FYS4160-2017
Problem sheet 7

NOTE: These notes are complementary to the discussion in the exercise sessions and are thus very condensed. If you want to discuss parts in more detail, please use our platform PIAZZA. If you spot some typos or mistakes, please email to *magdalena.kersting@fys.uio.no.*

Problem 25: Stress-energy tensor

Compare to section 1.9 in Carroll.

Problem 26: Noether's theorem Problem 21: Scalar field Lagrangian

Let's start with a sketch of the proof of Noether's theorem. First, recall that the principle of least action can be written as

$$\delta S = \int \mathrm{d}^4x \Big[\frac{\partial \mathcal{L}}{\partial \Phi_a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\delta\Phi_a\Big] \tag{1}$$

$$= \int \mathrm{d}^4x \Big[\left(\frac{\partial \mathcal{L}}{\partial \Phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\right)\right)\delta\Phi_a + \underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\delta\Phi_a\right)}_{\text{does not contribute}}\Big] = 0 \tag{2}$$

Now we consider an infinitesimal transformation $\delta\Phi_a$. We say that this is a symmetry of the theory if the Lagrangian changes by a total derivative $\delta\mathcal{L}(\Phi_a) = \partial_\mu \mathcal{J}^\mu$ for a set of functions $\mathcal{J}^\mu$. Because of (2), we have for the transformation of $\mathcal{L}$ under an arbitrary change of field $\delta\Phi_a$

$$\delta\mathcal{L} = \Big[\frac{\partial \mathcal{L}}{\partial \Phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\right)\Big]\delta\Phi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\delta\Phi_a\right). \tag{3}$$

If the Euler-Lagrange equation is satisfied, the first term vanishes and we get

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\delta\Phi_a\right). \tag{4}$$

Setting

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\delta\Phi_a - \mathcal{J}^\mu \tag{5}$$

this yields

$$\partial_\mu J^\mu = 0. \tag{6}$$

If the Lagrangian is invariant under $\delta\Phi_a$ (i.e. $\delta\mathcal{L} = 0$), then $\mathcal{J}^\mu = 0$ and J^μ contains only the first term on the right-hand side of (5).
Now, let's have a look at a space-time translation $x^\mu \longrightarrow x^\mu + a^\mu$ that corresponds to $\Phi_a(x) \longrightarrow \Phi_a(x+a) = \Phi_a(x) + a^\nu\partial_\nu\Phi_a(x)$ and $\mathcal{L} \longrightarrow \mathcal{L} + a^\mu\partial_\mu\mathcal{L}$. Noether's theorem gives us four conserved currents $T^\mu_\nu = (J^\mu)_\nu$ (one for each translation a^μ). From the above (5) we get

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}\partial_\nu\Phi_a - \delta^\mu_\nu\mathcal{L}. \tag{7}$$

For the Lagrangian of a scalar field $\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - V(\Phi)$, we get

$$T^{\mu\nu} = (\partial^\mu \Phi)(\partial^\nu \Phi) - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho \Phi)^2 + \eta^{\mu\nu} V. \tag{8}$$

Problem 27: Einstein-Hilbert action

a) The Hilbert action in terms of Riemann tensor can be written as $S_H = \int d^4x \sqrt{-g} R = \int d^4x \sqrt{-g} R_{\mu\nu} g^{\mu\nu} = \int d^4x \sqrt{-g} R^\lambda{}_{\mu\lambda\nu} g^{\mu\nu}$. Now, we can see that the action contains second derivatives of the metric through the Riemann tensor. Let's only keep the terms with second derivatives in $R^\lambda{}_{\mu\lambda\nu}$, then

$$\begin{aligned} R^\lambda{}_{\mu\lambda\nu} &= \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \text{1st order derivatives} \\ &= \frac{1}{2}\partial_\lambda \left(g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})\right) - \frac{1}{2}\partial_\nu \left(g^{\lambda\rho}(\partial_\mu g_{\lambda\rho})\right) + \text{1st order derivatives} \\ &= \frac{1}{2} g^{\lambda\rho} \left(\partial_\lambda \partial_\mu g_{\nu\rho} + \partial_\lambda \partial_\nu g_{\rho\mu} - \partial_\lambda \partial_\rho g_{\mu\nu} - \partial_\mu \partial_\nu g_{\lambda\rho}\right) + \text{1st order derivatives} \end{aligned} \tag{9}$$

Now putting this back in the action yields

$$\begin{aligned} S_H &= \int d^4x \sqrt{-g} R^\lambda{}_{\mu\lambda\nu} g^{\mu\nu} \\ &= \int d^4x \sqrt{-g} \frac{1}{2} g^{\mu\nu} g^{\lambda\rho} \left(\partial_\lambda \partial_\mu g_{\nu\rho} + \partial_\lambda \partial_\nu g_{\rho\mu} - \partial_\lambda \partial_\rho g_{\mu\nu} - \partial_\mu \partial_\nu g_{\lambda\rho}\right) + \text{1st order derivatives.} \end{aligned} \tag{10}$$

Now the four remaining terms can be cast into terms containing only first derivative using the product rule, for example,

$$\int d^4x \sqrt{-g} \frac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\lambda \partial_\mu g_{\nu\rho} = \int d^4x \partial_\lambda \left(\sqrt{-g} \frac{1}{2} g^{\mu\nu} g^{\lambda\rho} \partial_\mu g_{\nu\rho} \right) - \int d^4x \partial_\lambda \left(\sqrt{-g} \frac{1}{2} g^{\mu\nu} g^{\lambda\rho} \right) \partial_\mu g_{\nu\rho}. \tag{11}$$

The first term doesn't contribute and we are left with terms only containing first derivatives of the metric.

b) We make use of the fact that the difference of two connections is a tensor and that the variation of the Christoffel symbol is such a difference of two connections. In the problem session we have seen how our interpretation of the Christoffel connection being correction terms for the partial derivative can make us see why the difference of two connections is a tensor. Similarly, (and slightly nicer to write down ;)) if we have two different covariant derivatives, ∇_μ and $\hat{\nabla}_\mu$, with associated connection coefficients $\Gamma^\lambda_{\mu\nu}$ and $\hat{\Gamma}^\lambda_{\mu\nu}$, then

$$S^\mu_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\mu\nu}$$

is a tensor, because for an arbitrary vector field V^λ, both $\nabla_\mu V^\lambda$ and $\hat{\nabla}_\mu V^\lambda$ are tensors. Thus, their difference must be a tensor as well:

$$\begin{aligned} \nabla_\mu V^\lambda - \hat{\nabla}_\mu V^\lambda &= \partial_\mu V^\lambda + \Gamma^\lambda_{\mu\nu} V^\lambda - \partial_\mu V^\lambda + \hat{\Gamma}^\lambda_{\mu\nu} V^\lambda \\ &= S^\mu_{\mu\nu} \end{aligned}$$