<u>Mean Value Theorem</u>

- **Goals**: Explore how properties of $f(x)$ determine properties of $f'(x)$ and vice-versa, using Rolle's Theorem and the Mean Value Theorem.
- **Rolle's Theorem**: Let f be a function that satisfies the following:
  1. f is continuous on $[a, b]$.
  2. f is differentiable on (a, b).
  3. $f(a) = f(b)$.

  Then there is a number c in (a, b) such that $f'(c) = 0$.

Ex 1) Verify that $f(x) = 3x^2 - 12x + 5$ satisfies the hypotheses of Rolle's theorem on $[1,3]$. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

Ex 2) Show that the equation $2x + \cos x = 0$ has exactly one real root (Hint: Use the Intermediate Value Theorem on $[-1,0]$).

- **Mean Value Theorem**: Let f be a function that satisfies the following:
    1. f is continuous on $[a, b]$.
    2. f is differentiable on (a, b).

  Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \text{ or equivalently, } f'(c)(b-a) = f(b) - f(a).$$

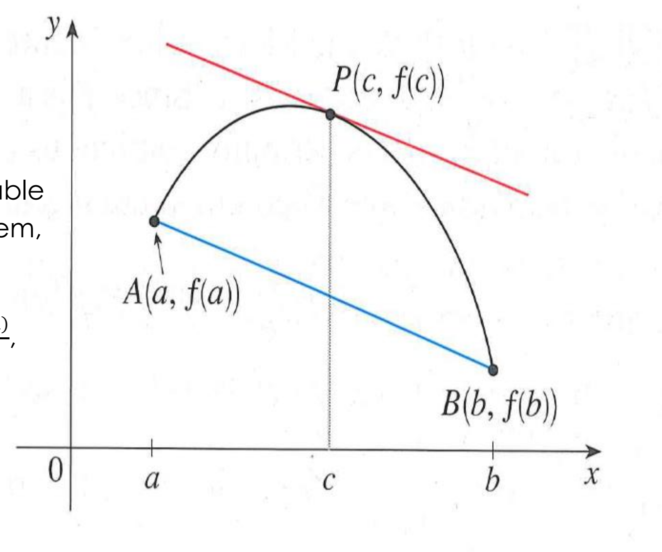
    - Graphically, there is a number c in (a, b) where the tangent line at $(c, f(c))$ and secant line from $(a, f(a))$ to $(b, f(b))$ have the same slope.

Proof: The equation of the secant line is

$g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$. Define $h(x)$ by
$h(x) = f(x) - g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$.

Then $h(x)$ is continuous on $[a, b]$, $h(x)$ is differentiable on (a, b), and $h(a) = 0 = h(b)$. So by Rolle's Theorem, there is some c in (a, b) such that $h'(c) = 0$.

But $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Thus $0 = f'(c) - \frac{f(b)-f(a)}{b-a}$, i.e. $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Ex 3) Verify that the function $f(x) = \frac{1}{x}$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1,3]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Ex 4) At 9:00AM, a car's odometer reads 40,500 mi. At 10:30AM it reads 40,635. Show that at some time between 9:00 and 10:30 the car was traveling at exactly 90mph.

- **Theorem**: If $f'(x) = 0$ for all x in an interval (a, b), then $f(x)$ is constant on (a, b).

Proof: Let x_1 and x_2 be in (a, b) with $x_1 < x_2$. Then $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). So the Mean Value Theorem holds. But $f'(c) = 0$ for the number c satisfying the conclusion of the Mean Value Theorem. So

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$0 = f(x_2) - f(x_1)$$

$$f(x_1) = f(x_2)$$

- **Corollary**: If $f'(x) = g'(x)$ for all x in an interval (a, b), then there is some number k such that $f(x) = g(x) + k$ on (a, b).