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REVIEW

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Review

A hundred years with the cosmological constant

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Abstract

The main points in the history of the cosmological constant are briefly discussed. As a conceptual background, useful for teaching of physics at an elementary college and university level, Newton's theory formulated locally in terms of the Poisson equation is presented, and it is shown how it is modified by the introduction of the cosmological constant. The different physical interpretations of the cosmological constant, as introduced by Einstein in 1917 and interpreted by Lemaître in 1934, are presented. Energy conservation in an expanding universe dominated by vacuum energy is discussed. The connection between the cosmological constant and the quantum mechanical vacuum energy is mentioned, together with the problem that a quantum mechanical calculation of the density of the vacuum energy gives a vastly too large value of the cosmological constant. The article is concluded by reviewing a solution of this problem that was presented on May 11, 2017.

Keywords: gravity, spacetime, cosmological constant, general relativity

1. Einstein's introduction of the cosmological constant

The cosmological constant was introduced into the general theory of relativity by Einstein [1] in 1917 in order to be able to construct a static universe model. It follows from the way he introduced it into the field equations that it was originally interpreted as expression of a natural tendency of empty space to expand.

In 1934 Lemaître [2] gave a new physical interpretation of the constant. According to him it represents the energy density of Lorentz invariant vacuum energy (LIVE). In a homogeneous, expanding universe with a perfect (non-viscous) fluid with no large scale temperature differences, and no processes generating heat, the expansion is adiabatic. As will be shown below, it then follows from the Friedmann equations that the density of LIVE is constant during the expansion, and hence that it can be represented by the cosmological constant.

Calculations by Zel'dovich [3] and Grøn [4] demonstrated that if all the components of the energy-momentum density tensor of a perfect fluid are Lorentz invariant, then the tensor must be proportional to the metric tensor, and from the relativity of simultaneity it then follows that the proportionality factor must be constant. This proportionality constant is essentially equal to the density of LIVE. From this it follows that the density of LIVE appears as a cosmological constant in Einstein's field equations.

The history of, and problems with, the cosmological constant have been reviewed in several great articles [5–7] with a large number of further references.

2. Newton's theory of gravitation

February 15, 1917 Einstein [1] presented an article that became the foundation of a new science: relativistic cosmology. Let us follow Einstein's thoughts in the first part of this article. He started by discussing Newtonian gravitational theory.

The main laws of this theory are a law for the gravitational force and a law relating how bodies move when acted upon by forces. Assume that a spherical body, for example the Sun, with mass m_1 is at the centre of a system of spherical coordinates, and that there is a particle with mass m_2 at a distance r from the centre. Then Newton's law of gravity says that there acts an attractive gravitational force between the bodies, given by

$$F = \frac{Gm_1 m_2}{r^2}, \tag{1}$$

where G is Newton's gravitational constant. Furthermore, Newton's second law says that force equals mass times acceleration, and Newton's first law that if no forces act on a body it will remain at rest if it was originally at rest. Both in Newton's theory and in the general theory of relativity an inertial reference frame is defined as a frame in which Newton's first law is valid. According to Newton's theory an inertial frame is a non-rotating frame that either is at rest or moves with constant velocity. Such frames have unlimited extension according to Newton's theory. In Einstein's theory gravitation is not reckoned as a force, and hence an inertial frame is a non-rotating frame in free fall. Due to tidal forces an inertial frame has a local character according to Einstein's theory, where 'local' means that the extension in space and time is so small that tidal forces cannot be measured within the frame.

The potential energy of a body is equal to the work performed on the body in order to move it from a level with zero potential energy to its present position. The *potential* is the potential energy per mass unit. Since the equation of motion is independent of the position with zero potential energy, this position can be selected freely. In the situation described above it is usual to choose the zero level infinitely far from the centre of the central body. Then the potential at a distance r from the centre is

$$\phi = -\frac{Gm_1}{r}. \tag{2}$$

Since the gravitational equations of Einstein's theory have the form of differential equations, it simplifies the comparison of the Newtonian limit of Einstein's theory with Newton's theory to write the law of gravity in Newton's theory as a differential equation. Then Newton's law of gravity takes the form of a Poisson equation [8]

$$\nabla^2 \phi = 4\pi G \rho, \tag{3}$$

where ρ is the mass density at the point with potential ϕ, and the ∇-operator in spherical coordinates, applied to the gravitational potential, takes the form

$$\nabla \phi = \frac{\mathrm{d}\phi}{\mathrm{d}r}\mathbf{e}_{\mathrm{r}}, \quad \nabla^2 \phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\phi}{\mathrm{d}r}\right). \tag{4}$$

