On the Representability of a Function by a Trigonometric Series[1]

By

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G ö t t i n g e n,
i n t h e D i e t e r i c h s c h e n B u c h h a n d l u n g.
1867.

Communicated from the author's estate by R. D e d e k i n d[2]

The following essay on trigonometric series consists of two essentially different parts. The first part contains a history of the investigations and views on arbitrary (graphically given) functions and their representability by trigonometric series. In compiling them, I have had the privilege of using a few hints from the famous

[1] [*Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe, translated by Michael Rogers, 2022.*]

[2] This treatise was submitted in the year 1854 by the author for his Habilitation to the philosophical faculty at the University of Göttingen. Although the author, as it seems, did not intend its publication, the publication of it herewith in a completely unchanged form seems sufficiently justified both by the great interest of the subject matter itself and by the most important principles of infinitesimal analysis laid down in its treatment.
Braunschweig, July 1867.

R. D e d e k i n d.

mathematician† to whom we owe the first thorough work on this subject. In the second part I deliver an investigation on the representability of a function by a trigonometric series, which also includes as yet unfinished cases. It was necessary to preface it with a short essay on the concept of a definite integral and the extent of its validity.[Q1]

† *Presumably Dirichlet.*

[Q1] *What appear to be the key phrases or concepts, which Riemann will write about? Is each one familiar to you?*

History of the question of the representability of an arbitrarily given function by a trigonometric series

1.

The so-called trigonometric series of Fourier, i.e. the series of the form

$$a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots$$

$$\tfrac{1}{2}\, b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots$$

plays a significant roll in the part of mathematics where completely arbitrary functions occur;[Q2] yes, it can be claimed with reason that the most essential advances in this part of mathematics, so important for physics, have been dependent on a clearer insight into the nature of these series. Right from the first mathematical investigations, which led to the consideration of arbitrary functions, arose the question whether such a completely arbitrary function can be expressed by a series of the above form.

[Q2] *What is an arbitrary function? How arbitrary can it be?*

It happened in the middle of the last century on occasion of investigations into vibrating strings, with which the most famous mathematicians were then occupied. Their views on our subject cannot be well presented without entering into this problem.

Under certain conditions that are approximately true in reality, it is well known that the form of a tight string vibrating in a plane, where x is the distance of an indeterminate point from its starting point, y its distance at time t from the rest position, is defined by the partial differential equation†

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \, \frac{\partial^2 y}{\partial x^2}$$

† *Riemann writes the wave equation with d instead of ∂*

where α is independent of t and of x due to the equal thickness of the string everywhere.

The first who gave a general solution of this differential equation was d'Alembert. He showed[3] that each function of x and t, which, when set to y, satisfies the equation identically, must be contained in the form

$$f(x+\alpha t)+\varphi(x-\alpha t)\,,$$

as results through the introduction of the independent variables $x+\alpha t$, $x-\alpha t$ instead of x, t, whereby⋆

⋆*From the multivariable chain rule*

$$\frac{\partial^2 y}{\partial t^2}-\alpha^2\,\frac{\partial^2 y}{\partial x^2}\quad\text{becomes}\quad 4\,\frac{\partial\dfrac{\partial y}{\partial\,(x+\alpha t)}}{\partial\,(x-\alpha t)}$$

Except for this partial differential equation, which results from the general laws of motion, y must now fulfill the condition at the fixed points of the string to always be $=0$; thus one has, if at one of the points $x=0$ and at the other $x=l$,

$$f(\alpha t)=-\varphi(-\alpha t),\;\; f(l+\alpha t)=-\varphi(l-\alpha t)$$

and consequently[Q3]

[Q3] *Can you think of such a function as $f(z)$? What is the most general form of $f(z)$ you can think of?*

$$f(z)=-\varphi(-z)=-\varphi(l-(l+z))=f(2l+z)$$

$$y=f(\alpha t+x)-f(\alpha t-x)$$

After d'Alembert had accomplished this for the general solution of the problem, he was occupied in the continuation[4] of his treatise with the equation $f(z)=f(2l+z)$; i.e. he sought analytic expressions which remain unchanged when z increases by $2l$.

It was a major contribution of Euler, who in the following year gave in the Berlin treatises[5] a new presentation of d'Alembert's work, that recognized more correctly the nature of the condition, which the function $f(z)$ must satisfy. He remarked that according to the nature of the problem the movement of the string is completely determined when at any instant the form of the string and the velocity of each point (thus y and $\partial y/\partial t$ are given, and shows that when one thinks these two functions are determined by arbitrarily drawn curves, then d'Alembert's function $f(z)$ may always

[3] Mémoires de l'académie de Berlin. 1747. **pag. 214.**
[4] Ibid. **pag. 220.**
[5] Mémoires de l'académie de Berlin. 1748. pag. 69. [= **E119** (Latin) or **E140** (French) — Transl.]

be found by a simple geometric construction. In fact, one assumes that for $t = 0$, if $y = g(x)$ and $\partial y/\partial t = h(x)$, then one obtains for the value of x between 0 and l

$$f(x) - f(-x) = g(x), \quad f(x) + f(-x) = \frac{1}{\alpha} \int h(x)\, dx$$

and consequently the function $f(x)$ between $-l$ and l; hence its value for each other value of z results by means of the equation $f(z) = f(2l + z)$. This is, presented in abstract but now generally accepted terms, Euler's determination of the function $f(z)$.

However, against this extension of his method by Euler, d'Alembert immediately objected,[6] because his method necessarily presupposes that y can be analytically expressed in t and x.

Before Euler's reply to this happened, there appeared a third treatment of the subject by Daniel Bernoulli,[7] quite different from the two. Even before d'Alembert, Taylor[8] had seen that if one sets $y = \sin\frac{n\pi x}{l}\cos\frac{n\pi\alpha t}{l}$ for n an integer $\frac{\partial^2 y}{\partial t^2} = \alpha^2\frac{\partial^2 y}{\partial x^2}$ and at the same time y is always equal to 0 for $x = 0$ and for $x = l$. From this, he explained the physical fact that a string, aside from the fundamental tone, could also give the fundamental tone of a string 1/2, 1/3, 1/4,... as long (being similarly procured), and treats his particular solution as general, i.e. he believed that, if the integer n were determined from the pitch of the tone, vibrating string would always at least be very nearly expressed by the equation. The observation that a string could give different tones at the same time now led Bernoulli to the remark that the string (in theory) could also vibrate according to the equation

$$y = \sum a_n \sin\frac{n\pi x}{l}\cos\frac{n\pi\alpha}{l}(t - \beta_n)\,,$$

and because from this equation all observed modifications of the phenomenon can be explained, he held it to be the most general.[9] To support this view, he investigated the

[6] Mémoires de l'académie de Berlin. 1750. **pag. 358.** In effect one cannot, it seems to me, express y analytically in a manner more general than supposing it a function of t and x. But in this supposition one finds the solution of the problem only for the case where the different figures of the vibrating cord can be contained in one and the same equation. [Transl. from French.]

[7] Mémoires de l'académie de Berlin. 1753. **pag. 147.**

[8] Taylor **de methodo incrementorum**.

[9] l.c. p. 157. art. XIII.

vibrations of a massless taut thread weighted at individual points with finite masses, and showed that the same vibrations could always be decomposed into a number of such vibrations equal to the number of points, each of which takes the same length of time for all masses.

These works of Bernoulli prompted a new essay by Euler which was printed immediately after them in the Memoirs of the Berlin Academy.[10] He holds in it, firmly opposing d'Alembert,[11] that the function $f(z)$ can be an entirely arbitrary one between the limits $-l$ and l, and remarks,[12] that Bernoulli's solution (which he had already previously established as a special one) then is general and only then is general when the series

$$a_1 \sin \frac{x\pi}{l} + a_2 \sin \frac{2x\pi}{l} + a_3 \sin \frac{3x\pi}{l} + \cdots$$
$$\tfrac{1}{2}\, b_0 + b_1 \cos \frac{x\pi}{l} + b_2 \cos \frac{2x\pi}{l} + b_3 \cos \frac{3x\pi}{l} + \cdots$$

can represent for the abscissa x the ordinate of completely arbitrary curve between the abscissas 0 and l. Now no one doubted at that time that all transformations which one could undertake with an analytical expression — be it finite or infinite — were valid for any values of indeterminate magnitudes, or at least became inapplicable only in very special cases. It seems, therefore, impossible to represent an algebraic curve or any analytically given, nonperiodic curve by the above expression, and Euler believed he needed to decide the question about Bernoulli.

The dispute between Euler and d'Alembert was however still unresolved. This prompted a young, then yet little known mathematician, Lagrange, to seek a solution to the problem in a completely new way, by which he reached Euler's results. He undertook[13] to determine the vibrations of a massless thread that is weighted with a finite, indefinite number of equal masses at equal distances, and then examined how these oscillations change as the number of masses increases to infinity. With whatever skill, with what effort of analytical tricks he carried out the first part of this

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[10] Mémoires de l'académie de Berlin. 1753. **pag. 196.**
[11] l.c. pag. 214.
[12] l.c. art. III–X
[13] Miscellanea Taurinensia. Tom. I. Recherches sur la nature et la propagation du son. [= **Oeuvres, v. 1, p. 39**.]

investigation, the transition from the finite to the infinite left much to be desired, so that d'Alembert, in a work which he placed at the head of his *opuscules mathématiques,* could continue to win for his solution the glory of the greatest generality. The views of famous mathematicians at the time were and therefore remain divided on this matter, because even in later works each in essence retained their point of view.

