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Problem 2

$$(*) \quad \Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^s_{\vec{p}} u^s(p) e^{-i\vec{p}\cdot\vec{x}} + b^{s+}_{\vec{p}} v^s(p) e^{i\vec{p}\cdot\vec{x}} \right)$$

a) $a^s_{\vec{p}}$ is an anihilation operator ($a^{s+}_{\vec{p}}$ is the corresponding creation operator) and particles created by $a^{s+}_{\vec{p}}$ is refered to as fermions. $b^{s+}_{\vec{p}}$ is also a creation operator ($b^s_{\vec{p}}$ is the corresponding annihilation operator) and particles created by $b^{s+}_{\vec{p}}$ is refered to as anti-fermions. The creation and anihilation operators obey the anti-comutation relations $\{a^r_{\vec{p}}, a^{s+}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s+}_{\vec{q}}\} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\delta^{rs}$. $u^s(p)$ and $v^s(p)$ are spinors for fermions and antifermions respectively. The superscript s refers to spin (up or down). properties of spinors: $u^+u = 2E_{\vec{p}}\xi^+\xi$ (where ξ is a two-component spinor. In general they take the form $u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$, $v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}$, where ξ^s and η^s is two basises of two-component spinors. $E_{\vec{p}}$ is the energy given by $E_{\vec{p}} = \sqrt{p^2+m^2}$.

b) To arrive at the quantized Dirac field we started with considering Lagrangian and Hamiltonian mechanics, but substituting the localized particle with a classical field where density perturbations in the field may be viewed as a particle. The field has a conserved charge $Q = \int d^3x\, j^0$ if $\partial_\mu j = 0$; $j^\mu = \mathcal{J}^\mu - \Delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}$.

↑ conserved current

Then we use a first order Lorentz invariant equation, as a starting point, to find $\psi(x)$:

Lorentz invariant $(i\gamma^\mu \partial_\mu - m)\psi = 0$ $\iff$ $(\partial^2 - m^2)\psi = 0$

Dirac equation — Klein-Gordon equation

Then we define $\bar{\Psi} = \psi^\dagger \gamma^0$ to be able to multiply two Dirac spinors (ψ) to form a Lorentz scalar. ($\bar{\Psi} \to \psi^\dagger(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger)\gamma^0$

$\Rightarrow \bar{\Psi} \to \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}$ since $S^{\mu\nu}$ both commutes and anticommutes with γ^0)

Then $\bar{\Psi}\psi$ is a Lorentz scalar.

Now the Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\psi$$

[illegible] ...ation ψ can be writen as a linear

Since ψ obeys the KG equation ψ can be writen as a linear combination of plane waves:

$$\psi(x) = u(p)\, e^{-i\vec{p}\cdot\vec{x}} + v(p)\, e^{i\vec{p}\cdot\vec{x}},$$

where $u(p)$ and $v(p)$ are the spinors discussed in 2a). In the rest frame $u(p) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$. In general it takes the form mentioned in 2a, but an alternative expression is $u(p) = \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}(E+m-\vec{p}\cdot\vec{\sigma})\,\xi\\ (E+m+\vec{p}\cdot\vec{\sigma})\,\xi\end{pmatrix}$.

Then we postulated the anti-commutation relations

$\{\psi_a(\vec{x}), \psi_b^+(\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$ and $\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi_a^+(\vec{x}), \psi_b^+(\vec{y})\} = 0$.

$\psi(\vec{x})$ can be expanded in terms of annihilation operators $a_{\vec{p}}^s$ and $b_{\vec{p}}^s$ as

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(a_{\vec{p}}^s\, u(p)\, e^{-i\vec{p}\cdot\vec{x}} + \tilde{b}_{\vec{p}}^s\, v(p)\, e^{i\vec{p}\cdot\vec{x}}\right)$$

where the creation and anihilation operators obey the anti commutation relations $\{a_{\vec{p}}^r, a_{\vec{q}}^{s+}\} = \{\tilde{b}_{\vec{p}}^r, \tilde{b}_{\vec{q}}^{s+}\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,\delta^{rs}$.

symmetric between $\tilde{b}_{\vec{p}}^r$ and $\tilde{b}_{\vec{q}}^{s+}$ so

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^s_{\vec{p}}\, u(p)\, e^{-i\vec{p}\cdot\vec{x}} + b^{s+}_{\vec{p}}\, v(p)\, e^{i\vec{p}\cdot\vec{x}} \right)$$

$\Rightarrow$ second term in Hamiltonian changes sign!

When we choose $|0\rangle$ as the state anihilated by $a^s_{\vec{p}}$ and $b^s_{\vec{p}}$ all excitations of $|0\rangle$ have positive energy!

The Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} \left(a^{s+}_{p}\, a^s_{\vec{p}} + b^{s+}_{p}\, b^s_{p} \right).$$