Here $\mathbf{e}_{\mathrm{r}}$ is a radial unit vector. If there is no mass at the position where the potential is to be calculated, the gravitational equation reduces to Laplace's equation

$$\nabla^2 \phi = 0. \tag{5}$$

The potential (2) is a solution of the Laplace equation at a point with distance r from the centre of a spherical body with mass m_1. The acceleration of a test particle that is acted upon by gravity only, i.e. the *acceleration of gravity*, is minus the gradient of the potential,

$$\mathbf{g} = -\nabla \phi. \tag{6}$$

Hence Newton's theory of gravitation can be summarized as follows. Mass determines the potential according to Poisson's equation, and the potential determines the acceleration of gravity from the gradient of the potential, i.e. *mass there determines motion here.*

When Einstein was about to construct a static universe model, he discovered that a universe filled by matter, which was originally static, would collapse under attractive gravity due to the matter. As a remedy of this Einstein changed the theory of gravitation and introduced a cosmological constant Λ into it. For Einstein Λ represented a natural tendency of space to expand so that attractive gravity due to matter was necessary to keep the universe in static equilibrium. Einstein concluded his article by writing: 'Λ is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars.'

The Dutch astronomer Willem de Sitter became inspired by Einstein's article and presented, also a hundred years ago, a new universe model [9]. It was empty and was a solution of Einstein's field equations with a cosmological constant. The spacetime of this model is called *the de Sitter spacetime* and corresponds to the Minkowski spacetime of the theory without a cosmological constant. De Sitter described this universe model with reference to a rigid, static reference frame. In this frame his solution looked like a static universe model. However, there was a field of gravity in this frame, making freely moving particles fall away from each other. If one used these freely falling particles as reference particles for space, and introduced a reference frame expanding together with the free particles, the solution appeared as an expanding universe model, which was later presented as the steady state model of the universe [10, 11].

In order to obtain some intuition about the physical significance of the cosmological constant, we shall now discuss how Newton's theory of gravity is modified by the introduction of the cosmological constant.

3. Modified Newtonian theory of gravity with a cosmological constant

Einstein began his cosmology article [1] by discussing the modified Newtonian theory of gravity with a cosmological constant. He then replaced equation (3) by

$$\nabla^2 \phi - \Lambda \phi = 4\pi G \rho, \tag{7}$$

where $\Lambda > 0$ is the cosmological constant. This is, however, not the correct limit of the general theory of relativity with a cosmological constant. The correct equation is [12]

$$\nabla^2\phi + \Lambda = 4\pi G\rho. \tag{8}$$

Let us study the physical significance of Λ by considering a space where $\rho = 0$ at all places outside the central body. Then the solution of equation (8) is

$$\phi = -\frac{Gm_1}{r} - \frac{\Lambda}{6}r^2. \tag{9}$$

The acceleration of gravity is

$$\mathbf{g} = -\frac{Gm_1}{r^2}\mathbf{e}_\mathrm{r} + \frac{\Lambda}{3}r\,\mathbf{e}_\mathrm{r}. \tag{10}$$

At a small distance from the central body its attractive gravity dominates, and a free particle falls towards it. At a great distance the repulsion due to the cosmological constant dominates, and a free particle falls outwards. The surface with vanishing acceleration of gravity has a radius r_0 given by

$$\frac{\Lambda}{3}r_0 = \frac{Gm_1}{r_0^2}, \tag{11}$$

This can be understood in the light of Lemaître's physical interpretation of the cosmological constant, which we need here, but will be discussed in a historical context below. Lemaître proposed [2] in 1934 that the cosmological constant represents the density of a quantum mechanical vacuum energy, and he argued that if it is impossible to measure velocity relative to the vacuum energy, then it will have constant density during the adiabatic expansion of a spatially homogeneous universe. The cosmological constant represents the constant density, ρ_Λ, of this LIVE[1] according to