Finally, in order to put together the views they had developed on the occasion of this problem about arbitrary functions and their representability by a trigonometric series, Euler first introduced these functions into analysis and, based on geometric notions, applied the infinitesimal calculus to them. Lagrange[14] believed Euler's results (his geometrical construction of the waveform) to be correct; but for him Euler's geometric treatment of these functions was not enough. D'Alembert,[15] on the other hand, went into Euler's conception of differential calculus and restricted himself to contesting the correctness of his results, because in the case of completely arbitrary functions, one cannot know whether their differential quotients are continuous. As for Bernoulli's solution, all three agreed not to consider it general; but while d'Alembert, in order to be able to explain Bernoulli's solution as less general than his own, had to assert that even an analytically given periodic function cannot always be represented by a trigonometric series, Lagrange[16] believed he could prove this possibility.

2.

Almost fifty years passed without any significant progress being made in the question of the analytical representability of arbitrary functions. Then a remark of Fourier's threw a new light on this subject; a new epoch in the development of this part of mathematics began, which soon also manifested itself externally in great extensions of mathematical physics. Fourier remarked that in the trigonometric series

$$f(x) = \begin{cases} \phantom{\tfrac{1}{2} b_0 + {}} a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \\ \tfrac{1}{2} b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \end{cases}$$

the coefficients may be determined by the formulas

[14] Opuscules mathématiques p. d'Alembert. Tome premier. 1761. **pag. 16. art. VII–XX.**
[15] Opuscules mathématiques. Tome I. 1761. **pag. 42. art. XXIV.**
[16] Misc. Taur. Tom. III. Pars math. pag. 221. art. XXV [= **Oeuvres, v. I, p. 514**.]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx\,.$$

He saw that this method of determination would also remain applicable when the function $f(x)$ is completely arbitrarily given; he let $f(x)$ be a so-called discontinuous function† (the ordinate of a broken line for the abscissa x) and thus obtained a series which in fact always gave the value of the function.

† *The standard German word for discontinuous is* unstetig *but here the word is* discontinuirliche

As Fourier in one of his first works on heat, which he presented to the French Academie[17] (21. Dec. 1807), first stated the theorem, that a completely arbitrarily (graphically) given function may be expressed by a trigonometric series, this assertion was so unexpected for the gray-old Lagrange that he opposed it most resolutely. There is said to be a document[18] about this in the archives of the Paris Academy. Nevertheless, whenever Poisson uses the trigonometric series to represent arbitrary function, he refers[19] to a place in Lagrange's work on the vibrating strings where this mode of representation is said to be found. To refute this assertion, which can only be explained[20] by the well-known rivalry between Fourier and Poisson, we feel compelled to return once more to Lagrange's treatise; for nothing published is found about that event in the Academy.

One finds in fact in the place cited by Poisson[21] the formula:

$$y = 2\int Y \sin X\pi\,dX \times \sin x\pi + 2\int Y \sin 2X\pi\,dX \times \sin 2x\pi$$
$$+\,2\int Y \sin 3X\pi\,dX \times \sin 3x\pi + \text{ etc. } + 2\int Y \sin nX\pi\,dX \times \sin nx\pi\,,$$

of the kind that when $x = X$, one will have $y = Y$, Y being the ordinate that corresponds to the abscissa X.†

† *The text after the formula is quoted: "de sorte que, lorsque x=X, on aura y=Y, Y étant l'ordonnée qui répond à l'abscisse X."*

This formula now looks exactly like Fourier's series, so that confusion is easily possible at a cursory glance; but semblance is only because Lagrange used the $\int dX$

[17] Bulletin des sciences p. la soc. philomatique **Tome I. p. 112.**
[18] According to an oral communication from Professor Dirichlet.
[19] Among others in the common Traité de mécanique Nro. 323. p. 638.
[20] The account in the bulletin des sciences of the paper submitted by Fourier to the Academy is by Poisson.
[21] Misc. Taur. Tom. III. Pars math. pag. 261. [= **Oeuvres, v. I, p. 553.**]

sign where he would have used the $\sum \mathit{\Delta} X$ sign today. It gives a solution to the problem of determining the finite sine series

$$a_1 \sin x\pi + a_2 \sin 2x\pi + a_3 \sin 3x\pi + \cdots + a_n \sin nx\pi$$

in such a way that for the values $\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}$ of x, which Lagrange indefinitely denotes by X, one obtains given values. Had Lagrange let n become infinitely large in this formula, he would certainly have reached Fourier's result. But if one reads through his treatise, one sees that he is far from believing that a completely arbitrary function can really be represented by an infinite sine series. Rather, he had undertaken all the work only because he believed these arbitrary functions could not be expressed by a formula, and of the trigonometric series, he believed that it could represent any analytically given periodic function. Of course, it now seems inconceivable to us that Lagrange should not have arrived at Fourier's series from his summation formula; but this is explained by the fact that the quarrel between Euler and d'Alembert had formed in him beforehand a definite view of the course to be taken. He believed that the vibration problem for a indefinite finite number of masses had to be completely solved before he applied his ideas about limits. These require a fairly extensive investigation,[22] which would have been unnecessary if he had known Fourier's series.

It is true that Fourier correctly recognized[23] the nature of the trigonometric series; since then they have been widely used in mathematical physics to represent arbitrary functions, and in each individual case one can easily convince oneself that the Fourier series really does converge to the value of the function; but it was a long time before this important theorem was generally proved.

The proof which Cauchy gave in a paper read to the Paris Academy on 27 Feb. 1826,[24] is insufficient, as Dirichlet has shown.[25]⋆ Cauchy assumes that if one substitutes a complex argument $x + yi$ for x in the arbitrarily given periodic function $f(x)$, this function is finite for each value of y. But this *only* takes place if the function is equal to a constant quantity. It is easy to see, however, that this assumption is not

⋆*Note the importance of sharing mistakes in advancing knowledge.*

[22] Misc. Taur. **Tom. III. Pars math. p. 251**
[23] Bulletin d. sc. **Tom. I. p. 115.** Les coefficient a, a', a'',... étant ainsi déterminés &c.
[24] Mémoires de l'ac. d. sc. de Paris. **Tom. VI. p. 603.**
[25] Crelle Journal für die Mathematik. **Bd. IV. p. 157 & 158.**

necessary for further deductions. Here it is enough if there is a function $f(x+yi)$ that is finite for all positive values of y and whose real part for $y=0$ will be equal to the given periodic function $f(x)$. If one wants to presuppose this theorem, which is in fact correct,[26] the path taken by Cauchy certainly leads to the goal, just as conversely this theorem can be derived from Fourier's series.

3.

Only in January 1829 there appeared a treatise by Dirichlet in Crelle's journal,[27] in which for functions that admit integration everywhere and do not have an infinite number of maxima and minima, the question of their representability by trigonometric series was in all rigor decided.

The knowledge of the path to be taken to solve this problem came to him from the insight that infinite series fall into two essentially different classes,* depending on whether they remain convergent or not, if all terms are made positive. In the former, the terms can be moved at will, but the value of the latter depends on the order of the terms. In fact, if in a series of the second class one designates the positive terms in sequence by

*= *Absolutely or conditionally convergent*

$$a_1,\ a_2,\ a_3, \ldots,$$

and the negative by

$$-b_1,\ -b_2,\ -b_3, \ldots,$$

it is clear that both $\sum a$ and $\sum b$ must be infinite; for if both were finite, the series would converge even after making the signs equal; but if *one* were infinite, the series would diverge. Evidently the series can now attain an arbitrarily given value C by a suitable arrangement of the terms.* For if one alternately takes positive terms of the series until its value becomes greater than C and negative terms until its value becomes less than C, then the deviation from C will never amount to more than the value of the term preceding the last sign change. Since both the quantities a and the quantities b ultimately become infinitely small as the index increases, the deviations

**Riemann's Rearrangement Theorem*

[26] The proof is found in the **Inauguraldissertation** of the author.
[27] **Bd. IV. pag. 157.**

from C will also become arbitrarily small if one proceeds sufficiently far in the series, i.e. the series converges to C.

The laws of finite sums only apply to series of the first class; only they can really be considered the epitome of their terms, not the series of the second class; a circumstance which was overlooked by the mathematicians of the last century, chiefly because the series which progress in increasing powers of a variable quantity,* to speak generally (i.e. apart from individual values of this quantity), belong to the first class.

**Recall that power series are absolutely convergent within its radius of convergence.*

The Fourier series obviously does not necessarily belong to the first class; their convergence could not, therefore, be deduced from the law according to which the terms decrease, as Cauchy had tried in vain to do.[28] Moreover, it had to be shown that the finite series

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin\alpha\, d\alpha\ \sin x + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin 2\alpha\, d\alpha\ \sin 2x + \cdots$$
$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin n\alpha\, d\alpha\ \sin nx$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, d\alpha + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos\alpha\, d\alpha\ \cos x + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos 2\alpha\, d\alpha\ \cos 2x + \cdots$$
$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos n\alpha\, d\alpha\ \cos nx$$

or, what is the same thing, the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,\frac{\sin\dfrac{2n+1}{2}(x-\alpha)}{\sin\dfrac{x-\alpha}{2}}\, d\alpha$$

infinitely approaches the value $f(x)$ as n increases to infinity.

Dirichlet supports his proof with two theorems:

1) When $0 < c \leq \frac{\pi}{2}$, $\int_0^c \varphi(\beta)\,\dfrac{\sin(2n+1)\,\beta}{\sin\beta}\, d\beta$ approaches the value $\frac{\pi}{2}\,\varphi(0)$, as n increases ultimately to infinity;

[28] Dirichlet in Crelle's Journal. **Bd. IV. pag. 158.** Quoi qu'il en soit de cette première observation,...à mesure que n croit.

