19.47°$)maximum at $t = 0.707\left(\frac{\sqrt{2}}{2}\right)$.

$\theta \neq \frac{\pi}{2}$ for any $t$.




**42.** $$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[\sqrt{t + \Delta t}\,\mathbf{i} + \frac{3}{t + \Delta t}\mathbf{j} - 2(t + \Delta t)\mathbf{k}\right] - \left[\sqrt{t}\,\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k}\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left[\frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t}\mathbf{i} + \frac{\frac{3}{t + \Delta t} - \frac{3}{t}}{\Delta t}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \lim_{\Delta t \to 0} \left[\frac{\Delta t}{\Delta t\left(\sqrt{t + \Delta t} + \sqrt{t}\right)}\mathbf{i} + \frac{-3\Delta t}{(t + \Delta t)t(\Delta t)}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \lim_{\Delta t \to 0} \left[\frac{1}{\sqrt{t + \Delta t} + \sqrt{t}}\mathbf{i} - \frac{3}{(t + \Delta t)t}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \frac{1}{2\sqrt{t}}\mathbf{i} - \frac{3}{t^2}\mathbf{j} - 2\mathbf{k}$$

**44.** $$\int \left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right) dt = t^4\mathbf{i} + 3t^2\mathbf{j} - \frac{8}{3}t^{3/2}\mathbf{k} + \mathbf{C}$$

**46.** $$\int \left[\ln t\,\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right] dt = (t\ln t - t)\mathbf{i} + \ln t\,\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

(Integration by parts)

**48.** $$\int [e^t\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}]\, dt = e^t\mathbf{i} - \cos t\,\mathbf{j} + \sin t\,\mathbf{k} + \mathbf{C}$$

**50.** $\int [e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j}]\,dt = \dfrac{e^{-t}}{2}(-\sin t - \cos t)\mathbf{i} + \dfrac{e^{-t}}{2}(-\cos t + \sin t)\mathbf{j} + \mathbf{C}$

**52.** $\int_{-1}^{1}\left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}\right)dt = \left[\dfrac{t^2}{2}\mathbf{i}\right]_{-1}^{1} + \left[\dfrac{t^4}{4}\mathbf{j}\right]_{-1}^{1} + \left[\dfrac{3}{4}t^{4/3}\mathbf{k}\right]_{-1}^{1} = \mathbf{0}$

**54.** $\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k})\,dt = \left[\dfrac{t^2}{2}\mathbf{i}\right]_0^2 + \left[e^t\mathbf{j}\right]_0^2 - \left[(t-1)e^t\mathbf{k}\right]_0^2$

$= 2\mathbf{i} + (e^2 - 1)\mathbf{j} - (e^2 + 1)\mathbf{k}$

**56.** $\mathbf{r}(t) = \int\left(3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}\right)dt = t^3\mathbf{j} + 4t^{3/2}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = C = \mathbf{i} + 2\mathbf{j}$

$\mathbf{r}(t) = \mathbf{i} + (2 + t^3)\mathbf{j} + 4t^{3/2}\mathbf{k}$

**58.** $\mathbf{r}''(t) = -4\cos t\mathbf{i} - 3\sin t\mathbf{k}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 3\cos t\mathbf{k} + \mathbf{C}_1$

$\mathbf{r}'(0) = 3\mathbf{k} = 3\mathbf{k} + \mathbf{C}_1 \implies \mathbf{C}_1 = \mathbf{0}$

$\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{k} + \mathbf{C}_2$

$\mathbf{r}(0) = 4\mathbf{i} + \mathbf{C}_2 = 4\mathbf{j} \implies \mathbf{C}_2 = 4\mathbf{j} - 4\mathbf{i}$

$\mathbf{r}(t) = (4\cos t - 4)\mathbf{i} + 4\mathbf{j} + 3\sin t\mathbf{k}$

**60.** $\mathbf{r}(t) = \int\left[\dfrac{1}{1+t^2}\mathbf{i} + \dfrac{1}{t^2}\mathbf{j} + \dfrac{1}{t}\mathbf{k}\right]dt = \arctan t\mathbf{i} - \dfrac{1}{t}\mathbf{j} + \ln t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \dfrac{\pi}{4}\mathbf{i} - \mathbf{j} + \mathbf{C} = 2\mathbf{i} \implies \mathbf{C} = \left(2 - \dfrac{\pi}{4}\right)\mathbf{i} + \mathbf{j}$

$\mathbf{r}(t) = \left[2 - \dfrac{\pi}{4} + \arctan t\right]\mathbf{i} + \left(1 - \dfrac{1}{t}\right)\mathbf{j} + \ln t\mathbf{k}$

**62.** To find the integral of a vector-valued function, you integrate each component function separately. The constant of integration **C** is a constant vector.

**64.** The graph of $\mathbf{u}(t)$ does not change position relative to the $xy$-plane.

**66.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\begin{aligned}
\mathbf{r}(t) \pm \mathbf{u}(t) &= [x_1(t) \pm x_2(t)]\mathbf{i} + [y_1(t) \pm y_2(t)]\mathbf{j} + [z_1(t) \pm z_2(t)]\mathbf{k}\\
D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] &= [x_1'(t) \pm x_2'(t)]\mathbf{i} + [y_1'(t) \pm y_2'(t)]\mathbf{j} + [z_1'(t) \pm z_2'(t)]\mathbf{k}\\
&= [x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}] \pm [x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}]\\
&= \mathbf{r}'(t) \pm \mathbf{u}'(t)
\end{aligned}$$

**68.** Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\begin{aligned}
\mathbf{r}(t) \times \mathbf{u}(t) &= [y_1(t)z_2(t) - z_1(t)y_2(t)]\mathbf{i} - [x_1(t)z_2(t) - z_1(t)x_2(t)]\mathbf{j} + [x_1(t)y_2(t) - y_1(t)x_2(t)]\mathbf{k}\\
D_t[\mathbf{r}(t) \times \mathbf{u}(t)] &= [y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)]\mathbf{j} +\\
&\qquad [x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)]\mathbf{k}\\
&= \{[y_1(t)z_2'(t) - z_1(t)y_2'(t)]\mathbf{i} - [x_1(t)z_2'(t) - z_1(t)x_2'(t)]\mathbf{j} + [x_1(t)y_2'(t) - y_1(t)x_2'(t)]\mathbf{k}\} +\\
&\qquad \{[y_1'(t)z_2(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1'(t)z_2(t) - z_1'(t)x_2(t)]\mathbf{j} + [x_1'(t)y_2(t) - y_1'(t)x_2(t)]\mathbf{k}\}\\
&= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)
\end{aligned}$$

**70.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{r}'(t) = [y(t)z'(t) - z(t)y'(t)]\mathbf{i} - [x(t)z'(t) - z(t)x'(t)]\mathbf{j} + [x(t)y'(t) - y(t)x'(t)]\mathbf{k}$$

$$\begin{aligned} D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] &= [y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)]\mathbf{i} - [x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)]\mathbf{j} + \\ &\quad [x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)]\mathbf{k} \\ &= [y(t)z''(t) - z(t)y''(t)]\mathbf{i} - [x(t)z''(t) - z(t)x''(t)]\mathbf{j} + [x(t)y''(t) - y(t)x''(t)]\mathbf{k} = \mathbf{r}(t) \times \mathbf{r}''(t) \end{aligned}$$

**72.** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant, then:

$$\begin{aligned} x^2(t) + y^2(t) + z^2(t) &= C \\ D_t[x^2(t) + y^2(t) + z^2(t)] &= D_t[C] \\ 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) &= 0 \\ 2[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)] &= 0 \\ 2[\mathbf{r}(t) \cdot \mathbf{r}'(t)] &= 0 \end{aligned}$$

Therefore, $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

**74.** False

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$$

(See Theorem 11.2, part 4)

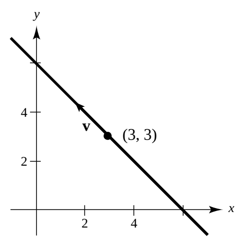
## Section 11.3 Velocity and Acceleration

**2.** $\mathbf{r}(t) = (6 - t)\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 6 - t,\ y = t,\ y = 6 - x$




**4.** $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

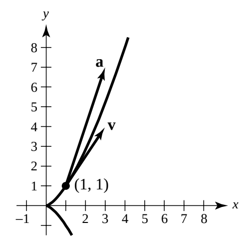
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j}$

$x = t^2, y = t^3 \quad x = y^{2/3}$

At $(1, 1)$, $t = 1$.

$\mathbf{v}(1) = 2\mathbf{i} + 3\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$




**6.** $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{v}(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j}$

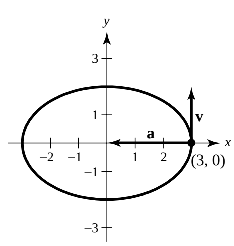
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 2\sin t\mathbf{j}$

$x = 3\cos t, y = 2\sin t, \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ Ellipse

At $(3, 0)$, $t = 0$.

$\mathbf{v}(0) = 2\mathbf{j}$

$\mathbf{a}(0) = -3\mathbf{i}$




**8.** $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -e^{-t}, e^t\rangle$

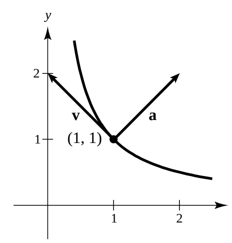
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle e^{-t}, e^t\rangle$

$x = e^{-t} = \dfrac{1}{e^t},\ y = e^t,\ y = \dfrac{1}{x}$

At $(1, 1)$, $t = 0$.

$\mathbf{v}(0) = \langle -1, 1\rangle = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(0) = \langle 1, 1\rangle = \mathbf{i} + \mathbf{j}$




**10.** $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$

$\mathbf{v}(t) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{16 + 16 + 4} = 6$

$\mathbf{a}(t) = \mathbf{0}$

**12.** $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$

$\mathbf{v}(t) = 3\mathbf{i} + \mathbf{j} + \frac{1}{2}t\mathbf{k}$

$s(t) = \sqrt{9 + 1 + \frac{1}{4}t^2} = \sqrt{10 + \frac{1}{4}t^2}$

$\mathbf{a}(t) = \frac{1}{2}\mathbf{k}$

**14.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{i} + \mathbf{j} + 3\sqrt{t}\,\mathbf{k}$

$s(t) = \sqrt{4t^2 + 1 + 9t} = \sqrt{4t^2 + 9t + 1}$

$\mathbf{a}(t) = 2\mathbf{i} + \frac{3}{2\sqrt{t}}\mathbf{k}$

**16.** $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$

$\mathbf{v}(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$

$s(t) = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}}$

$= e^t\sqrt{3}$

$\mathbf{a}(t) = -2e^t \sin t\mathbf{i} + 2e^t \cos t\mathbf{j} + e^t\mathbf{k}$

**18.** (a) $\mathbf{r}(t) = \left\langle t, \sqrt{25 - t^2}, \sqrt{25 - t^2} \right\rangle, t_0 = 3$

$\mathbf{r}'(t) = \left\langle 1, \frac{-t}{\sqrt{25 - t^2}}, \frac{-t}{\sqrt{25 - t^2}} \right\rangle$

$\mathbf{r}'(3) = \left\langle 1, -\frac{3}{4}, -\frac{3}{4} \right\rangle$

$x = 3 + t,\ y = z = 4 - \frac{3}{4}t$

(b) $\mathbf{r}(3 + 0.1) \approx \left\langle 3 + 0.1, 4 - \frac{3}{4}(0.1), 4 - \frac{3}{4}(0.1) \right\rangle$

$= \langle 3.100, 3.925, 3.925 \rangle$

**20.** $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}$

$\mathbf{v}(t) = \int (2\mathbf{i} + 3\mathbf{k})\,dt = 2t\mathbf{i} + 3t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = 4\mathbf{j} \implies \mathbf{v}(t) = 2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}(t) = \int (2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k})\,dt = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 0 \implies \mathbf{r}(t) = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k}$

$\mathbf{r}(2) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

**22.** $\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}, \mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{i}$

$\mathbf{v}(t) = \int (-\cos t\,\mathbf{i} - \sin t\,\mathbf{j})\,dt = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{j} + \mathbf{C} = \mathbf{j} + \mathbf{k} \implies \mathbf{C} = \mathbf{k}$

$\mathbf{v}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = \int (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})\,dt$

$= \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{i} \implies \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$

**24.** (a) The speed is increasing.

(b) The speed is decreasing.

**26.** $\mathbf{r}(t) = (900 \cos 45°)t\,\mathbf{i} + [3 + (900 \sin 45°)t - 16t^2]\mathbf{j}$

$= 450\sqrt{2}t\,\mathbf{i} + \left(3 + 450\sqrt{2}t - 16t^2\right)\mathbf{j}$

The maximum height occurs when $y'(t) = 450\sqrt{2} - 32t = 0$, which implies that $t = \left(225\sqrt{2}\right)/16$.
The maximum height reached by the projectile is

$$y = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2 = \frac{50{,}649}{8} = 6331.125 \text{ feet.}$$

The range is determined by setting $y(t) = 3 + 450\sqrt{2}t - 16t^2 = 0$ which implies that

$$t = \frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32} \approx 39.779 \text{ seconds.}$$

Range: $x = 450\sqrt{2}\left(\frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32}\right) \approx 25{,}315.500$ feet

**28.** 50 mph $= \dfrac{220}{3}$ ft/sec

$$\mathbf{r}(t) = \left(\frac{220}{3}\cos 15°\right)t\mathbf{i} + \left[5 + \left(\frac{220}{3}\sin 15°\right)t - 16t^2\right]\mathbf{j}$$

The ball is 90 feet from where it is thrown when

$$x = \frac{220}{3}\cos 15°t = 90 \implies t = \frac{27}{22\cos 15°} \approx 1.2706 \text{ seconds.}$$

The height of the ball at this time is

$$y = 5 + \left(\frac{220}{3}\sin 15°\right)\left(\frac{27}{22\cos 15°}\right) - 16\left(\frac{27}{22\cos 15°}\right)^2 \approx 3.286 \text{ feet.}$$

**30.** $y = x - 0.005x^2$

From Exercise 34 we know that $\tan\theta$ is the coefficient of $x$. Therefore, $\tan\theta = 1$, $\theta = (\pi/4)$ rad $= 45°$. Also

$$\frac{16}{v_0^2}\sec^2\theta = \text{negative of coefficient of } x^2$$

$$\frac{16}{v_0^2}(2) = 0.005 \text{ or } v_0 = 80 \text{ ft/sec}$$

$$\mathbf{r}(t) = \left(40\sqrt{2}t\right)\mathbf{i} + \left(40\sqrt{2}t - 16t^2\right)\mathbf{j}. \text{ Position function.}$$

When $40\sqrt{2}t = 60$,

$$t = \frac{60}{40\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

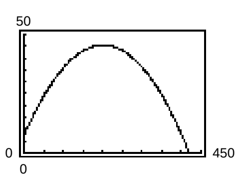
$$\mathbf{v}(t) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 32t\right)\mathbf{j}$$

$$\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 24\sqrt{2}\right)\mathbf{j} = 8\sqrt{2}(5\mathbf{i} + 2\mathbf{j}) \text{ direction}$$

$$\text{Speed} = \left\|\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right)\right\| = 8\sqrt{2}\sqrt{25 + 4} = 8\sqrt{58} \text{ ft/sec}$$

**32.** Wind: 8 mph $= \dfrac{176}{15}$ ft/sec

$$\mathbf{r}(t) = \left(140(\cos 22°)t - \frac{176}{15}\right)\mathbf{i} + (2.5 + (140\sin 22°)t - 16t^2)\mathbf{j}$$




When $x = 375$, $t \approx 2.98$ and $y \approx 16.7$ feet.

Thus, the ball clears the 10-foot fence.

**34.** $h = 7$ feet, $\theta = 35°$, 30 yards $= 90$ feet

$\mathbf{r}(t) = (v_0 \cos 35°)t\mathbf{i} + [7 + (v_0 \sin 35°)t - 16t^2]\,\mathbf{j}$

(a) $v_0 \cos 35° t = 90$ when $7 + (v_0 \sin 35°)t - 16t^2 = 4$

$$t = \frac{90}{v_0 \cos 35°}$$

$$7 + (v_0 \sin 35°)\left(\frac{90}{v_0 \cos 35°}\right) - 16\left(\frac{90}{v_0 \cos 35°}\right)^2 = 4$$

$$90 \tan 35° + 3 = \frac{129{,}600}{v_0^2 \cos^2 35°}$$

$$v_0^2 = \frac{129{,}600}{\cos^2 35°(90 \tan 35° + 3)}$$

$$v_0 \approx 54.088 \text{ feet per second}$$

(b) The maximum height occurs when

$$y'(t) = v_0 \sin 35° - 32t = 0.$$

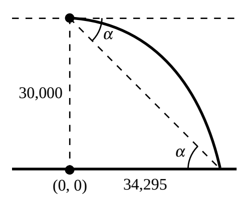
$$t = \frac{v_0 \sin 35°}{32} \approx 0.969 \text{ second}$$

At this time, the height is $y(0.969) \approx 22.0$ feet.

(c) $x(t) = 90 \implies (v_0 \cos 35°)t = 90$

$$t = \frac{90}{54.088 \cos 35°} \approx 2.0 \text{ seconds}$$

**36.** Place the origin directly below the plane. Then $\theta = 0$, $v_0 = 792$ and




$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + (30{,}000 + (v_0 \sin \theta)t - 16t^2)\mathbf{j}$$
$$= 792t\mathbf{i} + (30{,}000 - 16t^2)\mathbf{j}$$
$$\mathbf{v}(t) = 792\mathbf{i} - 32t\mathbf{j}.$$

At time of impact, $30{,}000 - 16t^2 = 0 \implies t^2 = 1875 \implies t \approx 43.3$ seconds.

$$\mathbf{r}(43.3) = 34{,}294.6\mathbf{i}$$
$$\mathbf{v}(43.3) = 792\mathbf{i} - 1385.6\mathbf{j}$$
$$\|\mathbf{v}(43.3)\| = 1596 \text{ ft/sec} = 1088 \text{ mph}$$
$$\tan \alpha = \frac{30{,}000}{34{,}294.6} \approx 0.8748 \implies \alpha \approx 0.7187(41.18°)$$

**38.** From Exercise 37, the range is

$$x = \frac{v_0^2}{32} \sin 2\theta.$$

Hence, $x = 150 = \dfrac{v_0^2}{32} \sin(24°) \implies v_0^2 = \dfrac{4800}{\sin 24°} \implies v_0 \approx 108.6$ ft/sec.

**40.** (a) $\mathbf{r}(t) = t(v_0 \cos \theta)\mathbf{i} + (tv_0 \sin \theta - 16t^2)\mathbf{j}$

$t(v_0 \sin \theta - 16t) = 0$ when $t = \dfrac{v_0 \sin \theta}{16}$.

Range: $x = v_0 \cos \theta\left(\dfrac{v_0 \sin \theta}{32}\right) = \left(\dfrac{v_0^2}{32}\right) \sin 2\theta$

The range will be maximum when

$$\frac{dx}{dt} = \left(\frac{v_0^2}{32}\right) 2 \cos 2\theta = 0$$

or

$$2\theta = \frac{\pi}{2}, \quad \theta = \frac{\pi}{4} \text{ rad.}$$

(b) $y(t) = tv_0 \sin \theta - 16t^2$

$\dfrac{dy}{dt} = v_0 \sin \theta - 32t = 0$ when $t = \dfrac{v_0 \sin \theta}{32}$.

Maximum height:

$$y\left(\frac{v_0 \sin \theta}{32}\right) = \frac{v_0^2 \sin^2 \theta}{32} - 16\,\frac{v_0^2 \sin^2 \theta}{32^2} = \frac{v_0^2 \sin^2 \theta}{64}$$

Minimum height when $\sin \theta = 1$, or $\theta = \dfrac{\pi}{2}$.

**42.** $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j}$

$= (v_0 \cos 8°)t\mathbf{i} + [(v_0 \sin 8°)t - 4.9t^2]\mathbf{j}$

$x = 50$ when $(v_0 \cos 8°)t = 50 \implies t = \dfrac{50}{v_0 \cos 8°}$. For this value of $t$, $y = 0$:

$$(v_0 \sin 8°)\left(\frac{50}{v_0 \cos 8°}\right) - 4.9\left(\frac{50}{v_0 \cos 8°}\right)^2 = 0$$

$$50 \tan 8° = \frac{(4.9)(2500)}{v_0^2 \cos^2 8°} \implies v_0^2 = \frac{(4.9)50}{\tan 8° \cos^2 8°} \approx 1777.698$$

$$\implies v_0 \approx 42.2 \text{ m/sec}$$

**44.** $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$\mathbf{v}(t) = b\omega[(1 - \cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}]$

Speed $= \|\mathbf{v}(t)\| = \sqrt{2}b\omega\sqrt{1 - \cos \omega t}$ and has a maximum value of $2b\omega$ when $\omega t = \pi, 3\pi, \ldots$ .

55 mph $= 80.67$ ft/sec $= 80.67$ rad/sec $= \omega$ since (since $b = 1$)

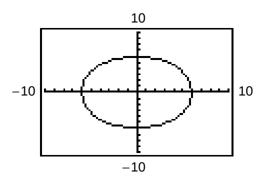
Therefore, the maximum speed of a point on the tire is twice the speed of the car:

$2(80.67)$ ft/sec $= 110$ mph

**46.** (a) Speed $= \|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}$

$= \sqrt{b^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} = b\omega$

(b)




The graphing utility draws the circle faster for greater values of $\omega$.

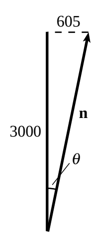
**48.** $\|\mathbf{a}(t)\| = b\omega^2\|\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\| = b\omega^2$

**50.** $\|\mathbf{v}(t)\| = 30$ mph $= 44$ ft/sec

$\omega = \dfrac{\|\mathbf{v}(t)\|}{b} = \dfrac{44}{300}$ rad/sec

$\|\mathbf{a}(t)\| = b\omega^2$

$F = m(b\omega^2) = \dfrac{3000}{32}(300)\left(\dfrac{44}{300}\right)^2 = 605$ lb




Let $\mathbf{n}$ be normal to the road.

$\|\mathbf{n}\| \cos \theta = 3000$

$\|\mathbf{n}\| \sin \theta = 605$

Dividing the second equation by the first:

$$\tan \theta = \frac{605}{3000}$$

$$\theta = \arctan\left(\frac{605}{3000}\right) \approx 11.4°.$$

**52.** $h = 6$ feet, $v_0 = 45$ feet per second, $\theta = 42.5°$. From Exercise 47,

$$t = \frac{45 \sin 42.5° + \sqrt{(45)^2 \sin^2 42.5° + 2(32)(6)}}{32} \approx 2.08 \text{ seconds.}$$

At this time, $x(t) \approx 69.02$ feet.

**54.** $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, $m$ and $b$ are constants.

$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$

$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$

$\mathbf{s}(t) = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = C$, $C$ is a constant.

Thus, $x'(t) = \dfrac{C}{\sqrt{1 + m^2}}$

$x''(t) = 0$

$\mathbf{a}(t) = x''(t)\mathbf{i} + mx''(t)\mathbf{j} = \mathbf{0}$.

**56.** $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$\mathbf{r}_2(t) = \mathbf{r}_1(2t)$

Velocity: $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$

Acceleration: $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$

In general, if $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$, then:

Velocity: $\mathbf{r}_3'(t) = \omega\mathbf{r}_1'(\omega t)$

Acceleration: $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

## Section 11.4 Tangent Vectors and Normal Vectors

**2.** $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 16t^2}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{9t^4 + 16t^2}}(3t^2\mathbf{i} + 4t\mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{9 + 16}}(3\mathbf{i} + 4\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

**4.** $\mathbf{r}(t) = 6\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{36\sin^2 t + 4\cos^2 t}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-6\sin t\mathbf{i} + 2\cos t\mathbf{j}}{\sqrt{36\sin^2 t + 4\cos^2 t}}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{-3\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{36(3/4) + (1/4)}} = \frac{1}{\sqrt{28}}\left(-3\sqrt{3}\mathbf{i} + \mathbf{j}\right)$$

**6.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

When $t = 1$, $\mathbf{r}'(t) = \mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}$ $\left[t = 1 \text{ at } \left(1, 1, \frac{4}{3}\right)\right]$.

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{\sqrt{5}}{5}(2\mathbf{i} + \mathbf{j})$$

Direction numbers: $a = 2, b = 1, c = 0$

Parametric equations: $x = 2t + 1, y = t + 1, z = \frac{4}{3}$

**8.** $\mathbf{r}(t) = \left\langle t, t, \sqrt{4 - t^2}\right\rangle$

$\mathbf{r}'(t) = \left\langle 1, 1, -\dfrac{t}{\sqrt{4 - t^2}}\right\rangle$

When $t = 1$, $\mathbf{r}'(1) = \left\langle 1, 1, -\dfrac{1}{\sqrt{3}}\right\rangle$, $\left[t = 1 \text{ at } \left(1, 1, \sqrt{3}\right)\right]$.

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sqrt{21}}{7}\left\langle 1, 1, -\frac{1}{\sqrt{3}}\right\rangle$$

Direction numbers: $a = 1$, $b = 1$, $c = -\dfrac{1}{\sqrt{3}}$

Parametric equations: $x = t + 1$, $y = t + 1$,

$$z = -\frac{1}{\sqrt{3}}t + \sqrt{3}$$

**10.** $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle$

$\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle$

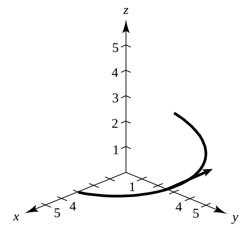
When $t = \dfrac{\pi}{6}$, $\mathbf{r}'\left(\dfrac{\pi}{6}\right) = \left\langle\sqrt{3}, -1, 2\sqrt{3}\right\rangle$, $\left[t = \dfrac{\pi}{6} \text{ at } \left(1, \sqrt{3}, 1\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'(\pi/6)}{\|\mathbf{r}'(\pi/6)\|} = \frac{1}{4}\left\langle\sqrt{3}, -1, 2\sqrt{3}\right\rangle$$

Direction numbers: $a = \sqrt{3}$, $b = -1$, $c = 2\sqrt{3}$

Parametric equations: $x = \sqrt{3}t + 1$, $y = -t + \sqrt{3}$, $z = 2\sqrt{3}t + 1$

**12.** $\mathbf{r}(t) = 3\cos t\mathbf{i} + 4\sin t\mathbf{j} + \dfrac{1}{2}\mathbf{k}$




$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 4\cos t\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

When $t = \dfrac{\pi}{2}$, $\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -3\mathbf{i} + \dfrac{1}{2}\mathbf{k}$, $\left[t = \dfrac{\pi}{2} \text{ at } \left(0, 4, \dfrac{\pi}{4}\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'(\pi/2)}{\|\mathbf{r}'(\pi/2)\|} = \frac{2}{\sqrt{37}}\left(-3\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{1}{\sqrt{37}}(-6\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = -6$, $b = 0$, $c = 1$

Parametric equations: $x = -6t$, $y = 4$, $z = t + \dfrac{\pi}{4}$

**14.** $\mathbf{r}(t) = e^{-t}\mathbf{i} + 2\cos t\mathbf{j} + 2\sin t\mathbf{k}$, $t_0 = 0$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i} - 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, $\mathbf{r}'(0) = -\mathbf{i} + 2\mathbf{k}$, $\|\mathbf{r}'(0)\| = \sqrt{5}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{-\mathbf{i} + 2\mathbf{k}}{\sqrt{5}}$$

Parametric equations: $x(s) = 1 - s$, $y(s) = 2$, $z(s) = 2s$

$$\begin{aligned}\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0 + 0.1) &\approx \langle 1 - 0.1, 2, 2(0.1)\rangle \\ &= \langle 0.9, 2, 0.2\rangle\end{aligned}$$

**16.** $\mathbf{r}(0) = \langle 0, 1, 0\rangle$

$\mathbf{u}(0) = \langle 0, 1, 0\rangle$

Hence the curves intersect.

$\mathbf{r}'(t) = \langle 1, -\sin t, \cos t\rangle$, $\mathbf{r}'(0) = \langle 1, 0, 1\rangle$

$$\mathbf{u}'(s) = \left\langle -\sin s\cos s - \cos s, -\sin s\cos s - \cos s, \frac{1}{2}\cos 2s + \frac{1}{2}\right\rangle$$

$\mathbf{u}'(0) = \langle -1, 0, 1\rangle$

$$\cos\theta = \frac{\mathbf{r}'(0)\cdot\mathbf{u}'(0)}{\|\mathbf{r}'(0)\|\,\|\mathbf{u}'(0)\|} = 0 \implies \theta = \frac{\pi}{2}$$

**18.** $\mathbf{r}(t) = t\mathbf{i} + \dfrac{6}{t}\mathbf{j}$, $t = 3$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{6}{t^2}\mathbf{j}$

$$\begin{aligned}\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} &= \frac{1}{\sqrt{1 + (36/t^4)}}\left(\mathrm{i} - \frac{6}{t^2}\mathbf{j}\right) \\ &= \frac{t^2}{\sqrt{t^4 + 36}}\left(\mathbf{i} - \frac{6}{t^2}\mathbf{j}\right)\end{aligned}$$

$$\mathbf{T}'(t) = \frac{72t}{(t^4 + 36)^{3/2}}\mathbf{i} + \frac{12t^3}{(t^4 + 36)^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$$

**20.** $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}$, $t = -\dfrac{\pi}{4}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t\mathbf{i} + 2\cos t\mathbf{j}}{\sqrt{\sin^2 t + 4\cos^2 t}}$$

The unit normal vector is perpendicular to this vector and points toward the $z$-axis:

$$\mathbf{N}(t) = \frac{-2\cos t\mathbf{i} - \sin t\mathbf{j}}{\sqrt{\sin^2 t + 4\cos^2 t}}.$$

**22.** $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$

$\mathbf{a}(t) = \mathbf{O}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

**24.** $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{2t\mathbf{j}}{2t} = \mathbf{j}$$

$\mathbf{T}'(t) = \mathbf{O}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

**26.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ t = 1$

$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j},\ \mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{i},\ \mathbf{a}(1) = 2\mathbf{i}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{4t^2 + 4}}(2t\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{t^2 + 1}}(t\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{1}{(t^2 + 1)^{3/2}}\mathbf{i} + \dfrac{-t}{(t^2 + 1)^{3/2}}\mathbf{j}}{\dfrac{1}{t^2 + 1}}$$

$$= \frac{1}{\sqrt{t^2 + 1}}(\mathbf{i} + t\mathbf{j})$$

$$\mathbf{N}(1) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

**28.** $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$\mathbf{v}(t) = -a\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$

$\mathbf{v}(0) = b\omega\mathbf{j}$

$\mathbf{a}(t) = -a\omega^2\cos(\omega t)\mathbf{i} - b\omega^2\sin(\omega t)\mathbf{j}$

$\mathbf{a}(0) = -a\omega^2\mathbf{i}$

$$\mathbf{T}(0) = \frac{\mathbf{v}(0)}{\|\mathbf{v}(0)\|} = \mathbf{j}$$

Motion along $\mathbf{r}(t)$ is counterclockwise. Therefore,

$\mathbf{N}(0) = -\mathbf{i}$.

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = a\omega^2$

**30.** $\mathbf{r}(t_0) = (\omega t_0 - \sin\omega t_0)\mathbf{i} + (1 - \cos\omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = \omega[(1 - \cos\omega t_0)\mathbf{i} + (\sin\omega t_0)\mathbf{j}]$

$\mathbf{a}(t_0) = \omega^2[(\sin\omega t_0)\mathbf{i} + (\cos\omega t_0)\mathbf{j}]$

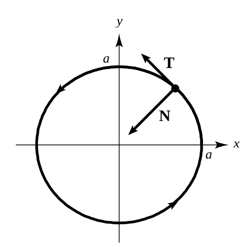
$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(1 - \cos\omega t_0)\mathbf{i} + (\sin\omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos\omega t_0}}$$

Motion along $\mathbf{r}$ is clockwise. Therefore, $\mathbf{N} = \dfrac{(\sin\omega t_0)\mathbf{i} - (1 - \cos\omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos\omega t_0}}$.

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\omega^2\sin\omega t_0}{\sqrt{2}\sqrt{1 - \cos\omega t_0}} = \frac{\omega^2}{\sqrt{2}}\sqrt{1 + \cos\omega t_0}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{\omega^2}{\sqrt{2}}\sqrt{1 - \cos\omega t_0}$$

**32.** $\mathbf{T}(t)$ points in the direction that $\mathbf{r}$ is moving. $\mathbf{N}(t)$ points in the direction that $\mathbf{r}$ is turning, toward the concave side of the curve.



**34.** If the angular velocity $\omega$ is halved,

$$a_{\mathbf{N}} = a\left(\frac{\omega}{2}\right)^2 = \frac{a\omega^2}{4}.$$

$a_{\mathbf{N}}$ is changed by a factor of $\frac{1}{4}$.

**36.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ t_0 = \dfrac{\pi}{4}$

$$x = 2\cos t,\ y = 2\sin t \implies x^2 + y^2 = 4$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

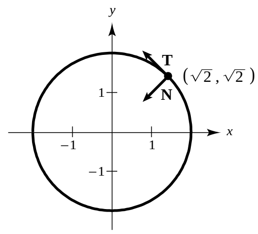
$$\mathbf{T}(t) = \frac{1}{2}(-2\sin t\mathbf{i} + 2\cos t\mathbf{j}) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} - \mathbf{j})$$




**38.** $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{v}(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_{\mathbf{T}}$, $a_{\mathbf{N}}$ are not defined.

**40.** $\mathbf{r}(t) = e^t\sin t\,\mathbf{i} + e^t\cos t\mathbf{j} + e^t\mathbf{k}$

$$\mathbf{v}(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (-e^t\sin t + e^t\cos t)\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2e^t\cos t\mathbf{i} - 2e^t\sin t\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}(0) = \frac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \sqrt{3}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \sqrt{2}$$

**42.** $\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 6t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(2) = \mathbf{i} + 12\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = 6\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1 + 37t^2}}(\mathbf{i} + 6t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(2) = \frac{1}{\sqrt{149}}(\mathbf{i} + 12\mathbf{j} + 2\mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\dfrac{1}{(1 + 37t^2)^{3/2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]}{\dfrac{\sqrt{37}}{1 + 37^2}}$$

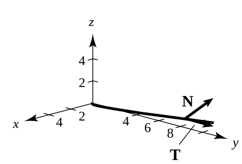
$$= \frac{1}{\sqrt{37}\sqrt{1 + 37t^2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(2) = \frac{1}{\sqrt{37}\sqrt{149}}[-74\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$= \frac{1}{\sqrt{5513}}(-74\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{74}{\sqrt{149}}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{37}{\sqrt{5513}} = \frac{\sqrt{37}}{\sqrt{149}}$$



**44.** The unit tangent vector points in the direction of motion.

**46.** If $a_{\mathbf{T}} = 0$, then the speed is constant.

**48.** (a) $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$

$$\mathbf{v}(t) = \langle -\pi \sin \pi t + \pi \sin \pi t + \pi^2 t \cos \pi t, \pi \cos \pi t - \pi \cos \pi t + \pi^2 t \sin \pi t \rangle = \langle \pi^2 t \cos \pi t, \pi^2 t \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \cos \pi t - \pi^3 t \sin \pi t, \pi^2 \sin \pi t + \pi^3 t \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle \cos \pi t, \sin \pi t \rangle$$

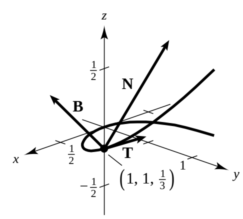
$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \cos \pi t(\pi^2 \cos \pi t - \pi^3 t \sin \pi t) + \sin \pi t(\pi^2 \sin \pi t + \pi^3 t \cos \pi t) = \pi^2$$

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{\pi^4(1 + \pi^2 t^2) - \pi^4} = \pi^3 t$$

When $t = 1$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = \pi^3$. When $t = 2$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = 2\pi^3$.

(b) Since $a_{\mathbf{T}} = \pi^2 > 0$ for all values of $t$, the speed is increasing when $t = 1$ and $t = 2$.

**50.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^3}{3}\mathbf{k}$, $t_0 = 1$




$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}}(\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k})$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}\sqrt{1 + t^2 + t^4}}[(-2t - t^3)\mathbf{i} + (1 - t^4)\mathbf{j} + (t + 2t^3)\mathbf{k}]$$

$$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{6}\sqrt{3}}(-3\mathbf{i} + 3\mathbf{k}) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$$

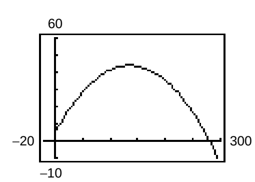
$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

**52.** (a) $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \dfrac{1}{2}gt^2\right]\mathbf{j}$

$$= (100 \cos 30°)t\mathbf{i} + [5 + (100 \sin 30°)t - 16t^2]\mathbf{j}$$

$$= 50\sqrt{3}t\mathbf{i} + [5 + 50t - 16t^2]\mathbf{j}$$

(b)




Maximum height $\approx 44.0625$

Range $\approx 279.0325$

(c) $\mathbf{v}(t) = 50\sqrt{3}\mathbf{i} + (50 - 32t)\mathbf{j}$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{2500(3) + (50 - 32t)^2}$$

$$= 4\sqrt{64t^2 - 200t + 625}\mathbf{a}(t) = -32\mathbf{j}$$

(d)

| $t$ | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|
| Speed | 93.04 | 88.45 | 86.63 | 87.73 | 91.65 | 98.06 |

**—CONTINUED—**

**52. —CONTINUED—**

(e) $$\mathbf{T}(t) = \frac{25\sqrt{3}\mathbf{i} + (25 - 16t)\mathbf{j}}{2\sqrt{64t^2 - 200t + 625}}$$

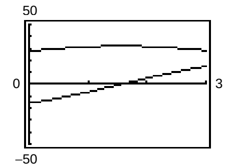
$$\mathbf{N}(t) = \frac{(25 - 16t)\mathbf{i} - 25\sqrt{3}}{2\sqrt{64t^2 - 200t + 625}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{16(16t - 25)}{\sqrt{64t^2 - 200t + 625}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{400\sqrt{3}}{\sqrt{64t^2 - 200t + 625}}$$

$$a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = -32\mathbf{j}$$

(f)




The speed is increasing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs.

**54.** 600 mph = 880 ft/sec

$$\mathbf{r}(t) = 880t\mathbf{i} + (-16t^2 + 36{,}000)\mathbf{j}$$

$$\mathbf{v}(t) = 880\mathbf{i} - 32t\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{880\mathbf{i} - 32t\mathbf{j}}{16\sqrt{4t^2 + 3025}} = \frac{55\mathbf{i} - 2t\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

Motion along $\mathbf{r}$ is clockwise, therefore

$$\mathbf{N}(t) = \frac{-2t\mathbf{i} - 55\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{64t}{\sqrt{4t^2 + 3025}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1760}{\sqrt{4t^2 + 3025}}$$

**56.** $$\mathbf{r}(t) = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$$

$$\mathbf{v}(t) = (-r\omega\sin\omega t)\mathbf{i} + (r\omega\cos\omega t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = r\omega\sqrt{1} = r\omega = v$$

$$\mathbf{a}(t) = (-r\omega^2\cos\omega t)\mathbf{i} - (r\omega^2\sin\omega t)\mathbf{j}$$

$$\|\mathbf{a}(t)\| = r\omega^2$$

(a) $F = m\|\mathbf{a}(t)\| = m(r\omega^2) = \dfrac{m}{r}(r^2\omega^2) = \dfrac{mv^2}{r}$

(b) By Newton's Law:

$$\frac{mv^2}{r} = \frac{GMm}{r^2},\ v^2 = \frac{GM}{r},\ v = \sqrt{\frac{GM}{r}}$$

**58.** $v = \sqrt{\dfrac{9.56 \times 10^4}{4200}} \approx 4.77$ mi/sec

**60.** Let $x$ = distance from the satellite to the center of the earth ($x = r + 4000$). Then:

$$v = \frac{2\pi x}{t} = \frac{2\pi x}{24(3600)} = \sqrt{\frac{9.56 \times 10^4}{x}}$$

$$\frac{4\pi^2x^2}{(24)^2(3600)^2} = \frac{9.56 \times 10^4}{x}$$

$$x^3 = \frac{(9.56 \times 10^4)(24)^2(3600)^2}{4\pi^2} \implies x \approx 26{,}245 \text{ mi}$$

$$v \approx \frac{2\pi(26{,}245)}{24(3600)} \approx 1.92 \text{ mi/sec} \approx 6871 \text{ mph}$$

**62.** $$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

$y(t) = m(x(t)) + b$, $m$ and $b$ are constants.

$$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$$

$$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = |x'(t)|\sqrt{1 + m^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\pm(\mathbf{i} + m\mathbf{j})}{\sqrt{1 + m^2}}, \text{ constant}$$

Hence, $\mathbf{T}'(t) = \mathbf{0}$.

**64.** $$\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$$

$$= (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \cdot (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})$$

$$= a_{\mathbf{T}}^2\|\mathbf{T}\|^2 + 2a_{\mathbf{T}}a_{\mathbf{N}}\mathbf{T} \cdot \mathbf{N} + a_{\mathbf{N}}^2\|\mathbf{N}\|^2$$

$$= a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2$$

$$a_{\mathbf{N}}^2 = \|\mathbf{a}\|^2 - a_{\mathbf{T}}^2$$

Since $a_{\mathbf{N}} > 0$, we have $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$.

## Section 11.5 Arc Length and Curvature

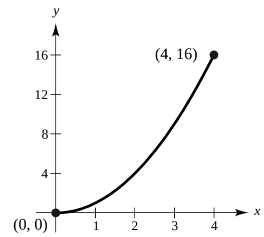
**2.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{k}$

$$\frac{dx}{dt} = 1, \ \frac{dy}{dt} = 0, \ \frac{dz}{dt} = 2t$$

$$s = \int_0^4 \sqrt{1 + 4t^2}\, dt$$

$$= \frac{1}{4}\left[2t\sqrt{1 + 4t^2} + \ln\left|2t + \sqrt{1 + 4t}\right|\right]_0^4$$

$$= \frac{1}{4}\left[8\sqrt{65} + \ln\left(8 + \sqrt{65}\right)\right] \approx 16.819$$

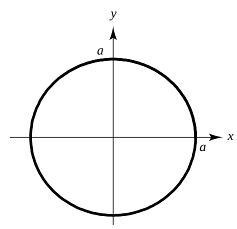



**4.** $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$

$$\frac{dx}{dt} = -a\sin t, \ \frac{dy}{dt} = a\cos t$$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} a\, dt = \Big[at\Big]_0^{2\pi} = 2\pi a$$




**6.** (a) $\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left[(v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2$$

$$y'(t) = v_0\sin\theta - gt = 0 \text{ when } t = \frac{v_0\sin\theta}{g}.$$

Maximum height when $\sin\theta = 1$, or $\theta = \dfrac{\pi}{2}$.

(b) $y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2 = 0 \implies t = \dfrac{2v_0\sin\theta}{g}$

Range: $x(t) = (v_0\cos\theta)\left(\dfrac{2v_0\sin\theta}{g}\right) = \dfrac{{v_0}^2}{g}\sin^2\theta$

The range $x(t)$ is a maximum for $\sin 2\theta = 1$, or $\theta = \dfrac{\pi}{4}$.

(c) $x'(t) = v_0\cos\theta$

$$y'(t) = v_0\sin\theta - gt$$

$$x'(t)^2 + y'(t)^2 = {v_0}^2\cos^2\theta + (v_0\sin\theta - gt)^2$$

$$= {v_0}^2\cos^2\theta + {v_0}^2\sin^2\theta - 2{v_0}^2 g\sin\theta t + g^2t^2$$

$$= {v_0}^2 - 2v_0 g\sin\theta t + g^2t^2$$

$$s(\theta) = \int_0^{2v_0\sin\theta/g}\left[{v_0}^2 - 2v_0 g\sin\theta t + g^2t^2\right]^{1/2}_{dt}$$

Since $v_0 = 96$ ft/sec, we have

$$s(\theta) = \int_0^{6\sin\theta}\left[96^2 - (6144\sin\theta)t + 1024t^2\right]^{1/2}_{dt}.$$

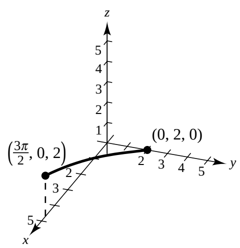
Using a computer algebra system, $s(\theta)$ is a maximum for $\theta \approx 0.9855 \approx 56.5°$.

**8.** $\mathbf{r}(t) = \langle 3t, 2\cos t, 2\sin t\rangle$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -2\sin t, \frac{dz}{dt} = 2\cos t$$

$$s = \int_0^{\pi/2} \sqrt{3^2 + (-2\sin t)^2 + (2\cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{13}\, dt = \sqrt{13}t\Big]_0^{\pi/2} = \frac{\sqrt{13}\pi}{2}$$

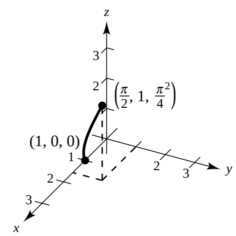



**10.** $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle$

$$\frac{dx}{dt} = t\cos t, \frac{dy}{dt} = t\sin t, \frac{dz}{dt} = 2t$$

$$s = \int_0^{\pi/2} \sqrt{(t\cos t)^2 + (t\sin t)^2 + (2t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2}\, dt = \sqrt{5}\frac{t^2}{2}\Big]_0^{\pi/2} = \frac{\sqrt{5}\pi^2}{8}$$




**12.** $\mathbf{r}(t) = \sin \pi t\mathbf{i} + \cos \pi t\mathbf{j} + t^3\mathbf{k}$

$$\frac{dx}{dt} = \pi\cos \pi t, \frac{dy}{dt} = -\pi\sin \pi t, \frac{dz}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{(\pi\cos \pi t)^2 + (-\pi\sin \pi t)^2 + (3t^2)^2}\, dt$$

$$= \int_0^2 \sqrt{\pi^2 + 9t^4}\, dt \approx 11.15$$

**14.** $\mathbf{r}(t) = 6\cos\left(\frac{\pi t}{4}\right)\mathbf{i} + 2\sin\left(\frac{\pi t}{4}\right)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2$

(a) $\mathbf{r}(0) = 6\mathbf{i} = \langle 6, 0, 0\rangle$

$\mathbf{r}(2) = 2\mathbf{j} + 2\mathbf{k} = \langle 0, 2, 2\rangle$

distance $= \sqrt{6^2 + 2^2 + 2^2} = \sqrt{44} = 2\sqrt{11} \approx 6.633$

(b) $\mathbf{r}(0) = \langle 6, 0, 0\rangle$

$\mathbf{r}(0.5) = \langle 5.543, 0.765, 0.5\rangle$

$\mathbf{r}(1.0) = \langle 4.243, 1.414, 1.0\rangle$

$\mathbf{r}(1.5) = \langle 2.296, 1.848, 1.5\rangle$

$\mathbf{r}(2.0) = \langle 0, 2, 2\rangle$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain

$$s = \int_0^2 \|\mathbf{r}'(t)\|\, dt \approx 7.0105.$$

**16.** $\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2\right\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$

$$= \int_0^t \sqrt{(4u\sin u)^2 + (4u\cos u)^2 + (3u)^2}\, du = \int_0^t \sqrt{16u + 9u^2}\, du = \int_0^t 5u\, du = \frac{5}{2}t^2$$

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**16. —CONTINUED—**

(b) $t = \sqrt{\dfrac{2s}{5}}$

$$x = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)$$

$$y = 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)$$

$$z = \frac{3}{2}\left(\sqrt{\frac{2s}{5}}\right)^2 = \frac{3s}{5}$$

$$\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$$

(c) When $s = \sqrt{5}$:

$$x = 4\left(\sin\sqrt{\frac{2\sqrt{5}}{5}} - \sqrt{\frac{2\sqrt{5}}{5}}\cos\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx -6.956$$

$$y = 4\left(\cos\sqrt{\frac{2\sqrt{5}}{5}} + \sqrt{\frac{2\sqrt{5}}{5}}\sin\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx 14.169$$

$$z = \frac{3\sqrt{5}}{5} \approx 1.342$$

$(-6.956, 14.169, 1.342)$

When $s = 4$:

$$x = 4\left(\sin\sqrt{\frac{8}{5}} - \sqrt{\frac{8}{5}}\cos\sqrt{\frac{8}{5}}\right) \approx 2.291$$

$$y = 4\left(\cos\sqrt{\frac{8}{5}} + \sqrt{\frac{8}{5}}\sin\sqrt{\frac{8}{5}}\right) \approx 6.029$$

$$z = \frac{12}{5} = 2.4$$

$(2.291, 6.029, 2.400)$

(d) $$\|\mathbf{r}'(s)\| = \sqrt{\left(\frac{4}{5}\sin\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{4}{5}\cos\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$$

**18.** $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(s) = \mathbf{i}$ and $\|\mathbf{r}'(s)\| = 1$

$\mathbf{T}(s) = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \implies K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

**20.** $$\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$$

$$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{4}{5}\sin\sqrt{\frac{2s}{5}}\mathbf{i} + \frac{4}{5}\cos\sqrt{\frac{2s}{5}}\mathbf{j} + \frac{3}{5}\mathbf{k}$$

$$\mathbf{T}'(s) = \frac{4}{25}\sqrt{\frac{5}{2s}}\cos\sqrt{\frac{2s}{5}}\mathbf{i} - \frac{4}{25}\sqrt{\frac{5}{2s}}\sin\sqrt{\frac{2s}{5}}\mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{4}{25}\sqrt{\frac{5}{2s}} = \frac{2\sqrt{10s}}{25s}$$

**22.** $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{j}$

$\mathbf{T}(t) = \mathbf{j}$

$\mathbf{T}'(t) = 0$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

**24.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j}$$
$$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$$
$$\mathbf{a}(t) = 2\mathbf{j}$$
$$\mathbf{a}(1) = 2\mathbf{j}$$
$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{1 + 4t^2}}$$
$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2}}(-2t\mathbf{i} + \mathbf{j})$$
$$\mathbf{N}(1) = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$$
$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{2}{5\sqrt{5}}$$

**26.** $\mathbf{r}(t) = 2\cos \pi t\mathbf{i} + \sin \pi t\mathbf{j}$

$$\mathbf{r}'(t) = -2\pi \sin \pi t\mathbf{i} + \pi \cos \pi t\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}$$
$$\mathbf{T}(t) = \frac{-2\sin \pi t\mathbf{i} + \cos \pi t\mathbf{j}}{\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$
$$\mathbf{T}'(t) = \frac{-2\pi \cos \pi t\mathbf{i} - 4\pi \sin \pi t\mathbf{j}}{(4\sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{2\pi}{4\sin^2 \pi t + \cos^2 \pi t}}{\pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$
$$= \frac{2}{(4\sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$

**28.** $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$$\mathbf{r}'(t) = -a\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$$
$$\mathbf{T}(t) = \frac{-a\sin(\omega t)\mathbf{i} + b\cos(\omega t)\mathbf{j}}{\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$
$$\mathbf{T}'(t) = \frac{-ab^2\omega\cos(\omega t)\mathbf{i} - a^2b\omega\sin(\omega t)\mathbf{j}}{[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)]^{3/2}}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{ab\omega}{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}{\omega\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$
$$= \frac{ab}{[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)]^{3/2}}$$

**30.** $\mathbf{r}(t) = \langle a(\omega t - \sin \omega t),\ a(1 - \cos \omega t)\rangle$

From Exercise 22, Section 11.4, we have:

$$\mathbf{a} \cdot \mathbf{N} = \frac{a\omega^2}{\sqrt{2}} \cdot \sqrt{\mathbf{1} - \cos \omega t}$$
$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$$
$$= \frac{\left(\dfrac{a\omega^2}{\sqrt{2}}\right)\sqrt{1 - \cos \omega t}}{2a^2\omega^2(1 - \cos \omega t)} = \frac{\sqrt{2}}{4a\sqrt{1 - \cos \omega t}}$$

**32.** $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{r}'(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{T}(t) = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$\mathbf{T}'(t) = \mathbf{0}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

**34.** $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$$
$$\mathbf{T}(t) = \frac{4t\mathbf{i} + \mathbf{j} + t\mathbf{k}}{\sqrt{1 + 17t^2}}$$
$$\mathbf{T}'(t) = \frac{4\mathbf{i} - 17t\mathbf{j} + \mathbf{k}}{(1 + 17t^2)^{3/2}}$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{289t^2 + 17}}{(1 + 17t^2)^{3/2}}/(1 + 17t^2)^{1/2}$$
$$= \frac{\sqrt{17}}{(1 + 17t^2)^{3/2}}$$

**36.** $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k}$

$$\mathbf{r}'(t) = (-e^t\sin t + e^t\cos t)\mathbf{i} + (e^t\cos t + e^t\sin t)\mathbf{j} + e^t\mathbf{k}$$
$$\mathbf{T}(t) = \frac{1}{\sqrt{3}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \mathbf{k}]$$
$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$$
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\left(1/\sqrt{3}\right)\sqrt{(-\cos t - \sin t)^2 + (-\sin t + \cos t)^2}}{\sqrt{3}e^t} = \frac{\sqrt{2}}{3e^t}$$

**38.** $y = mx + b$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

**40.** $y = 2x + \dfrac{4}{x}, x = 1$

$y' = 2 - \dfrac{4}{x^2}, y'(1) = -2$

$y'' = \dfrac{8}{x^3}, y''(1) = 8$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{8}{(1 + 4)^{3/2}} = \frac{8}{5^{3/2}}$$

$\dfrac{1}{K} = \dfrac{5^{3/2}}{8}$ (radius of curvature)

**42.** $y = \dfrac{3}{4}\sqrt{16 - x^2}$

$y' = \dfrac{-9x}{16y}$

$y'' = \dfrac{-[9 + (16y')^2]}{16y}$

At $x = 0$: $y' = 0$

$y'' = -\dfrac{3}{16}$

$$K = \left|\frac{-3/16}{(1 + 0^2)^{3/2}}\right| = \frac{3}{16}$$

$\dfrac{1}{K} = \dfrac{16}{3}$ (radius of curvature)

**44.** (a) $y = \dfrac{4x^2}{x^2 + 3}$

$y' = \dfrac{24x}{(x^2 + 3)^2}$

$y'' = \dfrac{72(1 - x^2)}{(x^2 + 3)^3}$

At $x = 0$: $y' = 0$

$y'' = \dfrac{72}{27} = \dfrac{8}{3}$

$$K = \frac{8/3}{(1 + 0^2)^{3/2}} = \frac{8}{3}$$

$r = \dfrac{1}{K} = \dfrac{3}{8}$

Center: $\left(0, \dfrac{3}{8}\right)$

Equation: $x^2 + \left(y - \dfrac{3}{8}\right)^2 = \dfrac{9}{64}$

(b) The circles have different radii since the curvature is different and

$r = \dfrac{1}{K}$.

**46.** $y = \ln x, \quad x = 1$

$y' = \dfrac{1}{x}, \quad y'' = -\dfrac{1}{x^2}$

$y'(1) = 1, \quad y''(1) = -1$

$$K = \frac{|-1|}{(1 + (1)^2)^{3/2}} = \frac{1}{2^{3/2}}, \; r = \frac{1}{K} = 2^{3/2} = 2\sqrt{2}$$

The slope of the tangent line at $(1, 0)$ is $y'(1) = 1$.

The slope of the normal line is $-1$.

Equation of normal line: $y = -(x - 1) = -x + 1$

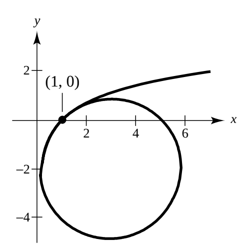
The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(1, 0)$.

$$\sqrt{(1 - x)^2 + (0 - y)^2} = 2\sqrt{2}$$
$$(1 - x)^2 + (x - 1)^2 = 8$$
$$2x^2 - 4x + 2 = 8$$
$$2(x^2 - 2x - 3) = 0$$
$$2(x - 3)(x + 1) = 0$$
$$x = 3 \text{ or } x = -1$$

Since the circle is below the curve, $x = 3$ and $y = -2$.

Center of circle: $(3, -2)$

Equation of circle: $(x - 3)^2 + (y + 2)^2 = 8$



**48.** $y = \dfrac{1}{3}x^3, \qquad x = 1$

$y' = x^2, \quad y''(1) = 2x$

$y'(1) = 1, \quad y''(1) = 2$

$$K = \frac{2}{(1+1)^{3/2}} = \frac{1}{\sqrt{2}},\ r = \frac{1}{K} = \sqrt{2}$$

The slope of the tangent line at $\left(1, \frac{1}{3}\right)$ is $y'(1) = 1$.

The slope of the normal line is $-1$.

Equation of normal line: $y - \frac{1}{3} = -(x-1)$ or $y = -x + \frac{4}{3}$

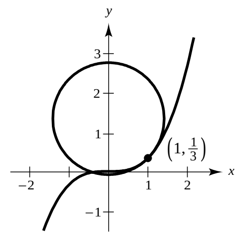
The center of the circle is on the normal line $\sqrt{2}$ units away from the point $\left(1, \frac{1}{3}\right)$.

$$\sqrt{(1-x)^2 + \left(\tfrac{1}{3} - y\right)^2} = \sqrt{2}$$

$$(1-x)^2 + (x-1)^2 = 2$$

$$(x-1)^2 = 1$$

$$x = 0 \text{ or } x = 2$$

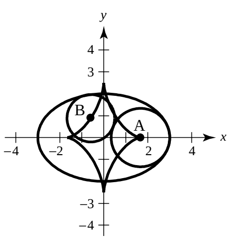



Since the circle is above the curve, $x = 0$ and $y = \frac{4}{3}$.

Center of circle: $\left(0, \frac{4}{3}\right)$

Equation of circle: $x^2 + \left(y - \frac{4}{3}\right)^2 = 2$

**50.**




**52.** $y = x^3,\ y' = 3x^2,\ y'' = 6x$

$$K = \left|\frac{6x}{(1+9x^4)^{3/2}}\right|$$

(a) $K$ is maximum at $\left(\dfrac{1}{\sqrt[4]{45}}, \dfrac{1}{\sqrt[4]{45^3}}\right), \left(\dfrac{-1}{\sqrt[4]{45}}, \dfrac{-1}{\sqrt[4]{45^3}}\right)$.

(b) $\displaystyle\lim_{x\to\infty} K = 0$

**54.** $y = \ln x,\ y' = \dfrac{1}{x},\ y'' = -\dfrac{1}{x^2}$

$$K = \left|\frac{-1/x^2}{[1 + (1/x)^2]^{3/2}}\right| = \frac{x}{(x^2+1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{-2x^2+1}{(x^2+1)^{5/2}}$$

(a) $K$ has a maximum when $x = \dfrac{1}{\sqrt{2}}$.

(b) $\displaystyle\lim_{x\to\infty} K = 0$

**56.** $y = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|-\cos x|}{(1 + \sin^2 x)^{3/2}} = 0 \text{ for } x = \frac{\pi}{2} + K\pi.$$

Curvature is 0 at $\left(\dfrac{\pi}{2} + K\pi, 0\right)$.

**58.** $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$y'' = \frac{e^x + e^{-x}}{2} = \cosh x$$

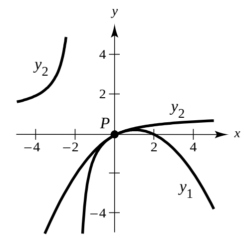
$$K = \frac{|\cosh x|}{[1 + (\sinh x)^2]^{3/2}} = \frac{\cosh x}{(\cosh^2 x)^{3/2}} = \frac{1}{\cosh^2 x} = \frac{1}{y^2}$$

**60.** See page 828.

**62.** $K = \dfrac{|y''|}{[1 + (y')^2]^{3/2}}$

At the smooth relative extremum $y' = 0$, so $K = |y''|$. Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum $(0, 0)$. The curvature is positive for any other point of the curvature.

**64.** $y_1 = ax(b - x),\ y_2 = \dfrac{x}{x + 2}$




We observe that $(0, 0)$ is a solution point to both equations. Therefore, the point $P$ is the origin.

$$y_1 = ax(b - x), \quad y_1' = a(b - 2x), \quad y_1'' = -2a$$

$$y_2 = \frac{x}{x+2}, \quad y_2' = \frac{2}{(x+2)^2}, \quad y_2'' = \frac{-4}{(x+2)^3}$$

At $P$,

$$y_1'(0) = ab \text{ and } y_2'(0) = \frac{2}{(0+2)^2} = \frac{1}{2}.$$

Since the curves have a common tangent at $P$, $y_1'(0) = y_2'(0)$ or $ab = \frac{1}{2}$. Therefore, $y_1'(0) = \frac{1}{2}$. Since the curves have the same curvature at $P$, $K_1(0) = K_2(0)$.

$$K_1(0) = \left|\frac{y_1''(0)}{[1 + (y_1(0))^2]^{3/2}}\right| = \left|\frac{-2a}{[1 + (1/2)^2]^{3/2}}\right|$$

$$K_2(0) = \left|\frac{y_2''(0)}{[1 + (y_2(0))^2]^{3/2}}\right| = \left|\frac{-1/2}{[1 + (1/2)^2]^{3/2}}\right|$$

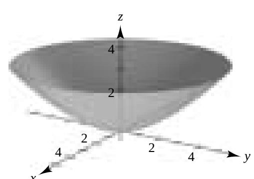
Therefore, $2a = \pm\frac{1}{2}$ or $a = \pm\frac{1}{4}$. In order that the curves intersect at only one point, the parabola must be concave downward. Thus,

$$a = \frac{1}{4} \quad \text{and} \quad b = \frac{1}{2a} = 2.$$

$$y_1 = \frac{1}{4}x(2 - x) \quad \text{and} \quad y_2 = \frac{x}{x+2}$$

**66.** $y = \dfrac{1}{4}x^{8/5},\ 0 \le x \le 5$

(a)




(rotated about $y$-axis)

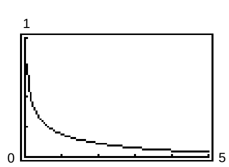
(b) $V = \displaystyle\int_0^5 2\pi x\left(\frac{1}{4}x^{8/5}\right) dx$ (shells)

$$= \frac{\pi}{2}\int_0^5 x^{13/5}\,dx = \frac{\pi}{2}\left[\frac{x^{18/5}}{18/5}\right]_0^5$$

$$= \frac{5\pi}{36}5^{18/5} \approx 143.25 \text{ cm}^3$$

(c) $y' = \dfrac{2}{5}x^{3/5},\ y'' = \dfrac{6}{25}x^{-2/5} = \dfrac{6}{25x^{2/5}}$

$$K = \frac{\dfrac{6}{25x^{2/5}}}{\left[1 + \dfrac{4}{25}x^{6/5}\right]^{3/2}} = \frac{6}{25x^{2/5}\left[1 + \dfrac{4}{25}x^{6/5}\right]^{3/2}}$$




(d) No, the curvature approaches $\infty$ as $x \to 0^+$. Hence, any spherical object will hit the sides of the goblet before touching the bottom $(0, 0)$.

