Solutions to final exam in FYS4170

Problem 1

a)
$$
\begin{aligned}
\not{p}\not{p} &= p_\mu p_\nu \gamma^\mu \gamma^\nu && | \quad \text{sum symmetric in } \mu, \nu \\
&= p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\}/2 \\
&= p_\mu p_\nu g^{\mu\nu} \mathbf{1} = m^2 \mathbf{1}
\end{aligned}
$$

b) Let us first collect some useful properties of the projectors P_R and P_L:

$$
\begin{aligned}
P_R P_L &= \left(1 - (\gamma^5)^2\right)/4 = 0 = P_L P_R \\
P_{R,L}^2 &= (1 \pm \gamma^5)^2/4 = \left(1 \pm 2\gamma^5 + (\gamma^5)^2\right)/4 = P_{R,L} \\
\gamma^\mu P_{R,L} &= \gamma^\mu (1 \pm \gamma^5)/2 = (1 \mp \gamma^5)/2 \gamma^\mu = P_{L,R} \gamma^\mu
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathrm{Tr}[(\not{p} + m) P_R (\not{p} - m) P_L] &= \mathrm{Tr}[(\not{p} + m) \not{p} P_L] \\
&= \mathrm{Tr}[\not{p}\not{p} P_L] && | \quad \mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^5] = 0 \\
&= \frac{1}{2} \mathrm{Tr}[\not{p}\not{p}] \\
&= \frac{m^2}{2} \mathrm{Tr}[\mathbf{1}] = 2m^2
\end{aligned}
$$

c)
$$
\begin{aligned}
\mathrm{Tr}[(\not{p} + m) P_R (\not{p} - m) P_R] &= -\mathrm{Tr}[(\not{p} + m) m P_R] \\
&= -\mathrm{Tr}[m^2 P_L] && | \quad \mathrm{Tr}[\gamma^5] = 0 \\
&= -2m^2
\end{aligned}
$$

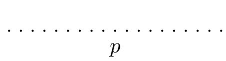
d)
$$
\begin{aligned}
\gamma^\mu \gamma^\rho \gamma^\sigma \gamma_\mu &= \left(2g^{\mu\rho} - \gamma^\rho \gamma^\mu\right) \gamma^\sigma \gamma_\mu \\
&= 2\gamma^\sigma \gamma^\rho - \gamma^\rho \left(2g^{\mu\sigma} - \gamma^\sigma \gamma^\mu\right) \gamma_\mu && | \quad \gamma^\mu \gamma_\mu = 4 \\
&= 2\left\{\gamma^\sigma, \gamma^\rho\right\} = 4g^{\sigma\rho} \mathbf{1}
\end{aligned}
$$

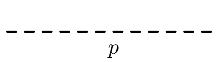
Problem 2

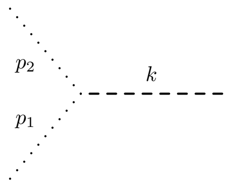
There was unfortunately a typo in the definition of the Lagrangian: the 3-point coupling involving derivatives was printed with the wrong overall sign, i.e.

$$
\mathcal{L} \supset -\varepsilon \frac{2\lambda}{M} \phi \left(\partial_\mu \phi\right) \left(\partial^\mu \Phi\right)
$$

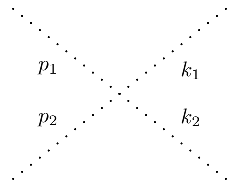
with $\varepsilon = 1$ (in the examination paper) vs. $\varepsilon = -1$ (intended). This really matters for the calculation of the amplitude in 2b), but not for any other part of problem 2. Obviously, any attempts to solve 2b) were thus treated very generously.

a)  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$

 $= \dfrac{i}{p^2 - M^2 + i\epsilon}$



$$= i\left(-2\lambda M - \varepsilon\tfrac{2\lambda}{M}(-ip_1 - ip_2)\cdot(-ik)\right) = -2i\lambda M\left(1 + \varepsilon\tfrac{k^2}{M^2}\right)$$



