THE SPHERICAL COLLAPSE MODEL IN A UNIVERSE WITH COSMOLOGICAL CONSTANT

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We generalize the spherical collapse model for the formation of dark matter halos to apply in a universe with arbitrary positive cosmological constant. We calculate the critical condition for collapse of an overdense region and give exact values of the characteristic densities and redshifts of its evolution.

1 Introduction

The spherical collapse model was first developed by Gunn & Gott [1] for a flat universe with no cosmological constant. It assumes that the process of formation of bound objects in the universe can be at first approximation described by evolution of an uniformly overdense spherical region in otherwise smooth background. Despite its simplicity, the model has been widely used to explain properties of a single bound object via extensions such as the spherical infall model as well as statistical properties of different classes of objects via Press-Schechter-like formalisms. Given the presently mounting evidence for cosmological constant we generalize the model here to include its effect.

2 The cosmological model

The evolution of the scale factor $a = R/R_0 = 1/(1+z)$ is governed by equation

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{H_0}{f(a)}, \quad \text{where} \quad f(a) = \left[1 + \Omega_0 \left(\frac{1}{a} - 1\right) + \lambda_0 (a^2 - 1)\right]^{-1/2}. \tag{1}$$

The parameters Ω_0 and λ_0 are the present values of Ω and $\lambda = \Lambda/(3H^2)$ where $\Lambda = \mathrm{const}$ is the standard cosmological constant. The evolution of these parameters with redshift z is given by $\Omega(z) = \Omega_0(1+z)^3[H_0/H(z)]^2$ and $\lambda(z) = \lambda_0[H_0/H(z)]^2$ where $[H(z)/H_0]^2 = \Omega_0(1+z)^3 - (\Omega_0+\lambda_0-1)(1+z)^2+\lambda_0$.

The linear growth of density fluctuations is given by

$$D(a) = \frac{5\Omega_0}{2af(a)} \int_0^a f^3(a)\mathrm{d}a \tag{2}$$

where $f(a)$ was defined in Eq. (1). The expression in (2) is normalized so that for $\Omega = 1$ and $\lambda = 0$ we have $D(a) = a$. For arbitrary parameters (Ω_0, λ_0), expansion of the right-hand side of Eq. (2) around $a = 0$ gives

$$D(a) = a + O(a^2). \tag{3}$$

3 Evolution of the overdense region

We assume that at some time $t_{\rm i}$ corresponding to redshift $z_{\rm i}$ the region of proper radius $r_{\rm i}$ is overdense by $\Delta_{\rm i} = {\rm const}$ with respect to the background, that is it encloses a mass $M(r_{\rm i}) = 4\pi\rho_{\rm b,i} r_{\rm i}^3 (1+\Delta_{\rm i})/3$, where $\rho_{\rm b,i}$ is the background density at $t_{\rm i}$.

Evolution of this region is governed by the familiar energy equation

$$\frac{1}{2}\left(\frac{{\rm d}r}{{\rm d}t}\right)^2 - \frac{GM}{r} - \frac{\Lambda r^2}{6} = E \tag{4}$$

which, with a new variable $s = r/r_{\rm i}$, can be rewritten in the form

$$\frac{{\rm d}s}{{\rm d}t} = \frac{H_{\rm i}}{g(s)}, \quad \text{where} \quad g(s) = \left[1 + \Omega_{\rm i}\left(\frac{1}{s} - 1\right) + \lambda_{\rm i}(s^2 - 1)\right]^{-1/2} \tag{5}$$

and we used the notation $H_{\rm i} = H(z_{\rm i})$, $\Omega_{\rm i} = \Omega(z_{\rm i})$ and $\lambda_{\rm i} = \lambda(z_{\rm i})$.

Assuming conservation of energy we find that the maximum expansion radius $r_{\rm ta}$ (or equivalently, $s_{\rm ta} = r_{\rm ta}/r_{\rm i}$) of the overdense region must obey the condition

$$b_1 s_{\rm ta}^3 + b_2 s_{\rm ta} + b_3 = 0 \tag{6}$$

where $b_1 = \lambda_{\rm i}$, $b_2 = 1 - \Omega_{\rm i}(1+\Delta_{\rm i}) - \lambda_{\rm i}$ and $b_3 = \Omega_{\rm i}(1+\Delta_{\rm i})$. The only solution to Eq. (6) which is real, positive and reproduces the $\lambda_0 = 0$ case in the limit of small λ_0 is

$$s_{\rm ta} = \frac{2}{\sqrt{3}}\left(\frac{-b_2}{b_1}\right)^{1/2} \cos\left(\frac{\phi - 2\pi}{3}\right) \tag{7}$$

with $\phi = \arccos[x/(x^2+y^2)^{1/2}]$, $x = -9 b_1^{1/2} b_3$ and $y = [3(-4b_2^3 - 27 b_1 b_3^2)]^{1/2}$, while for $\lambda_0 = 0$ we simply get $s_{\rm ta} = -b_3/b_2$.