2) When $0 < b < c \leq \frac{\pi}{2}$, $\int_b^c \varphi(\beta)\, \dfrac{\sin(2n+1)\,\beta}{\sin\beta}\, d\beta$ approaches the value 0, as n increases ultimately to infinity;

provided that the function $\varphi(\beta)$ either always decreases or always increases between the limits of these integrals.

With the help of these two theorems, if the function f does not go from increasing to decreasing or from decreasing to increasing infinitely often, the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, \frac{\sin\dfrac{2n+1}{2}(x-\alpha)}{\sin\dfrac{x-\alpha}{2}}\, d\alpha$$

can obviously be decomposed into a *finite* number of terms, one of which[29] tends to $\frac{1}{2}f(x+0)$, another to $\frac{1}{2}f(x-0)$, but the others converge to 0 as n grows to infinity.

From this it follows that a trigonometric series can represent *any* periodically repeating function after an interval 2π, which

1) admits integration everywhere,
2) does not have an infinite number of maxima and minima and
3) where its value jumps, assumes the mean value between the limit values on both sides.

A function that has the first two properties but not the third obviously cannot be represented by a trigonometric series; for the trigonometric series that represents it except for the discontinuities, would differ from it at the points of discontinuity themselves. But if and when a function that does not fulfill the first two conditions can be represented by a trigonometric series remains undecided by this investigation.

Through this work of Dirichet's a great number of important analytic investigations were given a firm foundation. He was successful, as he brought into full light the point where Euler erred, in handling a question that had occupied so many distinguished mathematicians for more than seventy years (since 1753). In fact, for all cases of nature, which are the only ones at stake, they were completely finished; for

[29] It is not difficult to prove that the value of a function f, which does not have infinitely many maxima and minima, always, both when the argument value decreases and when it increases equal to x, either approaches fixed limits $f(x+0)$ and $f(x-0)$ (according to Dirichlet's notation in Dove's Repertorium der Physik. **Bd. 1. pag. 170**), or must become infinitely large.

however great our ignorance may be as to how the forces and states of matter vary infinitesimally with place and time, we can safely assume that the functions, to which Dirichlet's investigation does not extend, do not occur in nature.

Nevertheless, these unsolved cases by Dirichlet seem to merit attention for two reasons.

There stands firstly, as Dirichlet himself remarks at the end of his treatise, this subject is most closely connected with the principles of infinitesimal calculus, and can serve to bring these principles to greater clarity and certainty. In this respect, the treatment of them has an immediate interest.

Secondly, however, the applicability of Fourier's series is not limited to physical investigations; it is now also used with success in a field of pure mathematics, number theory, and here those functions whose representability by a trigonometric series Dirichlet has not investigated seem directly to be of importance.

At the end of his treatise, Dirichlet freely promises to come back to these cases later, but this promise has so far remained unfulfilled. The work of Dirsksen and Bessel on the cosine and sine series does not complete this either; they stand rather below Dirichlet's in rigor and generality. Dirksen's essay,[30] which was written almost entirely at the same time as it and was obviously written without knowledge of it, strikes the right approach in general but contains a few inaccuracies in detail. For apart from the fact that in a special case[31] he finds an incorrect result for the sum of the series, he supports himself in a side note by a series expansion that is only possible in certain cases,[32] so that his proof is complete only for functions with everywhere-finite first differential quotients. Bessel[33] seeks to simplify Dirichlet's proof. But the changes in this proof afford no essential simplification in the conclusions, but only serve to clothe it in more familiar concepts, while its rigor and generality suffer considerably.

The question of the representability of a function by a trigonometric series has thus far only been decided under the two conditions, that the function admits integration everywhere and does not have infinitely many Maxima and Minima. If the latter

[30] Crelle's Journal. **Bd. IV. pag. 170.**
[31] l. c. Formel 22.
[32] l. c. Art. 3.
[33] Schumacher. Astronomische Nachrichten. **Nro. 374 (Bd. 16. p. 229).**

assumption is not made, then both Dirichlet's integral theorems are insufficient to decide the question; but if the first ceases to apply, Fourier's determination of the coefficients is not applicable. In the following, where this question is to be examined, the path taken is conditioned by this, as will be seen; such a direct way as Dirichlet's is, in the nature of things, not possible after all.

On the concept of a definite integral and the extent of its validity.

4.

The indefiniteness that still prevails in some fundamental points of the theory of definite integrals requires us to set out in advance some things about the concept of a definite integral and the extent of its validity.

So first: What does one understand by $\int_a^b f(x)\,dx$?

In order to establish this, we assume between a and b, following each other in size, a sequence of values x_1, x_2,..., x_{n-1} and, for the sake of brevity, denote $x_1 - a$ by δ_1, $x_2 - x_1$ by δ_2,..., $b - x_{n-1}$ by δ_n, and by ε a positive proper fraction. Then the value of the sum

$$S = \delta_1 f(a + \varepsilon_1 \delta_1) + \delta_2 f(x_1 + \varepsilon_2 \delta_2) + \delta_3 f(x_2 + \varepsilon_3 \delta_3) + \cdots + \delta_n f(x_{n-1} + \varepsilon_n \delta_n)$$

will depend on the choice of the intervals δ and the quantities ε. If it now has the property, however δ and ε may be chosen, of infinitely approaching a fixed limit A as soon as all δ become infinitely small, then this value is called $\int_a^b f(x)\,dx$.

If it does not have this property, then $\int_a^b f(x)\,dx$ has no meaning. However, attempts have been made in several cases to attach a meaning to this sign even then, and among these extensions of the notion of a definite integral, *one* is accepted by all mathematicians. Namely, when the function $f(x)$ becomes infinitely large as the argument approaches an individual value c in the interval (a, b), then obviously the sum S, whatever degree of smallness one can prescribe to δ, can attain any value whatsoever; thus it has no limit value, and $\int_a^b f(x)\,dx$ would by the above have no meaning.⋆ But when $\int_a^{c-\alpha_1} f(x)\,dx + \int_{c+\alpha_2}^b f(x)\,dx$, as α_1 and α_2 become infinitely small, approaches a fixed limit, then one understands by $\int_a^b f(x)\,dx$ this limit value.

⋆*Improper integrals*

The establishment by Cauchy of the concept of the definite integral in other cases where, according to the basic concept there is no such thing, may be useful for individual classes of investigations; however, they are not introduced generally and are hardly suitable for this purpose, if only because of their great arbitrariness.

5.

Secondly, we now investigate the extent of the validity of this concept or the question: in which cases does a function admit an integration and in which not?

We first consider the integral concept in a narrower sense, i.e. we posit beforehand that the sum S, when all δ become infinitely small, converges. We denote thus the largest variation of the function between a and x_1, i.e. the difference between its greatest and least values in this interval, by D_1, between x_1 and x_2 by D_2,..., between x_{n-1} and b by D_n, so that

$$\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$$

must with the quantities δ become infinitely small. We assume further that, so long as all δ remain less than d, the greatest value that the sum can have is $\varDelta$; then $\varDelta$ will be a function of d alone, which decreases with d and with this quantity becomes infinitely small. If the total size of the intervals in which the variation is greater than σ is $= s$, then the contribution of these intervals to the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ is obviously $\geq \sigma s$. Therefore one has

$$\sigma s \leq \delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n \leq \varDelta\,, \quad \text{consequently } s \leq \frac{\varDelta}{\sigma}\,.$$

$\frac{\varDelta}{\sigma}$ can now, when σ is given, always be made arbitrarily small by a suitable choice of d; the same applies, therefore, to s, and it follows thus:

*In order for the sum S to converge when all δ become infinitely small, it is necessary, in addition to the finiteness of the function $f(x)$, that the total magnitude of the intervals in which the variations are $> \sigma$, whatever σ is, can be made arbitrarily small by a suitable choice of d. *Theorem

This theorem can also be reversed:

*When the function $f(x)$ is always finite, and by infinite decrease of all the quantities δ the total magnitude s of the intervals in which the variation of the function *Theorem

$f(x)$ is greater than a given magnitude σ always in the end becomes infinitely small, then the sum S converges when all δ become infinitely small.

Then these intervals in which the variation is $> \sigma$ make a contribution to the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ less than s multiplied by the greatest variation of the function between a and b, which is finite; the remaining intervals make a contribution $< \sigma\,(b-a)$. Obviously one can first assume σ arbitrarily small and then still determine the size of the intervals such that s will also be arbitrarily small, whereby the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ can be given any arbitrary smallness, and consequently the value of the sum S can be enclosed in arbitrarily narrow limits.

Thus we have found conditions which are necessary and sufficient so that the sum S converges by an infinite decrease of the quantities δ and thus can in a narrower sense be said to be an integral of the function between a and b.

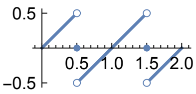
If the integral concept is extended as above, it is obvious that the last of the two conditions found is still necessary so that integration is possible everywhere; replacing the condition that the function is always finite, however, isas the condition that the function becomes infinite *only* when the argument approaches *individual* values,⋆ and that a certain limit value results when the limits of integration infinitely approach these values.

⋆*I.e., isolated vertical asymptotes at which the improper integral is convergent.*

6.

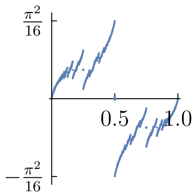
Now that we have investigated the conditions for the possibility of a definite integral in general, i.e. without making any special assumptions about the nature of the function to be integrated, this investigation will be partly applied to special cases and partly carried out further, namely first for the functions which are infinitely often discontinuous between any two limits, no matter how narrow.