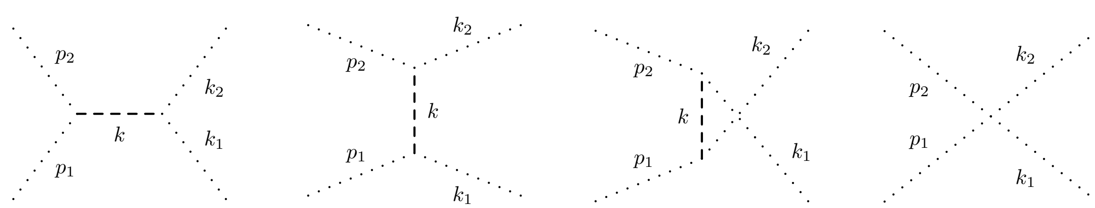
$$= i\left(-\lambda^2\frac{4!}{2} + \frac{2\cdot 4\lambda^2}{M^2}(k_1\cdot k_2 + k_1\cdot p_1 + k_1\cdot p_2 + k_2\cdot p_1 + k_2\cdot p_2 + p_1\cdot p_2)\right)$$

$$= -4i\lambda^2\left(3 + \frac{1}{M^2}\left[(p1+p2)^2 + (p1+k1)^2 + (p2+k1)^2\right]\right)$$

Notes:

- all momenta are here taken to be ingoing, hence we replaced $\partial_\mu \to -ik_\mu$ (this affects the final form of the 4-point, but not of the 3-point vertex)
- momentum conservation in the vertices was used to get rid of $p_1 + p_2$ (for the 3-point vertex) and k_2 (for the 4-point vertex)
- in general, the momenta do not have to be on shell (hence one cannot replace expressions like k^2 with an external mass)
- For the 2nd term in the 4-point vertex, the factor of four arises from two possible contractions of ϕ^2 times another two for $(\partial\phi)^2$.

b) There are four diagrams that contribute:



Using the Feynman rules derived above, the total amplitude is thus given by (note that now p_1, p_2 are ingoing while k_1, k_2 are outgoing!)

$$\begin{aligned}
i\mathcal{M} = &\frac{i}{(p_1+p_2)^2 - M^2}(-2i\lambda M)^2\left(1 + \varepsilon\frac{(p_1+p_2)^2}{M^2}\right)^2 \\
&+ \frac{i}{(p_1-k_1)^2 - M^2}(-2i\lambda M)^2\left(1 + \varepsilon\frac{(p_1-k_1)^2}{M^2}\right)^2 \\
&+ \frac{i}{(p_2-k_1)^2 - M^2}(-2i\lambda M)^2\left(1 + \varepsilon\frac{(p_2-k_1)^2}{M^2}\right)^2 \\
&- 12i\lambda^2 - \frac{4i\lambda^2}{M^2}\left[(p1+p2)^2 + (p1-k1)^2 + (p2-k1)^2\right] \\
= &-\frac{4i\lambda^2}{M^2}\left(\frac{(M^2+\varepsilon s)^2}{s-M^2} + \frac{(M^2+\varepsilon t)^2}{t-M^2} + \frac{(M^2+\varepsilon u)^2}{u-M^2}\right) \\
&- 12i\lambda^2 - \frac{4i\lambda^2}{M^2}\left[s+t+u\right]
\end{aligned}$$

For $\varepsilon = -1$, this simplifies to $\mathcal{M} = -8\lambda^2 M^{-2}[s+t+u] = -32\lambda^2 m^2/M^2$ (while for $\varepsilon = +1$ the expression becomes untractable once squared). For the differential cross section in the center-of-mass frame, we then have[1]

