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`http://www.uio.no/studier/emner/matnat/fys/FYS4160/v17/`

Lecture spring 2017:

General Relativity

Problem sheet 10

⇝ These problems are scheduled for discussion on **Thursday, 6 April** *and* **20 April 2017**. (recall that there are no lectures or exercise sessions during the week before Easter)

Problem 33

Calculate the area of the horizon for a) a Schwarzschild and b) a Kerr black hole by first deriving the induced metric on the horizon! How could you immediately have obtained the answer to a) ?

Problem 34

Assume a spaceship that was supposed to study the environment of a Schwarzschild black hole, hovering at a distance $R_{\rm ship}$ slightly above the horizon. Now it is time to return home, but since it has used up almost all its fuel the only chance for the crew to escape is by ejecting part of the rest mass of the ship into the black hole.

a) In order for the crew to escape with the maximum rest mass, they will want to escape radially. Which (minimal) values of the conserved quantities E and L describe the resulting geodesic, and what is hence the minimally require 4-velocity in this case?

b) Use momentum conservation to derive the 4-momentum of the ejected fragment. From the result, derive the largest fraction f of the initial rest mass of the space craft that can escape to infinity. What happens to this fraction as the radius $R_{\rm ship}$ of the mission approaches the Schwarzschild radius of the black hole?

Problem 35

For a star with constant density and coordinate radius R, the resulting metric is given by

$$ds^2 = -\left(1 - \frac{2GM(r)}{r}\right) dt^2 + \left(1 - \frac{2GM(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,,$$

where

$$M(r) = \begin{cases} M_\star (r/R)^3 & \text{for } 0 < r < R \\ M_\star & \text{for } r > R \end{cases}$$

and $M_\star$ is the total mass of the star. Assume $R > 2GM_\star$ and consider the geodesics of photons.

a) Are there any singularities of the metric?

b) Write down the Killing vectors of this spacetime. You may consider planar orbits with $\theta = \pi/2$ (why?); write down the two conserved quantities for such orbits.

c) Derive an expression for $dr/d\lambda$ in terms of an effective potential $V_{\text{eff}}(r)$. Sketch $V_{\text{eff}}(r)$ and (qualitatively) describe the photon orbits; how do they differ from the Schwarzschild case?

d) Calculate the coordinate time t for a photon to travel from the center of the star at $r = 0$ to the surface at $r = R$. For $r \gg GM$, what is the relative delay compared to a photon travelling in flat spacetime? What is the numerical value of this fraction for the sun?