**68.** $s = \dfrac{c}{\sqrt{K}}$

$y = \dfrac{1}{3}x^3$

$y' = x^2$

$y'' = 2x$

$K = \left|\dfrac{2x}{(1 + x^4)^{3/2}}\right|$

When $x = 1$: $K = \dfrac{1}{\sqrt{2}}$

$$s = \frac{c}{\sqrt{1/\sqrt{2}}} = \sqrt[4]{2}c$$

$$30 = \sqrt[4]{2}c \implies c = \frac{30}{\sqrt[4]{2}}$$

At $x = \dfrac{3}{2}$, $K = \dfrac{3}{[1 + (81/16)]^{3/2}} \approx 0.201$

$$s\left(\frac{3}{2}\right) = \frac{c}{\sqrt{K}} = \frac{30/\sqrt[4]{2}}{\sqrt{K}} \approx 56.27 \text{ mi/hr.}$$

**70.** $r(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = f(\theta)\cos\theta\mathbf{i} + f(\theta)\sin\theta\mathbf{j}$

$x(\theta) = f(\theta)\cos\theta$

$y(\theta) = f(\theta)\sin\theta$

$x'(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$y'(\theta) = f(\theta)\cos\theta + f'(\theta)\sin\theta$

$x''(\theta) = -f(\theta)\cos\theta - f'(\theta)\sin\theta - f'(\theta)\sin\theta + f''(\theta)\cos\theta = -f(\theta)\cos\theta - 2f'(\theta)\sin\theta + f''(\theta)\cos\theta$

$y''(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta + f'(\theta)\cos\theta + f''(\theta)\sin\theta = -f(\theta)\sin\theta + 2f'(\theta)\cos\theta + f''(\theta)\sin\theta$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|f^2(\theta) - f(\theta)f''(\theta) + 2(f'(\theta))^2|}{[f^2(\theta) + (f'(\theta))^2]^{3/2}} = \frac{|r^2 - rr'' + 2(r')^2|}{[r^2 + (r')^2]^{3/2}}$$

**72.** $r = \theta$

$r' = 1$

$r'' = 0$

$K = \dfrac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \dfrac{2 + \theta^2}{(1 + \theta^2)^{3/2}}$

**74.** $r = e^\theta$

$r' = e^\theta$

$r'' = e^\theta$

$K = \dfrac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \dfrac{2e^{2\theta}}{(2e^{2\theta})^{3/2}} = \dfrac{1}{\sqrt{2}e^\theta}$

**76.** At the pole, $r = 0$.

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2(r')^2|}{|r'|^3} = \frac{2}{|r'|}$$

**78.** $r = 6\cos 3\theta$

$r' = -18\sin 3\theta$

At the pole,

$$\theta = \frac{\pi}{6},\ r'\left(\frac{\pi}{6}\right) = -18,$$

and

$$K = \frac{2}{|r'(\pi/6)|} = \frac{2}{|-18|} = \frac{1}{9}.$$

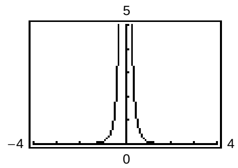
**80.** $x(t) = t^3,\ x'(t) = 3t^2,\ x''(t) = 6t$

$y(t) = \frac{1}{2}t^2,\ y'(t) = t,\ y''(t) = 1$

$$K = \frac{|(3t^2)(1) - (t)(6t)|}{[(3t^2)^2 + (t)^2]^{3/2}}$$

$$= \frac{3t^2}{|t^3|(9t^2 + 1)^{3/2}} = \frac{3}{|t|(9t^2 + 1)^{3/2}}$$

$K \to 0$ as $t \to \pm\infty$




**82.** (a) $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$

$\mathbf{v}(t) = 6t\mathbf{i} + (3 - 3t^2)\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = 3(1 + t^2),\ \frac{d^2s}{dt^2} = 6t$$

$$K = \frac{2}{3(1 + t^2)^2}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 6t$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{3(1 + t^2)^2} \cdot 9(1 + t^2)^2 = 6$$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = \sqrt{5t^2 + 1}$$

$$\frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}(5t^2 + 1) = \frac{\sqrt{5}}{\sqrt{5t^2 + 1}}$$

**84.** (a) $K = \|\mathbf{T}'(s)\| = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right\|$, by the Chain Rule

$$= \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

(b)
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{r}'(t)}{ds/dt}$$

$$\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

$$\mathbf{r}''(t) = \left(\frac{d^2s}{dt^2}\right)\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)[\mathbf{T}(t) \times \mathbf{T}(t)] + \left(\frac{ds}{dt}\right)^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

Since $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$ and $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, we have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t) \times \mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t)\|\,\|\mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2(1)K\|\mathbf{r}'(t)\| \quad \text{from (a)}$$

Therefore, $\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = K.$

(c)
$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)^3\|} = \frac{\dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}}{\|\mathbf{r}'(t)\|^2} = \frac{\dfrac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|}}{\|\mathbf{r}'(t)\|^2} = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|^2}$$

**86.** $\mathbf{F} = m\mathbf{a} \implies m\mathbf{a} = \dfrac{-GmM}{r^3}\mathbf{r}$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Since $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Since $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$$\left(\frac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

**88.**
$$\begin{aligned}
\frac{d}{dt}\left[\frac{\mathbf{r}'}{GM} \times \mathbf{L} - \frac{\mathbf{r}}{r}\right] &= \frac{1}{GM}[\mathbf{r}' \times \mathbf{0} + \mathbf{r}'' \times \mathbf{L}] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right) \times [\mathbf{r} \times \mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\
&= \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0}
\end{aligned}$$

Thus, $\left(\dfrac{\mathbf{r}'}{GM}\right) \times \mathbf{L} - \left(\dfrac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

**90.** $\|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$

Let: $\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

$$\mathbf{r}' = r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt} \qquad \left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt}\right)$$

Then: $\mathbf{r} \times \mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\theta & r\sin\theta & 0 \\ -r\sin\theta\dfrac{d\theta}{dt} & r\cos\theta\dfrac{d\theta}{dt} & 0 \end{vmatrix}$

$$= r^2\frac{d\theta}{dt}\mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2\frac{d\theta}{dt}.$$

**92.** Let $P$ denote the period. Then

$$A = \int_0^P \frac{dA}{dt}\,dt = \frac{1}{2}\|\mathbf{L}\|P.$$

Also, the area of an ellipse is $\pi ab$ where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\begin{aligned}
\pi ab &= \frac{1}{2}\|\mathbf{L}\|P \\
P &= \frac{2\pi ab}{\|\mathbf{L}\|} \\
P^2 &= \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}(a^2 - c^2) = \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}a^2(1 - e^2) \\
&= \frac{4\pi^2a^4}{\|\mathbf{L}\|^2}\left(\frac{ed}{a}\right) = \frac{4\pi^2ed}{\|\mathbf{L}\|^2}a^3 \\
&= \frac{4\pi^2(\|\mathbf{L}\|^2/GM)}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2}{GM}a^3 = Ka^3
\end{aligned}$$

## Review Exercises for Chapter 11

**2.** $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + \mathbf{k}$

(a) Domain: $[0, 4)$ and $(4, \infty)$

(b) Continuous except at $t = 4$

**4.** $\mathbf{r}(t) = (2t + 1)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$

(a) Domain: $(-\infty, \infty)$

(b) Continuous for all $t$

**6.** (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}\mathbf{k}$

(c) $\mathbf{r}(s - \pi) = 3\cos(s - \pi)\mathbf{i} + (1 - \sin(s - \pi))\mathbf{j} - (s - \pi)\mathbf{k}$

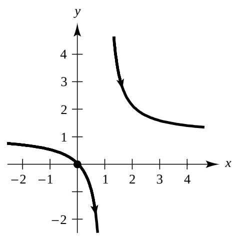
(d) $\mathbf{r}(\pi + \Delta t) - \mathbf{r}(\pi) = (3\cos(\pi + \Delta t)\mathbf{i} + (1 - \sin(\pi + \Delta t))\mathbf{j} - (\pi + \Delta t)\mathbf{k}) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k})$

$= (-3\cos\Delta t + 3)\mathbf{i} + \sin\Delta t - \Delta t\mathbf{k}$

**8.** $\mathbf{r}(t) = t\mathbf{i} + \dfrac{t}{t-1}\mathbf{j}$

$x(t) = t,\ y(t) = \dfrac{t}{t-1}$

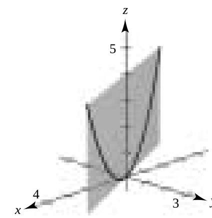
$y = \dfrac{x}{x-1}$




**10.** $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 2t,\ y = t,\ z = t^2,$
$y = \frac{1}{2}x,\ z = y^2$

| $t$ | 0 | 1 | $-1$ | 2 |
|---|---|---|---|---|
| $x$ | 0 | 2 | $-2$ | 4 |
| $y$ | 0 | 1 | $-1$ | 2 |
| $z$ | 0 | 1 | 1 | 4 |

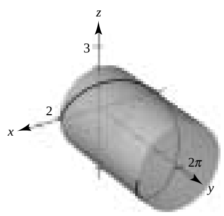



**12.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + t\mathbf{j} + 2\sin t\mathbf{k}$

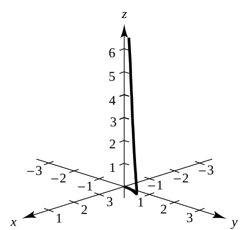
$x = 2\cos t,\ y = t,\ z = 2\sin t$

$x^2 + z^2 = 4$

| $t$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|
| $x$ | 2 | 0 | $-2$ | 0 |
| $y$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| $z$ | 0 | 2 | 0 | $-2$ |




**14.** $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{1}{4}t^3\mathbf{k}$




**16.** One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i}, \qquad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, \qquad 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \qquad 0 \le t \le 4$

**18.** The $x$- and $y$-components are $2\cos t$ and $2\sin t$. At

$t = \dfrac{3\pi}{2},$

the staircase has made $\frac{3}{4}$ of a revolution and is 2 meters high. Thus, one answer is

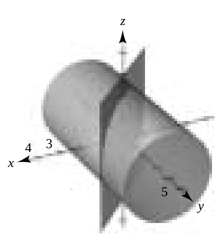
$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \dfrac{4}{3\pi}t\mathbf{k}.$

**20.** $x^2 + z^2 = 4,\ x - y = 0,\ t = x$

$x = t,\ y = t,\ z = \pm\sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\mathbf{k}$




**22.** $\lim_{t\to 0}\left(\dfrac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^t\mathbf{k}\right) = \left(\lim_{t\to 0}\dfrac{2\cos 2t}{1}\right)\mathbf{i} + \mathbf{j} + \mathbf{k}$

$= 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

**24.** $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k},\ \mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \dfrac{1}{t}\mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}$

(b) $\mathbf{r}''(t) = -\sin t\mathbf{i} - \cos t\mathbf{j}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 2$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -\sin t\mathbf{i} - \cos t\mathbf{j} + \left(\dfrac{1}{t} - 2t\right)\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -\cos t\mathbf{i} + \sin t\mathbf{j} + \left(-\dfrac{1}{t^2} - 2\right)\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{1 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{t}{\sqrt{1 + t^2}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \left(\dfrac{1}{t}\cos t - t\cos t\right)\mathbf{i} - \left(\dfrac{1}{t}\sin t - t\sin t\right)\mathbf{j}$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(-\frac{1}{t}\sin t - \frac{1}{t^2}\cos t + t\sin t - \cos t\right)\mathbf{i} - \left(\frac{1}{t}\cos t - \frac{1}{t^2}\sin t - t\cos t - \sin t\right)\mathbf{j}$$

**26.** The graph of $\mathbf{u}$ is parallel to the $yz$-plane.

**28.** $\displaystyle\int (\ln t\mathbf{i} + t\ln t\mathbf{j} + \mathbf{k})\,dt = (t\ln t - t)\mathbf{i} + \frac{t^2}{4}(-1 + 2\ln t)\mathbf{j} + t\mathbf{k} + \mathbf{C}$

**30.** $\displaystyle\int (t\mathbf{j} + t^2\mathbf{k}) \times (\mathbf{i} + t\mathbf{j} + t\mathbf{k})\,dt = \int [(t^2 - t^3)\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}]\,dt = \left(\frac{t^3}{3} - \frac{t^4}{4}\right)\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

**32.** $\displaystyle\mathbf{r}(t) = \int (\sec t\mathbf{i} + \tan t\mathbf{j} + t^2\mathbf{k})\,dt = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \left(\dfrac{t^3}{3} + 3\right)\mathbf{k}$

**34.** $\displaystyle\int_0^1 (\sqrt{t}\mathbf{j} + t\sin t\,\mathbf{k})\,dt = \left[\frac{2}{3}t^{3/2}\mathbf{j} + (\sin t - t\cos t)\mathbf{k}\right]_0^1 = \frac{2}{3}\mathbf{j} + (\sin 1 - \cos 1)\mathbf{k}$

**36.** $\displaystyle\int_{-1}^1 (t^3\mathbf{i} - \arcsin t\mathbf{j} - t^2\mathbf{k})\,dt = \left[\frac{t^4}{4}\mathbf{i} - \left(t\arcsin t + \sqrt{1 - t^2}\right)\mathbf{j} - \frac{t^3}{3}\mathbf{k}\right]_{-1}^1$

$= -\dfrac{2}{3}\mathbf{k}$

**38.** $\mathbf{r}(t) = \langle t, -\tan t, e^t\rangle$

$\mathbf{r}'(t) = \mathbf{v}(t) = \langle 1, -\sec^2 t, e^t\rangle$

$\|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$

$\mathbf{r}''(t) = \mathbf{a}(t) = \langle 0, -2\sec^2 t \cdot \tan t, e^t\rangle$

**40.** $\mathbf{r}(t) = \langle 3\cosh t, \sinh t, -2t\rangle,\ t_0 = 0$

$\mathbf{r}'(t) = \langle 3\sinh t, \cosh t, -2\rangle$

$\mathbf{r}'(0) = \langle 0, 1, -2\rangle$ direction numbers

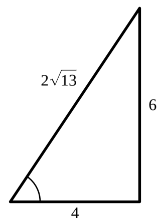
Since $\mathbf{r}(0) = \langle 3, 0, 0\rangle$, the parametric equations are $x = 3$, $y = t$, $z = -2t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0.1) \approx \langle 3, 0.1, -0.2\rangle$

**42.** Range $= 4 = \dfrac{v_0^2}{16}\sin\theta\cos\theta$

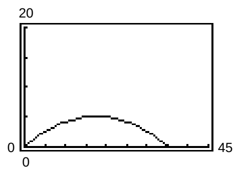
$$= \frac{v_0^2}{16}\cdot\frac{6}{2\sqrt{13}}\cdot\frac{4}{2\sqrt{13}} = \frac{3v_0^2}{104}$$

$$\frac{416}{3} = v_0^2 \implies v_0 \approx 11.776 \text{ ft/sec}$$



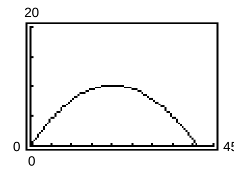

**44.** $\mathbf{r}(t) = [(v_0\cos\theta)t]\mathbf{i} + \left[(v_0\sin\theta)t - \frac{1}{2}(9.8)t^2\right]\mathbf{j}$

(a) $\mathbf{r}(t) = [(20\cos 30°)t]\mathbf{i} + [(20\sin 30°)t - 4.9t^2]\mathbf{j}$



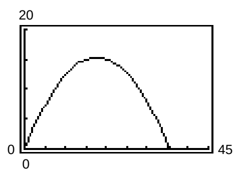

Maximum height $\approx$ 5.1 m; Range $\approx$ 35.3 m

(b) $\mathbf{r}(t) = [(20\cos 45°)t]\mathbf{i} + [(20\sin 45°)t - 4.9t^2]\mathbf{j}$




Maximum height $\approx$ 10.2 m; Range $\approx$ 40.8 m

(c) $\mathbf{r}(t) = [(20\cos 60°)t]\mathbf{i} + [(20\sin 60°)t - 4.9t^2]\mathbf{j}$




Maximum height $\approx$ 15.3 m; Range $\approx$ 35.3 m

(Note that 45° gives the longest range)

**46.** $\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 3\mathbf{j}$

$\|\mathbf{v}\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \dfrac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$\mathbf{N}(t)$ does not exist

$\mathbf{a}\cdot\mathbf{T} = 0$

$\mathbf{a}\cdot\mathbf{N}$ does not exist

**48.** $\mathbf{r}(t) = 2(t+1)\mathbf{i} + \dfrac{2}{t+1}\mathbf{j}$

$$\mathbf{v}(t) = 2\mathbf{i} - \frac{2}{(t+1)^2}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{2\sqrt{(t+1)^4+1}}{(t+1)^2}$$

$$\mathbf{a}(t) = \frac{4}{(t+1)^3}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(t+1)^2\mathbf{i} - \mathbf{j}}{\sqrt{(t+1)^4+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + (t+1)^2\mathbf{j}}{\sqrt{(t+1)^4+1}}$$

$$\mathbf{a}\cdot\mathbf{T} = \frac{-4}{(t+1)^3\sqrt{(t+1)^4+1}}$$

$$\mathbf{a}\cdot\mathbf{N} = \frac{4(t+1)^2}{(t+1)^3\sqrt{(t+1)^4+1}}$$

$$= \frac{4}{(t+1)\sqrt{(t+1)^4+1}}$$

**50.**
$$\mathbf{r}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j}$$
$$\mathbf{v}(t) = \mathbf{r}'(t) = (-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}$$
$$\|\mathbf{v}(t)\| = \text{speed} = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2} = \sqrt{t^2+1}$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = (-t\cos t - 2\sin t)\mathbf{i} + (-t\sin t + 2\cos t)\mathbf{j}$$
$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}}{\sqrt{t^2+1}}$$
$$\mathbf{N}(t) = \frac{-(t\cos t + \sin t)\mathbf{i} + (-t\sin t + \cos t)\mathbf{j}}{\sqrt{t^2+1}}$$
$$\mathbf{a}(t)\cdot\mathbf{T}(t) = \frac{t}{\sqrt{t^2+1}}$$
$$\mathbf{a}(t)\cdot\mathbf{N}(t) = \frac{t^2+2}{\sqrt{t^2+1}}$$

**52.**
$$\mathbf{r}(t) = (t-1)\mathbf{i} + t\mathbf{j} + \frac{1}{t}\mathbf{k}$$
$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{1}{t^2}\mathbf{k}$$
$$\|\mathbf{v}(t)\| = \frac{\sqrt{2t^4+1}}{t^2}$$
$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{k}$$
$$\mathbf{T}(t) = \frac{t^2\mathbf{i} + t^2\mathbf{j} - \mathbf{k}}{\sqrt{2t^4+1}}$$
$$\mathbf{N}(t) = \frac{\mathbf{i} + \mathbf{j} + 2t^2\mathbf{k}}{\sqrt{2}\sqrt{2t^4+1}}$$
$$\mathbf{a}\cdot\mathbf{T} = \frac{-2}{t^3\sqrt{2t^4+1}}$$
$$\mathbf{a}\cdot\mathbf{N} = \frac{4}{t\sqrt{2}\sqrt{2t^4+1}}$$

**54.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k},\ x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

When $t = 2,\ x = 2,\ y = 4,\ z = \frac{16}{3}$.

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$$

Direction numbers when $t = 2,\ a = 1,\ b = 4,\ c = 8$

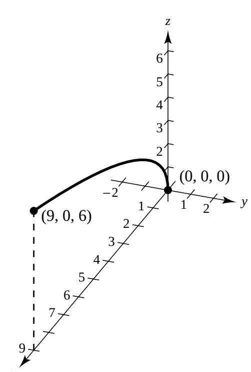
$$x = t + 2,\ y = 4t + 4,\ z = 8t + \frac{16}{3}$$

**56.** Factor of 4

**58.** $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k},\ 0 \le t \le 3$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^3 \sqrt{4t^2+4}\,dt$$
$$= \left[\ln\left|\sqrt{t^2+1} + t\right| + t\sqrt{t^2+1}\right]_0^3$$
$$= \ln\left(\sqrt{10} + 3\right) + 3\sqrt{10} \approx 11.3053$$

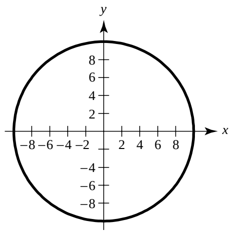


**60.** $\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}$

$\mathbf{r}'(t) = -10\sin t\mathbf{i} + 10\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 10$

$$s = \int_0^{2\pi} 10\,dt = 20\pi$$

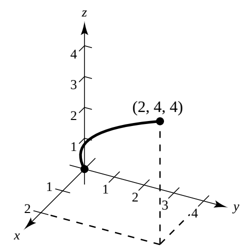



**62.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{5 + 4t^2}\,dt$$

$$= \sqrt{21} + \frac{5}{4}\ln 5 - \frac{5}{4}\ln\left(\sqrt{105} - 4\sqrt{5}\right) \approx 6.2638$$




**64.** $\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t\rangle,\ 0 \le t \le \dfrac{\pi}{2}$

$x'(t) = \langle 2t\sin t, 2t\cos t, 1\rangle,\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{4t^2 + 1}\,dt$$

$$= \sqrt{17} - \frac{1}{4}\ln\left(\sqrt{17} - 4\right) \approx 4.6468$$

**66.** $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{k},\ 0 \le t \le \pi$

$\mathbf{r}'(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (-e^t\sin t + e^t\cos t)\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{(e^t\cos t + e^t\sin t)^2 + (-e^t\sin t + e^t\cos t)^2}$

$= \sqrt{2}e^t$

$$s = \int_0^{\pi} \|\mathbf{r}'(t)\|\,dt$$

$$= \sqrt{2}\int_0^{\pi} e^t\,dt = \Big[\sqrt{2}e^t\Big]_0^{\pi} = \sqrt{2}(e^{\pi} - 1)$$

**68.** $\mathbf{r}(t) = 2\sqrt{t}\mathbf{i} + 3t\mathbf{j}$

$$\mathbf{r}'(t) = \frac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t} + 9} = \sqrt{\frac{1 + 9t}{t}}$$

$$\mathbf{r}''(t) = -\frac{1}{2}t^{-3/2}\mathbf{i}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{t}} & 3 & 0 \\ -\frac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k};\ \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1 + 9t)^{3/2}/t^{3/2}} = \frac{3}{2(1 + 9t)^{3/2}}$$

**70.** $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} - 5\sin t\mathbf{j} + 5\cos t\mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{29}$

$\mathbf{r}''(t) = 5\cos t\mathbf{j} - 5\sin t\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5\sin t & 5\cos t \\ 0 & -5\cos t & -5\sin t \end{vmatrix} = 25\mathbf{i} + 10\sin t\mathbf{j} - 10\cos t\mathbf{k}$$

$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

**72.** $y = e^{-x/2}$

$y' = -\frac{1}{2}e^{-x/2}, y'' = \frac{1}{4}e^{-x/2}$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\frac{1}{4}e^{-x/2}}{\left[1 + \frac{1}{4}e^{-x}\right]^{3/2}}$$

At $x = 0$, $K = \frac{1/4}{(5/4)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}} = \frac{2\sqrt{5}}{25}$, $r = \frac{5\sqrt{5}}{2}$.

**74.** $y = \tan x$

$y' = \sec^2 x$

$y'' = 2\sec^2 x \tan x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|2\sec^2 x \tan x|}{[1 + \sec^4 x]^{3/2}}$$

At $x = \frac{\pi}{4}$, $K = \frac{4}{5^{3/2}} = \frac{4}{5\sqrt{5}} = \frac{4\sqrt{5}}{25}$ and $r = \frac{5\sqrt{5}}{4}$.

## Problem Solving for Chapter 11

**2.** $x^{2/3} + y^{2/3} = a^{2/3}$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$y' = \frac{-y^{1/3}}{x^{1/3}}$ Slope at $P(x, y)$.

$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}$

$\mathbf{r}'(t) = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\text{i} = |3\cos t \sin t|$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\cos t\mathbf{i} + \sin t\mathbf{j}$

$\mathbf{T}'(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

$Q(0, 0, 0)$ origin

$P = (\cos^3 t, \sin^3 t, 0)$ on curve.

$$\overrightarrow{PQ} \times \mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos^3 t & \sin^3 t & 0 \\ -\cos t & \sin t & 0 \end{vmatrix}$$

$$= (\cos^3 t \sin t - \sin^3 t \cos t)\mathbf{k}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{T}\|}{\|\mathbf{T}\|} = |3\cos t \sin t|$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{|3\cos t \sin t|}$$

Thus, the radius of curvature, $\frac{1}{K}$, is three times the distance from the origin to the tangent line.

**4.** Bomb: $\mathbf{r}_1(t) = \langle 5000 + 400t, 3200 - 16t^2 \rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2 \rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \implies t = 10$

Projectile will travel 5 seconds:

$5(v_0 \sin\theta) - 16(25) = 1600$

$v_0 \sin\theta = 400.$

Horizontal position:

At $t = 10$, bomb is at $5000 + 400(10) = 9000$.

At $t = 5$, projectile is at $(v_0 \cos\theta)5$.

Thus,

$5v_0 \cos\theta = 9000$

$v_0 \cos\theta = 1800.$

Combining, $\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{1800} \implies \tan\theta = \frac{2}{9} \implies \theta \approx 12.5°$.

$v_0 = \frac{1800}{\cos\theta} \approx 1843.9$ ft/sec

**6.** $r = 1 - \cos\theta$

$r' = \sin\theta$

$$s(t) = \int_{\pi}^{t} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta = \int_{\pi}^{t} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_{\pi}^{t} 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_{\pi}^{t} = -4\cos\frac{t}{2}$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2\sin^2\theta - (1-\cos\theta)(\cos\theta) + (1-\cos\theta)^2|}{8\sin^3\dfrac{\theta}{2}}$$

$$= \frac{|3 - 3\cos\theta|}{8\sin^3\dfrac{\theta}{2}}$$

$$= \frac{3}{4}\,\frac{\sin^2\dfrac{\theta}{2}}{\sin^3\dfrac{\theta}{2}} = \frac{3}{4\sin\dfrac{\theta}{2}}$$

$$\rho = \frac{1}{K} = \frac{4\sin\dfrac{\theta}{2}}{3}$$

$$s^2 + 9\rho^2 = 16\cos^2\frac{\theta}{2} + 16\sin^2\frac{\theta}{2} = 16$$

**8.** (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ position vector

$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$

$$\frac{d\mathbf{r}}{dt} = \left[\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right]\mathbf{i} + \left[\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right]\mathbf{j}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2}\cos\theta - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}\right]\mathbf{i}$$

$$+ \left[\frac{d^2r}{dt^2}\sin\theta + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}\right]$$

$a_r = \mathbf{a}\cdot\mathbf{u}_r = \mathbf{a}\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$

$$= \left[\frac{d^2r}{dt^2}\cos^2\theta - 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 - r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$+ \left[\frac{d^2r}{dt^2}\sin^2\theta + 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$a_\theta = \mathbf{a}\cdot u_\theta = \mathbf{a}\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = 2\frac{dr}{dr}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$\mathbf{a} = (\mathbf{a}\cdot\mathbf{u}_r)\mathbf{u}_r + (\mathbf{a}\cdot\mathbf{u}_\theta)u_\theta$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\mathbf{u}_\theta$$

**—CONTINUED—**

**8. —CONTINUED—**

(b) $\mathbf{r} = 42{,}000 \cos\left(\frac{\pi t}{12}\right)\mathbf{i} + 42{,}000 \sin\left(\frac{\pi t}{12}\right)\mathbf{j}$

$$r = 42{,}000, \frac{dr}{dt} = 0, \frac{d^2r}{dt^2} = 0$$

$$\frac{d\theta}{dt} = \frac{\pi}{12}, \frac{d^2\theta}{dt^2} = 0$$

Therefore, $\mathbf{a} = -42000\left(\frac{\pi}{12}\right)^2 \mathbf{u}_r = -\frac{875}{3}\pi^2\mathbf{u}_r$.

Radial component: $-\frac{875}{3}\pi^2$

Angular component: 0

**10.** $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} - \mathbf{k}, t = \frac{\pi}{4}$

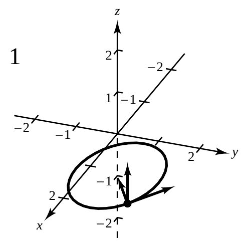
$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$\mathbf{T} = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{T}' = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{k}$




At $t = \frac{\pi}{4}, \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$$

**12.** $y = \frac{1}{32}x^{5/2}$

$$y' = \frac{5}{64}x^{3/2}$$

$$y'' = \frac{15}{128}x^{1/2}$$

$$K = \left|\frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}}\right|$$

At the point $(4, 1)$, $K = \frac{120}{(89)^{3/2}} \implies r = \frac{1}{K} = \frac{(89)^{3/2}}{120} \approx 7$.

**14.** (a) Eliminate the parameter to see that the Ferris wheel has a radius of 15 meters and is centered at $16\mathbf{j}$. At $t = 0$, the friend is located at $\mathbf{r}_1(0) = \mathbf{j}$, which is the low point on the Ferris wheel.

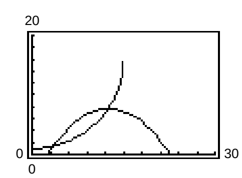
(b) If a revolution takes $\Delta t$ seconds, then

$$\frac{\pi(t + \Delta t)}{10} = \frac{\pi t}{10} + 2\pi$$

and so $\Delta t = 20$ seconds. The Ferris wheel makes three revolutions per minute.

(c) The initial velocity is $r'_2(t_0) = -8.03\mathbf{i} + 11.47\mathbf{j}$. The speed is $\sqrt{8.03^2 + 11.47^2} \approx 14$ m/sec. The angle of inclination is $\arctan(11.47/8/03) \approx 0.96$ radians or 55°.

(d) Although you may start with other values, $t_0 = 0$ is a fine choice. The graph at the right shows two points of intersection. At $t = 3.15$ sec the friend is near the vertex of the parabola, which the object reaches when




$$t - t_0 = -\frac{11.47}{2(-4.9)} \approx 1.17 \text{ sec.}$$

Thus, after the friend reaches the low point on the Ferris wheel, wait $t_0 = 2$ sec before throwing the object in order to allow it to be within reach.

(e) The approximate time is 3.15 seconds after starting to rise from the low point on the Ferris wheel. The friend has a constant speed of $\|\mathbf{r}'_1(t)\| = 15$ m/sec. The speed of the object at that time is

$$\|\mathbf{r}'_2(3.15)\| = \sqrt{8.03^2 + [11.47 - 9.8(3.15 - 2)]^2} \approx 8.03 \text{ m/sec.}$$

# CHAPTER 12
# Functions of Several Variables

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# Functions of Several Variables

## Section 12.1 Introduction to Functions of Several Variables

Solutions to Even-Numbered Exercises

**2.** $xz^2 + 2xy - y^2 = 4$

No, $z$ is not a function of $x$ and $y$. For example, $(x, y) = (1, 0)$ corresponds to both $z = \pm 2$

**4.** $z + x \ln y - 8 = 0$

$z = 8 - x \ln y$

Yes, $z$ is a function of $x$ and $y$.

**6.** $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0) = 4$

(b) $f(0, 1) = 4 - 0 - 4 = 0$

(c) $f(2, 3) = 4 - 4 - 36 = -36$

(d) $f(1, y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e) $f(x, 0) = 4 - x^2 - 0 = 4 - x^2$

(f) $f(t, 1) = 4 - t^2 - 4 = -t^2$

**8.** $g(x, y) = \ln|x + y|$

(a) $g(2, 3) = \ln|2 + 3| = \ln 5$

(b) $g(5, 6) = \ln|5 + 6| = \ln 11$

(c) $g(e, 0) = \ln|e + 0| = 1$

(d) $g(0, 1) = \ln|0 + 1| = 0$

(e) $g(2, -3) = \ln|2 - 3| = \ln 1 = 0$

(f) $g(e, e) = \ln|e + e| = \ln 2e$

$= \ln 2 + \ln e = (\ln 2) + 1$

**10.** $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(0, 5, 4) = \sqrt{0 + 5 + 4} = 3$

(b) $f(6, 8, -3) = \sqrt{6 + 8 - 3} = \sqrt{11}$

**12.** $V(r, h) = \pi r^2 h$

(a) $V(3, 10) = \pi(3)^2(10) = 90\pi$

(b) $V(5, 2) = \pi(5)^2(2) = 50\pi$

**14.** $g(x, y) = \displaystyle\int_x^y \frac{1}{t}\, dt$

(a) $g(4, 1) = \displaystyle\int_4^1 \frac{1}{t}\, dt = \Big[\ln|t|\Big]_4^1 = -\ln 4$

(b) $g(6, 3) = \displaystyle\int_6^3 \frac{1}{t}\, dt = \Big[\ln|t|\Big]_6^3 = \ln 3 - \ln 6 = \ln\left(\frac{1}{2}\right)$

**16.** $f(x, y) = 3xy + y^2$

(a) $$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{[3(x + \Delta x)y + y^2] - (3xy + y^2)}{\Delta x}$$

$$= \frac{3xy + 3(\Delta x)y + y^2 - 3xy - y^2}{\Delta x} = \frac{3(\Delta x)y}{\Delta x} = 3y,\ \Delta x \neq 0$$

(b) $$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[3x(y + \Delta y) + (y + \Delta y)^2] - (3xy + y^2)}{\Delta y}$$

$$= \frac{3xy + 3x(\Delta y) + y^2 + 2y(\Delta y) + (\Delta y)^2 - 3xy - y^2}{\Delta y}$$

$$= \frac{\Delta y(3x + 2y + \Delta y)}{\Delta y} = 3x + 2y + \Delta y,\ \Delta y \neq 0$$

**18.** $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Domain: $4 - x^2 - 4y^2 \geq 0$

$$x^2 + 4y^2 \leq 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} \leq 1$$

$$\left\{(x, y): \frac{x^2}{4} + \frac{y^2}{1} \leq 1\right\}$$

Range: $0 \leq z \leq 2$

**20.** $f(x, y) = \arccos \dfrac{y}{x}$

Domain: $\left\{(x, y): -1 \leq \dfrac{y}{x} \leq 1\right\}$

Range: $0 \leq z \leq \pi$

**22.** $f(x, y) = \ln(xy - 6)$

Domain: $xy - 6 > 0$

$$xy > 6$$

$$\{(x, y): xy > 6\}$$

Range: all real numbers

**24.** $z = \dfrac{xy}{x - y}$

Domain: $\{(x, y): x \neq y\}$

Range: all real numbers

**26.** $f(x, y) = x^2 + y^2$

Domain: $\{(x, y)$: $x$ is any real number, $y$ is any real number$\}$

Range: $z \geq 0$

**28.** $g(x, y) = x\sqrt{y}$

Domain: $\{(x, y): y \geq 0\}$

Range: all real numbers

**30.** (a) Domain: $\{(x, y)$: $x$ is any real number, $y$ is any real number$\}$

Range: $-2 \leq z \leq 2$

(b) $z = 0$ when $x = 0$ which represents points on the $y$-axis.

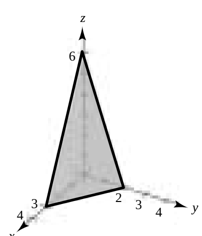
(c) No. When $x$ is positive, $z$ is negative. When $x$ is negative, $z$ is positive. The surface does not pass through the first octant, the octant where $y$ is negative and $x$ and $z$ are positive, the octant where $y$ is positive and $x$ and $z$ are negative, and the octant where $x$, $y$ and $z$ are all negative.

**32.** $f(x, y) = 6 - 2x - 3y$

Plane

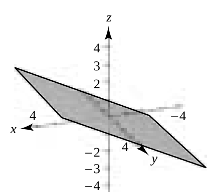
Domain: entire $xy$-plane

Range: $-\infty < z < \infty$




**34.** $g(x, y) = \frac{1}{2}x$

Plane: $z = \frac{1}{2}x$

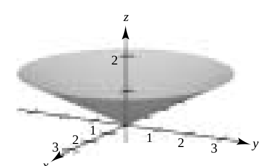



**36.** $z = \dfrac{1}{2}\sqrt{x^2 + y^2}$

Cone

Domain of $f$: entire $xy$-plane

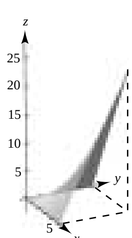
Range: $z \geq 0$




**38.** $f(x, y) = \begin{cases} xy, & x \geq 0,\ y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

Domain of $f$: entire $xy$-plane

Range: $z \geq 0$

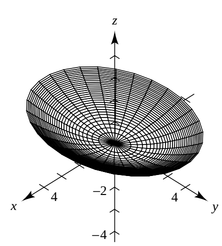



**40.** $f(x, y) = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

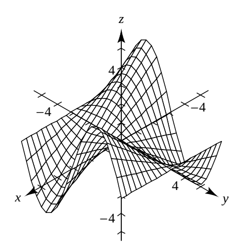
Semi-ellipsoid

Domain: set of all points lying on or inside the ellipse $(x^2/9) + (y^2/16) = 1$

Range: $0 \leq z \leq 1$

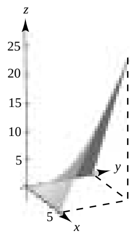



**42.** $f(x, y) = x \sin y$



**44.** $f(x, y) = xy, x \geq 0, y \geq 0$

(a)

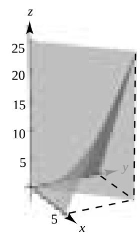



(b) $g$ is a vertical translation of $f$ 3 units downward

(c) $g$ is a reflection of $f$ in the $xy$-plane

(d) The graph of $g$ is lower than the graph of $f$. If $z = f(x, y)$ is on the graph of $f$, then $\frac{1}{2}z$ is on the graph of $g$.

(e)




**46.** $z = e^{1-x^2+y^2}$

Level curves:

$$c = e^{1-x^2+y^2}$$

$$\ln c = 1 - x^2 + y^2$$

$$x^2 - y^2 = 1 - \ln c$$

Hyperbolas centered at $(0, 0)$

Matches (d)

**48.** $z = \cos\left(\dfrac{x + 2y^2}{4}\right)$

Level curves:

$$c = \cos\left(\frac{x^2 + 2y^2}{4}\right)$$

$$\cos^{-1} c = \frac{x^2 + 2y^2}{4}$$

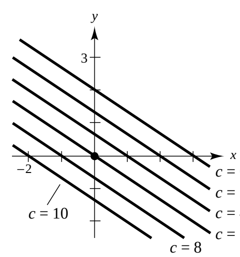
$$x^2 + 2y^2 = 4 \cos^{-1} c$$

Ellipses

Matches (a)

**50.** $f(x, y) = 6 - 2x - 3y$

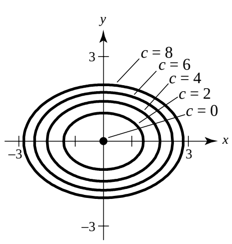
The level curves are of the form $6 - 2x - 3y = c$ or $2x + 3y = 6 - c$. Thus, the level curves are straight lines with a slope of $-\frac{2}{3}$.




**52.** $f(x, y) = x^2 + 2y^2$

The level curves are ellipses of the form

$x^2 + 2y^2 = c$ (except $x^2 + 2y^2 = 0$ is the point $(0, 0)$).

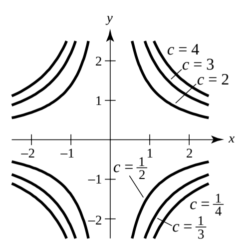



**54.** $f(x, y) = e^{xy/2}$

The level curves are of the form

$$e^{xy/2} = c, \text{ or } \ln c = \frac{xy}{2}.$$

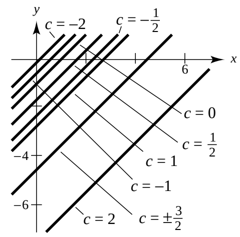
Thus, the level curves are hyperbolas.



**56.** $f(x, y) = \ln(x - y)$

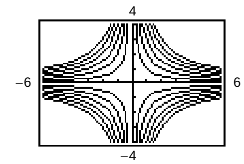
The level curves are of the form

$$c = \ln(x - y)$$
$$e^c = x - y$$
$$y = x - e^c$$

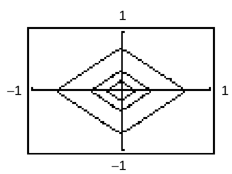



Thus, the level curves are parallel lines of slope 1 passing through the fourth quadrant.

**58.** $f(x, y) = |xy|$




**60.** $h(x, y) = 3\sin(|x| + |y|)$




**62.** The graph of a function of two variables is the set of all points $(x, y, z)$ for which $z = f(x, y)$ and $(x, y)$ is in the domain of $f$. The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, for $c$, a constant.

**64.** $f(x, y) = \dfrac{x}{y}$

The level curves are the lines

$$c = \frac{x}{y} \text{ or } y = \frac{1}{c}x$$

These lines all pass through the origin.

**66.** The surface could be an ellipsoid centered at $(0, 1, 0)$. One possible function is

$$f(x, y) = x^2 + \frac{(y - 1)^2}{4} = 1.$$

**68.** $A(r, t) = 1000e^{rt}$

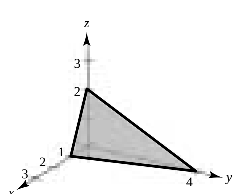
| | Number of years | | | |
|---|---|---|---|---|
| Rate | 5 | 10 | 15 | 20 |
| 0.08 | \$1491.82 | \$2225.54 | \$3320.12 | \$4953.03 |
| 0.10 | \$1648.72 | \$2718.28 | \$4481.69 | \$7389.06 |
| 0.12 | \$1822.12 | \$3320.12 | \$6049.65 | \$11,023.18 |
| 0.14 | \$2013.75 | \$4055.20 | \$8166.17 | \$16,444.65 |

**70.** $f(x, y, z) = 4x + y + 2z$

$c = 4$

$4 = 4x + y + 2z$

Plane



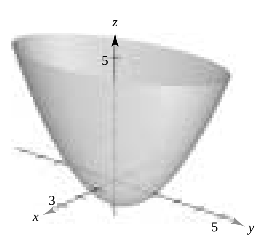

**72.** $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z$

$c = 1$

$1 = x^2 + \frac{1}{4}y^2 - z$

Elliptic paraboloid

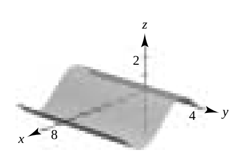
Vertex: $(0, 0, -1)$




**74.** $f(x, y, z) = \sin x - z$

$c = 0$

$0 = \sin x - z$ or $z = \sin x$



**76.** $W(x, y) = \dfrac{1}{x - y},\ y < x$

(a) $W(15, 10) = \dfrac{1}{15 - 10} = \dfrac{1}{5}$ hr $= 12$ min

(b) $W(12, 9) = \dfrac{1}{12 - 9} = \dfrac{1}{3}$ hr $= 20$ min

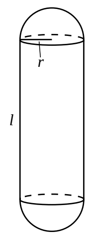
(c) $W(12, 6) = \dfrac{1}{12 - 6} = \dfrac{1}{6}$ hr $= 10$ min

(d) $W(4, 2) = \dfrac{1}{4 - 2} = \dfrac{1}{2}$ hr $= 30$ min

**78.** $f(x, y) = 100x^{0.6}y^{0.4}$

$$f(2x, 2y) = 100(2x)^{0.6}(2y)^{0.4}$$
$$= 100(2)^{0.6}x^{0.6}(2)^{0.4}y^{0.4} = 100(2)^{0.6}(2)^{0.4}x^{0.6}y^{0.4} = 2[100x^{0.6}y^{0.4}] = 2f(x, y)$$

**80.** $V = \pi r^2 l + \dfrac{4}{3}\pi r^3 = \dfrac{\pi r^2}{3}(3l + 4r)$




**82.** (a)

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 |
|---|---|---|---|---|---|---|
| $z$ | 12.7 | 14.8 | 17.1 | 18.5 | 21.1 | 25.8 |
| Model | 13.09 | 14.79 | 16.45 | 18.47 | 21.38 | 25.78 |

(b) $x$ has the greater influence because its coefficient $(0.143)$ is larger than that of $y(0.024)$.

(c) $f(x, 25) = 0.143x + 0.024(25) + 0.502$

$= 0.143x + 1.102$

This function gives the shareholder's equity $z$ in terms of net sales $x$ and assumes constant assets of $y = 25$.

**84.** Southwest

**86.** Latitude and land versus ocean location have the greatest effect on temperature.

**88.** True

**90.** True

## Section 12.2 Limits and Continuity

**2.** Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |x - 4| < \varepsilon$ whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} = \sqrt{(x - 4)^2 + (y + 1)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x - 4)^2 + (y + 1)^2} < \delta = \varepsilon$, we have

$$\sqrt{(x - 4)^2} < \varepsilon$$
$$|x - 4| < \varepsilon.$$

**4.** $$\lim_{(x, y)\to(a, b)} \left[\frac{4f(x, y)}{g(x, y)}\right] = \frac{4\left[\lim_{(x, y)\to(a, b)} f(x, y)\right]}{\lim_{(x, y)\to(a, b)} g(x, y)} = \frac{4(5)}{3} = \frac{20}{3}$$

**6.** $\displaystyle\lim_{(x,y)\to(a,b)}\left[\frac{f(x,y)-g(x,y)}{f(x,y)}\right] = \frac{\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} g(x,y)}{\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)} = \frac{5-3}{5} = \frac{2}{5}$

**8.** $\displaystyle\lim_{(x,y)\to(0,0)}(5x+y+1) = 0+0+1 = 1$

Continuous everywhere

**10.** $\displaystyle\lim_{(x,y)\to(1,1)}\frac{x}{\sqrt{x+y}} = \frac{1}{\sqrt{1+1}} = \frac{\sqrt{2}}{2}$

Continuous for $x+y>0$

**12.** $\displaystyle\lim_{(x,y)\to(\pi/4,2)} y\cos(xy) = 2\cos\frac{\pi}{2} = 0$

Continuous everywhere

**14.** $\displaystyle\lim_{(x,y)\to(1,1)}\frac{xy}{x^2+y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$

**16.** $\displaystyle\lim_{(x,y,z)\to(2,0,1)} xe^{yz} = 2e^0 = 2$

Continuous everywhere

**18.** $f(x,y) = \dfrac{x^2}{(x^2+1)(y^2+1)}$

$\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2}{(x^2+1)(y^2+1)} = \frac{0}{(0+1)(0+1)} = 0$

Continuous everywhere

**20.** $\displaystyle\lim_{(x,y)\to(0,0)}\left[1-\frac{\cos(x^2+y^2)}{x^2+y^2}\right] = -\infty$

The limit does not exist.

Continuous except at $(0, 0)$

**22.** $f(x,y) = \dfrac{y}{x^2+y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

| $(x, y)$ | $(1, 1)$ | $(0.5, 0.5)$ | $(0.1, 0.1)$ | $(0.01, 0.01)$ | $(0.001, 0.001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{2}$ | 1 | 5 | 50 | 500 |

Path: $y = x$

| $(x, y)$ | $(1, 0)$ | $(0.5, 0)$ | $(0.1, 0)$ | $(0.01, 0)$ | $(0.001, 0)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | 0 | 0 | 0 | 0 | 0 |

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function tends to infinity.

**24.** $f(x,y) = \dfrac{2x-y^2}{2x^2+y}$

Continuous except at $(0, 0)$

Path: $y = 0$

| $(x, y)$ | $(1, 0)$ | $(0.25, 0)$ | $(0.01, 0)$ | $(0.001, 0)$ | $(0.000001, 0)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | 1 | 4 | 100 | 1000 | 1,000,000 |

Path: $y = x$

| $(x, y)$ | $(1, 1)$ | $(0.25, 0.25)$ | $(0.01, 0.01)$ | $(0.001, 0.001)$ | $(0.0001, 0.0001)$ |
|---|---|---|---|---|---|
| $f(x, y)$ | $\frac{1}{3}$ | 1.17 | 1.95 | 1.995 | 2.0 |

The limit does not exist because along the line $y = 0$ the function tends to infinity, whereas along the line $y = x$ the function tends to 2.

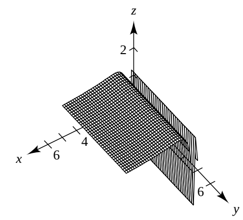
**26.** $\lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{4x^2y^2}{(x^2+y^2)} = 0$

Hence, $\lim\limits_{(x,\,y)\to(0,\,0)} f(x, y) = \lim\limits_{(x,\,y)\to(0,\,0)} g(x, y) = 0.$

$f$ is continuous at $(0, 0)$, whereas $g$ is not continuous at $(0, 0)$.

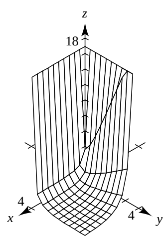
**28.** $\lim\limits_{(x,\,y)\to(0,\,0)} \left(\sin \dfrac{1}{x} + \cos \dfrac{1}{x}\right)$

Does not exist



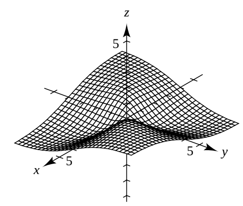

**30.** $\lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{x^2+y^2}{x^2y}$

Does not exist




**32.** $f(x, y) = \dfrac{2xy}{x^2+y^2+1}$

The limit equals 0.




**34.** $\lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{xy^2}{x^2+y^2} = \lim\limits_{r\to 0} \dfrac{(r\cos\theta)(r^2\sin^2\theta)}{r^2} = \lim\limits_{r\to 0} (r\cos\theta\sin^2\theta) = 0$

**36.** $\lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{x^2y^2}{x^2+y^2} = \lim\limits_{r\to 0} \dfrac{r^4\cos^2\theta\sin^2\theta}{r^2} = \lim\limits_{r\to 0} r^2\cos^2\theta\sin^2\theta = 0$

**38.** $f(x, y, z) = \dfrac{z}{x^2+y^2-9}$

Continuous for $x^2 + y^2 \neq 9$

**40.** $f(x, y, z) = xy\sin z$

Continuous everywhere

**42.**
$$\begin{aligned} f(t) &= \frac{1}{t} \\ g(x, y) &= x^2 + y^2 \\ f(g(x, y)) &= f(x^2+y^2) \\ &= \frac{1}{x^2+y^2} \end{aligned}$$

Continuous except at $(0, 0)$

**44.**
$$\begin{aligned} f(t) &= \frac{1}{4-t} \\ g(x, y) &= x^2 + y^2 \\ f(g(x, y)) &= f(x^2+y^2) = \frac{1}{4-x^2-y^2} \end{aligned}$$

Continuous for $x^2 + y^2 \neq 4$

**46.** $f(x, y) = x^2 + y^2$

(a) $$\lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{[(x+\Delta x)^2 + y^2] - (x^2+y^2)}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0} (2x + \Delta x) = 2x$$

(b) $$\lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{[x^2 + (y+\Delta y)^2] - (x^2+y^2)}{\Delta y}$$
$$= \lim_{\Delta y\to 0} \frac{2y\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y\to 0} (2y + \Delta y) = 2y$$

**48.** $f(x, y) = \sqrt{y}(y + 1)$

(a) $\displaystyle\lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sqrt{y}(y + 1) - \sqrt{y}(y + 1)}{\Delta x} = 0$

(b) $$\begin{aligned}\lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} &= \lim_{\Delta y\to 0} \frac{(y + \Delta y)^{3/2} + (y + \Delta y)^{1/2} - (y^{3/2} + y^{1/2})}{\Delta y}\\ &= \lim_{\Delta y\to 0} \frac{(y + \Delta y)^{3/2} - y^{3/2}}{\Delta y} + \lim_{\Delta y\to 0} \frac{(y + \Delta y)^{1/2 -} y^{1/2}}{\Delta y}\\ &= \frac{3}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \qquad \text{(L'Hôpital's Rule)}\\ &= \frac{3y + 1}{2\sqrt{y}}\end{aligned}$$

**50.** See the definition on page 854.

**52.** $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2 + y^2}{xy}$

(a) Along $y = ax$: $\displaystyle\lim_{(x, ax)\to(0, 0)} \frac{x^2 + (ax)^2}{x(ax)}$

$$= \lim_{x\to 0} \frac{x^2(1 + a^2)}{ax^2} = \frac{1 + a^2}{a},\ a \neq 0$$

If $a = 0$, then $y = 0$ and the limit does not exist.

(b) Along $y = x^2$: $\displaystyle\lim_{(x, x^2)\to(0, 0)} \frac{x^2 + (x^2)^2}{x(x^2)} = \lim_{x\to 0} \frac{1 + x^2}{x}$

limit does not exist

(c) No, the limit does not exist. Different paths result in different limits.

**54.** Given that $f(x, y)$ is continuous, then $\displaystyle\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b) < 0$, which means that for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon$ whenever

$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$

Let $\varepsilon = |f(a, b)|/2$, then $f(x, y) < 0$ for every point in the corresponding $\delta$ neighborhood since

$$|f(x, y) - f(a, b)| < \frac{|f(a, b)|}{2} \Rightarrow -\frac{|f(a, b)|}{2} < f(x, y) - f(a, b) < \frac{|f(a, b)|}{2}$$

$$\Rightarrow \frac{3}{2}f(a, b) < f(x, y) < \frac{1}{2}f(a, b) < 0.$$

**56.** False. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

See Exercise 21.

**58.** True

## Section 12.3 Partial Derivatives

**2.** $f_y(-1, -2) < 0$

**4.** $f_x(-1, -1) = 0$

**6.** $f(x, y) = x^2 - 3y^2 + 7$

$f_x(x, y) = 2x$

$f_y(x, y) = -6y$

**8.** $z = 2y^2\sqrt{x}$

$$\frac{\partial z}{\partial x} = \frac{y^2}{\sqrt{x}}$$

$$\frac{\partial z}{\partial y} = 4y\sqrt{x}$$

**10.** $z = y^3 - 4xy^2 - 1$

$$\frac{\partial z}{\partial x} = -4y^2$$

$$\frac{\partial z}{\partial y} = 3y^2 - 8xy$$

**12.** $z = xe^{x/y}$

$$\frac{\partial z}{\partial x} = \frac{x}{y}e^{x/y} + e^{x/y} = e^{x/y}\left(\frac{x}{y} + 1\right)$$

$$\frac{\partial z}{\partial y} = xe^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x^2}{y^2}e^{x/y}$$

**14.** $z = \ln\sqrt{xy} = \frac{1}{2}\ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{y}{xy} = \frac{1}{2x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{x}{xy} = \frac{1}{2y}$$

**16.** $z = \ln(x^2 - y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 - y^2}(2x) = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{x^2 - y^2}$$

**18.** $f(x, y) = \dfrac{xy}{x^2 + y^2}$

$$f_x(x, y) = \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2}$$

**20.** $g(x, y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2)$

$$g_x(x, y) = \frac{1}{2}\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$g_y(x, y) = \frac{1}{2}\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

**22.** $f(x, y) = \sqrt{2x + y^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(2x + y^3)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^3}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(2x + y^3)^{-1/2}(3y^2) = \frac{3y^2}{2\sqrt{2x + y^3}}$$

**24.** $z = \sin 3x \cos 3y$

$$\frac{\partial z}{\partial x} = 3\cos 3x \cos 3y$$

$$\frac{\partial z}{\partial y} = -3\sin 3x \sin 3y$$

**26.** $z = \cos(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = -2x\sin(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = -2y\sin(x^2 + y^2)$$

**28.**
$$\begin{aligned} f(x, y) &= \int_x^y (2t + 1)\,dt + \int_y^x (2t - 1)\,dt \\ &= \int_x^y (2t + 1)\,dt - \int_x^y (2t - 1)\,dt \\ &= \int_x^y 2\,dt = \Big[2t\Big]_x^y = 2y - 2x \end{aligned}$$

$$f_x(x, y) = -2$$

$$f_y(x, y) = 2$$

**30.** $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x)y + y^2 - x^2 + 2xy - y^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2y) = 2(x - y) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{x^2 - 2x(y + \Delta y) + (y + \Delta y)^2 - x^2 + 2xy - y^2}{\Delta y} = \lim_{\Delta y \to 0} (-2x + 2y + \Delta y) = 2(y - x) \end{aligned}$$

**32.** $f(x, y) = \dfrac{1}{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + y} - \dfrac{1}{x + y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{1}{x + y + \Delta y} - \dfrac{1}{x + y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{-1}{(x + y + \Delta y)(x + y)} = \frac{-1}{(x + y)^2}$$

**34.** $h(x, y) = x^2 - y^2$

$h_x(x, y) = 2x$

At $(-2, 1)$: $h_x(-2, 1) = -4$

$h_y(x, y) = -2y$

At $(-2, 1)$: $h_y(-2, 1) = -2$

**36.** $z = \cos(2x - y)$

$\dfrac{\partial z}{\partial x} = -2\sin(2x - y)$

At $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$, $\dfrac{\partial z}{\partial x} = -2\sin\left(\dfrac{\pi}{6}\right) = -1$

$\dfrac{\partial z}{\partial y} = -\sin(2x - y)(-1) = \sin(2x - y)$

At $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$, $\dfrac{\partial z}{\partial y} = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

**38.** $f(x, y) = \arccos(xy)$

$f_x(x, y) = \dfrac{-y}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, $f_x$ is undefined.

$f_y(x, y) = \dfrac{-x}{\sqrt{1 - x^2y^2}}$

At $(1, 1)$, $f_y$ is undefined.

**40.** $f(x, y) = \dfrac{6xy}{\sqrt{4x^2 + 5y^2}}$

$f_x(x, y) = \dfrac{30y^3}{(4x^2 + 5y^2)^{3/2}}$

At $(1, 1)$, $f_x(1, 1) = \dfrac{30}{27} = \dfrac{10}{9}$

$f_y(x, y) = \dfrac{24x^3}{(4x^2 + 5y^2)^{3/2}}$

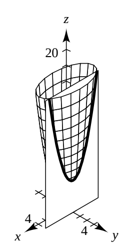
At $(1, 1)$, $f_y(1, 1) = \dfrac{8}{9}$

**42.** $z = x^2 + 4y^2,\ y = 1,\ (2, 1, 8)$

Intersecting curve: $z = x^2 + 4$

$\dfrac{\partial z}{\partial x} = 2x$

At $(2, 1, 8)$: $\dfrac{\partial z}{\partial x} = 2(2) = 4$

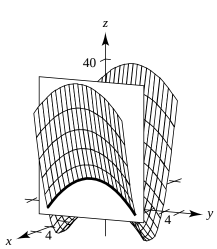



**44.** $z = 9x^2 - y^2,\ x = 1,\ (1, 3, 0)$

Intersecting curve: $z = 9 - y^2$

$\dfrac{\partial z}{\partial y} = -2y$

At $(1, 3, 0)$: $\dfrac{\partial z}{\partial y} = -2(3) = -6$




**46.** $f_x(x, y) = 9x^2 - 12y,\ f_y(x, y) = -12x + 3y^2$

$f_x = f_y = 0$: $9x^2 - 12y = 0 \implies 3x^2 = 4y$

$3y^2 - 12x = 0 \implies y^2 = 4x$

Solving for $x$ in the second equation, $x = y^2/4$, you obtain $3(y^2/4)^2 = 4y$.

$$3y^4 = 64y \implies y = 0 \text{ or } y = \frac{4}{3^{1/3}}$$

$$\implies x = 0 \text{ or } x = \frac{1}{4}\left(\frac{16}{3^{2/3}}\right)$$

Points: $(0, 0)$, $\left(\dfrac{4}{3^{2/3}}, \dfrac{4}{3^{1/3}}\right)$

**48.** $f_x(x, y) = \dfrac{2x}{x^2 + y^2 + 1} = 0 \Rightarrow x = 0$

$f_y(x, y) = \dfrac{2y}{x^2 + y^2 + 1} = 0 \Rightarrow y = 0$

Points: $(0, 0)$

**50.** (a) The graph is that of $f_x$.

(b) The graph is that of $f_y$.

**52.** $w = \dfrac{3xz}{x + y}$

$\dfrac{\partial w}{\partial x} = \dfrac{(x + y)(3z) - 3xz}{(x + y)^2} = \dfrac{3yz}{(x + y)^2}$

$\dfrac{\partial w}{\partial y} = \dfrac{-3xz}{(x + y)^2}$

$\dfrac{\partial w}{\partial z} = \dfrac{3x}{x + y}$

**54.** $G(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$G_x(x, y, z) = \dfrac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$

$G_y(x, y, z) = \dfrac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$

$G_z(x, y, z) = \dfrac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$

**56.** $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

$f_x(x, y, z) = 6xy - 5yz$

$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$

$f_z(x, y, z) = -5xy + 20yz$

**58.** $z = x^4 - 3x^2y^2 + y^4$

$\dfrac{\partial z}{\partial x} = 4x^3 - 6xy^2$

$\dfrac{\partial^2 z}{\partial x^2} = 12x^2 - 6y^2$

$\dfrac{\partial^2 z}{\partial y \partial x} = -12xy$

$\dfrac{\partial z}{\partial y} = -6x^2y + 4y^3$

$\dfrac{\partial^2 z}{\partial y^2} = -6x^2 + 12y^2$

$\dfrac{\partial^2 z}{\partial x \partial y} = -12xy$

**60.** $z = \ln(x - y)$

$\dfrac{\partial z}{\partial x} = \dfrac{1}{x - y}$

$\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{1}{(x - y)^2}$

$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{1}{(x - y)^2}$

$\dfrac{\partial z}{\partial y} = \dfrac{-1}{x - y} = \dfrac{1}{y - x}$

$\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{1}{(x - y)^2}$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{(x - y)^2}$

Therefore, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

**62.** $z = 2xe^y - 3ye^{-x}$

$\dfrac{\partial z}{\partial x} = 2e^y + 3ye^{-x}$

$\dfrac{\partial^2 z}{\partial x^2} = -3ye^{-x}$

$\dfrac{\partial^2 z}{\partial y \partial x} = 2e^y + 3ye^{-x}$

$\dfrac{\partial z}{\partial y} = 2xe^y - 3e^{-x}$

$\dfrac{\partial^2 z}{\partial y^2} = 2xe^y$

$\dfrac{\partial^2 z}{\partial x \partial y} = 2e^y + 3e^{-x}$

**64.** $z = \sin(x - 2y)$

$\dfrac{\partial z}{\partial x} = \cos(x - 2y)$

$\dfrac{\partial^2 z}{\partial x^2} = -\sin(x - 2y)$

$\dfrac{\partial^2 z}{\partial y \partial x} = 2\sin(x - 2y)$

$\dfrac{\partial z}{\partial y} = -2\cos(x - 2y)$

$\dfrac{\partial^2 z}{\partial y^2} = -4\sin(x - 2y)$

$\dfrac{\partial^2 z}{\partial x \partial y} = 2\sin(x - 2y)$

**66.** $z = \sqrt{9 - x^2 - y^2}$

$\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{9 - x^2 - y^2}}$

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{y^2 - 9}{(9 - x^2 - y^2)^{3/2}}$

$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{-xy}{(9 - x^2 - y^2)^{3/2}}$

$\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{9 - x^2 - y^2}}$

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^2 - 9}{(9 - x^2 - y^2)^{3/2}}$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{-xy}{(9 - x^2 - y^2)^{3/2}}$

Therefore, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

$\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$ if $x = y = 0$

**68.** $z = \dfrac{xy}{x-y}$

$$\frac{\partial z}{\partial x} = \frac{y(x-y) - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x-y)^2(-2y) + y^2(2)(x-y)(-1)}{(x-y)^4} = \frac{-2xy}{(x-y)^3}$$

$$\frac{\partial z}{\partial y} = -\frac{x(x-y) + xy}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x-y)^2(2x) - x^2(2)(x-y)}{(x-y)^4} = \frac{-2xy}{(x-y)^3}$$

There are no points for which $z_x = z_y = 0$.

**70.** $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

$f_x(x, y, z) = 2x - 3y$

$f_y(x, y, z) = -3x + 4z$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = -3$

$f_{yx}(x, y, z) = -3$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

**72.** $f(x, y, z) = \dfrac{2z}{x+y}$

$$f_x(x, y, z) = \frac{-2z}{(x+y)^2}$$

$$f_y(x, y, z) = \frac{-2z}{(x+y)^2}$$

$$f_{yy}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{xy}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{yx}(x, y, z) = \frac{4z}{(x+y)^3}$$

$$f_{yyx}(x, y, z) = \frac{-12z}{(x+y)^4}$$

$$f_{xyy}(x, y, z) = \frac{-12z}{(x+y)^4}$$

$$f_{yxy}(x, y, z) = \frac{-12z}{(x+y)^4}$$

**74.** $z = \sin x\left(\dfrac{e^y - e^{-y}}{2}\right)$

$$\frac{\partial z}{\partial x} = \cos x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$\frac{\partial z}{\partial y} = \sin x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \sin x\left(\frac{e^y - e^{-y}}{2}\right)$$

Therefore,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right) + \sin x\left(\frac{e^y - e^{-y}}{2}\right) = 0.$$

**76.** $z = \arctan \dfrac{y}{x}$

From Exercise 53, we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} = 0.$$

**78.** $z = \sin(wct)\sin(wx)$

$$\frac{\partial z}{\partial t} = wc\cos(wct)\sin(wx)$$

$$\frac{\partial^2 z}{\partial t^2} = -w^2c^2\sin(wct)\sin(wx)$$

$$\frac{\partial z}{\partial x} = w\sin(wct)\cos(wx)$$

$$\frac{\partial^2 z}{\partial x^2} = -w^2\sin(wct)\sin(wx)$$

Therefore, $\dfrac{\partial^2 z}{\partial t^2} = c^2\dfrac{\partial^2 z}{\partial x^2}$.

**80.** $z = e^{-t}\sin\frac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial z}{\partial x} = \frac{1}{c}e^{-t}\cos\frac{x}{c}$$

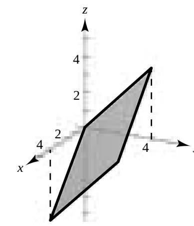
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2}e^{-t}\sin\frac{x}{c}$$

Therefore, $\frac{\partial z}{\partial t} = c^2\frac{\partial^2 z}{\partial x^2}$.

**82.** If $z = f(x, y)$, then to find $f_x$ you consider $y$ constant and differentiate with respect to $x$. Similarly, to find $f_y$, you consider $x$ constant and differentiate with respect to $y$.

**84.** The plane $z = -x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} < 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$




**86.** In this case, the mixed partials are equal, $f_{xy} = f_{yx}$.

See Theorem 12.3.

**88.** $f(x, y) = 200x^{0.7}y^{0.3}$

(a) $\frac{\partial f}{\partial x} = 140x^{-0.3}y^{0.3} = 140\left(\frac{y}{x}\right)^{0.3}$

At $(x, y) = (1000, 500)$, $\frac{\partial f}{\partial x} = 140\left(\frac{500}{1000}\right)^{0.3} = 140\left(\frac{1}{2}\right)^{0.3} \approx 113.72$

(b) $\frac{\partial f}{\partial x} = 60x^{0.7}y^{-0.7} = 60\left(\frac{x}{y}\right)^{0.7}$

At $(x, y) = (1000, 500)$, $\frac{\partial f}{\partial x} = 60\left(\frac{1000}{500}\right)^{0.7} = 60(2)^{0.7} \approx 97.47$

**90.** $$V(I, R) = 1000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$$

$$V_I(I, R) = 10{,}000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9\left[-\frac{1 + 0.10(1 - R)}{(1 + I)^2}\right] = -10{,}000\frac{[1 + 0.10(1 - R)]^{10}}{(1 + I)^{11}}$$

$V_I(0.03, 0.28) \approx -14{,}478.99$

$$V_R(I, R) = 10{,}000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9\left[\frac{-0.10}{1 + I}\right] = -1000\frac{[1 + 0.10(1 - R)]^9}{(1 + I)^{10}}$$

$V_R(0.03, 0.28) \approx -1391.17$

The rate of inflation has the greater negative influence on the growth of the investment. (See Exercise 61 in Section 12.1.)

**92.** $A = 0.885t - 22.4h + 1.20th - 0.544$

(a) $\frac{\partial A}{\partial t} = 0.885 + 1.20h$

$$\frac{\partial A}{\partial t}(30°, 0.80) = 0.885 + 1.20(0.80) = 1.845$$

$$\frac{\partial A}{\partial h} = -22.4 + 1.20t$$

$$\frac{\partial A}{\partial h}(30°, 0.80) = -22.4 + 1.20(30°) = 13.6$$

(b) The humidity has a greater effect on $A$ since its coefficient $-22.4$ is larger than that of $t$.

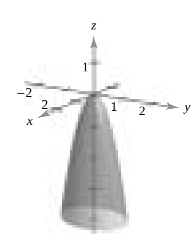
**94.** $U = -5x^2 + xy - 3y^2$

(a) $U_x = -10x + y$

(b) $U_y = x - 6y$

(c) $U_x(2, 3) = -17$ and $U_y(2, 3) = -16$. The person should consume one more unit of $y$ because the rate of decrease of satisfaction is less for $y$.

(d)



**96.** (a) $\dfrac{\partial z}{\partial x} = -1.55x + 22.15$

$\dfrac{\partial^2 z}{\partial x^2} = -1.55$

$\dfrac{\partial z}{\partial y} = 0.014y - 0.54$

$\dfrac{\partial^2 z}{\partial y^2} = 0.014$

(b) Concave downward $\left(\dfrac{\partial^2 z}{\partial x^2} < 0\right)$

The rate of increase of Medicare expenses $(z)$ is declining with respect to worker's compensation expenses $(x)$.

(c) Concave upward $\left(\dfrac{\partial^2 z}{\partial y^2} > 0\right)$

The rate of increase of Medicare expenses $(z)$ is increasing with respect to public assistance expenses $(y)$.

**98.** False

Let $z = x + y + 1$.