$$\Lambda = 8\pi G\rho_\Lambda. \tag{12}$$

It follows from equations (11) and (12) that the transition radius between attraction and repulsion is

$$r_0 = \left(\frac{3m_1}{8\pi\rho_\Lambda}\right)^{1/3}. \tag{13}$$

The mass of the LIVE inside a spherical surface with radius r is

$$m_\Lambda = \frac{4\pi}{3}\rho_\Lambda r^3. \tag{14}$$

Combining equations (13) and (14) we see that in Newtonian theory, where the gravitational action of pressure is neglected, the radius of a spherical surface with vanishing acceleration of gravity is given by the condition that the mass of the LIVE inside the surface is equal to the mass of the body at the centre, $m_\Lambda(r_0) = m_1$.

As an illustration we calculate the transition radius for the Solar system in the present universe. Then the density of the LIVE is $\rho_\Lambda = \Omega_\Lambda\rho_{\mathrm{cr}}$, where the mass parameter of LIVE is $\Omega_\Lambda = 0.7$ and the critical density is $\rho_{\mathrm{cr}} \approx 10^{-26}\,\mathrm{kg\,m^{-3}}$. The mass of the Sun is $m_1 = 2 \times 10^{30}$ kg. Inserting these quantities into equation (13) gives a transition radius of approximately 300 l.y. Since the distances in the solar system are much smaller than this, the cosmological constant is of negligible significance for solar system gravitational effects [13].

[1] The term 'LIVE' for Lorentz Invariant Vacuum Energy was not used by Lemaître. I think it was introduced in the article referred to in [20] below.

At distances much larger than r_0 the first term on the right-hand side of equation (10) can be neglected and the equation of motion of a free particle takes the form

$$\ddot{r} - \frac{\Lambda}{3} r = 0. \tag{15}$$

The solution of this equation representing motion of a free particle away from the observer at the origin is

$$r(t) = r(0) \mathrm{e}^{H\,t}, \quad H = \sqrt{\Lambda/3}, \tag{16}$$

where H is the Hubble parameter.

In the theory of relativity free particles such as the one with a motion described by equation (16) are reference particles defining space. Hence the relativistic interpretation of equation (16) is that space expands exponentially, like in the steady state universe.

If the universe is filled by matter with a uniform density ρ_0, the matter may be at rest in an equilibrium between the attractive gravity of the matter and the repulsive gravity of LIVE. In this case the solution of equation (8) is the constant potential

$$\phi = \frac{4\pi G \rho_0}{\Lambda}. \tag{17}$$

However, as was shown by Arthur Eddington [14] in 1930, this equilibrium is unstable.

4. Relativistic expanding universe models

The first expanding relativistic universe models with matter were constructed by the Russian researcher Alexander Friedmann in two epochal articles [15, 16] that appeared in 1922 and 1924. He constructed solutions of Einstein's field equations both with and without a cosmological constant. The solutions are the foundations of the present models of the universe. Friedmann did not have good enough observational data at his disposal to test his models against observations.

Georges Lemaître was a Belgian pastor who was also a mathematician and a creative cosmologist. In addition he had updated knowledge about astronomical observations. During the years from 1927 to 1935 he presented a series of cosmology articles that laid the foundation for the modern Big Bang theory of the universe. The first one [17] was written in French and published in a Belgian journal that nearly nobody outside Belgium read. Translated to English the title reads: *A Homogeneous Universe of Constant Mass and Growing Radius Accounting for the Radial Velocity of Extragalactic Nebulae*. Note that what was called 'nebulae' are called 'galaxies' today.

In this heading Lemaître presents an interpretation of the redshift of the spectral lines in light from remote galaxies, which has become the standard interpretation in present cosmology—namely that the redshift is due to the expansion of the universe, i.e. that the expanding space [18] stretches the waves while they move from the emitter to the observer. It is not a Doppler effect due to motion of galaxies *through* space.

