torsdag 27. oktober 2016 20.29

Problem 7

$$d\sigma = \frac{1}{2|\vec{p}_i|} \frac{d^3p}{(2\pi)^3} \frac{1}{2E_f} |\mathcal{M}(p_i \to p_f)|^2 \, 2\pi \, \delta(E_f - E_i)$$

a)

- $$d\sigma = \frac{1}{16\pi^2 |\vec{p}_i|} d^3p \frac{1}{E_f} \delta(E_f - E_i)$$

- $$d^3p = dp_f \, p_f^2 \, d\Omega$$

$$E_f = \sqrt{p_f^2 + m^2} \quad , \quad E_i = \sqrt{p_i^2 + m^2}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2 |\vec{p}_i|} dp_f \frac{p_f^2}{E_f} \delta(E_f - E_i) \quad |\mathcal{M}(p_i \to p_f)|^2$$

$$= \frac{1}{16\pi^2 |\vec{p}_i|} dp_f \underbrace{\frac{p_f^2}{\sqrt{p_f^2 + m^2}}}_{g(p_f)} \delta(\underbrace{\sqrt{p_f^2 + m^2} - \sqrt{p_i^2 + m^2}}_{f(p_f)}) \quad |\mathcal{M}(p_i \to p_f)|^2$$

$$\frac{1}{|f'(p_i)|} g(p_i)$$

$$f'(p_i) = \frac{d}{dp_f}\left(\sqrt{p_f^2+m^2} - \sqrt{p_i^2+m^2}\right)$$

$$= \frac{p_f}{\sqrt{p_f^2+m^2}}$$

$$\Rightarrow \quad |f'(p_i)| = \frac{|p_i|}{\left|\sqrt{p_i^2+m^2}\right|}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2 |\vec{p}_i|} \left(\frac{|\vec{p}_i|}{\sqrt{p_i^2+m^2}}\right)^{-1} \frac{p_i^2}{\sqrt{p_i^2+m^2}} \left|M(p_i \to p_f)\right|^2$$

$$= \frac{1}{16\pi^2} \frac{\vec{p}_i^{\,2}}{\cancel{|\vec{p}_i|^2}} \left|M(p_i \to p_f)\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \left|M(p_i \to p_f)\right|^2 \qquad \square$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} |M(p_i \to p_f)| \qquad \square$$

b) $A^\mu(x) = \left(\frac{Ze}{4\pi r}, \vec{0} \right)$ gives the Coulomb potential and this has the fourier transform

$$\tilde{A}^0(q) = \int d^3x \, e^{-i\vec{q}\cdot\vec{x}} A^0(x)$$

$$= \int_0^\infty r^2 dr \int_{-1}^{+1} d[\cos\theta] \int_0^{2\pi} d\phi \, e^{-i|\vec{q}| r \cos\theta} \frac{Ze}{4\pi r}$$

$$= \frac{Ze}{2} \int_0^\infty r \, dr \int_{-1}^{+1} d[\cos\theta] \, e^{-i|\vec{q}| r\cos\theta}$$

$$= \frac{Ze}{2i|\vec{q}|} \int_0^\infty dr \left(e^{-i|\vec{q}|r} - e^{i|\vec{q}|r} \right)$$

* since this integral will give $e^{i\infty}$ and $e^{-i\infty}$ after integrating we regulate with the term $e^{-\mu r}$, and let $\mu \to 0$ when we are done

$$= \lim_{\mu \to 0} \frac{Ze}{2i|\vec{q}|} \int_0^\infty dr\, e^{-\mu r} \left(e^{-i|\vec{q}|r} - e^{i|\vec{q}|r} \right)$$

$$= \lim_{\mu \to 0} - \frac{Ze}{2i|\vec{q}|} \left(\frac{e^{(i|\vec{q}| - \mu)r}}{i|\vec{q}| - \mu} + \frac{e^{-(i|\vec{q}| + \mu)r}}{i|\vec{q}| + \mu} \right) \Bigg|_0^\infty$$

