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Problem 4

Conserved charge given by

$$Q = \int d^3x \, j_V^0$$

$$j_V^0 = \bar{\Psi} \gamma^0 \psi = \psi^\dagger \gamma^0 \gamma^0 \psi$$

$$\Rightarrow j_V^0 = \psi^\dagger \psi$$

$$\Rightarrow Q = \int d^3x \, \psi^\dagger \psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a_{\vec{p}}^{s\dagger} u^{s\dagger}(p) e^{i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^s v^{s\dagger}(p) e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\times \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \sum_s \left(a_{\vec{q}}^s u^s(q) e^{-i\vec{q}\cdot\vec{x}} + b_{\vec{q}}^{s\dagger} v^s(q) e^{i\vec{q}\cdot\vec{x}} \right)$$

$$\sum_{s,r} \left(a_{\vec{p}}^{s\dagger} u^\dagger(p) e^{i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^s v^\dagger(p) e^{-i\vec{p}\cdot\vec{x}} \right) \times \left(a_{\vec{q}}^r u^r(q) e^{-i\vec{q}\cdot\vec{x}} + b_{\vec{q}}^{r\dagger} v^r(q) e^{i\vec{q}\cdot\vec{x}} \right)$$

$$= \left(a_{\vec{p}}^{s\dagger} a_{\vec{q}}^s \, \delta^{(3)}(\vec{p} - \vec{q}) u^{s\dagger}(p) u^r(q) + b_{\vec{p}}^s b_{\vec{q}}^{s\dagger} \, \delta(\vec{q} - \vec{p}) v^{s\dagger}(p) v^r(q) \right) (2\pi)^3$$

$$= \left(a_{\vec p}^{s\,\dagger} a_{\vec q}^{s}\, \delta^{(3)}(\vec p - \vec q)\, u^{s\dagger}(p) u^{r}(q) + b_{\vec p}^{s} b_{\vec q}^{s\,\dagger}\, \delta(\vec q - \vec p)\, v^{s\dagger}(p) v^{r}(q) \right) (2\pi)^3$$

remember normalization of $u(p)$ and $v(p)$:

$$v^{r\dagger}(p)\, v^{s}(p) = 2 E_{\vec p} \delta^{rs} \qquad u^{s\dagger}(p)\, u^{r} = 2 E_{\vec p}\, \delta^{rs}$$

Now the Dirac delta functions impose $\vec p = \vec q$, then the spinor normalization factors take care of $\frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} = \frac{1}{2 E_{\vec p}}$, giving

$$\Rightarrow Q = \int \frac{d^3 p}{(2\pi)^3} \left(a_{\vec p}^{s\,\dagger} a_{\vec p}^{s} + b_{\vec p}^{s} b_{\vec p}^{s\,\dagger} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left(a_{\vec p}^{s\,\dagger} a_{\vec p}^{s} - b_{\vec p}^{s\,\dagger} b_{\vec p}^{s} \right)$$

(ignoring an infinite constant)

$$Q\, |\vec p\rangle_{\text{fermion}} = \underline{|\vec p\rangle}$$

$$Q\, |\vec p\rangle_{\text{anti-fermion}} = \underline{-|\vec p\rangle}$$

i.e. the charge conserved of j^μ_V for a fermion $+1$ and for an anti-fermion the conserved charge of j^μ_V is -1.