Start with (no interaction)

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr_1^2}-\frac{\hbar^2}{2m}\frac{d^2}{dr_2^2}+\frac{1}{2}kr_1^2+\frac{1}{2}kr_2^2\right)u(r_1,r_2)=E\,u(r_1,r_2)$$

change variables

$$\vec{r}=\vec{r}_1-\vec{r}_2 \quad \wedge \quad \vec{R}=\frac{1}{2}(\vec{r}_1+\vec{r}_2) \Rightarrow \vec{r}_2=2\vec{R}-\vec{r}_1$$

$$2\vec{r}_2=2\vec{R}-\vec{r}$$

$$\vec{r}_2=\vec{R}-\vec{r}/2$$

$$\vec{r}_1=\vec{r}+\vec{r}_2$$

$$\frac{1}{2}kr_1^2+\frac{1}{2}kr_2^2=\frac{1}{4}kr^2+kR^2$$

In QM

$$\vec{p}=-i\hbar\nabla$$

momentum:

Relative momentum:

$$\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$$

center of mass momentum $\quad \vec{P} = \vec{P}_1 + \vec{P}_2$

kinetic energy

$$\frac{P_1^2}{2m} + \frac{P_2^2}{2m} = \frac{p^2}{m} + \frac{P^2}{4m}$$

$$= -\frac{\hbar^2}{m}\nabla_r^2 - \frac{\hbar^2}{4m}\nabla_R^2$$

$$\left(-\frac{\hbar^2}{m}\frac{d^2}{dr^2} + \frac{1}{4}kr^2\right)u(r) = E_r$$

$$\left(-\frac{\hbar^2}{4m}\frac{d^2}{dR^2} + kR^2\right)u(R) = E_R$$

$$E = E_1 + E_2 = E_r + E_R$$