Since these functions have not yet been considered anywhere, it will be good to start with a definite example. For the sake brevity, let (x) denote the excess of x from the nearest whole number,* or, when x lies in the middle between two and this condition is ambiguous, the mean value of both values $\frac{1}{2}$ and $-\frac{1}{2}$, thus zero; further, by n an integer; by p an odd integer; and then form the series⋆

*


⋆*Once it was easier to typeset nn than n^2*

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \cdots = \sum_{1,\infty} \frac{(nx)}{nn}\,;$$



The graph of $f(x)$

thus it is easy to see this series converges for every value of x;* its value always approaches a fixed limit value when the argument value, continually decreasing as well as when continually increasing, becomes equal to x; in particular, when $x = \dfrac{p}{2n}$ (where p, n are relatively prime numbers)

$$f(x+0) = f(x) - \frac{1}{2nn}(1 + \tfrac{1}{9} + \tfrac{1}{25} + \cdots) = f(x) - \frac{\pi\pi}{16nn}$$

$$f(x-0) = f(x) + \frac{1}{2nn}(1 + \tfrac{1}{9} + \tfrac{1}{25} + \cdots) = f(x) + \frac{\pi\pi}{16nn}$$

but otherwise, generally $f(x+0) = f(x)$, $f(x-0) = f(x)$.⋆

⋆$f(x+0) = \lim_{t \to x^+} f(t)$, $f(x-0) = \lim_{t \to x^-} f(t)$

This function is thus not continuous for each rational value of x, which is a fraction expressed in the least numbers with an even denominator, and thus not continuous infinitely often between every two limits no matter how narrow, so that, however, the number of jumps that are greater than a given quantity is always finite. It admits an integration throughout. In fact, in addition to its finiteness, two properties suffice for this purpose, that it has a limit value $f(x+0)$ and $f(x-0)$ on both sides for every value of x and that the number of jumps that are greater than or equal to a given quantity σ, is always finite. Then if we apply our investigation above, d can always be assumed, as a result of both these circumstances, to be so small that in all intervals that do not contain these jumps, the variation is less than σ, and that the total size of the intervals that contain these jumps becomes arbitrarily small.

It deserves to be noticed that the functions that do not have infinitely many maxima and minima (to which, by the way, the one just considered does not belong), where they do not become infinite, always possess both these properties, and therefore everywhere, where they do not become infinite, admits an integration, as can easily be shown directly.

Now to the case where the function $f(x)$ to be integrated becomes infinitely large for an individual value, to consider more specifically, we assume that this happens for $x = 0$, so that with decreasing positive x its value eventually increases beyond any given limit.

It can be easily shown that $x\,f(x)$ with x decreasing from a finite limit a, cannot continuously remain greater than a finite size c. Because then $\int_x^a f(x)\,dx > c\int_x^a \dfrac{dx}{x}$

would be greater than $c\left(\log\frac{1}{x}-\log\frac{1}{a}\right)$, which quantity with decreasing x ultimately increases to infinity. Thus it must be that $x\,f(x)$, when this function does not have infinitely many maxima and minima in the neighborhood of $x=0$, necessarily becomes infinitely small, in order that $f(x)$ can be capable of integration. When on the other hand $f(x)\,x^{\alpha}=\dfrac{f(x)\,dx\,(1-\alpha)}{d(x^{1-\alpha})}$ with a value of $\alpha<1$ will become with x infinitely small, so it is clear that the integral by infinite diminishment of the lower limit converges.

Likewise, one finds that in the case of the convergence of the integral, the functions

$$f(x)\,x\log\frac{1}{x}=\frac{f(x)\;dx}{-d\log\log\frac{1}{x}}\,,\;f(x)\,x\log\frac{1}{x}\log\log\frac{1}{x}=\frac{f(x)\;dx}{-d\log\log\log\frac{1}{x}}\,,\dots$$

$$f(x)\,x\log\frac{1}{x}\log\log\frac{1}{x}\dots\log^{n-1}\frac{1}{x}\log^{n}\frac{1}{x}=\frac{f(x)\;dx}{-d\log^{n+1}\frac{1}{x}}\,,$$

cannot continuously remain larger than a finite size when x decreases from a finite limit, and thus, if they do not have an infinite number of maxima and minima, must become infinitely small with x; that, on the other hand, the integral $\int f(x)\,dx$ converges with infinite diminishment of the lower limit as soon as

$$f(x)\,x\cdot\log\frac{1}{x}\dots\log^{n-1}\frac{1}{x}\left(\log^{n}\frac{1}{x}\right)^{\alpha}=\frac{f(x)\;dx\;(1-\alpha)}{-d\left(\log^{n}\frac{1}{x}\right)^{1-\alpha}}$$

becomes infinitely small for $\alpha>1$ with x.

But if the function $f(x)$ has infinitely many maxima and minima, then nothing can be determined about the order of their infinite growth. In fact, if we assume that the function is given according to its absolute value on which alone the order of infinite growth depends, then by appropriate determination of the sign one can always cause the integral $\int f(x)\;dx$ to converge as the lower limit decreases infinitely. As an example of such a function that becomes infinite in such a way that its order (the order of $\frac{1}{x}$ taken as a unit) is infinitely large, the function

$$\frac{d\left(x\cos(e^{\frac{1}{x}})\right)}{dx}=\cos e^{\frac{1}{x}}+\frac{1}{x}\,e^{\frac{1}{x}}\;\sin e^{\frac{1}{x}}$$

may serve.

That may be enough about this subject, which basically belongs to another area; we now go to our actual task, a general investigation of the representability of a function by a trigonometric series.

Investigation of the representability of a function by a trigonometric series without special assumptions about the nature of the function.

7.

The previous work on this subject had the purpose of proving Fourier series for the cases occurring in nature; the proof could therefore be begun for a function assumed quite arbitrarily, and later the path of the function could be subjected to arbitrary restrictions for the sake of proof, provided they did not interfere with that purpose. For our purpose, it may be subjected only to the conditions necessary for the representability of the function; therefore, sufficient ones for representability must first be selected. So while previous work has shown that if a function has this and that property, then it can be represented by a Fourier series, we must start from the inverted question: If a function can be represented by a trigonometric series, what follows about its path, about the change in its value with a continuous change in the argument? Thus we consider the series

$$a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots$$

$$\tfrac{1}{2}\, b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots$$

or, if for brevity we put

$$\tfrac{1}{2} b_0 = A_0,\ a_1 \sin x + b_1 \cos x = A_1,\ a_2 \sin 2x + b_2 \cos 2x = A_2, \dots ,$$

the series becomes

$$A_0 + A_1 + A_2 + \cdots .$$

We denote this expression by Ω and its value by $f(x)$,* so that this function exists for those values of x where the series converges.

**Defintion of Ω and $f(x)$*

For the convergence of a series, it is necessary the its terms ultimately become infinitely small.* If the coefficients a_n, b_n decrease with increasing n to infinity, then the terms of the series Ω ultimately become infinitely small for each value of x; otherwise this can only happen for special values of x. It is necessary to treat both cases separately.

*Nth term test

8.

Thus we assume first that the terms of the series Ω ultimately become infinitely small for each value of x.

Under this assumption the series

$$C + C'x + A_0\frac{xx}{2} - A_1 - \frac{A_2}{4} - \frac{A_3}{9} \cdots = F(x)\,,$$

which one obtains from Ω by twice integrating each term with respect to x, converges for each value of x. Its value $F(x)$ changes continuously with x, and this function F of x consequently admits integration everywhere.

To see both — the convergence of the series and the continuity of the function $F(x)$ — denote the sum of the terms through $-\frac{A_n}{nn}$ by N,⋆ the remainder of the series, i.e. the series

⋆N = partial sum

$$-\frac{A_{n+1}}{(n+1)^2} - \frac{A_{n+2}}{(n+2)^2} - \cdots$$

by R,⋆ and the greatest value of A_m for $m > n$ by ε.⋆ Then the value of R, however far one may continue this series, remains,⋆ apart from the sign,

⋆R = tail of the series
⋆$\varepsilon = \max_{m>n} A_m$
⋆Compare with the Integral Test

$$< \varepsilon\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right) < \frac{\varepsilon}{n}$$

and can thus be contained in arbitrarily narrow limits, when n is taken to be sufficiently large; consequently the series converges. Further, the function $F(x)$ is continuous; i.e. its change can be given any smallness when one prescribes the corresponding change of x to be sufficiently small.⋆ Then the change of $F(x)$ consists of the change

⋆Definition of continuity

of R and of N together; obviously, one can now assume *first* that n is so great that R, whatever x is, and consequently also the change in R for each change in x, becomes arbitrarily small, and *then* assume the change in x so small that the change in N becomes arbitrarily small.

It will be good to preface some theorems about this function $F(x)$, the proofs of which would interrupt the thread of the investigation.

Theorem 1. In the case that the series Ω converges, then *Theorem 1*

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta}$$

converges to the same value as the series, when α and β become infinitely small in such a way that their ratio is finite.