Lemaître also showed that the velocity due to the expansion of the universe is proportional to the distance from the observer,

$$v = H\,\hat{r}. \tag{18}$$

He estimated the distances to the galaxies using Hubble's assumption that they all had the same luminosity. The value he found for the Hubble parameter was $H = 192\,\mathrm{km\,s^{-1}}$ per million l.y. With constant expansion velocity this value of H gives an age of only 1.7 billion

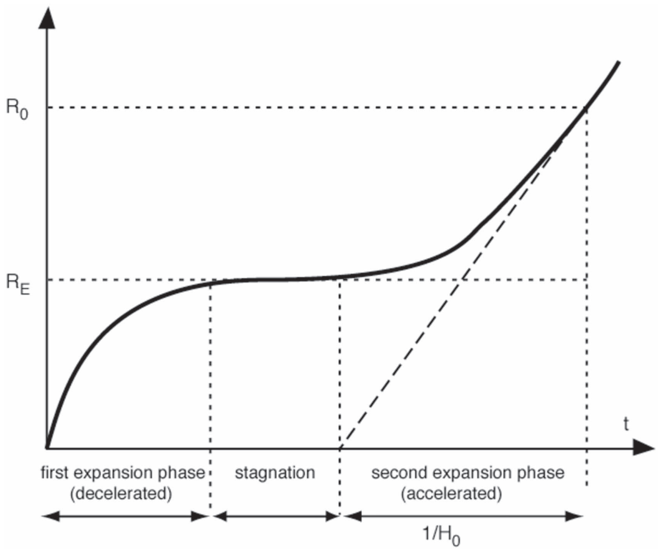

Figure 1. The evolution of the cosmic distance with time in Lemaître's universe model of 1931. The quantity $1/H_0$ where H_0 is the present value of the Hubble parameter, is the so-called Hubble age of the universe. It is the age that the universe would have had if it expanded with the present velocity during its whole history.

years for the universe. This was less than the age of the oldest stars. This was called the age problem of the universe model.

In 1931 Lemaître published an article [19] where he presented a new universe model that solved the age problem. He introduced a spherical universe with finite space but no boundaries and wrote that we may imagine that the space appears as a first atom. It was a universe filled by cold matter, and it was a solution of Einstein's field equations with a cosmological constant. In this model the universe came into existence a finite time ago in a cosmic explosion. This was the first Big Bang model of the universe. The radius of the universe was originally zero. The subsequent expansion took place in three eras (figure 1). The universe came into existence with infinitely large expansion velocity. The velocity then decreased due to the matter's attractive gravity. Then the universe entered the second so-called 'hesitating era' or 'stagnation' with very slow expansion. In this era there was close to equilibrium between attractive gravity due to the matter and repulsion associated with the cosmological constant. This era had a duration that could be adjusted by a proper choice of the value of Λ so that the age of the universe was in agreement with the requirements from observational data. Finally the universe entered a third era with accelerated expansion dominated by the cosmological constant. Lemaître wrote that we without doubt live in the third era. Hence Lemaître not only predicted the expansion of the universe, but also that there is accelerated expansion in the present era. His model is surprisingly similar to the present standard model of the universe [20].

In 1934 Lemaître introduced a new interpretation of Λ. He wrote [2]: 'The theory of relativity suggests that, when we identify gravitational mass and energy, we have to introduce a constant. Everything happens as though the energy in vacuum would be different from zero. In order that it shall not be possible to measure motion relative to the vacuum, we have to associate a pressure to the energy density of the vacuum. This is essentially the meaning of the cosmological constant.' Lemaître also indicated that the density of the Lorentz invariant vacuum energy (LIVE), was constant during the expansion of the universe. This made it possible to let the cosmological constant represent the energy density of LIVE.

5. The two cosmological constant problems

According to the general theory of relativity the relationship between the cosmological constant and the density of the vacuum energy is given in equation (12), and the gravitational Tolman–Whittaker mass density is [21–23]

$$\rho_{\text{grav}} = \rho + 3p/c^2. \tag{19}$$

In 1968 the Russian physicist Yakov B Zel'dovich [3] developed Lemaître's interpretation further and showed by explicit calculation that the requirement that all of the components of the energy-momentum tensor of the quantum mechanical vacuum energy are Lorentz invariant leads to the result that the energy-momentum tensor is proportional to the metric tensor. If the vacuum energy is described as a perfect fluid with mass density ρ_{V}, pressure p_{V} and four-velocity components u_μ, the energy-momentum density tensor has the form

$$T_{\mu\nu} = (\rho_{\text{V}} + p_{\text{V}}/c^2)u_\mu u_\nu + p_{\text{V}} g_{\mu\nu}. \tag{20}$$

The vacuum energy has the equation of state

$$p_{\text{V}} = -\rho_{\text{V}} c^2, \tag{21}$$

and a gravitational mass density

$$\rho_{\text{V grav}} = -2\rho_{\text{V}} < 0. \tag{22}$$

The acceleration of gravity is proportional to ρ_{grav}. This means that LIVE causes repulsive gravitation.