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi^2 s}|\mathcal{M}|^2 = \frac{16}{\pi^2 s}\lambda^4 \frac{m^4}{M^4}$$

c) $M > 2m$ is required by energy conservation for the decay $\Phi \to \phi\phi$. To lowest order, there is only a single diagram, namely the direct coupling of these three fields, see 1a). Using the vertex rule derived above, we have

$$|\mathcal{M}|^2 = 4\lambda^2 M^2 \left(1 + \varepsilon \frac{k^2}{M^2}\right)^2 = 4\lambda^2 M^2 \left(1+\varepsilon\right)^2$$

The decay rate is then given by

$$\Gamma = \frac{1}{2M}\int d\Pi_2 \, |\mathcal{M}|^2 = \frac{1}{32\pi M}\sqrt{1 - 4\frac{m^2}{M^2}}\, |\mathcal{M}|^2 \,,$$

where in the last step it was used that $\int d\Pi_2 = \frac{1}{2}\int_{-1}^{1} d(\cos\theta)\frac{1}{16\pi}\frac{2|\mathbf{p}_{\mathrm{cm}}|}{E_{\mathrm{cm}}}$ and $|\mathbf{p}_{\mathrm{cm}}| = \sqrt{(p^0_{\mathrm{cm}})^2 - m^2} = \sqrt{(M/2)^2 - m^2}$. Note the factor of $1/2$ in the expression for the phase-space integral (because of identical final state particles).

d) The Lagrangian is symmetric under $\phi \to -\phi$. Because only ϕ is charged under this $\mathcal{Z}_2$ symmetry, it is stable *to all orders in perturbation theory*. The simplest way to visualize this is by noting that all vertex rules involve $\phi\phi$ *pairs*; it is thus impossible to construct a diagram with an odd number of external ϕ states.

Problem 3

a) This Lagrangian is symmetric under $U(1)$ gauge transformations:

$$\begin{aligned}\phi &\to e^{iQ_\phi \alpha(x)}\phi\\ \psi &\to e^{iQ_\psi \alpha(x)}\psi\\ B_\mu &\to B_\mu - \frac{1}{g}\partial_\mu \alpha(x)\,,\end{aligned}$$

where $\alpha(x)$ is any real function. A mass term of the form $m_B^2 B_\mu B^\mu$ would violate this symmetry.

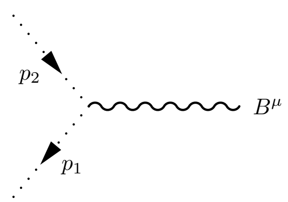
On top of that, the left-and right-handed components of the fermion field can be rotated independently:

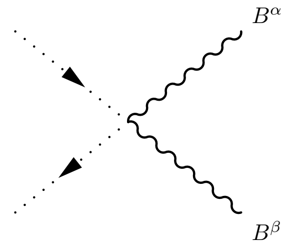
$$\begin{aligned}\psi_L &\to e^{i\beta}\psi_L\\ \psi_R &\to e^{i\gamma}\psi_R,\end{aligned}$$

[1] This expression can be obtained from an expression that was derived in the lecture, $d\sigma/dt = (64\pi s\,|\mathbf{p}_{\mathrm{CMS}}|^2)^{-1}\,|\mathcal{M}|^2$, by noting that $dt/d\cos\theta = 2\,|\mathbf{p}_1|\,|\mathbf{k}_1|$ – which is $2\,|\mathbf{p}_{\mathrm{CMS}}|^2$ *in the CMS.*

with any values for β, γ. This is known as 'chiral symmetry'; a fermion mass term $m_\psi \overline{\psi}_R \psi_L + h.c.$ would violate this symmetry.

b)



