E. Lindelöf, Sur l'application de la méthode des approximations successives aux équations différentielles ordinaires du premier ordre, *Comptes rendus hebdomadaires des séances de l'Académie des sciences* **116** (1894), pp. 454–457.

Translated by Michael Rogers.

CORRESPONDENCE.

MATHEMATICAL ANALYSIS. — *On the application of the method of successive approximations to the ordinary differential equations of the first order.* Note by M. Ernest Lindelöf, presented by M. Picard.

“ **1.** Consider a differential equation of the first order

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

“ We suppose that the function $f(x, y)$ is finite and continuous for all real values of x and y satisfying the inequalities $|x| < a$ and $|y| < b$. Further, one will have for $|y'| < b$, x and y lying in the same intervals,

$$|f(x, y') - f(x, y)| < k\,|y' - y|\,,^{*} \tag{2}$$

[*] *This condition is called a* **Lipschitz condition**.

k being a positive constant.

“ To find the integral of the equation (1) that vanishes at $x = 0$, one has, after the method of M. Picard[1], to form a sequence of functions $y_1, y_2, ..., y_n$, defined by the equations

$$\frac{dy_1}{dx} = f(x, 0), \quad \frac{dy_2}{dx} = f(x, y_1), \quad ..., \frac{dy_n}{dx} = f(x, y_{n-1}),$$

the constants of integration being determined so that these functions vanish at $x = 0$. M. Picard shows that the series

$$y_1 + (y_2 - y_1) + (y_3 - y_2) + \cdots + (y_n - y_{n-1}) + \cdots, \tag{3}$$

[1] *Traité d'Analyse,*, **II**, p. 301; *Sur l'application des méthodes d'approximations successives à l'étude de certaines équations differentielles ordinarires* (*Journal de Mathématiques,* p. 217; 1893).

converges uniformly and represents the integral we seek when x remains in the interval $(-h, h)$, h being the least of the three quantities

$$a, \frac{b}{M}, \frac{1}{k}, \tag{4}$$

M designating the maximum value of $|f(x,y)|$ for $|x| < a$ and $|y| < b$.

“ By modifying the proof of M. Picard, one can show that the last term, $\frac{1}{k}$, of the sequence (4) can be suppressed.

** The first two inequalities below on the far right should be $\leq$, but the ones for $n \geq 3$ should be strict ($<$).*

“ Let, in effect, M_0 be the maximum value of $|f(x,0)|$ for $|x| < a$ and denote by h_1 the least of the quantities a and $\frac{b}{M}$. So that $|x| < h_1$, the values of $y_1, y_2, \ldots$ will remain comprised between the limits $-b$ and $+b$ and one will have therefore

$$\begin{array}{rcllll}
|y_1| & = & \left|\int_0^x f(x,0)\, dx\right| & & < & \frac{M_0}{k}\,|k\,x| \\
|y_2 - y_1| & = & \left|\int_0^x [f(x,y_1) - f(x,0)]\; dx\right| & < \left|\int_0^x k\,|y_1|\, dx\right| & < & \frac{M_0}{k}\,\frac{|k\,x|^2}{2!} \\
\ldots & & & & & \ldots \\
|y_n - y_{n-1}| & = & \left|\int_0^x [f(x,y_{n-1}) - f(x,y_{n-2})]\; dx\right| & < \left|\int_0^x k\,|y_{n-1} - y_{n-2}|\, dx\right| & < & \frac{M_0}{k}\,\frac{|k\,x|^n}{n!} \\
\ldots & & & & & \ldots
\end{array}$$

“ The series (3) will therefore certainly be convergent and will represent the integral sought in the whole interval $(-h_1, h_1)$.[2]

“ One can still replace the limit $\frac{b}{M}$ by another which will be, in many cases, less restrictive than that. The preceding inequalities show us, in effect, that one will have $|y_i| < b$ $(i = 1, 2, \ldots)$, provided that one has

$$|x| < a, \quad \frac{M_0}{k}\,(e^{|kx|} - 1) < b\,,$$

conditions which are fulfilled in the interval $(-h_2, h_2)$, h_2 being the least of the quantities

[2] Since this Note has been presented to the Academy, I have learned that this same result has already been found, although in a less direct way, by M. I. Bendixon in Stockholm (*Öfversigt at Kongl. Vetenskaps-Akademiens föhandlingar,* 8 November 1893.)

$$a \quad \text{and} \quad \frac{1}{k}\log\left(1+\frac{k\,b}{M_0}\right).$$

One can thus confirm that the series (3) converges in the interval $(-h_2, h_2)$.