Note $\mathrm{Re}\left(e^{i\infty - \infty} \right) = 0$

$$= \lim_{\mu \to 0} - \frac{Ze}{2i|\vec{q}|} \left(\frac{1}{i|\vec{q}| - \mu} + \frac{1}{i|\vec{q}| + \mu} \right)$$

$$= \underline{\frac{Ze}{|\vec{q}|^2}}$$

The scattering amplitude for electrons become

$$iM = \bar{u}^r(p_f)\left(-ie\gamma^0 \tilde{A}_0(p_f - p_i)\right) u^s(p_i)$$

$$= -i \frac{Ze^2}{|\vec{q}|^2} \bar{u}^r(p_f)\gamma^0 u^s(p_i)$$

$$\underline{iM = -i \frac{Ze^2}{|\vec{q}|^2} u^{r\dagger}(p_f)\, u^s(p_i)}$$

And for positrons

$$\underline{iM = -i \frac{Ze^2}{|\vec{q}|^2} v^{r\dagger}(p_i)\, v^s(p_f)}$$

c)

We write $|\bar{M}| = \frac{1}{2} \sum_{r,s} |M|^2$

We write $\overline{|M|^2} = \frac{1}{2} \sum_{r,s} |M|^2$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \frac{Z^2 e^4}{|\vec{q}|^4} \frac{1}{2} \sum_{r,s} \left| \bar{u}^r(p_f) \gamma^0 u^s(p_i) \right|^2$$

$$\alpha^2 = \frac{e^4}{16\pi^2}$$

$$= \frac{\alpha^2 Z^2}{2|\vec{q}|^4} \mathrm{Tr}\left[(\not{p}_f + m) \gamma^0 (\not{p}_i + m) \gamma^0 \right]$$

(P. 134 P&S for $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho]$)

$$= \frac{\alpha^2 Z^2}{2|\vec{q}|^4} \left(\mathrm{Tr}\left[\not{p}_f \gamma^0 \not{p}_i \gamma^0 \right] + m^2 \mathrm{Tr}\left[\gamma^0 \gamma^0 \right] \right)$$

$$= \frac{2\alpha^2 Z^2}{|\vec{q}|^4} \left[p_{i\mu} p_{f\nu} \left(g^{\mu 0} g^{0\nu} - g^{\mu\nu} g^{00} + g^{\mu 0} g^{0\nu} \right) + m^2 \right]$$

$$= \frac{2\alpha^2 Z^2}{|\vec{q}|^4} \left(2E_i E_f - p_i \cdot p_f + m^2 \right)$$

$$= \frac{2\alpha^2 Z^2}{|\vec{q}|^4} \left(E_i E_f + \vec{p}_i \cdot \vec{p}_f + m^2 \right)$$

Now we use that $E_i = E_f = E$ and $|\vec{p}_i| = |\vec{p}_f| = |p| = v E$

v is the velocity of the scattering electron. Then:

$$p_i \cdot p_f = v^2 E^2 \cos\theta = v^2 E^2 \left(1 - 2 \sin^2 \frac{\theta}{2}\right)$$

and

$$q^2 = |p_f - p_i|^2 = 2 v^2 E^2 (1 - \cos\theta) = 4 v^2 E^2 \sin^2 \frac{\theta}{2}$$

We then have

$$\frac{d\sigma}{d\Omega} = 2 \alpha^2 Z^2 \left(\frac{1}{4 v^2 E^2 \sin^2 \frac{\theta}{2}}\right)^2 \underbrace{\left[E^2 + v^2 E^2 \left(\cancel{1} - 2 \sin^2 \frac{\theta}{2}\right) + E^2 (1 - \cancel{v^2})\right]}_{= E^2\left(2 - 2 v^2 \sin^2 \frac{\theta}{2}\right) = 2E^2\left(1 - \sin^2 \frac{\theta}{2}\right)}$$

$$= \frac{\alpha^2 Z^2}{4 v^4 E^2} \frac{1 - v^2 \sin^2 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}}$$

This is the Mott formula!

Taking the non-relativistic limit, i.e. $v \to 0$ and $E \to m$ gives

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4 m^2 v^4 \sin^4 \frac{\theta}{2}}$$

This is the Rutherford formula.