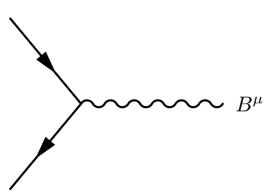


The above photon-scalar couplings[2] result from expanding

$$\begin{aligned}\mathcal{L} \supset |D\phi|^2 &= (\partial_\mu \phi + igQ_\phi B_\mu \phi)(\partial^\mu \phi^* - igQ_\phi B^\mu \phi^*) \\ &\supset g^2 Q_\phi^2 g^{\mu\nu} B_\mu B_\nu \phi\phi^* + igQ_\phi B_\mu (\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) \\ &= g^2 Q_\phi^2 g^{\mu\nu} B_\mu B_\nu \phi\phi^* - gQ_\phi (p'^\mu + p^\mu) \phi\phi^* B_\mu \,,\end{aligned}$$

where the last step follows from the (arbitrary) convention that assigns momenta as shown in the figure: p is the momentum of an ingoing scalar, which we know satisfies $\phi(x)|p\rangle = e^{-ipx}$ because $\phi(x) \propto \int d^3p \left(a_{\mathbf{p}} e^{-ipx} + b^\dagger_{\mathbf{p}} e^{ipx}\right)$. Likewise, p' is the momentum of an outgoing scalar with $\langle p'|\phi^*(x) = e^{ipx}$. (Note that the *same* rules result for an anti-scalar with *out*going momentum $(-p)$ and *in*going momentum $(-p')$!). From this, one can read off the Feynman rule for the four-point coupling as $ig^2 Q_\phi^2 2g^{\mu\nu}$ and the Feynman rule for the three-point coupling as $-igQ_\phi(p_1 + p_2)^\mu$.

The fermion scalar coupling $\mathcal{L}_{B\psi\overline{\psi}} = -gQ_\psi \overline{\psi} B_\mu \gamma^\mu P_R \psi$ derives from the covariant derivative acting on ψ. Hence the Feynman rule is



$$= \quad -igQ_\psi \gamma^\mu (1 + \gamma_5)/2$$

c) The field ϕ has to be expanded around the classical minimum energy solution $\langle \phi \rangle$, which has the consequence that the $U(1)$ symmetry is spontaneously broken, resulting in a massive real scalar and a massless real scalar (the Goldstone boson). Because this symmetry is a *gauge* symmetry, the Goldstone boson is 'eaten' by the gauge field A_μ, which thereby acquires a mass. This is known as the *Higgs mechanism*.

The photon mass term results from the term $\mathcal{L} \supset |D\phi|^2$. It is most easily derived by assuming that ϕ is *real*-valued – which can always be achieved via a

[2] Note that the scalar propagators need to be assigned a direction ('charge flow') to distinguish scalar and anti-scalar!

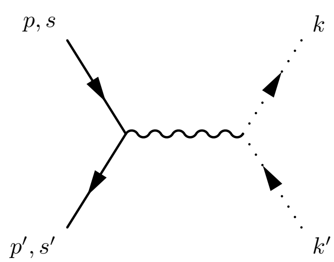
suitable gauge transformation $\alpha(x)$ (such that $\exp[iQ_\phi\alpha(x)]\phi$ is real). Then:

$$\begin{aligned}|D\phi|^2 &= (\partial_\mu\phi + igQ_\phi B_\mu\phi)(\partial^\mu\phi - igQ_\phi B^\mu\phi)\\ &= (\partial_\mu\phi)^2 + g^2Q_\phi^2\phi^2 B_\mu B^\mu\end{aligned}$$

For $\phi \to \langle\phi\rangle \equiv v$, the second term thus looks like a photon mass term, $\mathcal{L} \supset \frac{1}{2}m_B^2 B_\mu B^\mu$, with $m_A = \sqrt{2}gQ_\phi v$.

The fermion can *not* acquire a mass this way. Adding the simplest possible coupling between scalar field and fermions, $\mathcal{L} \supset \frac{y}{M}\phi\phi^*\overline{\psi}\psi = \frac{y}{M}\phi\phi^*(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L)$ breaks the chiral symmetry already before spontaneous symmetry breaking (note the difference to the case of the standard model electroweak theory!). Also the $U(1)$ gauge symmetry cannot be restored: the problem is that all fields (ϕ, ψ_R, ψ_L) carry different charges; any operator connecting those fields would thus have to contain *pairs* of the respective fields and their conjugate (to make it gauge-invariant) – but $\overline{\psi}_R\psi_R = \overline{\psi}_L\psi_L = 0$.