In fact, we will handle

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta}$$
$$= A_0 + A_1\frac{\sin\alpha}{\alpha}\frac{\sin\beta}{\beta} + A_2\frac{\sin 2\alpha}{2\alpha}\frac{\sin 2\beta}{2\beta} + A_3\frac{\sin 3\alpha}{3\alpha}\frac{\sin 3\beta}{3\beta} + \cdots$$

or, in the simpler case $\alpha = \beta$ first,

$$\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{4\alpha\alpha} = A_0 + A_1\left(\frac{\sin\alpha}{\alpha}\right)^2 + A_2\left(\frac{\sin 2\alpha}{2\alpha}\right)^2 + \cdots$$

If the infinite series $A_0+A_1+A_2+\cdots = f(x)$, the series $A_0+A_1+\cdots+A_{n-1} = f(x)+\varepsilon_n$, then for an arbitrarily given quantity δ, it must be possible to give a value m for n when $n > m$, we will have $\varepsilon_n < \delta$. We assume now that α is so small that $m\alpha < \pi$, and further by means of the substitution $A_n = \varepsilon_{n+1} - \varepsilon_n$, we put $\sum\limits_{0,\infty}\left(\frac{\sin n\alpha}{n\alpha}\right)^2 A_n$ in the form $f(x) + \sum\limits_{1,\infty}\varepsilon_n\left\{\left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2 - \left(\frac{\sin n\alpha}{n\alpha}\right)^2\right\}$, and we partition these terms of this latter infinite series in three parts,

1) the terms from index 1 to m inclusive,
2) from index $m+1$ to the largest integer s below $\frac{\pi}{\alpha}$,
3) from $s+1$ to infinity,

then the first part consists of a finite number of constantly changing terms and can therefore arbitrarily approached its limit 0, when α is allowed to become sufficiently small; the second part is, because the factor of ϵ_n is always positive, and obviously, apart from the sign,

$$< \delta \left\{ \left(\frac{\sin(n-1)\alpha}{(n-1)\alpha} \right)^2 - \left(\frac{\sin n\alpha}{n\alpha} \right)^2 \right\} ;$$

finally, to confine within bounds the third part, one decomposes the general term in

$$\varepsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{(n-1)\alpha} \right)^2 - \left(\frac{\sin(n-1)\alpha}{n\alpha} \right)^2 \right\}$$

$$\text{and } \varepsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{n\alpha} \right)^2 - \left(\frac{\sin n\alpha}{n\alpha} \right)^2 \right\} = -\,\varepsilon_n \frac{\sin(2n-1)\alpha \sin\alpha}{(n\alpha)^2} ;$$

thus it is clear that it is

$$< \delta \left\{ \frac{1}{(n-1)^2 \alpha\alpha} - \frac{1}{nn\,\alpha\alpha} \right\} + \delta \frac{1}{nn\alpha} ,$$

and consequently the sum from $n = s+1$ to $n = \infty$ is

$$< \delta \left\{ \frac{1}{(s\alpha)^2} + \frac{1}{s\alpha} \right\} ,$$

which value becomes

$$\delta \left\{ \frac{1}{\pi\pi} + \frac{1}{\pi} \right\}$$

for an infinitely small α.

The series

$$\sum \varepsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{(n-1)\alpha} \right)^2 - \left(\frac{\sin(n-1)\alpha}{n\alpha} \right)^2 \right\}$$

therefore approaches with decreasing α a limit value that cannot exceed

$$\delta \left\{ 1 + \frac{1}{\pi} + \frac{1}{\pi\pi} \right\} ,$$

thus must be zero, and consequently

$$\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{4\alpha\alpha}, \text{which} = f(x)+\sum\varepsilon_n\left\{\left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2-\left(\frac{\sin n\alpha}{n\alpha}\right)^2\right\}$$

converges with α diminishing infinitely, whereby our theorem for the case $\beta = \alpha$ is proven.

To prove generally, let

$$F(x+\alpha+\beta) - F(2x+) + F(x-\alpha-\beta) = (\alpha+\beta)^2(f(x)+\delta_1)$$

$$F(x+\alpha-\beta) - F(2x+) + F(x-\alpha+\beta) = (\alpha-\beta)^2(f(x)+\delta_2)\,,$$

from which

$$F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)$$
$$= 4\alpha\beta f(x) + (\alpha+\beta)^2\delta_1 - (\alpha-\beta)^2\delta_2\,.$$

In consequence of what has just been proved, now δ_1 and δ_2 will become infinitely small whenever α and β become infinitely small; thus also will

$$\frac{(\alpha+\beta)^2}{4\alpha\beta}\,\delta_1 - \frac{(\alpha-\beta)^2}{4\alpha\beta}\,\delta_2$$

be infinitely small, if the coefficients of δ_1 and δ_2 do not become infinitely large, which does not happen if as the same time $\dfrac{\beta}{\alpha}$ remains finite; and then consequently

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)}{4\alpha\beta} \text{ converges to } f(x)\,, \text{ q.e.d.}$$

Theorem 2. *Theorem 2*

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha}$$

always becomes infinitely small with α.

In order to prove this, one partitions the series

$$\sum A_n\left(\frac{\sin n\alpha}{n\alpha}\right)^2$$

into three groups, of which the first contains all terms through a fixed index m of which A_n always remains less than ε, the second all subsequent terms for which $n\alpha <$ a fixed quantity c, and the third includes the rest of the series. It is easy to see that, if α diminishes infinitely, the sum of the first group remains finite, i.e. $<$ a fixed quantity Q; that of the second $< \varepsilon\,\dfrac{c}{\alpha}$; that of the third

$$< \varepsilon \sum_{c<n\alpha} \frac{1}{nn\,\alpha\alpha} < \frac{\varepsilon}{\alpha c}\,.$$

Consequently

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha}\,,\ \text{which } = 2\alpha \sum A_n \left(\frac{\sin n\alpha}{n\alpha}\right)^2\,,$$
$$< 2\left(Q\alpha + \varepsilon(c+\frac{1}{c})\right)\,,$$

from which the theorem to be proved follows.

Theorem 3. If one denotes by b and c two arbitrary constants, the greater by c, and by $\lambda(x)$ a function that, along with its first differential quotient, is always continuous between b and c and equals zero at the limits, and of which the second differential quotient does not have an infinite number of maxima and minima, then the integral, *Theorem 3*

$$\mu\mu \int_b^c F(x)\ \cos\mu(x-a)\ \lambda(x)\,dx\,,$$

if μ grows infinitely, ultimately becomes smaller than any given magnitude.

One sets for $F(x)$ its expression as a series; thus one obtains for

$$\mu\mu \int_b^c F(x)\ \cos\mu(x-a)\ \lambda(x)\,dx\,,$$

the series $(\Phi)^\star$ **Definition of Φ*

$$\mu\mu \int_b^c \left(C + C'x + A_0\frac{xx}{2}\right)\ \cos\mu(x-a)\ \lambda(x)\,dx$$
$$-\sum_{1,\infty} \frac{\mu\mu}{nn} \int_b^c A_n\ \cos\mu(x-a)\ \lambda(x)\,dx\,,$$

Now can $A_n \cos\mu(x-a)$ obviously be expressed as a combination of $\cos(\mu+n)(x-a)$, $\sin(\mu+n)(x-a)$, $\cos(\mu-n)(x-a)$, $\sin(\mu-n)(x-a)$, and one denotes similarly the sum of both first terms by $B_{\mu+n}$, the sum of both last terms by $B_{\mu-n}$, so that one has $\cos\mu(x-a)\,A_n = B_{\mu+n} + B_{\mu-n}$,

$$\frac{d^2 B_{\mu+n}}{dx^2} = -(\mu+n)^2 B_{\mu+n}, \quad \frac{d^2 B_{\mu-n}}{dx^2} = -(\mu-n)^2 B_{\mu-n}$$

and $B_{\mu+n}$ and $B_{\mu+n}$ become with increasing n, whatever x may be, ultimately infinitely small.

The general term of the series Φ

$$-\frac{\mu\mu}{nn}\int_b^c A_n \cos\mu(x-a)\;\lambda(x)\;dx$$

will therefore

$$= \frac{\mu^2}{n^2(\mu+n)^2}\int_b^c \frac{d^2 B_{\mu+n}}{dx^2}\;\lambda(x)\;dx + \frac{\mu^2}{n^2(\mu-n)^2}\int_b^c \frac{d^2 B_{\mu-n}}{dx^2}\;\lambda(x)\;dx$$

or by partial integration twice,* considering first $\lambda(x)$, then $\lambda'(x)$ as constant[Q4]

$$= \frac{\mu^2}{n^2(\mu+n)^2}\int_b^c B_{\mu+n}\;\lambda''(x)\;dx + \frac{\mu^2}{n^2(\mu-n)^2}\int_b^c B_{\mu-n}\;\lambda''(x)\;dx\,,$$

*= *integration by parts; in* $\int u\,dv$, *the factor* u *is the "constant".*
[Q4] *Can you figure out why it is called a constant?*

since $\lambda(x)$ and $\lambda'(x)$ and therefore the terms coming out of the integral sign at the boundaries become $= 0$.

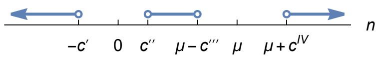
One is now easily convinced that $\int_b^c B_{\mu+n}\;\lambda''(x)\;dx$ when μ increases to infinity, whatever n may be, will become infinitely small; then this expression is equal to a combination of the integrals

$$\int_b^c \cos(\mu\pm n)(x-a)\;\lambda''(x)\;dx\,, \quad \int_b^c \cos(\mu\pm n)(x-a)\;\lambda''(x)\;dx$$

and when $\mu\pm n$ becomes infinitely large, then these integrals become infinitely small, but if not, because n then becomes infinitely large, their coefficients in this expression become infinitely small.