In a homogeneous universe model with no generation of heat, there is cosmic adiabatic expansion. Let U be the energy inside a volume V co-moving with the expanding space, and a the scale factor. Then

$$V = (4\pi/3)a^3, \quad U = \rho\, c^2 V. \tag{23}$$

The first law of thermodynamics for adiabatic expansion,

$$\mathrm{d}U + p\mathrm{d}V = 0, \tag{24}$$

then takes the form

$$ac^2 \mathrm{d}\rho + 3(\rho c^2 + p)\mathrm{d}a = 0. \tag{25}$$

Hence, for LIVE, obeying the equation of state (21), equation (25) reduces to $\mathrm{d}\rho_V = 0$, showing that LIVE has constant mass density during the cosmic expansion. In may be noted that equation (25) is valid in the general theory of relativity. It is an expression of energy conservation that follows as a consequence of Einstein's field equations.

One may wonder in what way energy can be conserved in a LIVE-dominated expanding universe. Since the energy density is constant there is more and more energy inside a surface co-moving with the expanding space. Consider the work performed at a co-moving surface. Since the pressure is negative the work makes energy flow in the opposite direction to that of the motion. Hence the exterior region performs a positive work on the interior region. So the work represents a transport of energy in the inwards direction through the surface. Looking at larger and larger surfaces, and taking the limit of a surface with an infinitely great radius, we end up with the conception of a universe where the infinitely far region is giving away energy to the regions at a finite distance from an observer at the origin. Hence there is a difficulty with global energy conservation in relativistic universe models.

There exists another point of view that may make global energy conservation possible. In a LIVE-dominated universe there is repulsive gravitation. Also in a homogeneous universe the introduction of an observer, say at the spatial origin, represents a breaking of the homogeneous symmetry. The LIVE energy falls down in the repulsive gravitational field around the observer and hence loses gravitational energy during the expansion of the universe. Energy conservation implies that the loss of gravitational energy of LIVE inside a comoving surface is equal to the increase of vacuum energy inside the surface.

Zel'dovich tried to calculate the energy density of LIVE. The answer turned out to be infinitely great, showing that the calculation did not provide a valid result. If the calculation is modified by assuming a 'cut off' at Planck dimensions, one arrives at the Planck density, $\rho_P = 5.2 \times 10^{96}$ kg m^{-3}. On the other hand the average density of the cosmic matter is $\rho = 8.5 \times 10^{-27}$ kg m^{-3}. Hence the calculated density is a factor of 10^{120} times too great. This enormous conflict between theory and observation is called the first cosmological constant problem.

The second cosmological constant problem is that observations indicate that the cosmological constant does not vanish, but has an extremely small value compared to the value that came out of the quantum theoretical calculation. The physicists consider it easier to explain that the cosmological constant is equal to zero than to explain that it is very small.

Until May 2017 both of these problems were unsolved.

6. Solution of the first cosmological constant problem

The physicists Wang, Zhu and Unruh presented, on May 11, 2017, a 34 page article [24] in *Physical Review* D with a detailed calculation of the density of the quantum mechanical vacuum energy. They had, for the first time, performed a combined quantum mechanical and general relativistic calculation of the density of the vacuum energy. The result turned out to be different from that of previous calculations where the theory of relativity had not been taken into account as rigorously as in the new calculation. I will briefly review their main points in a qualitative way.

If one had a microscope making it possible to see spacetime on the Planck scale, i.e. with a space resolution of 10^{-35} m, and one also could observe fluctuations with a duration 10^{-43} s, one would see a fluctuating 'spacetime foam' where the space locally shifted between being in a state of collapse and expansion. Also the energy density of the vacuum energy and the sign of the gravitational mass density, as given in equation (19), would fluctuate wildly. Gravity would fluctuate locally between being attractive and repulsive. Hence, although the gravitational effects of quantum fluctuations are great on the Planck scale, they are nearly averaged out on a macroscopic scale.