**100.** True

**102.** $f(x, y) = \displaystyle\int_x^y \sqrt{1 + t^3}\, dt$

By the Second Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\int_x^y \sqrt{1 + t^3}\, dt = -\frac{d}{dx}\int_y^x \sqrt{1 + t^3}\, dt = -\sqrt{1 + x^3}$$

$$\frac{\partial f}{\partial y} = \frac{d}{dy}\int_x^y \sqrt{1 + t^3}\, dt = \sqrt{1 + y^3}.$$

## Section 12.4 Differentials

**2.** $z = \dfrac{x^2}{y}$

$$dz = \frac{2x}{y}\, dx - \frac{x^2}{y^2}\, dy$$

**4.** $w = \dfrac{x + y}{z - 2y}$

$$dw = \frac{1}{z - 2y}\, dx + \frac{z + 2x}{(z - 2y)^2}\, dy - \frac{x + y}{(z - 2y)^2}\, dz$$

**6.** $z = \left(\dfrac{1}{2}\right)\left(e^{x^2+y^2} - e^{-x^2-y^2}\right)$

$$dz = 2x\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right) dx + 2y\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right) dy = \left(e^{x^2+y^2} + e^{-x^2-y^2}\right)(x\, dx + y\, dy)$$

**8.** $w = e^y \cos x + z^2$

$$dw = -e^y \sin x\, dx + e^y \cos x\, dy + 2z\, dz$$

**10.** $w = x^2yz^2 + \sin yz$

$$dw = 2xyz^2\, dx + (x^2z^2 + z\cos yz)\, dy + (2x^2yz + y\cos yz)\, dz$$

**12.** (a) $f(1, 2) = \sqrt{5} \approx 2.2361$

$f(1.05, 2.1) = \sqrt{5.5125} \approx 2.3479$

$\Delta z = 0.11180$

(b) $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\, dx + \dfrac{y}{\sqrt{x^2 + y^2}}\, dy$

$= \dfrac{x\, dx + y\, dy}{\sqrt{x^2 + y^2}} = \dfrac{0.05 + 2(0.1)}{\sqrt{5}} \approx 0.11180$

**14.** (a) $f(1, 2) = e^2 \approx 7.3891$

$f(1.05, 2.1) = 1.05e^{2.1} \approx 8.5745$

$\Delta z = 1.1854$

(b) $dz = e^y\, dx + xe^y\, dy$

$= e^2(0.05) + e^2(0.1) \approx 1.1084$

**16.** (a) $f(1, 2) = \dfrac{1}{2} = 0.5$

$$f(1.05, 2.1) = \frac{1.05}{2.1} = 0.5$$

$$\Delta z = 0$$

(b) $dz = \dfrac{1}{y}\,dx - \dfrac{x}{y^2}\,dy$

$$= \frac{1}{2}(0.05) - \frac{1}{4}(0.1) = 0$$

**18.** Let $z = x^2(1 + y)^3$, $x = 2$, $y = 9$, $dx = 0.03$, $dy = -0.1$. Then: $dz = 2x(1 + y)^3\,dx + 3x^2(1 + y)^2\,dy$

$$(2.03)^2(1 + 8.9)^3 - 2^2(1 + 9)^3 \approx 2(2)(1 + 9)^3(0.03) + 3(2)^2(1 + 9)^2(-0.1) = 0$$

**20.** Let $z = \sin(x^2 + y^2)$, $x = y = 1$, $dx = 0.05$, $dy = -0.05$. Then: $dz = 2x\cos(x^2 + y^2)\,dx + 2y\cos(x^2 + y^2)\,dy$

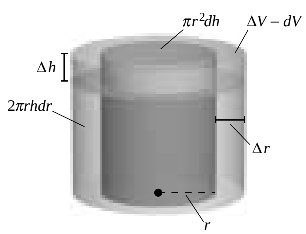
$$\sin[(1.05)^2 + (0.95)^2] - \sin 2 \approx 2(1)\cos(1^2 + 1^2)(0.05) + 2(1)\cos(1^2 + 1^2)(-0.05) = 0$$

**22.** In general, the accuracy worsens as $\Delta x$ and $\Delta y$ increase.

**24.** If $z = f(x, y)$, then $\Delta z \approx dz$ is the propagated error, and $\dfrac{\Delta z}{z} \approx \dfrac{dz}{z}$ is the relative error.

**26.** $V = \pi r^2 h$

$$dV = 2\pi rh\,dr + \pi r^2\,dh$$




**28.** $S = \pi r\sqrt{r^2 + h^2}$

$$r = 8,\ h = 20$$

$$\frac{dS}{dr} = \pi(r^2 + h^2)^{1/2} + \pi r^2(r^2 + h^2)^{-1/2}$$

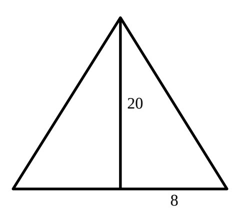
$$= \frac{\pi(r^2 + h^2) + \pi r^2}{(r^2 + h^2)^{1/2}} = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}$$

$$\frac{dS}{dh} = \pi r(r^2 + h^2)^{-1/2}h = \pi\frac{rh}{\sqrt{r^2 + h^2}}$$

$$dS = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}\,dr + \pi\frac{rh}{\sqrt{r^2 + h^2}}\,dh$$

$$= \frac{\pi}{\sqrt{r^2 + h^2}}[(2r^2 + h^2)\,dr + (rh)\,dh]$$

$$S(8, 20) = 541.3758$$




| $\Delta r$ | $\Delta h$ | $dS$ | $\Delta S$ | $\Delta S - dS$ |
|---|---|---|---|---|
| 0.1 | 0.1 | 10.0341 | 10.0768 | 0.0427 |
| 0.1 | $-0.1$ | 5.3671 | 5.3596 | $-0.0075$ |
| 0.001 | 0.002 | 0.12368 | 0.12368 | $0.683 \times 10^{-5}$ |
| $-0.0001$ | 0.0002 | $-0.00303$ | $-0.00303$ | $-0.286 \times 10^{-7}$ |

**30.** $\dfrac{\partial C}{\partial v} = 0.0817\left[(3.71)\dfrac{1}{2}v^{-1/2} - 0.25\right](T - 91.4)$

$= \left[\dfrac{0.1516}{v^{1/2}} - 0.0204\right](T - 91.4)$

$\dfrac{\partial C}{\partial T} = 0.0817\left(3.71\sqrt{v} + 5.81 - 0.25v\right)$

$dC = C_v\,dv + C_T\,dT$

$= \left(\dfrac{0.1516}{23^{1/2}} - 0.0204\right)(8 - 91.4)(\pm 3) + 0.0817\left(3.71\sqrt{23} + 5.81 - 0.25(23)\right)(\pm 1)$

$= \pm 2.79 \pm 1.46 = \pm 4.25$ Maximum propagated error

$\dfrac{dC}{C} = \dfrac{\pm 4.25}{-30.24} \approx \pm 0.14$

**32.** $(x, y) = (8.5, 3.2)$, $|dx| \le 0.05$, $|dy| \le 0.05$

$r = \sqrt{x^2 + y^2} \Longrightarrow dr = \dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy$

$= \dfrac{8.5}{\sqrt{8.5^2 + 3.2^2}}\,dx + \dfrac{3.2}{\sqrt{8.5^2 + 3.2^2}}\,dy \approx 0.9359\,dx + 0.3523\,dy$

$|dr| \le (1.288)(0.05) \approx 0.064$

$\theta = \arctan\left(\dfrac{y}{x}\right) \Longrightarrow d\theta = \dfrac{-\dfrac{y}{x^2}}{1 + \left(\dfrac{y}{x}\right)^2}\,dx + \dfrac{\dfrac{1}{x}}{1 + \left(\dfrac{y}{x}\right)^2}\,dy$

$= \dfrac{-y}{x^2 + y^2}\,dx + \dfrac{x}{x^2 + y^2}\,dy = \dfrac{-3.2}{8.5^2 + 3.2^2}\,dx + \dfrac{8.5}{8.5^2 + 3.2^2}\,dy$

Using the worst case scenario, $dx = -0.05$ and $dy = 0.05$, you see that

$|d\theta| \le 0.00194 + 0.00515 = 0.0071.$

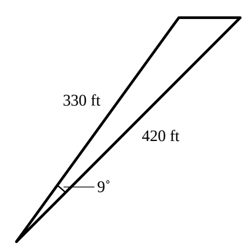
**34.** $a = \dfrac{v^2}{r}$

$da = \dfrac{2v}{r}\,dv - \dfrac{v^2}{r^2}\,dr$

$\dfrac{da}{a} = 2\dfrac{dv}{v} - \dfrac{dr}{r} = 2(0.03) - (-0.02) = 0.08 = 8\%$

**Note:** The maximum error will occur when $dv$ and $dr$ differ in signs.

**36.** (a) Using the Law of Cosines:




$a^2 = b^2 + c^2 - 2bc\cos A$

$= 330^2 + 420^2 - 2(330)(420)\cos 9°$

$a \approx 107.3$ ft.

(b) $a = \sqrt{b^2 + 420^2 - 2b(420)\cos\theta}$

$da = \dfrac{1}{2}\left[b^2 + 420^2 - 840b\cos\theta\right]^{-1/2}\left[(2b - 840\cos\theta)\,db + 840b\sin\theta\,d\theta\right]$

$= \dfrac{1}{2}\left[330^2 + 420^2 - 840(330)\left(\cos\dfrac{\pi}{20}\right)\right]^{-1/2}\left[\left(2(330) - 840\cos\dfrac{\pi}{20}\right)(6) + 840(330)\left(\sin\dfrac{\pi}{20}\right)\left(\dfrac{\pi}{180}\right)\right]$

$\approx \dfrac{1}{2}[11512.79]^{-1/2}[\pm 1774.79] \approx \pm 8.27$ ft

**38.** $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$$R = \frac{R_1R_2}{R_1 + R_2}$$

$$dR_1 = \Delta R_1 = 0.5$$

$$dR_2 = \Delta R_2 = -2$$

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\,dR + \frac{\partial R}{\partial R_2}\,dR_2 = \frac{{R_2}^2}{(R_1 + R_2)^2}\,\Delta R_1 + \frac{{R_1}^2}{(R_1 + R_2)^2}\,\Delta R_2$$

When $R_1 = 10$ and $R_2 = 15$, we have $\Delta R \approx \dfrac{15^2}{(10 + 15)^2}(0.5) + \dfrac{10^2}{(10 + 15)^2}(-2) = -0.14$ ohm.

**40.** $T = 2\pi\sqrt{\dfrac{L}{g}}$

$$dg = \Delta g = 32.24 - 32.09 = 0.15$$

$$dL = \Delta L = 2.48 - 2.5 = -0.02$$

$$\Delta T \approx dT = \frac{\partial T}{\partial g}\,dg + \frac{\partial T}{\partial L}\,dL = -\frac{\pi}{g}\sqrt{\frac{L}{g}}\,\Delta g + \frac{\pi}{\sqrt{Lg}}\,\Delta L$$

When $g = 32.09$ and $L = 2.5$, we have $\Delta T \approx -\dfrac{\pi}{32.09}\sqrt{\dfrac{2.5}{32.09}}(0.15) + \dfrac{\pi}{\sqrt{(2.5)(32.09)}}(-0.02) \approx -0.0111$ sec.

**42.** $z = f(x, y) = x^2 + y^2$

$$\begin{aligned}\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\ &= x^2 + 2x(\Delta x) + (\Delta x)^2 + y^2 + 2y(\Delta y) + (\Delta y)^2 - (x^2 + y^2)\\ &= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y)\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = \Delta x \text{ and } \epsilon_2 = \Delta y.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

**44.** $z = f(x, y) = 5x - 10y + y^3$

$$\begin{aligned}\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\ &= 5x + 5\Delta x - 10y - 10\Delta y + y^3 + 3y^2(\Delta y) + 3y(\Delta y)^2 + (\Delta y)^3 - (5x - 10y + y^3)\\ &= 5(\Delta x) + (3y^2 - 10)(\Delta y) + 0(\Delta x) + (3y(\Delta y) + (\Delta y)^2)\,\Delta y\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = 0 \text{ and } \epsilon_2 = 3y(\Delta y) + (\Delta y)^2.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.

**46.** $f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(0, 0) = \displaystyle\lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$

$f_y(0, 0) = \displaystyle\lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$

Thus, the partial derivatives exist at $(0, 0)$.

(b) Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{5x^3}{2x^3} = \frac{5}{2}$.

Along the line $x = 0$, $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

Thus, $f$ is not continuous at $(0, 0)$. Therefore $f$ is not differentiable at $(0, 0)$.

(See Theorem 12.5)

## Section 12.5 Chain Rules for Functions of Several Variables

**2.** $w = \sqrt{x^2 + y^2}$

$x = \cos t, y = e^t$

$$\frac{dw}{dt} = \frac{x}{\sqrt{x^2 + y^2}}(-\sin t) + \frac{y}{\sqrt{x^2 + y^2}}e^t$$

$$= \frac{-x\sin t + ye^t}{\sqrt{x^2 + y^2}} = \frac{-\cos t\sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}$$

**4.** $w = \ln\dfrac{y}{x}$

$x = \cos t$

$y = \sin t$

$$\frac{dw}{dt} = \left(\frac{-1}{x}\right)(-\sin t) + \left(\frac{1}{y}\right)(\cos t)$$

$$= \tan t + \cot t = \frac{1}{\sin t\cos t}$$

**6.** $w = \cos(x - y),\ x = t^2,\ y = 1$

(a) $\dfrac{dw}{dt} = -\sin(x - y)(2t) + \sin(x - y)(0)$

$= -2t\sin(x - y) = -2t\sin(t^2 - 1)$

(b) $w = \cos(t^2 - 1),\ \dfrac{dw}{dt} = -2t\sin(t^2 - 1)$

**8.** $w = xy\cos z$

$x = t$

$y = t^2$

$z = \arccos t$

(a) $\dfrac{dw}{dt} = (y\cos z)(1) + (x\cos z)(2t) + (-xy\sin z)\left(-\dfrac{1}{\sqrt{1 - t^2}}\right)$

$= t^2(t) + t(t)(2t) - t(t^2)\sqrt{1 - t^2}\left(\dfrac{-1}{\sqrt{1 - t^2}}\right) = t^3 + 2t^3 + t^3 = 4t^3$

(b) $w = t^4,\ \dfrac{dw}{dt} = 4t^3$

**10.** $w = xyz,\ x = t^2,\ y = 2t,\ z = e^{-t}$

(a) $\dfrac{dw}{dt} = yz(2t) + xz(2) + (xy)(-e^{-t})$

$= (2t)(e^{-t})(2t) + (t^2)(e^{-t})(2) + (t^2)(2t)(-e^{-t})$

$= 2t^2e^{-t}(2 + 1 - t) = 2t^2e^{-t}(3 - t)$

(b) $w = (t^2)(2t)(e^{-t}) = 2t^3e^{-t}$

$\dfrac{dw}{dt} = (2t^3)(-e^{-t}) + (e^{-t})(6t^2) = 2t^2e^{-t}(-t + 3)$

**12.** Distance $= f(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left[48 + \left(\sqrt{3} - \sqrt{2}\right)\right]^2 + \left[48t\left(1 - \sqrt{2}\right)\right]^2}$

$= 48t\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}}$

$f'(t) = 48\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}} = f'(1)$

**14.** $w = \dfrac{x^2}{y},$

$x = t^2,$

$y = t + 1,$

$t = 1$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2x}{y}(2t) + \frac{-x^2}{y^2}(1)$$

$$= \frac{2t^2(2t)}{t+1} - \frac{t^4}{(t+1)^2}$$

$$= \frac{(t+1)(4t^3) - t^4}{(t+1)^2}$$

$$= \frac{3t^4 + 4t^3}{(t+1)^2}$$

$$\frac{d^2w}{dt^2} = \frac{(t+1)^2(12t^3 + 12t^2) - (3t^4 + 4t^3)2(t+1)}{(t+1)^4}$$

At $t = 1$: $\dfrac{d^2w}{dt^2} = \dfrac{4(24) - (7)(4)}{16} = \dfrac{68}{16} = 4.25$

**16.** $w = y^3 - 3x^2y$

$x = e^s$

$y = e^t$

$$\frac{\partial w}{\partial s} = -6xy(e^s) + (3y^2 - 3x^2)(0) = -6e^{2s+t}$$

$$\frac{\partial w}{\partial t} = -6xy(0) + (3y^2 - 3x^2)(e^t)$$

$$= 3e^t(e^{2t} - e^{2s})$$

When $s = 0$ and $t = 1$, $\dfrac{\partial w}{\partial s} = -6e$ and $\dfrac{\partial w}{\partial t} = 3e(e^2 - 1)$.

**18.** $w = \sin(2x + 3y)$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2\cos(2x + 3y) + 3\cos(2x + 3y)$$

$$= 5\cos(2x + 3y) = 5\cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2\cos(2x + 3y) - 3\cos(2x + 3y)$$

$$= -\cos(2x + 3y) = -\cos(5s - t)$$

When $s = 0$ and $t = \dfrac{\pi}{2}$, $\dfrac{\partial w}{\partial s} = 0$ and $\dfrac{\partial w}{\partial t} = 0$.

**20.** $w = \sqrt{25 - 5x^2 - 5y^2}$, $x = r\cos\theta$, $y = r\sin\theta$

(a) $$\frac{\partial w}{\partial r} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}}\cos\theta + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}}\sin\theta$$

$$= \frac{-5r\cos^2\theta - 5r\sin^2\theta}{\sqrt{25 - 5x^2 - 5y^2}} = \frac{-5r}{\sqrt{25 - 5r^2}}$$

$$\frac{\partial w}{\partial \theta} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}}(-r\sin\theta) + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}}(r\cos\theta)$$

$$= \frac{-5r^2\sin^2\theta\cos\theta - 5r^2\sin\theta\cos\theta}{\sqrt{25 - 5x^2 - 5y^2}} = 0$$

(b) $w = \sqrt{25 - 5r^2}$

$$\frac{\partial w}{\partial r} = \frac{-5r}{\sqrt{25 - 5r^2}};\ \frac{\partial w}{\partial \theta} = 0$$

**22.** $w = \dfrac{yz}{x}, x = \theta^2, y = r + \theta, z = r - \theta$

(a) $\dfrac{\partial w}{\partial r} = \dfrac{-yz}{x^2}(0) + \dfrac{z}{x}(1) + \dfrac{y}{x}(1) = \dfrac{z + y}{x} = \dfrac{2r}{\theta^2}$

$$\frac{\partial w}{\partial \theta} = \frac{-yz}{x^2}(2\theta) + \frac{z}{x}(1) + \frac{y}{x}(-1)$$
$$= \frac{-(r + \theta)(r - \theta)}{\theta^4}(2\theta) + \frac{(r - \theta) - (r + \theta)}{\theta^2}$$
$$= \frac{2(\theta^2 - r^2)}{\theta^3} - \frac{2}{\theta} = \frac{-2r^2}{\theta^3}$$

(b) $w = \dfrac{yz}{x} = \dfrac{(r + \theta)(r - \theta)}{\theta^2} = \dfrac{r^2}{\theta^2} - 1$

$$\frac{\partial w}{\partial r} = \frac{2r}{\theta^2}$$
$$\frac{\partial w}{\partial \theta} = \frac{-2r^2}{\theta^3}$$

**24.** $w = x \cos yz,\; x = s^2, y = t^2,\; z = s - 2t$

$$\frac{\partial w}{\partial s} = \cos(yz)(2s) - xz\sin(yz)(0) - xy\sin(yz)(1)$$
$$= \cos(st^2 - 2t^3)2s - s^2t^2\sin(st^2 - 2t^3)$$
$$\frac{\partial w}{\partial t} = \cos(yz)(0) - xz\sin(yz)(2t) - xy\sin(yz)(-2)$$
$$= -2s^2t(s - 2t)\sin(st^2 - 2t^3) + 2s^2t^2\sin(st^2 - 2t^3)$$
$$= (6s^2t^2 - 2s^3t)\sin(st^2 - 2t^3)$$

**26.** $w = x^2 + y^2 + z^2, x = t\sin s, y = t\cos s, z = st^2$

$$\frac{\partial w}{\partial s} = 2x + \cos s + 2y(-t\sin s) + 2z(t^2)$$
$$= 2t^2\sin s\cos s - 2t^2\sin s\cos s + 2st^4 = 2st^4$$
$$\frac{\partial w}{\partial t} = 2x\sin s + 2y\cos s + 2z(2st)$$
$$= 2t\sin^2 s + 2t\cos^2 s + 4s^2t^3 = 2t + 4s^2t^3$$

**28.** $\cos x + \tan xy + 5 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{-\sin x + y\sec^2 xy}{x\sec^2 xy}$$

**30.** $\dfrac{x}{x^2 + y^2} - y^2 - 6 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$
$$= -\frac{(y^2 - x^2)/(x^2 + y^2)^2}{(-2xy)/(x^2 + y^2)^2 - 2y}$$
$$= \frac{y^2 - x^2}{2xy + 2y(x^2 + y^2)^2}$$
$$= \frac{y^2 - x^2}{2xy + 2yx^4 + 4x^2y^3 + 2y^5}$$

**32.** $F(x, y, z) = xz + yz + xy$

$$F_x = z + y$$
$$F_y = z + x$$
$$F_z = x + y$$
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y + z}{x + y}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x + z}{x + y}$$

**34.** $F(x, y, z) = e^x\sin(y + z) - z$

$$F_x = e^x\sin(y + z)$$
$$F_y = e^x\cos(y + z)$$
$$F_z = e^x\cos(y + z) - 1$$
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{e^x\sin(y + z)}{1 - e^x\cos(y + x)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x\cos(y + z)}{1 - e^x\cos(y + z)}$$

**36.** $x + \sin(y + z) = 0$

(i) $1 + \dfrac{\partial z}{\partial x}\cos(y + z) = 0$ implies

$$\frac{\partial z}{\partial x} = -\frac{1}{\cos(y + z)} = -\sec(y + z).$$

(ii) $\left(1 + \dfrac{\partial z}{\partial y}\right)\cos(y + z) = 0$ implies $\dfrac{\partial z}{\partial y} = -1.$

**38.** $x \ln y + y^2 z + z^2 - 8 = 0$

(i) $\dfrac{\partial z}{\partial x} = \dfrac{-F_x(x, y, z)}{F_z(x, y, z)} = \dfrac{-\ln y}{y^2 + 2z}$

(ii) $\dfrac{\partial z}{\partial y} = \dfrac{-F_y(x, y, z)}{F_z(x, y, z)} = -\dfrac{\dfrac{x}{y} + 2yz}{y^2 + 2z} = -\dfrac{x + 2y^2 z}{y^3 + 2yz}$

**40.** $x^2 + y^2 - z^2 - 5yw + 10w^2 - 2 = F(x, y, z, w)$

$F_x = 2x, F_y = 2y - 5w, F_z = 2z, F_w = -5y + 20w$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{-2x}{-5y + 20w} = \frac{2x}{5y - 20w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{5w - 2y}{20w - 5y}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = \frac{2z}{5y - 20w}$$

**42.** $F(x, y, z, w) = w - \sqrt{x - y} - \sqrt{y - z} = 0$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{1}{2}\frac{(x - y)^{-1/2}}{1} = \frac{1}{2\sqrt{x - y}}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-1}{2}(x - y)^{-1/2} + \frac{1}{2}(y - z)^{-1/2}$$

$$= \frac{-1}{2\sqrt{x - y}} + \frac{1}{2\sqrt{y - z}}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y - z}}$$

**44.** $f(x, y) = x^3 - 3xy^2 + y^3$

$$f(tx, ty) = (tx)^3 - 3(tx)(ty)^2 + (ty)^3$$

$$= t^3(x^3 - 3xy^2 + y^3) = t^3 f(x, y)$$

Degree: 3

$$xf_x(x, y) + yf_y(x, y) = x(3x^2 - 3y^2) + y(-6xy + 3y^2)$$

$$= 3x^3 - 9xy^2 + 3y^3 = 3f(x, y)$$

**46.** $f(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)^2}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{x^2}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

$$xf_x(x, y) + yf_y(x, y) = x\left[\frac{x^3 + 2xy^2}{(x^2 + y^2)^{3/2}}\right] + y\left[\frac{-x^2 y}{(x^2 + y^2)^{3/2}}\right]$$

$$= \frac{x^4 + x^2 y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}}$$

$$= \frac{x^2}{\sqrt{x^2 + y^2}} = f(x, y)$$

**48.** $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s}$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$ (Page 878)

**50.** $\dfrac{dy}{dx} = -\dfrac{f_x(x, y)}{f_y(x, y)}$

$$\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y(x, y, z)}{f_z(x, y, z)}$$

**52.** (a) $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right) = \pi(12)[2(36)(6) + 12(-4)] = 4608\pi \text{ in.}^3/\text{min}$$

(b) $S = 2\pi r(r + h)$

$$\frac{dS}{dt} = 2\pi\left[(2r + h)\frac{dr}{dt} + r\frac{dh}{dt}\right] = 2\pi[(24 + 36)(6) + 12(-4)] = 624\pi \text{ in.}^2/\text{min}$$

**54.** (a) $V = \frac{\pi}{3}(r^2 + rR + R^2)h$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[(2r + R)h\frac{dr}{dt} + (r + 2R)h\frac{dR}{dt} + (r^2 + rR + R^2)\frac{dh}{dt}\right]$$

$$= \frac{\pi}{3}\left[[2(15) + 25](10)(4) + [15 + 2(25)](10)(4) + [(15)^2 + (15)(25) + (25)^2](12)\right]$$

$$= \frac{\pi}{3}(19{,}500) = 6{,}500\pi \text{ cm}^3/\text{min}$$

(b) $S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$

$$\frac{dS}{dt} = \pi\left\{\left[\sqrt{(R - r)^2 + h^2} - (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dr}{dt} + \left[\sqrt{(R - r)^2 + h^2} + (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dR}{dt} + (R + r)\frac{h}{\sqrt{(R - r)^2 + h^2}}\frac{dh}{dt}\right\}$$

$$= \pi\left\{\left[\sqrt{(25 - 15)^2 + 10^2} - (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + \left[\sqrt{(25 - 15)^2 + 10^2} + (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + (25 + 15)\frac{10}{\sqrt{(25 - 15)^2 + 10^2}}(12)\right\}$$

$$= 320\sqrt{2}\pi \text{ cm}^2/\text{min}$$

**56.** $pV = mRT$

$$T = \frac{1}{mR}(pV)$$

$$\frac{dT}{dt} = \frac{1}{mR}\left[V\frac{dp}{dt} + p\frac{dV}{dt}\right]$$

**58.** $g(t) = f(xt, yt) = t^n f(x, y)$

Let $u = xt,\ v = yt,$ then

$$g'(t) = \frac{\partial f}{\partial u}\cdot\frac{du}{dt} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dt} = \frac{\partial f}{\partial u}x + \frac{\partial f}{\partial v}y$$

and $g'(t) = nt^{n-1}f(x, y)$.

Now, let $t = 1$ and we have $u = x,\ v = y$. Thus,

$$\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y = nf(x, y).$$

**60.** $w = (x - y)\sin(y - x)$

$$\frac{\partial w}{\partial x} = -(x - y)\cos(y - x) + \sin(y - x)$$

$$\frac{\partial w}{\partial y} = (x - y)\cos(y - x) - \sin(y - x)$$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

**62.** $w = \arctan\frac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

$$= \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) = \arctan(\tan\theta) = \theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{\partial w}{\partial x} = \frac{-y}{x^2 + y^2},\ \frac{\partial w}{\partial y} = \frac{x}{x^2 + y^2},\ \frac{\partial w}{\partial r} = 0,\ \frac{\partial w}{\partial \theta} = 1$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \frac{y^2}{(x^2 + y^2)^2} + \frac{x^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{r^2}$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 = 0 + \frac{1}{r^2}(1) = \frac{1}{r^2}$$

Therefore, $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$

**64.** Note first that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}.$$

$$\frac{\partial u}{\partial r} = \frac{x}{x^2 + y^2}\cos\theta + \frac{y}{x^2 + y^2}\sin\theta = \frac{r\cos^2\theta + r\sin^2\theta}{r^2} = \frac{1}{r}$$

$$\frac{\partial v}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r\sin\theta) + \frac{x}{x^2 + y^2}(r\cos\theta) = \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2} = 1$$

Thus, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{-y}{x^2 + y^2}\cos\theta + \frac{x}{x^2 + y^2}\sin\theta = \frac{-r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^2} = 0$$

$$\frac{\partial u}{\partial \theta} = \frac{x}{x^2 + y^2}(-r\sin\theta) + \frac{y}{x^2 + y^2}(r\cos\theta) = \frac{-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}{r^2} = 0$$

Thus, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

## Section 12.6 Directional Derivatives and Gradients

**2.** $f(x, y) = x^3 - y^3, \mathbf{v} = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$$\nabla f(x, y) = 3x^2\mathbf{i} - 3y^2\mathbf{j}$$

$$\nabla f(4, 3) = 48\mathbf{i} - 27\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(4, 3) = \nabla f(4, 3) \cdot \mathbf{u} = 24\sqrt{2} - \frac{27}{2}\sqrt{2} = \frac{21}{2}\sqrt{2}$$

**4.** $f(x, y) = \dfrac{x}{y}$

$$\mathbf{v} = -\mathbf{j}$$

$$\nabla f(x, y) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$$

$$\nabla f(1, 1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{j}$$

$$D_{\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = 1$$

**6.** $g(x, y) = \arccos xy, \mathbf{v} = \mathbf{i} + 5\mathbf{j}$

$$\nabla g(x, y) = \frac{-y}{\sqrt{1 - (xy)^2}}\mathbf{i} + \frac{-x}{\sqrt{1 - (xy)^2}}\mathbf{j}$$

$$\nabla g(1, 0) = -\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{26}}\mathbf{i} + \frac{5}{\sqrt{26}}\mathbf{j}$$

$$D_{\mathbf{u}} g(1, 0) = \nabla g(1, 0) \cdot \mathbf{u} = \frac{-5}{\sqrt{26}} = \frac{-5\sqrt{26}}{26}$$

**8.** $h(x, y) = e^{-(x^2 + y^2)}$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla h = -2xe^{-(x^2 + y^2)}\mathbf{i} - 2ye^{-(x^2 + y^2)}\mathbf{j}$$

$$\nabla h(0, 0) = \mathbf{0}$$

$$D_{\mathbf{u}} h(0, 0) = \nabla h(0, 0) \cdot \mathbf{u} = 0$$

**10.** $f(x, y, z) = x^2 + y^2 + z^2$

$$\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(1, 2, -1) = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$$

$$D_{\mathbf{u}} f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u} = -\frac{6}{7}\sqrt{14}$$

**12.** $h(x, y, z) = xyz$

$$\mathbf{v} = \langle 2, 1, 2 \rangle$$

$$\nabla h = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\nabla h(2, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}} h(2, 1, 1) = \nabla h(2, 1, 1) \cdot \mathbf{u} = \frac{8}{3}$$

**14.** $f(x, y) = \dfrac{y}{x+y}$

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\nabla f = -\frac{y}{(x+y)^2}\mathbf{i} + \frac{x}{(x+y)^2}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{\sqrt{3}y}{2(x+y)^2} - \frac{x}{2(x+y)^2}$$

$$= -\frac{1}{2(x+y)^2}\left(\sqrt{3}y + x\right)$$

**16.** $g(x, y) = xe^y$

$$\mathbf{u} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla g = e^y\,\mathbf{i} + xe^y\,\mathbf{j}$$

$$D_{\mathbf{u}} g = -\frac{1}{2}e^y + \frac{\sqrt{3}}{2}xe^y = \frac{e^y}{2}\left(\sqrt{3}x - 1\right)$$

**18.** $f(x, y) = \cos(x + y)$

$$\mathbf{v} = \frac{\pi}{2}\mathbf{i} - \pi\mathbf{j}$$

$$\nabla f = -\sin(x+y)\mathbf{i} - \sin(x+y)\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} f = -\frac{1}{\sqrt{5}}\sin(x+y) + \frac{2}{\sqrt{5}}\sin(x+y)$$

$$= \frac{1}{\sqrt{5}}\sin(x+y) = \frac{\sqrt{5}}{5}\sin(x+y)$$

At $(0, \pi)$, $D_{\mathbf{u}} f = 0$.

**20.** $g(x, y, z) = xye^z$

$$\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = ye^z\,\mathbf{i} + xe^z\,\mathbf{j} + xye^z\,\mathbf{k}$$

At $(2, 4, 0)$, $\nabla g = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u} = -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{8}{\sqrt{5}}$$

**22.** $g(x, y) = 2xe^{y/x}$

$$\nabla g(x, y) = \left(-\frac{2y}{x}e^{y/x} + 2e^{y/x}\right)\mathbf{i} + 2e^{y/x}\mathbf{j}$$

$$\nabla g(2, 0) = 2\mathbf{i} + 2\mathbf{j}$$

**24.** $z = \ln(x^2 - y)$

$$\nabla z(x, y) = \frac{2x}{x^2 - y}\mathbf{i} - \frac{1}{x^2 - y}\mathbf{j}$$

$$\nabla z(2, 3) = 4\mathbf{i} - \mathbf{j}$$

**26.** $w = x\tan(y + z)$

$$\nabla w(x, y, z) = \tan(y+z)\mathbf{i} + x\sec^2(y+z)\mathbf{j} + x\sec^2(y+z)\mathbf{k}$$

$$\nabla w(4, 3, -1) = \tan 2\mathbf{i} + 4\sec^2 2\mathbf{j} + 4\sec^2 2\mathbf{k}$$

**28.** $\overrightarrow{PQ} = -2\mathbf{i} + 7\mathbf{j}$, $\mathbf{u} = -\dfrac{2}{\sqrt{53}}\mathbf{i} + \dfrac{7}{\sqrt{53}}\mathbf{j}$

$$\nabla f(x, y) = 6x\mathbf{i} - 2y\mathbf{j},\ \nabla f(3, 1) = 18\mathbf{i} - 2\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{36}{\sqrt{53}} - \frac{14}{\sqrt{53}} = -\frac{50}{\sqrt{53}} = -\frac{50\sqrt{53}}{53}$$

**30.** $\overrightarrow{PQ} = \dfrac{\pi}{2}\mathbf{i} + \pi\mathbf{j}$, $\mathbf{u} = \dfrac{1}{\sqrt{5}}\mathbf{i} + \dfrac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla f(x, y) = 2\cos 2x\cos y\mathbf{i} - \sin 2x\sin y\mathbf{j}$$

$$\nabla f(0, 0) = 2\mathbf{i}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

**32.** $$h(x, y) = y\cos(x - y)$$
$$\nabla h(x, y) = -y\sin(x - y)\mathbf{i} + [\cos(x - y) + y\sin(x - y)]\mathbf{j}$$
$$\nabla h\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{6}\mathbf{i} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)\mathbf{j}$$
$$\left\|\nabla h\left(0, \frac{\pi}{3}\right)\right\| = \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\sqrt{3}\pi + 3\pi^2}{36}} = \frac{\sqrt{3\left(2\pi^2 - 2\sqrt{3}\pi + 3\right)}}{6}$$

**34.** $$g(x, y) = ye^{-x^2}$$
$$\nabla g(x, y) = -2xye^{-x^2}\mathbf{i} + e^{-x^2}\mathbf{j}$$
$$\nabla g(0, 5) = \mathbf{j}$$
$$\|\nabla g(0, 5)\| = 1$$

**36.** $$w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$$
$$\nabla w = \frac{1}{\left(\sqrt{1 - x^2 - y^2 - z^2}\right)^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$
$$\nabla w(0, 0, 0) = \mathbf{0}$$
$$\|\nabla w(0, 0, 0)\| = 0$$

**38.** $$w = xy^2z^2$$
$$\nabla w = y^2z^2\mathbf{i} + 2xyz^2\mathbf{j} + 2xy^2z\mathbf{k}$$
$$\nabla w(2, 1, 1) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$
$$\|\nabla w(2, 1, 1)\| = \sqrt{33}$$

**For Exercises 40–46, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta$.**

**40.** (a) $D_{\pi/4} f(3, 2) = -\left(\frac{1}{3}\right)\frac{\sqrt{2}}{2} - \left(\frac{1}{2}\right)\frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{12}$

(b) $D_{2\pi/3} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{2 - 3\sqrt{3}}{12}$

**42.** (a) $$\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{i} + \mathbf{j})$$
$$D_\mathbf{u} f = \nabla f \cdot \mathbf{u}$$
$$= -\left(\frac{1}{3}\right)\frac{1}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{12}$$

(b) $$\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$$
$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$
$$\mathbf{u} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$
$$D_\mathbf{u} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

**44.** $$\nabla f = -\left(\frac{1}{3}\right)\mathbf{i} - \left(\frac{1}{2}\right)\mathbf{j}$$

**46.** $$\nabla f = -\frac{1}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$$
$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{13}}(-2\mathbf{i} - 3\mathbf{j})$$

Therefore, $\mathbf{u} = \left(1/\sqrt{13}\right)(3\mathbf{i} - 2\mathbf{j})$ and $D_\mathbf{u} f(3, 2) = \nabla f \cdot \mathbf{u} = 0$. $\nabla f$ is the direction of greatest rate of change of $f$. Hence, in a direction orthogonal to $\nabla f$, the rate of change of $f$ is 0.

**For Exercises 48 and 50, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x\cos\theta - 2y\sin\theta = -2(x\cos\theta + y\sin\theta)$.**

**48.** (a) $D_{-\pi/4} f(1, 2) = -2\left(\dfrac{\sqrt{2}}{2} - \sqrt{2}\right) = \sqrt{2}$

(b) $D_{\pi/3} f(1, 2) = -2\left(\dfrac{1}{2} + \sqrt{3}\right) = -\left(1 + 2\sqrt{3}\right)$

**50.** $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$$\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \frac{1}{\sqrt{5}}(-\mathbf{i} - 2\mathbf{j})$$

Therefore,

$\mathbf{u} = \left(1/\sqrt{5}\right)(-2\mathbf{i} + \mathbf{j})$ and

$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = 0.$

**52.** (a) In the direction of the vector $\mathbf{i} + \mathbf{j}$.

(b) $\nabla f = \dfrac{1}{2}y\dfrac{1}{2\sqrt{x}}\mathbf{i} + \dfrac{1}{2}\sqrt{x}\,\mathbf{j} = \dfrac{y}{4\sqrt{x}}\mathbf{i} + \dfrac{1}{2}\sqrt{x}\,\mathbf{j}$

$\nabla f(1, 2) = \dfrac{1}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

(Same direction as in part (a).)

(c) $-\nabla f = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, the direction opposite that of the gradient.

**54.** (a) $f(x, y) = \dfrac{8y}{1 + x^2 + y^2} = 2$

$\implies 4y = 1 + x^2 + y^2$

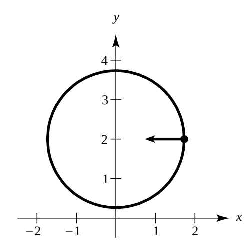
$4 = y^2 - 4y + 4 + x^2 + 1$

$(y - 2)^2 + x^2 = 3$

Circle: center: $(0, 2)$, radius: $\sqrt{3}$

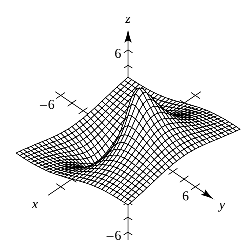
(b) $\nabla f = \dfrac{-16xy}{(1 + x^2 + y^2)^2}\mathbf{i} + \dfrac{8 + 8x^2 - 8y^2}{(1 + x^2 + y^2)^2}\mathbf{j}$

$\nabla f\left(\sqrt{3}, 2\right) = \dfrac{-\sqrt{3}}{2}\mathbf{i}$




(c) The directional derivative of $f$ is 0 in the directions $\pm\mathbf{j}$.

(d)




**56.** $f(x, y) = 6 - 2x - 3y$

$c = 6,\ P = (0, 0)$

$\nabla f(x, y) = -2\mathbf{i} - 3\mathbf{j}$

$6 - 2x - 3y = 6$

$0 = 2x + 3y$

$\nabla f(0, 0) = -2\mathbf{i} - 3\mathbf{j}$

**58.** $f(x, y) = xy$

$c = -3,\ P = (-1, 3)$

$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$

$xy = -3$

$\nabla f(-1, 3) = 3\mathbf{i} - \mathbf{j}$

**60.** $3x^2 - 2y^2 = 1$

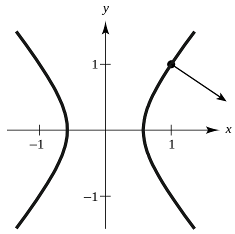
$f(x, y) = 3x^2 - 2y^2$

$\nabla f(x, y) = 6x\mathbf{i} - 4y\mathbf{j}$

$\nabla f(1, 1) = 6\mathbf{i} - 4\mathbf{j}$

$$\frac{\nabla f(1,1)}{\|\nabla f(1,1)\|} = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$$

$$= \frac{\sqrt{13}}{13}(3\mathbf{i} - 2\mathbf{j})$$




**62.** $xe^y - y = 5$

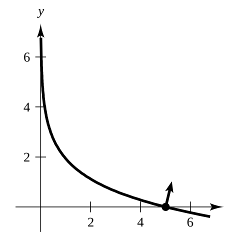
$f(x, y) = xe^y - y$

$\nabla f(x, y) = e^y\mathbf{i} + (xe^y - 1)\mathbf{j}$

$\nabla f(5, 0) = \mathbf{i} + 4\mathbf{j}$

$$\frac{\nabla f(5,0)}{\|\nabla f(5,0)\|} = \frac{1}{\sqrt{17}}(\mathbf{i} + 4\mathbf{j})$$

$$= \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$$




**64.** $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$

$\nabla h = -0.002x\mathbf{i} - 0.008y\mathbf{j}$

$\nabla h(500, 300) = -\mathbf{i} - 2.4\mathbf{j}$ or

$5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$

**66.** The directional derivative gives the slope of a surface at a point in an arbitrary direction $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

**68.** See the definition, page 887.

**70.** The gradient vector is normal to the level curves.

See Theorem 12.12.

**72.** The wind speed is greatest at $A$.

**74.** $T(x, y) = 100 - x^2 - 2y^2,\quad P = (4, 3)$

$$\frac{dx}{dt} = -2x \qquad \frac{dy}{dt} = -4y$$

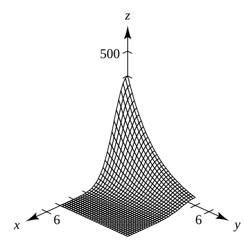
$x(t) = C_1e^{-2t} \qquad y(t) = C_2e^{-4t}$

$4 = x(0) = C_1 \qquad 3 = y(0) = C_2$

$x(t) = 4e^{-2t} \qquad y(t) = 3e^{-4t}$

$$\frac{3x^2}{16} = e^{-4t} = y \implies u = \frac{3}{16}x^2$$

**76.** (a)




(b) $\nabla T(x, y) = 400e^{-(x^2+y)/2}\left[(-x)\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

$\nabla T(3, 5) = 400e^{-7}\left[-3\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

There will be no change in directions perpendicular to the gradient: $\pm(\mathbf{i} - 6\mathbf{j})$

(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i} - \frac{1}{2}\mathbf{j}$

**78.** False

$D_{\mathbf{u}} f(x, y) = \sqrt{2} > 1$ when

$$\mathbf{u} = \left(\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sin\frac{\pi}{4}\right)\mathbf{j}.$$

**80.** True

## Section 12.7 Tangent Planes and Normal Lines

**2.** $F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$

$$x^2 + y^2 + z^2 = 25$$

Sphere, radius 5, centered at origin.

**4.** $F(x, y, z) = 16x^2 - 9y^2 + 144z = 0$

$16x^2 - 9y^2 + 144z = 0$ Hyperbolic paraboloid

**6.** $F(x, y, z) = x^2 + y^2 + z^2 - 11$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$
$$\nabla F(3, 1, 1) = 6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{44}}(6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$
$$= \frac{1}{\sqrt{11}}(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{11}}{11}(3\mathbf{i} + \mathbf{j} + \mathbf{k})$$

**8.** $F(x, y, z) = x^3 - z$

$$\nabla F(x, y, z) = 3x^2\mathbf{i} - \mathbf{k}$$
$$\nabla F(2, 1, 8) = 12\mathbf{i} - \mathbf{k}$$
$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{145}}(12\mathbf{i} - \mathbf{k})$$
$$= \frac{\sqrt{145}}{145}(12\mathbf{i} - \mathbf{k})$$

**10.** $F(x, y, z) = x^2 + 3y + z^3 - 9$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 3\mathbf{j} + 3z^2\mathbf{k}$$
$$\nabla F(2, -1, 2) = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$$
$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$$

**12.** $F(x, y, z) = ze^{x^2-y^2} - 3$

$$\nabla F(x, y, z) = 2xze^{x^2-y^2}\mathbf{i} - 2yze^{x^2-y^2}\mathbf{j} + e^{x^2-y^2}\mathbf{k}$$
$$\nabla F(2, 2, 3) = 12\mathbf{i} - 12\mathbf{j} + \mathbf{k}$$
$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{17}(12\mathbf{i} - 12\mathbf{j} + \mathbf{k})$$

**14.** $F(x, y, z) = \sin(x - y) - z - 2$

$$\nabla F(x, y, z) = \cos(x - y)\mathbf{i} - \cos(x - y)\mathbf{j} - \mathbf{k}$$
$$\nabla F\left(\frac{\pi}{3}, \frac{\pi}{6}, -\frac{3}{2}\right) = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}$$
$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{10}}\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}\right)$$
$$= \frac{1}{\sqrt{10}}\left(\sqrt{3}\,\mathbf{i} - \sqrt{3}\,\mathbf{j} - 2\mathbf{k}\right)$$
$$= \frac{\sqrt{10}}{10}\left(\sqrt{3}\,\mathbf{i} - \sqrt{3}\,\mathbf{j} - 2\mathbf{k}\right)$$

**16.** $f(x, y) = \dfrac{y}{x}$, $(1, 2, 2)$

$$F(x, y, z) = \frac{y}{x} - z$$

$F_x(x, y, z) = -\dfrac{y}{x^2}$ $\qquad F_y(x, y, z) = \dfrac{1}{x}$ $\qquad F_z(x, y, z) = -1$

$F_x(1, 2, 2) = -2$ $\qquad F_y(1, 2, 2) = 1$ $\qquad F_z(1, 2, 2) = -1$

$$-2(x - 1) + (y - 2) - (z - 2) = 0$$
$$-2x + y - z + 2 = 0$$
$$2x - y + z = 2$$

**18.** $g(x, y) = \arctan\dfrac{y}{x}$, $(1, 0, 0)$

$$G(x, y, z) = \arctan\frac{y}{x} - z$$

$G_x(x, y, z) = \dfrac{-(y/x^2)}{1 + (y^2/x^2)} = \dfrac{-y}{x^2 + y^2}$ $\qquad G_y(x, y, z) = \dfrac{1/x}{1 + (y^2/x^2)} = \dfrac{x}{x^2 + y^2}$ $\qquad G_z(x, y, z) = -1$

$G_x(1, 0, 0) = 0$ $\qquad G_y(1, 0, 0) = 1$ $\qquad G_z(1, 0, 0) = -1$

$$y - z = 0$$

**20.** $f(x, y) = 2 - \frac{2}{3}x - y, \ (3, -1, 1)$

$F(x, y, z) = 2 - \frac{2}{3}x - y - z$

$F_x(x, y, z) = -\frac{2}{3}, \qquad F_y(x, y, z) = -1, \qquad F_z(x, y, z) = -1$

$$-\frac{2}{3}(x - 3) - (y + 1) - (z - 1) = 0$$

$$-\frac{2}{3}x - y - z + 2 = 0$$

$$2x + 3y + 3z = 6$$

**22.** $z = x^2 - 2xy + y^2, (1, 2, 1)$

$F(x, y, z) = x^2 - 2xy + y^2 - z$

$F_x(x, y, z) = 2x - 2y \qquad F_y(x, y, z) = -2x + 2y \qquad F_z(x, y, z) = -1$

$F_x(1, 2, 1) = -2 \qquad F_y(1, 2, 1) = 2 \qquad F_z(1, 2, 1) = -1$

$$-2(x - 1) + 2(y - 2) - (z - 1) = 0$$

$$-2x + 2y - z - 1 = 0$$

$$2x - 2y + z = -1$$

**24.** $h(x, y) = \cos y, \ \left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$H(x, y, z) = \cos y - z$

$H_x(x, y, z) = 0 \qquad H_y(x, y, z) = -\sin y \qquad H_z(x, y, z) = -1$

$H_x\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = 0 \qquad H_y\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \qquad H_z\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -1$

$$-\frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$$

$$-\frac{\sqrt{2}}{2}y - z + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} = 0$$

$$4\sqrt{2}y + 8z = \sqrt{2}(\pi + 4)$$

**26.** $x^2 + 2z^2 = y^2, \ (1, 3, -2)$

$F(x, y, z) = x^2 - y^2 + 2z^2$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 4z$

$F_x(1, 3, -2) = 2 \qquad F_y(1, 3, -2) = -6 \qquad F_z(1, 3, -2) = -8$

$$2(x - 1) - 6(y - 3) - 8(z + 2) = 0$$

$$(x - 1) - 3(y - 3) - 4(z + 2) = 0$$

$$x - 3y - 4z = 0$$

**28.** $x = y(2z - 3), (4, 4, 2)$

$F(x, y, z) = x - 2yz + 3y$

$F_x(x, y, z) = 1 \qquad F_y(x, y, z) = -2z + 3 \qquad F_z(x, y, z) = -2y$

$F_x(4, 4, 2) = 1 \qquad F_y(4, 4, 2) = -1 \qquad F_z(4, 4, 2) = -8$

$$(x - 4) - 1(y - 4) - 8(z - 2) = 0$$

$$x - y - 8z = -16$$

$$-x + y + 8z = 16$$

**30.** $x^2 + y^2 + z^2 = 9$, $(1, 2, 2)$

$F(x, y, z) = x^2 + y^2 + z^2 - 9$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = 2z$

$F_x(1, 2, 2) = 2 \qquad F_y(1, 2, 2) = 4 \qquad F_z(1, 2, 2) = 4$

Direction numbers: 1, 2, 2

Plane: $(x - 1) + 2(y - 2) + 2(z - 2) = 0$, $x + 2y + 2z = 9$

Line: $\dfrac{x - 1}{1} = \dfrac{y - 2}{2} = \dfrac{z - 2}{2}$

**32.** $x^2 - y^2 + z^2 = 0$, $(5, 13, -12)$

$F(x, y, z) = x^2 - y^2 + z^2$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 2z$

$F_x(5, 13, -12) = 10 \qquad F_y(5, 13, -12) = -26 \qquad F_z(x, y, z) = -24$

Direction numbers: $5, -13, -12$

Plane

$$\begin{aligned} 5(x - 5) - 13(y - 13) - 12(z + 12) &= 0 \\ 5x - 13y - 12z &= 0 \end{aligned}$$

Line: $\dfrac{x - 5}{5} = \dfrac{y - 13}{-13} = \dfrac{z + 12}{-12}$

**34.** $xyz = 10$, $(1, 2, 5)$

$F(x, y, z) = xyz - 10$

$F_x(x, y, z) = yz \qquad F_y(x, y, z) = xz \qquad F_z(x, y, z) = xy$

$F_x(1, 2, 5) = 10 \qquad F_y(1, 2, 5) = 5 \qquad F_z(1, 2, 5) = 2$

Direction numbers: 10, 5, 2

Plane: $10(x - 1) + 5(y - 2) + 2(z - 5) = 0$, $10x + 5y + 2z = 30$

Line: $\dfrac{x - 1}{10} = \dfrac{y - 2}{5} = \dfrac{z - 5}{2}$

**36.** See the definition on page 897.

**38.** For a sphere, the common object is the center of the sphere. For a right circular cylinder, the common object is the axis of the cylinder.

**40.** $F(x, y, z) = x^2 + y^2 - z \qquad G(x, y, z) = 4 - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \qquad \nabla G(x, y, z) = -\mathbf{j} - \mathbf{k}$

$\nabla F(2, -1, 5) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \qquad \nabla G(2, -1, 5) = -\mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

Direction numbers: $1, 4, -4$, $\dfrac{x - 2}{1} = \dfrac{y + 1}{4} = \dfrac{z - 5}{-4}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \, \|\nabla G\|} = \dfrac{3}{\sqrt{21}\sqrt{2}} = \dfrac{3}{\sqrt{42}} = \dfrac{\sqrt{42}}{14}$; not orthogonal

**42.** $F(x, y, z) = \sqrt{x^2 + y^2} - z$ $\qquad G(x, y, z) = 5x - 2y + 3z = 22$

$\nabla F(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\nabla F(3, 4, 5) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \frac{2}{5}\mathbf{i} - \frac{34}{5}\mathbf{j} - \frac{26}{5}\mathbf{k}$$

Direction numbers: $1, -17, -13$

$$\frac{x - 3}{1} = \frac{y - 4}{-17} = \frac{z - 5}{-13} \text{ Tangent line}$$

$$\cos\theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{-(8/5)}{\sqrt{2}\sqrt{38}} = \frac{-8}{5\sqrt{76}} \text{ Not orthogonal}$$

**44.** $F(x, y, z) = x^2 + y^2 - z$ $\qquad G(x, y, z) = x + y + 6z - 33$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

$\nabla F(1, 2, 5) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ $\qquad \nabla G(1, 2, 5) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

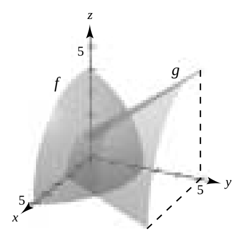
(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\mathbf{i} - 13\mathbf{j} - 2\mathbf{k}$

Direction numbers: $25, -13, -2$, $\dfrac{x - 1}{25} = \dfrac{y - 2}{-13} = \dfrac{z - 5}{-2}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = 0$; orthogonal

**46.** (a) $f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$

$g(x, y) = \dfrac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}$




(b)
$$\begin{aligned} f(x, y) &= g(x, y) \\ 16 - x^2 - y^2 + 2x - 4y &= \frac{1}{2}(1 - 3x^2 + y^2 + 6x + 4y) \\ 32 - 2x^2 - 2y^2 + 4x - 8y &= 1 - 3x^2 + y^2 + 6x + 4y \\ x^2 - 2x + 31 &= 3y^2 + 12y \\ (x^2 - 2x + 1) + 42 &= 3(y^2 + 4y + 4) \\ (x - 1)^2 + 42 &= 3(y + 2)^2 \end{aligned}$$

To find points of intersection, let $x = 1$. Then

$$\begin{aligned} 3(y + 2)^2 &= 42 \\ (y + 2)^2 &= 14 \\ y &= -2 \pm \sqrt{14} \end{aligned}$$

$\nabla f\left(1, -2 + \sqrt{14}\right) = -\sqrt{2}\mathbf{j}$, $\nabla g\left(1, -2 + \sqrt{14}\right) = \left(1/\sqrt{2}\right)\mathbf{j}$. The normals to $f$ and $g$ at this point are $-\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are orthogonal.

Similarly, $\nabla f\left(1, -2 - \sqrt{14}\right) = \sqrt{2}\,\mathbf{j}$ and $\nabla g\left(1, -2 - \sqrt{14}\right) = \left(-1/\sqrt{2}\right)\mathbf{j}$ and the normals are $\sqrt{2}\,\mathbf{j} - \mathbf{k}$ and $\left(-1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are also orthogonal.

(c) No, showing that the surfaces are orthogonal at 2 points does not imply that they are orthogonal at every point of intersection.

**48.** $F(x, y, z) = 2xy - z^3, (2, 2, 2)$

$\nabla F = 2y\mathbf{i} + 2x\mathbf{j} - 3z^2\mathbf{k}$

$\nabla F(2, 2, 2) = 4\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 2) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 2)\|} = \frac{|-12|}{\sqrt{176}} = \frac{3\sqrt{11}}{11}$$

$$\theta = \arccos\left(\frac{3\sqrt{11}}{11}\right) \approx 25.24°$$

**50.** $F(x, y, z) = x^2 + y^2 - 5, \ (2, 1, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$

$$\cos\theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$$\theta = \arccos 0 = 90°$$

**52.** $F(x, y, z) = 3x^2 + 2y^2 - 3x + 4y - z - 5$

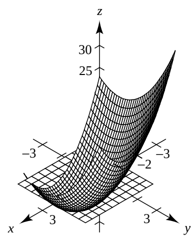
$\nabla F(x, y, z) = (6x - 3)\mathbf{i} + (4y + 4)\mathbf{j} - \mathbf{k}$

$6x - 3 = 0, \ x = \frac{1}{2}$

$4y + 4 = 0, \ y = -1$

$z = 3\left(\frac{1}{2}\right)^2 + 2(-1)^2 - 3\left(\frac{1}{2}\right) + 4(-1) - 5 = -\frac{31}{4}$

$\left(\frac{1}{2}, -1, -\frac{31}{4}\right)$




**54.** $T(x, y, z) = 100 - 3x - y - z^2, \ (2, 2, 5)$

$\frac{dx}{dt} = -3 \qquad \frac{dy}{dt} = -1 \qquad \frac{dz}{dt} = -2z$

$x(t) = -3t + C_1 \qquad y(t) = -t + C_2 \qquad z(t) = C_3e^{-2t}$

$x(0) = C_1 = 2 \qquad y(0) = C_2 = 2 \qquad z(0) = C_3 = 5$

$x = -3t + 2 \qquad y = -t + 2 \qquad z = 5e^{-2t}$

**56.** $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1$

$F_x(x, y, z) = \frac{2x}{a^2}$

$F_y(x, y, z) = \frac{2y}{b^2}$

$F_z(x, y, z) = \frac{-2z}{c^2}$

Plane: $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} - \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$$

**58.** $z = xf\left(\frac{y}{x}\right)$

$F(x, y, z) = xf\left(\frac{y}{x}\right) - z$

$F_x(x, y, z) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$

$F_y(x, y, z) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$

$F_x(x, y, z) = -1$

Tangent plane at $(x_0, y_0, z_0)$:

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) - (z - z_0) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x - x_0f\left(\frac{y_0}{x_0}\right) + y_0f'\left(\frac{y_0}{x_0}\right) + yf'\left(\frac{y_0}{x_0}\right) - y_0f'\left(\frac{y_0}{x_0}\right) - z + x_0f\left(\frac{y_0}{x_0}\right) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x + f'\left(\frac{y_0}{x_0}\right)y - z = 0$$

Therefore, the plane passes through the origin $(x, y, z) = (0, 0, 0)$.

**60.** $f(x, y) = \cos(x + y)$

$f_x(x, y) = -\sin(x + y)$ $\quad$ $f_y(x, y) = -\sin(x + y)$

$f_{xx}(x, y) = -\cos(x + y),$ $\quad$ $f_{yy}(x, y) = -\cos(x + y),$ $\quad$ $f_{xy}(x, y) = -\cos(x + y)$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0,0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2$

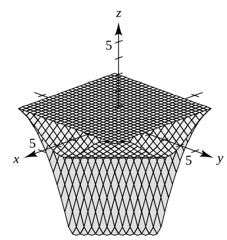
$= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - \frac{1}{2}y^2$. This is the second–degree Taylor polynomial for $\cos y$.
If $y = 0$, $P_2(x, 0) = 1 - \frac{1}{2}x^2$. This is the second–degree Taylor polynomial for $\cos x$.

(d)

| $x$ | $y$ | $f(x, y)$ | $P_1(x, y)$ | $P_2(x, y)$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 0.1 | 0.9950 | 1 | 0.9950 |
| 0.2 | 0.1 | 0.9553 | 1 | 0.9950 |
| 0.2 | 0.5 | 0.7648 | 1 | 0.7550 |
| 1 | 0.5 | 0.0707 | 1 | $-0.1250$ |

(e)




**62.** Given $z = f(x, y)$, then:

$$F(x, y, z) = f(x, y) - z = 0$$

$$\nabla F(x_0, y_0, z_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$$

$$\cos\theta = \frac{|\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}|}{\|\nabla F(x_0, y_0, z_0)\| \|\mathbf{k}\|}$$

$$= \frac{|-1|}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}$$

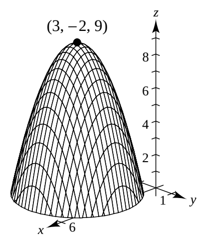
## Section 12.8 Extrema of Functions of Two Variables

**2.** $g(x, y) = 9 - (x - 3)^2 - (y + 2)^2 \le 9$

Relative maximum: $(3, -2, 9)$

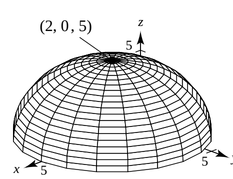
$g_x = -2(x - 3) = 0 \implies x = 3$

$g_y = -2(y + 2) = 0 \implies y = -2$




**4.** $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \le 5$

Relative maximum: $(2, 0, 5)$




**Check:** $f_x = -\dfrac{x - 2}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \implies x = 2$

$f_y = -\dfrac{y}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \implies y = 0$

$$f_{xx} = -\frac{25 - y^2}{[25 - (x - 2)^2 - y^2]^{3/2}},\ f_{yy} = -\frac{25 - (x - 2)^2}{[25 - (x - 2)^2 - y^2]^{3/2}},\ f_{xy} = -\frac{y(x - 2)}{[25 - (x - 2)^2 - y^2]^{3/2}}$$

At the critical point $(2, 0)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 0, 5)$ is a relative maximum.

**6.** $f(x, y) = -x^2 - y^2 + 4x + 8y - 11 = -(x - 2)^2 - (y - 4)^2 + 9 \le 9$

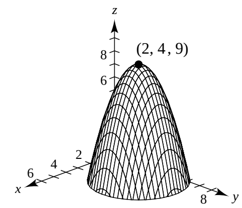
Relative maximum: $(2, 4, 9)$

**Check:** $f_x = -2x + 4 = 0 \implies x = 2$

$f_y = -2y + 8 = 0 \implies y = 4$

$f_{xx} = -2,\ f_{yy} = -2,\ f_{xy} = 0$

At the critical point $(2, 4)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 4, 9)$ is a relative maximum.




**8.** $f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62$

$$\left.\begin{aligned} f_x &= -2x + 10 = 0 \\ f_y &= -10y - 30 = 0 \end{aligned}\right\} x = 5, y = -3$$

$f_{xx} = -2, f_{yy} = -10, f_{xy} = 0$

At the critical point $(5, -3)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$.

Therefore, $(5, -3, 8)$ is a relative maximum.

**10.** $f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4$

$$\left.\begin{aligned} f_x &= 2x + 6y = 0 \\ f_y &= 6x + 20y - 4 = 0 \end{aligned}\right\}$$ Solving simultaneously yields $x = -6$ and $y = 2$.

$f_{xx} = 2,\ f_{yy} = 20,\ f_{xy} = 6$

At the critical point $(-6, 2)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-6, 2, 0)$ is a relative minimum.

**12.** $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3 = 0$ when $x = \frac{1}{2}$.

$f_y = -4y - 4 = 0$ when $y = -1$.

$f_{xx} = -6,\ f_{yy} = -4,\ f_{xy} = 0$

At the critical point $\left(\frac{1}{2}, -1\right)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(\frac{1}{2}, -1, \frac{31}{4}\right)$ is a relative maximum.

**14.** $h(x, y) = (x^2 + y^2)^{1/3} + 2$

$$\left.\begin{aligned} h_x &= \frac{2x}{3(x^2 + y^2)^{2/3}} = 0 \\ h_y &= \frac{2y}{3(x^2 + y^2)^{2/3}} = 0 \end{aligned}\right\} x = 0, y = 0$$

Since $h(x, y) \ge 2$ for all $(x, y)$, $(0, 0, 2)$ is a relative minimum.

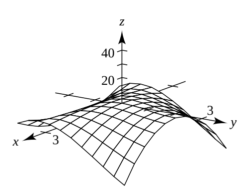
**16.** $f(x, y) = |x + y| - 2$

Since $f(x, y) \ge -2$ for all $(x, y)$, the relative minima of $f$ consist of all points $(x, y)$ satisfying $x + y = 0$.

**18.** $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$

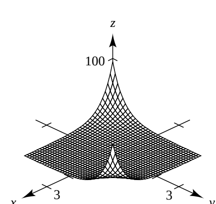
Relative maximum: $(0, 0, 1)$

Saddle points: $(0, 2, -3)$, $\left(\pm\sqrt{3}, -1, -3\right)$




**20.** $z = e^{xy}$

Saddle point: $(0, 0, 1)$



**22.** $g(x, y) = 120x + 120y - xy - x^2 - y^2$

$$\left.\begin{aligned} g_x &= 120 - y - 2x = 0 \\ g_y &= 120 - x - 2y = 0 \end{aligned}\right\} \quad \text{Solving simultaneously yields } x = 40 \text{ and } y = 40.$$

$g_{xx} = -2,\ g_{yy} = -2,\ g_{xy} = -1$

At the critical point $(40, 40)$, $g_{xx} < 0$ and $g_{xx}\, g_{yy} - (g_{xy})^2 > 0$. Therefore, $(40, 40, 4800)$ is a relative maximum.

**24.** $g(x, y) = xy$

$$\left.\begin{aligned} g_x &= y \\ g_y &= x \end{aligned}\right\} \quad x = 0 \text{ and } y = 0$$

$g_{xx} = 0,\ g_{yy} = 0,\ g_{xy} = 1$

At the critical point $(0, 0)$, $g_{xx}\, g_{yy} - (g_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

**26.** $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^2) + 1$

$$\left.\begin{aligned} f_x &= 2y - 2x^3 \\ f_y &= 2x - 2y^3 \end{aligned}\right\} \quad \begin{aligned} &\text{Solving by substitution yields 3 critical points:} \\ &(0, 0), (1, 1), (-1, -1) \end{aligned}$$

$f_{xx} = -6x^2, f_{yy} = -6y^2, f_{xy} = 2$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 < 0 \implies (0, 0, 1)$ saddle point.

At $(1, 1)$, $f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (1, 1, 2)$ relative maximum.

At $(-1, -1)$, $f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (-1, -1, 2)$ relative maximum.

**28.** $f(x, y) = \left(\frac{1}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$

$$\left.\begin{aligned} f_x &= (2x^3 - 2xy^2 - 3x)e^{1-x^2-y^2} = 0 \\ f_y &= (2x^2y - 2y^3 + y)e^{1-x^2-y^2} = 0 \end{aligned}\right\} \quad \text{Solving yields the critical points } (0, 0), \left(0, \pm\frac{\sqrt{2}}{2}\right), \left(\pm\frac{\sqrt{6}}{2}, 0\right).$$

$f_{xx} = (-4x^4 + 4x^2y^2 + 12x^2 - 2y^2 - 3)e^{1-x^2-y^2}$

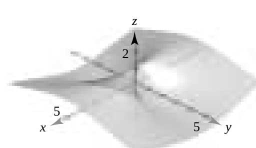
$f_{yy} = (4y^4 - 4x^2y^2 + 2x^2 - 8y^2 + 1)e^{1-x^2-y^2}$

$f_{xy} = (-4x^3y + 4xy^3 + 2xy)e^{1-x^2-y^2}$

At the critical point $(0, 0)$, $f_{xx}\, f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, e/2)$ is a saddle point. At the critical points $\left(0, \pm\sqrt{2}/2\right)$, $f_{xx} < 0$ and $f_{xx}\, f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(0, \pm\sqrt{2}/2, \sqrt{e}\right)$ are relative maxima. At the critical points $\left(\pm\sqrt{6}/2, 0\right)$, $f_{xx} > 0$ and $f_{xx}\, f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(\pm\sqrt{6}/2, 0, -\sqrt{e}/e\right)$ are relative minima.

**30.** $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2} \geq 0$. $z = 0$ if $x^2 = y^2 \neq 0$.

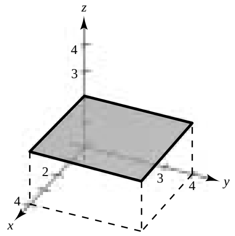
Relative minima at all points $(x, x)$ and $(x, -x)$, $x \neq 0$.



