THEORY OF ANALYTIC FUNCTIONS,

CONTAINING

The principles of the differential calculus, freed from any consideration of infinitely small or of vanishing, of limits or of fluxions, and reduced to algebraic analysis of finite quantities[1]

By J. L. LAGRANGE, de l'Institute national.

FIRST PART.

Exposition of the Theory, with its principal usages in Analysis.

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1. One calls a *function* of one or several quantities any expression for calculating in which these quantities enter in any manner, mixed or not with other quantities that one regards as having given and invariable values, while the quantities of the function may receive any values possible. Therefore in the function one considers only the quantities that are supposed variable, without any regard for the constants that may be mixed with them.

2. To mark a function of a sole variable such as x, we will simply precede this variable with the letter or characteristic f, or F; but when on would designate the function of a quantity composed of this variable, such as x^2 or $a + bx$ or etc., one encloses this quantity between two parentheses. Therefore fx designates a function of x, $f(x^2)$, $f(a + bx)$, etc. designate functions of x^2, $a + bx$, etc.

To mark a function of two independent variables such as x, y, we will write $f(x, y)$ and so on.

When we would like to employ other characteristics to mark functions, we will take care to note it.

The word *function* was employed by the first analysts to designate in general the powers of the same quantity. Since then, one has extended the meaning of this word

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[1] J.L. Lagrange, *Théorie des fonctions analytiques*, Translated by Michael Rogers, 2022.

to any quantity formed in any manner of another quantity. *Leibniz* and *Bernoulli* were the first to use it in this general sense, and today it is generally adopted.

3. Let us consider, then, a function fx of some variable x. If in place of x one puts $x+i$, i being any indeterminate quantity, it will become $f(x+i)$; and by the theory of series, on would could develop it in a sequence of this form $fx+p\,i+q\,i^2+r\,i^3+$ etc., in which the quantities p, q, r, etc., coefficients of the powers of i, will be new functions of x, derived from the primitive function fx, and independent of the quantity i.

4. The manner of deducing other functions from a given function, depending essentially on the primitive function, is of the utmost importance in analysis. The formation and calculation of these different functions are properly speaking the real subject of the new calculus, that is to say, of the calculus called *differential* or *fluxional*. The first geometers who employed the differential calculus, *Leibniz*, the *Bernoullis*, *l'Hopital*, etc. had based it on the consideration of infinitely small quantities of different orders, and on the assumption that one can consider and treat as equal quantities that differed from each other only by infinitely small quantities with respect to themselves. Content with arriving at exact results quickly and reliably by the procedures of this calculus, they did not bother to demonstrate its principles. Those who followed them, *Euler*, d'*Alembert*, etc. sought to make up for this fault, by showing, by particular applications, that the differences which are supposed to be infinitesimally small, must be absolutely zero; and that their ratios, which are the only quantities which actually enter into calculus, are nothing other than the limits of the ratios of finite differences, or indefinite.

But it is necessary to admit that this idea, although correct in itself, is not clear enough to serve as a principle for a science whose certainty must be based on evidence, and especially to be presented to beginners; moreover, it seems to me that since in the differential calculus, as it is used, infinitely small quantities are considered and calculated, or are assumed to be infinitely small themselves, the true metaphysics of this calculus consists in the fact that the error resulting from this false assumption is rectified or compensated for by that which arises from the very procedures of the calculus, according to which one retains in differentiation only infinitely small quantities of the same order. For example, in considering a curve as a polygon of an infinite number of sides each infinitely small, and of which the prolongation is a tangent of the curve, it is clear that one makes an erroneous assumption; but the error is found to be corrected in the calculation by the omission that one makes of the infinitely small quantities. It is this that can be seen easily in examples, but it perhaps would be difficult to give a general demonstration.

5. *Newton*, to avoid the assumption of the infinitely small, considered mathematical quantities as generated by movement, and it sought a method for determining directly the speeds or rather the ratio of the variable speeds with which these quantities are produced; it is this that one calls, after him, the *method of fluxions* or *the fluxional calculus*, because he names these speeds *fluxions of quantities.* This method or this calculus agrees in substance and in operations with the differential calculus, and differs only by the metaphysics that appear in effect clearer, because everyone has or believes to have an idea of speed. But, on the one hand, to introduce movement into a calculus that has only algebraic quantities as its subject is to introduce into it a foreign idea, and which obliges us to consider these quantities as lines traverse by a moveable object; and on the other, it is necessary to admit that one does not have even a very clear idea what the speed of a point is at each instant, when the speed is variable; and on can see by the learned Treatise of fluxion of *Maclaurin*, how difficult it is to rigorously demonstration the method of fluxions, and how many particular tricks must be employed to demonstrate the different parts of this method.

