ADJUNCTIONS

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1. Preliminaries

2. Adjunctions

2.1. **Definition of adjunction.** Adjunctions appear all around mathematics. We will see why they should be considered a foundational concept. MacLane [Lan78] wrote that *"adjunctions arise everywhere"*.

Definition 1. An **adjunction** from categories $\mathbf{X}$ to $\mathbf{Y}$ is a pair of functors $F\colon \mathbf{X} \to \mathbf{Y}$ and $G\colon \mathbf{Y} \to \mathbf{X}$ endowed with bijections $\varphi_{X,Y}\colon \mathbf{Y}(FX,Y) \cong \mathbf{X}(X,GY)$ such that, for every $f \in \mathbf{Y}(FX,Y)$, every $h \in \mathbf{X}(X',X)$ and every $k \in \mathbf{Y}(Y,Y')$,

$$\varphi(f) \circ h = \varphi(f \circ Fh), \quad \text{and} \quad Gk \circ \varphi(f) = \varphi(k \circ f).$$

These are called the *naturality* conditions. We say that F is *left-adjoint* to G and that G is *right-adjoint* to F, and we denote this as $F \dashv G$.

Lawvere [Law] proposed a different and maybe more intuitive way to write adjunctions as logical diagrams. An adjunction $F \dashv G$ can be written as

$$\frac{FX \xrightarrow{\ f\ } Y}{X \xrightarrow{\varphi(f)} GY}$$

to emphasize that, for each morphism $f\colon FX \to Y$, there exists a unique morphism $\varphi(f)\colon X \to GY$; written in a way that resembles bidirectional logical inference rules. Naturality, in this setting, means that precomposition and postcomposition of arrows are preserved by the *inference rule*. Given morphisms $h\colon X' \to X$ and $k\colon Y \to Y'$, naturality means precisely that the composed arrows of the following diagrams are adjoint to one another.

$$\dfrac{\begin{tikzcd} FX' \arrow[r, "Fh"] \arrow[rr, bend left, "f\circ Fh"] & FX \arrow[r, "f"] & Y \end{tikzcd}}{\begin{tikzcd} X' \arrow[r, "h"'] \arrow[rr, bend right, "\varphi(f)\circ h"'] & X \arrow[r, "\varphi(f)"'] & GY \end{tikzcd}} \qquad \dfrac{\begin{tikzcd} FX \arrow[r, "f"] \arrow[rr, bend left, "k\circ f"] & Y \arrow[r, "k"] & Y' \end{tikzcd}}{\begin{tikzcd} X \arrow[r, "\varphi(f)"'] \arrow[rr, bend right, "Gk\circ\varphi(f)"'] & GY \arrow[r, "Gk"'] & GY' \end{tikzcd}}$$

2.2. **Units and counits.** In the following two propositions, we will characterize all this information in terms of natural transformations made up of universal arrows.

Definition 2. Let $F \dashv G$ be an adjunction. We define the **unit** η to be a family of morphisms $\eta_X\colon X \to GFX$ defined by $\eta_X = \varphi(\mathrm{id}_{FX})$. We define the **counit** ε to be a family of morphisms $\varepsilon_Y\colon FGY \to Y$ defined by $\varepsilon_Y = \varphi^{-1}(\mathrm{id}_{GY})$.

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$$\frac{FX \xrightarrow{\mathrm{id}} FX}{X \xrightarrow{\eta_X} GFX} \qquad \frac{FGY \xrightarrow{\varepsilon_Y} Y}{GY \xrightarrow{\mathrm{id}} GY}$$

Proposition 3. *The unit and the counit satisfy the* triangle identities*.*

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] \arrow[dr, equal] & GFGY \arrow[d, "G\varepsilon_Y"] \\ & GY \end{tikzcd} \qquad \begin{tikzcd} FGFX \arrow[d, "\varepsilon_{FX}"'] & FX \arrow[l, "F\eta_X"'] \arrow[dl, equal] \\ FX & \end{tikzcd}$$

