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http://www.uio.no/studier/emner/matnat/fys/FYS4160/v17/

Lecture spring 2017:

General Relativity

Problem sheet 7

⇝ These problems are scheduled for discussion on **Thursday, 16 March 2016**.

Problem 25

Consider the stress-energy tensor for a perfect fluid,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + \eta^{\mu\nu} p$$

a) Use the conservation of this tensor, $\partial_\mu T^{\mu\nu} = 0$, in the non-relativistic limit to derive familiar results from classical hydrodynamics. In particular, derive the

   - continuity equation, $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$
   - Euler Equation, $\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho}$

   [Hint: Have a look at 1.9 in the book. ;) But make sure that you really understand each step!]

b) Now generalize the result to the Navier-Stokes equations, by allowing for off-diagonal (shear) terms in $T_{\mu\nu}$.

Problem 26

Noether's theorem states that for every *symmetry* in the Lagrangian – a field transformation that leaves the equations of motion invariant – there is a *conserved quantity.* There thus exists a 4-vector j_N^μ that is divergence-free, $\partial_\mu j_N^\mu = 0$, and the conserved quantity (the 'Noether charge') is given by $Q_N = \int d^3x\, j_N^0$. You are encouraged to look up the general proof (it's one of the most important results in theoretical physics – and actually not that difficult!), which allows you to compute j_N^μ for any such field transformation.

Let us now consider the specific case of space-time translations $x^\nu \to x^\nu + a^\nu$, where a^μ is a constant 4-vector. Any theory formulated in Minkowski space must be invariant under these transformations, so there must be *four* conserved currents $j_N^{(\nu)\,\mu} \equiv T^{\mu\nu}$ (one for each value of ν) and the corresponding charges are the total energy and 3-momentum of the system, respectively. The energy-momentum tensor, of any field $\phi^a(x)$, obtained in this way is given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial^\nu \phi^a - \eta^{\mu\nu} \mathcal{L} \,,$$

where $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a)$ is the Lagrangian (density). Now consider the general Lagrangian of a scalar field in an arbitrary potential, as seen in the lecture, and derive the stress-energy tensor! Try to give an interpretation of the individual terms that appear in the resulting expression.

Problem 27

As we will see shortly in the lecture, the field equations follow from varying the Einstein-Hilbert action,

$$S_H = \int d^4x \sqrt{-g} R \,.$$

a) The Lagrangian thus contains up to *second* derivatives of the fields (i.e. the metric). Why does one typically not encounter this situation in field theories? Why is this not a worry in this particular case? *[Hint: Show that the* action *actually only depends on first derivatives of the metric]*

b) In deriving the field equations from the above action, we used that the *variation* of the Christoffel symbols, $\delta \Gamma^\mu_{\rho\sigma}$, transforms like a tensor. Show that this is indeed the case!