The condition for the general solution (7) to be real is

$$\Delta_{\rm i} > \Delta_{\rm i,cr} = \frac{1}{\Omega_{\rm i}} p(\lambda_{\rm i}) - 1 \tag{8}$$

where

$$p(\lambda_{\rm i}) = 1 + \frac{5\lambda_{\rm i}}{4} + \frac{3\lambda_{\rm i}(8+\lambda_{\rm i})}{4q(\lambda_{\rm i})} + \frac{3q(\lambda_{\rm i})}{4} \tag{9}$$

and

$$q(\lambda_\mathrm{i}) = \{\lambda_\mathrm{i}[8 - \lambda_\mathrm{i}^2 + 20\lambda_\mathrm{i} + 8(1-\lambda_\mathrm{i})^{3/2}]\}^{1/3}. \tag{10}$$

$\Delta_{\mathrm{i,cr}}$ is the critical density for the overdense region to turn around. In the limit of $\lambda_0 \to 0$ we have $p(\lambda_\mathrm{i}) \to 1$ and we reproduce the well known condition

$$\Delta_\mathrm{i} > \Delta_{\mathrm{i,cr}}(\lambda_0 = 0) = \frac{1}{\Omega_\mathrm{i}} - 1. \tag{11}$$

After turn-around the proper size of the region $r(t)$ evolves almost independently of the presence of the cosmological constant and can be well approximated by the analytical solutions for $\lambda_0 = 0$: $r/r_\mathrm{ta} = (1-\cos\theta)/2$ and $t/t_\mathrm{coll} = (\theta - \sin\theta)/(2\pi)$ with $0 \le \theta \le 2\pi$.

4 The characteristic densities

Integrating equations (1) and (5) we get

$$\int_0^a f(a)\mathrm{d}a = H_0 t, \qquad \int_0^s g(s)\mathrm{d}s = H_\mathrm{i} t. \tag{12}$$

Eliminating t we obtain equations which can be used to calculate the scale factor at turn-around (a_ta) and collapse (a_coll) of a region with particular Δ_i at a given z_i

$$\int_0^{a_\mathrm{ta}} f(a)\mathrm{d}a = \frac{H_0}{H_\mathrm{i}} \int_0^{s_\mathrm{ta}} g(s)\mathrm{d}s \tag{13}$$

$$\int_0^{a_\mathrm{coll}} f(a)\mathrm{d}a = 2\frac{H_0}{H_\mathrm{i}} \int_0^{s_\mathrm{ta}} g(s)\mathrm{d}s \tag{14}$$

where s_ta is given by Eq. (7).

Assuming that the mass inside the overdense region does not change, the overdensity inside the sphere of size r with respect to the background density at any time is

$$\delta = \frac{\rho}{\rho_\mathrm{b}} - 1 = \frac{1}{s^3}\left(\frac{a}{a_\mathrm{i}}\right)^3 (1 + \Delta_\mathrm{i}) - 1. \tag{15}$$

At early times, $t \to 0$, we can expand the expressions on the left-hand sides of both equations in (12) around $a = 0$ and $s = 0$ respectively. Integrating term by term we can express time as a power series of a and s respectively. Inverting those series we obtain a and s as power series of $t^{2/3}$. Inserting them into Eq. (15) and replacing t-dependence with a we find

$$\delta = h(\Omega_0, \lambda_0, \Delta_\mathrm{i}, z_\mathrm{i})a + O(a^2), \tag{16}$$

