FYS4160-2017
Problem sheet 10

NOTE: These notes are complementary to the discussion in the exercise sessions and are thus very condensed. If you want to discuss parts in more detail, please use our platform PIAZZA. If you spot some typos or mistakes, please email to *magdalena.kersting@fys.uio.no.*

Problem 33: Area of a black hole

The induced metric at the horizon ($r = 2GM$) is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

and the area is

$$A_S = \int dA_S = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{-g} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; r^2 \sin\theta = 16\pi G^2 M^2, \tag{2}$$

which one would expect for a spherical body with radius $2GM$.
For the Kerr black hole, the induced metric at the outside horizon can be obtained by setting dt and dr to zero, because we are looking at a surface of constant radius and time (the metric is stationary). We obtain

$$ds^2 = \rho_+^2 d\theta^2 + \left(\frac{r_+^2 + a^2}{\rho_+}\right)^2 \sin^2\theta \; d\phi^2 \tag{3}$$

and the area is

$$A_K = \int dA_K = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{-g} = 4\pi(r_+^2 + a^2). \tag{4}$$

Note that we recover Schwarzschild for $a \longrightarrow 0$.

Problem 34: Spaceship hovering close to a black hole

a) The conserved quantities E and L provide a convenient way to understand orbits in the Schwarzschild geometry. For radial motion we have $L = 0$ and after the mass ejection, the conserved quantity E for the spaceship should have the same value as when it reaches $r \to \infty$, i.e., $E = 1$. From the discussion in section 5.4 and in particular equation 5.64 in Carroll, it follows

$$\begin{aligned}\left(\frac{dr}{d\tau}\right)^2_{R_{\text{ship}}} &= E^2 - \left(1 - \frac{2GM}{R_{\text{ship}}}\right)\left(\frac{L}{R^2_{\text{ship}}} + \epsilon\right) \\ &= \frac{2GM}{R_{\text{ship}}}\end{aligned} \tag{5}$$

where we have used that $\epsilon = 1$ for massive particles.
b) In this exercise we will see that the total rest mass is not conserved, while the 4-momentum is. We denote by m and u the rest mass and 4-velocity of the ship hovering at $r = R_{\text{ship}}$. Accordingly, m_{esc}, m_{ej}, u_{esc}, and u_{ej} denote the mass and velocity of the separated fragments.
We have

$$u^\alpha = \left(\left(1 - \frac{2MG}{r}\right)^{-1}, 0, 0, 0\right)$$

and from a) we see that

$$u^\alpha_{\text{esc}} = \left(\left(1 - \frac{2MG}{r}\right)^{-1}, \left(\frac{2MG}{r}\right)^{\frac{1}{2}}, 0, 0 \right).$$

Conservation of 3-momentum implies

$$0 = m_{\text{esc}} \cdot \sqrt{\frac{2MG}{r}} + m_{\text{ej}} u^r_{\text{ej}}$$

and thus

$$u^r_{\text{ej}} = -\left(\frac{m_{\text{esc}}}{m_{\text{ej}}}\right) \sqrt{\frac{2MG}{r}}.$$

Also,

$$u^t_{\text{ej}} = \left(1 - \frac{2MG}{r}\right)^{-1} \left(1 - \frac{2MG}{r} \left(1 - \left(\frac{m_{\text{esc}}}{m_{\text{ej}}}\right)^2\right)\right)^{\frac{1}{2}}.$$

Using conservation of energy leads to

$$\left(1 - \frac{2MG}{r}\right)^{\frac{1}{2}} = \left(\frac{m_{\text{esc}}}{m}\right) + \left(\frac{m_{\text{ej}}}{m}\right) \left(1 - \frac{2MG}{r} \left(1 - \left(\frac{m_{\text{esc}}}{m_{\text{ej}}}\right)^2\right)\right)^{\frac{1}{2}}.$$

One can check that m_{esc} is maximized for $m_{\text{ej}} = 0$. In the limit, $m_{\text{ej}} \to 0$, the spaceship cannot escape at $R_{\text{ship}} = 2GM$.

Problem 35: Photons around a black hole

For more details compare to section 5.4 in Carroll.

$$ds^2 = -(1 - 2GM(r)/r)dt^2 + (1 - 2GM(r)/r)^{-1} dr^2 + r^2 d\Omega^2 \tag{6}$$

a) For $r > R$, $1 - \frac{2GM_*}{r} > 0$, hence there is no singularity. And for $0 < r < R$, $1 - 2GM(r)/r = 1 - \frac{2GM_*}{R}\frac{r^2}{R^2} > 0$, again there is no singularity.

b) The Killing vectors are $K^\mu = (1, 0, 0, 0)$ and $R^\mu = (0, 0, 0, 1)$. The conserved quantities are,

$$E = -K \cdot U = (1 - 2GM(r)/r)\frac{dt}{d\lambda} \quad \text{and} \quad L = R \cdot U = r^2 \frac{d\phi}{d\lambda}. \tag{7}$$

We can choose an orbit in the equatorial plane, because conservation of angular momentum will constrain the motion to a plane.
c) For photons, $U \cdot U = 0$, hence,

$$\begin{aligned} &(1 - 2GM(r)/r)\left(\frac{dt}{d\lambda}\right)^2 - (1 - 2GM(r)/r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \\ \Longrightarrow\ & E^2 - \left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2}(1 - 2GM(r)/r) = 0 \\ \text{or}\quad & V_{\text{eff}} = \frac{L^2}{2r^2}(1 - 2GM(r)/r) \end{aligned} \tag{8}$$

d) The coordinate time is calculated by

$$\begin{aligned} t = \int dt &= \int_0^R \frac{1}{1 - 2GMr^2/R^3} dr \\ &= \left(\frac{R^3}{GM}\right)^{1/2} \tanh^{-1}\left(\left(\frac{2GM}{R}\right)^{1/2}\right) \end{aligned} \tag{9}$$

For $R \gg GM$, $t \to R\left(1 + \frac{2GM}{3R}\right)$, hence, $\delta t = \frac{2GM}{3R}$. For the sun, this yields $\delta t \sim 1.4\mu s$.