Proof. We are going to show that both the identity and the morphisms on the side of the triangle are adjoints to an intermediate morphism. Because the adjunction is an isomorphism, they must be the same.

$$\begin{array}{c} GY \xrightarrow{\eta_{GY}} GFGY \xrightarrow{G\varepsilon_Y} GY \\ \hline FGY \xrightarrow{\mathrm{id}_{FGY}} FGY \xrightarrow{\varepsilon_Y} Y \\ \hline GY \xrightarrow{\mathrm{id}_{GY}} GY \xrightarrow{\mathrm{id}_{GY}} GY \end{array} \qquad \begin{array}{c} FX \xrightarrow{F\eta_X} FGFY \xrightarrow{\varepsilon_{FX}} FX \\ \hline X \xrightarrow{\eta_X} GFX \xrightarrow{\mathrm{id}_{GFX}} GFX \\ \hline FX \xrightarrow{\mathrm{id}_{FX}} FX \xrightarrow{\mathrm{id}_{FX}} FX \end{array}$$

$\square$

2.3. Adjoints are unique.

Proposition 4. *Let $G\colon \mathbf{Y} \to \mathbf{X}$ be a functor with two adjoints, $\varphi\colon F \dashv G$ and $\varphi'\colon F' \dashv G$. Then they are naturally isomorphic, $F \cong F'$.*

Proof. We define $\theta_X \colon FX \to F'X$ to be $\theta_X = \varphi'(\eta)$. We also define $\theta'_X \colon F'X \to FX$ to be $\theta_X = \varphi(\eta')$. We only need to prove that they are mutually inverses.

$$\begin{array}{cl} F'Y \xrightarrow{\theta} FY \xrightarrow{\theta'} F'Y & \\ \cline{1-1} Y \xrightarrow{\eta_Y} GFY \xrightarrow{G\theta'_Y} GF'Y & (\varphi') \\ \cline{1-1} FY \xrightarrow{\mathrm{id}_{FY}} FY \xrightarrow{\theta'_Y} F'Y & (\varphi) \\ \cline{1-1} Y \xrightarrow{\mathrm{id}_{FY}} Y \xrightarrow{\eta'_Y} GF'Y & (\varphi) \\ \cline{1-1} F'Y \xrightarrow{\mathrm{id}_{F'Y}} F'Y \xrightarrow{\mathrm{id}_{F'Y}} F'Y & (\varphi') \end{array}$$

$\square$

2.4. Adjoints compose.

Theorem 5 (Composition of adjunctions)**.** *Given two adjunctions $\varphi\colon \mathbf{Y}(FX, Y) \cong \mathbf{X}(X, GY)$ and $\varphi'\colon \mathbf{Z}(F'Y, Z) \cong \mathbf{Y}(Y, G'Z)$ the composition gives an adjunction $\varphi \circ \varphi' \colon \mathbf{Z}(F'FX, Z) \cong \mathbf{X}(X, GG'Z)$.*

3. Examples

3.1. Free monoids. Let us consider the category **Mon** of monoids and monoid homomorphisms. The functor $U\colon \mathbf{Mon} \to \mathbf{Sets}$ that forgets monoid structure has a left adjoint $F\colon \mathbf{Sets} \to \mathbf{Mon}$ called the *free monoid functor*.

Proof. Let $A \in \mathbf{Sets}$ be any set and let $(M, e, \cdot) \in \mathbf{Mon}$ be a monoid.

$$\frac{FA \xrightarrow{f} M}{A \xrightarrow{\varphi(f)} UM}$$

Here FA is a monoid whose underlying set are words $\{a_1 \dots a_n \mid a_i \in A\}$, the unit is the empty word and the binary operation is the concatenation of words. Assume a morphism $f \colon FA \to M$ from the free monoid over A, we will show that it can be reconstructed from $\varphi(f) \colon A \to M$, its restriction to the set. □

References

[Lan78] Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, 1978.

[Law] F. William Lawvere. Use of Logical Operators in mathematics. *Lecture notes in Linear Algebra, 309.*