FYS4160-2017
Problem sheet 9

NOTE: These notes are complementary to the discussion in the exercise sessions and are thus very condensed. If you want to discuss parts in more detail, please use our platform PIAZZA. If you spot some typos or mistakes, please email to *magdalena.kersting@fys.uio.no.*

Problem 30: Eddington-Finkelstein coordinates

The null geodesic equation in the (t, r) coordinate system is,

$$\frac{dt}{dr} = \pm(1 - R_s/r)^{-1}, \tag{1}$$

where, $\pm$ denotes outgoing and incoming worldlines respectively.Integrating this equation we get,

$$t = \pm\left(r + R_s \ln\left(r/R_s - 1\right)\right) + \text{constant} = \pm r_* + \text{constant} \tag{2}$$

Hence, $u = t - r_* = \text{constant}$ and $v = t + r_* = \text{constant}$ describe the outgoing and the incoming null geodesics respectively.

For discussion on light-cones see Figures 5.7, 5.10 and 5.11 in the book. At $r = R_s$, in v (or u) one solution corresponds to the light trapped at R_s and the other shows the incoming (or outgoing) light path.

Problem 31: Object falling into a black hole

We have the line element (defining $2GM = R_s$), $ds^2 = -(1 - \mu/r)dt^2 + (1 - R_s/r)^{-1}dr^2 + r^2 d\Omega^2$, which for radial motion reduces to,

$$ds^2 = -(1 - R_s/r)dt^2 + (1 - R_s/r)^{-1}dr^2 \tag{3}$$

a) For the beacon the relation between the coordinates is,

$$1 = (1 - R_s/r)\left(\frac{dt}{d\tau}\right)^2 - (1 - R_s/r)^{-1}\left(\frac{dr}{d\tau}\right)^2. \tag{4}$$

Now as the metric is independent of t, we can say that $(1 - R_s/r)\frac{dt}{d\tau}$ is a conserved quantity (see eqn. 5.61). Also, at $r = r_*$, we know that the beacon is 'dropped', hence the initial velocity is zero, $dr/dt|_{r=r_*} = 0$. This implies, at r_*, $d\tau_{r_*} = \sqrt{1 - R_s/r_*}dt_{r_*}$ (from eqn. 4) and our conserved quantity has the value,

$$(1 - R_s/r)\frac{dt}{d\tau} = \sqrt{1 - R_s/r_*} \tag{5}$$

Substituting this result into eqn.4, we get,

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{R_s}{r} - \frac{R_s}{r_*}} \tag{6}$$

Note that we have to consider the negative result as dr decreases with time. And using chain rule we get the coordinate velocity,

$$\frac{dr}{dt} = -(1 - R_s/r)\sqrt{\frac{R_s(r_* - r)}{r(r_* - R_s)}} \tag{7}$$

b) For the comoving observer the proper time goes as, $dt_{obs2} = \sqrt{1 - R_s/r}dt$ and the proper distance measured is, $dr_{obs2} = (1 - R_s/r)^{-1/2}dr$. Therefore the proper speed measured by this observer is,

$$\left|\frac{dr_{obs2}}{dt_{obs2}}\right| = \sqrt{\frac{R_s(r_* - r)}{r(r_* - R_s)}} \tag{8}$$

c) The frequency of a photon traveling on a null geodesic $x^\mu(\lambda)$, as observed by an observer traveling with Four-velocity U^μ is (eqn 5.100 in the book),

$$\omega = -g_{\mu\nu} U^\mu \frac{dx^\nu}{d\lambda}. \tag{9}$$

The beacon observes the photon frequency to be,

$$\begin{aligned}
\omega_{em} &= -g_{\mu\nu} U_b^\mu \frac{dx^\nu}{d\lambda_\gamma} \\
&= (1 - R_s/r_{em}) \frac{dt}{d\tau_b} \frac{dt}{d\lambda_\gamma} - (1 - R_s/r)^{-1} \frac{dr}{d\tau_b} \frac{dr}{d\lambda_\gamma} \\
&= (1 - R_s/r_{em}) \frac{dt}{d\tau_b} \frac{dt}{d\lambda_\gamma} - \frac{dr}{d\tau_b} \frac{dt}{d\lambda_\gamma} \qquad \because \frac{dr}{d\lambda_\gamma} = +(1 - R_s/r) \frac{dt}{d\lambda_\gamma} \\
&= \frac{dt}{d\lambda_\gamma} \left(\sqrt{1 - R_s/r_*} + \sqrt{R_s/r_{em} - R_s/r_*} \right) \\
&= \frac{E_\gamma}{1 - R_s/r_{em}} \left(\sqrt{1 - R_s/r_*} + \sqrt{R_s/r_{em} - R_s/r_*} \right) \qquad \because E_\gamma = (1 - R_s/r) dt/d\lambda \text{ is a constant of motion}
\end{aligned} \tag{10}$$