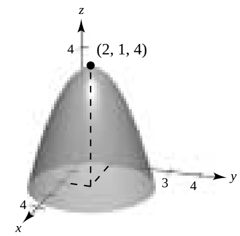

**32.** $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0$
$f$ has a relative maximum at $(x_0, y_0)$.

**34.** $f_{xx} > 0$ and $f_{xx}\, f_{yy} - (f_{xy})^2 = (25)(8) - 10^2 > 0$
$f$ has a relative minimum at $(x_0, y_0)$.

**36.** See Theorem 12.17.

**38.** 


Extrema at all $(x, y)$

**40.** 


Relative maximum

**42.** $A$ and $B$ are relative extrema. $C$ and $D$ are saddle points.

**44.** $d = f_{xx} f_{yy} - f_{xy}{}^2 < 0$ if $f_{xx}$ and $f_{yy}$ have opposite signs. Hence, $(a, b, f(a, b))$ is a saddle point. For example, consider $f(x, y) = x^2 - y^2$ and $(a, b) = (0, 0)$.

**46.** $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

$$\left.\begin{aligned} f_x &= 3x^2 - 12x + 12 = 0 \\ f_y &= 3y^2 + 18y + 27 = 0 \end{aligned}\right\} \text{Solving yields } x = 2 \text{ and } y = -3.$$

$f_{xx} = 6x - 12, f_{yy} = 6y + 18, f_{xy} = 0$

At $(2, -3)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(1, -2, 0)$ is a saddle point.

**48.** $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{x - 1}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \\ f_y &= \frac{y + 2}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \end{aligned}\right\} \text{Solving yields } x = 1 \text{ and } y = -2.$$

$$f_{xx} = \frac{(y + 2)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}}, \ f_{yy} = \frac{(x - 1)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}}, \ f_{xy} = \frac{(x - 1)(y + 2)}{[(x - 1)^2 + (y + 2)^2]^{3/2}}$$

At $(1, -2)$, $f_{xx} f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: $(1, -2, 0)$

**50.** $f(x, y) = (x^2 + y^2)^{2/3} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{4x}{3(x^2 + y^2)^{1/3}} \\ f_y &= \frac{4y}{3(x^2 + y^2)^{1/3}} \end{aligned}\right\} f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = \frac{4(x^2 + 3y^2)}{9(x^2 + y^2)^{4/3}}, \ f_{yy} = \frac{4(3x^2 + y^2)}{9(x^2 + y^2)^{4/3}}, \ f_{xy} = \frac{-8xy}{9(x^2 + y^2)^{4/3}}$$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: $(0, 0, 0)$

**52.** $f(x, y, z) = 4 - [x(y - 1)(z + 2)]^2 \leq 4$

$$\left.\begin{aligned} f_x &= -2x(y - 1)^2(z + 2)^2 = 0 \\ f_y &= -2x^2(y - 1)(z + 2)^2 = 0 \\ f_z &= -2x(y - 1)^2(z + 2) = 0 \end{aligned}\right\} \begin{array}{l} \text{Solving yields the critical points } (0, a, b), (c, 1, d), (e, f, -2). \\ \text{These points are all absolute maxima.} \end{array}$$

**54.** $f(x, y) = (2x - y)^2$

$f_x = 4(2x - y) = 0 \implies 2x = y$

$f_y = -2(2x - y) = 0 \implies 2x = y$

On the line $y = x + 1$, $0 \le x \le 1$,

$$f(x, y) = f(x) = (2x - (x + 1))^2 = (x - 1)^2$$

and the maximum is 1, the minimum is 0. On the line $y = -\frac{1}{2}x + 1$, $0 \le x \le 2$,

$$f(x, y) = f(x) = \left(2x - \left(-\tfrac{1}{2}x + 1\right)\right)^2 = \left(\tfrac{5}{2}x - 1\right)^2$$

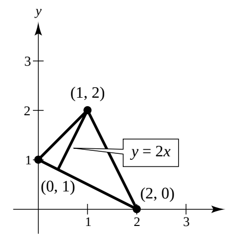
and the maximum is 16, the minimum is 0. On the line $y = -2x + 4$, $1 \le x \le 2$,

$$f(x, y) = f(x) = (2x - (-2x + 4))^2 = (4x - 4)^2$$

and the maximum is 16, the minimum is 0.

Absolute maximum: 16 at $(2, 0)$

Absolute minimum: 0 at $(1, 2)$ and along the line $y = 2x$.




**56.** $f(x, y) = 2x - 2xy + y^2$

$$\left.\begin{aligned} f_x &= 2 - 2y = 0 \implies y = 1 \\ f_y &= 2y - 2x = 0 \implies y = x \implies x = 1 \end{aligned}\right\} f(1, 1) = 1$$

On the line $y = 1$, $-1 \le x \le 1$,

$$f(x, y) = f(x) = 2x - 2x + 1 = 1.$$

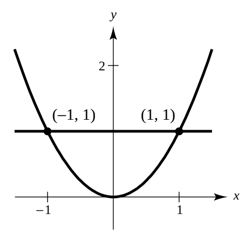
On the curve $y = x^2$, $-1 \le x \le 1$

$$f(x, y) = f(x) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$$

and the maximum is 1, the minimum is $-\frac{11}{16}$.

Absolute maximum: 1 at $(1, 1)$ and on $y = 1$

Absolute minimum: $-\frac{11}{16} = -0.6875$ at $\left(-\frac{1}{2}, \frac{1}{4}\right)$




**58.** $f(x, y) = x^2 + 2xy + y^2$, $R = \{(x, y): |x| \le 2, |y| \le 1\}$

$$\left.\begin{aligned} f_x &= 2x + 2y = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\} y = -x$$

$f(x, -x) = x^2 - 2x^2 + x^2 = 0$

Along $y = 1$, $-2 \le x \le 2$,

$$f = x^2 + 2x + 1, f' = 2x + 2 = 0 \implies x = -1, f(-2, 1) = 1, f(-1, 1) = 0, f(2, 1) = 9.$$

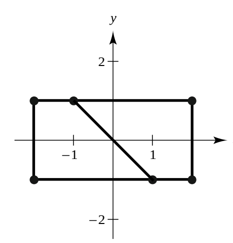
Along $y = -1$, $-2 \le x \le 2$,

$$f = x^2 - 2x + 1, f' = 2x - 2 = 0 \implies x = 1, f(-2, -1) = 9, f(1, -1) = 0, f(2, -1) = 1.$$

Along $x = 2$, $-1 \le y \le 1$, $f = 4 + 4y + y^2$, $f' = 2y + 4 \ne 0$.

Along $x = -2$, $-1 \le y \le 1$, $f = 4 - 4y + y^2$, $f' = 2y - 4 \ne 0$.

Thus, the maxima are $f(-2, -1) = 9$ and $f(2, 1) = 9$, and the minima are $f(x, -x) = 0$, $-1 \le x \le 1$.




**60.** $f(x, y) = x^2 - 4xy + 5$, $R = \left\{(x, y): 0 \le x \le 4, 0 \le y \le \sqrt{x}\right\}$

$$\left.\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= -4x = 0 \end{aligned}\right\} x = y = 0$$

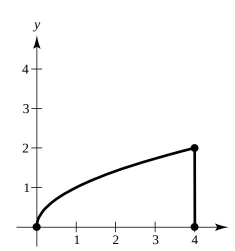
$f(0, 0) = 5$

Along $y = 0$, $0 \le x \le 4$, $f = x^2 + 5$ and $f(4, 0) = 21$.

Along $x = 4$, $0 \le y \le 2$, $f = 16 - 16y + 5$, $f' = -16 \ne 0$ and $f(4, 2) = -11$.

Along $y = \sqrt{x}$, $0 \le x \le 4$, $f = x^2 - 4x^{3/2} + 5$, $f' = 2x - 6x^{1/2} \ne 0$ on $[0, 4]$.

Thus, the maximum is $f(4, 0) = 21$ and the minimum is $f(4, 2) = -11$.



**62.** $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): x \ge 0, y \ge 0, x^2 + y^2 \le 1\}$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \implies x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} = 0 \implies y = 1 \text{ or } x = 0$$

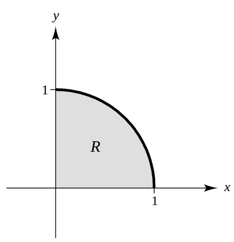
For $x = 0$, $y = 0$, also, and $f(0, 0) = 0$.

For $x = 1$ and $y = 1$, the point $(1, 1)$ is outside $R$.

For $x^2 + y^2 = 1$, $f(x, y) = f\left(x, \sqrt{1 - x^2}\right) = \dfrac{4x\sqrt{1 - x^2}}{2 + x^2 - x^4}$, and the maximum occurs at $x = \dfrac{\sqrt{2}}{2}$, $y = \dfrac{\sqrt{2}}{2}$.

Absolute maximum is $\dfrac{8}{9} = f\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)




**64.** False

Let $f(x, y) = x^4 - 2x^2 + y^2$.

Relative minima: $(\pm 1, 0, -1)$

Saddle point: $(0, 0, 0)$

## Section 12.9 Applications of Extrema of Functions of Two Variables

**2.** A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from $(1, 2, 3)$ to a point on the plane is given by

$$S = (x - 1)^2 + (y - 2)^2 + (9 - 2x - 3y)^2$$
$$S_x = 2(x - 1) + 2(9 - 2x - 3y)(-2)$$
$$S_y = 2(y - 2) + 2(9 - 2x - 3y)(-3).$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 19$$
$$6x + 10y = 29.$$

Solving simultaneously, we have $x = \frac{16}{14}$, $y = \frac{31}{14}$, $z = \frac{43}{14}$ and the distance is

$$\sqrt{\left(\frac{16}{14} - 1\right)^2 + \left(\frac{31}{14} - 2\right)^2 + \left(\frac{43}{14} - 3\right)^2} = \frac{1}{\sqrt{14}}.$$

**4.** A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 0, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + y^2 + (x^2 + y^2)^2$$
$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$
$$S_y = 2y + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$
$$2y^3 + 2x^2y + y = 0.$$

Solving as in Exercise 3, we have $x \approx 1.235$, $y = 0$, $z \approx 1.525$ and the distance is

$$\sqrt{(1.235 - 5)^2 + (1.525)^2} \approx 4.06.$$

**6.** Since $x + y + z = 32$, $z = 32 - x - y$. Therefore,

$$P = xy^2z = 32xy^2 - x^2y^2 - xy^3$$
$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0$$
$$P_y = 64xy - 2x^2y - 3xy^2 = y(64x - 2x^2 - 3xy) = 0.$$

Ignoring the solution $y = 0$ and substituting $y = 32 - 2x$ into $P_y = 0$, we have

$$64x - 2x^2 - 3x(32 - 2x) = 0$$
$$4x(x - 8) = 0.$$

Therefore, $x = 8$, $y = 16$, and $z = 8$.

**8.** Let $x$, $y$, and $z$ be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 1$, we have

$$S = x^2 + y^2 + (1 - x - y)^2$$
$$\left.\begin{aligned} S_x &= 2x - 2(1 - x - y) = 0 \\ S_y &= 2y - 2(1 - x - y) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 1 \\ x + 2y &= 1. \end{aligned}$$

Solving simultaneously yields $x = \frac{1}{3}$, $y = \frac{1}{3}$, and $z = \frac{1}{3}$.

**10.** Let $x$, $y$, and $z$ be the length, width, and height, respectively. Then $C_0 = 1.5xy + 2yz + 2xz$ and $z = \dfrac{C_0 - 1.5xy}{2(x + y)}$. The volume is given by

$$V = xyz = \frac{C_0 xy - 1.5x^2y^2}{2(x + y)}$$
$$V_x = \frac{y^2(2C_0 - 3x^2 - 6xy)}{4(x + y)^2}$$
$$V_y = \frac{x^2(2C_0 - 3y^2 - 6xy)}{4(x + y)^2}.$$

In solving the system $V_x = 0$ and $V_y = 0$, we note by the symmetry of the equations that $y = x$. Substituting $y = x$ into $V_x = 0$ yields

$$\frac{x^2(2C_0 - 9x^2)}{16x^2} = 0,\ 2C_0 = 9x^2,\ x = \frac{1}{3}\sqrt{2C_0},\ y = \frac{1}{3}\sqrt{2C_0},\ \text{and } z = \frac{1}{4}\sqrt{2C_0}.$$

**12.** Consider the sphere given by $x^2 + y^2 + z^2 = r^2$ and let a vertex of the rectangular box be $\left(x, y, \sqrt{r^2 - x^2 - y^2}\right)$. Then the volume is given by

$$V = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$$
$$V_x = 8\left(xy\frac{-x}{\sqrt{r^2 - x^2 - y^2}} + y\sqrt{r^2 - x^2 - y^2}\right) = \frac{8y}{\sqrt{r^2 - x^2 - y^2}}(r^2 - 2x^2 - y^2) = 0$$
$$V_y = 8\left(xy\frac{-y}{\sqrt{r^2 - x^2 - y^2}} + x\sqrt{r^2 - x^2 - y^2}\right) = \frac{8x}{\sqrt{r^2 - x^2 - y^2}}(r^2 - x^2 - 2y^2) = 0.$$

Solving the system

$$2x^2 + y^2 = r^2$$
$$x^2 + 2y^2 = r^2$$

yields the solution $x = y = z = r/\sqrt{3}$.

**14.** Let $x$, $y$, and $z$ be the length, width, and height, respectively. Then the sum of the two perimeters of the two cross sections is given by

$$(2x + 2z) + (2y + 2z) = 108 \text{ or } x = 54 - y - 2z.$$

The volume is given by

$$V = xyz = 54yz - y^2z - 2yz^2$$
$$V_y = 54z - 2yz - 2z^2 = z(54 - 2y - 2z) = 0$$
$$V_z = 54y - y^2 - 4yz = y(54 - y - 4z) = 0.$$

Solving the system $2y + 2z = 54$ and $y + 4z = 54$, we obtain the solution

$x = 18$ inches, $y = 18$ inches, and $z = 9$ inches.

**16.** $A = \frac{1}{2}[(30 - 2x) + (30 - 2x) + 2x\cos\theta]x\sin\theta$

$$= 30x\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta$$

$$\frac{\partial A}{\partial x} = 30\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta = 0$$

$$\frac{\partial A}{\partial \theta} = 30\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) = 0$$

From $\frac{\partial A}{\partial x} = 0$ we have $15 - 2x + x\cos\theta = 0 \implies \cos\theta = \frac{2x - 15}{x}$.

From $\frac{\partial A}{\partial \theta} = 0$ we obtain

$$30x\left(\frac{2x - 15}{x}\right) - 2x^2\left(\frac{2x - 15}{x}\right) + x^2\left(2\left(\frac{2x - 15}{x}\right)^2 - 1\right) = 0$$

$$30(2x - 15) - 2x(2x - 15) + 2(2x - 15)^2 - x^2 = 0$$

$$3x^2 - 30x = 0$$

$$x = 10$$

Then $\cos\theta = \frac{1}{2} \implies \theta = 60°$.

**18.** $P(p, q, r) = 2pq + 2pr + 2qr$.

$p + q + r = 1$ implies that $r = 1 - p - q$.

$$P(p, q) = 2pq + 2p(1 - p - q) + 2q(1 - p - q)$$
$$= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2$$
$$= -2pq + 2p + 2q - 2p^2 - 2q^2$$

$$\frac{\partial P}{\partial p} = -2q + 2 - 4p;\ \frac{\partial P}{\partial q} = -2p + 2 - 4q$$

Solving $\frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0$ gives

$$q + 2p = 1$$
$$p + 2q = 1$$

and hence $p = q = \frac{1}{3}$ and

$$P\left(\frac{1}{3}, \frac{1}{3}\right) = -2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right)$$

$$= \frac{6}{9} = \frac{2}{3}.$$

**20.** $R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2$

$$R_{p_1} = 515 + 1.5p_2 - 3p_1 = 0$$
$$R_{p_2} = 805 + 1.5p_1 - p_2 = 0$$
$$3p_1 - 1.5p_2 = 515$$
$$-1.5p_1 + p_2 = 805$$

Solving this system yields $p_1 = \$2296.67$, $p_2 = \$4250$.

**22.** $S = d_1 + d_2 + d_3 = \sqrt{(0 - 0)^2 + (y - 0)^2} + \sqrt{(0 - 2)^2 + (y - 2)^2} + \sqrt{(0 + 2)^2 + (y - 2)^2}$

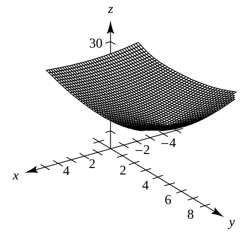
$$= y + 2\sqrt{4 + (y - 2)^2}$$

$\frac{dS}{dy} = 1 + \frac{2(y - 2)}{\sqrt{4 + (y - 2)^2}} = 0$ when $y = 2 - \frac{2\sqrt{3}}{3} = \frac{6 - 2\sqrt{3}}{3}$.

The sum of the distance is minimized when $y = \frac{2\left(3 - \sqrt{3}\right)}{3} \approx 0.845$.

**24.** (a) $S = \sqrt{(x+4)^2 + y^2} + \sqrt{(x-1)^2 + (y-6)^2} + \sqrt{(x-12)^2 + (y-2)^2}$

The surface appears to have a minimum near $(x, y) = (1, 5)$.




(b) $$S_x = \frac{x+4}{\sqrt{(x+4)^2 + y^2}} + \frac{x-1}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{x-12}{\sqrt{(x-12)^2 + (y-2)^2}}$$

$$S_y = \frac{y}{\sqrt{(x+4)^2 + y^2}} + \frac{y-6}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{y-2}{\sqrt{(x-12)^2 + (y-2)^2}}$$

(c) Let $(x_1, y_1) = (1, 5)$. Then

$-\nabla S(1, 5) = 0.258\mathbf{i} + 0.03\mathbf{j}$

Direction $\approx 6.6°$

(d) $t \approx 0.94$ $x_2 \approx 1.24$ $y_2 \approx 5.03$

(e) $t \approx 3.56,$ $x_3 \approx 1.24,$ $y_3 \approx 5.06,$

$t \approx 1.04,$ $x_4 \approx 1.23,$ $y_4 \approx 5.06$

**Note:** Minimum occurs at $(x, y) = (1.2335, 5.0694)$

(f) $-\nabla S(x, y)$ points in the direction that $S$ *decreases* most rapidly.

**26.** See the last paragraph on page 915 and Theorem 12.18.

**28.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| $-3$ | 0 | 0 | 9 |
| $-1$ | 1 | $-1$ | 1 |
| 1 | 1 | 1 | 1 |
| 3 | 2 | 6 | 9 |
| $\sum x_i = 0$ | $\sum y_i = 4$ | $\sum x_i y_i = 6$ | $\sum x_i^2 = 20$ |

$$a = \frac{4(6) - 0(4)}{4(20) - (0)^2} = \frac{3}{10},\ b = \frac{1}{4}\left[4 - \frac{3}{10}(0)\right] = 1,$$

$$y = \frac{3}{10}x + 1$$

(b) $$S = \left(\frac{1}{10} - 0\right)^2 + \left(\frac{7}{10} - 1\right)^2 + \left(\frac{13}{10} - 1\right)^2 + \left(\frac{19}{10} - 2\right)^2$$

$$= \frac{1}{5}$$

**30.** (a)

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 3 | 0 | 0 | 9 |
| 1 | 0 | 0 | 1 |
| 2 | 0 | 0 | 4 |
| 3 | 1 | 3 | 9 |
| 4 | 1 | 4 | 16 |
| 4 | 2 | 8 | 16 |
| 5 | 2 | 10 | 25 |
| 6 | 2 | 12 | 36 |
| $\sum x_i = 28$ | $\sum y_i = 8$ | $\sum x_i y_i = 37$ | $\sum x_i^2 = 116$ |

$$a = \frac{8(37) - (28)(8)}{8(116) - (28)^2} = \frac{72}{144} = \frac{1}{2},\ b = \frac{1}{8}\left[8 - \frac{1}{2}(28)\right] = -\frac{3}{4}, y = \frac{1}{2}x - \frac{3}{4}$$

(b) $$S = \left(\frac{3}{4} - 0\right)^2 + \left(-\frac{1}{4} - 0\right)^2 + \left(\frac{1}{4} - 0\right)^2 + \left(\frac{3}{4} - 1\right)^2 + \left(\frac{5}{4} - 1\right)^2 + \left(\frac{5}{4} - 2\right)^2 + \left(\frac{7}{4} - 2\right)^2 + \left(\frac{9}{4} - 2\right)^2 = \frac{3}{2}$$

**32.** (1, 0), (3, 3), (5, 6)

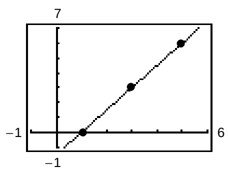
$\sum x_i = 9, \quad \sum y_i = 9,$

$\sum x_i y_i = 39, \quad \sum x_i^2 = 35$

$$a = \frac{3(39) - 9(9)}{3(35) - (9)^2} = \frac{36}{24} = \frac{3}{2}$$

$$b = \frac{1}{3}\left[9 - \frac{3}{2}(9)\right] = -\frac{9}{6} = -\frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2}$$




**34.** (6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); $n = 6$

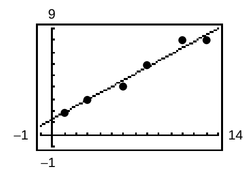
$\sum x_i = 42 \quad \sum y_i = 31$

$\sum x_i y_i = 275 \quad \sum x_i^2 = 400$

$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

$$b = \frac{1}{6}\left(31 - \frac{29}{53}42\right) = \frac{425}{318} \approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$




**36.** (a) (1.00, 450), (1.25, 375), (1.50, 330)

$\sum x_i = 3.75, \sum y_i = 1{,}155, \sum x_i^2 = 4.8125,$

$\sum x_i y_i = 1{,}413.75$

$$a = \frac{3(1{,}413.75) - (3.75)(1{,}155)}{3(4.8125) - (3.75)^2} = -240$$

$$b = \frac{1}{3}[1{,}155 - (-240)(3.75)] = 685$$

$$y = -240x + 685$$

(b) When $x = 1.40$, $y = -240(1.40) + 685 = 349$.

**38.** (a) $y = 1.8311x - 47.1067$

(b) For each 1 point increase in the percent $(x)$, $y$ increases by about 1.83 (slope of line).

**40.** $S(a, b) = \sum_{i=1}^{n}(ax_i + b - y_i)^2$

$$S_a(a, b) = 2a\sum_{i=1}^{n}x_i^2 + 2b\sum_{i=1}^{n}x_i - 2\sum_{i=1}^{n}x_i y_i$$

$$S_b(a, b) = 2a\sum_{i=1}^{n}x_i + 2nb - 2\sum_{i=1}^{n}y_i$$

$$S_{aa}(a, b) = 2\sum_{i=1}^{n}x_i^2$$

$$S_{bb}(a, b) = 2n$$

$$S_{ab}(a, b) = 2\sum_{i=1}^{n}x_i$$

$S_{aa}(a, b) > 0$ as long as $x_i \neq 0$ for all $i$. (**Note:** If $x_i = 0$ for all $i$, then $x = 0$ is the least squares regression line.)

$$d = S_{aa}S_{bb} - S_{ab}^2 = 4n\sum_{i=1}^{n}x_i^2 - 4\left(\sum_{i=1}^{n}x_i\right)^2 = 4\left[n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2\right] \geq 0 \text{ since } n\sum_{i=1}^{n}x_i^2 \geq \left(\sum_{i=1}^{n}x_i\right)^2.$$

As long as $d \neq 0$, the given values for $a$ and $b$ yield a minimum.

**42.** $(-4, 5),\ (-2, 6),\ (2, 6),\ (4, 2)$

$\sum x_i = 0$

$\sum y_i = 19$

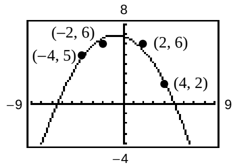
$\sum x_i^2 = 40$

$\sum x_i^3 = 0$

$\sum x_i^4 = 544$

$\sum x_i y_i = -12$

$\sum x_i^2 y_i = 160$




$544a + 40c = 160,\ 40b = -12,\ 40a + 4c = 19$

$a = -\frac{5}{24},\ b = -\frac{3}{10},\ c = \frac{41}{6},\ y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$

**44.** $(0, 10), (1, 9), (2, 6), (3, 0)$

$\sum x_i = 6$

$\sum y_i = 25$

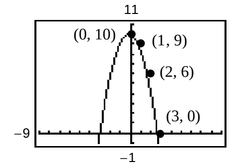
$\sum x_i^2 = 14$

$\sum x_i^3 = 36$

$\sum x_i^4 = 98$

$\sum x_i y_i = 21$

$\sum x_i^2 y_i = 33$




$98a + 36b + 14c = 33$

$36a + 14b + 6c = 21$

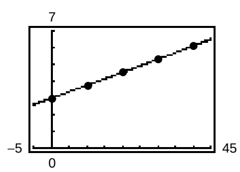
$14a + 6b + 4c = 25$

$a = -\frac{5}{4},\ b = \frac{9}{20},\ c = \frac{199}{20},\ y = -\frac{5}{4}x^2 + \frac{9}{20}x + \frac{199}{20}$

**46.** (a) $y = 0.078x + 2.96$

(b) $y = 0.0001429x^2 + 0.07229x + 2.9886$

(c)




(d) For the linear model, $x = 50$ gives $y \approx 6.86$ billion.

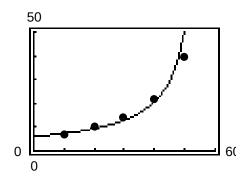
For the quadratic model, $x = 50$ gives $y \approx 6.96$ billion.

As you extrapolate into the future, the quadratic model increases more rapidly.

**48.** (a) $\frac{1}{y} = ax + b = -0.0029x + 0.1640$

$$y = \frac{1}{-0.0029x + 0.1640}$$

(b)




(c) No. For $x = 60$, $y \approx -100$. Note that there is a vertical asymptote at $x \approx 56.6$.

## Section 12.10 Lagrange Multipliers

**2.** Maximize $f(x, y) = xy$.

Constraint: $2x + y = 4$

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = 2\lambda\,\mathbf{i} + \lambda\,\mathbf{j}$

$y = 2\lambda$

$x = \lambda$

$2x + y = 4 \implies 4\lambda = 4$

$\lambda = 1,\ x = 1,\ y = 2$

$f(1, 2) = 2$

**4.** Minimize $f(x, y) = x^2 + y^2$.

Constraint: $2x + 4y = 5$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = 2\lambda\,\mathbf{i} + 4\lambda\,\mathbf{j}$

$2x = 2\lambda \implies x = \lambda$

$2y = 4\lambda \implies y = 2\lambda$

$2x + 4y = 5 \implies 10\lambda = 5$

$\lambda = \frac{1}{2},\ x = \frac{1}{2},\ y = 1$

$f\left(\frac{1}{2}, 1\right) = \frac{5}{4}$

**6.** Maximize $f(x, y) = x^2 - y^2$.

Constraint: $2y - x^2 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\,\mathbf{i} + 2\lambda\,\mathbf{j}$

$2x = -2x\lambda \implies x = 0$ or $\lambda = -1$

If $x = 0$, then $y = 0$ and $f(0, 0) = 0$.

If $\lambda = -1$,

$$-2y = 2\lambda = -2 \implies y = 1 \implies x^2 = 2 \implies x = \sqrt{2}.$$

$f\left(\sqrt{2}, 1\right) = 2 - 1 = 1$ Maximum.

**8.** Minimize $f(x, y) = 3x + y + 10$.

Constraint: $x^2y = 6$

$\nabla f = \lambda \nabla g$

$3\mathbf{i} + \mathbf{j} = 2xy\lambda\,\mathbf{i} + x^2\lambda\,\mathbf{j}$

$$\left.\begin{aligned} 3 = 2xy\lambda &\implies \lambda = \frac{3}{2xy} \\ 1 = x^2\lambda &\implies \lambda = \frac{1}{x^2} \end{aligned}\right\} 3x^2 = 2xy \implies y = \frac{3x}{2} \quad (x \neq 0)$$

$$x^2y = 6 \implies x^2\left(\frac{3x}{2}\right) = 6$$

$$x^3 = 4$$

$$x = \sqrt[3]{4},\ y = \frac{3\sqrt[3]{4}}{2}$$

$$f\left(\sqrt[3]{4}, \frac{3\sqrt[3]{4}}{2}\right) = \frac{9\sqrt[3]{4} + 20}{2}$$

**10.** **Note:** $f(x, y) = \sqrt{x^2 + y^2}$ is minimum when $g(x, y)$ is minimum.

Minimize $g(x, y) = x^2 + y^2$.

Constraint: $2x + 4y = 15$

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 4\lambda \end{aligned}\right\} y = 2x$$

$2x + 4y = 15 \implies 10x = 15$

$$x = \frac{3}{2},\ y = 3$$

$$f\left(\frac{3}{2}, 3\right) = \sqrt{g\left(\frac{3}{2}, 3\right)} = \frac{3\sqrt{5}}{2}$$

**12.** Minimize $f(x, y) = 2x + y$.

Constraint: $xy = 32$

$$\left.\begin{aligned} 2 &= y\lambda \\ 1 &= x\lambda \end{aligned}\right\} y = 2x$$

$xy = 32 \implies 2x^2 = 32$

$$x = 4, y = 8$$

$f(4, 8) = 16$

**14.** Maximize or minimize $f(x, y) = e^{-xy/4}$.

Constraint: $x^2 + y^2 \leq 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} -(y/4)e^{-xy/4} &= 2x\lambda \\ -(x/4)e^{-xy/4} &= 2y\lambda \end{aligned}\right\} \implies x^2 = y^2$$

$$x^2 + y^2 = 1 \implies x = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= -(y/4)e^{-xy/4} = 0 \\ f_y &= -(x/4)e^{-xy/4} = 0 \end{aligned}\right\} \implies x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4},\ f_{yy} = \frac{x^2}{16}e^{-xy/4},\ f_{xy} = e^{-xy}\left[\frac{1}{16}xy - \frac{1}{4}\right]$$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$.

Saddle point: $f(0, 0) = 1$

Combining the two cases, we have a maximum of $e^{1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$ and a minimum of $e^{-1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

**16.** Maximize $f(x, y, z) = xyz$.

Constraint: $x + y + z = 6$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 6 \implies x = y = z = 2$

$f(2, 2, 2) = 8$

**18.** Minimize $x^2 - 10x + y^2 - 14y + 70$

Constraint: $x + y = 10$

$$\left.\begin{aligned} 2x - 10 &= \lambda \\ 2y - 14 &= \lambda \\ x + y &= 8 \end{aligned}\right\} \begin{aligned} x &= (1/2)(\lambda + 10) \\ y &= (1/2)(\lambda + 14) \end{aligned}$$

$$\begin{aligned} x + y &= \frac{1}{2}(\lambda + 10) + \frac{1}{2}(\lambda + 14) \\ &= \lambda + 12 = 8 \implies \lambda = -4 \end{aligned}$$

Then $x = 3, y = 5$.

$f(3, 5) = 9 - 30 + 25 - 70 + 70 = 4$

**20.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraints: $x + 2z = 6$

$x + y = 12$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$

$$\left.\begin{aligned} 2x &= \lambda + \mu \\ 2y &= \mu \\ 2z &= 2\lambda \end{aligned}\right\} 2x = 2y + z$$

$x + 2z = 6 \implies z = \dfrac{6 - x}{2} = 3 - \dfrac{x}{2}$

$x + y = 12 \implies y = 12 - x$

$2x = 2(12 - x) + \left(3 - \dfrac{x}{2}\right) \implies \dfrac{9}{2}x = 27 \implies x = 6$

$x = 6, z = 0$

$f(6, 6, 0) = 72$

**22.** Maximize $f(x, y, z) = xyz$.

Constraints: $x^2 + z^2 = 5$

$x - 2y = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(2x\mathbf{i} + 2z\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j})$

$yz = 2x\lambda + \mu$

$xz = -2\mu \implies \mu = -\dfrac{xy}{2}$

$xy = 2z\lambda \implies \lambda = \dfrac{xy}{2z}$

$x^2 + z^2 = 5 \implies z = \sqrt{5 - x^2}$

$x - 2y = 0 \implies y = \dfrac{x}{2}$

$yz = 2x\left(\dfrac{xy}{2z}\right) - \dfrac{xz}{2}$

$$\frac{x\sqrt{5 - x^2}}{2} = \frac{x^3}{2\sqrt{5 - x^2}} - \frac{x\sqrt{5 - x^2}}{2}$$

$$x\sqrt{5 - x^2} = \frac{x^3}{2\sqrt{5 - x^2}}$$

$2x(5 - x^2) = x^3$

$0 = 3x^3 - 10x = x(3x^2 - 10)$

$x = 0$ or $x = \sqrt{\dfrac{10}{3}}, y = \dfrac{1}{2}\sqrt{\dfrac{10}{3}}, z = \sqrt{\dfrac{5}{3}}$

$$f\left(\sqrt{\frac{10}{3}}, \frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{5\sqrt{15}}{9}$$

**Note:** $f\left(0, 0, \sqrt{5}\right) = 0$ does not yield a maximum.

**24.** Minimize the square of the distance $f(x, y) = x^2 + (y - 10)^2$ subject to the constraint $(x - 4)^2 + y^2 = 4$.

$$\left.\begin{aligned} 2x &= 2(x-4)\lambda \\ 2(y-10) &= 2y\lambda \end{aligned}\right\} \frac{x}{x-4} = \frac{y-10}{y} \implies y = -\frac{5}{2}x + 10$$

$$(x-4)^2 + y^2 = 4 \implies (x^2 - 8x + 16) + \left(\frac{25}{4}x^2 - 50x + 100\right) = 4$$

$$\frac{29}{4}x^2 - 58x + 112 = 0$$

Using a graphing utility, we obtain $x \approx 3.2572$ and $x \approx 4.7428$ or, by the Quadratic Formula,

$$x = \frac{58 \pm \sqrt{58^2 - 4(29/4)(112)}}{2(29/4)} = \frac{58 \pm 2\sqrt{29}}{29/2} = 4 \pm \frac{4\sqrt{29}}{29}.$$

Using the smaller value, we have $x = 4\left(1 - \dfrac{\sqrt{29}}{29}\right)$ and $y = \dfrac{10\sqrt{29}}{29} \approx 1.8570.$

The point on the circle is $\left[4\left(1 - \dfrac{\sqrt{29}}{29}\right), \dfrac{10\sqrt{29}}{29}\right]$

and the desired distance is $d = \sqrt{16\left(1 - \dfrac{\sqrt{29}}{29}\right)^2 + \left(\dfrac{10\sqrt{29}}{29} - 10\right)^2} \approx 8.77.$

The larger $x$-value does not yield a minimum.

**26.** Minimize the square of the distance

$$f(x, y, z) = (x - 4)^2 + y^2 + z^2$$

subject to the constraint $\sqrt{x^2 + y^2} - z = 0$.

$$\left.\begin{aligned} 2(x-4) &= \frac{x}{\sqrt{x^2+y^2}}\lambda = \frac{x}{z}\lambda \\ 2y &= \frac{y}{\sqrt{x^2+y^2}}\lambda = \frac{y}{z}\lambda \\ 2z &= -\lambda \end{aligned}\right\} \begin{aligned} 2(x-4) &= -2x \\ 2y &= -2y \end{aligned}$$

$$\sqrt{x^2 + y^2} - z = 0,\ x = 2,\ y = 0,\ z = 2$$

The point on the plane is $(2, 0, 2)$ and the desired distance is

$$d = \sqrt{(2-4)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

**28.** Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 - z^2 = 0$ and $x + 2z = 4$.

$$0 = 2x\lambda + \mu$$
$$0 = 2y\lambda \implies y = 0$$
$$1 = -2z\lambda + 2\mu$$
$$x^2 + y^2 - z^2 = 0$$
$$x + 2z = 4 \implies x = 4 - 2z$$
$$(4 - 2z)^2 + 0^2 - z^2 = 0$$
$$3z^2 - 16z + 16 = 0$$
$$(3z - 4)(z - 4) = 0$$
$$z = \tfrac{4}{3} \text{ or } z = 4$$

The maximum value of $f$ occurs when $z = 4$ at the point of $(-4, 0, 4)$.

**30.** See explanation at the bottom of page 922.

**32.** Maximize $V(x, y, z) = xyz$ subject to the constraint $1.5xy + 2xz + 2yz = C$.

$$\left.\begin{aligned} yz &= (1.5y + 2z)\lambda \\ xz &= (1.5x + 2z)\lambda \\ xy &= (2x + 2y)\lambda \end{aligned}\right\} x = y \text{ and } z = \frac{3}{4}x$$

$$1.5xy + 2xz + 2yz = C \implies 1.5x^2 + \frac{3}{2}x^2 + \frac{3}{2}x^2 = C$$

$$x = \frac{\sqrt{2C}}{3}$$

Volume is maximum when

$$x = y = \frac{\sqrt{2C}}{3} \quad \text{and} \quad z = \frac{\sqrt{2C}}{4}.$$

**34.** Minimize $A(\pi, r) = 2\pi rh + 2\pi r^2$ subject to the constraint $\pi r^2 h = V_0$.

$$\left.\begin{aligned} 2\pi h + 4\pi r &= 2\pi rh\lambda \\ 2\pi r &= \pi r^2\lambda \end{aligned}\right\} h = 2r$$

$$\pi r^2 h = V_0 \implies 2\pi r^3 = V_0$$

Dimensions: $r = \sqrt[3]{\dfrac{V_0}{2\pi}}$ and $h = 2\sqrt[3]{\dfrac{V_0}{2\pi}}$

**36.** (a) Maximize $P(x, y, z) = xyz$ subject to the constraint

$$x + y + z = S.$$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = S \implies x = y = z = \frac{S}{3}$$

Therefore,

$$xyz \le \left(\frac{S}{3}\right)\left(\frac{S}{3}\right)\left(\frac{S}{3}\right), \; x, y, z > 0$$

$$xyz \le \frac{S^3}{27}$$

$$\sqrt[3]{xyz} \le \frac{S}{3}$$

$$\sqrt[3]{xyz} \le \frac{x + y + z}{3}.$$

(b) Maximize $P = x_1 x_2 x_3 \ldots x_n$ subject to the constraint

$$\sum_{i=1}^{n} x_i = S.$$

$$\left.\begin{aligned} x_2 x_3 \ldots x_n &= \lambda \\ x_1 x_3 \ldots x_n &= \lambda \\ x_1 x_2 \ldots x_n &= \lambda \\ &\vdots \\ x_1 x_2 x_3 \ldots x_{n-1} &= \lambda \end{aligned}\right\} x_1 = x_2 = x_3 = \cdots = x_n$$

$$\sum_{i=1}^{n} x_i = S \implies x_1 = x_2 = x_3 = \cdots = x_n = \frac{S}{n}$$

Therefore,

$$x_1 x_2 x_3 \ldots x_n \le \left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\left(\frac{S}{n}\right) \cdots \left(\frac{S}{n}\right), \; x_i \ge 0$$

$$x_1 x_2 x_3 \ldots x_n \le \left(\frac{S}{n}\right)^n$$

$$\sqrt[n]{x_1 x_2 x_3 \ldots x_n} \le \frac{S}{n}$$

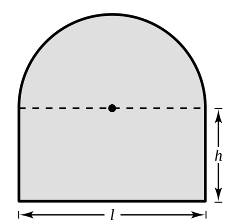
$$\sqrt[n]{x_1 x_2 x_3 \ldots x_n} \le \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

**38.** Case 1: Minimize $P(l, h) = 2h + l + \left(\frac{\pi l}{2}\right)$ subject to the constraint $lh + \left(\frac{\pi l^2}{8}\right) = A$.

$$1 + \frac{\pi}{2} = \left(h + \frac{\pi l}{4}\right)\lambda$$

$$2 = l\lambda \implies \lambda = \frac{2}{l}, 1 + \frac{\pi}{2} = \frac{2h}{l} + \frac{\pi}{2}$$

$$l = 2h$$




Case 2: Minimize $A(l, h) = lh + \left(\frac{\pi l^2}{8}\right)$ subject to the constraint $2h + l + \left(\frac{\pi l}{2}\right) = P$.

$$h + \frac{\pi l}{4} = \left(\perp + \frac{\pi}{2}\right)\lambda$$

$$l = 2\lambda \implies \lambda = \frac{l}{2}, h + \frac{\pi l}{4} = \frac{l}{2} + \frac{\pi l}{4}$$

$$h = \frac{l}{2} \text{ or } l = 2h$$

**40.** Maximize $T(x, y, z) = 100 + x^2 + y^2$ subject to the constraints $x^2 + y^2 + z^2 = 50$ and $x - z = 0$.

$$\left.\begin{aligned} 2x &= 2x\lambda + \mu \\ 2y &= 2y\lambda \\ 0 &= 2z\lambda - \mu \end{aligned}\right\}$$

If $y \ne 0$, then $\lambda = 1$ and $\mu = 0$, $z = 0$.

Thus, $x = z = 0$ and $y = \sqrt{50}$.

$$T\left(0, \sqrt{50}, 0\right) = 100 + 50 = 150$$

If $y = 0$, then $x^2 + z^2 = 2x^2 = 50$ and $x = z = \sqrt{50}/2$.

$$T\left(\frac{\sqrt{50}}{2}, 0, \frac{\sqrt{50}}{2}\right) = 100 + \frac{50}{4} = 112.5$$

Therefore, the maximum temperature is 150.

**42.** Maximize $P(x, y) = 100x^{0.4}y^{0.6}$

Constraint: $48x + 36y = 100{,}000$.

$$40x^{-0.6}y^{0.6} = 48\lambda \implies \left(\frac{y}{x}\right)^{0.6} = \frac{48\lambda}{40}$$

$$60x^{0.4}y^{-0.4} = 36\lambda \implies \left(\frac{x}{y}\right)^{0.4} = \frac{36\lambda}{60}$$

$$\left(\frac{y}{x}\right)^{0.6}\left(\frac{y}{x}\right)^{0.4} = \left(\frac{48\lambda}{40}\right)\left(\frac{60}{36\lambda}\right)$$

$$\frac{y}{x} = 2 \implies y = 2x$$

$$48x + 36y(2x) = 100{,}000 \implies x = \frac{2500}{3}, y = \frac{5000}{3}$$

$$P\left(\frac{2500}{3}, \frac{5000}{3}\right) \approx \$126{,}309.71.$$

**44.** Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.6}y^{0.4} = 20{,}000$.

$$48 = 60x^{-0.4}y^{0.4}\lambda \implies \left(\frac{y}{x}\right)^{0.4} = \frac{48}{60\lambda}$$

$$36 = 40x^{0.6}y^{-0.6}\lambda \implies \left(\frac{x}{y}\right)^{0.6} = \frac{36}{40\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4}\left(\frac{y}{x}\right)^{0.6} = \left(\frac{48}{60\lambda}\right)\left(\frac{40\lambda}{36}\right)$$

$$\frac{y}{x} = \frac{8}{9} \implies y = \frac{8}{9}x$$

$$100x^{0.6}y^{0.4} = 20{,}000 \implies x^{0.6}\left(\frac{8}{9}x\right)^{0.4} = 200$$

$$x = \frac{200}{(8/9)^{0.4}} \approx 209.65$$

$$y = \frac{8}{9}\left[\frac{200}{(8/9)^{0.4}}\right] \approx 186.35$$

Therefore, $C(209.65, 186.35) = \$16{,}771.94$.

**46.** $f(x, y) = ax + by,\ x, y > 0$

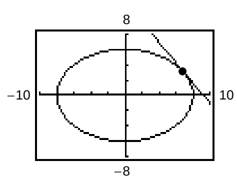
Constraint: $\dfrac{x^2}{64} + \dfrac{y^2}{36} = 1$

(a) Level curves of $f(x, y) = 4x + 3y$ are lines of form

$$y = -\frac{4}{3}x + C.$$

Using $y = -\dfrac{4}{3}x + 12.3$, you obtain

$$x \approx 7,\ y \approx 3, \quad \text{and} \quad f(7, 3) = 28 + 9 = 37.$$




Constraint is an ellipse.

(b) Level curves of $f(x, y) = 4x + 9y$ are lines of form

$$y = -\frac{4}{9}x + C.$$

Using $y = -\dfrac{4}{9}x + 7$, you obtain

$$x \approx 4,\ y \approx 5.2, \text{ and } f(4, 5.2) = 62.8.$$

## Review Exercises for Chapter 12

**2.** Yes, it is the graph of a function.

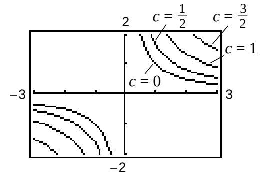
**4.** $f(x, y) = \ln xy$

The level curves are of the form

$$c = \ln xy$$

$$e^c = xy.$$

The level curves are hyperbolas.

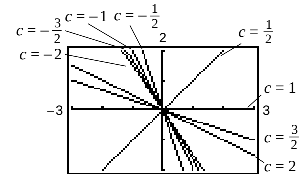



**6.** $f(x, y) = \dfrac{x}{x + y}$

The level curves are of the form

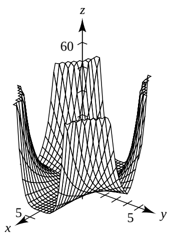
$$c = \frac{x}{x + y}$$

$$y = \left(\frac{1 - c}{c}\right)x.$$



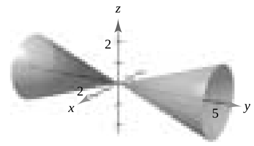

The level curves are passing through the origin with slope $\dfrac{1 - c}{c}$.

**8.** $g(x, y) = |y|^{1+|x|}$




**10.** $f(x, y, z) = 9x^2 - y^2 + 9z^2 = 0$

Elliptic cone




**12.** $\lim_{(x, y)\to(1, 1)} \dfrac{xy}{x^2 - y^2}$

Does not exist

Continuous except when $y = \pm x$.

**14.** $\lim_{(x, y)\to(0, 0)} \dfrac{y + xe^{-y^2}}{1 + x^2} = \dfrac{0 + 0}{1 + 0} = 0$

Continuous everywhere

**16.** $f(x, y) = \dfrac{xy}{x + y}$

$$f_x = \frac{y(x + y) - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2}$$

$$f_y = \frac{x^2}{(x + y)^2}$$

**18.** $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

**20.** $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

**22.** $f(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$f_x = -\frac{1}{2}(1 - x^2 - y^2 - z^2)^{-3/2}(-2x)$$

$$= \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_y = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_z = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

**24.** $u(x, t) = c(\sin akx) \cos kt$

$$\frac{\partial u}{\partial x} = akc(\cos akx) \cos kt$$

$$\frac{\partial u}{\partial t} = -kc(\sin akx) \sin kt$$

**26.** $z = x^2 \ln(y + 1)$

$\dfrac{\partial z}{\partial x} = 2x \ln(y + 1)$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial x} = 0$.

Slope in $x$-direction.

$\dfrac{\partial z}{\partial y} = \dfrac{x^2}{1 + y}$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial y} = 4$.

Slope in $y$-direction.

**28.** $h(x, y) = \dfrac{x}{x + y}$

$$h_x = \frac{y}{(x + y)^2}$$

$$h_y = \frac{-x}{(x + y)^2}$$

$$h_{xx} = \frac{-2y}{(x + y)^3}$$

$$h_{yy} = \frac{2x}{(x + y)^3}$$

$$h_{xy} = \frac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

$$h_{yx} = \frac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

**30.** $g(x, y) = \cos(x - 2y)$

$$g_x = -\sin(x - 2y)$$

$$g_y = 2 \sin(x - 2y)$$

$$g_{xx} = -\cos(x - 2y)$$

$$g_{yy} = -4 \cos(x - 2y)$$

$$g_{xy} = 2 \cos(x - 2y)$$

$$g_{yx} = 2 \cos(x - 2y)$$

**32.** $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -6x$$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

**34.** $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

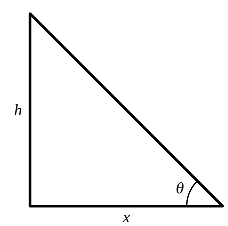
Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

**36.** $z = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$= \left[\frac{\sqrt{x^2 + y^2}\,y - xy\left(x/\sqrt{x^2 + y^2}\right)}{x^2 + y^2}\right]dx + \left[\frac{\sqrt{x^2 + y^2}\,x - xy\left(y/\sqrt{x^2 + y^2}\right)}{x^2 + y^2}\right]dy = \frac{y^3}{(x^2 + y^2)^{3/2}}\,dx + \frac{x^3}{(x^2 + y^2)^{3/2}}\,dy$$

**38.** From the accompanying figure we observe




$$\tan\theta = \frac{h}{x} \text{ or } h = x\tan\theta$$

$$dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial \theta}\,d\theta = \tan\theta\,dx + x\sec^2\theta\,d\theta.$$

Letting $x = 100$, $dx = \pm\dfrac{1}{2}$, $\theta = \dfrac{11\pi}{60}$, and $d\theta = \pm\dfrac{\pi}{180}$.

(Note that we express the measurement of the angle in radians.) The maximum error is approximately

$$dh = \tan\left(\frac{11\pi}{60}\right)\left(\pm\frac{1}{2}\right) + 100\sec^2\left(\frac{11\pi}{60}\right)\left(\pm\frac{\pi}{180}\right) \approx \pm 0.3247 \pm 2.4814 \approx \pm 2.81 \text{ feet.}$$

**40.** $A = \pi r\sqrt{r^2 + h^2}$

$$dA = \left(\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}}\right)dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}\,dh$$

$$= \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}}\,dr + \frac{\pi rh}{\sqrt{r^2 + h^2}}\,dh = \frac{\pi(8 + 25)}{\sqrt{29}}\left(\pm\frac{1}{8}\right) + \frac{10\pi}{\sqrt{29}}\left(\pm\frac{1}{8}\right) = \pm\frac{43\pi}{8\sqrt{29}}$$

**42.** $u = y^2 - x$, $x = \cos t$, $y = \sin t$

Chain Rule: $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$

$$= -1(-\sin t) + 2y(\cos t)$$

$$= \sin t + 2(\sin t)\cos t$$

$$= \sin t(1 + 2\cos t)$$

Substitution: $u = \sin^2 t - \cos t$

$$\frac{du}{dt} = 2\sin t\cos t + \sin t = \sin t(1 + 2\cos t)$$

**44.** $w = \dfrac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$

Chain Rule: $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= \frac{y}{z}(2) + \frac{x}{z}(t) - \frac{xy}{z^2}(2)$$

$$= \frac{2rt}{2r - t} + \frac{(2r + t)t}{2r - t} - \frac{2(2r + t)(rt)}{(2r - t)^2}$$

$$= \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{y}{z}(1) + \frac{x}{z}(r) = \frac{xy}{z^2}(-1)$$

$$= \frac{4r^2t - rt^2 + 4r^3}{(2r - t)^2}$$

Substitution: $$w = \frac{xy}{z} = \frac{(2r + t)(rt)}{2r - t} = \frac{2r^2t + rt^2}{2r - t}$$

$$\frac{\partial w}{\partial r} = \frac{4r^2t - 4rt^2 - t^3}{(2r - t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{4r^2t - rt^2 - 4r^3}{(2r - t)^2}$$

**46.** $$xz^2 - y \sin z = 0$$

$$2xz\frac{\partial z}{\partial x} + z^2 - y \cos z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{z^2}{y \cos z - 2xz}$$

$$2xz\frac{\partial z}{\partial y} - y \cos z\frac{\partial z}{\partial y} - \sin z = 0$$

$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y \cos z}$$

**48.** $$f(x, y) = \frac{1}{4}y^2 - x^2$$

$$\nabla f = -2x\mathbf{i} + \frac{1}{2}y\mathbf{j}$$

$$\nabla f(1, 4) = -2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{v} = \frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f(1, 4) = \nabla f(1, 4) \cdot \mathbf{u} = -\frac{4\sqrt{5}}{5} + \frac{2\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5}$$

**50.** $$w = 6x^2 + 3xy - 4y^2z$$

$$\nabla w = (12x + 3y)\mathbf{i} + (3x - 8yz)\mathbf{j} + (-4y^2)\mathbf{k}$$

$$\nabla w(1, 0, 1) = 12\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{v} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 0, 1) = \nabla w(1, 0, 1) \cdot \mathbf{u}$$

$$= 4\sqrt{3} + \sqrt{3} + 0 = 5\sqrt{3}$$

**52.** $$z = \frac{x^2}{x - y}$$

$$\nabla z = \frac{x^2 - 2xy}{(x - y)^2}\mathbf{i} + \frac{x^2}{(x - y)^2}\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4$$

**54.** $$z = x^2y$$

$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

**56.** $4y \sin x - y^2 = 3$

$$f(x, y) = 4y \sin x - y^2$$

$$\nabla f(x, y) = 4y \cos x\mathbf{i} + (4 \sin x - 2y)\mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = 2\mathbf{j}$$

Normal vector: $\mathbf{j}$

**58.** $F(x, y, z) = y^2 + z^2 - 25 = 0$

$$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$3(y - 3) + 4(z - 4) = 0 \quad \text{or} \quad 3y + 4z = 25,$$

and the equation of the normal line is

$$x = 2, \ \frac{y - 3}{3} = \frac{z - 4}{4}.$$

**60.** $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

Therefore, the equation of the tangent plane is

$(x - 1) + 2(y - 2) + 2(z - 2) = 0$ or

$x + 2y + 2z = 9,$

and the equation of the normal line is

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-2}{2}.$$

**62.** $F(x, y, z) = y^2 + z - 25 = 0$

$G(x, y, z) = x - y = 0$

$\nabla F = 2y\mathbf{i} + \mathbf{k}$

$\nabla G = \mathbf{i} - \mathbf{j}$

$\nabla F(4, 4, 9) = 8\mathbf{i} + \mathbf{k}$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 8\mathbf{k}$$

Therefore, the equation of the tangent line is

$$\frac{x-4}{1} = \frac{y-4}{1} = \frac{z-9}{-8}.$$

**64.** (a) $f(x, y) = \cos x + \sin y, \quad f(0, 0) = 1$

$f_x = -\sin x, \quad f_x(0, 0) = 0$

$f_y = \cos y, \quad f_y(0, 0) = 1$

$P_1(x, y) = 1 + y$

(b) $f_{xx} = -\cos x, \quad f_{xx}(0, 0) = -1$

$f_{yy} = -\sin y, \quad f_{yy}(0, 0) = 0$

$f_{xy} = 0, \quad f_{xy}(0, 0) = 0$

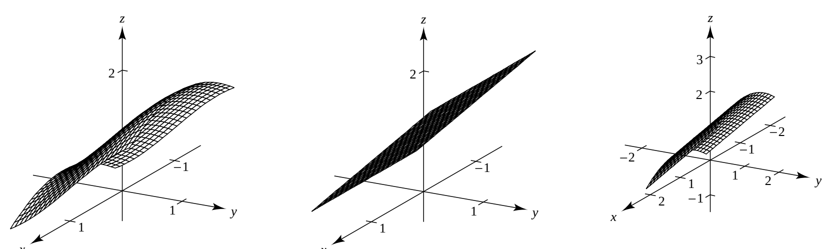
$P_2(x, y) = 1 + y - \frac{1}{2}x^2$

(c) If $y = 0$, you obtain the 2nd degree Taylor polynomial for $\cos x$.

(d)

| $x$ | $y$ | $f(x, y)$ | $P_1(x, y)$ | $P_2(x, y)$ |
|---|---|---|---|---|
| 0 | 0 | 1.0 | 1.0 | 1.0 |
| 0 | 0.1 | 1.0998 | 1.1 | 1.1 |
| 0.2 | 0.1 | 1.0799 | 1.1 | 1.095 |
| 0.5 | 0.3 | 1.1731 | 1.3 | 1.175 |
| 1 | 0.5 | 1.0197 | 1.5 | 1.0 |

(e)



The accuracy lessens as the distance from $(0, 0)$ increases.

**66.** $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$f_x = 4x + 6y + 8 = 0$

$f_y = 6x + 18y = 0, \; x = -3y$

$4(-3y) + 6y = -8 \implies y = \frac{4}{3}, \; x = -4$

$f_{xx} = 4$

$f_{yy} = 18$

$f_{xy} = 6$

$f_{xx}f_{yy} - (f_{xy})^2 = 4(18) - (6)^2 = 36 > 0$ Therefore, $\left(-4, \frac{4}{3}, -2\right)$ is a relative minimum.

**68.** $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$

$z_x = 50 - 0.3x^2 - 20 = 0, \ x = \pm 10$

$z_y = 50 - 0.15y^2 - 20.6 = 0, \ y = \pm 14$

Critical Points: $(10, 14), \ (10, -14), \ (-10, 14), \ (-10, -14)$

$z_{xx} = -0.6x, \ z_{yy} = -0.3y, \ z_{xy} = 0$

At $(10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(-4.2) - 0^2 > 0, \ z_{xx} < 0.$

$(10, 14, 199.4)$ is a relative maximum.

At $(10, -14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(4.2) - 0^2 < 0.$

$(10, -14, -349.4)$ is a saddle point.

At $(-10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (6)(-4.2) - 0^2 < 0.$

$(-10, 14, -200.6)$ is a saddle point.

At $(-10, -14)$, $z_{xx}z_{yy} - (x_{xy})^2 = (6)(4.2) - 0^2 > 0, \ z_{xx} < 0.$

$(-10, -14, -749.4)$ is a relative minimum.

**70.** The level curves indicate that there is a relative extremum at $A$, the center of the ellipse in the second quadrant, and that there is a saddle point at $B$, the origin.

**72.** Minimize $C(x_1, x_2) = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2$ subject to the constraint $x_1 + x_2 = 1000$.

$$\left.\begin{aligned} 0.50x_1 + 10 &= \lambda \\ 0.30x_2 + 12 &= \lambda \end{aligned}\right\} 5x_1 - 3x_2 = 20$$

$$\begin{aligned} x_1 + x_2 = 1000 \implies 3x_1 + 3x_2 &= 3000 \\ 5x_1 - 3x_2 &= 20 \\ \hline 8x_1 &= 3020 \\ x_1 &= 377.5 \\ x_2 &= 622.5 \end{aligned}$$

$C(377.5, 622.5) = 104{,}997.50$

**74.** Minimize the square of the distance:

$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (x^2 + y^2 - 0)^2.$$

$$\left.\begin{aligned} f_x &= 2(x - 2) + 2(x^2 + y^2)2x = 0 \\ f_y &= 2(y - 2) + 2(x^2 + y^2)2y = 0 \end{aligned}\right\} \begin{aligned} x - 2 + 2x^3 + 2xy^2 &= 0 \\ y - 2 + 2y^3 + 2x^2y &= 0 \end{aligned}$$

Clearly $x = y$ and hence: $4x^3 + x - 2 = 0$. Using a computer algebra system, $x \approx 0.6894$.
Thus, $(\text{distance})^2 = (0.6894 - 2)^2 + (0.6894 - 2)^2 + [2(0.6894)^2]^2 \approx 4.3389$.
distance $\approx 2.08$

**76.** (a) $(25, 28), \ (50, 38), \ (75, 54), \ (100, 75), \ (125, 102)$

$$\sum x_i = 375, \quad \sum y_i = 297, \quad \sum x_i^2 = 34{,}375, \quad \sum x_i^3 = 3{,}515{,}625$$
$$\sum x_i^4 = 382{,}421{,}875, \quad \sum x_i y_i = 26{,}900, \quad \sum x_i^2 y_i = 2{,}760{,}000,$$

$$\begin{aligned} 382{,}421{,}875a + 3{,}515{,}625b + 34{,}375c &= 2{,}760{,}000 \\ 3{,}515{,}625a + 34{,}375b + 375c &= 26{,}900 \\ 34{,}375a + 375b + 5c &= 297 \end{aligned}$$

$a \approx 0.0045, \ b \approx 0.0717, \ c \approx 23.2914, \ y \approx 0.0045x^2 + 0.0717x + 23.2914$

(b) When $x = 80$ km/hr, $y \approx 57.8$ km.

**78.** Optimize $f(x, y) = x^2y$ subject to the constraint $x + 2y = 2$.

$$\left.\begin{aligned} 2xy &= \lambda \\ x^2 &= 2\lambda \end{aligned}\right\} x^2 = 4xy \implies x = 0 \text{ or } x = 4y$$

$x + 2y = 2$

If $x = 0$, $y = 1$. If $x = 4y$, then $y = \frac{1}{3}$, $x = \frac{4}{3}$.

Maximum: $f\left(\frac{4}{3}, \frac{1}{3}\right) = \frac{16}{27}$

Minimum: $f(0, 1) = 0$

# Problem Solving for Chapter 12

**2.** $V = \dfrac{4}{3}\pi r^3 + \pi r^2 h$

Material $= M = 4\pi r^2 + 2\pi rh$

$V = 1000 \implies h = \dfrac{1000 - (4/3)\pi r^3}{\pi r^2}$

Hence,

$$\begin{aligned} M &= 4\pi r^2 + 2\pi r\left(\frac{1000 - (4/3)\pi r^3}{\pi r^2}\right) \\ &= 4\pi r^2 + \frac{2000}{r} - \frac{8}{3}\pi r^2 \end{aligned}$$

$$\frac{dM}{dr} = 8\pi r - \frac{2000}{r^2} - \frac{16}{3}\pi r = 0$$

$$8\pi r - \frac{16}{3}\pi r = \frac{2000}{r^2}$$

$$r^3\left(\frac{8}{3}\pi\right) = 2000$$

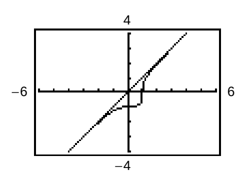
$$r^3 = \frac{750}{\pi} \implies r = 5\left(\frac{6}{\pi}\right)^{1/3}.$$

Then, $h = \dfrac{1000 - (4/3)\pi(750/\pi)}{\pi r^2} = 0$.

The tank is a sphere of radius $r = 5\left(\dfrac{6}{\pi}\right)^{1/3}$.

**4.** (a) As $x \to \pm\infty$, $f(x) = (x^3 - 1)^{1/3} \to x$ and hence

$$\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to-\infty} [f(x) - g(x)] = 0.$$




(b) Let $(x_0, (x_0{}^3 - 1)^{1/3})$ be a point on the graph of $f$.

The line through this point perpendicular to $g$ is

$$y = -x + x_0 + \sqrt[3]{x_0{}^3 - 1}.$$

This line intersects $g$ at the point

$$\left(\frac{1}{2}\left[x_0 + \sqrt[3]{x_0{}^3 - 1}\right], \frac{1}{2}\left[x_0 + \sqrt[3]{x_0{}^3 - 1}\right]\right).$$

The square of the distance between these two points is

$$h(x_0) = \frac{1}{2}\left(x_0 - \sqrt[3]{x_0{}^3 - 1}\right)^2.$$

$h$ is a maximum for $x_0 = \dfrac{1}{\sqrt[3]{2}}$. Hence, the point on $f$ farthest from $g$ is $\left(\dfrac{1}{\sqrt[3]{2}}, -\dfrac{1}{\sqrt[3]{2}}\right)$.

**6.** Heat Loss $= H = k(5xy + xy + 3xz + 3xz + 3yz + 3yz)$

$$= k(6xy + 6xz + 6yz)$$

$V = xyz = 1000 \implies z = \dfrac{1000}{xy}.$

Then $H = 6k\left(xy + \dfrac{1000}{y} + \dfrac{1000}{x}\right).$

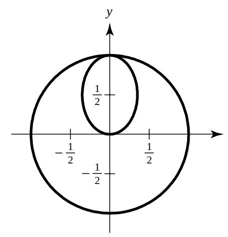
Setting $H_x = H_y = 0$, you obtain $x = y = z = 10$.

**8.** (a) $T(x, y) = 2x^2 + y^2 - y + 10 = 10$

$$2x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

$$2x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{x^2}{1/8} + \frac{(y - (1/2))^2}{1/4} = 1 \quad \text{ellipse}$$




(b) On $x^2 + y^2 = 1$, $T(x, y) = T(y) = 2(1 - y^2) + y^2 - y + 10 = 12 - y^2 - y$

$$T'(y) = -2y - 1 = 0 \implies y = -\frac{1}{2}, x = \pm\frac{\sqrt{3}}{2}.$$

Inside: $T_x = 4x - 0$, $T_y = 2y - 1 = 0 \implies \left(0, \dfrac{1}{2}\right)$

$T\left(0, \dfrac{1}{2}\right) = \dfrac{39}{4}$ minimum

$T\left(\pm\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right) = \dfrac{49}{4}$ maximum

**10.** $x = r\cos\theta, y = r\sin\theta, z = z$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta \quad \text{Similarly,}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

$$\frac{\partial^2 u}{\partial \theta^2} = (-r\sin\theta)\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial x\partial z}\frac{\partial z}{\partial \theta}\right] - r\frac{\partial u}{\partial x}\cos\theta$$

$$+ (r\cos\theta)\left[\frac{\partial^2 u}{\partial y\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial y\partial z}\frac{\partial z}{\partial \theta}\right] - r\frac{\partial u}{\partial y}\sin\theta$$

$$= \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\frac{\partial^2 u}{\partial x\partial y}r^2\sin\theta\cos\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta$$

Similarly, $\dfrac{\partial^2 u}{\partial r^2} = \dfrac{\partial^2 u}{\partial x^2}\cos^2\theta + \dfrac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\dfrac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta.$

Now observe that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \left[\frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta\right] + \frac{1}{r}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]$$

$$+ \left[\frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta - \frac{1}{r}\frac{\partial u}{\partial x}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial y}\sin\theta\right] + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Thus, Laplaces equation in cylindrical coordinates, is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

**12.** (a) $d = \sqrt{x^2 + y^2} = \sqrt{\left(32\sqrt{2}t\right)^2 + \left(32\sqrt{2}t - 16t^2\right)^2}$

$= \sqrt{4096t^2 - 1024\sqrt{2}t^3 + 256t^4}$

$= 16t\sqrt{t^2 - 4\sqrt{2}t + 16}$

(b) $\dfrac{dd}{dt} = \dfrac{32\left(t^2 - 3\sqrt{2}t + 8\right)}{\sqrt{t^2 - 4\sqrt{2}t + 16}}$

(c) When $t = 2$:

$$\frac{dd}{dt} = \frac{32\left(12 - 6\sqrt{2}\right)}{\sqrt{20 - 8\sqrt{2}}} \approx 38.16 \text{ ft/sec}$$

(d) $\dfrac{d^2d}{dt^2} = \dfrac{32\left(t^3 - 6\sqrt{2}t^2 + 36t - 32\sqrt{12}\right)}{\left(t^2 - 4\sqrt{2}t + 16\right)^{3/2}} = 0$

when $t \approx 1.943$ seconds. No. The projectile is at its maximum height when $t = \sqrt{2}$.