But not quite. The calculations of Wang, Zhu and Unruh showed that there is an asymmetry between the attractive and repulsive phases of the fluctuations, so that the expanding phases last an extremely small time longer than the contracting phases. On a macroscopic scale this summarizes to a repulsion. The average is not zero.

Their new picture of the physics giving rise to a non-vanishing cosmological constant implies that the average *gravitational* mass density of the vacuum fluctuations is vastly smaller than the mass density of the vacuum energy. These two properties of the quantum mechanical vacuum fluctuations in curved spacetime explain both that the cosmological constant is non-zero and that it is small.

However, the authors were not able to calculate how small. In their calculation there appears a proportionality constant between the gravitational mass density and the mass density itself, and although the authors were able to show that it is small, its value could not be determined from their calculation using quantum field theory in curved spacetime described by classical general relativity. Presumably a full quantum gravity theory is needed to calculate the value of the cosmological constant. Hence, the authors have solved the first cosmological problem, but not the second.

Nevertheless, if a consensus is reached that their calculation and results are valid, then their new picture of the connection between the cosmological constant, the accelerated expansion of the universe and the quantum mechanical vacuum fluctuations, amounts to a breakthrough of new understanding of the gravitational effects of the quantum mechanical vacuum fluctuations.

7. Conclusion

The cosmological constant has been with us for a hundred years [25]. (The recent preprint 25, which appeared after the present article was submitted to the *European Journal of Physics*, is a rather non-technical historical account, which should be easily accessible to beginning college and university students of physics. Also it contains a large number of further references). It was introduced by Einstein in 1917 as the simplest extension of the original version of the field equations of the general theory of relativity in order to make it possible to construct a static universe model. Its physical meaning was not stated explicitly by Einstein.

However, from the way it appeared in the equations it follows that it belonged to the geometrical part of the equations, not the part representing the properties of matter. The effect of the cosmological constant was to cause to free particles to accelerate away from each other. Hence the original interpretation of the cosmological constant was implicitly that it represented a natural tendency for empty space to expand exponentially, although it was introduced in the context of a static space.

In 1934 Lemaître introduced a new interpretation of the cosmological constant. He argued that if motion is truly relative, then it must be impossible to measure velocity relative to the vacuum energy. Otherwise the vacuum energy would constitute some sort of 'ether' that might be used to define a state of absolute rest. This means that all physical properties of the vacuum energy must be Lorentz invariant. One can show [4] that this means that the energy-momentum tensor of the vacuum energy must be proportional to the metric tensor, where the proportionality constant is in fact the cosmological constant. So according to Lemaître the cosmological constant represents the energy density of LIVE.

The cosmological constant obtained a new importance in 1998 when it was discovered that the universe has been in a state of accelerated expansion for the last five billion years [26–30]. This means that in this era the universe has been dominated by some sort of dark energy causing repulsive gravitation. It follows from equation (19) that this energy must be in a state with negative pressure (strain) having $p < -(1/3)\rho c^2$. The simplest type of dark energy is LIVE with $p = -\rho c^2$ and a constant energy density represented by Λ. The presently favoured universe model is the so-called ΛCDM model with 70% dark energy, 26% cold dark matter and 4% ordinary matter.

However, even if LIVE should be due to quantum fluctuations, we do not have a workable understanding of this energy. Trying to calculate the density of this type of energy has given a value of the cosmological constant vastly larger than permitted by cosmic observations. A promising calculation has been published in 2017, a hundred years after the cosmological constant was introduced. The cosmological constant represents gravitational repulsion. The new calculation takes into account that in the quantum fluctuations expansion and contraction and also repulsion and attraction interchange. Wang *et al* [24] have shown that the macroscopic average that gives rise to the cosmological constant is small even if the density of the vacuum energy due to the quantum fluctuations is large. This may be a first step towards a realistic calculation of the magnitude of the cosmological constant.

However it is possible that in order to carry through such a calculation, and fully understand the cosmological constant, one needs a quantum gravitational theory.

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