d)



The above diagram is the only one that contributes to this process. Spontaneous symmetry breaking does *not* affect any of the vertex rules derived in b), but it gives a mass to the 'photon' propagator.[3] The amplitude is thus given by

$$\begin{aligned}i\mathcal{M} &= \overline{u}(p)(-igQ_\psi\gamma^\mu P_R)v(p')\frac{-ig_{\mu\nu}}{(p+p')^2 - m_B^2}(-igQ_\phi)(k-k')^\nu\\ &= i\frac{g^2Q_\psi Q_\phi}{s-m_B^2}\overline{u}(p)(\not{k} - \not{k}')P_R v(p')\,.\end{aligned}$$

Taking into account the initial state spin degrees of freedom, we thus have (note that the fermions are massless!)

$$\begin{aligned}\overline{|\mathcal{M}|}^2 &= \frac{1}{4}\frac{g^4Q_\psi^2Q_\phi^2}{(s-m_B^2)^2}\mathrm{Tr}\left[\not{p}(\not{k}-\not{k}')P_R\not{p}'P_L(\not{k}-\not{k}')\right]\\ &= \frac{1}{8}\frac{g^4Q_\psi^2Q_\phi^2}{(s-m_B^2)^2}\mathrm{Tr}\left[\not{p}(\not{k}-\not{k}')\not{p}'(\not{k}-\not{k}')(1+\gamma^5)\right]\\ &= \frac{1}{2}\frac{g^4Q_\psi^2Q_\phi^2}{(s-m_B^2)^2}\left[2p\cdot(k-k')p'\cdot(k-k') - p\cdot p'(k-k')^2\right]\end{aligned}$$

[3] We did not actually quantize such theories in the lecture, so I will not deduct any points for using a wrong propgator – as long as you made clear that it should contain a mass term. In the expression that follows, the Feynman gauge is chosen.

The γ^5 contribution in the last step vanishes because $\epsilon^{\mu\nu\rho\sigma}p^1_\mu p^2_\nu p^3_\rho p^4_\sigma = 0$ whenever the four p^i_μ are not linearly independent (which they are not because of momentum conservation). To continue, we use again $p + p' = k + k'$ to find

$$\begin{aligned}
p\cdot k &= -\frac{1}{2}[(p-k)^2 - m_\psi^2 - m_\phi^2] = \frac{1}{2}[-t + m_\psi^2 + m_\phi^2] = p'\cdot k' \\
p\cdot p' &= \frac{1}{2}s - m_\psi^2, \qquad\qquad k\cdot k' = \frac{1}{2}s - m_\phi^2 \\
p\cdot k' &= p\cdot(p+p'-k) = \frac{1}{2}[s + t - m_\psi^2 - m_\phi^2] = p'\cdot k
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{|\mathcal{M}|}^2 &= \frac{1}{2}\frac{g^4Q_\psi^2Q_\phi^2}{(s-m_B^2)^2}\left[-2(p\cdot k - p\cdot k')^2 - p\cdot p'(k-k')^2\right] \\
&= \frac{1}{2}\frac{g^4Q_\psi^2Q_\phi^2}{(s-m_B^2)^2}\left[-2(-\frac{s}{2} - t + m_\psi^2 + m_\phi^2)^2 - \left(\frac{s}{2} - m_\psi^2\right)\left(-s + 4m_\phi^2\right)\right] \\
&= \frac{g^4Q_\psi^2Q_\phi^2}{(s-m_B^2)^2}\left[m_\psi^2 s - 4t\left(\frac{s}{2} + \frac{t}{2} - m_\psi^2 - m_\phi^2\right) - 2\left(m_\psi^2 + m_\phi^2\right)\right] \\
&= \frac{g^4Q_\psi^2Q_\phi^2}{(s-m_B^2)^2}\left[m_\psi^2 s + 2\left(tu - m_\psi^2 - m_\phi^2\right)\right]
\end{aligned}$$