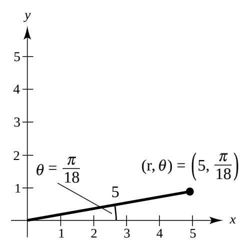
**14.** Given that $f$ is a differentiable function such that $\nabla f(x_0, y_0) = \mathbf{0}$, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Therefore, the tangent plane is $-(z - z_0) = 0$ or $z = z_0 = f(x_0, y_0)$ which is horizontal.

**16.** $(r, \theta) = \left(5, \dfrac{\pi}{18}\right)$

$dr = \pm 0.05$, $d\theta = \pm 0.05$

$x = r\cos\theta = 5\cos\dfrac{\pi}{18} \approx 4.924$

$y = r\sin\theta = 5\sin\dfrac{\pi}{18} \approx 0.868$




(a) $dx$ should be more effected by changes in $r$.

$$dx = (\cos\theta)dr + (-r\sin\theta)d\theta$$
$$\approx (0.985)dr - 0.868\, d\theta$$

$dx$ is more effected by changes in $r$ because $0.985 > 0.868$.

(b) $dy$ should be more effected by changes in $\theta$.

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$
$$\approx 0.174\, dr + 4.924\, d\theta$$

$dy$ is more effected by $\theta$ because $4.924 > 0.174$.

**18.** $\dfrac{\partial u}{\partial t} = \dfrac{1}{2}[-\cos(x - t) + \cos(x + t)]$

$\dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{2}[-\sin(x - t) - \sin(x + t)]$

$\dfrac{\partial u}{\partial x} = \dfrac{1}{2}[\cos(x - t) + \cos(x + t)]$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{2}[-\sin(x - t) - \sin(x + t)]$

Then, $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$.

# CHAPTER 13
# Multiple Integration

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# Multiple Integration

## Section 13.1 Iterated Integrals and Area in the Plane

Solutions to Even-Numbered Exercises

**2.** $\displaystyle\int_x^{x^2} \frac{y}{x}\,dy = \left[\frac{1}{2}\frac{y^2}{x}\right]_x^{x^2} = \frac{1}{2}\left(\frac{x^4}{x} - \frac{x^2}{x}\right) = \frac{x}{2}(x^2 - 1)$

**4.** $\displaystyle\int_0^{\cos y} y\,dx = \Big[yx\Big]_0^{\cos y} = y\cos y$

**6.** $\displaystyle\int_{x^3}^{\sqrt{x}} (x^2 + 3y^2)\,dy = \Big[x^2y + y^3\Big]_{x^3}^{\sqrt{x}} = \left(x^2\sqrt{x} + \left(\sqrt{x}\right)^3\right) - (x^2x^3 + (x^3)^3) = x^{5/2} + x^{3/2} - x^5 - x^9$

**8.** $\displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx = \left[\frac{1}{3}x^3 + y^2x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 2\left[\frac{1}{3}(1-y^2)^{3/2} + y^2(1-y^2)^{1/2}\right] = \frac{2\sqrt{1-y^2}}{3}(1 + 2y^2)$

**10.** $$\int_y^{\pi/2} \sin^3 x\cos y\,dx = \int_y^{\pi/2} (1 - \cos^2 x)\sin x\cos y\,dx$$
$$= \left[\left(-\cos x + \frac{1}{3}\cos^3 x\right)\cos y\right]_y^{\pi/2} = \left(\cos y - \frac{1}{3}\cos^3 y\right)\cos y$$

**12.** $$\int_{-1}^{1}\int_{-2}^{2}(x^2 - y^2)\,dy\,dx = \int_{-1}^{1}\left[x^2y - \frac{y^3}{3}\right]_{-2}^{2} dx = \int_{-1}^{1}\left[2x^2 - \frac{8}{3} + 2x^2 - \frac{8}{3}\right] dx$$
$$= \int_{-1}^{1}\left(4x^2 - \frac{16}{3}\right) dx = \left[\frac{4x^3}{3} - \frac{16}{3}x\right]_{-1}^{1} = \left(\frac{4}{3} - \frac{16}{3}\right) - \left(-\frac{4}{3} + \frac{16}{3}\right) = -8$$

**14.** $$\int_{-4}^{4}\int_{0}^{x^2}\sqrt{64 - x^3}\,dy\,dx = \int_{-4}^{4}\left[y\sqrt{64 - x^3}\right]_0^{x^2} dx$$
$$= \int_{-4}^{4}\sqrt{64 - x^3}\,x^2\,dx = \left[-\frac{2}{9}(64 - x^3)^{3/2}\right]_{-4}^{4} = 0 + \frac{2}{9}(128)^{3/2} = \frac{2048}{9}\sqrt{2}$$

**16.** $$\int_0^2\int_y^{2y}(10 + 2x^2 + 2y^2)\,dx\,dy = \int_0^2\left[10x + \frac{2x^3}{3} + 2y^2x\right]_y^{2y} dy = \int_0^2\left[\left(20y + \frac{16}{3}y^3 + 4y^3\right) - \left(10y + \frac{2}{3}y^3 + 2y^3\right)\right] dy$$
$$= \int_0^2\left[10y + \frac{14}{3}y^3 + 2y^3\right] dy = \left[5y^2 + \frac{7y^4}{6} + \frac{y^4}{2}\right]_0^2 = 20 + \frac{56}{3} + 8 = \frac{140}{3}$$

**18.** $$\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy = \int_0^2\Big[3xy\Big]_{3y^2-6y}^{2y-y^2} dy = 3\int_0^2(8y^2 - 4y^3)\,dy = \left[3\left(\frac{8}{3}y^3 - y^4\right)\right]_0^2 = 16$$

**20.** $$\int_0^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^2\theta\,d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{2}$$

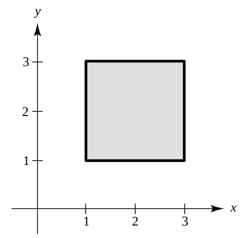
**22.** $$\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\Big[r^3\sin\theta\Big]_0^{\cos\theta} d\theta$$
$$= \int_0^{\pi/4}\cos^3\sin\theta\,d\theta = \left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/4} = -\frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right] = \frac{3}{16}$$

**24.** $\displaystyle\int_0^3\int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx = \int_0^3 \left[x^2 \arctan y\right]_0^\infty dx = \int_0^3 x^2\left(\frac{\pi}{2}\right)dx = \left[\frac{\pi}{2}\cdot\frac{x^3}{3}\right]_0^3 = \frac{9\pi}{2}$

**26.** $\displaystyle\int_0^\infty\int_0^\infty xye^{-(x^2+y^2)}\,dx\,dy = \int_0^\infty\left[-\frac{1}{2}ye^{-(x^2+y^2)}\right]_0^\infty dy = \int_0^\infty \frac{1}{2}ye^{-y^2}\,dy = \left[-\frac{1}{4}e^{-y^2}\right]_0^\infty = \frac{1}{4}$

**28.** $\displaystyle A = \int_1^3\int_1^3 dy\,dx = \int_1^3\left[y\right]_1^3 dx = \int_1^3 2\,dx = \left[2x\right]_1^3 = 4$

$\displaystyle A = \int_1^3\int_1^3 dx\,dy = \int_1^3\left[x\right]_1^3 dy = \int_1^3 2\,dy = \left[2y\right]_1^3 = 4$



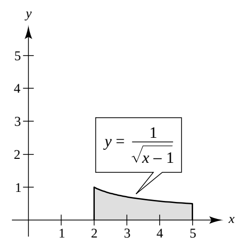
**30.** $\displaystyle A = \int_2^5\int_0^{1/\sqrt{x-1}} dy\,dx = \int_2^5\left[y\right]_0^{1/\sqrt{x-1}} dx = \int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \left[2\sqrt{x-1}\right]_2^5 = 2$

$\displaystyle A = \int_0^{1/2}\int_2^5 dx\,dy + \int_{1/2}^1\int_2^{1+(1/y^2)} dx\,dy$

$\displaystyle = \int_0^{1/2}\left[x\right]_2^5 dy + \int_{1/2}^1\left[x\right]_2^{1+(1/y^2)} dy$

$\displaystyle = \int_0^{1/2} 3\,dy + \int_{1/2}^1\left(\frac{1}{y^2}-1\right)dy$

$\displaystyle = \left[3y\right]_0^{1/2} + \left[-\frac{1}{y}-y\right]_{1/2}^1 = 2$




**32.** $\displaystyle A = \int_0^2\int_0^{\sqrt{4-x^2}} dy\,dx$

$\displaystyle = \int_0^2 \sqrt{4-x^2}\,dx$

$\displaystyle = 4\int_0^{\pi/2}\cos^2\theta\,d\theta$

$\displaystyle = 2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$

$\displaystyle = \left[2\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi$

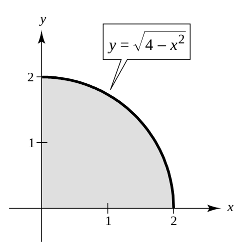
$\left(x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta\right)$

$\displaystyle A = \int_0^2\int_0^{\sqrt{4-y^2}} dx\,dy = \int_0^2\sqrt{4-y^2}\,dy$

$\displaystyle = 4\int_0^{\pi/2}\cos^2\theta\,d\theta = 2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$

$\displaystyle = \left[2\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi$

$\left(y = 2\sin\theta,\ dy = 2\cos\theta\,d\theta,\ \sqrt{4-y^2} = 2\cos\theta\right)$




**34.** $\displaystyle A = \int_0^4\int_{x^{3/2}}^{2x} dy\,dx$

$\displaystyle = \int_0^4\left[y\right]_{x^{3/2}}^{2x} dx$

$\displaystyle = \int_0^4 (2x - x^{3/2})\,dx$

$\displaystyle = \left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4$

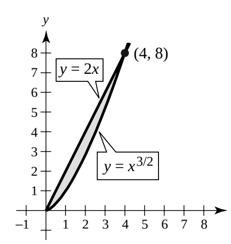
$\displaystyle = 16 - \frac{2}{5}(32) = \frac{16}{5}$

$\displaystyle A = \int_0^8\int_{y/2}^{y^{2/3}} dx\,dy$

$\displaystyle = \int_0^8\left(y^{2/3}-\frac{y}{2}\right)dy$

$\displaystyle = \left[\frac{3}{5}y^{5/3} - \frac{y^2}{4}\right]_0^8$

$\displaystyle = \frac{3}{5}(32) - 16 = \frac{16}{5}$



**36.** $A = \int_0^3 \int_0^x dy\, dx + \int_3^9 \int_0^{9/x} dy\, dx$

$= \int_0^3 \Big[ y \Big]_0^x dx + \int_3^9 \Big[ y \Big]_0^{9/x} dx = \int_0^3 x\, dx + \int_3^9 \frac{9}{x}\, dx$

$= \left[ \frac{1}{2}x^2 \right]_0^3 + \Big[ 9 \ln x \Big]_3^9 = \frac{9}{2} + 9(\ln 9 - \ln 3)$

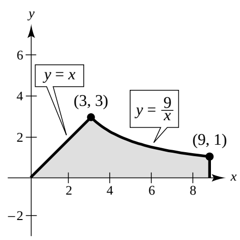
$= \frac{9}{2}(1 + \ln 9)$

$A = \int_0^1 \int_y^9 dx\, dy + \int_1^3 \int_y^{9/y} dx\, dy$

$= \int_0^1 \Big[ x \Big]_y^9 dy + \int_1^3 \Big[ x \Big]_y^{9/y} dy$

$= \int_0^1 (9 - y)\, dy + \int_1^3 \left( \frac{9}{y} - y \right) dy$

$= \left[ 9y - \frac{1}{2}y^2 \right]_0^1 + \left[ 9 \ln y - \frac{1}{2}y^2 \right]_1^3 = \frac{9}{2}(1 + \ln 9)$



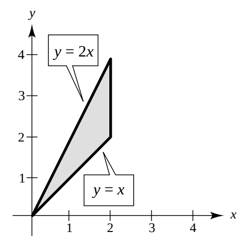

**38.** $A = \int_0^2 \int_{y/2}^y dx\, dy + \int_2^4 \int_{y/2}^2 dx\, dy$

$= \int_0^2 \frac{y}{2}\, dy + \int_2^4 \left( 2 - \frac{y}{2} \right) dy$

$= \left[ \frac{y^2}{4} \right]_0^2 + \left[ 2y - \frac{y^2}{4} \right]_2^4$

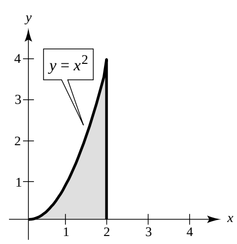
$= 1 + (4 - 3) = 2$

$A = \int_0^2 \int_x^{2x} dy\, dx = \int_0^2 (2x - x)\, dx = \left[ \frac{x^2}{2} \right]_0^2 = 2$



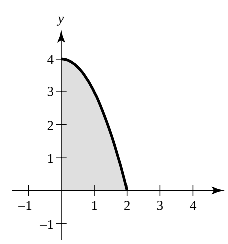

**40.** $\int_0^4 \int_{\sqrt{y}}^2 f(x, y)\, dx\, dy, \;\; \sqrt{y} \le x \le 2, \;\; 0 \le y \le 4$

$= \int_0^2 \int_0^{x^2} f(x, y)\, dy\, dx$



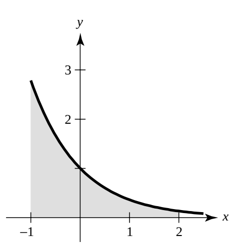

**42.** $\int_0^2 \int_0^{4-x^2} f(x, y)\, dy\, dx, \; 0 \le y \le 4 - x^2, \; 0 \le x \le 2$

$= \int_0^4 \int_0^{\sqrt{4-y}} f(x, y)\, dx\, dy$



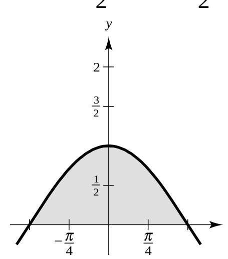

**44.** $\int_{-1}^2 \int_0^{e^{-x}} f(x, y)\, dy\, dx, \; 0 \le y \le e^{-x}, \; -1 \le x \le 2$

$= \int_0^{e^{-2}} \int_{-1}^2 f(x, y)\, dx\, dy + \int_{e^{-2}}^e \int_{-1}^{-\ln y} f(x, y)\, dx\, dy$

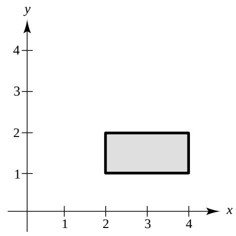



**46.** $\int_{-\pi/2}^{\pi/2} \int_0^{\cos x} f(x, y)\, dy\, dx, \; 0 \le y \le \cos x, \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$= \int_0^1 \int_{-\arccos y}^{\arccos y} f(x, y)\, dx\, dy$

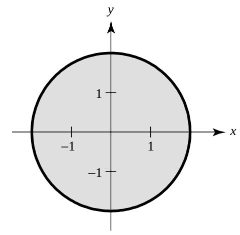


**48.** $\displaystyle\int_1^2\int_2^4 dx\,dy = \int_2^4\int_1^2 dy\,dx = 2$



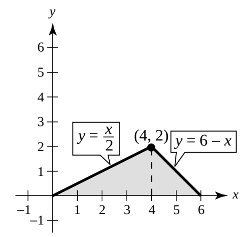
**50.** $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx = \int_{-2}^{2}\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) dx = 4\pi$

$\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\,dy = 4\pi$



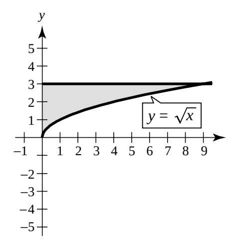
**52.** $\displaystyle\int_0^4\int_0^{x/2} dy\,dx + \int_4^6\int_0^{6-x} dy\,dx = \int_0^4 \frac{x}{2}\,dx + \int_4^6 (6-x)\,dx = 4 + 2 = 6$

$\displaystyle\int_0^2\int_{2y}^{6-y} dx\,dy = \int_0^2 (6-3y)\,dy = \left[6y - \frac{3y^2}{2}\right]_0^2 = 6$

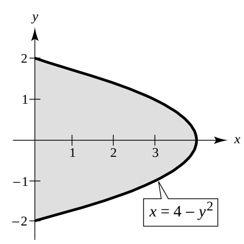



**54.** $\displaystyle\int_0^9\int_{\sqrt{x}}^3 dy\,dx = \int_0^9 \left(3 - \sqrt{x}\right) dx = \left[3x - \frac{2}{3}x^{3/2}\right]_0^9 = 27 - 18 = 9$

$\displaystyle\int_0^3\int_0^{y^2} dx\,dy = \int_0^3 y^2\,dy = \left[\frac{y^3}{3}\right]_0^3 = 9$




**56.** $\displaystyle\int_{-2}^{2}\int_0^{4-y^2} dx\,dy = \int_0^4\int_{-\sqrt{4-x}}^{\sqrt{4-x}} dy\,dx = \frac{32}{3}$



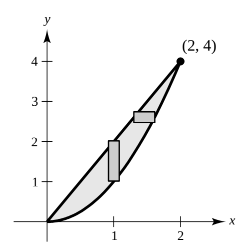

**58.** The first integral arises using vertical representative rectangles. The second integral arises using horizontal representative rectangles.

$$\int_0^2\int_{x^2}^{2x} x \sin y\,dy\,dx = \int_0^2 \left(-x\cos(2x) + x\cos(x^2)\right) dx$$

$$= -\frac{1}{4}\cos(4) - \frac{1}{2}\sin(4) + \frac{1}{4}$$

$$\int_0^4\int_{y/2}^{\sqrt{y}} x \sin y\,dx\,dy = \int_0^4 \left(\frac{1}{2}y\sin(y) - \frac{1}{8}y^2\sin(y)\right) dy$$

$$= -\frac{1}{4}\cos(4) - \frac{1}{2}\sin(4) + \frac{1}{4}$$



**60.** $\int_0^2 \int_x^2 e^{-y^2}\,dy\,dx = \int_0^2 \int_0^y e^{-y^2}\,dx\,dy$

$$= \int_0^2 \left[xe^{-y^2}\right]_0^y dy = \int_0^2 ye^{-y^2}\,dy = \left[-\frac{1}{2}e^{-y^2}\right]_0^2 = -\frac{1}{2}(e^{-4}) + \frac{1}{2}e^0 = \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \approx 0.4908$$

**62.** $\int_0^2 \int_{y^2}^4 \sqrt{x}\sin x\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} \sqrt{x}\sin x\,dy\,dx$

$$= \int_0^4 \left[y\sqrt{x}\sin x\right]_0^{\sqrt{x}} dx = \int_0^4 x\sin x\,dx = \left[\sin x - x\cos x\right]_0^4 = \sin 4 - 4\cos 4 \approx 1.858$$

**64.** $\int_0^1 \int_y^{2y} \sin(x + y)\,dx\,dy = \frac{\sin 2}{2} - \frac{\sin 3}{3} \approx 0.408$

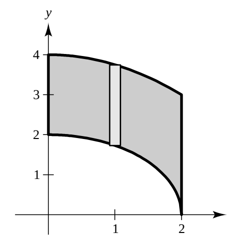
**66.** $\int_0^a \int_0^{a-x} (x^2 + y^2)\,dy\,dx = \frac{a^4}{6}$

**68.** (a) $y = \sqrt{4 - x^2} \Leftrightarrow x = \sqrt{4 - y^2}$

$y = 4 - \frac{x^2}{4} \Leftrightarrow x = \sqrt{16 - 4y}$

(b) $\int_0^2 \int_{\sqrt{4-y^2}}^2 \frac{xy}{x^2 + y^2 + 1}\,dx\,dy + \int_2^3 \int_0^2 \frac{xy}{x^2 + y^2 + 1}\,dx\,dy + \int_3^4 \int_0^{\sqrt{16-4y}} \frac{xy}{x^2 + y^2 + 1}\,dx\,dy$

(c) Both orders of integration yield 1.11899.




**70.** $\int_0^2 \int_x^2 \sqrt{16 - x^3 - y^3}\,dy\,dx \approx 6.8520$

**72.** $\int_0^{\pi/2} \int_0^{1+\sin\theta} 15\theta r\,dr\,d\theta = \frac{45\pi^2}{32} + \frac{135}{8} \approx 30.7541$

**74.** A region is vertically simple if it is bounded on the left and right by vertical lines, and bounded on the top and bottom by functions of $x$. A region is horizontally simple if it is bounded on the top and bottom by horizontal lines, and bounded on the left and right by functions of $y$.

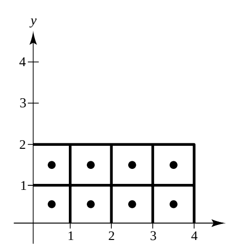
**76.** The integrations might be easier. See Exercise 59-62.

**78.** False, let $f(x, y) = x$.

## Section 13.2 Double Integrals and Volume

**For Exercises 2 and 4, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are**

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right).$$




**2.** $f(x, y) = \frac{1}{2}x^2y$

$$\sum_{i=1}^{\infty} f(x_i, y_i)\Delta x_i\,\Delta y_i = \frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16} + \frac{3}{16} + \frac{27}{16} + \frac{75}{16} + \frac{147}{16} = 21$$

$$\int_0^4 \int_0^2 \frac{1}{2}x^2y\,dy\,dx = \int_0^4 \left[\frac{x^2y^2}{4}\right]_0^2 dx = \int_0^4 x^2\,dx = \left.\frac{x^3}{3}\right]_0^4 = \frac{64}{3} \approx 21.3$$

**4.** $f(x, y) = \dfrac{1}{(x+1)(y+1)}$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{4}{9} + \frac{4}{15} + \frac{4}{21} + \frac{4}{27} + \frac{4}{15} + \frac{4}{25} + \frac{4}{35} + \frac{4}{45} = \frac{7936}{4725} \approx 1.680$$

$$\int_0^4\int_0^2 \frac{1}{(x+1)(y+1)}\,dy\,dx = \int_0^4 \left[\frac{1}{x+1}\ln(y+1)\right]_0^2 dx$$

$$= \int_0^4 \frac{\ln 3}{x+1}\,dx = \Big[\ln 3 \cdot \ln(x+1)\Big]_0^4 = (\ln 3)(\ln 5) \approx 1.768$$

**6.** $\displaystyle\int_0^2\int_0^2 f(x, y)\,dy\,dx \approx 4 + 2 + 8 + 6 = 20$

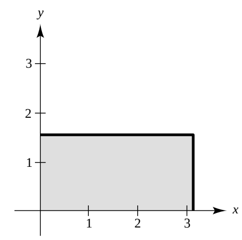
**8.** $$\int_0^{\pi}\int_0^{\pi/2} \sin^2 x \cos^2 y\,dy\,dx = \int_0^{\pi}\left[\frac{1}{2}\sin^2 x\left(y + \frac{1}{2}\sin 2y\right)\right]_0^{\pi/2} dx$$

$$= \int_0^{\pi}\frac{1}{2}\sin^2 x\left(\frac{\pi}{2}\right) dx$$

$$= \frac{\pi}{8}\int_0^{\pi}(1 - \cos 2x)\,dx$$

$$= \left[\frac{\pi}{8}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\pi}$$

$$= \frac{\pi^2}{8}$$

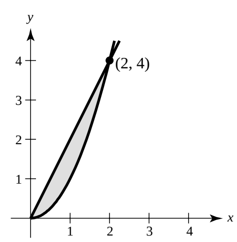


**10.** $$\int_0^4\int_{(1/2)y}^{\sqrt{y}} x^2y^2\,dx\,dy = \int_0^4\left[\frac{x^3y^2}{3}\right]_{(1/2)y}^{\sqrt{y}} dy$$

$$= \int_0^4\left(\frac{y^{7/2}}{3} - \frac{y^5}{24}\right) dy$$

$$= \left[\frac{2y^{9/2}}{27} - \frac{y^6}{144}\right]_0^4$$

$$= \frac{1024}{27} - \frac{256}{9} = \frac{256}{27}$$

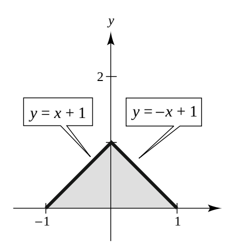



**12.** $$\int_0^1\int_{y-1}^{0} e^{x+y}\,dx\,dy + \int_0^1\int_0^{1-y} e^{x+y}\,dx\,dy = \int_0^1\Big[e^{x+y}\Big]_{y-1}^{0} dy + \int_0^1\Big[e^{x+y}\Big]_0^{1-y} dy$$

$$= \int_0^1 (e - e^{2y-1})\,dy$$

$$= \left[ey - \frac{1}{2}e^{2y-1}\right]_0^1$$

$$= \frac{1}{2}(e + e^{-1})$$

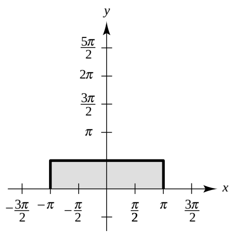


**14.** $\displaystyle\int_0^{\pi/2}\int_{-\pi}^{\pi} \sin x \sin y \, dx\, dy = \int_{-\pi}^{\pi}\int_0^{\pi/2} \sin x \sin y \, dy\, dx$

$$= \int_{-\pi}^{\pi}\Big[-\sin x \cos y\Big]_0^{\pi/2} dx$$

$$= \int_{-\pi}^{\pi} \sin x \, dx$$

$$= 0$$



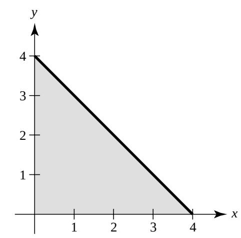

**16.** $\displaystyle\int_0^4\int_0^{4-x} xe^y \, dy\, dx = \int_0^4\int_0^{4-y} xe^y \, dx\, dy$

For the first integral, we obtain:

$$\int_0^4\Big[xe^y\Big]_0^{4-x} - x\, dx = \int_0^{4-x}(xe^4 - x)dx$$

$$= \Big[-e^{4-x}(1+x) - \frac{x^2}{2}\Big]_0^4$$

$$= (-5-8) + e^4 = e^4 - 13.$$

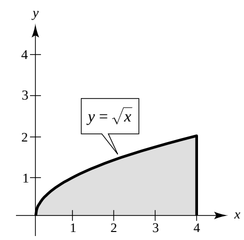



**18.** $\displaystyle\int_0^2\int_{y^2}^4 \frac{y}{1+x^2}\, dx\, dy = \int_0^4\int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dy\, dx$

$$= \frac{1}{2}\int_0^4 \Big[\frac{y^2}{1+x^2}\Big]_0^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^4 \frac{x}{1+x^2}\, dx$$

$$= \Big[\frac{1}{4}\ln(1+x^2)\Big]_0^4 = \frac{1}{4}\ln(17)$$



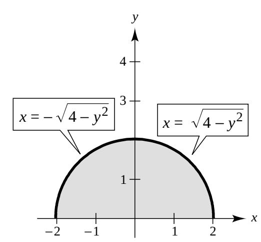

**20.** $\displaystyle\int_0^2\int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2+y^2)\, dx\, dy$

$$= \int_{-2}^2\int_0^{\sqrt{4-x^2}} (x^2+y^2)\, dy\, dx$$

$$= \int_{-2}^2\Big[x^2y + \frac{1}{3}y^3\Big]_0^{\sqrt{4-x^2}} dx$$

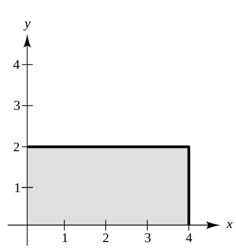
$$= \int_{-2}^2\Big[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2}\Big] dx$$

$$= \Big[-\frac{x}{4}(4-x^2)^{3/2} + \frac{1}{2}\Big(x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}\Big) + \frac{1}{12}\Big[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24\arctan\frac{x}{2}\Big]\Big]_{-2}^2 = 4\pi$$

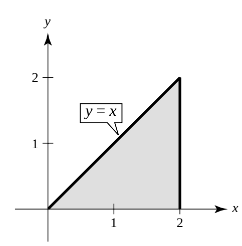



**22.** $\displaystyle\int_0^4\int_0^2 (6-2y)\, dy\, dx = \int_0^4\Big[6y - y^2\Big]_0^2 dx$

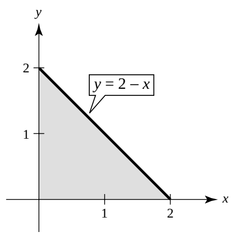
$$= \int_0^4 8\, dx = 32$$



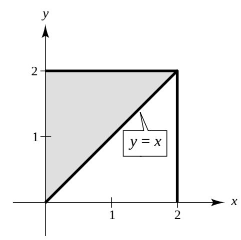

**24.** $\displaystyle\int_0^2\int_0^x 4\, dy\, dx = \int_0^2 4x\, dx = 2x^2\Big]_0^2 = 8$



**26.** $$\int_0^2\int_0^{2-x}(2-x-y)\,dy\,dx = \int_0^2\left[2y - xy - \frac{y^2}{2}\right]_0^{2-x}dx$$
$$= \int_0^2 \frac{1}{2}(2-x)^2\,dx$$
$$= -\frac{1}{6}(x-2)^3\Big]_0^2 = \frac{4}{3}$$



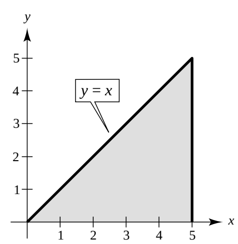

**28.** $$\int_0^2\int_0^y (4-y^2)\,dx\,dy = \int_0^2 (4y - y^3)\,dy$$
$$= \left[2y^2 - \frac{y^4}{4}\right]_0^2$$
$$= 4$$



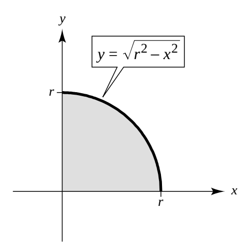

**30.** $$\int_0^\infty\int_0^\infty e^{-(x+y)/2}\,dy\,dx = \int_0^\infty\left[-2e^{-(x+y)/2}\right]_0^\infty dx = \int_0^\infty 2e^{-x/2}\,dx = \left[-4e^{-x/2}\right]_0^\infty = 4$$

**32.** $$\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx = \frac{1}{3}$$

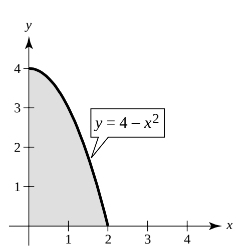
**34.** $$V = \int_0^5\int_0^x x\,dy\,dx$$
$$= \int_0^5\left[xy\right]_0^x dx = \int_0^5 x^2\,dx$$
$$= \left[\frac{1}{3}x^3\right]_0^5 = \frac{125}{3}$$



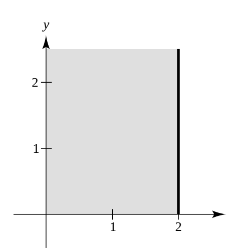

**36.** $$V = 8\int_0^r\int_0^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2}\,dy\,dx$$
$$= 4\int_0^r\left[\left[y\sqrt{r^2-x^2-y^2} + (r^2-x^2)\arcsin\frac{y}{\sqrt{r^2-x^2}}\right]\right]_0^{\sqrt{r^2-x^2}}dx$$
$$= 4\left(\frac{\pi}{2}\right)\int_0^r (r^2-x^2)\,dx$$
$$= \left[2\pi\left(r^2x - \frac{1}{3}x^3\right)\right]_0^r$$
$$= \frac{4\pi r^3}{3}$$




**38.** $$V = \int_0^2\int_0^{4-x^2}(4-x^2)\,dy\,dx$$
$$= \int_0^2 (4-x^2)(4-x^2)\,dx$$
$$= \int_0^2 (16 - 8x^2 + x^4)\,dx$$
$$= \left[16x - 8\frac{x^3}{3} + \frac{x^5}{5}\right]_0^2$$
$$= 32 - \frac{64}{3} + \frac{32}{5} = \frac{256}{15}$$




**40.** $$V = \int_0^2\int_0^\infty \frac{1}{1+y^2}\,dy\,dx$$
$$= \int_0^2\left[\arctan y\right]_0^\infty dx$$
$$= \int_0^2 \frac{\pi}{2}\,dx$$
$$= \left[\frac{\pi x}{2}\right]_0^2 = \pi$$

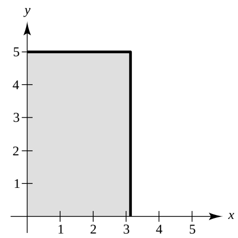


**42.** $V = \int_0^5 \int_0^{\pi} \sin^2 x \, dx \, dy$

$= \int_0^5 \frac{\pi}{2} \, dy$

$= \left[\frac{\pi}{2} y\right]_0^5$

$= \frac{5\pi}{2}$




**44.** $V = \int_0^9 \int_0^{\sqrt{9-y}} \sqrt{9-y} \, dx \, dy = \frac{81}{2}$

**46.** $V = \int_0^{16} \int_0^{4-\sqrt{y}} \ln(1 + x + y) \, dx \, dy \approx 38.25$

**48.** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

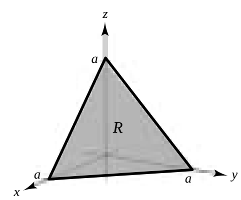
$V = \iint_R f(x, y) \, dA = \int_0^a \int_0^{b[1-(x/a)]} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy \, dx$

$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b[1-(x/a)]} dx$

$= c\int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{xb}{a}\left(1 - \frac{x}{a}\right) - \frac{b^2}{2b}\left(1 - \frac{x}{a}\right)^2\right] dx$

$= c\left[-\frac{ab}{2}\left(1 - \frac{x}{a}\right)^2 - \frac{x^2 b}{2a} + \frac{x^3 b}{3a^2} + \frac{ab}{6}\left(1 - \frac{x}{a}\right)^3\right]_0^a$

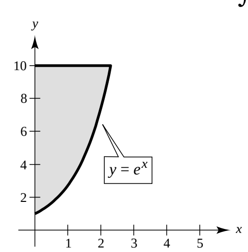
$= c\left[\left(-\frac{ab}{2} + \frac{ab}{3}\right) - \left(-\frac{ab}{2} + \frac{ab}{6}\right)\right] = \frac{abc}{6}$




**50.** $\int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y} \, dy \, dx = \int_1^{10} \int_0^{\ln y} \frac{1}{\ln y} \, dx \, dy$

$= \int_1^{10} \left[\frac{x}{\ln y}\right]_0^{\ln y} dy$

$= \int_1^{10} dy = \left[y\right]_1^{10} = 9$



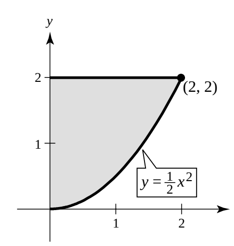

**52.** $\int_0^2 \int_{(1/2)x^2}^2 \sqrt{y} \cos y \, dy \, dx = \int_0^2 \int_0^{\sqrt{2y}} \sqrt{y} \cos y \, dx \, dy$

$= \int_0^2 \sqrt{y} \cos y \sqrt{2y} \, dy$

$= \sqrt{2} \int_0^2 y \cos y \, dy$

$= \sqrt{2}\left[\cos y + y \sin y\right]_0^2$

$= \sqrt{2}[\cos 2 + 2 \sin 2 - 1]$




**54.** Average $= \frac{1}{8}\int_0^4 \int_0^2 xy \, dy \, dx = \frac{1}{8}\int_0^4 2x \, dx = \left[\frac{x^2}{8}\right]_0^4 = 2$

**56.** Average $= \frac{1}{1/2}\int_0^1 \int_x^1 e^{x+y} \, dy \, dx = 2\int_0^1 e^{x+1} - e^{2x} \, dx$

$= 2\left[e^{x+1} - \frac{1}{2}e^{2x}\right]_0^1 = 2\left[e^2 - \frac{1}{2}e^2 - e + \frac{1}{2}\right]$

$= e^2 - 2e + 1$

$= (e - 1)^2$

**58.** The second is integrable. The first contains $\int \sin y^2\,dy$ which does not have an elementary antiderivation.

**60.** (a) The total snowfall in the county $R$.

(b) The average snowfall in $R$.

**62.** Average $= \dfrac{1}{150}\displaystyle\int_{45}^{60}\int_{40}^{50}\left[192x + 576y - x^2 - 5y^2 - 2xy - 5000\right] dx\,dy \approx 13{,}246.67$

**64.** $f(x, y) \geq 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^2\int_0^2 \frac{1}{4}xy\,dy\,dx = \int_0^2 \frac{x}{2}\,dx = 1$$

$$P(0 \leq x \leq 1, 1 \leq y \leq 2) = \int_0^1\int_1^2 \frac{1}{4}xy\,dy\,dx = \int_0^1 \frac{3x}{8}\,dx = \frac{3}{16}.$$

**66.** $f(x, y) \geq 0$ for all $(x, y)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^{\infty}\int_0^{\infty} e^{-x-y}\,dy\,dx$$

$$= \int_0^{\infty} \lim_{b\to\infty}\Big[-e^{-x-y}\Big]_0^b\,dx = \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = 1$$

$$P(0 \leq x \leq 1, x \leq y \leq 1) = \int_0^1\int_x^1 e^{-x-y}\,dy\,dx = \int_0^1 \Big[e^{-x-y}\Big]_x^1\,dx = \int_0^1 (e^{-2x} - e^{-x-1})\,dx$$

$$= \left[-\frac{1}{2}e^{-2x} + e^{-x-1}\right]_0^1 = \frac{1}{2}e^{-2} - e^{-1} + \frac{1}{2} = \frac{1}{2}(e^{-1} - 1)^2 \approx 0.1998.$$

**68.** Sample Program for TI-82:

Program: DOUBLE

: Input A
: Input B
: Input M
: Input C
: Input D
: Input N
: $0 \to V$
: $(\mathrm{B} - \mathrm{A})/\mathrm{M} \to \mathrm{G}$
: $(\mathrm{D} - \mathrm{C})/\mathrm{N} \to \mathrm{H}$
: For (I, 1, M, 1)
: For (J, 1, N, 1)
: $\mathrm{A} + 0.5\mathrm{G}(2\mathrm{I} - 1) \to \mathrm{x}$
: $\mathrm{C} + 0.5\mathrm{H}(2\mathrm{J} - 1) \to \mathrm{y}$
: $\mathrm{V} + \sin\left(\sqrt{\mathrm{x} + \mathrm{y}}\right) \times \mathrm{G} \times \mathrm{H} \to \mathrm{V}$
: End
: End
: Disp V

**70.** $\displaystyle\int_0^2\int_0^4 20e^{-x^3/8}\,dy\,dx \qquad m = 10, n = 20$

(a) 129.2018

(b) 129.2756

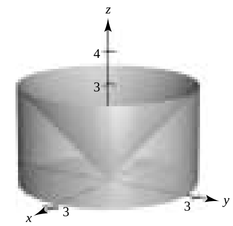
**72.** $\displaystyle\int_1^4\int_1^2 \sqrt{x^3+y^3}\,dx\,dy \qquad m = 6, n = 4$

(a) 13.956

(b) 13.9022

**74.** $V \approx 50$

Matches a.




**76.** True

**78.** $\displaystyle\int_1^2 e^{-xy}\,dy = \left[-\frac{1}{x}e^{-xy}\right]_1^2 = \frac{e^{-x}-e^{-2x}}{x}$

Thus,

$$\begin{aligned}\int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx &= \int_0^\infty\int_1^2 e^{-xy}\,dx\,dy\\ &= \int_1^2\int_0^\infty e^{-xy}\,dx\,dy\\ &= \int_1^2\left[-\frac{e^{-xy}}{y}\right]_0^\infty dy\\ &= \int_1^2 \frac{1}{y}\,dy = \Big[\ln y\Big]_1^2 = \ln 2.\end{aligned}$$

## Section 13.3 Change of Variables: Polar Coordinates

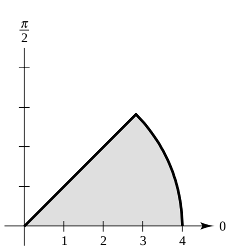
**2.** Polar coordinates

**4.** Rectangular coordinates

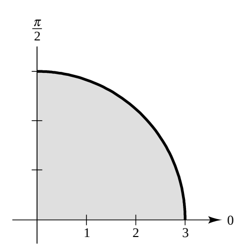
**6.** $R = \{(r, \theta):\ 0 \le r \le 4\sin\theta, 0 \le \theta \le \pi\}$

**8.** $R = \{(r, \theta):\ 0 \le r \le r\cos 3\theta, 0 \le \theta \le \pi\}$

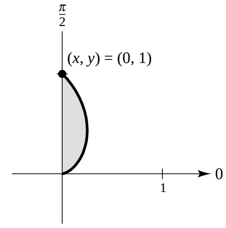
**10.** $$\begin{aligned}\int_0^{\pi/4}\int_0^4 r^2\sin\theta\cos\,dr\,d\theta &= \int_0^{\pi/4}\left[\frac{r^3}{3}\sin\theta\cos\theta\right]_0^4 d\theta\\ &= \left[\left(\frac{64}{3}\right)\frac{\sin^2\theta}{2}\right]_0^{\pi/4}\\ &= \frac{16}{3}\end{aligned}$$




**12.** $$\begin{aligned}\int_0^{\pi/2}\int_0^3 re^{-r^2}\,dr\,d\theta &= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^3 d\theta\\ &= \left[-\frac{1}{2}(e^{-9}-1)\theta\right]_0^{\pi/2}\\ &= \frac{\pi}{4}\left(1-\frac{1}{e^9}\right)\end{aligned}$$



**14.** $$\int_0^{\pi/2}\int_0^{1-\cos\theta}(\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\left[(\sin\theta)\frac{r^2}{2}\right]_0^{1-\cos\theta}d\theta$$
$$= \int_0^{\pi/2}\frac{\sin\theta}{2}(1-\cos\theta)^2\,d\theta$$
$$= \left[\frac{1}{6}(1-\cos(\theta))^3\right]_0^{\pi/2} = \frac{1}{6}$$



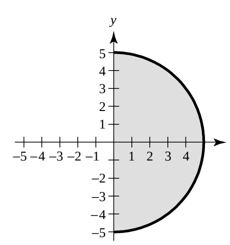

**16.** $$\int_0^a\int_0^{\sqrt{a^2-x^2}}x\,dy\,dx = \int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\cos\theta\,d\theta = \left[\frac{a^3}{3}\sin\theta\right]_0^{\pi/2} = \frac{a^3}{3}$$

**18.** $$\int_0^2\int_y^{\sqrt{8-y^2}}\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/4}\int_0^{2\sqrt{2}}r^2\,dr\,d\theta$$
$$= \int_0^{\pi/4}\frac{\left(2\sqrt{2}\right)^3}{3}\,d\theta = \left[\frac{\left(2\sqrt{2}\right)^3}{3}\theta\right]_0^{\pi/4} = \frac{\left(2\sqrt{2}\right)^3}{3}\cdot\frac{\pi}{4} = \frac{4\sqrt{2}\pi}{3}$$

**20.** $$\int_0^4\int_0^{\sqrt{4y-y^2}}x^2\,dx\,dy = \int_0^{\pi/2}\int_0^{4\sin\theta}r^3\cos^2\theta\,dr\,d\theta = \int_0^{\pi/2}64\sin^4\theta\cos^2\,d\theta$$
$$= 64\int_0^{\pi/2}(\sin^4\theta - \sin^6\theta)\,d\theta = \frac{64}{6}\left[\sin^5\theta\cos\theta - \frac{\sin^3\theta\cos\theta}{4} + \frac{3}{8}(\theta - \sin\theta\cos\theta)\right]_0^{\pi/2} = 2\pi$$

**22.** $$\int_0^{\left(5\sqrt{2}\right)/2}\int_0^x xy\,dy\,dx + \int_{\left(5\sqrt{2}\right)/2}^5\int_0^{\sqrt{25-x^2}}xy\,dy\,dx = \int_0^{\pi/4}\int_0^5 r^3\sin\theta\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi/4}\frac{625}{4}\sin\theta\cos\,d\theta$$
$$= \left[\frac{625}{8}\sin^2\theta\right]_0^{\pi/4}$$
$$= \frac{625}{16}$$

**24.** $$\int_{-\pi/2}^{\pi/2}\int_0^5 e^{-r^2/2}r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[-e^{-r^2/2}\right]_0^5 d\theta$$
$$= \int_{-\pi/2}^{\pi/2}(1-e^{-25/2})\,d\theta$$
$$= \left[(1-e^{-25/2})\theta\right]_{-\pi/2}^{\pi/2} = \pi(1-e^{-25/2})$$




**26.** $$\int_0^3\int_0^{\sqrt{9-x^2}}(9-x^2-y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^3(9-r^2)r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^3(9r-r^3)\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^3 d\theta = \frac{81}{4}\int_0^{\pi/2}d\theta = \frac{81\pi}{8}$$

**28.** $V = 4\int_0^{\pi/2}\int_0^1 (r^2 + 3)r\,dr\,d\theta = 4\int_0^{\pi/2}\left(\frac{r^4}{4} + \frac{3r^2}{2}\right)_0^1 d\theta$

$$= 4\int_0^{\pi/4} \frac{7}{4}\,d\theta$$

$$= 7\left(\frac{\pi}{4}\right) = \frac{7\pi}{4}$$

**30.** $\displaystyle\int_R\int \ln(x^2 + y^2)\,dA = \int_0^{2\pi}\int_1^2 (\ln r^2)r\,dr\,d\theta$

$$= 2\int_0^{2\pi}\int_1^2 r\ln r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{r^2}{4}(-1 + 2\ln r)\right]_1^2 d\theta$$

$$= 2\int_0^{2\pi}\left(\ln 4 - \frac{3}{4}\right)d\theta$$

$$= 4\pi\left(\ln 4 - \frac{3}{4}\right)$$

**32.** $V = \displaystyle\int_0^{2\pi}\int_1^4 \sqrt{16 - r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_1^4 d\theta = \int_0^{2\pi} 5\sqrt{15}\,d\theta = 10\sqrt{15}\pi$

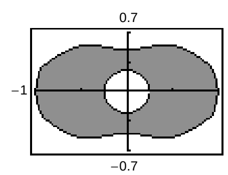
**34.** $x^2 + y^2 + z^2 = a^2 \Longrightarrow z = \sqrt{a^2 - (x^2 + y^2)} = \sqrt{a^2 - r^2}$

$V = 8\displaystyle\int_0^{\pi/2}\int_0^a \sqrt{a^2 - r^2}\,r\,dr\,d\theta$ (8 times the volume in the first octant)

$$= 8\int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta$$

$$= 8\int_0^{\pi/2}\frac{a^3}{3}\,d\theta = \left[\frac{8a^3}{3}\theta\right]_0^{\pi/2} = \frac{4\pi a^3}{3}$$

**36.** $\dfrac{-9}{4(x^2 + y^2 + 9)} \le z \le \dfrac{9}{4(x^2 + y^2 + 9)};\quad \dfrac{1}{4} \le r \le \dfrac{1}{2}(1 + \cos^2\theta)$



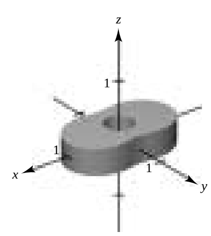

(a) $\dfrac{-9}{4r^2 + 36} \le z \le \dfrac{9}{4r^2 + 36}$

(b) Perimeter $= \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$

$$r = \frac{1}{2}(1 + \cos^2\theta) = \frac{1}{2} + \frac{1}{2}\cos^2\theta$$

$$\frac{dr}{d\theta} = -\cos\theta\sin\theta$$

Perimeter $= 2\displaystyle\int_0^\pi \sqrt{\frac{1}{4}(1 + \cos^2\theta)^2 + \cos^2\theta\sin^2\theta}\,d\theta \approx 5.21$




(c) $V = 2\displaystyle\int_0^{2\pi}\int_{1/4}^{1/2(1+\cos^2\theta)} \frac{9}{4r^2 + 36}\,r\,dr\,d\theta \approx 0.8000$

**38.** $A = \displaystyle\int_0^{2\pi}\int_2^4 r\,dr\,d\theta = \int_0^{2\pi} 6\,d\theta = 12\pi$

**40.** $\displaystyle\int_0^{2\pi}\int_0^{2+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(2 + \sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4 + 4\sin\theta + \sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(4 + 4\sin\theta + \frac{1 - \cos 2\theta}{2}\right)d\theta$

$$= \frac{1}{2}\left[4\theta - 4\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{1}{2}[8\pi - 4 + \pi + 4] = \frac{9\pi}{2}$$

**42.** $8\int_0^{\pi/4}\int_0^{3\cos 2\theta} r\,dr\,d\theta = 4\int_0^{\pi/4} 9\cos^2 2\theta\,d\theta = 18\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta = 18\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{9\pi}{2}$

**44.** See Theorem 13.3.

**46.** (a) Horizontal or polar representative elements

(b) Polar representative element

(c) Vertical or polar

**48.** (a) The volume of the subregion determined by the point $(5, \pi/16, 7)$ is base $\times$ height $= (5 \cdot 10 \cdot \pi/8)(7)$. Adding up the 20 volumes, ending with $(45 \cdot 10 \cdot \pi/8)(12)$, you obtain

$$\begin{aligned} V &\approx 10 \cdot \frac{\pi}{8}[5(7+9+9+5) + 15(8+10+11+8) + 25(10+14+15+11) \\ &\qquad + 35(12+15+18+16) + 45(9+10+14+12)] \\ &= \frac{5\pi}{4}[150 + 555 + 1250 + 2135 + 2025] \approx \frac{5\pi}{4}[6115] \approx 24013.5 \text{ ft}^3 \end{aligned}$$

(b) $(56)(24013.5) = 1{,}344{,}759$ pounds

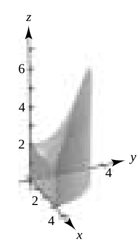
(c) $(7.48)(24103.5) \approx 179{,}621$ gallons

**50.** $\int_0^{\pi/4}\int_0^4 5e^{\sqrt{r\theta}}\, r\,dr\,d\theta \approx 87.130$

**52.** Volume $=$ base $\times$ height

$\approx \frac{9}{4}\pi \times 3 \approx 21$

Answer (a)




**54.** True

**56.** (a) Let $u = \sqrt{2}x$, then $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2/2}\frac{1}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\left(\sqrt{2\pi}\right) = \sqrt{\pi}.$

(b) Let $u = 2x$, then $\int_{-\infty}^{\infty} e^{-4x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2}\frac{1}{2}\,du = \frac{1}{2}\sqrt{\pi}.$

**58.** $\int_0^{\infty}\int_0^{\infty} ke^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2}\int_0^{\infty} ke^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{k}{2}e^{-r^2}\right]_0^{\infty} d\theta = \int_0^{\pi/2}\frac{k}{2}\,d\theta = \frac{k\pi}{4}$

For $f(x, y)$ to be a probability density function,

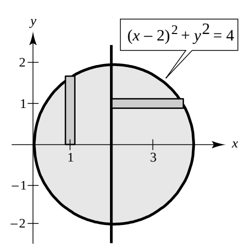
$$\frac{k\pi}{4} = 1$$

$$k = \frac{4}{\pi}.$$

**60.** (a) $4\int_0^2\int_2^{2+\sqrt{4-y^2}} f\,dx\,dy$

(b) $4\int_0^2\int_0^{\sqrt{4-(x-2)^2}} f\,dy\,dx$

(c) $2\int_0^{\pi/2}\int_0^{4\cos\theta} f\,r\,dr\,d\theta$



# Section 13.4 Center of Mass and Moments of Inertia

**2.** $m = \int_0^3 \int_0^{9-x^2} xy\, dy\, dx = \int_0^3 \left[\frac{xy^2}{2}\right]_0^{9-x^2} dx$

$$= \int_0^3 \frac{x(9-x^2)^2}{2}\, dx$$

$$= \left[-\frac{1}{4}\frac{(9-x^2)^3}{3}\right]_0^3$$

$$= -\frac{1}{12}(0 - 9^3) = \frac{243}{4}$$

**4.** $m = \int_0^3 \int_3^{3+\sqrt{9-x^2}} xy\, dy\, dx = \int_0^3 \left[x\frac{y^2}{2}\right]_3^{3+\sqrt{9-x^2}} dx$

$$= \int_0^3 \frac{x}{2}\left(\left(3+\sqrt{9-x^2}\right) - 9\right) dx$$

$$= \frac{1}{2}\int_0^3 \left[6x\sqrt{9-x^2} + 9x - x^3\right] dx$$

$$= \frac{1}{2}\left[-2(9-x^2)^{3/2} + \frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3$$

$$= \frac{1}{2}\left[\frac{81}{2} - \frac{81}{4} + 54\right] = \frac{297}{8}$$

**6.** (a) $m = \int_0^a \int_0^b kxy\, dy\, dx = \frac{ka^2b^2}{4}$

$M_x = \int_0^a \int_0^b kxy^2\, dy\, dx = \frac{ka^2b^3}{6}$

$M_y = \int_0^a \int_0^b kx^2y\, dy\, dx = \frac{ka^3b^2}{6}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^3b^2/6}{ka^2b^2/4} = \frac{2}{3}a$

$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^3/6}{ka^2b^2/4} = \frac{2}{3}b$

(b) $m = \int_0^a \int_0^b k(x^2+y^2)\, dy\, dx = \frac{kab}{3}(a^2+b^2)$

$M_x = \int_0^a \int_0^b k(x^2y + y^3)\, dy\, dx = \frac{kab^2}{12}(2a^2+3b^2)$

$M_y = \int_0^a \int_0^b k(x^3 + xy^2)\, dy\, dx = \frac{ka^2b}{12}(3a^2+2b^2)$

$\bar{x} = \frac{M_y}{m} = \frac{(ka^2b/12)(3a^2+2b^2)}{(kab/3)(a^2+b^2)} = \frac{a}{4}\left(\frac{3a^2+2b^2}{a^2+b^2}\right)$

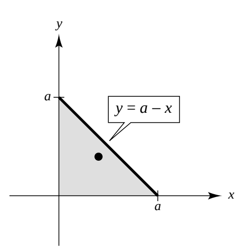
$\bar{y} = \frac{M_x}{m} = \frac{(kab^2/12)(2a^2+3b^2)}{(kab/3)(a^2+b^2)} = \frac{b}{4}\left(\frac{2a^2+3b^2}{a^2+b^2}\right)$

**8.** (a) $m = \frac{a^2k}{2}$

$M_x = \int_0^a \int_0^{a-x} ky\, dy\, dx = \frac{ka^3}{6}$

$M_y = M_x$ by symmetry

$\bar{x} = \bar{y} = \frac{M_x}{m} = \frac{ka^3/6}{ka^2/2} = \frac{a}{3}$




**—CONTINUED—**

**8. —CONTINUED—**

(b) $$m = \int_0^a \int_0^{a-x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^{a-x} dx = \int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3\right] dx = \frac{a^4}{6}$$

$$M_y = \int_0^a \int_0^{a-x} (x^3 + xy^2)\, dy\, dx$$

$$= \int_0^a \left(ax^3 - x^4 + \frac{1}{3}a^3x - a^2x^2 + ax^3 - \frac{1}{3}x^4\right) dx = \frac{1}{3}\int_0^a (a^3x - 3a^2x^2 + 6ax^3 - 4x^4)\, dx = \frac{a^5}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{a^5/15}{a^4/6} = \frac{2a}{5}$$

$\bar{y} = \dfrac{2a}{5}$ by symmetry

**10.** The $x$-coordinate changes by $h$ units horizontally and $k$ units vertically. This is not true for variable densities.

**12.** (a) $$m = \int_0^a \int_0^{\sqrt{a^2-x^2}} k\, dy\, dx = \frac{k\pi a^2}{4}$$

$$M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} kx\, dy\, dx$$

$$= k\int_0^a x\sqrt{a^2 - x^2}\, dx$$

$$= \left[-\frac{k}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{ka^3}{3}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3/3}{k\pi a^2/4} = \frac{4a}{3\pi}$$

$\bar{y} = \dfrac{4a}{3\pi}$ by symmetry

(b) $$m = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^a kr^3\, dr\, d\theta = \frac{ka^4\pi}{8}$$

$$M_x = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)\, y\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^a kr^4 \sin\theta\, dr\, d\theta = \frac{ka^5}{5}$$

$M_y = M_x$ by symmetry

$$\bar{x} = \bar{y} = \frac{M_y}{m} = \frac{ka^5}{5} \cdot \frac{8}{ka^4\pi} = \frac{8a}{5\pi}$$

**14.** $$m = \int_0^2 \int_0^{x^3} kx\, dy\, dx = \int_0^2 kx^4\, dx = \frac{32k}{5}$$

$$M_x = \int_0^2 \int_0^{x^3} kxy\, dy\, dx = 16k$$

$$M_y = \int_0^2 \int_0^{x^3} kx^2\, dy\, dx = \frac{32k}{3}$$

$$\bar{x} = \frac{M_y}{m} = \frac{32k}{3} \cdot \frac{5}{32k} = \frac{5}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{16k}{32k}(5) = \frac{5}{2}$$

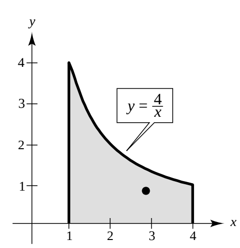
**16.** $$m = \int_1^4 \int_0^{4/x} kx^2\, dy\, dx = 30k$$

$$M_x = \int_1^4 \int_0^{4/x} kx^2\, y\, dy\, dx = 24k$$

$$M_y = \int_1^4 \int_0^{4/x} kx^3\, dy\, dx = 84k$$

$$\bar{x} = \frac{M_y}{m} = \frac{84k}{30k} = \frac{14}{5}$$

$$\bar{y} = \frac{M_y}{m} = \frac{24k}{30k} = \frac{4}{5}$$

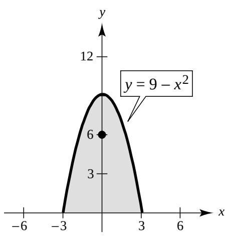


**18.** $\bar{x} = 0$ by symmetry

$$m = \int_{-3}^{3}\int_{0}^{9-x^2} ky^2\,dy\,dx = \frac{23{,}328k}{35}$$

$$M_x = \int_{-3}^{3}\int_{0}^{9-x^2} ky^3\,dy\,dx = \frac{139{,}968k}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{139{,}968k}{35}\cdot\frac{35}{23{,}328k} = 6$$



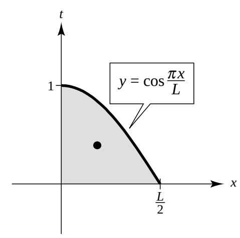

**20.** $$m = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} k\,dy\,dx = \frac{kL}{\pi}$$

$$M_x = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} ky\,dy\,dx = \frac{kL}{8}$$

$$M_y = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} kx\,dy\,dx = \frac{L^2(\pi - 2)k}{2\pi^2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{L^2(\pi-2)k}{2\pi^2}\cdot\frac{\pi}{kL} = \frac{L(\pi-2)}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kL}{8}\cdot\frac{\pi}{kL} = \frac{\pi}{8}$$



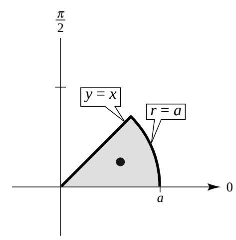

**22.** $$m = \iint_R k\sqrt{x^2+y^2}\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^2\,dr\,d\theta = \frac{ka^3\pi}{12}$$

$$M_x = \iint_R k\sqrt{x^2+y^2}\,y\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^3 \sin\theta\,d\theta = \frac{ka^4\left(2-\sqrt{2}\right)}{8}$$

$$M_y = \iint_R k\sqrt{x^2+y^2}\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^3 \cos\theta\,d\theta = \frac{ka^4\sqrt{2}}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^4\sqrt{2}}{8}\cdot\frac{12}{ka^3\pi} = \frac{3\sqrt{2}a}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^4\left(2-\sqrt{2}\right)}{8}\cdot\frac{12}{ka^3\pi} = \frac{3\left(2-\sqrt{2}\right)a}{2\pi}$$



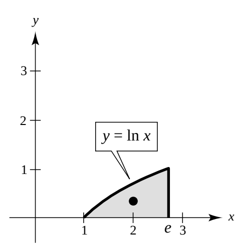

**24.** $$m = \int_{1}^{e}\int_{0}^{\ln x} \frac{k}{x}\,dy\,dx = \frac{k}{2}$$

$$M_x = \int_{1}^{e}\int_{0}^{\ln x} \frac{k}{x}y\,dy\,dx = \frac{k}{6}$$

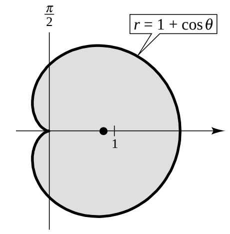
$$M_y = \int_{1}^{e}\int_{0}^{\ln x} \frac{k}{x}x\,dy\,dx = k$$

$$\bar{x} = \frac{M_y}{m} = \frac{k}{1}\cdot\frac{2}{k} = 2$$

$$\bar{y} = \frac{M_x}{m} = \frac{k}{6}\cdot\frac{2}{k} = \frac{1}{3}$$



**26.** $\bar{y} = 0$ by symmetry




$$m = \iint_R k\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} kr\,dr\,d\theta = \frac{3\pi k}{2}$$

$$M_y = \iint_R kx\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} kr^2\cos\theta\,dr\,d\theta$$

$$= \frac{k}{3}\int_0^{2\pi} \cos\theta(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= \frac{k}{3}\int_0^{2\pi} \left[\cos\theta + \frac{3}{2}(1 + \cos^2\theta) + 3\cos\theta(1 - \sin^2\theta) + \frac{1}{4}(1 + \cos 2\theta)^2\,d\theta\right.$$

$$= \frac{5k\pi}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5k\pi}{4}\cdot\frac{2}{3k\pi} = \frac{5}{6}$$

**28.** $m = \displaystyle\int_0^b\int_0^{h-(hx/b)} dy\,dx = \frac{bh}{2}$

$$I_x = \int_0^b\int_0^{h-(hx/b)} y^2\,dy\,dx = \frac{bh^3}{12}$$

$$I_y = \int_0^b\int_0^{h-(hx/b)} x^2\,dy\,dx = \frac{b^3h}{12}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h/12}{bh/2}} = \frac{b}{\sqrt{6}} = \frac{\sqrt{6}}{6}b$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3/12}{bh/2}} = \frac{h}{\sqrt{6}} = \frac{\sqrt{6}}{6}h$$

**30.** $m = \dfrac{\pi a^2}{2}$

$$I_x = \iint_R y^2\,dA = \int_0^{\pi}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{a^4\pi}{8}$$

$$I_y = \iint_R x^2\,dA = \int_0^{\pi}\int_0^a r^3\cos^2\theta\,dr\,d\theta = \frac{a^4\pi}{8}$$

$$I_0 = I_x + I_x = \frac{a^4\pi}{8} + \frac{a^4\pi}{8} = \frac{a^4\pi}{4}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{8}\cdot\frac{2}{\pi a^2}} = \frac{a}{2}$$

**32.** $m = \pi ab$

$$I_x = 4\int_0^a\int_0^{(b/a)\sqrt{a^2-x^2}} y^2\,dy\,dx$$

$$= 4\int_0^a \frac{b^3}{3a^3}(a^2 - x^2)^{3/2}\,dx = \frac{4b^3}{3a^3}\int_0^a \left[a^2\sqrt{a^2 - x^2} - x^2\sqrt{a^2 - x^2}\right]dx$$

$$= \frac{4b^3}{3a^3}\left[\frac{a^2}{2}\left(x\sqrt{a^2 - x^2} + a^2\arcsin\frac{x}{a}\right) - \frac{1}{8}\left[x(2x^2 - a^2)\sqrt{a^2 - x^2} + a^4\arcsin\frac{x}{a}\right]\right]_0^a = \frac{ab^3\pi}{4}$$

$$I_y = 4\int_0^b\int_0^{(a/b)\sqrt{b^2-y^2}} x^2\,dx\,dy = \frac{a^3b\pi}{4}$$

$$I_0 = I_y + I_x = \frac{a^3b\pi}{4} + \frac{ab^3\pi}{4} = \frac{ab\pi}{4}(a^2 + b^2)$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{a^3b\pi}{4}\cdot\frac{1}{\pi ab}} = \frac{a}{2}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{ab^3\pi}{4}\cdot\frac{1}{\pi ab}} = \frac{b}{2}$$

**34.** $\rho = ky$

$$m = 2k\int_0^a\int_0^{\sqrt{a^2-x^2}} y\,dy\,dx$$

$$= k\int_0^a (a^2 - x^2)\,dx = \frac{2ka^3}{3}$$

$$I_x = k\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} y^3\,dy\,dx = \frac{4ka^5}{15}$$

$$I_y = k\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx = \frac{2ka^5}{15}$$

$$I_0 = I_x + I_y = \frac{2ka^5}{5}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2ka^5/15}{2ka^3/3}} = \sqrt{\frac{a^2}{5}} = \frac{a}{\sqrt{5}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{4ka^5/15}{2ka^3/3}} = \sqrt{\frac{2a^2}{5}} = \frac{2a}{\sqrt{10}}$$

**36.** $\rho = kxy$

$$m = k\int_0^1\int_{x^2}^x xy\,dy\,dx = \frac{k}{2}\int_0^1 (x^3 - x^5)\,dx = \frac{k}{24}$$

$$I_x = k\int_0^1\int_{x^2}^x xy^3\,dy\,dx = \frac{k}{4}\int_0^1 (x^5 - x^9)\,dx = \frac{k}{60}$$

$$I_y = k\int_0^1\int_{x^2}^x x^3y\,dy\,dx = \frac{k}{2}\int_0^1 (x^5 - x^7)\,dx = \frac{k}{48}$$

$$I_0 = I_x + I_y = \frac{9k}{240} = \frac{3k}{80}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k/48}{k/24}} = \frac{1}{\sqrt{2}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}\sqrt{\frac{k/60}{k/24}} = \sqrt{\frac{2}{5}}$$

**38.** $\rho = x^2 + y^2$

$$m = \int_0^1\int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,dy\,dx = \frac{6}{35}$$

$$I_x = \int_0^1\int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,y^2\,dy\,dx = \frac{158}{2079}$$

$$I_y = \int_0^1\int_{x^2}^{\sqrt{x}} (x^2 + y^2)x^2\,dy\,dx = \frac{158}{2079}$$

$$I_0 = I_x + I_y = \frac{316}{2079}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{158}{2079}\cdot\frac{35}{6}} = \sqrt{\frac{395}{891}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \bar{\bar{x}} = \sqrt{\frac{395}{891}}$$

**40.** $\rho = ky$

$$m = 2\int_0^2\int_{x^3}^{4x} ky\,dy\,dx = \frac{512k}{21}$$

$$I_x = 2\int_0^2\int_{x^3}^{4x} ky^3\,dy\,dx = \frac{32{,}768k}{65}$$

$$I_y = 2\int_0^2\int_{x^3}^{4x} kx^2\,y\,dy\,dx = \frac{2048k}{45}$$

$$I_0 = I_x + I_y = \frac{321{,}536k}{585}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2048k}{45}\cdot\frac{21}{512k}} = \sqrt{\frac{28}{15}} = \frac{2\sqrt{105}}{15}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32{,}768k}{65}\cdot\frac{21}{512k}} = \frac{8\sqrt{1365}}{65}$$

**42.** $I = \displaystyle\int_0^4\int_0^2 k(x-6)^2\,dy\,dx = \int_0^4 2k(x-6)^2\,dx = \left[\frac{2k}{3}(x-6)^3\right]_0^4 = \frac{416k}{3}$

**44.** $I = \displaystyle\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} ky(y-a)^2\,dy\,dx$

$$= \int_{-a}^a k\left[\frac{y^4}{4} - \frac{2ay^3}{3} + \frac{a^2y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^a k\left[\frac{1}{4}(a^4 - 2a^2x^2 + x^4) - \frac{2a}{3}\left(a^2\sqrt{a^2-x^2} - x^2\sqrt{a^2-x^2}\right) + \frac{a^2}{2}(a^2 - x^2)\right] dx$$

$$= k\left[\frac{1}{4}\left(a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right) - \frac{2a}{3}\left[\frac{a^2}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right)\right.\right.$$

$$\left.\left. - \frac{1}{8}\left(x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4\arcsin\frac{x}{a}\right)\right] + \frac{a^2}{2}\left(a^2x - \frac{x^3}{3}\right)\right]_{-a}^a$$

$$= 2k\left[\frac{1}{4}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) - \frac{2a}{3}\left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right) + \frac{a^2}{2}\left(a^3 - \frac{a^3}{3}\right)\right] = 2k\left(\frac{7a^5}{15} - \frac{a^5\pi}{8}\right) = ka^5\left(\frac{56 - 15\pi}{60}\right)$$

**46.** $I = \int_{-2}^{2}\int_{0}^{4-x^2} k(y-2)^2\,dy\,dx = \int_{-2}^{2}\left[\frac{k}{3}(y-1)^3\right]_0^{4-x^2} dx = \int_{-2}^{2}\frac{k}{3}[(2-x^2)+8]\,dx$

$$= \frac{k}{3}\int_{-2}^{2}(16 - 12x^2 + 6x^4 - x^6)\,dx = \left[\frac{k}{3}\left(16x - 4x^3 + \frac{6}{5}x^5 - \frac{1}{7}x^7\right)\right]_{-2}^{2}$$

$$= \frac{2k}{3}\left(32 - 32 + \frac{192}{5} - \frac{128}{7}\right) = \frac{1408k}{105}$$

**48.** $\rho(x, y) = k|2 - x|$.

$(\bar{x}, \bar{y})$ will be the same.

