Problem 1

a)

$$\gamma^\mu \gamma_\mu = \gamma^\mu g_{\mu\nu} \gamma^\nu$$

$$= g_{\mu\nu} \gamma^\mu \gamma^\nu$$

$$= \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) \gamma^\mu \gamma^\nu$$

$$= \frac{1}{2}\left(g_{\mu\nu} \gamma^\mu \gamma^\nu + g_{\nu\mu} \gamma^\nu \gamma^\mu\right)$$

$$= \frac{1}{2}\left(g_{\mu\nu} \gamma^\mu \gamma^\nu + g_{\mu\nu} \gamma^\nu \gamma^\mu\right)$$

$$= \frac{1}{2} g_{\mu\nu} \{\gamma^\mu \gamma^\nu\}$$

$$= g_{\mu\nu} g^{\mu\nu} I = \underline{4\,\mathbb{1}} \quad \square$$

$\{\gamma^\mu \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}$

$\mathbb{1}$ (identity matrix)

b)

$$\gamma^\mu \gamma^\nu \gamma_\mu = \gamma^\mu (2 g^\nu_\mu \mathbb{1} - \gamma_\mu \gamma^\nu)$$

$$= 2\gamma^\mu g^\nu_\mu - \gamma^\mu \gamma_\mu \gamma^\nu$$

$$= 2\gamma^\nu - 4\gamma^\nu = \underline{-2\gamma^\nu} \quad \square$$

$\{\gamma^\nu \gamma_\mu\} = 2 g^\nu_\mu \mathbb{1}$

$\Rightarrow \boxed{\gamma^\nu \gamma_\mu = 2 g^\nu_\mu \mathbb{1} - \gamma_\mu \gamma^\nu}$

c)

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = \{\gamma^\mu, \gamma^\nu\} \gamma^\rho \gamma_\mu - \gamma^\nu \gamma^\mu \gamma^\rho \gamma_\mu$$

$$= 2 g^{\mu\nu} \gamma^\rho \gamma_\mu - \gamma^\nu \{\gamma^\mu, \gamma^\rho\} \gamma_\mu + \gamma^\nu \gamma^\rho \gamma^\mu \gamma_\mu$$

$$= 2 g^{\mu\nu} \gamma^\rho \gamma_\mu - \gamma^\nu 2 g^{\mu\rho} \gamma_\mu + 4 \gamma^\nu \gamma^\rho$$

$$= 2 \gamma^\rho \gamma^\nu - 2 \gamma^\nu \gamma^\rho + 4 \gamma^\nu \gamma^\rho$$

$$= 2 \gamma^\rho \gamma^\nu + 2 \gamma^\nu \gamma^\rho$$

$$= 2 (\gamma^\rho \gamma^\nu + \gamma^\nu \gamma^\rho)$$

$$= 2\{\gamma^1, \gamma^2\}$$ □