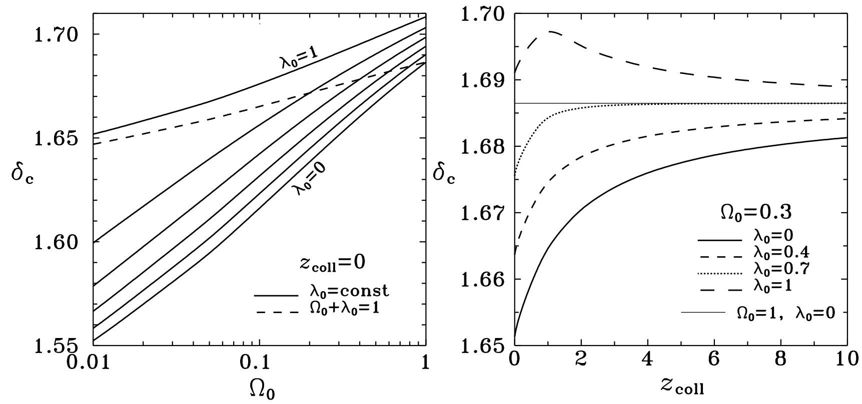

Figure 1: Left panel: parameter $\delta_{\rm c}$ as a function of Ω_0 for $z_{\rm coll} = 0$. Solid lines correspond from bottom to top to $\lambda_0 = 0, 0.2, 0.4, 0.6, 0.8$ and 1. The dashed line shows results for the flat case $\Omega_0 + \lambda_0 = 1$. Right panel: parameter $\delta_{\rm c}$ as a function of $z_{\rm coll}$ for four models with $\Omega_0 = 0.3$ and $\lambda_0 = 0, 0.4, 0.7$ and 1. The thin straight line marks the fiducial value $\delta_{\rm c} = 1.68647$ for $\Omega_0 = 1$, $\lambda_0 = 0$.

where

$$h(\Omega_0, \lambda_0, \Delta_{\rm i}, z_{\rm i}) = \frac{3}{5}\left[\frac{1-\Omega_0-\lambda_0}{\Omega_0} + \frac{[\Omega_{\rm i}(1+\Delta_{\rm i})+\lambda_{\rm i}-1](1+z_{\rm i})}{\Omega_{\rm i}(1+\Delta_{\rm i})^{2/3}}\right]. \qquad (17)$$

Given the behavior of the linear growth factor $D(a)$ in Eq. (3), we obtain the density contrast as predicted by linear theory

$$\delta_{\rm L} = h(\Omega_0, \lambda_0, \Delta_{\rm i}, z_{\rm i}) D(a). \qquad (18)$$

A particularly useful quantity is the linear density contrast at the moment of collapse i.e. when s reaches zero

$$\delta_{\rm c} = h[\Omega_0, \lambda_0, \Delta_{\rm i}(a_{\rm coll}), z_{\rm i}] D(a_{\rm coll}). \qquad (19)$$

$\Delta_{\rm i}(a_{\rm coll})$ in the above equation means that $\Delta_{\rm i}$ corresponding to $a_{\rm coll}$ has to be determined numerically for a given $z_{\rm i}$ from the Eq. (14). Numerical results for $\delta_{\rm c}$ are shown in Figure 1. In the special case of $\Omega_0 + \lambda_0 = 1$ we reproduce the results of Eke, Cole & Frenk [2], while for open universes with no cosmological constant our results match those of Lacey & Cole[3]. Although it is not obvious from Figure 1, in the limit of $\Omega_0 \to 0$ we have $\delta_{\rm c} \to 3/2$, independently of λ_0.

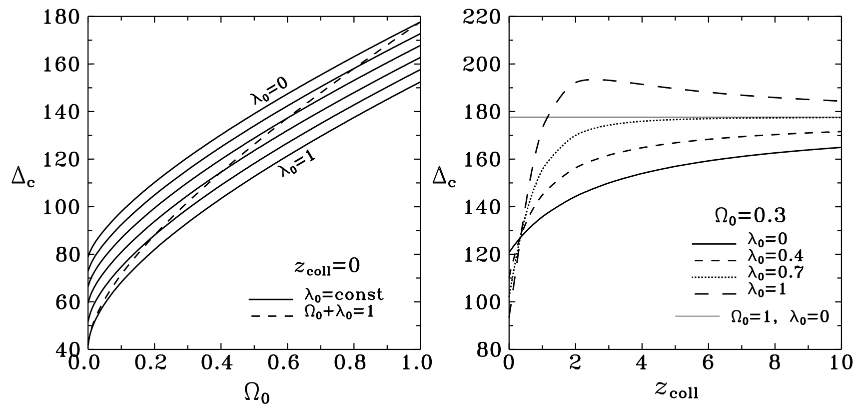

Figure 2: Left panel: parameter Δ_c as a function of Ω_0 for $z_{coll} = 0$. Solid lines correspond from top to bottom to $\lambda_0 = 0, 0.2, 0.4, 0.6, 0.8$ and 1. The dashed line shows results for the flat case $\Omega_0 + \lambda_0 = 1$. Right panel: parameter Δ_c as a function of z_{coll} for four models with $\Omega_0 = 0.3$ and $\lambda_0 = 0, 0.4, 0.7$ and 1. The thin straight line marks the fiducial value $\Delta_c = 177.653$ for $\Omega_0 = 1$, $\lambda_0 = 0$.