Problem 4

a) This Lagrangian describes an N-plet fermion field ψ charged under an arbitrary gauge group G, with interaction fields A^a_μ. The transformation of these fields under infinitesimal gauge transformations is as follows:

$$\begin{aligned}
\psi^i &\to \psi^i + i\alpha^a\,(t^a)^{ij}\,\psi^j \\
A^a_\mu &\to A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c\,,
\end{aligned}$$

where $\alpha^a(x)$ are arbitrary real-valued functions, t^a are the generators and f^{abc} the structure constants of G.

b)

$$\begin{aligned}
D_\mu &= \partial_\mu - igA^a_\mu t^a \\
F^a_{\mu\nu} &\equiv \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu
\end{aligned}$$

c) The mass term does not appear in the standard model. This is because

$$m\overline{\psi}\psi = m\overline{\psi}(P_R + P_L)\psi = m(\overline{\psi_R}\psi_L + \overline{\psi_L}\psi_R)\,,$$

where $\psi_{R,L} \equiv P_{R,L}\psi$. In the standard model, left-and right-handed fields are assigned different charges under $U(1)_Y$ and $SU(2)_L$ and thus transform differently under the underlying symmetry group $SU(2)_L \times U(1)$. In other words, a mass term would violate gauge invariance.

d)

- We noticed that the Lagrangian for a *free* fermion multiplet ψ, $\mathcal{L}_{\text{free}} = \overline{\psi}^i \,(i\partial\!\!\!/ - m)\,\psi^i$, is invariant under any unitary transformation $\psi \to V\psi$. This implies that the theory is *globally invariant* under any continuous group G of transformations that is represented by hermitian generators t^a (such that $V = \exp[i\alpha^a t^a]$, where α^i are real parameters).
- We then demanded that the theory should exhibit this symmetry in every space-time point independently, i.e. be *locally invariant* under $\psi(x) \to \exp[i\alpha^a(x)t^a]\psi(x)$ for any set of real *functions* $\alpha^a(x)$.
- While $m\overline{\psi}\psi$ is already locally symmetric, $i\overline{\psi}\partial\!\!\!/\psi$ is not. In oder to compensate for the extra terms induced by the derivate, we replaced the standard derivate ∂_μ by the covariant derivative D_μ as given in b). This forced us to introduce gauge fields with the transformation properties stated in a).
- It can then be shown that any globally symmetric function of ψ, $D\psi$, $F^a_{\mu\nu}$ and $DF^a_{\mu\nu}$ is also locally symmetric. The Yang-Mills Lagrangian contains all possible gauge-invariant terms up to dimension-4 operators, with the additional requirement that P and T are conserved.

e)

- The kinetic term for the gauge fields, described by the quadratic terms contained in $\frac{1}{4}(F^a_{\mu\nu})^2$, is singular – which makes the path-integral over the action ill-defined and does not allow to inverted the kinetic terms to derive the gauge field propagators.
- ξ is a parameter that determines which specific gauge is chosen for the calculation of an amplitude (e.g. $\xi = 1$ for Feynman gauge). Unlike individual diagrams, the final result must be independent of ξ (which in practice provides an important check for the validity of more complex calculations).
- The ghosts $\bar{c}, c$ are Grassmann-valued (i.e. anti-commuting) scalar fields. They thus show the wrong spin-statistics relation and are no real particles in the sense that they cannot appear in external states. They do appear in loop calculations, however, where they serve to cancel the unphysical time-like and longitudinal degrees of freedom of gauge bosons.
- While the formal expression $D_\mu = \partial_\mu - igA^a_\mu t^a$ is the same, the generators t^a are in general different: Ghosts transform in the adjoint representation of the gauge group, i.e. in the same representation as the field strengths – which implies e.g. that the number of ghosts and gauge fields is the same. The fermion multiplets, on the other hand, typically transform in a lower-dimensional representation (the representation with the lowest possible dimension – the so-called ‘fundamental’ representation – being the standard case).