Also *Newton* himself, in his book of Principles, preferred, as shorter, the method of the last ratios of vanishing quantities; and it must be admitted that it is to the principles of this method that the demonstrations relative to that of fluxions are reduced in the last analysis. But this method has, like that of limits of which we spoke above, and which is properly only the algebraic translation of it, the great inconvenience of considering quantities in the state where the cease to be, so to speak, quantities; for although one may conceive of the ratio of two quantities as long as they remain finite, this ratio no longer offers the mind a clear and precise idea, as soon as its two terms become both zero at the same time.

6. It is to prevent these difficulties that an able English geometer, who made important discoveries in analysis, lately proposed to substitute for the method of fluxions, until then scrupulously followed by all the English geometers, another method purely analytical, and analogous to the differential method, but in which, instead of employing only infinitely small differences or zeros of variable quantities, one first uses different values of these quantities, which one sets equal after having made disappear

by division the factor that this equality would make zero. By this means, one truly avoids the infinitely small and vanishing quantities; but the processes and applications of calculus are cumbersome and unnatural, and it must be admitted that this way of making the differential calculus more rigorous in its principles causes it to lose its principal advantages, the simplicity of the method and the ease of operations. *See* the work entitled: *The residual analysis, a new branch of the algebraic art by* **John Landen,** *London, 1764;* as well as the discourse published by the same author, in 1758, on the same subject.

These variations in the manner of establishing and presenting the principals of differential calculus, even in the name of this calculus, show, it seems to me, that we had not grasped the true theory, although we had first found the simplest and most convenient rules for the mechanism of operations.

7. In the memoire published among others by the academy of Berlin, in 1772, I advanced a theory of the development of functions in series containing the true principle of the differential calculus, freed of all consideration of the infinitely small, or of limits, and I proved with this theory the theorem of *Tailor* [Taylor], which one can consider as the fundamental principle of this calculus, and which had not yet been demonstrated except by the aid of this same calculation, or by the consideration of infinitely small differences.

Since then, *Arbogast* has presented to the academy of sciences a beautiful memoir, where the same idea is exposed with development and applications that pertain to it. This work must leave nothing to be desired on the object in question; but the author having not yet seen fit to publish it, and having found myself engaged by particular circumstances to develop the general principles of analysis, I recalled my old ideas on those of the differential calculus, and I have made new reflections tending to confirm and generalize them; determine to publish in only consideration of the usefulness it may be to those who study this important branch of analysis.

8. It may, moreover, appear surprising that this manner of looking at the differential calculus was not offered rather to geometers, and especially that it escaped *Newton*, inventor of the method of series and that of fluxions. But we observe in this regard that in effect, *Newton* had first employed the simple consideration of series for solving the third problem of the second book of the Principles, in which he seeks the

law of resistance necessary for a heavy body to describe freely a given curve; a problem which depends naturally on the differential or fluxional calculus. One knows that *Johann Bernoulli* found this solution false, in comparing it with that which results from the differential calculus; and his nephew *Nicolas* claimed that the error came from the fact that *Newton* had taken the third term of the convergent series in which he reduced the ordinate of the given curve, for the second differential of this ordinate, and the fourth for the third differential, whereas according to the rules of the differential calculus, these terms are, one only half, the other only the sixth part of the same differentials. (*See* the Memoires of the academy of sciences of 1711, and volume 1 of the Works of *Johann Bernoulli.*) *Newton*, without responding, abandoned entirely his first method, and gave in the second edition of the principles a different solution to the same problem, based on the same method of the differential calculus. Since then, no one has spoken of the application of the method of series to this kind of problem, except to warn of the mistake into which *Newton* had fallen, and to make felt the need to pay attention to the observation of *Nicolas Bernoulli.* (*See* l'Encyclopédie, the articles *differentiel*, *force.*) But we will show that this mistake does not come from the basis of the method, but simply from the fact that *Newton* did not take into account all the terms which had to be taken into account, and we will rectify in this way his first solution, which none of the commentators on the Principles mentioned.

9. We propose in this work to consider functions that arise from the development of any function. We will give the law of their formation and their derivation, and we will its use for the transformation of analytic expressions. We will then apply these derived functions to the main problems of geometry and mechanics that relate to the differential calculus, and we will thereby give to the solution of these problems the rigor of the old demonstrations. Finally, we will show the identity of this calculus of functions with the differential calculus properly speaking, and we will demonstrate by this means the principles and the known rules of the latter, in a manner independent of all assumptions and of all metaphysics.

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