Another useful quantity is the ratio of the density in the object to the critical density at virialization

$$\Delta_c = \frac{\rho_{vir}}{\rho_{crit}}(a_{coll}) = \frac{\Omega(a_{coll})}{s_{coll}^3} \left(\frac{a_{coll}}{a_i}\right)^3 [1 + \Delta_i(a_{coll})] \qquad (20)$$

where $s_{coll} = r_{coll}/r_i$ and r_{coll} is the effective final radius of the collapsed object. We assume that the object virializes at t_{coll}, the time corresponding to $s \to 0$. Application of the virial theorem in the presence of cosmological constant leads to the equation for the ratio of the final radius of the object to its turn-around radius $F = r_{coll}/r_{ta}$ obtained by Lahav *et al* [4]: $2\eta F^3 - (2 + \eta)F + 1 = 0$ where $\eta = 2\lambda_i s_{ta}^3/[\Omega_i(1 + \Delta_i)]$. Numerical results for Δ_c in different models are shown in Figure 2. Again, we agree with the results for the special cases of $\Omega_0 < 1$, $\lambda_0 = 0$ and $\Omega_0 + \lambda_0 = 1$ derived previously by Lacey & Cole [3] and Eke *et al* [2] respectively. It should be emphasized that the value of Δ_c is strongly model-dependent, contrary to the common assumption of $\Delta_c \approx 200$.

5 The characteristic redshifts

It is sometimes useful to be able to estimate the redshift of a particular stage of evolution of perturbation given its present overdensity as predicted by linear theory, $\delta_L(a = 1) = \delta_0$. For the redshift of collapse combining equations (18)

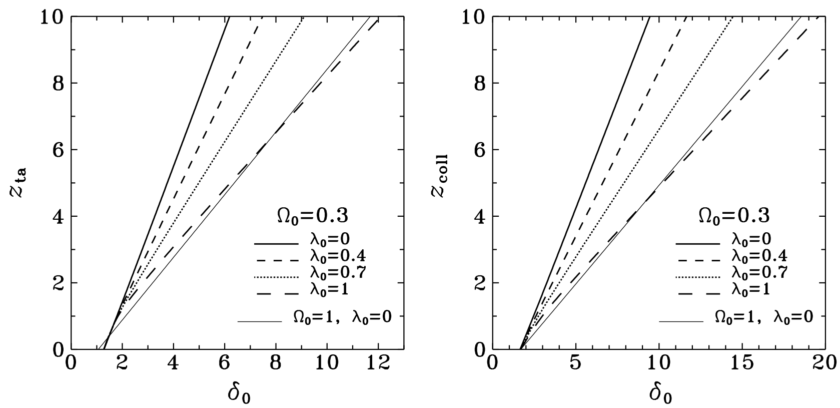

Figure 3: Redshift of turn-around (left panel) and collapse (right panel) of density fluctuation with present density contrast δ_0 for four models with $\Omega_0 = 0.3$ and $\lambda_0 = 0, 0.4, 0.7$ and 1. The thin straight line marks the fiducial case of $\Omega_0 = 1$, $\lambda_0 = 0$.

and (19) gives $\delta_0 = \delta_{\rm c}(a_{\rm coll})D(a=1)/D(a_{\rm coll})$. Using the previously obtained results for $\delta_{\rm c}$ and formula (2), we can calculate the present linear density contrast of fluctuation that collapsed at $z_{\rm coll}$. This relation can only be inverted analytically in the case of $\Omega_0 = 1$, $\lambda_0 = 0$ when we get $z_{\rm coll} = \delta_0/\delta_{\rm c} - 1$. In an analogous way one can obtain the redshifts of turn-around, $z_{\rm ta}$. Numerical results for both quantities are shown in Figure 3. Useful fitting formulae for those results will be presented in our forthcoming paper [5].

Acknowledgments

This work was supported by the Polish KBN grant 2P03D02319 and by the Israel Science Foundation grant 103/98.

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