The observer at rest at $r = r_*$ observes the frequency as,

$$\begin{aligned}
\omega_{obs} &= -g_{\mu\nu} U_{obs}^\mu \frac{dx^\nu}{d\lambda_\gamma} \\
&= (1 - R_s/r_*)(1 - R_s/r_*)^{-1/2} \frac{dt}{d\lambda_\gamma} \\
&= \frac{E_\gamma}{\sqrt{1 - R_s/r_*}}
\end{aligned} \tag{11}$$

Therefore,

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{\omega_{em}}{\omega_{obs}} = \frac{\sqrt{1 - R_s/r_*}}{1 - R_s/r_{em}} \left(\sqrt{1 - R_s/r_*} + \sqrt{R_s/r_{em} - R_s/r_*} \right) \tag{12}$$

d) The coordinate time required by the beacon to reach r_{em} is,

$$t_1 = \int_{r_{em}}^{r_*} (1 - R_s/r)^{-1} \sqrt{\frac{r(r_* - R_s)}{R_s(r_* - r)}} \tag{13}$$

and the coordinate time taken by the photon to reach back the observer is,

$$t_2 = \int_{r_{em}}^{r_*} (1 - R_s/r)^{-1} \tag{14}$$

Hence the total coordinate time is,

$$t_{obs} = t_1 + t_2 = \int_{r_{em}}^{r_*} (1 - R_s/r)^{-1} \left(1 + \sqrt{\frac{r(r_* - R_s)}{R_s(r_* - r)}} \right) \tag{15}$$

e) For late times, $r \to R_s$,

$$\lambda_{obs}/\lambda_{em} \to 2 \frac{1 - R_s/r_*}{1 - R_s/r_{em}} \propto \frac{r_* - R_s}{r_{em} - R_s} \tag{16}$$

and,

$$t_{obs} \sim 2t_1 = 2 \int_{r_{em}}^{r_*} (1 - R_s/r)^{-1} = 2 \left(r_* - r_{em} + R_s \ln \frac{r_* - R_s}{r_{em} - R_s} \right) \sim 2R_s \ln \frac{r_* - R_s}{r_{em} - R_s} \tag{17}$$

hence, $\lambda_{obs}/\lambda_{em} \propto e^{t_{obs}/T}$, where T is some constant in terms of R_s.

Problem 32: Reaching the singularity of a black hole

1. Inside the black hole,

$$\begin{aligned} 1 &= (2GM/r-1)^{-1}\frac{dr^2}{d\tau^2} - (2GM/r-1)\frac{dt^2}{d\tau^2} - r^2\frac{d\Omega^2}{d\tau^2} \\ \therefore \quad & (2GM/r-1)^{-1}\frac{dr^2}{d\tau^2} \geq 1 \qquad \text{or} \quad \left|\frac{dr}{d\tau}\right| \geq \sqrt{(2GM/r-1)} \end{aligned} \tag{18}$$

2. Maximal proper time would be achieved when, $d\tau = -(2GM/r-1)^{-1/2}dr$, Integrating this you get,

$$\tau = \sqrt{r(2GM-r)} + GM \arctan\left(\frac{GM-r}{\sqrt{r(2GM-r)}}\right) \tag{19}$$

which for $r \to 2GM$, $\tau \to GM\arctan(-\infty) = -\frac{\pi}{2}GM$, and for $r \to 0$, $\tau \to GM\arctan(\infty) = \frac{\pi}{2}GM$. Thus, $\tau_{max} = \pi GM$.

3. Restoring c in the expression ($GM \to GM/c^2$ and $\tau \to c\tau$) we get,

$$\implies \tau_{max} = \pi GM/c^3 \sim 5\pi M_\odot \mu s \tag{20}$$

4. Considering the motion along a geodesic, i.e., $(1-2GM/r)dt/d\tau = E$ =constant.

$$\begin{aligned} d\tau^2 &= (2GM/r-1)^{-1}dr^2 - (2GM/r-1)dt^2 \\ d\tau^2 &= (2GM/r-1)^{-1}dr^2 - (2GM/r-1)^{-1}E^2 d\tau^2 \\ (dr/d\tau)^2 &= E^2 + (2GM/r-1) \end{aligned} \tag{21}$$

Hence the maximal proper time is achieved for $E = 0$ for a free fall.