" It is well evident that this last interval will be, in certain cases, larger that the interval $(-h_1, h_1)$.

" **2.** The interval of convergence of the series (3) is, in general, limited. One can however indicate fairly extensive cases where this series is always convergent. Thus the integral will be finite and, what interests us more, will be represented by the same development for all values of x. This is what will take place in the two following cases.

" (a). The function $f(x, y)$ is continuous in the whole plane, and one can determine, for part of the plane contained between any two straight lines parallel to the y axis, a number k such that the inequality (2) is satisfied (the number k being able, moreover, to tend to infinity). Let, for example, the equation be

$$\frac{dy}{dx} = A(x) + B(x)y + C(x)\,\frac{y}{\sqrt{1+y^2}}\,,$$

$A(x)$, $B(x)$, $C(x)$ being any continuous functions for all values of x.

" (b). The function $f(x, y)$ is continuous in the whole plane; there exists a number k relative to any finite domain; in the band limited by any two parallels to the y axis, the absolute value of $f(x, y)$ remains less than a fixed number.

" **3.** Let us consider now the following case. Let $f(x, y)$ be a continuous function, positive for $x > 0$, $y > 0$, and suppose that, for any value x_1 of x, one can find a value y_1 such that, in the triangle formed by the straight lines

$$y = 0, \quad x = x_1, \quad y = \frac{y_1}{x_1}\,x, \tag{5}$$

one has $f(x, y) < \frac{y_1}{x_1}$. Then, in every finite domain there will be a corresponding number k. With these conditions supposed fulfilled, *the series (3) will be convergent for all positive values of x and will represent the integral of (1) that vanishes for $x = 0$.*

" Let us consider as an application the following equation

$$\frac{dy}{dx} = C + \frac{y^2}{a + x^2}\,, \tag{6}$$

where C and a are positive constants. In the triangle formed by the straight lines (5), the second member has for a maximum value, on putting $\frac{y_1}{x_1} = \alpha$,

$$C + \frac{\alpha^2}{1 + \frac{a}{x^2}},$$

a quantity smaller than $C + \alpha^2$ and which tends to this limit when x tends to infinity. Thus, for the integral of which the question is whether it is finite and continuous for all positive values of x, it is necessary and sufficient that on can find a real value for α satisfying the inequality $\alpha^2 + C \leq \alpha$, a condition that is realized for $C \leq \frac{1}{4}$. For these values of C, the integral curve is situated entirely between the two straight lines

$$y = 0 \quad \text{and} \quad y = \left(\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - C}\right) x$$

“ **4.** To conclude, here is another rather general theorem:

“ Let $f(x, y)$ be a continuous, positive function for $x > 0$, $y > 0$, and which is contantly increasing or constantly decreasing which y increases. I suppose moreover that there exists a number k relative to any finite domain. *If the equation (1) admits a finite continuous integral for $x > 0$, this will necessarily be furnished for us by the successive approximations of which the sequence will be convergent for all positive values of x.*

“ We have limited ourselves, in this study, to the case of only one equations and we have considered only the integral that vanishes for $x = 0$. However, it is evident that the preceding results extend to a system of any number of equations and to any initial values of the variables. ”

Observations on the preceding communication;
by M. **Émile Picard**.

" If one applies the method of successive approximations to the case where, in the equation

$$\frac{dy}{dx} = f(x, y)$$

the function f is holomorphic in x and y in the circles C and C' of radii a and b described around the points $x = 0$ and $y = 0$ as centers and has for a maximum absolute value M in these circles, the integral is found to be represented by the series

$$y_1 + (y_2 - y_1) + (y_3 - y_2) + \cdots + (y_n - y_{n-1}) + \cdots, \tag{7}$$

in which each term is holomorphic in the interior of a circle having the origin as center and a radius h, designating by h the least of the two quantities

$$a \quad \text{and} \quad \frac{b}{M}. \tag{8}$$

" In applying here the happy modification by M. Lindelöf in the preceding Note, one sees very easily that the series (7) will represent a holomorphic function in the interior of the circle of radius h, and in this fashion it will be found that the radius of convergence of the integral vanishing at $x = 0$ will be the least of the quantities (8). I have established the result at another time, but in a manner less rapid (*Bulletin des Sciences mathématiques*, 1888). One knows that the calculation of the limits leads to a smaller radius. "