For of the proof of our theorem, it is therefore obviously enough, if it is shown of the sum

$$\sum \frac{\mu^2}{(\mu - n)^2\, n^2}$$



extended over all integer values of n that satisfy the conditions* $n < -c'$, $c'' < n < \mu - c'''$, $\mu + c^{IV} < n$ for any positive values of the quantities c that it remains finite, if μ becomes infinitely large. The apart from the terms for which $-c' < n < c''$, $\mu - c''' < n < \mu + c^{IV}$, which obviously become infinitely small and are finite in number, the series $\varPhi$ remains obviously smaller than this sum multiplied by the greatest value of $\int_b^c B_{\mu \pm n}\, \lambda''(x)\, dx$, which becomes infinitely small.

*See plot above.

But now, if the quantities $c > 1$, the sum

$$\sum \frac{\mu^2}{(\mu - n)^2\, n^2} = \frac{1}{\mu} \sum \frac{\frac{1}{\mu}}{\left(1 - \frac{n}{\mu}\right)^2 \left(\frac{n}{\mu}\right)^2}$$

in the above limits, is smaller than

$$\frac{1}{\mu} \int \frac{dx}{(1-x)^2\, x^2}$$

extended from $-\infty$ to $-\dfrac{c'-1}{\mu}$, $\dfrac{c''-1}{\mu}$ to $1 - \dfrac{c''-1}{\mu}$, $1 + \dfrac{c^{IV}-1}{\mu}$ to ∞; for if one breaks down the entire interval from $-\infty$ to ∞, starting from zero, into intervals of size $\dfrac{1}{\mu}$, and if one replaces the function under the integral sign everywhere by the smallest value in each interval, then one obtains all terms of the series, since this function has no maximum anywhere between the integration limits.

If one carries out the integration, one obtains

$$\frac{1}{\mu} \int \frac{dx}{(1-x)^2\, x^2} = \frac{1}{\mu}\left(-\frac{1}{x} + \frac{1}{1-x} + 2\log x - 2\log(1-x)\right) + \text{const.}$$

and consequently a value between the above limits that does not become infinitely large with μ.

9.

With the help of these theorems, the following can be determined about the representability of a function by a trigonometric series, the terms of which ultimately become infinitely small for each argument value:

I. If a function $f(x)$, repeating periodically over an interval 2π, can be represented by a trigonometric series, of which the terms for each value of x ultimately become infinitely small, then there must be a continuous function $F(x)$ on which $f(x)$ depends in such a way that

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta},$$

if α and β become infinitely small and at the same time their ratio remains finite, converges to $f(x)$.

Further it must be that $\mu\mu\int_b^c F(x)\ \cos\mu(x-a)\ \lambda(x)\ dx$, if $\lambda(x)$ and $\lambda'(x)=0$ at the limits of the integral and are continuous everywhere between them, and $\lambda''(x)$ does not have infinitely many maxima and minima, ultimately become infinitely small with increasing μ.

II. Conversely, if these conditions are fulfilled, then there is a trigonometric series in which the coefficients ultimately become infinitely small and which represents the function wherever it converges.

Because if one determines the quantities C', A_0 in such a way that $F(x)-C'x-A_0\dfrac{xx}{2}$ is a function that repeats periodically after the interval 2π, and develops this further by Fourier's method in the trigonometric series

$$C-\frac{A_1}{1}-\frac{A_2}{4}-\frac{A_3}{9}-\cdots$$

by setting

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(t)-C't-A_0\frac{tt}{2}\right)\ dt=C$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(F(t)-C't-A_0\frac{tt}{2}\right)\cos n\,(x-t)\ dt=-\frac{A_n}{nn}$$

then

$$A_0=-\frac{nn}{\pi}\int_{-\pi}^{\pi}\left(F(t)-C't-A_0\frac{tt}{2}\right)\cos n\,(x-t)\ dt$$

must ultimately become infinitely small as n increases; from which it follows by Theorem 1 of the previous Art. that the series

$$A_0 + A_1 + A_2 + \cdots$$

converges to $f(x)$ wherever it converges.

III. If $b < x < c$, and $\rho(t)$ is such a function that $\rho(t)$ and $\rho'(t)$ for $t = b$ and $t = c$ have the value 0 and vary continuously between these values, $\rho''(t)$ does not have infinitely many maxima and minima, and further that for $t = x$, $\rho(t) = 1$, $\rho'(t) = 0$, $\rho''(t) = 0$, $\rho'''(t)$ and $\rho^{IV}(t)$ are only finite and continuous; then the difference between the series $A_0 + A_1 + A_2 + \cdots$ and the integral*

*Recall dd denotes d^2, i.e. the second derivative

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\, \frac{dd\, \dfrac{\sin \dfrac{2n+1}{2}(x-t)}{\sin \dfrac{(x-t)}{2}}}{dt^2}\, \rho(t)\, dt$$

ultimately becomes infinitely small with increasing n. The series $A_0 + A_1 + A_2 + \cdots$ therefore would converge or not converge according as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\, \frac{dd\, \dfrac{\sin \dfrac{2n+1}{2}(x-t)}{\sin \dfrac{(x-t)}{2}}}{dt^2}\, \rho(t)\, dt$$

ultimately approaches a fixed limit with increasing n or this does not happen.

In fact,*

*In a more modern style the sum would be over k and the summand in terms of k instead of n: $\sum_{k=1}^{n} \cdots$

$$A_1 + A_2 + \cdots + A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(F(t) - C't - A_0\,\frac{tt}{2}\right) \sum_{1,n} -nn \cos n\,(x-t)\; dt$$

or, since

$$2\sum_{1,n} -nn\cos n\,(x-t) = 2\sum_{1,n} \frac{d^2 \cos n\,(x-t)}{dt^2} = \frac{dd\, \dfrac{\sin \dfrac{2n+1}{2}(x-t)}{\sin \dfrac{(x-t)}{2}}}{dt^2}\ ,$$

thus

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(t) - C't - A_0\,\frac{tt}{2}\right)\frac{dd\,\dfrac{\sin\dfrac{2n+1}{2}(x-t)}{\sin\dfrac{(x-t)}{2}}}{dt^2}\,dt$$

But now by Theorem 3 of the previous Art.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(t) - C't - A_0\,\frac{tt}{2}\right)\frac{dd\,\dfrac{\sin\dfrac{2n+1}{2}(x-t)}{\sin\dfrac{(x-t)}{2}}}{dt^2}\,\lambda(t)\,dt$$

becomes infinitely small with an infinite increase of n, if $\lambda(t)$ together with its first differential quotient is continuous, $\lambda''(t)$ does not have infinitely many maxima and minima, and for $t = x$, $\lambda(t) = 1$, $\lambda'(t) = 0$, $\lambda''(t) = 0$, $\lambda'''(t)$ and $\lambda''''(t)$ are only finite and continuous.

If one sets $\lambda(t)$ outside the limits b, c equal to 1 and between these limits $= 1-\rho(t)$, which is obviously allowed, it follows that the difference between the series $A_1+\cdots+A_n$ and the integral

$$\frac{1}{2\pi}\int_{b}^{c}\left(F(t) - C't - A_0\,\frac{tt}{2}\right)\frac{dd\,\dfrac{\sin\dfrac{2n+1}{2}(x-t)}{\sin\dfrac{(x-t)}{2}}}{dt^2}\,\rho(t)\,dt$$

ultimately becomes infinitely small with increasing n. One is easily convinced by partial integration* that

*= *integration by parts*

$$\frac{1}{2\pi}\int_{b}^{c}\left(C't + A_0\,\frac{tt}{2}\right)\frac{dd\,\dfrac{\sin\dfrac{2n+1}{2}(x-t)}{\sin\dfrac{(x-t)}{2}}}{dt^2}\,\rho(t)\,dt$$

converges to A_0, when n becomes infinitely large, by which one obtains the above theorem.

10.

This investigation has shown that if the coefficients of the series Ω ultimately becomes infinitely small, then the convergence of the series for a certain value of x depends only on the behavior of the function $f(x)$ in the immediate neighborhood of this value.

Whether the coefficients of the series finally become infinitely small will in many cases not be determined from their integrals, but will have to be determined in some other way. *One* case deserves to be highlighted where this can be decided directly from the nature of the function, namely when the function $f(x)$ remains finite everywhere and allows integration.

In this case, when one breaks down the whole interval from $-\pi$ to π in sequence into pieces of magnitudes

$$\delta_1\,,\ \delta_2\,,\ \delta_3\,,\ldots,$$

and denotes by D_1 the largest variation of the function in first, by D_2 in second, etc.,

$$\delta_1 D_1 + \delta_2 D_2 + \delta_3 D_3 + \cdots,$$

become infinitely small as soon as all δ become infinitely small.

But if one decomposes the integral $\int_{-\pi}^{\pi} f(x) \sin n(x-a)\, dx$, in which form, apart from the factor $\frac{1}{\pi}$, the coefficients of the series are contained, or what is the same, $\int_a^{a+2\pi} f(x) \sin n(x-a)\, dx$ starting from $x = a$, into integrals of size $\frac{2\pi}{n}$, each of them makes a contribution to the sum that is smaller than $\frac{2}{n}$ multiplied by the largest variation in its interval, and their sum is therefore smaller than a quantity which with $\frac{2\pi}{n}$ must become infinitely small.