**50.** $\rho(x, y) = k(4 - x)(4 - y)$. Both $\bar{x}$ and $\bar{y}$ will decrease

**52.** $I_x = \int\int_R y^2\,\rho(x, y)\,dA$ Moment of inertia about $x$-axis.

$I_y = \int\int_R x^2\,\rho(x, y)\,dA$ Moment of inertia about $y$-axis.

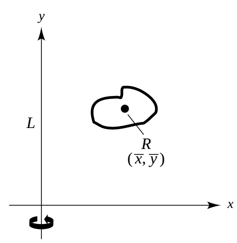
**54.** Orient the $xy$-coordinate system so that $L$ is along the $y$-axis and $R$ is in the first quadrant. Then the volume of the solid is

$$V = \int\int_R 2\pi x\,dA$$

$$= 2\pi \int\int_R x\,dA$$

$$= 2\pi\left(\frac{\int\int_R x\,dA}{\int\int_R dA}\right)\int\int_R dA$$

$$= 2\pi\bar{x}A.$$




By our positioning, $\bar{x} = r$. Therefore, $V = 2\pi rA$.

**56.** $\bar{y} = \frac{a}{2}, A = ab, h = L - \frac{a}{2}$

$$I_{\bar{y}} = \int_0^b\int_0^a \left(y - \frac{a}{2}\right)^2 dy\,dx = \frac{a^3b}{12}$$

$$y_a = \frac{a}{2} - \frac{a^3b/12}{[L - (a/2)]ab} = \frac{a(3L - 2a)}{3(2L - a)}$$

**58.** $\bar{y} = 0, A = \pi a^2, h = L$

$$I_{\bar{y}} = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^a r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\frac{a^4}{4}\sin^2\theta\,d\theta$$

$$= \frac{a^4\pi}{4}$$

$$y_a = -\frac{(a^4\pi/4)}{L\pi a^2} = -\frac{a^2}{4L}$$

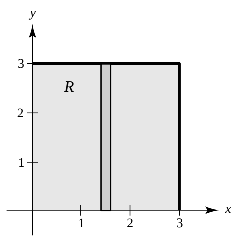
# Section 13.5 Surface Area

**2.** $f(x, y) = 15 + 2x - 3y$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_0^3 \int_0^3 \sqrt{14}\, dy\, dx = \int_0^3 3\sqrt{14}\, dx = 9\sqrt{14}$$




**4.** $f(x, y) = 10 + 2x - 3y$

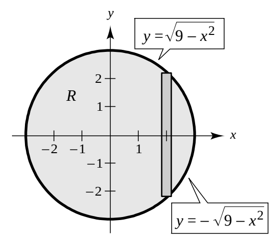
$R = \{(x, y): x^2 + y^2 \le 9\}$

$f_x = 2,\ f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{14}\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{14}\, r\, dr\, d\theta = 9\sqrt{14}\pi$$




**6.** $f(x, y) = y^2$

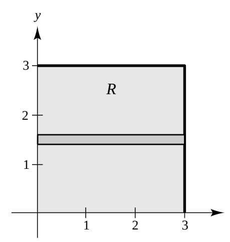
$R =$ square with vertices (0, 0), (3, 0), (0, 3), (3, 3)

$f_x = 0,\ f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4y^2}\, dx\, dy = \int_0^3 3\sqrt{1 + 4y^2}\, dy$$

$$= \left[\frac{3}{4}\left(2y\sqrt{1 + 4y^2} + \ln\left|2y + \sqrt{1 + 4y^2}\right|\right)\right]_0^3 = \frac{3}{4}\left(6\sqrt{37} + \ln\left|6 + \sqrt{37}\right|\right)$$




**8.** $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

$f_x = 0,\ f_y = y^{1/2}$

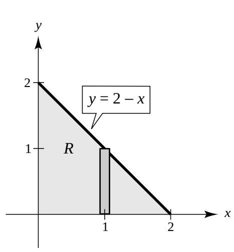
$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + y}$

$$S = \int_0^2 \int_0^{2-y} \sqrt{1 + y}\, dx\, dy = \int_0^2 \sqrt{1 + y}(2 - y)\, dy$$

$$= \left[2(1 + y)^{3/2} - \frac{2}{5}(1 + y)^{5/2}\right]_0^2$$

$$= 2 \cdot 3^{3/2} - \frac{2}{5} \cdot 3^{5/2} - 2 + \frac{2}{5}$$

$$= \frac{12}{5}\sqrt{3} - \frac{8}{5}$$




**10.** $f(x, y) = 9 + x^2 - y^2$

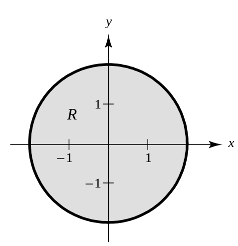
$f_x = 2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 1)\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$



**12.** $f(x, y) = xy$

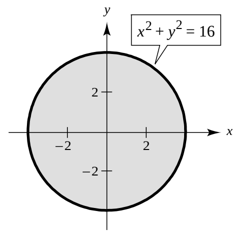
$R = \{(x, y): x^2 + y^2 \le 16\}$

$f_x = y, \; f_y = x$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 + x^2}$

$$S = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + y^2 + x^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^4 \sqrt{1 + r^2}\, r\, dr\, d\theta = \frac{2\pi}{3}\left(17\sqrt{17} - 1\right)$$




**14.** See Exercise 13.

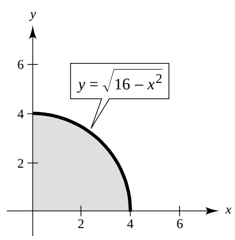
$$S = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx = \int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a^2$$

**16.** $z = 16 - x^2 - y^2$

$\sqrt{1 + f_y{}^2 + f_y{}^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^4\int_0^{\sqrt{16-x^2}} \sqrt{1 + 4(x^2 + y^2)}\, dy\, dx$$

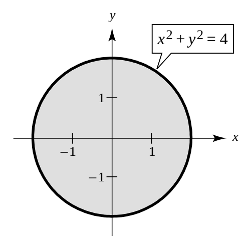
$$= \int_0^{\pi/2}\int_0^4 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{\pi}{24}\left(65\sqrt{65} - 1\right)$$




**18.** $z = 2\sqrt{x^2 + y^2}$

$$\sqrt{1 + f_x{}^2 + f_y{}^2} = \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} = \sqrt{5}$$

$$S = \int_0^{2\pi}\int_0^2 \sqrt{5}r\, dr\, d\theta = 4\pi\sqrt{5}$$

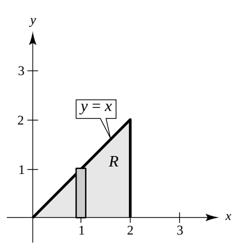



**20.** $f(x, y) = 2x + y^2$

$R =$ triangle with vertices $(0, 0), (2, 0), (2, 2)$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4y^2}$

$$S = \int_0^2\int_0^x \sqrt{5 + 4y^2}\, dy\, dx = \frac{1}{12}\left(21\sqrt{21} - 5\sqrt{5}\right)$$




**22.** $f(x, y) = x^2 + y^2$

$R = \{(x, y): \; 0 \le f(x, y) \le 16\}$

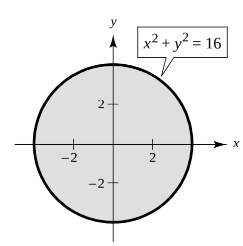
$0 \le x^2 + y^2 \le 16$

$f_x = 2x, \; f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^4 \sqrt{1 + 4r^2}\, dr\, d\theta = \frac{\left(65\sqrt{65} - 1\right)\pi}{6}$$



**24.** $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

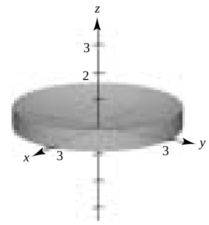
$f_x = x^{1/2} - \sin x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\sqrt{x} - \sin x\right)^2}$

$$S = \int_0^1 \int_0^1 \sqrt{1 + \left(\sqrt{x} - \sin x\right)^2}\, dy\, dx \approx 1.02185$$

**26.** Surface area $\approx (9\pi)$

Matches (c)




**28.** $f(x, y) = \frac{2}{5}y^{5/2}$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$f_x = 0,\ f_y = y^{3/2}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^3}$

$$S = \int_0^1 \int_0^1 \sqrt{1 + y^3}\, dx\, dy$$

$$= \int_0^1 \sqrt{1 + y^3}\, dy \approx 1.1114$$

**30.** $f(x, y) = x^2 - 3xy - y^2$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

$f_x = 2x - 3y,\ f_y = -3x - 2y = -(3x + 2y)$

$$\sqrt{1 + (f_x) + (f_y)^2} = \sqrt{1 + (2x - 3y)^2 + (3x + 2y)^2}$$

$$= \sqrt{1 + 13(x^2 + y^2)}$$

$$S = \int_0^4 \int_0^x \sqrt{1 + 13(x^2 + y^2)}\, dy\, dx$$

**32.** $f(x, y) = \cos(x^2 + y^2)$

$R = \left\{(x, y): x^2 + y^2 \le \dfrac{\pi}{2}\right\}$

$f_x = -2x \sin(x^2 + y^2),\ f_y = -2y \sin(x^2 + y^2)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2)} = \sqrt{1 + 4[\sin^2(x^2 + y^2)](x^2 + y^2)}$$

$$S = \int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} \int_{-\sqrt{(\pi/2) - x^2}}^{\sqrt{(\pi/2) - x^2}} \sqrt{1 + 4(x^2 + y^2)\sin^2(x^2 + y^2)}\, dy\, dx$$

**34.** $f(x, y) = e^{-x} \sin y$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

$f_x = -e^{-x} \sin y,\ f_y = e^{-x} \cos y$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y}$$

$$= \sqrt{1 + e^{-2x}}$$

$$S = \int_0^4 \int_0^x \sqrt{1 + e^{-2x}}\, dy\, dx$$

**36.** (a) Yes. For example, let $R$ be the square given by

$0 \le x \le 1, 0 \le y \le 1,$

and $S$ the square parallel to $R$ given by

$0 \le x \le 1, 0 \le y \le 1, z = 1.$

(b) Yes. Let $R$ be the region in part (a) and $S$ the surface given by $f(x, y) = xy$.

(c) No.

**38.** $f(x, y) = k\sqrt{x^2 + y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{k^2x^2}{x^2 + y^2} + \frac{k^2y^2}{x^2 + y^2}} = \sqrt{k^2 + 1}$$

$$S = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_R \sqrt{k^2 + 1}\, dA = \sqrt{k^2 + 1} \iint_R dA = A\sqrt{k^2 + 1} = \pi r^2 \sqrt{k^2 + 1}$$

**40.** (a) $z = \frac{-1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

(b) $V \approx 2(50)\int_0^{15}\left(-\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25\right) dy$

$= 100(266.25) = 26{,}625$ cubic feet

(c) $f(x, y) = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

$f_x = 0, f_y = -\frac{1}{25}y^2 + \frac{8}{25}y - \frac{16}{15}$

$S = 2\int_0^{50}\int_0^{15}\sqrt{1 + f_y^2 + f_x^2}\, dy\, dx \approx 3087.58$ sq ft

(d) Arc length $\approx 30.8758$

Surface area of roof $\approx 2(50)(30.8758) = 3087.58$ sq ft

**42.** False. The surface area will remain the same for any vertical translation.

## Section 13.6 Triple Integrals and Applications

**2.** $$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} x^2y^2z^2\, dx\, dy\, dz = \frac{1}{3}\int_{-1}^{1}\int_{-1}^{1}\left[x^3y^2z^2\right]_{-1}^{1} dy\, dz$$
$$= \frac{2}{3}\int_{-1}^{1}\int_{-1}^{1} y^2z^2\, dy\, dz = \frac{2}{9}\int_{-1}^{1}\left[y^3z^2\right]_{-1}^{1} dz = \frac{4}{9}\int_{-1}^{1} z^2\, dz = \left[\frac{4}{27}z^3\right]_{-1}^{1} = \frac{8}{27}$$

**4.** $$\int_0^9\int_0^{y/3}\int_0^{\sqrt{y^2-9x^2}} z\, dz\, dx\, dy = \frac{1}{2}\int_0^9\int_0^{y/3}(y^2 - 9x^2)\, dx\, dy$$
$$= \frac{1}{2}\int_0^9\left[xy^2 - 3x^3\right]_0^{y/3} dy = \frac{2}{18}\int_0^9 y^3\, dy = \left[\frac{1}{36}y^4\right]_0^9 = \frac{729}{4}$$

**6.** $$\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln z\, dy\, dz\, dx = \int_1^4\int_1^{e^2}\left[(\ln z)y\right]_0^{1/xz} dz\, dx = \int_1^4\int_1^{e^2}\frac{\ln z}{xz}\, dz\, dx$$
$$= \int_1^4\left[\frac{1}{x}\frac{(\ln z)^2}{2}\right]_1^{e^2} dx = \int_1^4\frac{2}{x}\, dx = \left[2\ln|x|\right]_1^4 = 2\ln 4$$

**8.** $$\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y}\sin y\, dz\, dx\, dy = \int_0^{\pi/2}\int_0^{y/2}\frac{\sin y}{y}\, dx\, dy = \frac{1}{2}\int_0^{\pi/2}\sin y\, dy = \left[-\frac{1}{2}\cos y\right]_0^{\pi/2} = \frac{1}{2}$$

**10.** $$\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} y\, dz\, dy\, dx = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}(4y - 2x^2y - 2y^3)\, dy\, dx = \frac{16\sqrt{2}}{15}$$

**12.** $$\int_0^3\int_0^{2-(2y/3)}\int_0^{6-2y-3z} ze^{-x^2y^2}\, dx\, dz\, dy = \int_0^6\int_0^{(6-x)/2}\int_0^{(6-x-2y)/3} ze^{-x^2y^2}\, dz\, dy\, dx$$
$$= \int_0^6\int_0^{3-(x/2)}\frac{1}{2}\left(\frac{6 - x - 2y}{3}\right)^2 e^{-x^2y^2}\, dy\, dx \approx 2.118$$

**14.** $$\int_0^3\int_0^{2x}\int_0^{9-x^2} dz\, dy\, dx$$

**16.** $z = \frac{1}{2}(x^2 + y^2) \implies 2z = x^2 + y^2$

$x^2 + y^2 + z^2 = 2z + z^2 = 80 \implies z^2 + 2z - 80 = 0 \implies (z - 8)(z + 10) = 0 \implies z = 8 \implies x^2 + y^2 = 2z = 16$

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{1/2(x^2+y^2)}^{\sqrt{80-x^2-y^2}} dz\,dy\,dx$$

**18.** $\displaystyle\int_0^1\int_0^1\int_0^{xy} dz\,dy\,dx = \int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$

**20.** $$4\int_0^6\int_0^{\sqrt{36-x^2}}\int_0^{36-x^2-y^2} dz\,dy\,dx = 4\int_0^6\int_0^{\sqrt{36-x^2}}(36 - x^2 - y^2)dy\,dx = 4\int_0^6\left[36y - x^2y - \frac{y^3}{3}\right]_0^{\sqrt{36-x^2}} dx$$

$$= 4\int_0^6\left[36\sqrt{36 - x^2} - x^2\sqrt{36 - x^2} - \frac{1}{3}(36 - x^2)^{3/2}\right]dx$$

$$= 4\left[9x\sqrt{36 - x^2} + 324 \arcsin\left(\frac{x}{6}\right) + \frac{1}{6}x(36 - x^2)^{3/2}\right]_0^6 = 4(162\pi) = 648\pi$$

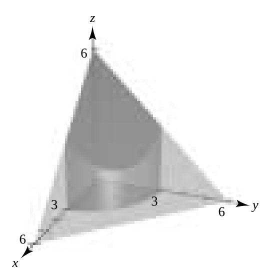
**22.** $$\int_0^2\int_0^{2-x}\int_0^{9-x^2} dz\,dy\,dx = \int_0^2\int_0^{2-x}(9 - x^2)\,dy\,dx = \int_0^2 (9 - x^2)(2 - x)\,dx$$

$$= \int_0^2 (18 - 9x - 2x^2 + x^3)\,dx = \left[18x - \frac{9}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^2 = \frac{50}{3}$$

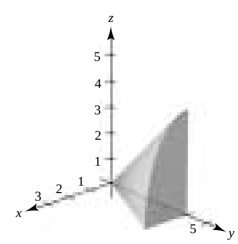
**24.** Top plane: $x + y + z = 6$

Side cylinder: $x^2 + y^2 = 9$

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{6-x-y} dz\,dx\,dy$$

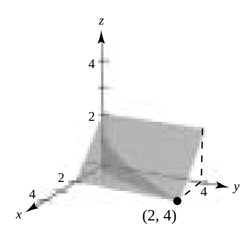



**26.** Elliptic cone: $4x^2 + z^2 = y^2$




$$\int_0^4\int_z^4\int_0^{\sqrt{y^2-z^2}/2} dx\,dy\,dz$$

**28.** $Q = \{(x, y, z): 0 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - x\}$




$$\iiint_Q xyz\,dV = \int_0^2\int_{x^2}^4\int_0^{2-x} xyz\,dz\,dy\,dx$$

$$= \int_0^4\int_0^{\sqrt{y}}\int_0^{2-x} xyz\,dz\,dx\,dy$$

$$= \int_0^2\int_0^{2-x}\int_{x^2}^4 xyz\,dy\,dz\,dx$$

$$= \int_0^2\int_0^{2-z}\int_{x^2}^4 xyz\,dy\,dx\,dz$$

$$= \int_0^2\int_0^{(2-z)^2}\int_0^{\sqrt{y}} xyz\,dx\,dy\,dz + \int_0^2\int_{(2-z)^2}^4\int_0^{2-z} xyz\,dx\,dy\,dz$$

$$= \int_0^4\int_0^{2-\sqrt{y}}\int_0^{\sqrt{y}} xyz\,dx\,dz\,dy + \int_0^4\int_{2-\sqrt{y}}^2\int_0^{2-z} dx\,dz\,dy\left(= \frac{104}{21}\right)$$

**30.** $Q = \{(x, y, z):\ 0 \le x \le 1, y \le 1 - x^2, 0 \le z \le 6\}$

$$\iiint_Q xyz\, dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^6 xyz\, dz\, dy\, dx$$
$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^6 xyz\, dz\, dx\, dy$$
$$= \int_0^1 \int_0^6 \int_0^{\sqrt{1-y^2}} xyz\, dx\, dz\, dy$$
$$= \int_0^6 \int_0^1 \int_0^{\sqrt{1-y^2}} xyz\, dx\, dy\, dz$$
$$= \int_0^1 \int_0^6 \int_0^{\sqrt{1-x^2}} xyz\, dy\, dz\, dx$$
$$= \int_0^6 \int_0^1 \int_0^{\sqrt{1-x^2}} xyz\, dy\, dx\, dz$$


**32.** $$m = k\int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y\, dz\, dy\, dz = \frac{125}{8}k$$
$$M_{xz} = k\int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y^2\, dz\, dy\, dx = \frac{125}{4}k$$
$$\bar{y} = \frac{M_{xz}}{m} = 2$$

**34.** $$m = k\int_0^b \int_0^{a[1-(y/b)]} \int_0^{c[1-(y/b)-(x/a)]} dz\, dx\, dy = \frac{kabc}{6}$$
$$M_{xz} = k\int_0^b \int_0^{a[1-(y/b)]} \int_0^{c[1-(y/b)-(x/a)]} y\, dz\, dx\, dy = \frac{kab^2c}{24}$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c/24}{kabc/6} = \frac{b}{4}$$

**36.** $$m = k\int_0^a \int_0^b \int_0^c z\, dz\, dy\, dx = \frac{kabc^2}{2}$$
$$M_{xy} = k\int_0^a \int_0^b \int_0^c z^2\, dz\, dy\, dx = \frac{kabc^3}{3}$$
$$M_{yz} = k\int_0^a \int_0^b \int_0^c xz\, dz\, dy\, dx = \frac{ka^2bc^2}{4}$$
$$M_{xz} = k\int_0^a \int_0^b \int_0^c yz\, dz\, dy\, dx = \frac{kab^2c^2}{4}$$
$$\bar{x} = \frac{M_{yz}}{m} = \frac{ka^2bc^2/4}{kabc^2/2} = \frac{a}{2}$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c^2/4}{kabc^2/2} = \frac{b}{2}$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{kabc^3/3}{kabc^2/2} = \frac{2c}{3}$$

**38.** $\bar{z}$ will be greater than $8/5$, whereas $\bar{x}$ and $\bar{y}$ will be unchanged.

**40.** $\bar{x}, \bar{y}$ and $\bar{z}$ will all be greater than their original values.

**42.** $$m = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y dz\,dy\,dx$$

$$= k\int_0^2 (4 - x^2)\,dx = \frac{16k}{3}$$

$$M_{yz} = k\int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_0^y x\,dz\,dy\,dx = 0$$

$$M_{xz} = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y y\,dz\,dy\,dx = 2k\pi$$

$$M_{xy} = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y z\,dz\,dy\,dx = k\pi$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{0}{16k/3} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{2k\pi}{16k/3} = \frac{3\pi}{8}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi}{16k/3} = \frac{3\pi}{16}$$

**44.** $\bar{x} = 0$

$$m = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} dz\,dy\,dx = 2k\int_0^2\int_0^1 \frac{1}{y^2+1}\,dy\,dx = 2k\left(\frac{\pi}{4}\right)\int_0^2 dx = k\pi$$

$$M_{xz} = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} y\,dz\,dy\,dx = 2k\int_0^2\int_0^1 \frac{y}{y^2+1}\,dy\,dx = k\int_0^2 (\ln 2)\,dx = k\ln 4$$

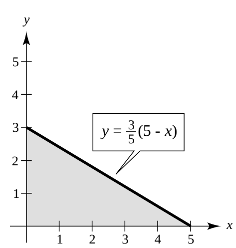
$$M_{xy} = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} z\,dz\,dy\,dx$$

$$= k\int_0^2\int_0^1 \frac{1}{(y^2+1)^2}\,dy\,dx = k\int_0^2\left[\frac{y}{2(y^2+1)} + \frac{1}{2}\arctan y\right]_0^1 dx = k\left(\frac{1}{4} + \frac{\pi}{8}\right)\int_0^2 dx = k\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{k\ln 4}{k\pi} = \frac{\ln 4}{\pi}$$

$$\bar{z} = \frac{M_{xy}}{m} = k\left(\frac{1}{2} + \frac{\pi}{4}\right)\Big/k\pi = \frac{2+\pi}{4\pi}$$

**46.** $f(x, y) = \dfrac{1}{15}(60 - 12x - 20y)$




$$m = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} dz\,dy\,dx = 10k$$

$$M_{yz} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} x\,dz\,dy\,dx = \frac{25k}{2}$$

$$M_{xz} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{-(1/15)(60-12x-20y)} y\,dz\,dy\,dx = \frac{15k}{2}$$

$$M_{xy} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} z\,dz\,dy\,dx = 10k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{25k/2}{10k} = \frac{5}{4}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{15k/2}{10k} = \frac{3}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{10k}{10k} = 1$$

**48.** (a) $I_{xy} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} z^2\,dz\,dy\,dx = \dfrac{ka^5}{12}$

$I_{xz} = I_{yz} = \dfrac{ka^5}{12}$ by symmetry

$I_x = I_y = I_z = \dfrac{ka^5}{12} + \dfrac{ka^5}{12} = \dfrac{ka^5}{6}$

(b) $I_{xy} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} z^2(x^2 + y^2)\,dz\,dy\,dx = \dfrac{a^3k}{12}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(x^2 + y^2)\,dy\,dx = \dfrac{a^7k}{72}$

$I_{xz} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} y^2(x^2 + y^2)\,dz\,dy\,dx = ka\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(x^2y^2 + y^4)\,dy\,dx = \dfrac{7ka^7}{360}$

$I_{yz} = I_{xz}$ by symmetry

$I_x = I_{xy} + I_{xz} = \dfrac{a^7k}{30}$

$I_y = I_{xy} + I_{yz} = \dfrac{a^7k}{30}$

$I_z = I_{yz} + I_{xz} = \dfrac{7ka^7}{180}$

**50.** (a) $I_{xy} = k\int_0^4\int_0^2\int_0^{4-y^2} z^3\,dz\,dy\,dx = k\int_0^4\int_0^2 \dfrac{1}{4}(4 - y^2)^4\,dy\,dx$

$= \dfrac{k}{4}\int_0^4\int_0^2 (256 - 256y^2 + 96y^4 - 16y^6 + y^8)\,dy\,dx$

$= \dfrac{k}{4}\int_0^4 \left[256y - \dfrac{256y^3}{3} + \dfrac{96y^5}{5} - \dfrac{16y^7}{7} + \dfrac{y^9}{9}\right]_0^2 dx = k\int_0^4 \dfrac{16{,}384}{945}\,dx = \dfrac{65{,}536k}{315}$

$I_{xz} = k\int_0^4\int_0^2\int_0^{4-y^2} y^2z\,dz\,dy\,dx = k\int_0^4\int_0^2 \dfrac{1}{2}y^2(4 - y^2)^2\,dy\,dx$

$= k\int_0^4\int_0^2 \dfrac{1}{2}(16y^2 - 8y^4 + y^6)\,dy\,dx = \dfrac{k}{2}\int_0^4\left[\dfrac{16y^3}{3} - \dfrac{8y^5}{5} + \dfrac{y^7}{7}\right]_0^2 dx = \dfrac{k}{2}\int_0^4 \dfrac{1024}{105}\,dx = \dfrac{2048k}{105}$

$I_{yz} = k\int_0^4\int_0^2\int_0^{4-y^2} x^2z\,dz\,dy\,dx = k\int_0^4\int_0^2 \dfrac{1}{2}x^2(4 - y^2)^2\,dy\,dx$

$= k\int_0^4\int_0^2 \dfrac{1}{2}x^2(16 - 8y^2 + y^4)\,dy\,dx = \dfrac{k}{2}\int_0^4 \left[x^2\left(16y - \dfrac{8y^3}{3} + \dfrac{y^5}{5}\right)\right]_0^2 dx = \dfrac{k}{2}\int_0^4 \dfrac{256}{15}x^2\,dx = \dfrac{8192k}{45}$

$I_x = I_{xz} + I_{xy} = \dfrac{2048k}{9},\ I_y = I_{yz} + I_{xy} = \dfrac{8192k}{21},\ I_z = I_{yz} + I_{xz} = \dfrac{63{,}488k}{315}$

**—CONTINUED—**

**50. —CONTINUED—**

(b) $I_{xy} = \int_0^4\int_0^2\int_0^{4-y^2} z^2(4-z)\,dz\,dy\,dx$

$$= k\int_0^4\int_0^2\int_0^{4-y^2} 4z^2\,dz\,dy\,dx - k\int_0^4\int_0^2\int_0^{4-y^2} z^3\,dz\,dy\,dx = \frac{32{,}768k}{105} - \frac{65{,}536k}{315} = \frac{32{,}768k}{315}$$

$$I_{xz} = \int_0^4\int_0^2\int_0^{4-y^2} y^2(4-z)\,dz\,dy\,dx$$

$$= k\int_0^4\int_0^2\int_0^{4-y^2} 4y^2\,dz\,dy\,dx - k\int_0^4\int_0^2\int_0^{4-y^2} y^2 z\,dz\,dy\,dx = \frac{1024k}{15} - \frac{2048k}{105} = \frac{1024k}{21}$$

$$I_{yz} = k\int_0^4\int_0^2\int_0^{4-y^2} x^2(4-z)\,dz\,dy\,dx$$

$$= k\int_0^4\int_0^2\int_0^{4-y^2} 4x^2\,dz\,dy\,dx - k\int_0^4\int_0^2\int_0^{4-y^2} x^2 z\,dz\,dy\,dx = \frac{4096k}{9} - \frac{8192k}{45} = \frac{4096k}{15}$$

$$I_x = I_{xz} + I_{xy} = \frac{48{,}128k}{315},\ I_y = I_{yz} + I_{xy} = \frac{118{,}784k}{315},\ I_z = I_{xz} + I_{yz} = \frac{11{,}264k}{35}$$

**52.** $I_{xy} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} z^2\,dz\,dy\,dx = \frac{b^3}{12}\int_{-c/2}^{c/2}\int_{-a/2}^{a/2} dy\,dx = \frac{1}{12}b^2(abc) = \frac{1}{12}mb^2$

$$I_{xz} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} y^2\,dz\,dy\,dx = b\int_{-c/2}^{c/2}\int_{-a/2}^{a/2} y^2\,dy\,dx = \frac{ba^3}{12}\int_{-c/2}^{c/2} dx = \frac{ba^3c}{12} = \frac{1}{12}a^2(abc) = \frac{1}{12}ma^2$$

$$I_{yz} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} x^2\,dz\,dy\,dx = ab\int_{-c/2}^{c/2} x^2\,dx = \frac{abc^3}{12} = \frac{1}{12}c^2(abc) = \frac{1}{12}mc^2$$

$$I_x = I_{xy} + I_{xz} = \frac{1}{12}m(a^2+b^2)$$

$$I_y = I_{xy} + I_{yz} = \frac{1}{12}m(b^2+c^2)$$

$$I_z = I_{xz} + I_{yz} = \frac{1}{12}m(a^2+c^2)$$

**54.** $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{4-x^2-y^2} kx^2(x^2+y^2)\,dz\,dy\,dx$

**56.** 6

**58.** Because the density increases as you move away from the axis of symmetry, the moment of intertia will increase.

## Section 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

**2.** $\int_0^{\pi/4}\int_0^2\int_0^{2-r} rz\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_0^2\left[\frac{rz^2}{2}\right]_0^{2-r} dr\,d\theta$

$$= \frac{1}{2}\int_0^{\pi/4}\int_0^2 (4r - 4r^2 + r^3)dr\,d\theta = \frac{1}{2}\int_0^{\pi/4}\left[2r^2 - \frac{4r^3}{3} + \frac{r^4}{4}\right]_0^2 d\theta = \frac{2}{3}\int_0^{\pi/4} d\theta = \frac{\pi}{6}$$

**4.** $\int_0^{\pi/2}\int_0^{\pi}\int_0^2 e^{-\rho^3}\rho^2\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi}\left[-\frac{1}{3}e^{-\rho^3}\right]_0^2 d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi}\frac{1}{3}(1-e^{-8})\,d\theta\,d\phi = \frac{\pi^2}{6}(1-e^{-8})$

**6.** $\displaystyle\int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\cos\theta}\rho^2\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi = \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4}\cos^3\theta\sin\phi\cos\phi\,d\theta\,d\phi$

$$= \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4}\sin\phi\cos\phi[\cos\theta(1-\sin^2\theta)]d\theta\,d\phi$$

$$= \frac{1}{3}\int_0^{\pi/4}\sin\phi\cos\phi\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/4}d\phi$$

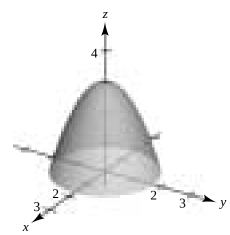
$$= \frac{5\sqrt{2}}{36}\int_0^{\pi/4}\sin\phi\cos\phi\,d\phi = \left[\frac{5\sqrt{2}}{36}\frac{\sin^2\phi}{2}\right]_0^{\pi/4} = \frac{5\sqrt{2}}{144}$$

**8.** $\displaystyle\int_0^{\pi/2}\int_0^{\pi}\int_0^{\sin\theta}(2\cos\phi)\rho^2\,d\rho\,d\theta\,d\phi = \frac{8}{9}$

**10.** $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{3-r^2}r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}}r(3-r^2)dr\,d\theta$

$$= \int_0^{2\pi}\left(\frac{3r^2}{2}-\frac{r^4}{4}\right)\Bigg]_0^{\sqrt{3}}d\theta$$

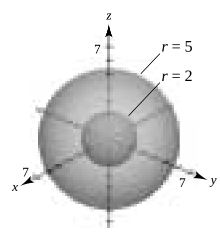
$$= \int_0^{2\pi}\frac{9}{4}d\theta = \frac{9\pi}{2}$$




**12.** $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_2^{5}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{117}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$

$$= \frac{117}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_0^{\pi}d\theta$$

$$= \frac{468\pi}{3}$$




**14.** (a) $\displaystyle\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{16-r^2}}r^2\,dz\,dr\,d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$

(b) $\displaystyle\int_0^{\pi/2}\int_0^{\pi/6}\int_0^{4}\rho^3\sin^2\phi\,d\rho\,d\phi\,d\theta + \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_4^{2\csc\phi}\rho^3\sin^2\phi\,d\rho\,d\phi\,d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$

**16.** (a) $\displaystyle\int_0^{\pi/2}\int_0^{1}\int_0^{\sqrt{1-r^2}}r\sqrt{r^2+z^2}\,dz\,dr\,d\theta = \frac{\pi}{8}$

(b) $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{\pi}{8}$

**18.** $\displaystyle V = \frac{2}{3}\pi(4)^3 + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}}\int_0^{r}r\,dz\,dr\,d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^{4}\int_0^{\sqrt{16-r^2}}r\,dz\,dr\,d\theta\right]$

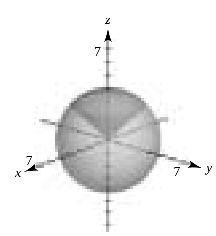
(Volume of lower hemisphere) + 4(Volume in the first octant)

$$V = \frac{128\pi}{3} + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}}r^2\,dr\,d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^{4}r\sqrt{16-r^2}\,dr\,d\theta\right]$$

$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \int_0^{\pi/2}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_{2\sqrt{2}}^{4}d\theta\right]$$

$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \frac{8\sqrt{2}\pi}{3}\right]$$

$$= \frac{128\pi}{3} + \frac{64\sqrt{2}\pi}{3} = \frac{64\pi}{3}\left(2+\sqrt{2}\right)$$



**20.** $$V = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left(r\sqrt{4-r^2} - r^2\right) dr\,d\theta$$
$$= \int_0^{2\pi}\left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$$
$$= \frac{8\pi}{3}\left(2-\sqrt{2}\right)$$

**22.** $$\int_0^{\pi/2}\int_0^2\int_0^{12e^{-r^2}} k\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 12ke^{-r^2}\,r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left[-6ke^{-r^2}\right]_0^2$$
$$= \int_0^{\pi/2}(-6ke^{-4} + 6k)\,d\theta$$
$$= 3k\pi(1-e^{-4})$$

**24.** $\bar{x} = \bar{y} = 0$ by symmetry

$m = \frac{1}{3}\pi r_0^2hk$ from Exercise 23

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta$$
$$= \frac{2kh^2}{r_0^2}\int_0^{\pi/2}\int_0^{r_0}(r_0^2r - 2r_0r^2 + r^3)dr\,d\theta$$
$$= \frac{2kh^2}{r_0^2}\left(\frac{r_0^4}{12}\right)\left(\frac{\pi}{2}\right) = \frac{kr_0^2\,h^2\pi}{12}$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{kr_0^2h^2\pi}{12}\left(\frac{3}{\pi r_0^2hk}\right) = \frac{h}{4}$$

**26.** $\rho = kz$

$\bar{x} = \bar{y} = 0$ by symmetry

$$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta$$
$$= \frac{1}{12}k\pi r_0^2h^2$$
$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} z^2r\,dz\,dr\,d\theta$$
$$= \frac{1}{30}k\pi r_0^2h^3$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r_0^2h^3/30}{k\pi r_0^2h^2/12} = \frac{2h}{5}$$

**28.** $$I_z = \iiint_Q (x^2+y^2)\rho(x,y,z)\,dV$$
$$= 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^4\,dz\,dr\,d\theta$$
$$= 4kh\int_0^{\pi/2}\int_0^{r_0}\frac{r_0-r}{r_0}r^4\,dr\,d\theta$$
$$= 4kh\int_0^{\pi/2}\left[\frac{r^5}{5} - \frac{r^6}{6r_0}\right]_0^{r_0} d\theta$$
$$= 4kh\int_0^{\pi/2}\left[\frac{r_0^5}{5} - \frac{r_0^5}{6}\right] d\theta$$
$$= 4kh\int_0^{\pi/2}\frac{1}{30}r_0^5\,d\theta$$
$$= 4kh\frac{1}{30}r_0^5\frac{\pi}{2}$$
$$= \frac{1}{15}r_0^5\pi kh$$

**30.** $m = k\pi a^2h$

$$I_z = 2k\int_0^{\pi/2}\int_0^{2a\sin\theta}\int_0^h r^3\,dz\,dr\,d\theta$$
$$= \frac{3}{2}k\pi a^4h$$
$$= \frac{3}{2}ma^2$$

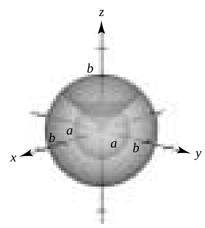
**32.** $V = 8\int_0^{\pi/4}\int_0^{\pi/2}\int_a^b \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$ (includes upper and lower cones)

$$= \frac{8}{3}(b^3 - a^3)\int_0^{\pi/4}\int_0^{\pi/2} \sin\phi \, d\theta \, d\phi$$

$$= \frac{4\pi}{3}(b^3 - a^3)\int_0^{\pi/4} \sin\phi \, d\phi$$

$$= \left[\frac{4\pi}{3}(b^3 - a^3)(-\cos\phi)\right]_0^{\pi/4}$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right)\frac{4\pi}{3}(b^3 - a^3) = \frac{2\pi}{3}\left(2 - \sqrt{2}\right)(b^3 - a^3)$$




**34.** $m = 8k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3 \sin^2\phi \, d\rho \, d\theta \, d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2} \sin^2\phi \, d\theta \, d\phi$$

$$= k\pi a^4\int_0^{\pi/2} \sin^2\phi \, d\phi$$

$$= \left[k\pi a^4\left(\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\right)\right]_0^{\pi/2}$$

$$= k\pi a^4 \frac{\pi}{4} = \frac{1}{4}k\pi^2 a^4$$

**36.** $\bar{x} = \bar{y} = 0$ by symmetry

$$m = k\left(\frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3\right) = \frac{2}{3}k\pi(R^3 - r^3)$$

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{1}{2}k(R^4 - r^4)\int_0^{\pi/2}\int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi$$

$$= \frac{1}{4}k\pi(R^4 - r^4)\int_0^{\pi/2} \sin 2\phi \, d\phi$$

$$= \left[-\frac{1}{8}k\pi(R^4 - r^4)\cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi(R^4 - r^4)$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi(R^4 - r^4)/4}{2k\pi(R^3 - r^3)/3} = \frac{3(R^4 - r^4)}{8(R^3 - r^3)}$$

**38.** $I_z = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

$$= \frac{4k}{5}(R^5 - r^5)\int_0^{\pi/2}\int_0^{\pi/2} \sin^3\phi \, d\theta \, d\phi$$

$$= \frac{2k\pi}{5}(R^5 - r^5)\int_0^{\pi/2} \sin\phi(1 - \cos^2\phi) \, d\phi$$

$$= \left[\frac{2k\pi}{5}(R^5 - r^5)\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\right]_0^{\pi/2}$$

$$= \frac{4k\pi}{15}(R^5 - r^5)$$

**40.** $x = \rho\sin\phi\cos\theta$ $\qquad \rho^2 = x^2 + y^2 + z^2$

$y = \rho\sin\phi\sin\theta$ $\qquad \tan\theta = \dfrac{y}{x}$

$z = \rho\cos\phi$ $\qquad \cos\phi = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

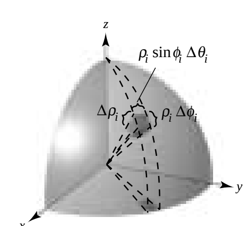
**42.** $\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$

**44.** (a) You are integrating over a cylindrical wedge. (b) You are integrating over a spherical block.

**46.** The volume of this spherical block can be determined as follows. One side is length $\Delta\rho$. Another side is $\rho\Delta\phi$. Finally, the third side is given by the length of an arc of angle $\Delta\theta$ in a circle of radius $\rho\sin\phi$. Thus:

$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\Delta\theta\rho\sin\phi)$$

$$= \rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta$$



# Section 13.8 Change of Variables: Jacobians

**2.** $x = au + bv$

$y = cu + dv$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = ad - cb$$

**4.** $x = uv - 2u$

$y = uv$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (v-2)u - vu = -2u$$

**6.** $x = u + a$

$y = v + a$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (0)(0) = 1$$

**8.** $x = \dfrac{u}{v}$

$y = u + v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{v}\right)(1) - (1)\left(-\frac{u}{v^2}\right) = \frac{1}{v} + \frac{u}{v^2} = \frac{u+v}{v^2}$$

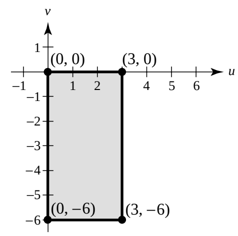
**10.** $x = \dfrac{1}{3}(4u - v)$

$y = \dfrac{1}{3}(u - v)$

$u = x - y$

$v = x - 4y$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 0)$ | $(0, 0)$ |
| $(4, 1)$ | $(3, 0)$ |
| $(2, 2)$ | $(0, -6)$ |
| $(6, 3)$ | $(3, -6)$ |




**12.** $x = \dfrac{1}{2}(u + v), \quad u = x - y$

$y = -\dfrac{1}{2}(u - v), \quad v = x + y$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{2}\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

| $(x, y)$ | $(u, v)$ |
|---|---|
| $(0, 1)$ | $(-1, 1)$ |
| $(2, 1)$ | $(1, 3)$ |
| $(1, 2)$ | $(-1, 3)$ |
| $(1, 0)$ | $(1, 1)$ |

$$\int_R\int 60xy\,dA$$

$$= \int_{-1}^{1}\int_{1}^{3} 60\left(\frac{1}{2}(u+v)\right)\left(-\frac{1}{2}(u-v)\right)\left(\frac{1}{2}\right) dv\,du$$

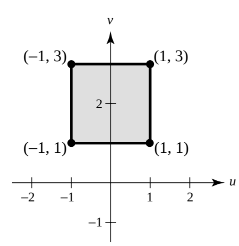
$$= \int_{-1}^{1}\int_{1}^{3} -\frac{15}{2}(v^2 - u^2)\,dv\,du$$

$$= \int_{-1}^{1}\left[-\frac{15}{2}\left(\frac{v^3}{3} - u^2v\right)\right]_1^3 du$$

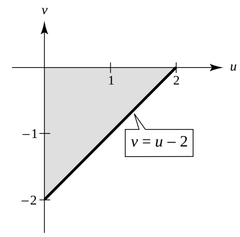
$$= \int_{-1}^{1}\frac{15}{2}\left(2u^2 - \frac{26}{3}\right) du$$

$$= \left[\frac{15}{2}\left(\frac{2}{3}u^3 - \frac{26}{3}u\right)\right]_{-1}^{1}$$

$$= 15\left(\frac{2}{3} - \frac{26}{3}\right) = -120$$



**14.** $x = \frac{1}{2}(u + v)$




$y = \frac{1}{2}(u - v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$

$$\iint_R 4(x + y)e^{x-y}\,dA = \int_0^2 \int_{u-2}^0 4ue^v\left(\frac{1}{2}\right) dv\,du$$

$$= \int_0^2 2u(1 - e^{u-2})\,du = 2\left[\frac{u^2}{2} - ue^{u-2} + e^{u-2}\right]_0^2 = 2(1 - e^{-2})$$

**16.** $x = \frac{u}{v}$

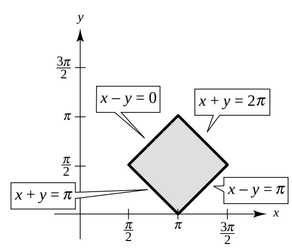
$y = v$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{v}$

$$\iint_R y \sin xy\,dA = \int_1^4 \int_1^4 v(\sin u)\frac{1}{v}\,dv\,du = \int_1^4 3 \sin u\,du = \Big[-3 \cos u\Big]_1^4 = 3(\cos 1 - \cos 4) \approx 3.5818$$

**18.** $u = x + y = \pi, \qquad v = x - y = 0$

$u = x + y = 2\pi, \qquad v = x - y = \pi$




$x = \frac{1}{2}(u + v), \qquad y = \frac{1}{2}(u - v)$

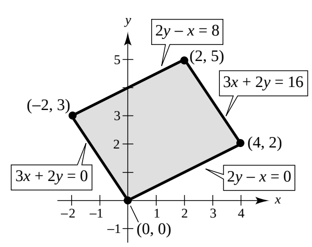
$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$

$$\iint_R (x + y)^2 \sin^2 (x - y)\,dA = \int_0^\pi \int_\pi^{2\pi} u^2 \sin^2 v\left(\frac{1}{2}\right) du\,dv$$

$$= \int_0^\pi \left[\frac{1}{2}\left(\frac{u^3}{3}\right)\frac{1 - \cos 2v}{2}\right]_\pi^{2\pi} dv = \left[\frac{7\pi^3}{12}\left(v - \frac{1}{2}\sin 2v\right)\right]_0^\pi = \frac{7\pi^4}{12}$$

**20.** $u = 3x + 2y = 0, \qquad v = 2y - x = 0$

$u = 3x + 2y = 16, \qquad v = 2y - x = 8$




$x = \frac{1}{4}(u - v), \qquad y = \frac{1}{8}(u + 3v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{4}\left(\frac{3}{8}\right) - \frac{1}{8}\left(-\frac{1}{4}\right) = \frac{1}{8}$

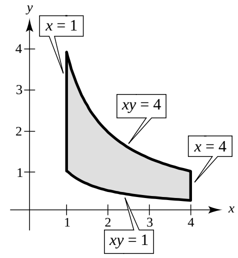
$$\iint_R (3x + 2y)(2y - x)^{3/2}\,dA = \int_0^8 \int_0^{16} uv^{3/2}\left(\frac{1}{8}\right) du\,dv$$

$$= \int_0^8 16v^{3/2}\,dv = \left(\frac{2}{5}\right)16v^{5/2}\Big]_0^8 = \frac{4096}{5}\sqrt{2}$$

**22.** $u = x = 1,\qquad v = xy = 1$

$u = x = 4,\qquad v = xy = 4$

$x = u,\qquad y = \dfrac{v}{u}$




$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{u}$$

$$\iint_R \frac{xy}{1 + x^2y^2}\,dA = \int_1^4\int_1^4 \frac{v}{1 + v^2}\left(\frac{1}{u}\right) dv\,du$$

$$= \int_1^4 \left[\frac{1}{2}\ln(1 + v^2)\right]_1^4 \frac{1}{u}\,du = \left[\frac{1}{2}[\ln 17 - \ln 2]\ln u\right]_1^4 = \frac{1}{2}\left(\ln\frac{17}{2}\right)(\ln 4)$$

**24.** (a) $f(x, y) = 16 - x^2 - y^2$

$R\colon \dfrac{x^2}{16} + \dfrac{y^2}{9} \le 1$

$V = \displaystyle\iint_R f(x, y)\,dA$

Let $x = 4u$ and $y = 3v$.

$$\iint_R (16 - x^2 - y^2)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (16 - 16u^2 - 9v^2)\,12dv\,du \qquad (\text{Let } u = r\cos\theta,\ v = r\sin\theta.)$$

$$= \int_0^{2\pi}\int_0^1 (16 - 16r^2\cos^2\theta - 9r^2\sin^2\theta)\,12r\,dr\,d\theta$$

$$= 12\int_0^{2\pi}\left[8r^2 - 4r^4\cos^2\theta - \frac{9}{4}r^4\sin^2\theta\right]_0^1 d\theta = 12\int_0^{2\pi}\left[8 - 4\cos^2\theta - \frac{9}{4}\sin^2\theta\right] d\theta$$

$$= 12\int_0^{2\pi}\left[8 - 4\left(\frac{1 + \cos 2\theta}{2}\right) - \frac{9}{4}\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta = 12\int_0^{2\pi}\left[\frac{39}{8} - \frac{7}{8}\cos 2\theta\right] d\theta$$

$$= 12\left[\frac{39}{8}\theta - \frac{7}{16}\sin 2\theta\right]_0^{2\pi} = 12\left[\frac{39\pi}{4}\right] = 117\pi$$

(b) $f(x, y) = A\cos\left[\dfrac{\pi}{2}\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}\right]$

$R\colon \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$

Let $x = au$ and $y = bv$.

$$\iint_R f(x, y)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} A\cos\left[\frac{\pi}{2}\sqrt{u^2 + v^2}\right] ab\,dv\,du$$

Let $u = r\cos\theta$, $v = r\sin\theta$.

$$Aab\int_0^{2\pi}\int_0^1 \cos\left[\frac{\pi}{2}r\right] r\,dr\,d\theta = Aab\left[\frac{2r}{\pi}\sin\left(\frac{\pi r}{2}\right) + \frac{4}{\pi^2}\cos\left(\frac{\pi r}{2}\right)\right]_0^1 (2\pi)$$

$$= 2\pi Aab\left[\left(\frac{2}{\pi} + 0\right) - \left(0 + \frac{4}{\pi^2}\right)\right] = \frac{4(\pi - 2)Aab}{\pi}$$

**26.** See Theorem 13.5.

**28.** $x = 4u - v,\ y = 4v - w,\ z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

**30.** $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1[r\cos^2\theta + r\sin^2\theta] = r$$

## Review Exercises for Chapter 13

**2.** $\displaystyle\int_y^{2y} (x^2 + y^2)\,dx = \left[\frac{x^3}{3} + xy^2\right]_y^{2y} = \frac{10y^3}{3}$

**4.** $\displaystyle\int_0^2\int_{x^2}^{2x} (x^2 + 2y)\,dy\,dx = \int_0^2 \left[x^2y + y^2\right]_{x^2}^{2x} dx = \int_0^2 (4x^2 + 2x^3 - 2x^4)\,dx = \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5\right]_0^2 = \frac{88}{15}$

**6.** $\displaystyle\int_0^{\sqrt{3}}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\,dy = 2\int_0^{\sqrt{3}} \sqrt{4 - y^2}\,dy = \left[y\sqrt{4 - y^2} + 4\arcsin\frac{y}{2}\right]_0^{\sqrt{3}} = \sqrt{3} + \frac{4\pi}{3}$

**8.** $\displaystyle\int_0^2\int_0^x dy\,dx + \int_2^3\int_0^{6-2x} dy\,dx = \int_0^2\int_y^{(6-y)/2} dx\,dy$

$$A = \int_0^2\int_y^{(6-y)/2} dx\,dy$$

$$= \frac{1}{2}\int_0^2 (6 - 3y)\,dy = \left[\frac{1}{2}\left(6y - \frac{3}{2}y^2\right)\right]_0^2 = 3$$

**10.** $\displaystyle\int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_{-1}^0\int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dy\,dx + \int_0^8\int_{3-\sqrt{9-y}}^{1+\sqrt{1+y}} dx\,dy + \int_8^9\int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx\,dy$

$$A = \int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_0^4 (8x - 2x^2)\,dx = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = \frac{64}{3}$$

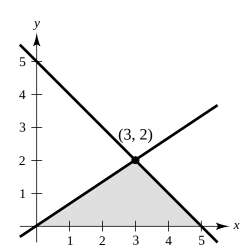
**12.** $\displaystyle A = \int_0^2\int_0^{y^2+1} dx\,dy = \int_0^1\int_0^2 dy\,dx + \int_1^5\int_{\sqrt{x-1}}^2 dy\,dx = \frac{14}{3}$

**14.** $\displaystyle A = \int_0^3\int_{-y}^{2y-y^2} dx\,dy = \int_{-3}^0\int_{-x}^{1+\sqrt{1-x}} dy\,dx + \int_0^1\int_{1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy\,dx = \frac{9}{2}$

**16.** Both integrations are over the common region $R$ shown in the figure. Analytically,

$$\int_0^2\int_{3y/2}^{5-y} e^{x+y}\,dx\,dy = \frac{2}{5} + \frac{8}{5}e^5$$

$$\int_0^3\int_0^{2x/3} e^{x+y}\,dy\,dx + \int_3^5\int_0^{5-x} e^{x+y}\,dy\,dx = \left(\frac{3}{5}e^5 - e^3 + \frac{2}{5}\right) + (e^5 + e^3) = \frac{8}{5}e^5 + \frac{2}{5}$$



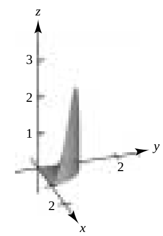
**18.** $V = \int_0^3 \int_0^x (x + y)\, dy\, dx$

$= \int_0^3 \left[xy + \frac{1}{2}y^2\right]_0^x dx$

$= \frac{3}{2}\int_0^3 x^2\, dx$

$= \left[\frac{1}{2}x^3\right]_0^3 = \frac{27}{2}$

**20.** Matches (c)



**22.** $\int_0^1 \int_0^x kxy\, dy\, dx = \int_0^1 \left[\frac{kxy^2}{2}\right]_0^x dx$

$= \int_0^1 \frac{kx^3}{2}\, dx$

$= \left[\frac{kx^4}{8}\right]_0^1 = \frac{k}{8}$

Since $k/8 = 1$, we have $k = 8$.

$P = \int_0^{0.5} \int_0^{0.25} 8xy\, dy\, dx = 0.03125$

**24.** False, $\int_0^1 \int_0^1 x\, dy\, dx \neq \int_1^2 \int_1^2 x\, dy\, dx$

**26.** True, $\int_0^1 \int_0^1 \frac{1}{1 + x^2 + y^2}\, dx\, dy < \int_0^1 \int_0^1 \frac{1}{1 + x^2}\, dx\, dy = \frac{\pi}{4}$

**28.** $\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\, dx\, dy = \int_0^{\pi/2} \int_0^4 r^3\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^4 d\theta = \int_0^{\pi/2} 64\, d\theta = 32\pi$

**30.** $V = 8\int_0^{\pi/2} \int_b^R \sqrt{R^2 - r^2}\, r\, dr\, d\theta$

$= -\frac{8}{3}\int_0^{\pi/2} \left[(R^2 - r^2)^{3/2}\right]_b^R d\theta$

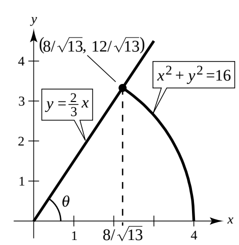
$= \frac{8}{3}(R^2 - b^2)^{3/2} \int_0^{\pi/2} d\theta$

$= \frac{4}{3}\pi(R^2 - b^2)^{3/2}$

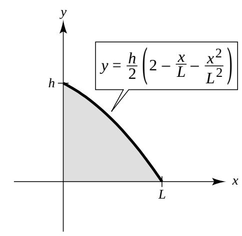
**32.** $\tan\theta = \frac{12\sqrt{13}}{8\sqrt{13}} = \frac{3}{2} \implies \theta \approx 0.9828$

The polar region is given by $0 \le r \le 4$ and $0 \le \theta \le 0.9828$. Hence,

$$\int_0^{\arctan(3/2)} \int_0^4 (r\cos\theta)(r\sin\theta) r\, dr\, d\theta = \frac{288}{13}$$



**34.** $m = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} dy\,dx = \frac{kh}{2}\int_0^L\left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)dx = \frac{7khL}{12}$




$$M_x = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} y\,dy\,dx$$

$$= \frac{kh^2}{8}\int_0^L\left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)^2 dx$$

$$= \frac{kh^2}{8}\int_0^L\left[4 - \frac{4x}{L} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right]dx$$

$$= \frac{kh^2}{8}\left[4x - \frac{2x^2}{L} - \frac{x^3}{L^2} + \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^L = \frac{kh^2}{8}\cdot\frac{17L}{10} = \frac{17kh^2L}{80}$$

$$M_y = k\int_0^L\int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} x\,dy\,dx$$

$$= \frac{kh}{2}\int_0^L\left(2x - \frac{x^2}{L} - \frac{x^3}{L^2}\right)dx = \frac{kh}{2}\left[x^2 - \frac{x^3}{3L} - \frac{x^4}{4L^2}\right]_0^L = \frac{kh}{2}\cdot\frac{5L^2}{12} = \frac{5khL^2}{24}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5khL^2}{24}\cdot\frac{12}{7khL} = \frac{5L}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{17kh^2L}{80}\cdot\frac{12}{7khL} = \frac{51h}{140}$$

**36.** $I_x = \iint_R y^2\,\rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} ky^3\,dy\,dx = \frac{16{,}384}{315}k$

$$I_y = \iint_R x^2\,\rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} kx^2y\,dy\,dx = \frac{512}{105}k$$

$$I_0 = I_x + I_y = \frac{16{,}384k}{315} + \frac{512k}{105} = \frac{17{,}920}{315}k = \frac{512}{9}k$$

$$m = \iint_R \rho(x, y)\,dA = \int_0^2\int_0^{4-x^2} ky\,dy\,dx = \frac{128}{15}k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/105}{128k/15}} = \sqrt{\frac{4}{7}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16{,}384k/315}{128k/15}} = \sqrt{\frac{128}{21}}$$

**38.** $f(x, y) = 16 - x - y^2$

$R = \{(x, y): 0 \le x \le 2,\ 0 \le y \le x\}$

$f_x = -1, f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4y^2}$

$$S = \int_0^2\int_y^2 \sqrt{2 + 4y^2}\,dx\,dy = \int_0^2\left[2\sqrt{2 + 4y^2} - y\sqrt{2 + 4y^2}\right]dy$$

$$= \left[\frac{1}{2}\left(2y\sqrt{2 + 4y^2} + 2\ln\left|2y + \sqrt{2 + 4y^2}\right|\right) - \frac{1}{12}(2 + 4y^2)^{3/2}\right]_0^2$$

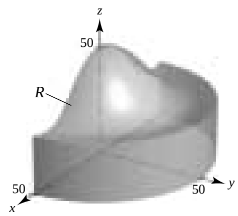
$$= \left[\frac{1}{2}\left(4\sqrt{18} + 2\ln\left|4 + \sqrt{18}\right|\right) - \frac{1}{12}\left(18\sqrt{18}\right)\right] - \left[\ln\sqrt{2} - \frac{2\sqrt{2}}{12}\right]$$

$$= 6\sqrt{2} + \ln\left|4 + 3\sqrt{2}\right| - \frac{9\sqrt{2}}{2} - \ln\sqrt{2} + \frac{\sqrt{2}}{6} = \frac{5\sqrt{2}}{3} + \ln\left|2\sqrt{2} + 3\right|$$

**40.** (a) Graph of

$$f(x, y) = z$$
$$= 25\left[1 + e^{-(x^2+y^2)/1000}\cos^2\left(\frac{x^2+y^2}{1000}\right)\right]$$

over region $R$




(b) Surface area $= \displaystyle\iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA$

Using a symbolic computer program, you obtain surface area $\approx$ 4,540 sq. ft.

**42.** $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{(x^2+y^2)/2}(x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_0^{r^2/2} r^3\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 r^5\,dr\,d\theta = \frac{16}{3}\int_0^{2\pi}d\theta = \frac{32\pi}{3}$$

**44.** $$\int_0^5\int_0^{\sqrt{25-x^2}}\int_0^{\sqrt{25-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^5\frac{\rho^2}{1+\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\Big[\rho - \arctan\rho\Big]_0^5\sin\phi\,d\phi\,d\theta$$
$$= \int_0^{\pi/2}\Big[(5 - \arctan 5)(-\cos\phi)\Big]_0^{\pi/2}d\theta = \frac{\pi}{2}(5 - \arctan 5)$$

**46.** $$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{4}{3}$$

**48.** $$V = 2\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{16-r^2} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r(16-r^2)\,dr\,d\theta$$
$$= 2\int_0^{\pi/2}(32\sin^2\theta - 4\sin^4\theta)\,d\theta = 8\int_0^{\pi/2}(8\sin^2\theta - \sin^4\theta)\,d\theta$$
$$= 8\left[4\theta - 2\sin 2\theta + \frac{1}{4}\sin^3\theta\cos\theta - \frac{3}{4}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{29\pi}{2}$$

**50.** $$m = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{2}{3}kca^3\int_0^{\pi/2}\sin\theta\,d\theta = \frac{2}{3}kca^3$$
$$M_{xz} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r^2\sin\theta\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{2}kca^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{8}\pi kca^4$$
$$M_{xy} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} rz\,dz\,dr\,d\theta = kc^2\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{4}kc^2a^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{16}\pi kc^2a^4$$
$$\bar{x} = 0$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{\pi kca^4/8}{2kca^3/3} = \frac{3\pi a}{16}$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{\pi kc^2a^4/16}{2kca^3/3} = \frac{3\pi ca}{32}$$

**52.** $m = \dfrac{500\pi}{3} - \displaystyle\int_0^3\int_0^{2\pi}\int_4^{\sqrt{25-r^2}} r\,dz\,d\theta\,dr = \dfrac{500\pi}{3} - \int_0^3\int_0^{2\pi}\left(r\sqrt{25-r^2} - 4r\right)d\theta\,dr$

$$= \frac{500\pi}{3} - 2\pi\left[-\frac{1}{3}(25-r^2)^{3/2} - 2r^2\right]_0^3 = \frac{500\pi}{3} - 2\pi\left[-\frac{64}{3} - 18 + \frac{125}{3}\right] = \frac{500\pi}{3} - \frac{14\pi}{3} = 162\pi$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = \int_0^{2\pi}\int_0^3\int_{-\sqrt{25-r^2}}^4 zr\,dz\,dr\,d\theta + \int_0^{2\pi}\int_3^5\int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3\left[8 - \frac{1}{2}(25-r^2)\right]r\,dr\,d\theta + 0$$

$$= \int_0^{2\pi}\int_0^3\left[\frac{1}{2}r^3 - \frac{9}{2}r\right]dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{8}r^4 - \frac{9}{4}r^2\right]_0^3 d\theta = \left[-\frac{81}{8}\theta\right]_0^{2\pi} = -\frac{81}{4}\pi$$

$$\bar{z} = \frac{M_{xy}}{m} = -\frac{81\pi}{4}\frac{1}{162\pi} = -\frac{1}{8}$$

**54.** $I_z = k\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_0^a \rho^2\sin^2\phi(\rho)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$= \frac{4k\pi a^6}{9}$$

**56.** $x^2 + y^2 + \dfrac{z^2}{a^2} = 1$

$$I_z = \iiint_Q (x^2+y^2)\,dV$$

$$= \int_{-a}^{a}\int_{-\sqrt{1-z^2-a^2}}^{\sqrt{1-z^2-a^2}}\int_{-\sqrt{1-y^2-z^2-a^2}}^{\sqrt{1-y^2-z^2-a^2}}(x^2+y^2)\,dx\,dy\,dz$$

$$= \frac{8}{15}\pi a$$

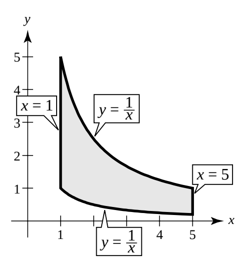
**58.** $\displaystyle\int_0^{\pi}\int_0^2\int_0^{1+r^2} r\,dz\,dr\,d\theta$

Since $z = 1 + r^2$ represents a paraboloid with vertex $(0, 0, 1)$, this integral represents the volume of the solid below the paraboloid and above the semi-circle $y = \sqrt{4 - x^2}$ in the $xy$-plane.

**60.** $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}$

$$= (2u)(-2v) - (2u)(2v) = -8uv$$

**62.** $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u} = 1\left(\dfrac{1}{u}\right) - 0 = \dfrac{1}{u}$




$x = u, y = \dfrac{v}{u} \implies u = x, v = xy$

| Boundary in $xy$-plane | Boundary in $uv$-plane |
|---|---|
| $x = 1$ | $u = 1$ |
| $x = 5$ | $u = 5$ |
| $xy = 1$ | $v = 1$ |
| $xy = 5$ | $v = 5$ |

$$\iint_R \frac{x}{1+x^2y^2}\,dA = \int_1^5\int_1^5 \frac{u}{1+u^2(v/u)^2}\left(\frac{1}{u}\right)du\,dv = \int_1^5\int_1^5 \frac{1}{1+v^2}\,du\,dv = \int_1^5 \frac{4}{1+v^2}\,dv$$

$$= 4\arctan v\Big]_1^5 = 4\arctan 5 - \pi$$

# Problem Solving for Chapter 13

**2.** $z = \dfrac{1}{c}(d - ax - by)$ Plane

$$f_x = -\frac{a}{c},\ f_y = -\frac{b}{c}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}$$

$$S = \iint_R \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}\, dA$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \iint_R dA$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{c} A(R)$$

**4.** $A$: $\displaystyle\int_0^{2\pi}\int_4^5 \left(\frac{r}{16} - \frac{r^2}{160}\right) r\, dr\, d\theta = \frac{1333\pi}{960} \approx 4.36 \text{ ft}^3$

$$B = \int_0^{2\pi}\int_9^{10} \left(\frac{r}{16} - \frac{r^2}{160}\right) r\, dr\, d\theta = \frac{523\pi}{960} \approx 1.71 \text{ ft}^3$$

The distribution is not uniform. Less water in region of greater area.

