1 Evolution

We start from the acceleration:

$$\ddot{r} = -\frac{GM}{r^2} + \frac{\Lambda}{3} r \tag{1}$$

First we rewrite $\ddot{r}$ in terms of $x = \ln(a)$:

$$\begin{aligned}
\frac{d^2}{dt^2} &= \frac{d}{dt}\left[\frac{da}{dt}\frac{d}{da}\right] \\
&= \frac{d^2 a}{dt^2}\frac{d}{da} + \left[\frac{da}{dt}\right]^2 \frac{d^2}{da^2} \\
&= \ddot{a}\frac{d}{dx}\frac{dx}{da} + \dot{a}^2 \frac{d}{da}\frac{d}{dx}\frac{dx}{da} \\
&= \frac{\ddot{a}}{a}\frac{d}{dx} + \dot{a}^2\left(\frac{d^2x}{da^2}\frac{d}{dx} + \frac{dx}{da}\frac{d^2}{dx^2}\frac{dx}{da}\right) \\
&= \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\frac{d}{dx} + \frac{\dot{a}^2}{a^2}\frac{d^2}{dx^2}.
\end{aligned} \tag{2}$$

Now applying this to r and using that $\frac{\dot{a}^2}{a^2} = H^2$ and the notation $' = \frac{d}{dx}$ we have

$$\left(\frac{\ddot{a}}{a} - H^2\right) r' + H^2 r'' = -\frac{GM}{r^2} + \frac{\Lambda}{3} r \tag{3}$$

or

$$H^2 r'' = -\frac{GM}{r^2} + \frac{\Lambda}{3} r - \left(\frac{\ddot{a}}{a} - H^2\right) r'. \tag{4}$$

Now we find the mass in terms of the initial overdensity δ_i, it is found through

$$M = \frac{4\pi}{3}\rho_m r^3$$

where $\rho_m = \rho_{c,0}\Omega_{m,0}(1+\delta_i)(\frac{r_i}{r a_i})^3$ and $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$, which gives

$$M = \frac{H_0^2}{2G}\left(\frac{r_i}{a_i}\right)^2 \Omega_{m,0}(1+\delta_i). \tag{5}$$

Inserting eq. (5), the unit less $\tilde{r} = \frac{r a_i}{r_i}$, which implies that $\tilde{r}' = \frac{r' a_i}{r_i}$ and $\tilde{r}'' = \frac{r'' a_i}{r_i}$, and $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ we get the equation

$$H^2 \tilde{r}'' = H_0^2\left(- \frac{\Omega_{m,0}(1+\delta_i)}{2\tilde{r}^2} + \Omega_\Lambda \tilde{r}\right) - \left(\frac{\ddot{a}}{a} - H^2\right)\tilde{r}'. \tag{6}$$

We now find

$$\begin{aligned}\ddot{a} = \frac{d\dot{a}}{dt} &= \frac{d}{dt}\left(aH_0^2\sqrt{\Omega_{m,0}a^{-3}+\Omega_\Lambda}\right) \\ &= a\left(H^2 - \frac{3}{2}H_0^2\Omega_{m,0}a^{-3}\right),\end{aligned} \tag{7}$$

which means that

$$\frac{\ddot{a}}{a} - H^2 = -\frac{3}{2}\frac{H_0^2\Omega_{m,0}}{a^3}. \tag{8}$$

Finally inserting (8) into (6) and dividing by $H^2 = H_0^2(\Omega_{m,0}+\Omega_\Lambda)$ for a flat background we get

$$\tilde{r}'' = \left(\Omega_\Lambda\tilde{r} - \frac{\Omega_{m,0}(1+\delta_i)}{2\tilde{r}^2} + \frac{3\Omega_{m,0}\tilde{r}'}{2a^3}\right)\left[\Omega_{m,0}a^{-3}+\Omega_\Lambda\right] \tag{9}$$

2 Virialization

We start with the potentials

$$U_m = -\frac{3GM^2}{5r} \tag{10}$$

and

$$U_\Lambda = -\frac{\Lambda M}{6}r^2 \tag{11}$$

for the gravitational and DE potential respectively. Now we use the relation for a virialized system with kinetic energy T and potential energy U

$$\begin{aligned}T &= \frac{1}{2}r\frac{\partial U}{\partial r} \\ &= \frac{1}{2}r\left(\frac{3GM^2}{5r^2} - \frac{\Lambda M}{3}r\right) \\ &= \frac{3GM^2}{10r} - \frac{\Lambda M}{6}r^2.\end{aligned} \tag{12}$$

Now using that $T = \frac{1}{2}M\dot{r}^2 = \frac{1}{2}H^2Mr'^2$ we get

$$H^2r'^2 = \frac{3GM}{5r} - \frac{\Lambda}{3}r^2. \tag{13}$$

Now we can insert $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ and eq. (5) for the mass. We also use the variable $\tilde{r}$ to get

$$H^2\tilde{r}'^2 = H_0^2\left(\frac{3}{10}\frac{\Omega_{m,0}(1+\delta_i)}{\tilde{r}} - \Omega_\Lambda\tilde{r}^2\right). \tag{14}$$

Which we can write as

$$\tilde{r}'^2 = \left(\frac{3\Omega_{m,0}(1+\delta_i)}{10\tilde{r}} - \Omega_\Lambda \tilde{r}^2\right)\left(\Omega_{m,0}a^{-3} + \Omega_\Lambda\right). \tag{15}$$