In fact: these integrals have the form

$$\int_{a+\frac{s}{n}2\pi}^{a+\frac{s+1}{n}2\pi} f(x)\ \sin n(x-a)\, dx$$

The sine will be positive in the first half, negative in the second. If one denotes the greatest value of $f(x)$ in the interval by M, the least by m; then it is obvious that one increases the integral when one replaces $f(x)$ with M in the first half and with m in the second half, but that one decreases the integral if one replaces $f(x)$ with m in the first half and by M in the second. In the first case, however, one gets the value $\frac{2}{n}(M-m)$, in the latter $\frac{2}{n}(m-M)$. Therefore the integral, apart from the sign, is less than $\frac{2}{n}(M-m)$ and the integral $\int_a^{a+2\pi} f(x)\sin n(x-a)\,dx$ is less than $\frac{2}{n}(M_1-m_1)+\frac{2}{n}(M_2-m_2)+\frac{2}{n}(M_3-m_3)+\cdots$, where one denotes by M_s the greatest, by m_s the least value of $f(x)$ in the s-th interval; however, this sum must, when $f(x)$ is capable of integration, become infinitely small, as soon as n becomes infinitely large and the size of the interval $\frac{2\pi}{n}$ infinitely small.

In the assumed case, therefore, the coefficients of the series ultimately become infinitely small.

11.

It remains now to investigate the case where the terms of the series Ω for the argument value x ultimately become infinitely small, without this happening for every argument value.

If one adds terms of the same rank in the series for the argument value $x+t$ and $x-t$, one obtains the series

$$2A_0 + 2A_1\cos t + 2A_2\cos 2t + \cdots,$$

in which the terms ultimately become infinitely small for each value of t and thus to which the previous investigation can be applied.

To this end, if we denote the value of the infinite series

$$C + C'x + A_0\frac{xx}{2} - A_1\frac{\cos t}{1} - A_2\frac{\cos 2t}{4} - A_3\frac{\cos 3t}{9} - \cdots$$

by $G(t)$, such that where the series for $F(x+t)$ and $F(x-t)$ converge, $\frac{F(x+t)+F(x-t)}{2}$ is $= G(t)$ everywhere, then the following results:

I. If the members of the series Ω finally become infinitely small for the argument value x, then

$$\mu\mu\int_b^c G(t)\;\cos\mu(t-a)\;\lambda(t)\;dt\,,$$

must, if λ denotes a function as above — Art. 9 — with increasing μ ultimately become infinitely small. The value of this integral is composed of the two components

$$\mu\mu\int_b^c \frac{F(x+t)}{2}\cos\mu(t-a)\;\lambda(t)\;dt \quad\text{and}\quad \mu\mu\int_b^c \frac{F(x-t)}{2}\cos\mu(t-a)\;\lambda(t)\;dt\,,$$

as long as these expressions have a value. The becoming infinitely small of it is therefore caused by the behavior of the function F at two points lying symmetrically on either side of x. It should be noted, however, that there must be places where each component is not infinitely small in itself; for otherwise the terms of the series would ultimately become infinitely small for each argument value. The contributions of the points lying symmetrically on both sides of x must then cancel each other out, so that their sum becomes infinitely small for an infinite μ. Hence it follows that the series Ω can converge only for such values of the quantity x at which the points, where $\mu\mu\int_b^c F(x)\;\cos\mu(x-a)\;\lambda(x)\;dx$ does not become infinitely small for an infinite μ, lie symmetrically [about x]. Apparently, therefore, only if the number of these points is infinitely large can the trigonometric series with coefficients that do not diminish infinitely converge for an infinite number of argument values.

Conversely, $A_n = -\,nn\,\frac{2}{\pi}\int_0^\pi \left(G(t) - A_0\,\frac{tt}{2}\right)\cos nt\;dt$ and thus becomes infinitely small with increasing n, if $\mu\mu\int_b^c G(t)\;\cos\mu(t-a)\;\lambda(t)\;dt$ always becomes infinitely small for an infinite μ.

II. When the terms of the series Ω ultimately become infinitely small for the argument value x, then it depends only on the path of the function $G(t)$ for an infinitely small t, whether the series converges or not, and indeed the difference between $A_0 + A_1 + \cdots + A_n$ and the integral

$$\frac{1}{\pi}\int_b^c G(t)\frac{dd\,\dfrac{\sin\dfrac{2n+1}{2}\,t}{\sin\dfrac{t}{2}}}{dt^2}\;\rho(t)\;dt$$

becomes ultimately infinitely small with increasing n, where b denotes a constant, no matter how small, between 0 and π, and $\rho(t)$ denotes a function such that $\rho(t)$ and

$\rho'(t)$ are always continuous and zero for $t = b$, $\rho''(t)$ does not have infinitely many maxima and minima, and for $t = 0$, $\rho(t) = 0$,† $\rho'(t) = 0$, $\rho''(t) = 0$, $\rho'''(t)$ and $\rho^{IV}(t)$ are only finite and continuous.

† $\rho(t)=1$ *earlier*

12.

The conditions for the representability of a function by a trigonometric series can admittedly still be somewhat restricted, and thus our investigations into the nature of the function can be carried out a little further without any special assumptions. Thus e.g., in the theorem last obtained, the condition that $\rho''(0) = 0$ can be omitted, if in the integral

$$\frac{1}{\pi}\int_b^c G(t)\frac{dd\,\dfrac{\sin\dfrac{2n+1}{2}t}{\sin\dfrac{t}{2}}}{dt^2}\,\rho(t)\,dt$$

$G(t)$ is replaced by $G(t) - G(0)$. But nothing essential is gained from this.

Therefore, as we turn to the consideration of special cases, we will first seek to give the investigation of a function that does not have an infinite number of maxima and minima the completion of which is possible, according to Dirichlet's work.

It has been remarked above that such a function can be integrated wherever it does not become infinite, and it is obvious that this can only occur for a finite number of argument values. Also Dirichlet's proof — that in the integral expression for the n-th member of the series and for the sum of its n first members, the contribution of all segments except those where the function becomes infinite and those infinitely close to the argument value of the series ultimately becomes infinitely small with increasing n; and that

$$\int_x^{x+b} f(t)\,\frac{\sin\dfrac{2n+1}{2}(x-t)}{\sin\dfrac{x-t}{2}}\,dt\,,$$

if $0 < b < \pi$ and $f(t)$ does not become infinite between limits of the integral, converges to $\pi\, f(x+0)$ for an infinite n — in fact leaves nothing to be desired, if one omits the unnecessary assumption that the function is continuous. So it only remains to investigate in which cases of these integral expressions the contribution of the locations, where the function becomes infinite, ultimately becomes infinitely small with increasing n. This investigation is not yet finished; but it is only occasionally shown by Dirichlet that this happens, provided that the function to be represented admits an integration, which is not necessary.

We have seen above that if the terms of the series Ω ultimately become infinitely small for each value of x, the function $F(x)$, whose second differential quotient is $f(x)$, must be finite and continuous, and that

$$\frac{F(x+\alpha) - 2F(x) + F(x-\alpha)}{\alpha}$$

always becomes infinitely small with α. Now, if $F'(x+t) - F'(x-t)^{\dagger}$ does not have infinitely many maxima and minima, then it must, when t becomes zero, converge towards a fixed limit value L or become infinitely large, and it is obvious that

† *erratum sic:* $F'(x-t)-F'(x-t)$

$$\frac{1}{\alpha}\int_0^{\alpha} \left(F'(x+t) - F'(x-t)\right)\, dt = \frac{F(x+\alpha) - 2F(x) + F(x-\alpha)}{\alpha}$$

then likewise must converge towards L or towards ∞ and can therefore only become infinitely small if $F'(x+t) - F'(x-t)$ converges towards zero. Therefore, when $f(x)$ becomes infinitely large for $x = a$, it must always be possible to integrate $f(a+t) - f(a-t)$ up to $t = 0$. It suffices for $\left(\int_b^{a-\varepsilon} + \int_{a+\varepsilon}^{c}\right) dx\, \left(f(x)\cos n\,(x-a)\right)$ to converge with decreasing ε and to become infinitely small with increasing n. Further because the function $F(x)$ is finite and continuous, so must $F'(x)$ up to $x = a$ admit an integration and $(x-a)F'(x)$ with $(x-a)$ become infinitely small, when this function does not have infinitely many maxima and minima; from which it follows that $\dfrac{d(x-a)F'(x)}{dx} = (x-a)f(x) + F'(x)$ and thus also $(x-a)f(x)$ can be integrated up to $x = a$. It is also possible, therefore, for $\int f(x)\sin n(x-a)\, dx$ to be integrated up to $x = a$, and so that the coefficients of the series ultimately become infinitely small, it is obviously necessary only that $\int_b^c f(x)\sin n(x-a)\, dx$, where $b < a < c$, ultimately become infinitely small with increasing n. If one sets $f(x)(x-a) = \varphi(x)$, then, when this function does not have infinitely many maxima and minima, for an infinite n

$$\int_b^c f(x)\sin n(x-a)\,dx = \int_b^c \frac{\varphi(x)}{x-a}\,\sin n(x-a)\,dx = \frac{\varphi(a+0)+\varphi(a-0)}{2}\,,$$

as Dirichlet has shown. Therefore $\varphi(a+t)+\varphi(a-t) = f(a+t)\,t - f(a-t)\,t$ must become infinitely small with t, and since $f(a+t)+f(a-t)$ can be integrated up to $t=0$ and consequently also $f(a+t)\,t + f(a-t)\,t$ must become infinitely small with t, so must $f(a+t)\,t$ as well as $f(a-t)\,t$ ultimately become infinitely small with decreasing t. Apart from functions that have infinitely many maxima and minima, it is therefore sufficient and necessary for the representability of the function $f(x)$ by a trigonometric series with coefficients diminishing infinitely that, when it becomes infinite for $x=a$, $f(a+t)\,t$ and $f(a-t)\,t$ can become infinitely small with t and $f(a+t)+f(a-t)$ can be integrated up to at $t=0$.