In one hour, the entire lawn receives

$$\int_0^{2\pi}\int_0^{10} \left(\frac{r}{16} - \frac{r^2}{160}\right) r\, dr\, d\theta = \frac{125\pi}{12} \approx 32.72 \text{ ft}^3.$$

**6.** (a) $V = \displaystyle\int_0^{2\pi}\int_0^2\int_2^{\sqrt{8-r^2}} r\, dz\, dr\, d\theta = \frac{8\pi}{3}\left(4\sqrt{2} - 5\right)$

(b) $V = \displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{2\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{8\pi}{3}\left(4\sqrt{2} - 5\right)$

**8.** Volume $\approx [5 + 6 + 5 + 5]4 = 84 \text{ m}^3$

**10.** Let $v = \ln\left(\dfrac{1}{x}\right),\ dv = -\dfrac{dx}{x}$.

$$e^v = \frac{1}{x},\ x = e^{-v},\ dx = -e^{-v}\, dv$$

$$\int_0^1 \sqrt{\ln(1/x)}\, dx = \int_\infty^0 \sqrt{v}\,(-e^{-v})\, dv = \int_0^\infty \sqrt{v}\, e^{-v}\, dv$$

Let $u = \sqrt{v},\ u^2 = v,\ 2u\, du = dv$.

$$\int_0^1 \sqrt{\ln(1/x)}\, dx = \int_0^\infty u\, e^{-u^2}(2u\, du) = 2\int_0^\infty u^2 e^{-u^2}\, du = 2\left(\frac{\sqrt{\pi}}{4}\right) = \frac{\sqrt{\pi}}{2} \quad \text{(PS \#9)}$$

**12.** Essay

**14.** The greater the angle between the given plane and the $xy$-plane, the greater the surface area. Hence:

$$z_2 < z_1 < z_4 < z_3$$

# CHAPTER 14
# Vector Analysis

# CHAPTER 14
# Vector Analysis

## Section 14.1 Vector Fields

Solutions to Even-Numbered Exercises

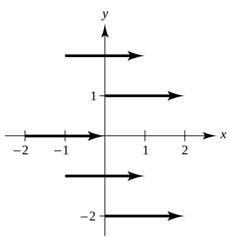
**2.** All vectors are parallel to $x$-axis.
Matches (d)

**4.** Vectors are in rotational pattern.
Matches (e)

**6.** Vectors along $x$-axis have no $x$-component.
Matches (f)

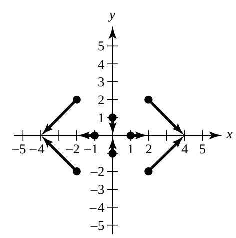
**8.** $\mathbf{F}(x, y) = 2\mathbf{i}$

$\|\mathbf{F}\| = 2$



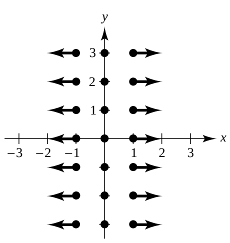
**10.** $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{x^2 + y^2}$



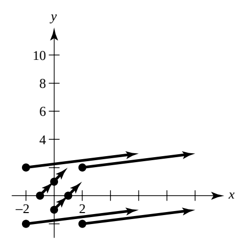
**12.** $\mathbf{F}(x, y) = x\mathbf{i}$

$\|\mathbf{F}\| = |x| = c$



**14.** $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + \mathbf{j}$

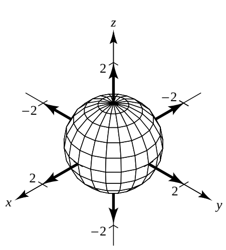
$\|\mathbf{F}\| = \sqrt{1 + (x^2 + y^2)^2}$



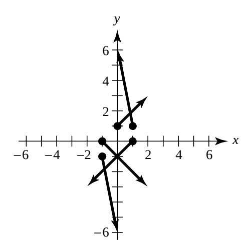
**16.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\|\mathbf{F}\| = \sqrt{x^2 + y^2 + z^2} = c$

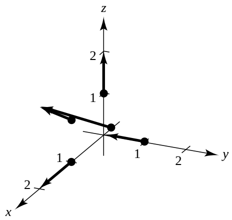
$x^2 + y^2 + z^2 = c^2$



**18.** $\mathbf{F}(x, y) = (2y - 3x)\mathbf{i} + (2y + 3x)\mathbf{j}$



**20.** $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$




**22.** $f(x, y) = \sin 3x \cos 4y$

$f_x(x, y) = 3 \cos 3x \cos 4y$

$f_y(x, y) = -4 \sin 3x \sin 4y$

$\mathbf{F}(x, y) = 3 \cos 3x \cos 4y\mathbf{i} - 4 \sin 3x \sin 4y\mathbf{j}$

**24.** $f(x, y, z) = \frac{y}{z} + \frac{z}{x} - \frac{xz}{y}$

$f_x(x, y, z) = -\frac{z}{x^2} - \frac{z}{y}$

$f_y(x, y, z) = \frac{1}{z} + \frac{xz}{y^2}$

$f_z(x, y, z) = -\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}$

$\mathbf{F}(x, y, z) = \left(-\frac{z}{x^2} - \frac{z}{y}\right)\mathbf{i} + \left(\frac{1}{z} + \frac{xz}{y^2}\right)\mathbf{j} + \left(-\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}\right)\mathbf{k}$

**26.** $g(x, y, z) = x \arcsin yz$

$g_x(x, y, z) = \arcsin yz$

$g_y(x, y, z) = \frac{xz}{\sqrt{1 - y^2z^2}}$

$g_z(x, y, z) = \frac{xy}{\sqrt{1 - y^2z^2}}$

$\mathbf{G}(x, y, z) = (\arcsin yz)\mathbf{i} + \frac{xz}{\sqrt{1 - y^2z^2}}\mathbf{j} + \frac{xy}{\sqrt{1 - y^2z^2}}\mathbf{k}$

**28.** $\mathbf{F}(x, y) = \frac{1}{x^2}(y\mathbf{i} - x\mathbf{j}) = \frac{y}{x^2}\mathbf{i} - \frac{1}{x}\mathbf{j}$

$M = y/x^2$ and $N = -(1/x)$ have continuous first partial derivatives for all $x \neq 0$.

$\frac{\partial N}{\partial x} = \frac{1}{x^2} = \frac{\partial M}{\partial y} \implies \mathbf{F}$ is conservative.

**30.** $\mathbf{F}(x, y) = \frac{1}{xy}(y\mathbf{i} - x\mathbf{j}) = \frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j}$

$M = 1/x$ and $N = -1/y$ have continuous first partial derivatives for all $x, y \neq 0$.

$\frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \implies \mathbf{F}$ is conservative.

**32.** $M = \frac{x}{\sqrt{x^2 + y^2}}, N = \frac{y}{\sqrt{x^2 + y^2}}$

$\frac{\partial N}{\partial x} = +\frac{-xy}{(x^2 + y^2)^{3/2}} = \frac{\partial M}{\partial y} \implies$ Conservative

**34.** $M = \frac{y}{\sqrt{1 - x^2y^2}}, N = \frac{-x}{\sqrt{1 - x^2y^2}}$

$\frac{\partial N}{\partial x} = +\frac{-1}{(1 - x^2y^2)^{3/2}} \neq \frac{\partial M}{\partial y} = \frac{1}{(1 - x^2y^2)^{3/2}}$

$\implies$ Not conservative

**36.** $\mathbf{F}(x, y) = \frac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$

$= \frac{1}{y}\mathbf{i} - \frac{2x}{y^2}\mathbf{j}$

$\frac{\partial}{\partial y}\left[\frac{1}{y}\right] = -\frac{1}{y^2}$

$\frac{\partial}{\partial x}\left[-\frac{2x}{y^2}\right] = -\frac{2}{y^2}$

Not conservative

**38.** $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$

$\frac{\partial}{\partial y}[3x^2y^2] = 6x^2y$

$\frac{\partial}{\partial x}[2x^3y] = 6x^2y$

Conservative

$f_x(x, y) = 3x^2y^2$

$f_y(x, y) = 2x^3y$

$f(x, y) = x^3y^2 + K$

**40.** $\mathbf{F}(x, y) = \frac{2y}{x}\mathbf{i} - \frac{x^2}{y^2}\mathbf{j}$

$\frac{\partial}{\partial y}\left[\frac{2y}{x}\right] = \frac{2}{x}$

$\frac{\partial}{\partial x}\left[-\frac{x^2}{y^2}\right] = -\frac{2x}{y^2}$

Not conservative

**42.** $\mathbf{F}(x, y) = \dfrac{2x}{(x^2 + y^2)^2}\mathbf{i} + \dfrac{2y}{(x^2 + y^2)^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2x}{(x^2 + y^2)^2}\right] = -\frac{8xy}{(x^2 + y^2)^3}$$

$$\frac{\partial}{\partial x}\left[\frac{2y}{(x^2 + y^2)^2}\right] = -\frac{8xy}{(x^2 + y^2)^3}$$

Conservative

$$f_x(x, y) = \frac{2x}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2y}{(x^2 + y^2)^2}$$

$$f(x, y) = -\frac{1}{x^2 + y^2} + K$$

**44.** $\mathbf{F}(x, y, z) = x^2z\,\mathbf{i} - 2xz\,\mathbf{j} + yz\,\mathbf{k},\ (2, -1, 3)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix}$$

$$= (z + 2x)\mathbf{i} - (0 - x^2)\mathbf{j} + (-2z - 0)\mathbf{k}$$

$$= (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$\textbf{curl F}\,(2, -1, 3) = 7\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$

**46.** $\mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k}),\ (3, 2, 0)$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{-xyz} & e^{-xyz} & e^{-xyz} \end{vmatrix} = (-xz + xy)e^{-xyz}\mathbf{i} - (-yz + xy)e^{-xyz}\mathbf{j} + (-yz + xz)e^{-xyz}\mathbf{k}$$

$\textbf{curl F}\,(3, 2, 0) = 6\mathbf{i} - 6\mathbf{j}$

**48.** $\mathbf{F}(x, y, z) = \dfrac{yz}{y - z}\mathbf{i} + \dfrac{xz}{x - z}\mathbf{j} + \dfrac{xy}{x - y}\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{y - z} & \dfrac{xz}{x - z} & \dfrac{xy}{x - y} \end{vmatrix}$$

$$= \left[\frac{x^2}{(x - y)^2} - \frac{x^2}{(x - z)^2}\right]\mathbf{i} - \left[\frac{-y^2}{(x - y)^2} - \frac{y^2}{(y - z)^2}\right]\mathbf{j} + \left[\frac{-z^2}{(x - z)^2} - \frac{-z^2}{(y - z)^2}\right]\mathbf{k}$$

$$= x^2\left[\frac{1}{(x - y)^2} - \frac{1}{(x - z)^2}\right]\mathbf{i} + y^2\left[\frac{1}{(x - y)^2} + \frac{1}{(y - z)^2}\right]\mathbf{j} + z^2\left[\frac{1}{(y - z)^2} - \frac{1}{(x - z)^2}\right]\mathbf{k}$$

**50.** $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} \end{vmatrix} = \frac{(y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

**52.** $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + \mathbf{k})$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^z & xe^z & e^z \end{vmatrix} = -xe^z\mathbf{i} + ye^z\mathbf{j} \neq \mathbf{0}$$

Not conservative

**54.** $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f(x, y, z) = xy^2z^3 + K$$

**56.** $\mathbf{F}(x, y, z) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{x^2 + y^2} & \dfrac{y}{x^2 + y^2} & 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y, z) = \frac{y}{x^2 + y^2}$$

$$f_z(x, y, z) = 1$$

$$f(x, y, z) = \int \frac{x}{x^2 + y^2}\,dx = \frac{1}{2}\ln(x^2 + y^2) + g(y, z) + K_1$$

$$f(x, y, z) = \int \frac{y}{x^2 + y^2}\,dy = \frac{1}{2}\ln(x^2 + y^2) + h(x, z) + K_2$$

$$f(x, y, z) = \int dz = z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{1}{2}\ln(x^2 + y^2) + z + K$$

**58.** $\mathbf{F}(x, y) = xe^x\mathbf{i} + ye^y\mathbf{j}$

$$\text{div}\,\mathbf{F}(x, y) = \frac{\partial}{\partial x}[xe^x] + \frac{\partial}{\partial y}[ye^y] = xe^x + e^x + ye^y + e^y = e^x(x + 1) + e^y(y + 1)$$

**60.** $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + xy\mathbf{j} + \ln(y^2 + z^2)\mathbf{k}$

$$\text{div}\,\mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\ln(x^2 + y^2)] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[\ln(y^2 + z^2)] = \frac{2x}{x^2 + y^2} + x + \frac{2z}{y^2 + z^2}$$

**62.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\text{div}\,\mathbf{F}(x, y, z) = 2xz + y$$

$$\text{div}\,\mathbf{F}(2, -1, 3) = 11$$

**64.** $\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\text{div}\,\mathbf{F}(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\text{div}\,\mathbf{F}(3, 2, 1) = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$$

**66.** See the definition of Conservative Vector Field on page 1011. To test for a conservative vector field, see Theorem 14.1 and 14.2.

**68.** See the definition on page 1016.

**70.** $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$$\mathbf{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & -xz^2 - x^2z & xy \end{vmatrix}$$

$$= (x + 2xz + x^2)\mathbf{i} - (y - y)\mathbf{j} + (-z^2 - 2xz - z)\mathbf{k}$$

$$= x(x + 2z + 1)\mathbf{i} - z(z + 2x + 1)\mathbf{k}$$

**72.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\textbf{curl}(\textbf{curl F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z + 2x & x^2 & -2z \end{vmatrix} = \mathbf{j} + 2x\mathbf{k}$$

**74.** $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 0$

**76.** $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$\operatorname{div}(\textbf{curl F}) = 2 - 2 = 0$

**78.** Let $f(x, y, z)$ be a scalar function whose second partial derivatives are continuous.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\textbf{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}$$

**80.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M & N & P \\ R & S & T \end{vmatrix} = (NT - PS)\mathbf{i} - (MT - PR)\mathbf{j} + (MS - NR)\mathbf{k}$$

$$\begin{aligned} \operatorname{div}(\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}(NT - PS) + \frac{\partial}{\partial y}(PR - MT) + \frac{\partial}{\partial z}(MS - NR) \\ &= N\frac{\partial T}{\partial x} + T\frac{\partial N}{\partial x} - P\frac{\partial S}{\partial x} - S\frac{\partial P}{\partial x} + P\frac{\partial R}{\partial y} + R\frac{\partial P}{\partial y} - M\frac{\partial T}{\partial y} - T\frac{\partial M}{\partial y} + M\frac{\partial S}{\partial z} + S\frac{\partial M}{\partial z} - N\frac{\partial R}{\partial z} - R\frac{\partial N}{\partial z} \\ &= \left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)R + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)S + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)T\right] - \left[M\left(\frac{\partial T}{\partial y} - \frac{\partial S}{\partial z}\right) + N\left(\frac{\partial R}{\partial z} - \frac{\partial T}{\partial x}\right) + P\left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right)\right] \\ &= (\textbf{curl F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\textbf{curl G}) \end{aligned}$$

**82.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\begin{aligned} \nabla \times (f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fM & fN & fP \end{vmatrix} \\ &= \left(\frac{\partial f}{\partial y}P + f\frac{\partial P}{\partial y} - \frac{\partial f}{\partial z}N - f\frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} - \frac{\partial f}{\partial z}M - f\frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}N + f\frac{\partial N}{\partial x} - \frac{\partial f}{\partial y}M - f\frac{\partial M}{\partial y}\right)\mathbf{k} \\ &= f\left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ M & N & P \end{vmatrix} = f[\nabla \times \mathbf{F}] + (\nabla f) \times \mathbf{F} \end{aligned}$$

**84.** Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\mathbf{curl\ F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\text{div}(\mathbf{curl\ F}) = \frac{\partial}{\partial x}\left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0 \qquad \text{(since the mixed partials are equal)}$$

**In Exercises 86 and 88, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.**

**86.** $$\frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\nabla\left(\frac{1}{f}\right) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} = \frac{-(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = -\frac{\mathbf{F}}{f^3}$$

**88.** $$w = \frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

Therefore $w = \dfrac{1}{f}$ is harmonic.

## Section 14.2 Line Integrals

**2.** $$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = \frac{x^2}{16}$$

$$\sin^2 t = \frac{y^2}{9}$$

$$x = 4\cos t$$

$$y = 3\sin t$$

$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$$

$$0 \le t \le 2\pi$$

**4.** $$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \frac{4}{5}t\mathbf{j}, & 0 \le t \le 5 \\ 5\mathbf{i} + (9 - t)\mathbf{j}, & 5 \le t \le 9 \\ (14 - t)\mathbf{i}, & 9 \le t \le 14 \end{cases}$$

**6.** $$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \le t \le 2 \\ (4 - t)\mathbf{i} + 4\mathbf{j}, & 2 \le t \le 4 \\ (8 - t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$$

**8.** $\mathbf{r}(t) = t\mathbf{i} + (2 - t)\mathbf{j},\ 0 \le t \le 2;\ \mathbf{r}'(t) = \mathbf{i} - \mathbf{j}$

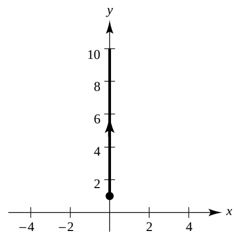
$$\int_C 4xy\,ds = \int_0^2 4t(2 - t)\sqrt{1 + 1}\,dt = 4\sqrt{2}\int_0^2 (2t - t^2)\,dt = 4\sqrt{2}\left[t^2 - \frac{t^3}{3}\right]_0^2 = 4\sqrt{2}\left(4 - \frac{8}{3}\right) = \frac{16\sqrt{2}}{3}$$

**10.** $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 3\mathbf{k},\ 0 \le t \le 2;\ \mathbf{r}'(t) = 12\mathbf{i} + 5\mathbf{j}$

$$\int_C 8xyz\,ds = \int_0^2 8(12t)(5t)(3)\sqrt{12^2 + 5^2 + 0^2}\,dt = \int_0^2 18{,}720t^2\,dt = 18{,}720\left[\frac{t^3}{3}\right]_0^2 = 49{,}920$$

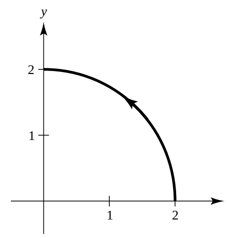
**12.** $\mathbf{r}(t) = t\mathbf{j},\ 1 \le t \le 10$

$$\int_C (x^2 + y^2)\,ds = \int_1^{10} [0 + t^2]\sqrt{0 + 1}\,dt$$
$$= \int_1^{10} t^2\,dt$$
$$= \left[\frac{1}{3}t^3\right]_1^{10} = 333$$



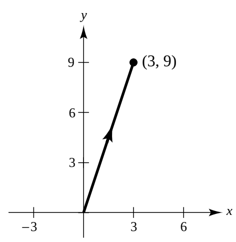

**14.** $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$$\int_C (x^2 + y^2)\,ds = \int_0^{\pi/2} [4\cos^2 t + 4\sin^2 t]\sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt$$
$$= \int_0^{\pi/2} 8\,dt = 4\pi$$




**16.** $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j},\ 0 \le t \le 3$

$$\int_C (x + 4\sqrt{y})\,ds = \int_0^3 \left(t + 4\sqrt{3t}\right)\sqrt{1 + 9}\,dt$$
$$= \left[\sqrt{10}\left(\frac{t^2}{2} + \frac{8\sqrt{3}}{3}t^{3/2}\right)\right]_0^3$$
$$= \frac{\sqrt{10}}{6}(27 + 144) = \frac{57\sqrt{10}}{2}$$




**18.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 2 \\ 2\mathbf{i} + (t-2)\mathbf{j}, & 2 \le t \le 4 \\ (6-t)\mathbf{i} + 2\mathbf{j}, & 4 \le t \le 6 \\ (8-t)\mathbf{j}, & 6 \le t \le 8 \end{cases}$

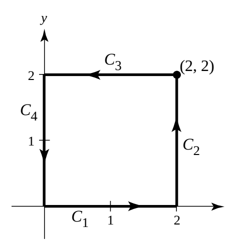
$$\int_{C_1} \left(x + 4\sqrt{y}\right)ds = \int_0^2 t\,dt = 2$$

$$\int_{C_2} \left(x + 4\sqrt{y}\right)ds = \int_2^4 \left(2 + 4\sqrt{t-2}\right)ds = 4 + \frac{16\sqrt{2}}{3}$$

$$\int_{C_3} \left(x + 4\sqrt{y}\right)ds = \int_4^6 \left((6-t) + 4\sqrt{2}\right)ds = 2 + 8\sqrt{2}$$

$$\int_{C_4} \left(x + 4\sqrt{y}\right)ds = \int_6^8 4\sqrt{8-t}\,ds = \frac{16\sqrt{2}}{3}$$

$$\int_C \left(x + 4\sqrt{y}\right)ds = 2 + 4 + \frac{16\sqrt{2}}{3} + 2 + 8\sqrt{2} + \frac{16\sqrt{2}}{3} = 8 + \frac{56}{3}\sqrt{2}$$



**20.** $\rho(x, y, z) = z$

$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 4\pi$$

$$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2)^2} = \sqrt{13}$$

$$\text{Mass} = \int_C \rho(x, y, z)\,ds = \int_0^{4\pi} 2t\sqrt{13}\,dt = 16\pi^2\sqrt{13}$$

**22.** $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$$C\colon \mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{F}(t) = 16\sin t\cos t\mathbf{i} + 4\sin t\mathbf{j}$$

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (-64\sin^2 t\cos t + 16\sin t\cos t)\,dt$$

$$= \left[-\frac{64}{3}\sin^3 t + 8\sin^2 t\right]_0^{\pi/2} = -\frac{40}{3}$$

**24.** $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$$C\colon\ \mathbf{r}(t) = t\mathbf{i} + \sqrt{4 - t^2}\mathbf{j},\ -2 \le t \le 2$$

$$\mathbf{F}(t) = 3t\mathbf{i} + 4\sqrt{4 - t^2}\mathbf{j}$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{t}{\sqrt{4 - t^2}}\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-2}^{2} (3t - 4t)\,dt = \left[-\frac{t^2}{2}\right]_{-2}^{2} = 0$$

**26.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$C\colon\ \mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k},\ 0 \le t \le \pi$$

$$\mathbf{F}(t) = 4\sin^2 t\mathbf{i} + 4\cos^2 t\mathbf{j} + \frac{1}{4}t^4\mathbf{k}$$

$$\mathbf{r}'(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} + t\mathbf{k}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi}\left(8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{4}t^5\right)dt$$

$$= \left[\frac{8}{3}\sin^3 t + \frac{8}{3}\cos^3 t + \frac{t^6}{24}\right]_0^{\pi}$$

$$= -\frac{8}{3} + \frac{\pi^6}{24} - \frac{8}{3} = \frac{\pi^6}{24} - \frac{16}{3}$$

**28.** $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k},\ 0 \le t \le 2$$

$$\mathbf{F}(t) = \frac{t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}}{\sqrt{2t^2 + e^{2t}}}$$

$$d\mathbf{r} = (\mathbf{i} + \mathbf{j} + e^t\mathbf{k})\,dt$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \frac{1}{\sqrt{2t^2 + e^{2t}}}(2t + e^{2t})\,dt \approx 6.91$$

**30.** $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

$C\colon\ x = \cos^3 t,\ y = \sin^3 t$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -3\cos^2 t\sin t\mathbf{i} + 3\sin^2 t\cos t\mathbf{j}$$

$$\mathbf{F}(t) = \cos^6 t\mathbf{i} - \cos^3 t\sin^3 t\mathbf{j}$$

$$\begin{aligned}\mathbf{F}\cdot\mathbf{r}' &= -3\cos^8 t\sin t - 3\cos^4 t\sin^5 t\\ &= -3\cos^4 t\sin t(\cos^4 t + \sin^4 t)\\ &= -3\cos^4 t\sin t[\cos^4 t + (1 - \cos^2 t)^2]\\ &= -3\cos^4 t\sin t(2\cos^4 t - 2\cos^2 t + 1)\\ &= -6\cos^8 t\sin t + 6\cos^6 t\sin t - 3\cos^4 t\sin t\end{aligned}$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} [-6\cos^8 t\sin t + 6\cos^6 t\sin t - 3\cos^4 t\sin t]\,dt$$

$$= \left[\frac{2\cos^9 t}{3} - \frac{6\cos^7 t}{7} + \frac{3\cos^5 t}{5}\right]_0^{\pi/2} = -\frac{43}{105}$$

**32.** $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

$C$: counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \le t \le \pi$$

$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$

$$\mathbf{F}(t) = -2\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j}$$

$$\mathbf{F}\cdot\mathbf{r}' = 4\sin^2 t - 4\cos^2 t = -4\cos 2t$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = -4\int_0^{\pi} \cos 2t\,dt = \Big[-2\sin 2t\Big]_0^{\pi} = 0$$

**34.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$C$: line from $(0, 0, 0)$ to $(5, 3, 2)$

$$\mathbf{r}(t) = 5t\,\mathbf{i} + 3t\,\mathbf{j} + 2t\,\mathbf{k}, \quad 0 \le t \le 1$$

$$\mathbf{r}'(t) = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{F}(t) = 6t^2\mathbf{i} + 10t^2\mathbf{j} + 15t^2\mathbf{k}$$

$$\mathbf{F}\cdot\mathbf{r}' = 90t^2$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 90t^2\,dt = 30$$

**36.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} \approx \frac{1-0}{3(4)}[5 + 4(4) + 2(4) + 4(6) + 11]$$

$$= \frac{16}{3}$$

| $(x, y)$ | $(0, 0)$ | $\left(\frac{1}{4}, \frac{1}{16}\right)$ | $\left(\frac{1}{2}, \frac{1}{4}\right)$ | $\left(\frac{3}{4}, \frac{9}{16}\right)$ | $(1, 1)$ |
|---|---|---|---|---|---|
| $\mathbf{F}(x, y)$ | $5\mathbf{i}$ | $3.5\mathbf{i} + \mathbf{j}$ | $2\mathbf{i} + 2\mathbf{j}$ | $1.5\mathbf{i} + 3\mathbf{j}$ | $\mathbf{i} + 5\mathbf{j}$ |
| $\mathbf{r}'(t)$ | $\mathbf{i}$ | $\mathbf{i} + 0.5\mathbf{j}$ | $\mathbf{i} + \mathbf{j}$ | $\mathbf{i} + 1.5\mathbf{j}$ | $\mathbf{i} + 2\mathbf{j}$ |
| $\mathbf{F}\cdot\mathbf{r}'$ | 5 | 4 | 4 | 6 | 11 |

**38.** $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

(a) $\mathbf{r}_1(t) = (t + 1)\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

$$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = (t + 1)^2t^2\mathbf{i} + (t + 1)t^3\mathbf{j}$$

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \left[(t+1)^2t^2 + 2t^4(t+1)\right]dt = \frac{256}{3}$$

(b) $\mathbf{r}_2(t) = (1 + 2\cos t)\mathbf{i} + 4\cos^2 t\,\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$$\mathbf{r}_2'(t) = -2\sin t\,\mathbf{i} - 8\cos t\sin t\,\mathbf{j}$$

$$\mathbf{F}(t) = (1 + 2\cos t)^2(4\cos^2 t)\mathbf{i} + (1 + 2\cos t)(8\cos^3 t)\mathbf{j}$$

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}\Big[(1 + 2\cos t)^2(4\cos^2 t)(-2\sin t) - 8\cos t\sin t(1 + 2\cos t)(8\cos^3 t)\,dt = -\frac{256}{5}$$

Both paths join $(1, 0)$ and $(3, 4)$. The integrals are negatives of each other because the orientations are different.

**40.** $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$

$C$: $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$

$$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$$

$$\mathbf{F}(t) = 3t^3\mathbf{i} + t\mathbf{j}$$

$$\mathbf{F}\cdot\mathbf{r}' = 3t^3 - 3t^3 = 0$$

Thus, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

**42.** $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$C$: $\mathbf{r}(t) = 3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$

$$\mathbf{r}'(t) = 3\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$$

$$\mathbf{F}(t) = 3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$$

$$\mathbf{F}\cdot\mathbf{r}' = 9\sin t\cos t - 9\sin t\cos t = 0$$

Thus, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

**44.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x,\ 0 \le x \le 2$

$$\int_C (x + 3y^2)\,dx = \int_0^2 (x + 75x^2)\,dx = \left[\frac{x^2}{2} + 25x^3\right]_0^2 = 202$$

**46.** $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x,\ dy = 5\,dx,\ 0 \le x \le 2$

$$\int_C (3y - x)\,dx + y^2\,dy = \int_0^2 (3(5x) - x)\,dx + (5x)^2 5\,dx = \int_0^2 (14x + 125x^2)\,dx$$

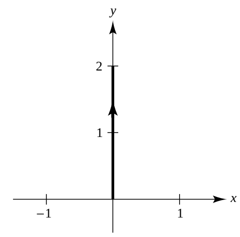
$$= \left[7x^2 + \frac{125}{3}x^3\right]_0^2 = 28 + \frac{125}{3}(8) = \frac{1084}{3}$$

**48.** $\mathbf{r}(t) = t\mathbf{j},\ 0 \le t \le 2$

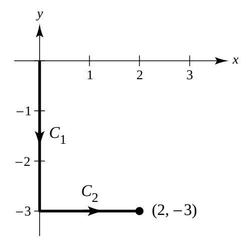
$x(t) = 0,\ y(t) = t$

$dx = 0,\ dy = dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^2 3t\,dt = \left[\frac{3}{2}t^2\right]_0^2 = 6$$




**50.** $\mathbf{r}(t) = \begin{cases} -t\mathbf{j}, & 0 \le t \le 3 \\ (t - 3)\mathbf{i} - 3\mathbf{j}, & 3 \le t \le 5 \end{cases}$




$C_1$: $x(t) = 0,\ y(t) = -t$

$dx = 0,\ dy = -dt$

$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 3t\,dt = \frac{27}{2}$$

$C_2$: $x(t) = t - 3,\ y(t) = -3$

$dx = dt,\ dy = 0$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^5 [2(t - 3) + 3]\,dt = \Big[(t - 3)^2 + 3t\Big]_3^5 = 10$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \frac{27}{2} + 10 = \frac{47}{2}$$

**52.** $x(t) = t,\ y(t) = t^{3/2},\ 0 \le t \le 4,\ dx = dt,\ dy = \frac{3}{2}t^{1/2}\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^4 \left[(2t - t^{3/2}) + (t + 3t^{3/2})\left(\frac{3}{2}t^{1/2}\right)\right] dt$$

$$= \int_0^4 \left(\frac{9}{2}t^2 + \frac{1}{2}t^{3/2} + 2t\right) dt = \left[\frac{3}{2}t^3 + \frac{1}{5}t^{5/2} + t^2\right]_0^4 = 96 + \frac{1}{5}(32) + 16 = \frac{592}{5}$$

**54.** $x(t) = 4\sin t,\ y(t) = 3\cos t,\ 0 \le t \le \frac{\pi}{2}$

$dx = 4\cos t\,dt,\ dy = -3\sin t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{\pi/2} (8\sin t - 3\cos t)(4\cos t)\,dt + (4\sin t + 9\cos t)(-3\sin t)\,dt$$

$$= \int_0^{\pi/2} (5\sin t\cos t - 12\cos^2 t - 12\sin^2 t)\,dt$$

$$= \left[\frac{5}{2}\sin^2 t - 12t\right]_0^{\pi/2} = \frac{5}{2} - 6\pi$$

**56.** $f(x, y) = y$

$C$: line from $(0, 0)$ to $(4, 4)$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 4$$
$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{2}$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^4 t\left(\sqrt{2}\right) dt = 8\sqrt{2}$$

**58.** $f(x, y) = x + y$

$C$: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^{\pi/2} (\cos t + \sin t)\, dt$$
$$= \Big[\sin t - \cos t\Big]_0^{\pi/2} = 2$$

**60.** $f(x, y) = y + 1$

$C$: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j},\ 0 \le t \le 1$$
$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^1 [2 - (1 - t)^2]\sqrt{1 + 4(1 - t)^2}\, dt$$
$$= 2\int_0^1 \sqrt{1 + 4(1 - t)^2}\, dt - \int_0^1 (1 - t)^2\sqrt{1 + 4(1 - t)^2}\, dt$$
$$= -\frac{1}{2}\Big[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\Big]_0^1$$
$$+ \frac{1}{64}\Big[2(1 - t)[2(4)(1 - t)^2 + 1]\sqrt{1 + 4(1 - t)^2} - \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\Big]_0^1$$
$$= \frac{1}{2}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] - \frac{1}{64}\left[18\sqrt{5} - \ln\left(2 + \sqrt{5}\right)\right]$$
$$= \frac{23}{32}\sqrt{5} + \frac{33}{64}\ln\left(2 + \sqrt{5}\right) = \frac{1}{64}\left[46\sqrt{5} + 33\ln\left(2 + \sqrt{5}\right)\right] \approx 2.3515$$

**62.** $f(x, y) = x^2 - y^2 + 4$

$C$: $x^2 + y^2 = 4$

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le 2\pi$$
$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$
$$\|\mathbf{r}'(t)\| = 2$$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^{2\pi} (4\cos^2 t - 4\sin^2 t + 4)(2)\, dt = 8\int_0^{2\pi} (1 + \cos 2t)\, dt = \left[8\left(t + \frac{1}{2}\sin 2t\right)\right]_0^{2\pi} = 16\pi$$

**64.** $f(x, y) = 20 + \frac{1}{4}x$

$C$: $y = x^{3/2}$, $0 \le x \le 40$

$\mathbf{r}(t) = t\mathbf{i} + t^{3/2}\mathbf{j}$, $0 \le t \le 40$

$\mathbf{r}'(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + \left(\frac{9}{4}\right)t}$

Lateral surface area: $\int_C f(x, y)\,ds = \int_0^{40}\left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\,dt$

Let $u = \sqrt{1 + \left(\frac{9}{4}\right)t}$, then $t = \frac{4}{9}(u^2 - 1)$ and $dt = \frac{8}{9}u\,du$.

$$\int_0^{40}\left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\,dt = \int_1^{\sqrt{91}}\left[20 + \frac{1}{9}(u^2 - 1)\right](u)\left(\frac{8}{9}u\right)du = \frac{8}{81}\int_1^{\sqrt{91}}(u^4 + 179u^2)\,du$$

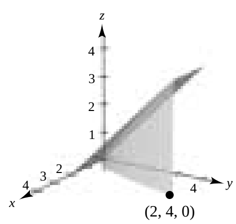
$$= \frac{8}{81}\left[\frac{u^5}{5} + \frac{179u^3}{3}\right]_1^{\sqrt{91}} = \frac{850{,}304\sqrt{91} - 7184}{1215} \approx 6670.12$$

**66.** $f(x, y) = y$

$C$: $y = x^2$ from $(0, 0)$ to $(2, 4)$

$S \approx 8$

Matches $c$.




**68.** $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy$

$M = 15(4 - x^2y) = 60 - 15x^2(c - cx^2)$

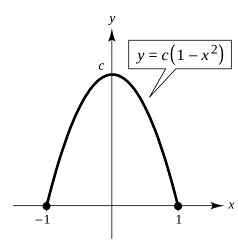
$N = -15xy = -15x(c - cx^2)$

$dx = dx$, $dy = -2cx\,dx$

$W = \int_{-1}^{1}\left[60 - 15x^2(c - cx^2) + (-15x(c - cx^2))(-2\,cx)\right]dx$

$= 120 - 4c + 8c^2$ (parabola)

$w' = 16c - 4 = 0 \implies c = \frac{1}{4}$ yields the minimum work, 119.5. Along the straight line path, $y = 0$, the work is 120.




**70.** See the definition, page 1024.

**72.** (a) Work $= 0$

(b) Work is negative, since against force field.

(c) Work is positive, since with force field.

**74.** False, the orientation of $C$ does not affect the form

$$\int_C f(x, y)\,ds.$$

**76.** False. For example, see Exercise 32.

## Section 14.3 Conservative Vector Fields and Independence of Path

**2.** $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j},\ 0 \le t \le 4$

$\mathbf{r}_1'(t) = \mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = (t^2 + t)\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^4 \left(t^2 + t - \frac{1}{2}\sqrt{t}\right) dt = \left[\frac{t^3}{3} + \frac{t^2}{2} - \frac{t^{3/2}}{3}\right]_0^4 = \frac{80}{3}$$

(b) $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j},\ 0 \le w \le 2$

$\mathbf{r}_2'(w) = 2w\mathbf{i} + \mathbf{j}$

$\mathbf{F}(w) = (w^4 + w^2)\mathbf{i} - w^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[2w(w^4 + w^2) - w^2\right] dw = \left[\frac{w^6}{3} + \frac{w^4}{2} - \frac{w^3}{3}\right]_0^2 = \frac{80}{3}$$

**4.** $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = (2 + t)\mathbf{i} + (3 - t)\mathbf{j},\ 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = (3 - t)\mathbf{i} + (2 + t)^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[(3 - t) - (2 + t)^2\right] dt = \left[-\frac{(3 - t)^2}{2} - \frac{(2 + t)^3}{3}\right]_0^3 = -\frac{69}{2}$$

(b) $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j},\ 1 \le w \le e^3$

$\mathbf{r}_2'(w) = \dfrac{1}{w}\mathbf{i} - \dfrac{1}{w}\mathbf{j}$

$\mathbf{F}(w) = (3 - \ln w)\mathbf{i} + (2 + \ln w)^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^{e^3} \left[(3 - \ln w)\left(\frac{1}{w}\right) - (2 + \ln w)^2\left(\frac{1}{w}\right)\right] dw = \left[-\frac{(3 - \ln w)^2}{2} - \frac{(2 + \ln w)^3}{3}\right]_1^{e^3} = -\frac{69}{2}$$

**6.** $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

$\dfrac{\partial N}{\partial x} = 30x^2y \qquad \dfrac{\partial M}{\partial y} = 30x^2y$

Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

**8.** $\mathbf{F}(x, y, z) = y \ln z\,\mathbf{i} - x \ln z\,\mathbf{j} + \dfrac{xy}{z}\mathbf{k}$

**curl F** $\neq \mathbf{0}$ so **F** is not conservative.

$\left(\dfrac{\partial P}{\partial y} = \dfrac{x}{z} \neq -\dfrac{x}{z} = \dfrac{\partial N}{\partial z}\right)$

**10.** $\mathbf{F}(x, y, z) = \sin(yz)\mathbf{i} + xz\cos(yz)\mathbf{j} + xy\sin(yz)\mathbf{k}$

**curl F** $\neq \mathbf{0}$, so **F** is not conservative.

**12.** $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} - (t - 3)\mathbf{j},\ 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = -(t - 3)e^{3t - t^2}\mathbf{i} + te^{3t - t^2}\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \left[-(t - 3)e^{3t - t^2} - te^{3t - t^2}\right] dt = \int_0^3 e^{3t - t^2}(3 - 2t)\, dt = \left[e^{3t - t^2}\right]_0^3 = e^0 - e^0 = 0$$

(b) $\mathbf{F}(x, y)$ is conservative since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = xye^{xy} + e^{xy}.$$

The potential function is $f(x, y) = e^{xy} + k$.

**14.** $\mathbf{F}(x, y) = xy^2\,\mathbf{i} + 2x^2y\,\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad 1 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

$\mathbf{F}(t) = \dfrac{1}{t}\mathbf{i} + 2t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 -\frac{1}{t}\,dt = \Big[-\ln|t|\Big]_1^3 = -\ln 3$$

(b) $\mathbf{r}_2(t) = (t+1)\mathbf{i} - \dfrac{1}{3}(t-3)\mathbf{j}, \quad 0 \le t \le 2$

$\mathbf{r}_2'(t) = \mathbf{i} - \dfrac{1}{3}\mathbf{j}$

$\mathbf{F}(t) = \dfrac{1}{9}(t+1)(t-3)^2\mathbf{i} - \dfrac{2}{3}(t+1)^2(t-3)\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[\frac{1}{9}(t+1)(t-3)^2 + \frac{2}{9}(t+1)^2(t-3)\right] dt$$
$$= \frac{1}{9}\int_0^2 (3t^3 - 7t^2 - 7t + 3)\,dt$$
$$= \frac{1}{9}\left[\frac{3t^4}{4} - \frac{7t^3}{3} - \frac{7t^2}{2} + 3t\right]_0^2 = -\frac{44}{27}$$

**16.** $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy$

Since $\partial M/\partial y = \partial N/\partial x = -3$, $\mathbf{F}(x, y) = (2x - 3y + 1)\mathbf{i} - (3x + y - 5)\mathbf{j}$ is conservative. The potential function is $f(x, y) = x^2 - 3xy - (y^2/2) + x + 5y + k$.

(a) and (d) Since $C$ is a closed curve, $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = 0$.

(b) $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,\,-1)}^{(0,\,1)} = 10$

(c) $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,\,1)}^{(2,\,e^2)} = \frac{1}{2}(3 - 2e^2 - e^4)$

**18.** $\displaystyle\int_C (x^2 + y^2)\,dx + 2xy\,dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$,

$\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$

is conservative. The potential function is

$f(x, y) = (x^3/3) + xy^2 + k$.

(a) $\displaystyle\int_C (x^2 + y^2)\,dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(0,\,0)}^{(8,\,4)} = \frac{896}{3}$

(b) $\displaystyle\int_C (x^2 + y^2)\,dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(2,\,0)}^{(0,\,2)} = -\frac{8}{3}$

**20.** $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

Since **curl F** $= \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = x + yz + k$.

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\mathbf{k}, \ 0 \le t \le \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[x + yz\Big]_{(1,\,0,\,0)}^{(-1,\,0,\,\pi^2)} = -2$$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}, \ 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[x + yz\Big]_{(1,\,0,\,0)}^{(-1,\,0,\,\pi^2)} = -2$$

**22.** $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}, \ 0 \le t \le \pi$

$\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3t^2\cos t\,\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi [\sin^2 t + \cos^2 t + 3t^2\cos t]\,dt = \int_0^\pi [1 + 3t^2\cos t]\,dt$$
$$= \Big[t\Big]_0^\pi + 3\Big[t^2\sin t\Big]_0^\pi - 6\int_0^\pi t\sin t\,dt = \Big[t + 3t^2\sin t - 6(\sin t - t\cos t)\Big]_0^\pi = -5\pi$$

**—CONTINUED—**

**22. —CONTINUED—**

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = -2\mathbf{i} + \pi\mathbf{k}$

$\mathbf{F}(t) = (1 - 2t)\mathbf{j} + 3\pi^2t^2(1 - 2t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 3\pi^3t^2(1 - 2t)\,dt = 3\pi^3\int_0^1 (t^2 - 2t^3)\,dt = 3\pi^3\left[\frac{t^3}{3} - \frac{t^4}{2}\right]_0^1 = -\frac{\pi^3}{2}$$

**24.** $\mathbf{F}(x, y, z) = y\sin z\,\mathbf{i} + x\sin z\,\mathbf{j} + xy\cos x\,\mathbf{k}$

(a) $\mathbf{r}_1(t) = t^2\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}_1'(t) = 2t\mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^4\cos t^2\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 0\,dt = 0$$

(b) $\mathbf{r}_2(t) = 4t\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = 4\mathbf{i} + 4\mathbf{j}$

$\mathbf{F}(t) = 16t^2\cos(4t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\,dt = 0$$

**26.** $$\int_C [2(x + y)\mathbf{i} + 2(x + y)\mathbf{j}] \cdot d\mathbf{r} = \Big[(x + y)^2\Big]_{(-3, 2)}^{(4, 3)} = 49$$

**28.** $$\int_C \frac{y\,dx - x\,dy}{x^2 + y^2} = \left[\arctan\left(\frac{x}{y}\right)\right]_{(1, 1)}^{(2\sqrt{3}, 2)} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

**30.** $$\int_C \frac{2x}{(x^2 + y^2)^2}\,dx + \frac{2y}{(x^2 + y^2)^2}\,dy = \left[-\frac{1}{x^2 + y^2}\right]_{(7, 5)}^{(1, 5)} = -\frac{1}{26} + \frac{1}{74} = \frac{-12}{481}$$

**32.** $$\int_C zy\,dx + xz\,dy + xy\,dz$$

**Note:** Since $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ is conservative and the potential function is $f(x, y, z) = xyz + k$, the integral is independent of path as illustrated below.

(a) $\Big[xyz\Big]_{(0, 0, 0)}^{(1, 1, 1)} = 1$

(b) $\Big[xyz\Big]_{(0, 0, 0)}^{(0, 0, 1)} + \Big[xyz\Big]_{(0, 0, 1)}^{(1, 1, 1)} = 0 + 1 = 1$

(c) $\Big[xyz\Big]_{(0, 0, 0)}^{(1, 0, 0)} + \Big[xyz\Big]_{(1, 0, 0)}^{(1, 1, 0)} + \Big[xyz\Big]_{(1, 1, 0)}^{(1, 1, 1)} = 0 + 0 + 1 = 1$

**34.** $$\int_C 6x\,dx - 4z\,dy - (4y - 20z)\,dz = \Big[3x^2 - 4yz + 10z^2\Big]_{(0, 0, 0)}^{(4, 3, 1)} = 46$$

**36.** $\mathbf{F}(x, y) = \dfrac{2x}{y}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$ is conservative.

$$\text{Work} = \left[\frac{x^2}{y}\right]_{(-3, 2)}^{(1, 4)} = \frac{1}{4} - \frac{9}{2} = -\frac{17}{4}$$

**38.** $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Since $\mathbf{F}(x, y, z)$ is conservative, the work done in moving a particle along any path from $P$ to $Q$ is

$$f(x, y, z) = \Big[a_1x + a_2y + a_3z\Big]_{P=(p_1, p_2, p_3)}^{Q=(q_1, q_2, q_3)}$$
$$= a_1(q_1 - p_1) + a_2(q_2 - p_2) + a_3(q_3 - p_3) = \mathbf{F} \cdot \overrightarrow{PQ}.$$

**40.** $\mathbf{F} = -150\mathbf{j}$

(a) $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j},\ 0 \le t \le 50$

$d\mathbf{r} = (\mathbf{i} - \mathbf{j})\,dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{50} 150\,dt = 7500 \text{ ft} \cdot \text{lbs}$$

(b) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2\mathbf{j}$

$d\mathbf{r} = \left(\mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j}\right)dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 6\int_0^{50} (50 - t)\,dt = 7500 \text{ ft} \cdot \text{lbs}$$

**42.** $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$

(a) $M = \dfrac{y}{x^2 + y^2}$

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$N = -\dfrac{x}{x^2 + y^2}$

$$\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Thus, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

(b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \pi$

$\mathbf{F} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$

$d\mathbf{r} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} (-\sin^2 t - \cos^2 t)\,dt = \Big[-t\Big]_0^{\pi} = -\pi$$

(c) $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j},\ 0 \le t \le \pi$

$\mathbf{F} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}$

$d\mathbf{r} = (-\sin t\,\mathbf{i} - \cos t\,\mathbf{j})\,dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} (\sin^2 t + \cos^2 t)\,dt$$
$$= \Big[t\Big]_0^{\pi} = \pi$$

(d) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le 2\pi$

$\mathbf{F} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$

$d\mathbf{r} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\,dt$$
$$= \Big[-t\Big]_0^{2\pi} = -2\pi$$

This does not contradict Theorem 14.7 since $\mathbf{F}$ is not continuous at $(0, 0)$ in $R$ enclosed by curve $C$.

(e) $$\nabla\left(\arctan\frac{x}{y}\right) = \frac{1/y}{1 + (x/y)^2}\mathbf{i} + \frac{-x/y^2}{1 + (x/y)^2}\mathbf{j}$$
$$= \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j} = \mathbf{F}$$

**44.** A line integral is independent of path if $\int_C \mathbf{F} \cdot d\mathbf{r}$ does not depend on the curve joining $P$ and $Q$. See Theorem 14.6

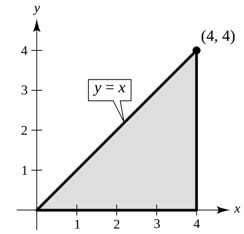
**46.** No, the amount of fuel required depends on the flight path. Fuel consumption is dependent on wind speed and direction. The vector field is not conservative.

**48.** True

**50.** False, the requirement is $\partial M/\partial y = \partial N/\partial x$.

## Section 14.4 Green's Theorem

**2.** $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \le t \le 8 \\ (12-t)\mathbf{i} + (12-t)\mathbf{j}, & 8 \le t \le 12 \end{cases}$

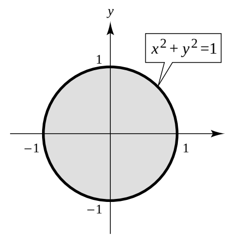



$$\int_C y^2\,dx + x^2\,dy = \int_0^4 [0\,dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16\,dt]$$

$$+ \int_8^{12} [(12-t)^2(-dt) + (12-t)^2(-dt)] = 0 + 64 - \frac{128}{3} = \frac{64}{3}$$

By Green's Theorem, $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_0^x (2x-2y)\,dy\,dx = \int_0^4 x^2\,dx = \frac{64}{3}.$

**4.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le 2\pi$




$$\int_C y^2\,dx + x^2\,dy = \int_0^{2\pi} [\sin^2 t(-\sin t\,dt) + \cos^2 t(\cos t\,dt)]$$

$$= \int_0^{2\pi} (\cos^3 t - \sin^3 t)\,dt$$

$$= \int_0^{2\pi} [\cos t(1-\sin^2 t) - \sin t(1-\cos^2 t)]\,dt$$

$$= \left[\sin t - \frac{\sin^3 t}{3} + \cos t - \frac{\cos^3 t}{3}\right]_0^{2\pi} = 0$$

By Green's Theorem,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x-2y)\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta - 2r\sin\theta)r\,dr\,d\theta = \frac{2}{3}\int_0^{2\pi} (\cos\theta - \sin\theta)\,d\theta = \frac{2}{3}(0) = 0.$$

**6.** $C$: boundary of the region lying between the graphs of $y = x$ and $y = x^3$

$$\int_C xe^y\,dx + e^x\,dy = \int_0^1 (xe^{x^3} + 3x^2e^x)\,dx + \int_1^0 (xe^x + e^x)\,dx \approx 2.936 - 2.718 \approx 0.22$$

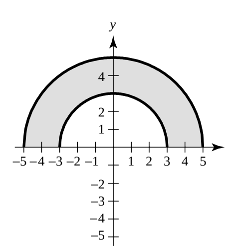
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1\int_{x^3}^{x} (e^x - xe^y)\,dy\,dx = \int_0^1 (xe^{x^3} - x^3e^x)\,dx \approx 0.22$$

**In Exercises 8 and 10, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$.**

**8.** Since $C$ is an ellipse with $a = 2$ and $b = 1$, then $R$ is an ellipse of area $\pi ab = 2\pi$. Thus, Green's Theorem yields

$$\int_C (y-x)\,dx + (2x-y)\,dy = \iint_R 1\,dA = \text{Area of ellipse} = 2\pi.$$

**10.** $R$ is the shaded region of the accompanying figure.




$$\int_C (y-x)\,dx + (2x-y)\,dy = \iint_R 1\,dA$$

$$= \text{Area of shaded region}$$

$$= \frac{1}{2}\pi[25 - 9] = 8\pi$$

**12.** The given curves intersect at (0, 0) and (9, 3). Thus, Green's Theorem yields

$$\int_C y^2\,dx + xy\,dy = \int_R\!\int (y - 2y)\,dA$$
$$= \int_0^9\int_0^{\sqrt{x}} -y\,dy\,dx = \int_0^9 \left[\frac{-y^2}{2}\right]_0^{\sqrt{x}} dx = \int_0^9 \frac{-x}{2}\,dx = \left[\frac{-x^2}{4}\right]_0^9 = -\frac{81}{4}$$

**14.** In this case, let $y = r\sin\theta$, $x = r\cos\theta$. Then $dA = r\,dr\,d\theta$ and Green's Theorem yields

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy = \int_R\!\int 4y\,dA = 4\int_0^{2\pi}\int_0^{1+\cos\theta} r\sin\theta r\,dr\,d\theta$$
$$= 4\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\sin\theta\,dr\,d\theta$$
$$= \frac{4}{3}\int_0^{2\pi} \sin\theta(1 + \cos\theta)^3\,d\theta$$
$$= \left[-\frac{(1+\cos\theta)^4}{3}\right]_0^{2\pi} = 0.$$

**16.** Since $\dfrac{\partial M}{\partial y} = -2e^x\sin 2y = \dfrac{\partial N}{\partial x}$ we have

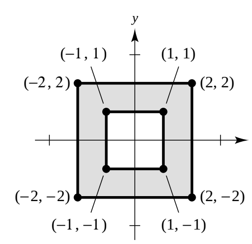
$$\int_R\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

**18.** By Green's Theorem,

$$\int_C \left(e^{-x^2/2} - y\right) dx + \left(e^{-y^2/2} + x\right) dy = \int_R\!\int 2\,dA = 2(\text{Area of } R) = 2\left[\pi(6)^2 - \pi(2)(3)\right] = 60\pi.$$

**20.** By Green's Theorem,

$$\int_C 3x^2e^y\,dx + e^y\,dy = \int_R\!\int -3x^2e^y\,dA$$
$$= \int_1^2\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_1^2 -3x^2e^y\,dy\,dx$$
$$+ \int_{-2}^{-1}\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_{-2}^{-1} -3x^2e^y\,dy\,dx$$
$$= -7(e^2 - e^{-2}) - 2(e^2 - e) - 7(e^2 - e^{-2}) - 2(e^{-1} - e^{-2})$$
$$= -16e^2 + 16e^{-2} + 2e - 2e^{-1}.$$




**22.** $\mathbf{F}(x, y) = (e^x - 3y)\mathbf{i} + (e^y + 6x)\mathbf{j}$

$C$: $r = 2\cos\theta$

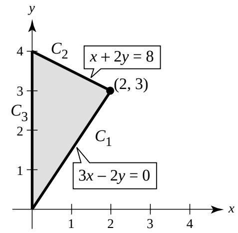
Work $= \displaystyle\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy = \int_R\!\int 9\,dA = 9\pi$ since $r = 2\cos\theta$ is a circle with a radius of one.

**24.** $\mathbf{F}(x, y) = (3x^2 + y)\mathbf{i} + 4xy^2\mathbf{j}$

$C$: boundary of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, $x = 4$

Work $= \displaystyle\int_C (3x^2 + y)\,dx + 4xy^2\,dy = \int_0^4\int_0^{\sqrt{x}} (4y^2 - 1)\,dy\,dx = \int_0^4 \left(\tfrac{4}{3}x^{3/2} - x^{1/2}\right) dx = \tfrac{176}{15}$

**26.** From the figure we see that




$$C_1\colon\ y = \frac{3}{2}x,\ dy = \frac{3}{2}\,dx,\ 0 \le x \le 2$$

$$C_2\colon\ y = -\frac{x}{2} + 4,\ dy = -\frac{1}{2}\,dx$$

$$C_3\colon\ x = 0,\ dx = 0.$$

$$A = \frac{1}{2}\int_0^2 \left(\frac{3}{2}x - \frac{3}{2}x\right)dx + \frac{1}{2}\int_2^0 \left(-\frac{1}{2}x + \frac{x}{2} - 4\right)dx + \frac{1}{2}(0)$$

$$= \frac{1}{2}\int_2^0 (-4)\,dx = 2\int_0^2 dx = 4$$

**28.** Since the loop of the folium is formed on the interval $0 \le t \le \infty$,

$$dx = \frac{3(1 - 2t^3)}{(t^3 + 1)^2}\,dt \text{ and } dy = \frac{3(2t - t^4)}{(t^3 + 1)^2}\,dt,$$

we have

$$A = \frac{1}{2}\int_0^\infty \left[\left(\frac{3t}{t^3 + 1}\right)\frac{3(2t - t^4)}{(t^3 + 1)^2} - \left(\frac{3t^2}{t^3 + 1}\right)\frac{3(1 - 2t^3)}{(t^3 + 1)^2}\right]dt$$

$$= \frac{9}{2}\int_0^\infty \frac{t^5 + t^2}{(t^3 + 1)^3}\,dt = \frac{9}{2}\int_0^\infty \frac{t^2(t^3 + 1)}{(t^3 + 1)^3}\,dt = \frac{3}{2}\int_0^\infty 3t^2(t^3 + 1)^{-2}\,dt = \left[\frac{-3}{2(t^3 + 1)}\right]_0^\infty = \frac{3}{2}.$$

**30.** See Theorem 14.9: $A = \frac{1}{2}\int_C x\,dy - y\,dx.$

**32.** (a) For the moment about the $x$-axis, $M_x = \iint_R y\,dA$. Let $N = 0$ and $M = -y^2/2$. By Green's Theorem,

$$M_x = \int_C -\frac{y^2}{2}\,dx = -\frac{1}{2}\int_C y^2\,dx \text{ and } \bar{y} = \frac{M_x}{2A} = -\frac{1}{2A}\int_C y^2\,dx.$$

For the moment about the $y$-axis, $M_y = \iint_R x\,dA$. Let $N = x^2/2$ and $M = 0$. By Green's Theorem,

$$M_y = \int_C \frac{x^2}{2}\,dy = \frac{1}{2}\int_C x^2\,dy \text{ and } \bar{x} = \frac{M_y}{2A} = \frac{1}{2A}\int_C x^2\,dy.$$

(b) By Theorem 14.9 and the fact that $x = r\cos\theta,\ y = r\sin\theta$, we have

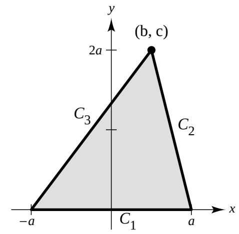
$$A = \frac{1}{2}\int x\,dy - y\,dx = \frac{1}{2}\int (r\cos\theta)(r\cos\theta)\,d\theta - (r\sin\theta)(-r\sin\theta)\,d\theta = \frac{1}{2}\int_C r^2\,d\theta.$$

**34.** Since $A =$ area of semicircle $= \frac{\pi a^2}{2}$, we have $\frac{1}{2A} = \frac{1}{\pi a^2}$. Note that $y = 0$ and $dy = 0$ along the boundary $y = 0$. Let $x = a\cos t,\ y = a\sin t,\ 0 \le t \le \pi$, then

$$\bar{x} = \frac{1}{\pi a^2}\int_0^\pi a^2\cos^2 t(a\cos t)\,dt = \frac{a}{\pi}\int_0^\pi \cos^3 t\,dt = \frac{a}{\pi}\int_0^\pi (1 - \sin^2 t)\cos t\,dt = \frac{a}{\pi}\left[\sin t - \frac{\sin^3 t}{3}\right]_0^\pi = 0$$

$$\bar{y} = \frac{-1}{\pi a^2}\int_0^\pi a^2\sin^2 t(-a\sin t\,dt) = \frac{a}{\pi}\int_0^\pi \sin^3 t\,dt = \frac{a}{\pi}\left[-\cos t + \frac{\cos^3 t}{3}\right]_0^\pi = \frac{4a}{3\pi}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4a}{3\pi}\right)$$




**36.** Since $A = \frac{1}{2}(2a)(c) = ac$, we have $\frac{1}{2A} = \frac{1}{2ac}$,

$C_1$: $y = 0$, $dy = 0$

$C_2$: $y = \dfrac{c}{b-a}(x-a)$, $dy = \dfrac{c}{b-a}\,dx$

$C_3$: $y = \dfrac{c}{b+a}(x+a)$, $dy = \dfrac{c}{b+a}\,dx$.

Thus,

$$\bar{x} = \frac{1}{2ac}\int_C x^2\,dy = \frac{1}{2ac}\left[\int_{-a}^{a} 0 + \int_a^b x^2 \frac{c}{b-a}\,dx + \int_b^{-a} x^2 \frac{c}{b+a}\,dx\right] = \frac{1}{2ac}\left[0 + \frac{2abc}{3}\right] = \frac{b}{3}$$

$$\bar{y} = \frac{-1}{2ac}\int_C y^2\,dx = \frac{-1}{2ac}\left[0 + \int_a^b \left(\frac{c}{b-a}\right)^2 (x-a)^2\,dx + \int_b^{-a}\left(\frac{c}{b+a}\right)^2 (x+a)^2\,dx\right]$$

$$= \frac{-1}{2ac}\left[\frac{c^2(b-a)}{3} - \frac{c^2(b+a)}{3}\right] = \frac{c}{3}.$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

**38.** $A = \dfrac{1}{2}\displaystyle\int_0^{\pi} a^2\cos^2 3\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi}\frac{1+\cos 6\theta}{2}\,d\theta = \frac{a^2}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi} = \frac{\pi a^2}{4}$

Note: In this case $R$ is enclosed by $r = a\cos 3\theta$ where $0 \le \theta \le \pi$.

**40.** In this case, $0 \le \theta \le 2\pi$ and we let

$$u = \frac{\sin\theta}{1+\cos\theta},\ \cos\theta = \frac{1-u^2}{1+u^2},\ d\theta = \frac{2\,du}{1+u^2}.$$

Now $u \Longrightarrow \infty$ as $\theta \Longrightarrow \pi$ and we have

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi}\frac{9}{(2-\cos\theta)^2}\,d\theta = 9\int_0^{\infty}\frac{\dfrac{2du}{1+u^2}}{4 - 4\left(\dfrac{1-u^2}{1+u^2}\right) + \dfrac{(1-u^2)^2}{(1+u^2)^2}} = 18\int_0^{\infty}\frac{1+u^2}{(1+3u^2)^2}\,du$$

$$= 18\int_0^{\infty}\frac{1/3}{1+3u^2}\,du + 18\int_0^{\infty}\frac{2/3}{(1+3u^2)^2}\,du = \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}\,u\right]_0^{\infty} + \frac{12}{\sqrt{3}}\left(\frac{1}{2}\right)\left[\frac{u}{1+3u^2} + \int\frac{\sqrt{3}}{1+3u^2}\,du\right]_0^{\infty}$$

$$= \frac{6}{\sqrt{3}}\left(\frac{\pi}{2}\right) + \frac{6}{\sqrt{3}}\left[\frac{u}{1+3u^2}\right]_0^{\infty} + \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}u\right]_0^{\infty} = \frac{3\pi}{\sqrt{3}} + 0 + \frac{3\pi}{\sqrt{3}} = 2\sqrt{3}\pi.$$

**42.** (a) Let $C$ be the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$.

$$y = \frac{y_2-y_1}{x_2-x_1}(x-x_1) + y_1$$

$$dy = \frac{y_2-y_1}{x_2-x_1}\,dx$$

$$\int_C -y\,dx + x\,dy = \int_{x_1}^{x_2}\left[-\frac{y_2-y_1}{x_2-x_1}(x-x_1) - y_1 + x\left(\frac{y_2-y_1}{x_2-x_1}\right)\right]dx = \int_{x_1}^{x_2}\left[x_1\left(\frac{y_2-y_1}{x_2-x_1}\right) - y_1\right]dx$$

$$= \left[\left[x_1\left(\frac{y_2-y_1}{x_2-x_1}\right) - y_1\right]x\right]_{x_1}^{x_2} = \left[x_1\left(\frac{y_2-y_1}{x_2-x_1}\right) - y_1\right](x_2-x_1)$$

$$= x_1(y_2-y_1) - y_1(x_2-x_1) = x_1y_2 - x_2y_1$$

**—CONTINUED—**

**42. —CONTINUED—**

(b) Let $C$ be the boundary of the region $A = \frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\int\int_R (1-(-1))\,dA = \int\int_R dA.$

Therefore,

$$\int\int_R dA = \frac{1}{2}\left[\int_{C_1} -y\,dx + x\,dy + \int_{C_2} -y\,dx + x\,dy + \cdots + \int_{C_n} -y\,dx + x\,dy\right]$$

where $C_1$ is the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$, $C_2$ is the line segment joining $(x_2, y_2)$ and $(x_3, y_3)$, . . . , and $C_n$ is the line segment joining $(x_n, y_n)$ and $(x_1, y_1)$. Thus,

$$\int\int_R dA = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)].$$

**44.** Hexagon: $(0, 0), (2, 0), (3, 2), (2, 4), (0, 3), (-1, 1)$

$A = \frac{1}{2}[(0-0) + (4-0) + (12-4) + (6-0) + (0+3) + (0-0)] = \frac{21}{2}$

**46.** Since $\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int\int_R \operatorname{div}\mathbf{F}\,dA$, then

$$\begin{aligned}\int_C fD_{\mathbf{N}}g\,ds &= \int_C f\nabla g\cdot\mathbf{N}\,ds\\ &= \int\int_R \operatorname{div}(f\nabla g)\,dA = \int\int_R (f\operatorname{div}(\nabla g) + \nabla f\cdot\nabla g)\,dA = \int\int_R (f\nabla^2 g + \nabla f\cdot\nabla g)\,dA.\end{aligned}$$

**48.** $\int_C f(x)\,dx + g(y)\,dy = \int\int_R \left[\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right]dA = \int\int_R (0-0)\,dA = 0$

## Section 14.5 Parametric Surfaces

**2.** $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\mathbf{k}$

$x^2 + y^2 = z^2$

Matches d.

**4.** $\mathbf{r}(u, v) = 4\cos u\,\mathbf{i} + 4\sin u\,\mathbf{j} + v\mathbf{k}$

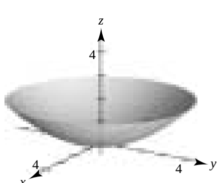
$x^2 + y^2 = 16$

Matches a.

**6.** $\mathbf{r}(u, v) = 2u\cos v\,\mathbf{i} + 2u\sin v\,\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$

$z = \frac{1}{2}u^2,\ x^2 + y^2 = 4u^2 \implies z = \frac{1}{8}(x^2 + y^2)$

Paraboloid



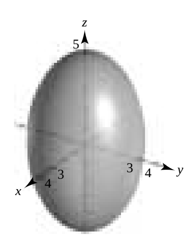

**8.** $\mathbf{r}(u, v) = 3\cos v\cos u\,\mathbf{i} + 3\cos v\sin u\,\mathbf{j} + 5\sin v\mathbf{k}$

$x^2 + y^2 = 9\cos^2 v\cos^2 u + 9\cos^2 v\sin^2 u = 9\cos^2 v$

$\frac{x^2 + y^2}{9} + \frac{z^2}{25} = \cos^2 v + \sin^2 v = 1$

$\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{25} = 1$

Ellipsoid

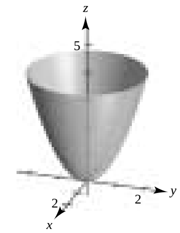


**For Exercises 10 and 12,**

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi.$$

**Eliminating the parameter yields**

$$z = x^2 + y^2,\ 0 \le z \le 4.$$




**10.** $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u^2\mathbf{j} + u \sin v\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$y = x^2 + z^2$

The paraboloid opens along the $y$-axis instead of the $z$-axis.

**12.** $\mathbf{s}(u, v) = 4u \cos v\mathbf{i} + 4u \sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

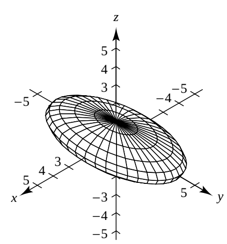
$z = \dfrac{x^2 + y^2}{16}$

The paraboloid is "wider." The top is now the circle $x^2 + y^2 = 64$. It was $x^2 + y^2 = 4$.

**14.** $\mathbf{r}(u, v) = 2 \cos v \cos u\mathbf{i} + 4 \cos v \sin u\mathbf{j} + \sin v\mathbf{k},$

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

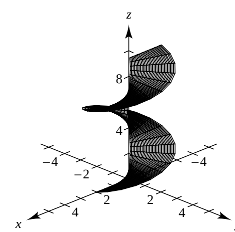
$\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{1} = 1$




**16.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k},$

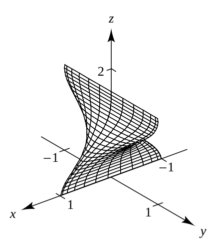
$0 \le u \le 1,\ 0 \le v \le 3\pi$

$z = \arctan\left(\dfrac{y}{x}\right)$




**18.** $\mathbf{r}(u, v) = \cos^3 u \cos v\mathbf{i} + \sin^3 u \sin v\mathbf{j} + u\mathbf{k},$

$0 \le u \le \dfrac{\pi}{2},\ 0 \le v \le 2\pi$




**20.** $z = 6 - x - y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (6 - u - v)\mathbf{k}$

**22.** $4x^2 + y^2 = 16$

$\mathbf{r}(u, v) = 2 \cos u\mathbf{i} + 4 \sin u\mathbf{j} + v\mathbf{k}$

**24.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$

$\mathbf{r}(u, v) = 3 \cos v \cos u\mathbf{i} + 2 \cos v \sin u\mathbf{j} + \sin v\mathbf{k}$

**26.** $z = x^2 + y^2$ inside $x^2 + y^2 = 9$.

$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + v^2\mathbf{k},\ 0 \le v \le 3$

**28.** Function: $y = x^{3/2},\ 0 \le x \le 4$

Axis of revolution: $x$-axis

$x = u,\ y = u^{3/2} \cos v,\ z = u^{3/2} \sin v$

$0 \le u \le 4,\ 0 \le v \le 2\pi$

**30.** Function: $z = 4 - y^2,\ 0 \le y \le 2$

Axis of revolution: $y$-axis

$x = (4 - u^2) \cos v,\ y = u,\ z = (4 - u^2) \sin v$

$0 \le u \le 2,\ 0 \le v \le 2\pi$

**32.** $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k},\ (1, 1, 1)$

$\mathbf{r}_u(u, v) = \mathbf{i} + \dfrac{v}{2\sqrt{uv}}\mathbf{k},\ \mathbf{r}_v(u, v) = \mathbf{j} + \dfrac{u}{2\sqrt{uv}}\mathbf{k}$

At $(1, 1, 1)$, $u = 1$ and $v = 1$.

$\mathbf{r}_u(1, 1) = \mathbf{i} + \dfrac{1}{2}\mathbf{k},\ \mathbf{r}_v(1, 1) = \mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$$\mathbf{N} = \mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{vmatrix} = -\tfrac{1}{2}\mathbf{i} - \tfrac{1}{2}\mathbf{j} + \mathbf{k}$$

Direction numbers: $1, 1, -2$

Tangent plane: $(x - 1) + (y - 1) - 2(z - 1) = 0$

$$x + y - 2z = 0$$

**34.** $\mathbf{r}(u, v) = 2u\cosh v\,\mathbf{i} + 2u\sinh v\,\mathbf{j} + \dfrac{1}{2}u^2\mathbf{k},$

$\mathbf{r}_u(u, v) = 2\cosh v\,\mathbf{i} + 2\sinh v\,\mathbf{j} + u\mathbf{k}$

$\mathbf{r}_v(u, v) = 2u\sinh v\,\mathbf{i} + 2u\cosh v\,\mathbf{j}$

At $(-4, 0, 2)$, $u = -2$ and $v = 0$.

$\mathbf{r}_u(-2, 0) = 2\mathbf{i} - 2\mathbf{k},\ \mathbf{r}_v(-2, 0) = -4\mathbf{j}$

$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -8\mathbf{i} - 8\mathbf{k}$

Direction numbers: $1, 0, 1$

Tangent plane: $(x + 4) + (z - 2) = 0$

$$x + z = -2$$

**36.** $\mathbf{r}(u, v) = 4u\cos v\,\mathbf{i} + 4u\sin v\,\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = 4\cos v\,\mathbf{i} + 4\sin v\,\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -4u\sin v\,\mathbf{i} + 4u\cos v\,\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos v & 4\sin v & 2u \\ -4u\sin v & 4u\cos v & 0 \end{vmatrix} = -8u^2\cos v\,\mathbf{i} - 8u^2\sin v\,\mathbf{j} + 16u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{64u^4 + 256u^2} = 8u\sqrt{u^2 + 4}$

$$A = \int_0^{2\pi}\int_0^2 8u\sqrt{u^2 + 4}\,du\,dv = \int_0^{2\pi}\left(\frac{128\sqrt{2}}{3} - \frac{64}{3}\right)dv = \frac{128\pi}{3}\left(2\sqrt{2} - 1\right)$$

**38.** $\mathbf{r}(u, v) = a\sin u\cos v\,\mathbf{i} + a\sin u\sin v\,\mathbf{j} + a\cos u\,\mathbf{k},\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a\cos u\cos v\,\mathbf{i} + a\cos u\sin v\,\mathbf{j} - a\sin u\,\mathbf{k}$

$\mathbf{r}_v(u, v) = -a\sin u\sin v\,\mathbf{i} + a\sin u\cos v\,\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -a\sin u\sin v & a\sin u\cos v & 0 \end{vmatrix} = a^2\sin^2 u\cos v\,\mathbf{i} + a^2\sin^2 u\sin v\,\mathbf{j} + a^2\sin u\cos u\,\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a^2\sin u$

$$A = \int_0^{2\pi}\int_0^{\pi} a^2\sin u\,du\,dv = 4\pi a^2$$

**40.** $\mathbf{r}(u, v) = (a + b\cos v)\cos u\,\mathbf{i} + (a + b\cos v)\sin u\,\mathbf{j} + b\sin v\,\mathbf{k},\ a > b,\ 0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = -(a + b\cos v)\sin u\,\mathbf{i} + (a + b\cos v)\cos u\,\mathbf{j}$

$\mathbf{r}_v(u, v) = -b\sin v\cos u\,\mathbf{i} - b\sin v\sin u\,\mathbf{j} + b\cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(a + b\cos v)\sin u & (a + b\cos v)\cos u & 0 \\ -b\sin v\cos u & -b\sin v\sin u & b\cos v \end{vmatrix}$$

$$= b\cos u\cos v(a + b\cos v)\mathbf{i} + b\sin u\cos v(a + b\cos v)\mathbf{j} + b\sin v(a + b\cos v)\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b\cos v)$

$$A = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v)\,du\,dv = 4\pi^2 ab$$

**42.** $\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + u\mathbf{j} + \sin u \sin v\mathbf{k},\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \cos u \cos v\mathbf{i} + \mathbf{j} + \cos u \sin v\mathbf{k}$

$\mathbf{r}_v(u, v) = -\sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = \sin u \cos v\mathbf{i} - \cos u \sin u\mathbf{j} + \sin u \sin v\mathbf{k}$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u\sqrt{1 + \cos^2 u}$

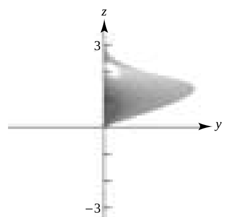
$$A = \int_0^{2\pi}\int_0^{\pi} \sin u\sqrt{1 + \cos^2 u}\, du\, dv = \pi\left[2\sqrt{2} + \ln\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right|\right]$$

**44.** See the definition, page 1055.

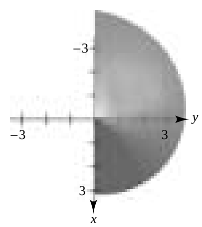
**46.** Graph of $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}$

$0 \le u \le \pi,\ 0 \le v \le \pi$ from

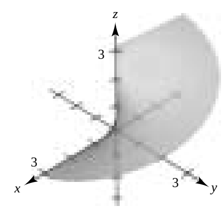
(a) $(10, 0, 0)$




(b) $(0, 0, 10)$




(c) $(10, 10, 10)$




**48.** $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k},\ 0 \le u \le 1, 0 \le v \le 3\pi$

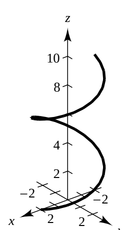
(a) If $u = 1$:

$\mathbf{r}(1, v) = 2 \cos v\mathbf{i} + 2 \sin v\mathbf{j} + v\mathbf{k}$

$x^2 + y^2 = 4$

$0 \le z \le 3\pi$

Helix



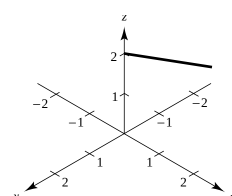

(b) If $v = \dfrac{2\pi}{3}$:

$\mathbf{r}\left(u, \dfrac{2\pi}{3}\right) = -u\mathbf{i} + \sqrt{3}u\mathbf{j} + \dfrac{2\pi}{3}\mathbf{k}$

$y = -\sqrt{3}x$

$z = \dfrac{2\pi}{3}$

Line




(c) If one parameter is held constant, the result is a **curve** in 3-space.

**50.** $x^2 + y^2 - z^2 = 1$

Let $x = u \cos v$, $y = u \sin v$, and $z = \sqrt{u^2 - 1}$. Then,

$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \dfrac{u}{\sqrt{u^2 - 1}}\mathbf{k}$

$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j}$.

At $(1, 0, 0)$, $u = 1$ and $v = 0$. $\mathbf{r}_u(1, 0)$ is undefined and $\mathbf{r}_v(1, 0) = \mathbf{j}$. The tangent plane at $(1, 0, 0)$ is $x = 1$.

**52.** $\mathbf{r}(u, v) = u\mathbf{i} + f(u)\cos v\,\mathbf{j} + f(u)\sin v\,\mathbf{k},\; a \le u \le b,\; 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + f'(u)\cos v\,\mathbf{j} + f'(u)\sin v\,\mathbf{k}$

$\mathbf{r}_v(u, v) = -f(u)\sin v\,\mathbf{j} + f(u)\cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u)\cos v & f'(u)\sin v \\ 0 & -f(u)\sin v & f(u)\cos v \end{vmatrix} = f(u)f'(u)\mathbf{i} - f(u)\cos v\,\mathbf{j} - f(u)\sin v\,\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = f(u)\sqrt{1 + [f'(u)]^2}$

$$A = \int_0^{2\pi}\int_a^b f(u)\sqrt{1 + [f'(u)]^2}\,du\,dv$$

$$= 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx \qquad (\text{since } u = x)$$

## Section 14.6 Surface Integrals

**2.** $S\colon z = 15 - 2x + 3y,\; 0 \le x \le 2,\; 0 \le y \le 4,\; \dfrac{\partial z}{\partial x} = -2,\; \dfrac{\partial z}{\partial y} = 3,\, dS = \sqrt{1 + 4 + 9}\,dy\,dx = \sqrt{14}\,dy\,dx$

$$\iint_S (x - 2y + z)\,dS = \int_0^2\int_0^4 (x - 2y + 15 - 2x + 3y)\sqrt{14}\,dy\,dx$$

$$= \sqrt{14}\int_0^2\int_0^4 (15 - x + y)\,dy\,dx = 128\sqrt{14}$$

**4.** $S\colon z = \dfrac{2}{3}x^{3/2},\; 0 \le x \le 1,\; 0 \le y \le x,\; \dfrac{\partial z}{\partial x} = x^{1/2},\; \dfrac{\partial z}{\partial y} = 0$

$$\iint_S (x - 2y + z)\,dS = \int_0^1\int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + (x^{1/2})^2 + (0)^2}\,dy\,dx$$

$$= \int_0^1\int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + x}\,dy\,dx$$

$$= \frac{2}{3}\int_0^1 x^{5/2}\sqrt{x + 1}\,dx$$

$$= \frac{2}{3}\left[\frac{1}{4}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\int_0^1 x^{3/2}\sqrt{1 + x}\,dx$$

$$= \left[\frac{1}{6}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\left(\frac{1}{3}\right)\left[x^{3/2}(1 + x)^{3/2}\right]_0^1 + \frac{5}{24}\int_0^1 x^{1/2}\sqrt{1 + x}\,dx$$

$$= \frac{\sqrt{2}}{3} - \frac{5\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{x + x^2}\,dx$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}\,dx$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\left(\frac{1}{2}\right)\left[\left(x + \frac{1}{2}\right)\sqrt{x^2 + x} - \frac{1}{4}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right|\right]_0^1$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{48}\left[\frac{3}{2}\sqrt{2} - \frac{1}{4}\ln\left|\frac{3}{2} + \sqrt{2}\right| + \frac{1}{4}\ln\left|\frac{1}{2}\right|\right]$$

$$= \frac{\sqrt{2}}{18} + \frac{15\sqrt{2}}{96} + \frac{5}{192}\ln\left|\frac{1}{3 + 2\sqrt{2}}\right| = \frac{61\sqrt{2}}{288} - \frac{5}{192}\ln\left|3 + 2\sqrt{2}\right| \approx 0.2536$$

**6.** $S$: $z = h$, $0 \le x \le 2$, $0 \le y \le \sqrt{4 - x^2}$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\iint_S dx\, dS = \int_0^2 \int_0^{\sqrt{4-x^2}} xy\, dy\, dx = \frac{1}{2}\int_0^2 x(4 - x^2)\, dx = \frac{1}{2}\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2$$

**8.** $S$: $z = \dfrac{1}{2}xy$, $0 \le x \le 4$, $0 \le y \le 4$, $\dfrac{\partial z}{\partial x} = \dfrac{1}{2}y$, $\dfrac{\partial z}{\partial y} = \dfrac{1}{2}x$

$$\iint_S xy\, dS = \int_0^4 \int_0^4 xy\sqrt{1 + \frac{y^2}{4} + \frac{x^2}{4}}\, dy\, dx = \frac{3904}{15} - \frac{160\sqrt{5}}{3}$$

**10.** $S$: $z = \cos x$, $0 \le x \le \dfrac{\pi}{2}$, $0 \le y \le \dfrac{x}{2}$

$$\iint_S (x^2 - 2xy)\, dS = \int_0^{\pi/2} \int_0^{x/2} (x^2 - 2xy)\sqrt{1 + \sin^2 x}\, dy\, dx = \int_0^{\pi/2} \frac{x^3}{4}\sqrt{1 + \sin^2 x}\, dx \approx 0.52$$

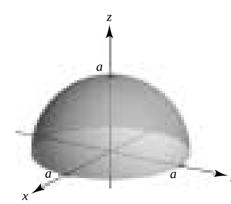
**12.** $S$: $z = \sqrt{a^2 - x^2 - y^2}$

$\rho(x, y, z) = kz$

$$m = \iint_S kz\, dS = \iint_R k\sqrt{a^2 - x^2 - y^2}\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\, dA$$

$$= \iint_R k\sqrt{a^2 - x^2 - y^2}\left(\frac{a}{\sqrt{a^2 - x^2 - y^2}}\right) dA$$

$$= \iint_R ka\, dA = ka\iint_R dA = ka(2\pi a^2) = 2ka^3\pi$$




**14.** $S$: $\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\mathbf{k}$, $0 \le u \le \dfrac{\pi}{2}$,
$0 \le v \le 2$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j}\| = 2$

$$\iint_S (x + y)\, dS = \int_0^2 \int_0^{\pi/2} (2\cos u + 2\sin u)2\, du\, dv = 16$$

**16.** $S$: $\mathbf{r}(u, v) = 4u\cos v\,\mathbf{i} + 4u\sin v\,\mathbf{j} + 3u\mathbf{k}$, $0 \le u \le 4$, $0 \le v \le \pi$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|-12u\cos v\,\mathbf{i} - 12u\sin v\,\mathbf{j} + 16u\mathbf{k}\| = 20u$

$$\iint_S (x + y)\, dS = \int_0^{\pi} \int_0^4 (4u\cos v + 4u\sin v)20u\, du\, dv = \frac{10{,}240}{3}$$

**18.** $f(x, y, z) = \dfrac{xy}{z}$

$S$: $z = x^2 + y^2$, $4 \le x^2 + y^2 \le 16$

$$\iint_S f(x, y, z)\, dS = \iint_S \frac{xy}{x^2 + y^2}\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx = \int_0^{2\pi} \int_2^4 \frac{r^2\sin\theta\cos\theta}{r^2}\sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_2^4 r\sqrt{1 + 4r^2}\sin\theta\cos\theta\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_2^4 \sin\theta\cos\theta\, d\theta$$

$$= \left[\frac{65\sqrt{65} - 17\sqrt{17}}{12}\left(\frac{\sin^2\theta}{2}\right)\right]_0^{2\pi} = 0$$

**20.** $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S$: $z = \sqrt{x^2 + y^2}$, $(x - 1)^2 + y^2 \le 1$

$$\iint_S f(x, y, z)\, dS = \iint_S \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2}\sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx$$

$$= \iint_S \sqrt{2(x^2 + y^2)}\sqrt{\frac{2(x^2 + y^2)}{x^2 + y^2}}\, dy\, dx$$

$$= 2\iint_S \sqrt{x^2 + y^2}\, dy\, dx = 2\int_0^{\pi}\int_0^{2\cos\theta} r^2\, dr\, d\theta$$

$$= \frac{16}{3}\int_0^{\pi} \cos^3\theta\, d\theta = \frac{16}{3}\int_0^{\pi}(1 - \sin^2\theta)\cos\theta\, d\theta$$

$$= \left[\frac{16}{3}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\right]_0^{\pi} = 0$$

**22.** $f(x, y, z) = x^2 + y^2 + z^2$

$S$: $x^2 + y^2 = 9$, $0 \le x \le 3$, $0 \le z \le x$

Project the solid onto the $xz$-plane; $y = \sqrt{9 - x^2}$.

$$\iint_S f(x, y, z)\, dS = \int_0^3\int_0^x \left[x^2 + (9 - x^2) + z^2\right]\sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}}\right)^2 + (0)^2}\, dz\, dx$$

$$= \int_0^3\int_0^x (9 + z^2)\frac{3}{\sqrt{9 - x^2}}\, dz\, dx = \int_0^3\left[\frac{3}{\sqrt{9 - x^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^x dx$$

$$= \int_0^3 \frac{3}{\sqrt{9 - x^2}}\left(9x + \frac{x^3}{3}\right) dx = \int_0^3 27x(9 - x^2)^{-1/2}\, dx + \int_0^3 x^3(9 - x^2)^{-1/2}\, dx$$

Let $u = x^2$, $dv = x(9 - x^2)^{-1/2}\, dx$, then $du = 2x\, dx$, $v = -\sqrt{9 - x^2}$.

$$= \left[-27\sqrt{9 - x^2}\right]_0^3 + \left[\left[-x^2\sqrt{9 - x^2}\right]_0^3 + \int_0^3 2x\sqrt{9 - x^2}\, dx\right]$$

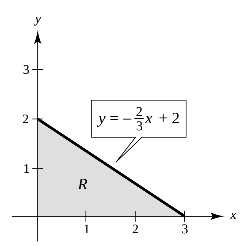
$$= \left[81 - \frac{2}{3}(9 - x^2)^{3/2}\right]_0^3 = 81 + 18 = 99$$

**24.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

$S$: $2x + 3y + z = 6$ (first octant)

$G(x, y, z) = 2x + 3y + z - 6$

$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

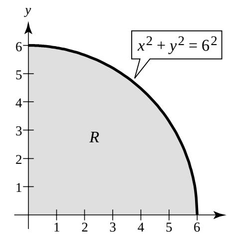



$$\iint_S \mathbf{F}\cdot\mathbf{N}\, dS = \iint_R \mathbf{F}\cdot\nabla G\, dA = \int_0^3\int_0^{-(2x/3)+2} (2x + 3y)\, dy\, dx$$

$$= \int_0^3\left[-\frac{4}{3}x^2 + 4x + \frac{3}{2}\left(-\frac{2}{3}x + 2\right)^2\right] dx$$

$$= \left[-\frac{4}{9}x^3 + 2x^2 - \frac{3}{4}\left(-\frac{2}{3}x + 2\right)^3\right]_0^3 = 12$$

**26.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$




$S$: $x^2 + y^2 + z^2 = 36$ (first octant)

$z = \sqrt{36 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{36 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{36 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{36 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

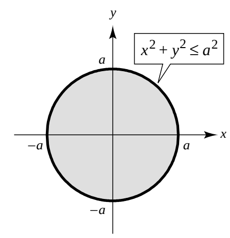
$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{36 - x^2 - y^2}} + \frac{y^2}{\sqrt{36 - x^2 - y^2}} + z = \frac{36}{\sqrt{36 - x^2 - y^2}}$$

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot \nabla G\, dA = \int_R\int \frac{36}{\sqrt{36 - x^2 - y^2}}\, dA$$

$$= \int_0^{\pi/2}\int_0^6 \frac{36}{\sqrt{36 - r^2}}\, r\, dr\, d\theta \quad \text{(improper)}$$

$$= 108\pi$$

**28.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$




$S$: $z = \sqrt{a^2 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{a^2 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} + \frac{y^2}{\sqrt{a^2 - x^2 - y^2}} - 2\sqrt{a^2 - x^2 - y^2} = \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}$$

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot \nabla G\, dA = \int_R\int \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^a \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\, dr\, d\theta - 2a^2\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= 3\left[\int_0^{2\pi}\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta\right] - 2a^2\int_0^{2\pi}\left[-\sqrt{a^2 - r^2}\right]_0^a d\theta$$

$$= 3\int_0^{2\pi} \frac{2}{3}a^3\, d\theta - 2a^2\int_0^{2\pi} a\, d\theta = 0$$

**30.** $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S$: $z = 1 - x^2 - y^2,\ z = 0$

$G(x, y, z) = z + x^2 + y^2 - 1$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$\mathbf{F} \cdot \nabla G = 2x(x + y) + 2y(y) + (1 - x^2 - y^2) = x^2 + 2xy + y^2 + 1$

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot \nabla G\, dA = \int_R\int (x^2 + 2xy + y^2 + 1)\, dA$$

$$= \int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos\theta\sin\theta + 1)r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{3}{4} + \frac{1}{2}\sin\theta\cos\theta\right)d\theta = \left[\frac{3}{4}\theta + \frac{\sin^2\theta}{4}\right]_0^{2\pi} = \frac{3\pi}{2}$$

The flux across the bottom $z = 0$ is zero.

**32.** A surface is orientable if a unit normal vector $N$ can be defined at every nonboundary point of $S$ in such a way that the normal vectors vary continuously over the surface $S$.

**34.** Orientable

**36.** $\mathbf{E} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$

$S$: $z = \sqrt{1 - x^2 - y^2}$

$$\iint_S \mathbf{E} \cdot \mathbf{N}\, dS = \iint_R \mathbf{E} \cdot (-g_x(x, y)\,\mathbf{i} - g_y(x, y)\,\mathbf{j} + \mathbf{k})\, dA$$

$$= \iint_R (yz\,\mathbf{i} + xz\,\mathbf{j} + xy\mathbf{k}) \cdot \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \iint_R \left(\frac{2xyz}{\sqrt{1 - x^2 - y^2}} + xy\right) dA = \iint_R 3xy\, dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3xy\, dy\, dx = 0$$

**38.** $x^2 + y^2 + z^2 = a^2$

$z = \pm\sqrt{a^2 - x^2 - y^2}$

$$m = 2\iint_S k\, dS = 2k\iint_R \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\, dA$$

$$= 2k\iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = 2ka\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= 2ka\Big[-\sqrt{a^2 - r^2}\Big]_0^a (2\pi) = 4\pi ka^2$$

$$I_z = 2\iint_S k(x^2 + y^2)\, dS$$

$$= 2k\iint_R (x^2 + y^2)\frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = 2ka\int_0^{2\pi}\int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\, dr\, d\theta \text{ (use integration by parts)}$$

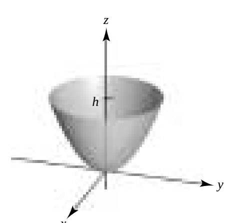
$$= 2ka\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a (2\pi)$$

$$= 2ka\left(\frac{2}{3}a^3\right)(2\pi) = \frac{2}{3}a^2(4\pi ka^2) = \frac{2}{3}a^2 m$$

Let $u = r^2,\ dv = r(a^2 - r^2)^{-1/2}\, dr,\ du = 2r\, dr,\ v = -\sqrt{a^2 - r^2}$.

**40.** $z = x^2 + y^2,\ 0 \le z \le h$

Project the solid onto the $xy$-plane.




$$I_z = \iint_S (x^2 + y^2)(1)\, dS$$

$$= \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{\sqrt{h}} r^2\sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= 2\pi\left[\frac{h}{12}(1 + 4h)^{3/2} - \frac{1}{120}(1 + 4h)^{5/2}\right] + \frac{2\pi}{120}$$

$$= \frac{(1 + 4h)^{3/2}\pi}{60}[10h - (1 + 4h)] + \frac{\pi}{60} = \frac{\pi}{60}[(1 + 4h)^{3/2}(6h - 1) + 1]$$

**42.** $S$: $z = \sqrt{16 - x^2 - y^2}$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

$$\int_S\int \rho\mathbf{F}\cdot\mathbf{N}\,dS = \int_R\int \rho\mathbf{F}\cdot(-g_x(x,y)\mathbf{i} - g_y(x,y)\mathbf{j} + \mathbf{k})\,dA$$

$$= \int_R\int 0.5\rho z\mathbf{k}\cdot\left[\frac{x}{\sqrt{16-x^2-y^2}}\mathbf{i} + \frac{y}{\sqrt{16-x^2-y^2}}\mathbf{j} + \mathbf{k}\right]dA$$

$$= \int_R\int 0.5\,\rho z\,dA = \int_R\int 0.5\rho\sqrt{16-x^2-y^2}\,dA$$

$$= 0.5\rho\int_0^{2\pi}\int_0^4 \sqrt{16-r^2}\,r\,dr\,d\theta = 0.5\rho\int_0^{2\pi}\frac{64}{3}\,d\theta = \frac{64\pi\rho}{3}$$

## Section 14.7 Divergence Theorem

**2. Surface Integral:** There are three surfaces to the cylinder.

Bottom: $z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F}\cdot\mathbf{N} = -z^2$

$$\int_{S_1}\int 0\,dS = 0$$

Top: $z = h$, $\mathbf{N} = \mathbf{k}$, $\mathbf{F}\cdot\mathbf{N} = z^2$

$$\int_{S_2}\int h^2\,dS = h^2\,(\text{Area of circle}) = 4\pi h^2$$

Side: $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}$, $0 \le u \le 2\pi$, $0 \le v \le h$

$$\mathbf{r}_u = -2\sin u\mathbf{i} + 2\cos u\mathbf{j},\ r_v = \mathbf{k}$$

$$\mathbf{r}_u\times\mathbf{r}_v = 2\cos u\mathbf{i} + 2\sin u\mathbf{j}$$

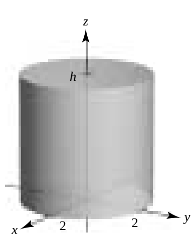
$$\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v) = 8\cos^2 u - 8\sin^2 u$$

$$\int_{S_3}\int \mathbf{F}\cdot\mathbf{N}\,dS = \int_0^h\int_0^{2\pi}(8\cos^2 u - 8\sin^2 u)\,du\,dv = 0$$

Therefore, $\displaystyle\int_S\int \mathbf{F}\cdot\mathbf{N}\,dS = 0 + 4\pi h^2 + 0 = 4\pi h^2$.

**Divergence Theorem:** $\operatorname{div}\mathbf{F} = 2 - 2 + 2z = 2z$

$$\iiint_Q 2z\,dV = \int_0^{2\pi}\int_0^2\int_0^h 2zr\,dz\,dr\,d\theta = 4\pi h^2.$$



**4.** $\mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$

$S$: surface bounded by the planes $y = 4$, $z = 4 - x$ and the coordinate planes

**Surface Integral:** There are five surfaces to this solid.

$z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = -(x + y)$

$$\int_{S_1}\int -(x + y)\, dS = \int_0^4\int_0^4 -(x + y)\, dy\, dx = -\int_0^4 (4x + 8)\, dx = -64$$

$y = 0$, $\mathbf{N} = -\mathbf{j}$, $\mathbf{F} \cdot \mathbf{N} = -z$

$$\int_{S_2}\int -z\, dS = \int_0^4\int_0^{4-x} -z\, dz\, dx = -\int_0^4 \frac{(4 - x)^2}{2}\, dx = -\frac{32}{3}$$

$y = 4$, $\mathbf{N} = \mathbf{j}$, $\mathbf{F} \cdot \mathbf{N} = z$

$$\int_{S_3}\int z\, dS = \int_0^4\int_0^{4-x} z\, dz\, dx = \int_0^4 \frac{(4 - x)^2}{2}\, dx = \frac{32}{3}$$

$x = 0$, $\mathbf{N} = -\mathbf{i}$, $\mathbf{F} \cdot \mathbf{N} = -xy$

$$\int_{S_4}\int -xy\, dS = \int_0^4\int_0^4 0\, dS = 0$$

$x + z = 4$, $\mathbf{N} = \dfrac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}$, $\mathbf{F} \cdot \mathbf{N} = \dfrac{1}{\sqrt{2}}[xy + x + y]$, $dS = \sqrt{2}\, dA$

$$\int_{S_5}\int \frac{1}{\sqrt{2}}[xy + x + y]\sqrt{2}\, dA = \int_0^4\int_0^4 (xy + x + y)\, dy\, dx = 128$$

Therefore, $\displaystyle\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = -64 - \tfrac{32}{3} + \tfrac{32}{3} + 0 + 128 = 64.$

**Divergence Theorem:** Since div $\mathbf{F} = y$, we have

$$\iiint_Q \text{div}\, \mathbf{F}\, dV = \int_0^4\int_0^4\int_0^{4-x} y\, dz\, dy\, dx = 64.$$

**6.** Since div $\mathbf{F} = 2xz^2 - 2 + 3xy$ we have

$$\begin{aligned}\iiint_Q \text{div}\, \mathbf{F}\, dV &= \int_0^a\int_0^a\int_0^a (2xz^2 - 2 + 3xy)\, dz\, dy\, dx = \int_0^a\int_0^a \left(\frac{2}{3}xa^3 - 2a + 3xya\right) dy\, dx \\ &= \int_0^a \left(\frac{2}{3}xa^4 - 2a^2 + \frac{3}{2}xa^3\right) dx \\ &= \frac{1}{3}a^6 - 2a^3 + \frac{3}{4}a^5.\end{aligned}$$

**8.** Since div $\mathbf{F} = y + z - y = z$, we have

$$\begin{aligned}\iiint_Q \text{div}\, \mathbf{F}\, dV &= \int_{-a}^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} z\, dz\, dy\, dx = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} zr\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^a \left[\frac{a^2r}{2} - \frac{r^3}{2}\right] dr\, d\theta = \int_0^{2\pi}\left[\frac{a^2r^2}{4} - \frac{r^4}{8}\right]_0^a d\theta = \int_0^{2\pi} \frac{a^4}{8}\, d\theta = \frac{\pi a^4}{4}.\end{aligned}$$

**10.** Since div $\mathbf{F} = xz$, we have

$$\iiint_Q xz\, dV = \int_0^4\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xz\, dx\, dy\, dz = \int_0^4\int_{-3}^3 \frac{z}{2}(0)\, dy\, dz = 0.$$

**12.** Since div $\mathbf{F} = y^2 + x^2 + e^z$, we have

$$\iiint\limits_Q (x^2 + y^2 + e^z)\, dV = \int_0^{16}\int_{-\sqrt{256-x^2}}^{\sqrt{256-x^2}}\int_{(1/2)\sqrt{x^2+y^2}}^{8} (x^2 + y^2 + e^z)\, dz\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{16}\int_{r/2}^{8} (r^2 + e^z)r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{16}\left(8r^3 + re^8 - \frac{1}{2}r^4 - re^{r/2}\right) dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{131{,}052}{5} + 100e^8\right) d\theta = \frac{262{,}104}{5}\pi + 200e^8\pi$$

**14.** Since div $\mathbf{F} = e^z + e^z + e^z = 3e^z$, we have

$$\iiint\limits_Q 3e^z\, dV = \int_0^6\int_0^4\int_0^{4-y} 3e^z\, dz\, dy\, dx = \int_0^6\int_0^4 3\left[e^{4-y} - 1\right] dy\, dx = \int_0^6 3(e^4 - 5)\, dx = 18(e^4 - 5).$$

**16.** div $\mathbf{F} = 2$

$$\iint\limits_S \mathbf{F}\cdot\mathbf{N}\, dS = \iiint\limits_Q \text{div}\,\mathbf{F}\, dV = \iiint\limits_Q 2\, dV.$$

The surface $S$ is the upper half of a hemisphere of radius 2. Since the volume is $\frac{1}{2}\left(\frac{4}{3}\pi(2^3)\right) = 16\pi/3$, you have

$$\iint\limits_S \mathbf{F}\cdot\mathbf{N}\, dS = 2(\text{Volume}) = \frac{32\pi}{3}.$$

**18.** Using the Divergence Theorem, we have

$$\iint\limits_S \mathbf{curl\ F}\cdot\mathbf{N}\, dS = \iiint\limits_Q \text{div}\,(\mathbf{curl F})\, dV$$

$$\mathbf{curl\ F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy\cos z & yz\sin x & xyz \end{vmatrix} = (xz - y\sin x)\mathbf{i} - (yz + xy\sin z)\mathbf{j} + (yz\cos x - x\cos z)\mathbf{k}.$$

Now, div $\mathbf{curl\ F}(x, y, z) = (z - y\cos x) - (z + x\sin z) + (y\cos x + x\sin z) = 0$. Therefore,

$$\iint\limits_S \mathbf{curl\ F}\cdot\mathbf{N}\, dS = \iiint\limits_Q \text{div}\,(\mathbf{curl\ F})\, dV = 0.$$

**20.** If div $\mathbf{F}(x, y, z) > 0$, then source.

If div $\mathbf{F}(x, y, z) < 0$, then sink.

If div $\mathbf{F}(x, y, z) = 0$, then incompressible.

**22.** $v = \int_0^a\int_0^a x\, dy\, dz = \int_0^a\int_0^a a\, dy\, dz = \int_0^a a^2\, dz = a^3$

Similarly, $\int_0^a\int_0^a y\, dz\, dx = \int_0^a\int_0^a z\, dx\, dy = a^3$.

**24.** If $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then div $\mathbf{F} = 0$.
Therefore,

$$\iint\limits_S \mathbf{F}\cdot\mathbf{N}\, dS = \iiint\limits_Q \text{div}\,\mathbf{F}\, dV = \iiint\limits_Q 0\, dV = 0.$$

**26.** If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then div $\mathbf{F} = 3$.

$$\frac{1}{\|\mathbf{F}\|}\iint\limits_S \mathbf{F}\cdot\mathbf{N}\, dS = \frac{1}{\|\mathbf{F}\|}\iiint\limits_Q \text{div}\,\mathbf{F}\, dV = \frac{1}{\|\mathbf{F}\|}\iiint\limits_Q 3\, dV = \frac{3}{\|\mathbf{F}\|}\iiint\limits_Q dV$$

**28.** $$\iint\limits_S (fD_{\mathbf{N}}g - gD_{\mathbf{N}}f)\, dS = \iint\limits_S fD_{\mathbf{N}}g\, dS - \iint\limits_S gD_{\mathbf{N}}f\, dS$$

$$= \iiint\limits_Q (f\nabla^2 g + \nabla f\cdot\nabla g)\, dV - \iiint\limits_Q (g\nabla^2 f + \nabla g\cdot\nabla f)\, dV = \iiint\limits_Q (f\nabla^2 g - g\nabla^2 f)\, dV$$

## Section 14.8 Stokes's Theorem

**2.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & x^2 \end{vmatrix} = -2x\mathbf{j}$$

**4.** $\mathbf{F}(x, y, z) = x \sin y\mathbf{i} - y \cos x\mathbf{j} + yz^2\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x \sin y & -y \cos x & yz^2 \end{vmatrix}$$
$$= z^2\mathbf{i} + (y \sin x - x \cos y)\mathbf{k}$$

**6.** $\mathbf{F}(x, y, z) = \arcsin y\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{j} + y^2\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \arcsin y & \sqrt{1 - x^2} & y^2 \end{vmatrix}$$
$$= 2y\mathbf{i} + \left[\frac{-x}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - y^2}}\right]\mathbf{k}$$
$$= 2y\mathbf{i} - \left[\frac{x}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - y^2}}\right]\mathbf{k}$$

**8.** In this case $C$ is the circle $x^2 + y^2 = 4,\ z = 0,\ dz = 0$.

**Line Integral:** $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y\,dx + x\,dy$

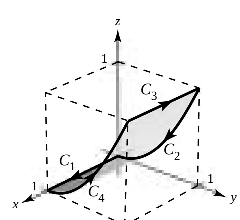
Let $x = 2 \cos t,\ y = 2 \sin t,$ then $dx = -2 \sin t\,dt,\ dy = 2 \cos t\,dt,$ and $\displaystyle\int_C -y\,dx + x\,dy = \int_0^{2\pi} 4\,dt = 8\pi.$

**Double Integral:** $F(x, y, z) = z + x^2 + y^2 - 4,\ \mathbf{N} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}},\ dS = \sqrt{1 + 4x^2 + 4y^2}\,dA$

$\mathbf{curl\ F} = 2\mathbf{k}$, therefore

$$\int\!\!\int (\mathbf{curl\ F}) \cdot \mathbf{N}dS = \int_R\!\!\int 2\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\,dy\,dx = 2\int_{-2}^{2} 2\sqrt{4 - x^2}\,dx$$
$$= 4\int_{-2}^{2} \sqrt{4 - x^2}\,dx = 2\left[x\sqrt{4 - x^2} + 4 \arcsin\frac{x}{2}\right]_{-2}^{2} = 8\pi.$$

**10. Line Integral:** From the accompanying figure we see that for




$C_1$: $y = 0,\ z = 0,\ dy = dz = 0$

$C_2$: $z = y^2,\ x = 0,\ dx = 0,\ dz = 2y\,dy$

$C_3$: $y = a,\ z = a^2,\ dy = dz = 0$

$C_4$: $z = y^2,\ x = a,\ dx = 0,\ dz = 2y\,dy.$

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C z^2\,dx + x^2\,dy + y^2\,dz$$
$$= \int_{C_1} 0 + \int_{C_2} 2y^3\,dy + \int_{C_3} a^4\,dx + \int_{C_4} a^2\,dy + y^2(2y)\,dy$$
$$= \int_0^a 2y^3\,dy + \int_0^a a^4\,dx + \int_a^0 a^2\,dy + \int_a^0 2y^3\,dy = \Big[a^4x\Big]_0^a + \Big[a^2y\Big]_a^0 = a^5 - a^3 = a^3(a^2 - 1).$$

—CONTINUED—

**10. —CONTINUED—**

**Double Integral:** Since $\mathbf{F}(x, y, z) = y^2 - z$, we have

$$\mathbf{N} = \frac{2y\mathbf{j} - \mathbf{k}}{\sqrt{1 + 4y^2}} \text{ and } dS = \sqrt{1 + 4y^2}\, dA.$$

Furthermore, $\mathbf{curl\ F} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$. Therefore,

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int (4yz - 2x)\, dA = \int_0^a\int_0^a (4y^2 - 2x)\, dy\, dx = \int_0^a (a^4 - 2ax)\, dx = \Big[a^4x - ax^2\Big]_0^a = a^3(a^2 - 1).$$

**12.** Let $A = (0, 0, 0)$, $B = (1, 1, 1)$, and $C = (0, 0, 2)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{k}$, and

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{2\mathbf{i} - 2\mathbf{j}}{2\sqrt{2}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

Hence, $F(x, y, z) = x - y$ and $dS = \sqrt{2}\, dA$. Since $\mathbf{curl\ F} = \dfrac{2x}{x^2 + y^2}\mathbf{k}$, we have $\displaystyle\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int 0\, dS = 0.$

**14.** $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}$, $S$: $9 - x^2 - y^2$, $z \le 0$

$\mathbf{curl\ F} = 4x\mathbf{i} + (4x - 4y)\mathbf{j}$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned}\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\int \left[8x^2 + 2y(4x - 4y)\right] dA \\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[8x^2 + 8xy - 8y^2\right] dy\, dx \\ &= \int_{-3}^{3}\left(16x^2\sqrt{9 - x^2} - \frac{16}{3}(9 - x^2)^{3/2}\right) dx = 0\end{aligned}$$

**16.** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}$, $S$: $z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz - 2z)\mathbf{i} + yz\mathbf{j}$$

$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\int \left[\frac{-z(x + 2)x}{\sqrt{4 - x^2 - y^2}} + \frac{y^2z}{\sqrt{4 - x^2 - y^2}}\right] dA \\ &= \int_R\int \left[-x(x + 2) + y^2\right] dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-x^2 - 2x + y^2)\, dy\, dx \\ &= \int_{-2}^{2}\left[-x^2y - 2xy + \frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= \int_{-2}^{2}\left[-2x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{2}{3}(4 - x^2)\sqrt{4 - x^2}\right] dx \\ &= \int_{-2}^{2}\left[-\frac{8}{3}x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{8}{3}\sqrt{4 - x^2}\right] dx \\ &= \left[-\frac{8}{3}\left(\frac{1}{8}\right)\left[x(2x^2 - 4)\sqrt{4 - x^2} + 16 \arcsin\frac{x}{2}\right] + \frac{4}{3}(4 - x^2)^{3/2} + \frac{8}{3}\left(\frac{1}{2}\right)\left[x\sqrt{4 - x^2} + 4 \arcsin\frac{x}{2}\right]\right]_{-2}^{2} \\ &= \left[\left(-\frac{1}{3}\right)(8\pi) + \frac{4}{3}(2\pi) + \frac{1}{3}(-8\pi) - \frac{4}{3}(-2\pi)\right] = 0\end{aligned}$$

**18.** $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 2 - 3y & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} + (y - 2x)\mathbf{j} - z\mathbf{k}$$

$S$: the first octant portion of $x^2 + z^2 = 16$ over $x^2 + y^2 = 16$

$G(x, y, z) = z - \sqrt{16 - x^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{16 - x^2}}\mathbf{i} + \mathbf{k}$$

$$\begin{aligned}
\int_S\!\!\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int \left[\frac{2xy}{\sqrt{16 - x^2}} - z\right] dA \\
&= \int_R\!\!\int \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right] dA \\
&= \int_0^4 \int_0^{\sqrt{16 - x^2}} \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right] dy\, dx \\
&= \int_0^4 \left[\frac{x}{\sqrt{16 - x^2}}y^2 - \sqrt{16 - x^2}\,y\right]_0^{\sqrt{16 - x^2}} dx \\
&= \int_0^4 \left[x\sqrt{16 - x^2} - (16 - x^2)\right] dx \\
&= \left[-\frac{1}{3}(16 - x^2)^{3/2} - 16x + \frac{x^3}{3}\right]_0^4 \\
&= \left(-64 + \frac{64}{3}\right) - \left(-\frac{64}{3}\right) = -\frac{64}{3}
\end{aligned}$$

**20.** $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$S$: the first octant portion of $z = x^2$ over $x^2 + y^2 = a^2$. We have

$$\mathbf{N} = \frac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}} \text{ and } dS = \sqrt{1 + 4x^2}\, dA.$$

$$\begin{aligned}
\int_S\!\!\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int xz\, dA = \int_R\!\!\int x^3\, dA \\
&= \int_0^a \int_0^{\sqrt{a^2 - x^2}} x^3\, dy\, dx \\
&= \int_0^a x^3\sqrt{a^2 - x^2}\, dx \\
&= \left[-\frac{1}{3}x^2(a^2 - x^2)^{3/2} - \frac{2}{15}(a^2 - x^2)^{5/2}\right]_0^a \\
&= \frac{2}{15}a^5
\end{aligned}$$

**22.** $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

$S: x^2 + y^2 = 1$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -z & 0 & y \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

Letting $\mathbf{N} = \mathbf{k}$, $\mathbf{curl\ F} \cdot \mathbf{N} = 0$ and $\displaystyle\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = 0.$

**24.** **curl F** measures the rotational tendency.

See page 1084.

**26.** $f(x, y, z) = xyz,\ g(x, y, z) = z,\ S: z = \sqrt{4 - x^2 - y^2}$

(a) $\nabla g(x, y, z) = \mathbf{k}$

$f(x, y, z)\nabla g(x, y, z) = xyz\mathbf{k}$

$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 0\mathbf{k},\ \ 0 \le t \le 2\pi$

$$\int_C [f(x, y, z)\nabla g(x, y, z)] \cdot d\mathbf{r} = 0$$

(b) $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\nabla g(x, y, z) = \mathbf{k}$

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & xz & xy \\ 0 & 0 & 1 \end{vmatrix} = xz\mathbf{i} - yz\mathbf{j}$$

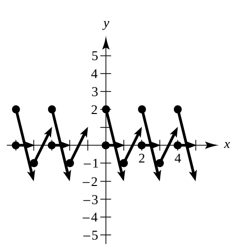
$$\mathbf{N} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}}\right)^2}\, dA = \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA$$

$$\begin{aligned} \int_S\int [\nabla f(x, y, z) \times \nabla g(x, y, z)] \cdot \mathbf{N}\, dS &= \int_S\int \left[\frac{x^2 z}{\sqrt{4 - x^2 - y^2}} - \frac{y^2 z}{\sqrt{4 - x^2 - y^2}}\right] \frac{2}{\sqrt{4 - x^2 - y^2}} dA \\ &= \int_S\int \frac{2(x^2 - y^2)}{\sqrt{4 - x^2 - y^2}}\, dA \\ &= \int_0^2\int_0^{2\pi} \frac{2r^2(\cos^2\theta - \sin^2\theta)}{\sqrt{4 - r^2}} r\, d\theta\, dr \\ &= \int_0^2 \left[\frac{2r^3}{\sqrt{4 - r^2}}\left(\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} dr = 0 \end{aligned}$$

## Review Exercises for Chapter 14

**2.** $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$




**4.** $f(x, y, z) = x^2 e^{yz}$

$$\begin{aligned} \mathbf{F}(x, y, z) &= 2xe^{yz}\,\mathbf{i} + x^2 z e^{yz}\,\mathbf{j} + x^2 y e^{yz}\,\mathbf{k} \\ &= xe^{yz}(2\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \end{aligned}$$

**6.** Since $\partial M/\partial y = -1/x^2 = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -y/x^2$ and $N = \partial U/\partial y = 1/x$, partial integration yields $U = (y/x) + h(y)$ and $U = (y/x) + g(x)$ which suggests that $U(x, y) = (y/x) + C$.

**8.** Since $\partial M/\partial y = -6y^2 \sin 2x = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -2y^3 \sin 2x$ and $N = \partial U/\partial y = 3y^2(1 + \cos 2x)$, we obtain $U = y^3 \cos 2x + h(y)$ and $U = y^3(1 + \cos 2x) + g(x)$ which suggests that $h(y) = y^3$, $g(x) = C$, and $U(x, y) = y^3(1 + \cos 2x) + C$.

**10.** Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = 6y \neq \frac{\partial P}{\partial y}$$

$\mathbf{F}$ is not conservative.

**12.** Since

$$\frac{\partial M}{\partial y} = \sin z = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = y \cos z \neq \frac{\partial P}{\partial x},$$

$\mathbf{F}$ is not conservative.

**14.** Since $\mathbf{F} = xy^2\mathbf{j} - zx^2\mathbf{k}$;

(a) $\text{div}\,\mathbf{F} = 2xy - x^2$

(b) $\textbf{curl}\,\mathbf{F} = 2xz\,\mathbf{j} + y^2\mathbf{k}$

**16.** Since $\mathbf{F} = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = 3 + 1 + 1 = 5$

(b) $\textbf{curl}\,\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

**18.** Since $\mathbf{F} = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$:

(a) $\text{div}\,\mathbf{F} = 2x - 2 \sin y \cos y$

(b) $\textbf{curl}\,\mathbf{F} = \mathbf{0}$

**20.** Since $\mathbf{F} = \dfrac{z}{x}\mathbf{i} + \dfrac{z}{y}\mathbf{j} + z^2\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = -\dfrac{z}{x^2} - \dfrac{z}{y^2} + 2z = z\left(2 - \dfrac{1}{x^2} - \dfrac{1}{y^2}\right)$

(b) $\textbf{curl}\,\mathbf{F} = -\dfrac{1}{y}\mathbf{i} + \dfrac{1}{x}\mathbf{j}$

**22.** (a) Let $x = 5t$, $y = 4t$, $0 \le t \le 1$, then $ds = \sqrt{41}\,dt$.

$$\int_C xy\,ds = \int_0^1 20t^2\sqrt{41}\,dt = \frac{20\sqrt{41}}{3}$$

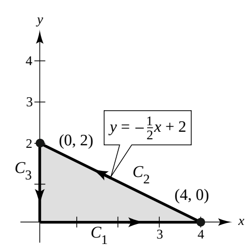
(b) $C_1$: $x = t$, $y = 0$, $0 \le t \le 4$, $ds = dt$

$C_2$: $x = 4 - 4t$, $y = 2t$, $0 \le t \le 1$, $ds = 2\sqrt{5}\,dt$

$C_3$: $x = 0$, $y = 2 - t$, $0 \le t \le 2$, $ds = dt$

Therefore, $\displaystyle\int_C xy\,ds = \int_0^4 0\,dt = \int_0^1 (8t - 8t^2)2\sqrt{5}\,dt + \int_0^2 0\,dt$

$$= 16\sqrt{5}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{8\sqrt{5}}{3}.$$




**24.** $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$, $\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = \sin t$

$$\int_C x\,ds = \int_0^{2\pi} (t - \sin t)\sqrt{(1 - \cos t)^2 + (\sin t)^2}\,dt = \int_0^{2\pi} (t - \sin t)\sqrt{2 - 2\cos t}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \left[t\sqrt{1 - \cos t} - \sin t\sqrt{1 - \cos t}\right] dt = \sqrt{2}\left[-\frac{2}{3}(1 - \cos t)^{3/2}\right]_0^{2\pi} + \sqrt{2}\int_0^{2\pi} t\sqrt{1 - \cos t}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} t\sqrt{1 - \cos t}\,dt$$

$$= 8\pi$$

**26.** $x = \cos t + t \sin t,\ y = \sin t - t \sin t,\ 0 \le t \le \dfrac{\pi}{2},\ dx = t \cos t\, dt,\ dy = (\cos t - t \cos t - \sin t)\, dt$

$$\int_C (2x - y)\, dx + (x + 3y)\, dy = \int_0^{\pi/2} \left[\sin t \cos t(5t^2 - 6t + 2) + \cos^2 t(t + 1) + \sin^2 t(2t - 3)\right] dt \approx 1.01$$

**28.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k},\ 0 \le t \le 4$

$x'(t) = 1,\ y'(t) = 2t,\ z'(t) = \dfrac{3}{2}t^{1/2}$

$$\int_C (x^2 + y^2 + z^2)\, ds = \int_0^4 (t^2 + t^4 + t^3)\sqrt{1 + 4t^2 + \frac{9}{4}t}\ dt \approx 2080.59$$

**30.** $f(x, y) = 12 - x - y$

$C$: $y = x^2$ from $(0, 0)$ to $(2, 4)$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$

Lateral surface area:

$$\int_C f(x, y)\, ds = \int_0^2 (12 - t - t^2)\sqrt{1 + 4t^2}\, dt \approx 41.532$$

**32.** $d\mathbf{r} = \left[(-4 \sin t)\mathbf{i} + 3 \cos t\,\mathbf{j}\right] dt$

$\mathbf{F} = (4 \cos t - 3 \sin t)\mathbf{i} + (4 \cos t + 3 \sin t)\mathbf{j},\ 0 \le t \le 2\pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (12 - 7 \sin t \cos t)\, dt = \left[12t - \frac{7 \sin^2 t}{2}\right]_0^{2\pi} = 24\pi$$

**34.** $x = 2 - t,\ y = 2 - t,\ z = \sqrt{4t - t^2},\ 0 \le t \le 2$

$$d\mathbf{r} = \left[-\mathbf{i} - \mathbf{j} + \frac{2 - t}{\sqrt{4t - t^2}}\mathbf{k}\right] dt$$

$$\mathbf{F} = \left(4 - 2t - \sqrt{4t - t^2}\right)\mathbf{i} + \left(\sqrt{4t - t^2} - 2 + t\right)\mathbf{j} + 0\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (t - 2)\, dt = \left[\frac{t^2}{2} - 2t\right]_0^2 = -2$$

**36.** Let $x = 2 \sin t,\ y = -2 \cos t,\ z = 4 \sin^2 t,\ 0 \le t \le \pi.$

$d\mathbf{r} = \left[(2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + (8 \sin t \cos t)\mathbf{k}\right] dt$

$\mathbf{F} = 0\mathbf{i} + 4\mathbf{j} + (2 \sin t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} (8 \sin t + 16 \sin^2 t \cos t)\, dt = \left[-8 \cos t + \frac{16}{3} \sin^3 t\right]_0^{\pi} = 16$$

**38.** $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (2x - y)\, dx + (2y - x)\, dy$

$\mathbf{r}(t) = (2 \cos t + 2t \sin t)\mathbf{i} + (2 \sin t - 2t \cos t)\mathbf{j},\ 0 \le t \le \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4\pi^2 + 4\pi$$

**40.** $\mathbf{r}(t) = 10 \sin t\mathbf{i} + 10 \cos t\mathbf{j} + \dfrac{2000/5280}{\pi/2} t\mathbf{k}, \quad 0 \le t \le \dfrac{\pi}{2}$

$$= 10 \sin t\mathbf{i} + 10 \cos t\mathbf{j} + \frac{25}{33\pi} t\mathbf{k}$$

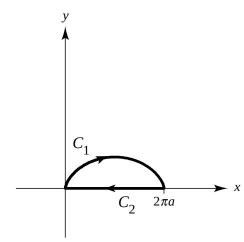
$\mathbf{F} = 20\mathbf{k}$

$$d\mathbf{r} = \left(10 \cos t\mathbf{i} - 10 \sin t\mathbf{j} + \frac{25}{33\pi}\mathbf{k}\right)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \frac{500}{33\pi}\, dt = \frac{250}{33} \text{ mi} \cdot \text{ton}$$

**42.** $\displaystyle\int_C y\,dx + x\,dy + \frac{1}{z}\,dz = \Big[xy + \ln|z|\Big]_{(0,0,1)}^{(4,4,4)} = 16 + \ln 4$

**44.** $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),\ 0 \le \theta \le 2\pi$




(a) $A = \dfrac{1}{2}\displaystyle\int_C x\,dy - y\,dx.$

Since these equations orient the curve backwards, we will use

$$A = \frac{1}{2}\int (y\,dx - x\,dy)$$

$$= \frac{1}{2}\int_0^{2\pi} \left[a^2(1 - \cos\theta)(1 - \cos\theta) - a^2(\theta - \sin\theta)(\sin\theta)\right] d\theta + \frac{1}{2}\int_0^{2\pi a} (0 - 0)\, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left[1 - 2\cos\theta + \cos^2\theta - \theta\sin\theta + \sin^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} (2 - 2\cos\theta - \theta\sin\theta)\, d\theta = \frac{a^2}{2}(6\pi) = 3\pi a^2.$$

(b) By symmetry, $\bar{x} = \pi a$. From Section 14.4,

$$\bar{y} = -\frac{1}{2A}\int_C y^2\,dx = \frac{1}{2A}\int_0^{2\pi} a^3(1 - \cos\theta)^2(1 - \cos\theta)\, d\theta = \frac{1}{2(3\pi a^2)} a^3(5\pi) = \frac{5}{6}a$$

**46.** $\displaystyle\int_C xy\,dx + (x^2 + y^2)\,dy = \int_0^2\int_0^2 (2x - x)\,dy\,dx$

$$= \int_0^2 2x\,dx = 4$$

**48.** $\displaystyle\int_C (x^2 - y^2)\,dx + 2xy\,dy = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y\,dy\,dx$

$$= \int_{-a}^{a} 0\,dx = 0$$

**50.** $\displaystyle\int_C y^2\,dx + x^{4/3}\,dy = \int_{-1}^{1}\int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \left(\frac{4}{3}x^{1/3} - 2y\right) dy\,dx$

$$= \int_{-1}^{1}\left[\frac{4}{3}x^{1/3}y - y^2\right]_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} dx$$

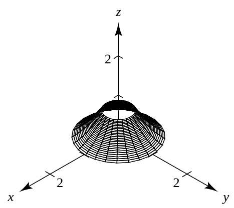
$$= \int_{-1}^{1} \frac{8}{3}x^{1/3}(1 - x^{2/3})^{3/2}\,dx$$

$$= \left[-\frac{8}{7}x^{2/3}(1 - x^{2/3})^{5/2} - \frac{16}{35}(1 - x^{2/3})^{5/2}\right]_{-1}^{1}$$

$$= 0$$

**52.** $\mathbf{r}(u, v) = e^{-u/4} \cos v\mathbf{i} + e^{-u/4} \sin v\mathbf{j} + \dfrac{u}{6}\mathbf{k}$

$0 \le u \le 4, \quad 0 \le v \le 2\pi$




**54.** $S$: $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le \pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}_u(u, v) = \mathbf{i} - \mathbf{j} + \cos v\mathbf{k}$

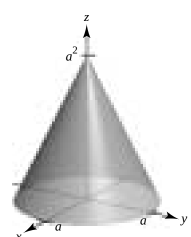
$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & \cos v \end{vmatrix} = \cos v\mathbf{i} - \cos v\mathbf{j} - 2\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2\cos^2 v + 4}$

$$\iint_S z\, dS = \int_0^{\pi}\int_0^2 \sin v\sqrt{2\cos^2 v + 4}\, du\, dv = 2\left[\sqrt{6} + \sqrt{2}\ln\left(\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}}\right)\right]$$

**56.** (a) $z = a\left(a - \sqrt{x^2 + y^2}\right), 0 \le z \le a^2$

$z = 0 \Longrightarrow x^2 + y^2 = a^2$




(b) $S$: $g(x, y) = z = a^2 - a\sqrt{x^2 + y^2}$

$\rho(x, y) = k\sqrt{x^2 + y^2}$

$$m = \iint_S e(x, y, z)\, dS$$

$$= \iint_R k\sqrt{x^2 + y^2}\,\sqrt{1 + g_x^{\,2} + g_y^{\,2}}\, dA$$

$$= k\iint_R \sqrt{x^2 + y^2}\,\sqrt{1 + \frac{a^2x^2}{x^2 + y^2} + \frac{a^2y^2}{x^2 + y^2}}\, dA$$

$$= k\iint_R \sqrt{a^2 + 1}\left(\sqrt{x^2 + y^2}\right) dA$$

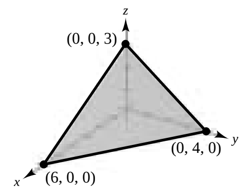
$$= k\sqrt{a^2 + 1}\int_0^{2\pi}\int_0^a r^2\, dr\, d\theta$$

$$= k\sqrt{a^2 + 1}\int_0^{2\pi} \frac{a^3}{3}\, d\theta$$

$$= \frac{2}{3}k\sqrt{a^2 + 1}\, a^3\pi$$

**58.** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$Q$: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$




**Surface Integral**: There are four surfaces for this solid.

$$z = 0 \qquad \mathbf{N} = -\mathbf{k}, \qquad \mathbf{F} \cdot \mathbf{N} = -z, \qquad \iint_{S_1} 0\, dS = 0$$

$$y = 0, \qquad \mathbf{N} = -\mathbf{j}, \qquad \mathbf{F} \cdot \mathbf{N} = -y, \qquad \iint_{S_2} 0\, dS = 0$$

$$x = 0, \qquad \mathbf{N} = -\mathbf{i}, \qquad \mathbf{F} \cdot \mathbf{N} = -x, \qquad \iint_{S_3} 0\, dS = 0$$

$$2x + 3y + 4z = 12, \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)}dA = \frac{\sqrt{29}}{4}\, dA$$

$$\iint_{S_4} \mathbf{N} \cdot \mathbf{F}\, dS = \frac{1}{4}\iint_R (2x + 3y + 4z)\, dy\, dx$$

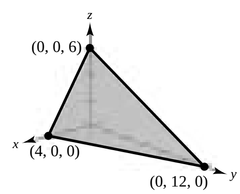
$$= \frac{1}{4}\int_0^6\int_0^{(12-2x)/3} 12\, dy\, dx = 3\int_0^6 \left(4 - \frac{2x}{3}\right) dx = 3\left[4x - \frac{x^2}{3}\right]_0^6 = 36$$

**Triple Integral:** Since div $\mathbf{F} = 3$, the Divergence Theorem yields.

$$\iiint_Q \text{div}\,\mathbf{F}\, dV = \iiint_Q 3\, dV = 3(\text{Volume of solid}) = 3\left[\frac{1}{3}(\text{Area of base})(\text{Height})\right] = \frac{1}{2}(6)(4)(3) = 36.$$

**60.** $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$

$S$: first octant portion of the plane $3x + y + 2z = 12$




**Line Integral:**

$$C_1: y = 0, \qquad dy = 0, \qquad z = \frac{12 - 3x}{2}, \qquad dz = -\frac{3}{2}\, dx$$

$$C_2: x = 0, \qquad dx = 0, \qquad z = \frac{12 - y}{2}, \qquad dz = -\frac{1}{2}\, dy$$

$$C_3: z = 0, \qquad dz = 0, \qquad y = 12 - 3x, \qquad dy = -3\, dx$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x - z)\, dx + (y - z)\, dy + x^2\, dz$$

$$= \int_{C_1} \left[x - \frac{12 - 3x}{2} + x^2\left(-\frac{3}{2}\right)\right] dx + \int_{C_2} \left[y - \frac{12 - y}{2}\right] dy + \int_{C_3} [x + (12 - 3x)(-3)]\, dx$$

$$= \int_4^0 \left(-\frac{3}{2}x^2 + \frac{5}{2}x - 6\right) dx + \int_0^{12} \left(\frac{3}{2}y - 6\right) dy + \int_0^4 (10x - 36)\, dx = 8$$

**Double Integral:** $G(x, y, z) = \dfrac{12 - 3x - y}{2} - z$

$$\nabla G(x, y, z) = -\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$\mathbf{curl\ F} = \mathbf{i} - (2x + 1)\mathbf{j}$$

$$\iint_S (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_0^4\int_0^{12-3x} (x - 1)\, dy\, dx = \int_0^4 (-3x^2 + 15x - 12)\, dx = 8$$

## Problem Solving for Chapter 14

**2.** (a) $z = \sqrt{1 - x^2 - y^2}, \dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{1 - x^2 - y^2}}, \dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{1 - x^2 - y^2}}$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{i} + z\mathbf{k}) = -25(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\mathbf{N} = \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}$$

$$= \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\,\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\,\mathbf{j} + \mathbf{k}\right)\sqrt{1 - x^2 - y^2}$$

$$= x\mathbf{i} + y\mathbf{j} + \sqrt{1 - x^2 - y^2}\,\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\text{Flux} = \iint_S -k\nabla T \cdot N\, dS$$

$$= k\iint_R 25(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\frac{1}{\sqrt{1 - x^2 - y^2}}\, dA$$

$$= k\iint_R \frac{25}{\sqrt{1 - x^2 - y^2}}\, dA$$

$$= 25k\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1 - r^2}}\, r\, dr\, d\theta = 50\pi k$$

(b) $\mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle$

$\mathbf{r_u} = \langle \cos u \cos v, \cos u \sin v, -\sin u\rangle$

$\mathbf{r_v} = \langle -\sin u \sin v, \sin u \cos v, 0\rangle$

$\mathbf{r_u} \times \mathbf{r_v} = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \sin^2 v + \sin u \cos u \cos^2 v\rangle$

$\|\mathbf{r_u} \times \mathbf{r_v}\| = \sin u$

$$\text{Flux} = 25k\int_0^{2\pi}\int_0^{\pi/2} \sin u\, du\, dv = 50\pi k$$

**4.** $\mathbf{r}(t) = \left\langle \dfrac{t^2}{2}, t, \dfrac{2\sqrt{2}t^{3/2}}{3}\right\rangle$

$\mathbf{r}'(t) = \left\langle t, 1, \sqrt{2}t^{1/2}\right\rangle, \|\mathbf{r}'(t)\| = t + 1$

$$\rho\, ds = \frac{1}{1 + t}(t + 1)\, dt = 1$$

$$I_y = \int_C (x^2 + z^2)\rho\, ds = \int_0^1 \left(\frac{t^4}{4} + \frac{8}{9}t^3\right) dt = \frac{49}{180}$$

$$I_x = \int_C (y^2 + z^2)\rho\, ds = \int_0^1 \left(t^2 + \frac{8}{9}t^3\right) dt = \frac{5}{9}$$

$$I_z = \int_C (x^2 + y^2)\rho\, ds = \int_0^1 \left(\frac{t^4}{4} + t^2\right) dt = \frac{23}{60}$$

**6.** $\frac{1}{2}\int_C x\,dy - y\,dx = 2\int_0^{\pi/2}\left[\frac{1}{2}\sin 2t\cos t - \sin t\cos 2t\right]dt = 2\left(\frac{2}{3}\right)$

Hence, the area is $4/3$.

**8.** $F(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$ is conservative.

$f(x, y) = x^3y^2$ potential function.

Work $= f(2, 4) - f(1, 1) = 8(16) - 1 = 127$

**10.** Area $= \pi ab$

$\mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j},\ 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j}$

$\mathbf{F} = -\frac{1}{2}b\sin t\,\mathbf{i} + \frac{1}{2}a\cos t\,\mathbf{j}$

$\mathbf{F}\cdot d\mathbf{r} = \left[\frac{1}{2}ab\sin^2 t + \frac{1}{2}ab\cos^2 t\right]dt = \frac{1}{2}ab$

$W = \int_0^{2\pi}\mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}ab(2\pi) = \pi ab$

Same as area.