A function $f(x)$ that does not have infinitely many maxima and minima, since $\mu\mu\int_b^c F(x)\ \cos\mu(x-a)\ \lambda(x)\ dx$ does not become infinitely small at only a finite number of locations for an infinite μ, can be represented also at only a finite number of argument values by a trigonometric series whose coefficients do not ultimately become infinitely small, on which it is unnecessary to linger longer.

13.

Concerning the functions which have infinitely many maxima and minima, it is probably not superfluous to remark that a function $f(x)$ which has infinitely many maxima and minima can be capable of integration everywhere without being representable by a Fourier series. This happens e.g. when $f(x)$ between 0 and 2π is equal to

$$\frac{d\left(x^\nu \cos\dfrac{1}{x}\right)}{dx} \text{ and } 0<\nu<\tfrac{1}{2}\,.$$

Then in the integral $\int_0^{2\pi} f(x)\cos n(x-a)\,dx$ with increasing n, the contribution of that place where x is close to $= \sqrt{\frac{1}{n}}$ becomes, to speak generally, ultimately infinitely large, so that the ratio of this integral to

$$\tfrac{1}{2}\sin\left(2\sqrt{n} - na + \frac{\pi}{4}\right)\sqrt{\pi}n^{\frac{1-2\nu}{4}}$$

converges to 1, as will be found in the way to be given shortly. In order to generalize the example, whereby the essence of the matter emerges more, set

$$\int f(x)\,dx = \varphi(x)\cos\psi(x)$$

and assume that for an infinitely small x, $\varphi(x)$ is infinitely small and $\psi(x)$ becomes infinitely large, and moreover that these functions and their differential quotients are continuous and do not have an infinite number of maxima and minima. Then one will have

$$f(x) = \varphi'(x)\cos\psi(x) - \varphi(x)\psi'(x)\sin\psi(x)$$

and $\int f(x)\cos n\,(x-a)\,dx$ equal to the sum of four integrals

$$\tfrac{1}{2}\int \varphi'(x)\cos(\psi(x) \pm n\,(x-a))\,dx\,,\ -\tfrac{1}{2}\int \varphi(x)\psi'(x)\sin(\psi(x) \pm n\,(x-a))\,dx$$

Now consider, $\psi(x)$ being assumed positive, the term

$$-\tfrac{1}{2}\int \varphi(x)\psi'(x)\sin(\psi(x) + n\,(x-a))\,dx$$

and examine in this integral the place where the sign changes of the sine follow each other most slowly. If one sets

$$\psi(x) + n(x-a) = y\,,$$

then this happens where $\dfrac{dy}{dx} = 0$, and thus one gets

$$\psi'(\alpha) + n = 0$$

for $x = \alpha$. Thus one examines the behavior of the integral

$$-\tfrac{1}{2}\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \varphi(x)\psi'(x)\sin y\,dx$$

in the case that ε becomes infinitely small for an infinite n and introduce y here as a variable. If one sets

$$\psi(\alpha) + n(\alpha - a) = \beta\,,$$

then for a sufficiently small ε, we will have

$$y = \beta + \psi''(\alpha)\frac{(x-\alpha)^2}{2} + \cdots$$

and in particular, $\psi''(\alpha)$ is positive, since $\psi(x)$ becomes positive infinity for an infinitely small x; further, we will have

$$\frac{dy}{dx} = \psi''(\alpha)(x-\alpha) = \pm\sqrt{2\psi''(\alpha)\,(y-\beta)}\,,\ \text{according as } x-\alpha \gtrless 0$$

and

$$-\tfrac{1}{2}\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \varphi(x)\psi'(x)\sin y\,dx = \tfrac{1}{2}\Big(\int_{\beta-\psi''(\alpha)\frac{\varepsilon\varepsilon}{2}}^{\beta} - \int_{\beta}^{\beta+\psi''(\alpha)\frac{\varepsilon\varepsilon}{2}}\Big)\Big(\sin y\,\frac{dy}{\sqrt{y-\beta}}\Big)\frac{\varphi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}$$

$$= -\int_0^{\psi''(\alpha)\frac{\varepsilon\varepsilon}{2}} \sin(y+\beta)\frac{dy}{\sqrt{y}}\cdot\frac{\varphi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}\,.$$

If one lets the magnitude ε decrease with increasing m in such a way that ψ becomes infinitely large, then if $\int_0^\infty \sin(y+\beta)\frac{dy}{\sqrt{y}}$, which is known to be equal to $\sin\left(\beta+\frac{\pi}{4}\right)\sqrt{\pi}$, is not zero, apart from magnitudes of lesser order

$$-\tfrac{1}{2}\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \varphi(x)\psi'(x)\sin(\psi(x)+n\,(x-a))\;dx = \sin\Big(\beta+\frac{\pi}{4}\Big)\frac{\sqrt{\pi}\;.\;\varphi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}\,.$$

Therefore, if this quantity does not become infinitely small, the ratio of $\int_0^{2\pi} f(x)\cos n\,(x-a)\;dx$ to this quantity, since its other components become infinitely small, will converge to 1 as n increases infinitely.

If one assumes that $\phi(x)$ and $\psi'(x)$ for an infinitely small x have powers of x of degrees that are equal, respectively, $\phi(x)$ to x^ν and $\psi'(x)$ to $x^{-\mu-1}$ so it must be that $\nu > 0$ and $\mu \geq 0$, then for an infinite n,

$$\frac{\varphi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}$$

has the same degree as $\alpha^{\nu-\frac{\mu}{2}}$ and therefore is not infinitely small if $\mu \geq 2\nu$. But in general, when $x\,\psi'(x)$ or, what is identical to it, when $\dfrac{\psi(x)}{\log x}$ is infinitely large for an infinitely small x, $\phi(x)$ can always be assumed to be such that for an infinitely small x, $\phi(x)$ is infinitely small but

$$\varphi(x)\frac{\psi'(x)}{\sqrt{2\psi''(x)}} = \frac{\varphi(x)}{\sqrt{-2\frac{d}{dx}\frac{1}{\psi'(x)}}} = \frac{\varphi(x)}{\sqrt{-2\lim\frac{1}{x\psi'(x)}}}$$

is infinitely large, and consequently $\int_x f(x)\ dx$ can be extended to $x = 0$, while $\int_0^{2\pi} f(x)\cos n\,(x-a)\ dx$ is not infinitely small for an infinite n. As one sees in the integral $\int_x f(x)\ dx$ with the infinite diminishment of x, the increments of the integral, although their ratio to the changes in x increases very rapidly, cancel each other out, because of the rapid sign change of the function $f(x)$; but the addition of the factor $\cos n(x-a)$ causes these increments to add up.

But just as the Fourier series cannot converge for a function despite the general possibility of integration and even its term can ultimately become infinitely large — so despite the general impossibility of integration of $f(x)$ between any two values, no matter how close, can there be infinitely many values of x for which the series Ω converges.

An example is furnished by (nx) in the meaning taken above (Art. 6), the function given by the series

$$\sum_{1,\infty}\frac{(nx)}{n}\,,$$

which exists for every rational value of x and can be represented by the trigonometric series*

$$\sum_{1,\infty}^{n}\frac{\sum^{\theta}-(-)1^{\theta}}{n\pi}\sin 2\,n\,x\,\pi\,,$$

where θ runs over all the divisors of n, but which is not contained in any interval between finite limits, however small, and consequently does not permit integration anywhere.

Another example is obtained if one sets positive values for $c_0,\ c_1,\ c_2, \ldots$ in the series

* $-(-)1^{\theta}=-(-1)^{\theta}$

$$\sum_{0,\infty} c_n \cos nnx \qquad \sum_{0,\infty} c_n \sin nnx \,,$$

which always decrease and ultimately become infinitely small, while $\sum_{1,n}^{s} c_s$ becomes infinitely large with n. For if the ratio of x to 2π is rational and, expressed in the least numbers, a fraction with the denominator m, then these series will obviously converge or increase to infinity, depending on whether $\sum_{0,m-1} c_n \cos nnx$, $\sum_{0,m-1} c_n \sin nnx$ are equal to zero or not equal to zero. Both cases, however, occur according to a well-known theorem of the division of a circle[34] between any two limits, no matter how narrow, for infinitely many values of x.

To just the same extent, the series Ω can also converge without the value of the series

$$C' + A_0 x - \sum \frac{\frac{dA_n}{dx}}{nn} \,,$$

which is obtained by integrating each term in Ω, being integrable over an interval of any size, no matter how small.

For example, if one expands the expression

$$\sum_{1,\infty} \frac{1}{n^3}(1-q^n) \log \left(\frac{-\log(1-q^n)}{q^n} \right) \,,$$

where the logarithms are to be taken in such a way that they vanish for $q = 0$, in increasing powers of q and in it set $q = e^{xi}$, then the imaginary part forms a trigonometric series, which, differentiated twice according to x, converges infinitely often in each size interval, while its first differential quotient becomes zero infinitely often.

To the same extent, i.e. infinitely often between two argument values, no matter how close, the trigonometric series can also converge even if its coefficients do not ultimately become infinitely small. A simple example of such a series is the infinite series $\sum_{1,\infty} \sin(n!)x\pi$,* where $n!$, as usual, $= 1 \cdot 2 \cdot 3 \cdots n$, which not only converges for every rational value of x, turning into a finite one, but also for an infinite number

**I.e.* $\sin(n!\,\pi\,x)$

[34] Disquis. ar. pag. 636 art. 356

of irrational ones, the simplest of which are $\sin 1$, $\cos 1$, $\frac{2}{e}$ and its multiples, odd multiples of e, $\frac{e-\frac{1}{e}}{4}$, etc.

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